2

SYSTEM DYNAMICS PROBLEMS WITH RATE PROPORTIONAL TO AMOUNT

MODULE 2.1

System Dynamics Tool—Tutorial 1

Download

From the textbook's website, download Tutorial 1 in PDF format for your system dynamics tool. We recommend that you work through the tutorial and answer all Quick Review Questions using the corresponding software.

Introduction

Dynamic systems, which change with time, are usually very complex, having many components, with involved relationships. Two examples are systems involving competition among different species for limited resources and the kinetics of enzymatic reactions.

With a system dynamics tool, we can model complex systems using diagrams and equations. Thus, such a tool helps us perform Step 2 of the modeling process—formulate a model—by helping us document our simplifying assumptions, variables, and units; establish relationships among variables and submodels; and record equations and functions. Then, a system dynamics tool can help us solve the model—Step 3 of the modeling process—by performing simulations using the model and generating tables and graphs of the results. We use this output to perform Step 4 of the modeling process—verify and interpret the model's solution. Often such examination leads us to change a model. With its graphical view and built-in functions, a system dynamics tool facilitates cycling back to an earlier step of the modeling process to simplify or refine a model. Once we have verified and validated a model, the tool's diagrams and equations from the design and the results from the simulation should be part of our report, which we do in Step 5 of the modeling process. The tool can even help us as we maintain the model (Step 6) by making corrections, improvements, or enhancements.

This first tutorial is available for download from the textbook's website for several different system dynamics tools. Tutorial 1 in your system of choice prepares you to perform basic modeling with such a tool, including the following:

- Diagramming a model
- Entering equations and values
- Running a simulation
- Constructing graphs
- Producing tables

The module gives examples and Quick Review Questions for you to complete and execute with your desired tool.

MODULE 2.2

Unconstrained Growth and Decay

Introduction

Many situations exist where the rate at which an amount is changing is proportional to the amount present. Such might be the case for a population of people, deer, or bacteria, for example. When money is compounded continuously, the rate of change of the amount is also proportional to the amount present. For a radioactive element, the amount of radioactivity decays at a rate proportional to the amount present. Similarly, the concentration of a chemical pollutant decays at a rate proportional to the concentration of pollutant present.

Rate of Change

We deal with rate of change every time we drive a car. Suppose our position (y) is a function (s) of time (t), so we write $y = s(t)$. Suppose also that we start driving on a straight road at time $t = 0$ hours (h) at position marker $s(0) = 10$ miles (mi; about 16.1 km), and at time $t = 2$ h we are at position $s(2) = 116$ mi (about 186.7 km). Our **average velocity**, or average rate of change of position with respect to time, is the **change in position** (Δs) over the **change in time** (Δt) and incorporates average speed as well as direction by its sign:

$$\textbf{average velocity} = \frac{\Delta s}{\Delta t} = \frac{116 \text{ mi} - 10 \text{ mi}}{2 \text{ h} - 0 \text{ h}} = \frac{106 \text{ mi}}{2 \text{ h}} = 53 \text{ mi/h}$$

or

$$\textbf{average velocity} = \frac{\Delta s}{\Delta t} = \frac{186.7 \text{ km} - 16.1 \text{ km}}{2 \text{ h} - 0 \text{ h}} = \frac{170.6 \text{ km}}{2 \text{ h}} = 85.3 \text{ km/h}$$

We probably are not driving at a constant rate of 53 mi/h (85.3 km/h), but sometimes we are moving faster and other times, slower. To obtain a more accurate measure of our velocity at time $t = 1$ h, we can use a smaller interval. For instance, at time $t = 1$ h, our position might be at marker $s(1) = 51.2$ mi, while a short time before at $t = 0.98$ h, our position was $s(0.98) = 50.0$ mi. As the following calculation shows, over this interval of 0.02 h (1.2 min), our average velocity is faster, 60 mi/h:

$$\text{average velocity} = \frac{\Delta s}{\Delta t} = \frac{51.2 \text{ mi} - 50 \text{ mi}}{1.00 \text{ h} - 0.98 \text{ h}} = \frac{1.2 \text{ mi}}{0.02 \text{ h}} = 60 \text{ mi/h}$$

or about 96.6 km/h.

Definition Suppose $s(t)$ is the position of an object at time t, where $a \le t \le b$. Then the **change in time**, $\boldsymbol{\Delta t}$, is $\boldsymbol{\Delta t = b - a}$; and the **change in position**, $\boldsymbol{\Delta s}$, is $\boldsymbol{\Delta s = s(b) - s(a)}$. Moreover, the **average velocity**, or the **average rate of change of $\boldsymbol{s}$ with respect to $\boldsymbol{t}$**, of the object from time $a = b - \Delta t$ to time b is

$$\textbf{average velocity} = \frac{\textbf{change in position}}{\textbf{change in time}} = \frac{\Delta s}{\Delta t}$$

$$= \frac{s(b) - s(a)}{b - a} = \frac{s(b) - s(b - \Delta t)}{\Delta t}$$

Quick Review Question 1

Suppose on a windless day someone standing on a bridge holds a ball over the side and tosses the ball straight up into the air. After reaching its highest point, the ball falls, eventually landing in the water. The ball's height in meters (m) above the water (y) is a function (s) of time (t) in seconds (s), or $y = s(t)$.

a. Determine the average velocity with units of the ball from $t = 1$ s to $t = 2$ s if $s(1) = 21.1$ m and $s(2) = 21.4$ m.
b. Determine the average velocity with units of the ball from $t = 1$ s to $t = 3$ s if $s(1) = 21.1$ m and $s(3) = 11.9$ m.
c. Using the notation of the definition of average velocity, for Part b determine the following, including units: b, $s(b)$, Δt, $b - \Delta t$, $s(b - \Delta t)$, Δs.

By making the interval smaller and smaller around the time $t = 1$ h, the average velocity calculation approaches our precise velocity at $t = 1$ h, or our **instantaneous rate of change of position with respect to time**, which is our odometer's reading. This instantaneous rate of change of s with respect to t is the **derivative** of s with respect to t, written as $\boldsymbol{s'(t)}$, or $\boldsymbol{\frac{dy}{dt}}$, or $\boldsymbol{dy/dt}$; and $\boldsymbol{s'(1)}$, or $\left.\boldsymbol{\frac{ds}{dt}}\right|_{t=1}$, indicates the derivative at time $t = 1$ h.

Definition The **instantaneous velocity**, or the **instantaneous rate of change of *s* with respect to *t***, at $t = b$ is the number the average velocity, $\frac{s(b) - s(b - \Delta t)}{\Delta t}$, approaches as Δt comes closer and closer to 0 (provided the ratio approaches a number). In this case, the **derivative of $y = s(t)$ with respect to *t*** at $t = b$, written $s'(b)$ or $\left.\frac{dy}{dt}\right|_{t=b}$, is the instantaneous velocity at $t = b$. In general, the **derivative of $y = s(t)$ with respect to *t*** is written as $s'(t)$, or $\frac{dy}{dt}$, or dy/dt.

A function, such as $y = s(t)$, can represent many things other than position. Moreover, we are not restricted to using symbols, such as s. For example, $Q(t)$ might represent a quantity (mass) of radioactive carbon-14 at time t, and the instantaneous rate of change of Q with respect to t, $Q'(t) = dQ/dt$, is the instantaneous rate of decay. As another example, $P(t)$ might symbolize a population at time t, so that $P'(t) = dP/dt$, is the rate of change of the population with respect to t.

Differential Equation

Continuing with the population example, suppose we have a population in which no individuals arrive or depart; the only change in the population comes from births and deaths. No constraints, such as competition for food or a predator, exist on growth of the population. When no limiting factor exists, we have the **Malthusian model** for unconstrained population growth, where the rate of change of the population is **directly proportional** ($\propto$) to the number of individuals in the population. If P represents the population and t represents time, then we have the following proportion:

$$\frac{dP}{dt} \propto P$$

For a positive growth rate, the larger the population, the greater the change in the population. With the same positive growth rate in two cities, say New York City and Spartanburg, S.C., the population of the larger New York City increases more in magnitude in a year than that of Spartanburg. In a later section of this module, "Unconstrained Decay," we consider a situation in which the rate is negative.

We write the preceding proportion in equation form as follows:

$$\frac{dP}{dt} = rP$$

The constant r is the **growth rate**, or **instantaneous growth rate**, or **continuous growth rate**, while dP/dt is the **rate of change of the population**.

In “System Dynamics Tool—Tutorial 1” (Module 2.1), we started with a bacterial population of size 100, an instantaneous growth rate of 10% = 0.10, and time measured in hours. Thus, we had

$$\frac{dP}{dt} = 0.10P$$

with $P_0 = 100$. The equation $\frac{dP}{dt} = 0.10P$ with the **initial condition** $P_0 = 100$ is a **differential equation** because it contains a derivative. A **solution** to this differential equation is a function, $P(t)$, whose derivative is $0.10P(t)$, with $P(0) = 100$. We begin by reconsidering this example from Tutorial 1 for reinforcement and a more in-depth examination of the concepts.

Definitions A **differential equation** is an equation that contains one or more derivatives. An **initial condition** is the value of the dependent variable when the independent variable is zero. A **solution** to a differential equation is a function that satisfies the equation and initial condition(s).

Difference Equation

Each diagram in Figure 2.2.1, developed with a choice of modeling tools and with the generic format employed by the text, depicts the situation with *population* indicating P, *growth_rate* representing r, and *growth* meaning dP/dt. A **stock** (**box variable**, or **reservoir**), such as *population*, accumulates with time. By contrast, a **converter** (**variable-auxiliary/constant**, or **formula**), such as *growth_rate*, does not accumulate but stores an equation or a constant. The growth is the additional number of organisms that join the population. Thus, a **flow** (**rate**), such as *growth*, is an activity that changes the magnitude of a stock and represents a derivative. Because both population and growth rate are necessary to determine the growth, we have **arrows** (**connectors**, or **arcs**) from these quantities to the flow indicator.

For a simulation with a system dynamics tool or a program we write, we consider time advancing in small, incremental steps. For time, t, and length of a time step, Δt, the **previous time** is $t - \Delta t$. Thus, if t is 7.75 s and Δt is 0.25 s, the previous time is 7.50 s. A system dynamics tool might call the change in time dt, DT, or something else instead of Δt. As some tools do to avoid confusion, we replace each blank in a diagram component name with an underscore when using the name in equations and discussions. For example, we employ *growth rate* in the diagrams of Figure 2.2.1 and the corresponding *growth_rate* in the following discussion. Regardless of the notation, with initial *population* = 100, *growth_rate* = 0.1, and *growth* = *growth_rate* * *population*, as in Figure 2.2.1, a system dynamics tool generates an equation similar to the following, where *population*(t) is the population at time t and *population*($t - \Delta t$) is the population at time $t - \Delta t$:

$$population(t) = population(t - \Delta t) + (growth) * \Delta t$$

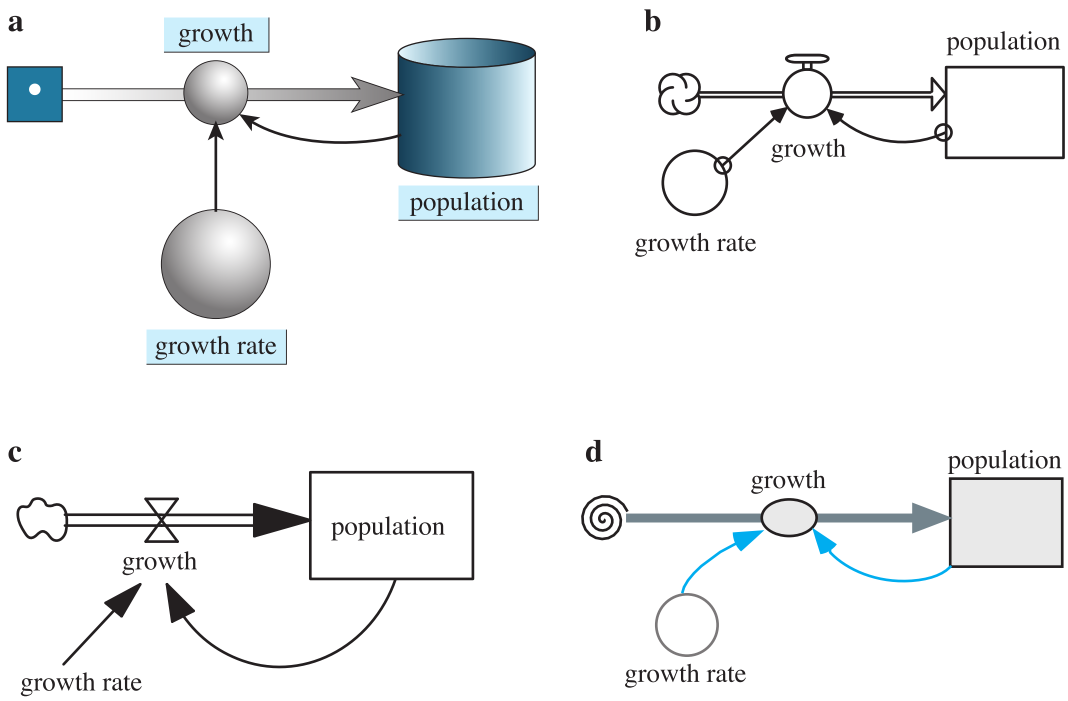

Figure 2.2.1 Diagrams of population models where growth rate is proportional to population: (a) *Berkeley Madonna*® (b) *STELLA*® (c) *Vensim PLE*® (d) Text's format

This equation, called a **finite difference equation**, indicates that the population at one time step is the population at the previous time step plus the change in population over that time interval:

$$\text{(new population)} = \text{(old population)} + \text{(change in population)}$$

or

$$population(t) = population(t - \Delta t) + \Delta population$$

where **Δpopulation** is a notation for the **change in population**. We approximate the change in the population over one time step, $\Delta population$ or $(growth) * \Delta t$, as the finite difference of the populations at one time step and at the previous time step, $population(t) - population(t - \Delta t)$. Thus, solving for *growth*, we have an approximation of the derivative dP/dt as follows:

$$growth = \frac{\Delta population}{\Delta t} = \frac{population(t) - population(t - \Delta t)}{\Delta t}$$

Computer programs and system dynamics tools employ such finite difference equations to solve differential equations.

Definition A **finite difference equation** is of the following form:

(new value) = (old value) + (change in value)

Such an equation is a discrete approximation to a differential equation.

Quick Review Question 2

Consider the differential equation $dQ/dt = -0.0004Q$, with $Q_0 = 200$.

a. Using delta notation, give a finite difference equation corresponding to the differential equation.
b. At time $t = 9.0$ s, give the time at the previous time step, where $\Delta t = 0.5$ s.
c. If $Q(t - \Delta t) = 199.32$ and $Q(t) = 199.28$, give ΔQ.

The *growth* is the *growth_rate* (r previously) times the current *population* (P previously). For example, we can show that the population at time $t = 0.025$ h is approximately *population*(0.025) = 100.250250 bacteria, so that *growth* is about *growth_rate* * *population*(0.025) = 0.1 * 100.250250 = 10.025025 bacteria per hour at that instant. For $\Delta t = 0.005$ h, the change in the population of bacteria to the next time step, 0.025 + 0.005 = 0.030 h, is approximately *growth* * Δt = 10.025025 * 0.005 = 0.050125 bacteria[1]. We calculate the population at time 0.030 h as follows:

$$\begin{aligned} population(0.030) &= population(0.025) + (growth \text{ at time } 0.025 \text{ h}) * \Delta t \\ &= 100.250250 + 10.025025 * 0.005 \\ &= 100.250250 + 0.050125 \\ &= 100.300375 \end{aligned}$$

Thus, we compute the value at the line $t = 0.030$ h of Table 2.2.1 using the previous line.

Quick Review Question 3

Evaluate *population*(0.045), the population at the next time interval after the end of Table 2.2.1, to six decimal places.

Table 2.2.1
Table of Estimated Populations, Where the Initial Population is 100, the Continuous Growth Rate is 10% per Hour, and the Time Step is 0.005 h

t	*population*(t)	=	*population*$(t - \Delta t)$	+	(*growth*)	*	Δt
0.000	100.000000						
0.005	100.050000	=	100.000000	+	10.000000	*	0.005
0.010	100.100025	=	100.050000	+	10.005000	*	0.005
0.015	100.150075	=	100.100025	+	10.010003	*	0.005
0.020	100.200150	=	100.150075	+	10.015008	*	0.005
0.025	100.250250	=	100.200150	+	10.020015	*	0.005
0.030	100.300375	=	100.250250	+	10.025025	*	0.005
0.035	100.350525	=	100.300375	+	10.030038	*	0.005
0.040	100.400701	=	100.350525	+	10.035053	*	0.005

[1] Computations in this model use **Euler's Method** for estimating values of a function. In Chapter 6, we examine this and two other techniques for numeric integration.

Because of compounding, the number of bacteria at $t = 1$ h is slightly more than 10% of 100, namely, 110.51. Table 2.2.2 lists the growth and the population on the hour for 20 h, and Figure 2.2.2 graphs the population versus time. The model states and the table and figure illustrate that as the population increases, the growth does, too.

The model gives an estimate of the population at various times. If the model is analytically correct, a simulation estimates the values for *growth* and *population*. Until computer round-off error (discussed in Module 5.2) causes the step size to be zero, it is usually the case that the smaller the step size, the more accurate will be the results. (In Exercise 9, we explore a situation where the smaller step size does not produce better results.) Because the additional computations resulting from a smaller step size cause the simulation to run longer, we often use a larger Δt during development and switch to a smaller Δt for more accurate results when the project is close to completion.

Table 2.2.2
Table of Estimated Growths and Populations, Reported on the Hour, Where the Initial Population is 100, the Growth Rate is 10%, and the Time Step is 0.005 h

t (h)	*growth*	*population*
0.000	10.00	100.00
1.000	11.05	110.51
2.000	12.21	122.13
3.000	13.50	134.98
4.000	14.92	149.17
5.000	16.49	164.85
6.000	18.22	182.18
7.000	20.13	201.34
8.000	22.25	222.51
9.000	24.59	245.90
10.000	27.18	271.76
11.000	30.03	300.33
12.000	33.19	331.91
13.000	36.68	366.81
14.000	40.54	405.38
15.000	44.80	448.00
16.000	49.51	495.11
17.000	54.72	547.16
18.000	60.47	604.69
19.000	66.83	668.27
20.000		738.54

Rule of Thumb Although the simulation takes longer because of more computation, it is usually more accurate to use a small step size (Δt), say, 0.01 or less.

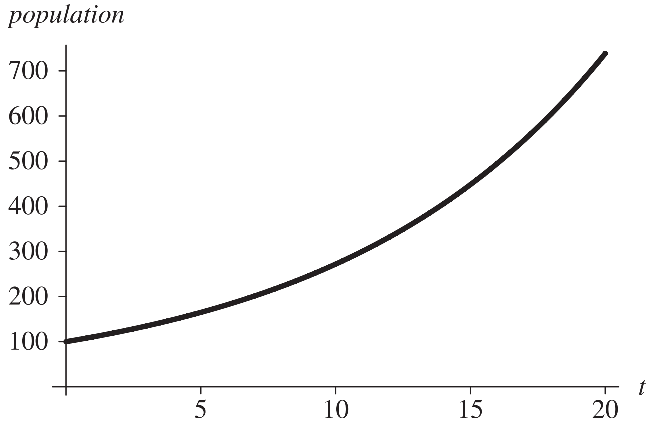

Figure 2.2.2 Graph of population versus time (hours) for the data in Table 2.2.2

Simulation Program

In developing a simulation program, we use statements similar to the preceding finite difference equations. We initialize constants, such as *growthRate*, *population*, Δt, and the length of time the simulation is to run (*simulationLength*), and we update the population repeatedly in a loop. The calculation for the total number of iterations (*numIterations*) of this loop is *simulationLength*/Δt. For example, if the simulation length is 10 h and Δt is 0.25 h, then the number of loop iterations is *numIterations* = 10/0.25 = 40. We have a loop index (i) go from 1 through *numIterations*. Inside the loop, we calculate time t as the product of i and Δt. For example, if Δt is 0.25 h, during the first iteration, the index i becomes 1 and the time is 1 * Δt = 0.25 h. On loop iteration i = 8, the time gets the value 8 * Δt = 8 * 0.25 h = 4.00 h.

Algorithm 1 contains **pseudocode**, or a structured English outline of the design, for generating and displaying the time, growth, and population at each time step. In the algorithm, a **left-facing arrow** (←) indicates assignment of the value of the expression on the right to the variable on the left. For example, *numIterations* ← *simulationLength*/Δt represents an assignment statement in which *numIterations* gets the value of *simulationLength*/Δt.

Algorithm 1 Algorithm for simulation of unconstrained growth

initialize *simulationLength*
initialize *population*
initialize *growthRate*
initialize length of time step Δt
numIterations ← *simulationLength*/Δt
for i going from 1 through *numIterations* do the following:
    growth ← *growthRate* * *population*
    population ← *population* + *growth* * Δt
    t ← i * Δt
    display t, *growth*, and *population*

If we do not need to display *growth* (derivative) at each step and the length of a step (Δt) is constant throughout the simulation, we can calculate the constant growth rate per step (*growthRatePerStep*) before the loop, as follows:

$$growthRatePerStep \leftarrow growthRate * \Delta t$$

Within the loop, we do not compute *growth* but estimate *population* as follows:

$$population \leftarrow population + growthRatePerStep * population$$

Thus, within the loop, we have two assignments instead of three and two multiplications instead of three, saving time in a lengthy simulation. The revised algorithm appears as Algorithm 2.

Algorithm 2 Alternative algorithm to Algorithm 1 for simulation of unconstrained growth that does not display *growth*

initialize *simulationLength*
initialize *population*
initialize *growthRate*
initialize Δt
growthRatePerStep* ← *growthRate* * Δ*t
numIterations ← *simulationLength*/Δ*t*
for *i* going from 1 through *numIterations* do the following:
  population* ← *population* + *growthRatePerStep* * *population
  t ← *i* * Δ*t*
  display *t* and *population*

Analytical Solution: Introduction

We can solve the preceding model analytically for unconstrained growth, which is the differential equation $\frac{dP}{dt} = 0.10P$ with initial condition $P_0 = 100$, as follows:

$$P = 100\, e^{0.10t}$$

The next three sections develop the analytical solution. The first section starts the explanation using indefinite integrals, while the second section begins the discussion using derivatives without using integrals. Thus, you may select the section that matches your calculus background. The third section completes the development of the analytical solution for both tracks. Those without calculus background may go immediately to the section "Completion of the Analytical Solution."

When it is possible to solve a problem analytically, we should usually do so. We have employed simulation of unconstrained growth with a system dynamic tool as an introduction to fundamental concepts and as a building block to more complex problems for which no analytical solutions exist.

Analytical Solution: Explanation with Indefinite Integrals (Optional)

We can solve the differential equation $\frac{dP}{dt} = 0.10P$ using a technique called **separation of variables**. First, we move all terms involving P to one side of the equation and all those involving t to the other. Leaving 0.10 on the right, we have the following:

$$\frac{1}{P}dP = 0.10\,dt$$

Then, we integrate both sides of the equation, as follows:

$$\int \frac{1}{P}dP = \int 0.10\,dt$$

$$\ln|P| = 0.10t + C \text{ for an arbitrary constant } C$$

We solve for $|P|$ by taking the exponential function of both sides and using the fact that the exponential and natural logarithmic functions are inverses of each other.

$$e^{\ln|P|} = e^{0.10t+C}$$

$$|P| = e^{0.10t}e^{C} = A\,e^{0.10t}$$

where $A = e^C$. Solving for P, we have

$$P = (\pm A)e^{0.10t}$$

or

$$P = ke^{0.10t}$$

where $k = (\pm A)$ is a constant.

Analytical Solution: Explanation with Derivatives (Optional)

We can solve the differential equation $\frac{dP}{dt} = 0.10P$ for P analytically by finding a function whose derivative is 0.10 times the function itself. The only functions that are their own derivative are exponential functions of the following form:

$$f(t) = ke^t, \text{ where } k \text{ is a constant}$$

For example, the derivative of $5e^t$ is $5e^t$. To obtain a factor of 0.10 through use of the chain rule, we have the general solution

$$P = ke^{0.10t}$$

For example, if $P = 5e^{0.10t}$, we have

$$\frac{dP}{dt} = \frac{d(5e^{0.10t})}{dt} = 5\frac{d(e^{0.10t})}{dt} = 5(0.10e^{0.10t}) = 0.10(5e^{0.10t}) = 0.10P$$

Completion of the Analytical Solution

Thus, the general solution to $\frac{dP}{dt} = 0.10P$ is $P = ke^{0.10t}$ for a constant k. Using the initial condition that $P_0 = 100$, we can determine a particular value of k and, thus, a particular solution of the form $P = ke^{0.10t}$. Substituting 0 for t and 100 for P, we have the following:

$$100 = ke^{0.10(0)} = ke^0 = k(1) = k$$

The constant is the initial population. For this example,

$$P = 100e^{0.10t}$$

Figure 2.2.3 displays the dramatic increase in the bacterial population as time advances.

In general, the solution to

$$\frac{dP}{dt} = rP \quad \text{with initial population } P_0$$

is

$$P = P_0e^{rt}$$

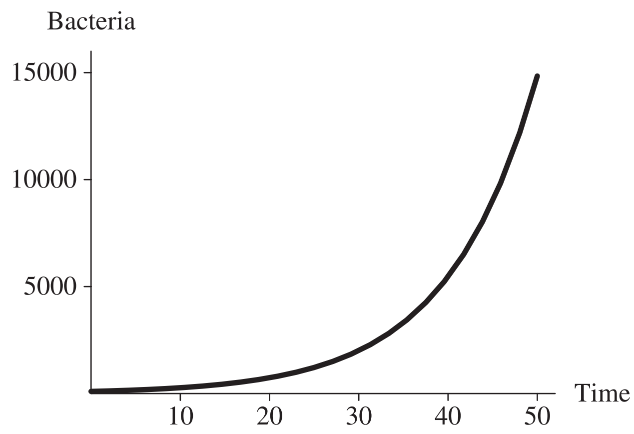

Figure 2.2.3 Bacterial population with a continuous growth rate of 10% per hour and an initial population of 100 bacteria

Quick Review Question 4

Give the solution of the differential equation

$$\frac{dP}{dt} = 0.03P\text{, where } P_0 = 57$$

The simulated values for the bacterial population are slightly less than those the model $P = 100e^{0.10t}$ determines. For example, after 20 h, a simulation may display, to two decimal places, a population of 738.54. However, $100e^{0.10(20)}$, expressed to two decimal places, is 738.91. The simulation compounds the population every step, and, in this case, the step size is $\Delta t = 0.005$ h. The analytic model compounds the population continuously; that is, as the step size goes to zero and the number of steps goes to infinity approaches, the simulated values approach the analytic solution.

Both the analytic model and simulation produce valid estimates of the population of bacteria. After 20 h, the number of bacteria will be an integer, not a decimal number, such as 738.54 or 738.91. Moreover, the population probably does not grow in an ideal fashion with a 10%-per-hour growth rate at every instant. Both the analytic model and the simulation produce estimates of the population at various times.

Further Refinement

We can refine the model further by having separate parameters for birth rate and death rate instead of the combined growth rate. Thus,

$$growth\_rate = birth\_rate - death\_rate$$

Unconstrained Decay

The rate of change of the mass of a radioactive substance is proportional to the mass of the substance, and the constant of proportionality is negative. Thus, the mass decays with time. For example, the constant of proportionality for radioactive carbon-14 is approximately –0.000120968. The continuous decay rate is about 0.0120968% per year, and the differential equation is as follows, where Q is the quantity (mass) of carbon-14:

$$\frac{dQ}{dt} = -0.000120968Q$$

As indicated in the section "Completion of the Analytical Solution," the analytical solution to this equation is

$$Q = Q_0 e^{-0.000120968t}$$

After 10,000 yr, only about 29.8% of the original quantity of carbon-14 remains, as the following shows:

$$Q = Q_0 e^{-0.000120968(10{,}000)} = 0.298292Q_0$$

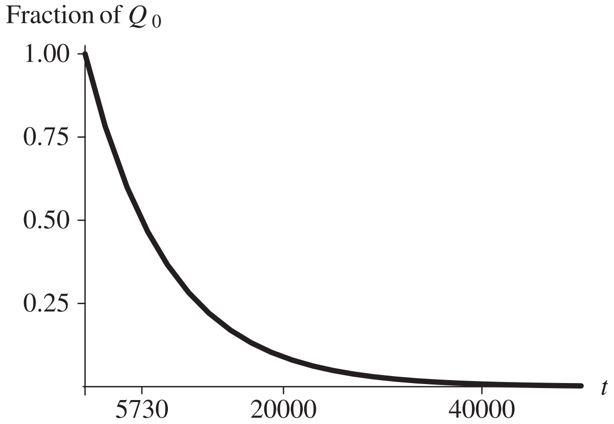

Figure 2.2.4 Exponential decay of radioactive carbon-14 as a fraction of the initial quantity Q_0, with time (t) in years

Figure 2.2.4 displays the decay of carbon-14 with time.

Carbon dating uses the amount of carbon-14 in an object to estimate the age of an object. All living organisms accumulate small quantities of carbon-14, but accumulation stops when the organism dies. For example, we can compare the proportion of carbon-14 in living bone to that in the bone of a mummy and estimate the age of the mummy using the model.

Example 1

Suppose the proportion of carbon-14 in a mummy is only about 20% of that in a living human. To estimate the age of the mummy, we use the preceding model with the information that $Q = 0.20Q_0$. Substituting into the analytical model, we have

$$0.20Q_0 = Q_0 e^{-0.000120968t}$$

After canceling Q_0, we solve for t by taking the natural logarithm of both sides of the equation. Because the natural logarithm and the exponential functions are inverses of each other, we have the following:

$$\ln(0.20) = \ln(e^{-0.000120968t}) = -0.000120968t$$

$$t = \ln(0.20)/(-0.000120968) \approx 13{,}305 \text{ yr}$$

We often express the rate of decay in terms of the half-life of the radioactive substance. The **half-life** is the period of time that it takes for the substance to decay to half of its original amount. Figure 2.2.5 illustrates that the half-life of radioactive carbon-14 is about 5730 yr. We can determine this value analytically as we did in Example 1 using 50% instead of 20%; $Q = 0.50Q_0$.

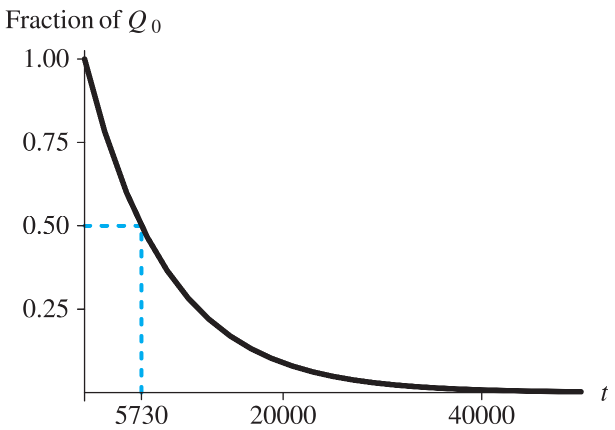

Figure 2.2.5 The half-life of radioactive carbon-14 indicated as 5730 yr

Definition The **half-life** is the period of time that it takes for a radioactive substance to decay to half of its original amount.

Quick Review Question 5

Radium-226 has a continuous decay rate of about 0.0427869% per year. Determine its half-life in whole years.

Reports for System Dynamics Models

The fifth step of the modeling process discussed in Module 1.2 is to "Report on the model." The following summarizes the items that would be included in a report for a system dynamics model:

a. **Analysis of the problem:** We begin by describing the problem, such as to model the growth of bacteria in media.
b. **Model design:** In this section, we should list simplifying assumptions, such as those in the section "Differential Equation"; equations, such as $\frac{dP}{dt} = 0.10P$ with $P_0 = 100$; reasoning for choices of constants, such as an instantaneous growth rate of 10%; the basic time step, such as hour; and other units. A diagram of the model, such as in Figure 2.2.1, is also appropriate to include.
c. **Model solution:** This part should contain the analytical solution or an algorithm, such as Algorithm 1.
d. **Results and conclusions:** Part d should include simulation tables, such as Table 2.2.2, and graphs, such as Figure 2.2.2. Moreover, the section should contain an explanation of verification accomplished by comparing the results to real data when available, descriptions of the outcomes of various scenar-

ios, a discussion of our conclusions with support from the results, and suggestions for model refinement.

e. Appendices: Usually, a copy of the file created with a system dynamics tool should be submitted with this report. Besides the model, this file should contain appropriate documentation, such as a text box with the authors' names, date, module and problem number, and problem description.

Exercises

Answers to marked exercises appear in the appendix "Answers to Selected Exercises."

1. a. For an initial population of 100 bacteria and a continuous growth rate of 10% per hour, determine the number of bacteria at the end of one week.

b. How long will it take the population to double?

2. a. Suppose the initial population of a certain animal is 15,000 and its continuous growth rate is 2% per year. Determine the population at the end of 20 yr.

b. Suppose we are performing a simulation of the population using a step size of 0.083 yr. Determine the growth and the population at the end of the first three time steps.

3. Adjust the model in Figure 2.2.1 to accommodate birth rate and death rate instead of just growth rate.

4. a. Newton's Law of Heating and Cooling states that the rate of change of the temperature (T) with respect to time (t) of an object is proportional to the difference between the temperatures of the object and of its surroundings. Suppose the temperature of the surroundings is 25 °C. Write the differential equation that models Newton's Law.

b. Solve this equation for T as a function of time t.

c. Suppose cold water at 6 °C is placed in a room that has temperature 25 °C. After 1 h, the temperature of the water is 20 °C. Determine all constants in the equation for T.

d. What is the temperature of the water after 15 minutes (min)?

e. How long will it take for the water to warm to room temperature?

5. a. Suppose someone, whose temperature is originally 37 °C, is murdered in a room that has constant temperature 25 °C. The temperature is measured as 28 °C when the body is found and at 27 °C 1 h later. How long ago was the murder committed from discovery of the body? See Exercise 4 for Newton's Law of Heating and Cooling.

b. Suppose we are performing a simulation using a step size of 0.004 h. Using the decay rate from Part a, determine the temperature at the end of the first three time steps after discovery of the body.

6. a. What proportion of the original quantity of carbon-14 is left after 30,000 yr?

b. If 60% is left, how old is the item?

7. a. The half-life of radioactive strontium-90 is 29 yr. Give the model for the quantity present as a function of time.

b. What proportion of strontium-90 is present after 10 yr?

c. After 50 yr?
d. How long will it take for the quantity to be 15% of the original amount?

8. Suppose an investment has approximately a continuous growth rate of 9.3%. Calculate analytically the value of an initial investment of $500 after

a. 10 yr **b.** 20 yr **c.** 30 yr **d.** 40 yr
d. How long will it take for the value to double?
e. How long to quadruple?

9. Suppose the amount of deposited ash, A, in millimeters (mm) is a function of time t in days. Suppose the model states that the rate of change of ash with respect to time is 4 mm/day and the initial quantity is 3 mm.

a. Using a step size of 0.5 days (da), estimate the amount of ash when $t = 1$ da.
b. Repeat Part a using a step size of 0.25 da.
c. Does the smaller step size change the result?
d. Solve the model for A.
e. What kind of function do you obtain?

Projects

For additional projects, see Module 7.1, "Radioactive Chains—Never the Same Again"; Module 7.2, "Turnover and Turmoil—Blood Cell Populations"; Module 7.3, "Deep Trouble—Ideal Gas Laws and Scuba Diving"; Module 7.4, "What Goes Around Comes Around—The Carbon Cycle"; after completion of "System Dynamics Tool: Tutorial 2," Module 7.9, "Transmission of Nerve Impulses: Learning from the Action Potential Heroes"; Module 7.12 "Mercury Pollution—Getting on Our Nerves."

1. Develop a model for Newton's Law of Heating and Cooling (see Exercise 4). Using this model, answer the questions of Exercises 4 and 5.

2. In 1854, Dr. John Snow, the father of epidemiology, identified a particular London water pump as the point source of the Broad Street cholera epidemic, which spread in a radial fashion from the pump. Model such a spread of disease assuming that the rate of change of the number of cases of cholera is proportional to the square root of the number of cases.

3. Develop a model for Exercise 8.

4. A young professional would like to save enough money to pay cash for a new car. Develop a model to determine when such a purchase will be possible. Take into account the following issues: The price of a new car is rising due to inflation. The buyer plans to trade in a car, which is depreciating. This person already has some savings and plans to make regular monthly payments. Thus, use a Δt value of 1 mo. Assume appropriate rates and values.

Develop a spreadsheet for each of Projects 5–8.

5. Exercise 2
6. Exercise 4
7. Exercise 5
8. Exercise 8

Answers to Quick Review Questions

1. a. Average velocity from 1 to 2 s = $\dfrac{s(2)-s(1)}{2-1}=\dfrac{21.4-21.1}{1}$ = 0.3 m/s

b. Average velocity from 1 to 3 s = $\dfrac{s(3)-s(1)}{3-1}=\dfrac{11.9-21.1}{2}$ = –4.6 m/s

c. $b = 3$ s, $s(b) = 11.9$ m, $\Delta t = 2$ s, $b - \Delta t = 1$ s, $s(b - \Delta t) = 21.1$ m, $\Delta s = 11.9 - 21.1 = -9.2$ m

2. a. $Q(t) = Q(t - \Delta t) + \Delta Q$, where $\Delta Q = -0.0004Q(t - \Delta t)\Delta t$ and $Q(0) = 200$

b. $t - \Delta t = 9.0 - 0.5 = 8.5$ s

c. $\Delta Q = 199.28 - 199.32 = -0.04$

3. 100.450901

growth = 100.400701 * 0.10 = 10.040070

Thus, *population*(0.045) = 100.400701 + 10.040070 * 0.005 = 100.450901

4. $P = 57e^{0.03t}$

5. 1620. Reasoning:

$$Q = Q_0 e^{-0.000427869t}$$

For $Q = 0.50Q_0$, $0.50Q_0 = Q_0 e^{-0.000427869t}$ or $0.50 = e^{-0.000427869t}$

$$\ln(0.50) = -0.000427869t$$

$$t = \ln(0.50)/(-0.000427869) = 1620$$

Reference

Zill, Dennis G. 2013. *A First Course in Differential Equations with Modeling* Applications, 10th ed. Belmont, CA. Brooks-Cole Publishing (Cengage Learning).