MODULE 7.1

Radioactive Chains—Never the Same Again

Prerequisite: Module 2.2, "Unconstrained Growth and Decay."

Introduction

The mass $Q(t)$ of a radioactive substance decays at a rate proportional to the mass of the substance (see the section "Unconstrained Decay" in Module 2.2, "Unconstrained Growth and Decay"). Thus, for positive **disintegration constant**, or **decay constant**, r, we have the following differential equation:

$$dQ/dt = -rQ(t)$$

and its difference equation counterpart:

$$\Delta Q = -rQ(t - \Delta t)\Delta t$$

In this module, we model the situation where one radioactive substance decays into another radioactive substance, forming a chain of such substances. For example, radioactive bismuth-210 decays to radioactive polonium-210, which in turn decays to lead-206. We consider the amounts of each substance as time progresses.

Modeling the Radioactive Chain

If a radioactive substance, *substanceA*, decays into substance *substanceB*, we say that *substanceA* is the **parent** of *substanceB* and that *substanceB* is the **child** of *substanceA*. If *substanceB* is also radioactive, *substanceB* is the parent of another substance, *substanceC*, and we have a **chain** of substances. Figure 7.1.1 depicts the situation where A, B, and C are the masses of radioactive substances, *substanceA*, *substanceB*, and *substanceC*, respectively; and different disintegration constants, *decay_rate_of_A* (a) and *decay_rate_of_B* (b), exist for each decay.

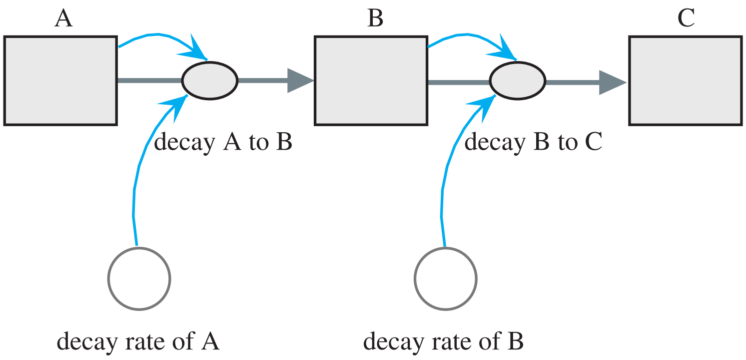

Figure 7.1.1 Chain of decays

Quick Review Question 1

Suppose A and B are the masses of *substanceA* and *substanceB*, respectively, at time t; ΔA and ΔB are the changes in these masses; and a and b are the positive disintegration constants.

a. Using these constants and variables along with arithmetic operators, such as minus and plus, give the difference equation for the change in the mass of *substanceA*, ΔA.
b. Through disintegration of *substanceA*, *substanceB*'s mass increases, while some of *substanceB* decays to *substanceC*. Give the difference equation for the change in the mass of *substanceB*, ΔB.
c. In Figure 7.1.1, where A, B, and C are the masses of three radioactive substances, give the formula as it appears in a system dynamics tool's equation for the flow *decay_A_to_B*.
d. Give the formula as it appears in a systems dynamics tool's equation for the flow *decay_B_to_C*.

The mass of *substanceA* that decays to *substanceB* is aA. Thus, in Figure 7.1.1, the flow *decay_A_to_B* contains the formula *decay_rate_of_A* * A. What *substanceA* loses, *substanceB* gains. However, *substanceB* decays to *substanceC* at a rate proportional to the mass of *substanceB*, bB. Consequently, in Figure 7.1.1, the flow *decay_B_to_C* contains the mass that flows from one stock to another, *decay_rate_of_B* * B. The total change in the mass of *substanceB* consists of the gain from *substanceA* minus the loss to *substanceC* with the result multiplied by the change in time, Δt:

$$\Delta B = (aA - bB)\,\Delta t$$

We consider the initial amounts of *substanceB* and *substanceC* to be zero.

Projects

1. **a.** With a system dynamics tool or a computer program, develop a model for a radioactive chain of three elements, from *substanceA* to *substanceB* to *substanceC*. Allow the user to designate constants. Generate a graph and a table for the amounts of *substanceA*, *substanceB*, and *substanceC* versus time. Answer the following questions using this model.

   b. Explain the shapes of the graphs.

   c. As the decay rate of A, a, increases from 0.1 to 1, describe how the time of the maximum total radioactivity changes. The total radioactivity is the sum of the change from *substanceA* to *substanceB* and the change from *substanceB* to *substanceC*, or the total number of disintegrations. Why?

   d. (The verification in Part d requires calculus.) With b being the decay rate of B, in several cases where $a < b$, observe that eventually we have the following approximation:

$$\frac{B}{A} \approx \frac{a}{b-a}$$

   With the ratio of the mass of *substanceB* (B) to the mass of *substanceA* (A) being almost constant, $a/(b - a)$, we say the system is in **transient equilibrium**. Eventually, *substanceA* and *substanceB* appear to decay at the same rate. Using the following material, verify this approximation:

   Find the exact solution to the differential equation for the rate of change of A with respect to time, $dA/dt = -aA$ (see the section "Analytic Solution" in Module 2.2, "Unconstrained Growth and Decay").

   Verify that $B = \frac{aA_0}{b-a}(e^{-at} - b^{-bt})$, where A_0 is the initial mass of *substanceA*, is a solution to the differential equation for the rate of change of B with respect to time (see the difference equation for ΔB). What number does e^{-at} approach as t goes to infinity? For $a < b$, which is smaller, e^{-at} or e^{-bt}? Thus, for large t, B is approximately equal to what?

   e. Using your model from Part a, observe in several cases where $a > b$ that the ratio of the mass of *substanceB* to the mass of *substanceA* does not approach a number. Thus, transient equilibrium (see Part d) does not occur in this case.

   f. (Requires calculus) Verify the observation from Part e analytically using work similar to that in Part d.

   g. If a is much smaller than b, we have $A \approx A_0$ and $B \approx \frac{aA_0}{b-a}$. With the two amounts being almost constant, we have a situation called **secular equilibrium**. Observe this phenomenon for the radioactive chain from radium-226 to radon-222 to polonium-218: $Ra^{226} \rightarrow Rn^{222} \rightarrow Po^{218}$, where the decay rate of Ra^{226}, a, is 0.00000117/da and the decay rate of Rn^{222}, b, is 0.181/da. Using your work from Part a, run the simulation for at least one year.

h. (Requires calculus) Show analytically that the approximations from Part g hold.

i. In the radioactive chain $Bi^{210} \rightarrow Po^{210} \rightarrow Pb^{206}$ (bismuth-210 to polonium-210 to lead-206), the decay rate of Bi^{210}, a, is 0.0137/da and the decay rate of Po^{210}, b, is 0.0051/da. Assuming the initial mass of Bi^{210} is 10^{-8} g and using your model from Part a, find, approximately, the maximum mass of Po^{210} and when the maximum occurs.

j. (Requires calculus) In Part d, we verified that $B = \frac{aA_0}{b-a}(e^{-at} - b^{-bt})$. Using this result, find analytically the maximum of mass of *substanceB* and when this maximum occurs.

k. Check your approximations of Part i using your solution to Part j.

l. For the chain in Part g, use your solution to Part j to find when the largest mass of Rn^{222} occurs.

m. For the chain in Part g, use your simulation of Part a to approximate the time when the largest mass of Rn^{222} occurs. How does your approximation compare with the analytical solution of Part l?

2. Develop a model for a chain of four elements. Perform simulations, observations, and analyses similar to those before. Discuss your results

Answers to Quick Review Question

1. a. $\Delta A = -aA\ \Delta t$

b. $\Delta B = (aA - bB)\Delta t$

c. *decay_A_to_B = decay_rate_of_A * A*

d. *decay_B_to_C = decay_rate_of_B * B*

Reference

Horelick, Brindell, and Sinan Koont. 1979, 1989. "Radioactive Chains: Parents and Children." *UMAP Module 234*. COMAP, Inc.