Stat1600 Binomial Distribution Examples

I. Emily hits 60% of her free throws in basketball games. She had 25 free throws in last week's game. Use this information to answer the next **two** questions.

1. What is the average number of hits?

   (a) 10
   (b) 15
   (c) 20
   (d) 25

   ANSWER: $\boxed{\text{(b)}}$

   REASON:

The average (or the expected value) is $n \times p = 25 \times 0.60 = 15$.

2. What is the standard deviation of Emily's hit ?

   (a) 6
   (b) 3
   (c) 3.2
   (d) $\sqrt{6}$

   ANSWER: $\boxed{\text{(d)}}$

   REASON:

The standard deviation is $\sqrt{np(1-p)} = \sqrt{25 \times 0.60 \times (1-0.60)} = \sqrt{6} = 2.45$.

II. In the previous question, suppose Emily had 7 free throws in yesterday's game.

1. What is the probability that she made at least 5 hits?

   (a) 0.2613
   (b) 0.1306
   (c) 0.0280
   (d) 0.1586
   (e) 0.4199

   ANSWER: $\boxed{\text{(e)}}$

   REASON:

Denote Y the hits in Emily's 7 free throws. The event that she made at least 5 hits is then

$$(Y \geq 5) = (Y = 5 \text{ or } 6 \text{ or } 7).$$

So,

$$P(Y \geq 5) = P(Y = 5) + P(Y = 6) + P(Y = 7)$$

$$\begin{aligned}
P(Y = 5) &= \frac{7!}{5!(7-5)!}(.6)^5(.4)^{7-5} = \frac{7 \times 6 \times 5!}{5! \times 2!}(.6)^5(.4)^2 \\
&= \frac{7 \times 6}{2 \times 1}(.6)^5(.4)^2 \text{ (note that 5! was factored out)} \\
&= 21 \times (.6)^5(.4)^2 = 0.2613 \\
P(Y = 6) &= \frac{7!}{6!(7-6)!}(.6)^6(.4)^{7-6} = \frac{7 \times 6!}{6! \times 1!}(.6)^6(.4)^1 \\
&= \frac{7}{1}(.6)^6(.4)^1 \text{ (note that 6! was factored out)} \\
&= 7 \times (.6)^6 \times .4 = 0.1306 \\
P(Y = 7) &= \frac{7!}{7!(7-7)!}(.6)^7(.4)^{7-7} = \frac{7!}{7! \times 0!}(.6)^7(.4)^0 \\
&= (.6)^7 \times 1 \text{ (note that 7! was factored out)} \\
&= 0.0280.
\end{aligned}$$

Note that, $0! = 1$ and $(.4)^0 = 1$ above.
It follows that the probability of interest is $0.2613 + 0.1306 + 0.0280 = 0.4199$.

III. A coin is flipped three times.

1. Denote X the number of heads turn out in the experiment. What is the the set of possible values of the variable X?

   (a) {1,2,3}
   (b) {0,1,2}
   (c) {0,1,2,3}
   (d) {0,1}
   (e) none of the previous

   ANSWER: $\boxed{\text{(c)}}$

   REASON:

Since X counts the occurrences of head in 3 flips of the coin, the set of possible values of X is:

$$\{0, 1, 2, 3\}.$$

That is, you could get 0 head, 1 head, 2 heads, or 3 heads.

2. Suppose the coin is biased (i.e., loaded) so that the probability that a head turns out in a flip is 0.6. What are the mean and the standard deviation of X?
   (a) $\mu = 1.8,\ \sigma = 0.8485$
   (b) $\mu = 1.2,\ \sigma = 0.8485$
   (c) $\mu = 1.8,\ \sigma = 0.72$
   (d) $\mu = 1.2,\ \sigma = 0.72$
   (e) none of the previous since μ, the average number of heads, must be an integer

   ANSWER: (a)

   REASON:

   The mean and the standard deviation are, respectively,

$$\mu = np = 3 \times 0.6 = 1.8,\ \sigma = \sqrt{np(1-p)} = \sqrt{3 \times 0.6 \times 0.4} = 0.8485.$$

3. What is the probability that the head turns out at least twice?
   (a) 0.784
   (b) 0.648
   (c) 0.352
   (d) 0.432
   (e) none of the previous

   ANSWER: (b)

   REASON:

   For $X \sim binomial(n = 3, p = 0.6)$, the probability that the heads turns out at least twice is:

$$\begin{aligned} P(X \geq 2) &= P(X = 2 \text{ or } X = 3) \\ &= P(X = 2) + P(X = 3) \\ &= \frac{3!}{2!(3-2)!}(.6)^2(.4)^{3-2} + \frac{3!}{3!(3-3)!}(.6)^3(.4)^{3-3} \\ &= \frac{3 \times 2!}{2!1!}(.6)^2(.4)^{3-2} + \frac{3!}{3!0!}(.6)^3(.4)^{3-3} \\ &= 3 \times (.6)^2 \times .4 + 1 \times (.6)^3 \times 1 \\ &= 0.432 + 0.216 = 0.648. \end{aligned}$$

   Note above, $0! = 1$, $(.4)^0 = 1$.

4. What is the probability that an odd number of heads turn out in 3 flips?
   (a) 0.496
   (b) 0.288
   (c) 0.504
   (d) 0.216
   (e) none of the previous

ANSWER: $\boxed{\text{(c)}}$

REASON:

The probability that an odd number of heads turn out in 3 flips is

$$P(X \text{ is odd}) = P(X = 1 \text{ or } 3) = P(X = 1) + P(X = 3) = 0.288 + 0.216 = 0.504.$$

Note that

$$P(X = 1) = \frac{3!}{1!(3-1)!}(.6)^1(.4)^{3-1} = \frac{3 \times 2!}{1 \times 2!}(.6)(.4)^2 = 3 \times .6 \times .16 = 0.288$$

IV. 1. According to the 2009 current Population Survey conducted by the U.S. Census Bureau, ***40%*** of the U.S. population 25 years old and over have completed a bachelor's degree or more. Given a random sample of ***50*** people 25 years old or over,

(a) (**10 points**) the number of people who have completed a bachelor's degree is expected to be around ______, give or take ______ or so.

ANSWER: $\boxed{\text{around 20 give or take 3.46}}$

CALCULATION/REASON: (show your work)

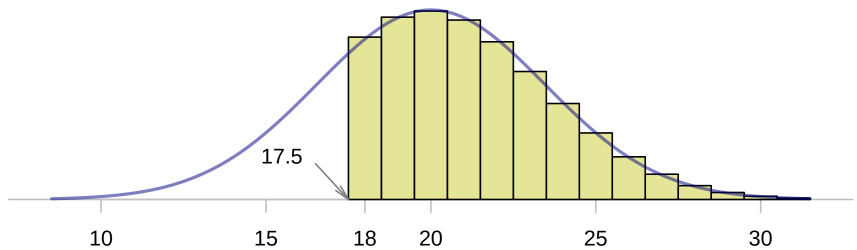
$\mu = n \times p = 50 \times 0.4 = 20$ and $\sigma = \sqrt{n \times p \times q} = \sqrt{50 \times 0.4 \times (1 - 0.4)} = 3.46.$

(b) (**10 points**) what is the chance that ***18 or more*** has completed a bachelors degree? (Hint: Draw rectangles representing the area of interest (***18 or more***) and use normal approximation.)

ANSWER: $\boxed{0.7642}$

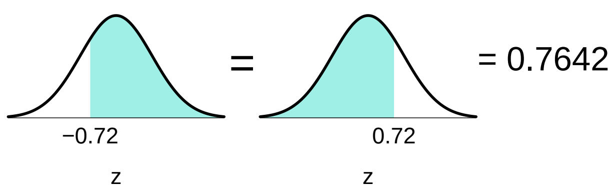
CALCULATION/REASON: (show your work)

The area of interest are the shaded rectangles as seen below and can be approximated by the area to the right of 17.5 ($= 18 - 0.5$) under the normal curve $N(\mu = 20, \sigma = 3.46)$:



Hence,

$$
\begin{aligned}
P(X \geq 18) &= P(X > 17.5) \approx P\left(Z > \frac{17.5 - 20}{3.46}\right) \\
&= P(Z > -0.72) = P(Z < 0.72) = 0.7642
\end{aligned}
$$



2. According to the 2009 current Population Survey conducted by the U.S. Census Bureau, ***240*** people classified their occupation as chef or head cook . Out of these ***240*** people, ***200*** were men and the rest women.

   (a) (**10 points**) What percentage of women are among chefs and head cooks?

   ANSWER: 16.7%

   CALCULATION/REASON: (show your work)

   Number of women chefs or head cooks is $X = 240 - 200 = 40$ and hence the proportion of women chefs or head cooks is

   $$\hat{p} = \frac{X}{n} = \frac{40}{240} = 0.167 \text{ or } 16.7\%$$

   (b) (**10 points**) Calculate the standard error for your estimate in (a)

   ANSWER: 2.4%

   CALCULATION/REASON: (show your work)

   The standard error is $SE(\hat{p}) = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\dfrac{0.167 \times (1 - 0.167)}{240}} = 0.024 \text{ or } 2.4\%$

   (c) (**10 points**) Calculate a 95% confidence interval for the true percentage.

ANSWER: $\boxed{(12\%,21.4\%)}$

CALCULATION/REASON: (show your work)

The margin of error is $ME = 1.96 \times SE(\hat{p}) = 1.96 \times 0.024 = 0.047$

So 95% c.i. for p is $(0.167 - 0.047, 0.167 + 0.047) = (0.120, 0.214)$.

That is, 95% c.i. for the true percentage is (12%,21.4%).