PROBABILITY EXERCISES

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1. Suppose a family contains two children of different ages, and we are interested in the gender of these children. Let F denote that a child is female and M that the child is male and let a pair such as FM denote that the older child is female and the younger is male. There are four points in the set S of possible observations:

$$S = \{FF, FM, MF, MM\}.$$

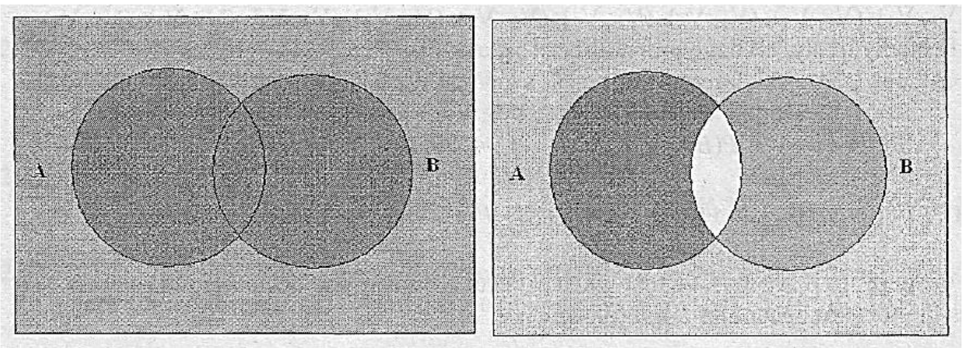
Let A denote the subset of possibilities containing no males; B, the subset containing two males; and C, the subset containing at least one male. List the elements of $A, B, C, A \cap B, \ A \cup B, A \cap C, A \cup C, B \cap C, B \cup C, and\ C \cap \bar{B}$

Solution:

$A = \{FF\}$, $B = \{MM\}$, $C = \{MF, FM, MM\}$. Then, $A \cap B = \emptyset$, $B \cap C = \{MM\}$, $C \cap \bar{B} = \{MF, FM\}$, $A \cup B = \{FF, MM\}$, $A \cup C = S$, $B \cup C = C$.

2. Draw Venn diagrams to verify DeMorgan's laws. That is, for any two sets A and B, $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$ and $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$.

Solution:



3. Use the identities A = A ∩ S and S = B ∪ $\bar{B}$ and a distributive law to prove that

a. A = (A ∩ B) ∪ (A ∩ $\bar{B}$).

b. If B ⊂ A then A = B ∪ (A ∩ $\bar{B}$).

c. Further, show that (A ∩ B) and (A ∩ $\bar{B}$) are mutually exclusive and therefore that A is the union of two mutually exclusive sets, (A ∩ B) and (A ∩ $\bar{B}$).

d. Also show that B and (A ∩ $\bar{B}$) are mutually exclusive and if B ⊂ A, A is the union of two mutually exclusive sets, B and (A ∩ $\bar{B}$).

Solution:

a. $(A\cap B)\cup(A\cap\bar{B})=A\cap(B\cup\bar{B})=A\cap S=A$.

b. $B\cup(A\cap\bar{B})=(B\cap A)\cup(B\cap\bar{B})=(B\cap A)=A$.

c. $(A\cap B)\cap(A\cap\bar{B})=A\cap(B\cap\bar{B})=\emptyset$. The result follows from part (a).

d. $B\cap(A\cap\bar{B})=A\cap(B\cap\bar{B})=\emptyset$. The result follows from part (b).

4. A group of five applicants for a pair of identical jobs consists of three men and two women. The employer is to select two of the five applicants for the jobs. Let S denote the set of all possible outcomes for the employer's selection. Let A denote the subset of outcomes corresponding to the selection of two men and B the subset corresponding to the selection of at least one woman. List the outcomes in $A, B, A \cup \bar{B}, A \cap B, and\ A \cap \bar{B}$. (Denote the different man and women by M_1, M_2, M_3 and W_1, W_2, respectively.)

Solution:

A = {two males} = {M_1, M_2), (M_1,M_3), (M_2,M_3)}

B = {at least one female} = {(M_1,W_1), (M_2,W_1), (M_3,W_1), (M_1,W_2), (M_2,W_2), (M_3,W_2), {W_1,W_2)}

$\bar{B}$ = {no females} = A $\quad A\cup B=S \quad A\cap B=\emptyset \quad A\cap\bar{B}=A$

5. A manufacturer has five seemingly identical computer terminals available for shipping. Unknown to her, two of the five are defective. A particular order calls for two of the terminals and is filled by randomly selecting two of the five that are available.

a. List the sample space for this experiment.

b. Let *A* denote the event that the order is filled with two nondefective terminals. List the sample points in *A*.

c. Construct a Venn diagram for the experiment that illustrates event *A*.

d. Assign probabilities to the simple events in such a way that the information about the experiment is used and the axioms in Definition 2.6 are met.

e. Find the probability of event *A*.

<u>Solution:</u>

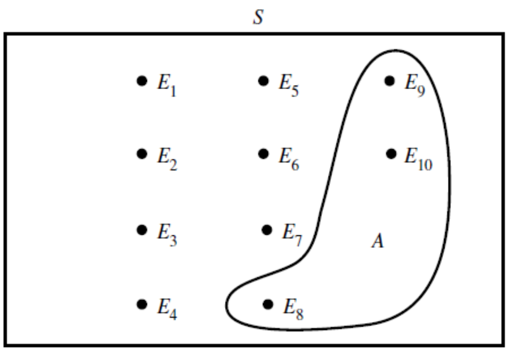
a. Let the two defective terminals be labeled D1 and D2 and let the three good terminals be labeled G1, G2, and G3. Any single sample point will consist of a list of the two terminals selected for shipment. The simple events may be denoted by

E1 = {D1, D2}, E5 = {D2, G1}, E8 = {G1, G2}, E10 = {G2, G3}.
E2 = {D1, G1}, E6 = {D2, G2}, E9 = {G1, G3},
E3 = {D1, G2}, E7 = {D2, G3},
E4 = {D1, G3},

b. Thus, there are ten sample points in S, and $S = \{E1, E2, \ldots, E10\}$.

Event A = {E8, E9, E10}.

c.



d. Because the terminals are selected at random, any pair of terminals is as likely to be selected as any other pair. Thus, $P(E_i) = 1/10$, for $i = 1, 2, \ldots, 10$, is a reasonable assignment of probabilities.

e. Because $A = E_8 \cup E_9 \cup E_{10}$, $P(A) = P(E_8) + P(E_9) + P(E_{10}) = 3/10$.

6. Every person's blood type is A, B, AB, or O. In addition, each individual either has the Rhesus (Rh) factor (+) or does not (−). A medical technician records a person's blood type and Rh factor. List the sample space for this experiment.

Solution:

$S = \{A+, B+, AB+, O+, A-, B-, AB-, O-\}$

7. A sample space consists of five simple events, E1, E2, E3, E4, and E5.

a. If $P(E1) = P(E2) = 0.15, P(E3) = 0.4, and\ P(E4) = 2P(E5)$, find the probabilities of E4 and E5.

b. If $P(E1) = 3P(E2) = 0.3$, find the probabilities of the remaining simple events if you know that the remaining simple events are equally probable.

Solution:

a. Since $P(S) = P(E_1) + \ldots + P(E_5) = 1$, $1 = .15 + .15 + .40 + 3P(E_5)$. So, $P(E_5) = .10$ and $P(E_4) = .20$.
b. Obviously, $P(E_3) + P(E_4) + P(E_5) = .6$. Thus, they are all equal to .2

8. Americans can be quite suspicious, especially when it comes to government conspiracies. On the question of whether the U.S. Air Force has withheld proof of the existence of intelligent life on other planets, the proportions of Americans with varying opinions are given in the table.

Opinion	Proportion
Very likely	.24
Somewhat likely	.24
Unlikely	.40
Other	.12

Suppose that one American is selected and his or her opinion is recorded.

a. What are the simple events for this experiment?

b. Are the simple events that you gave in part (a) all equally likely? If not, what are the probabilities that should be assigned to each?

c. What is the probability that the person selected finds it at least somewhat likely that the Air Force is withholding information about intelligent life on other planets?

Solution:

a. Denote the events as very likely (VL), somewhat likely (SL), unlikely (U), other (O).
b. Not equally likely: $P(\text{VL}) = .24$, $P(\text{SL}) = .24$, $P(\text{U}) = .40$, $P(\text{O}) = .12$.
c. $P(\text{at least SL}) = P(\text{SL}) + P(\text{VL}) = .48$.

9. An oil prospecting firm hits oil or gas on 10% of its drillings. If the firm drills two wells, the four possible simple events and three of their associated probabilities are as given in the accompanying table. Find the probability that the company will hit oil or gas

a. on the first drilling and miss on the second.

b. on at least one of the two drillings.

Simple Event	Outcome of First Drilling	Outcome of Second Drilling	Probability
E_1	Hit (oil or gas)	Hit (oil or gas)	.01
E_2	Hit	Miss	?
E_3	Miss	Hit	.09
E_4	Miss	Miss	.81

Solution:

a. Since the events are M.E., $P(S) = P(E_1) + \ldots + P(E_4) = 1$. So, $P(E_2) = 1 - .01 - .09 - .81 = .09$.
b. $P(\text{at least one hit}) = P(E_1) + P(E_2) + P(E_3) = .19$.

10. Hydraulic landing assemblies coming from an aircraft rework facility are each inspected for defects. Historical records indicate that 8% have defects in shafts only, 6% have defects in bushings only, and2%have defects in both shafts and bushings. One of the hydraulic assemblies is selected randomly. What is the probability that the assembly has

a. a bushing defect?

b. a shaft or bushing defect?

c. exactly one of the two types of defects?

d. neither type of defect?

Solution:

Let B = bushing defect, SH = shaft defect.

a. $P(\text{B}) = .06 + .02 = .08$

b. $P(\text{B or SH}) = .06 + .08 + .02 = .16$

c. $P(\text{exactly one defect}) = .06 + .08 = .14$

d. $P(\text{neither defect}) = 1 - \text{P(B or SH)} = 1 - .16 = .84$

11. A business office orders paper supplies from one of three vendors, V1, V2, or V3. Orders are to be placed on two successive days, one order per day. Thus, (V2, V3) might denote that vendor V2 gets the order on the first day and vendor V3 gets the order on the second day.

a. List the sample points in this experiment of ordering paper on two successive days.

b. Assume the vendors are selected at random each day and assign a probability to each sample point.

c. Let A denote the event that the same vendor gets both orders and B the event that V2 gets at least one order. Find $P(A), P(B), P(A \cup B)$, and $P(A \cap B)$ by summing the probabilities of the sample points in these events.

Solution:

a. $(V_1, V_1), (V_1, V_2), (V_1, V_3), (V_2, V_1), (V_2, V_2), (V_2, V_3), (V_3, V_1), (V_3, V_2), (V_3, V_3)$

b. if equally likely, all have probability of 1/9.

c. $A = \{\text{same vendor gets both}\} = \{(V_1, V_1), (V_2, V_2), (V_3, V_3)\}$

$B = \{\text{at least one V2}\} = \{(V_1, V_2), (V_2, V_1), (V_2, V_2), (V_2, V_3), (V_3, V_2)\}$

So, $P(A) = 1/3$, $P(B) = 5/9$, $P(A \cup B) = 7/9$, $P(A \cap B) = 1/9$.

12. Consider the problem of selecting two applicants for a job out of a group of five and imagine that the applicants vary in competence, 1 being the best, 2 second best, and so on, for 3, 4, and 5. These ratings are of course unknown to the employer. Define two events *A* and *B* as:

A: The employer selects the best and one of the two poorest applicants (applicants 1 and 4 or 1 and 5).

B: The employer selects at least one of the two best.

Find the probabilities of these events.

Solution:

The experiment involves randomly selecting two applicants out of five. Denote the selection of applicants 3 and 5 by {3, 5}. 2. The ten simple events, with $\{i, j\}$ denoting the selection of applicants *i* and *j* , are

$E1$: {1, 2}, $E5$: {2, 3}, $E8$: {3, 4}, $E10$: {4, 5}.

$E2$: {1, 3}, $E6$: {2, 4}, $E9$: {3, 5},

$E3$: {1, 4}, $E7$: {2, 5},

$E4$: {1, 5},

A random selection of two out of five gives each pair an equal chance for selection. Hence, we will assign each sample point the probability 1/10. That is,

$$P(Ei) = 1/10 = .1, i = 1, 2, \dots, 10.$$

Checking the sample points, we see that *B* occurs whenever $E1, E2, E3, E4, E5, E6, or\ E7$ occurs. Hence, these sample points are included in *B*. Finally, *P(B)* is equal to the sum of the probabilities of the sample points in *B*, or

$$P(B) = \sum_{i=1}^{7} P(E_i) = \sum_{i=1}^{7} .1 = .7.$$

Similarly, we see that event $A = E3 \cup E4\ and\ that\ P(A) = .1 + .1 = .2.$

13. A balanced coin is tossed three times. Calculate the probability that exactly two of the three tosses result in heads.

Solution:

The experiment consists of observing the outcomes (heads or tails) for each of three tosses of a coin. A simple event for this experiment can be symbolized by a three-letter sequence of H's and T 's, representing heads and tails, respectively. The first letter in the sequence represents the observation on the first coin. The second letter represents the observation on the second coin, and so on.The eight simple events in S are

E1 : HHH, E2 : HHT, E3 : HT H, E4 : T HH, E5 : HTT, E6 : T HT, E7 : TT H, E8 : TTT.

Because the coin is balanced, we would expect the simple events to be equally likely; that is, $P(Ei) = 1/8,\ i = 1, 2, \ldots, 8$. The event of interest, A, is the event that exactly two of the tosses result in heads. An examination of the sample points will verify that $A = \{E2, E3, E4\}$. Finally, $P(A) = P(E2) + P(E3) + P(E4) = 1/8 + 1/8 + 1/8 = 3/8$.

14. The odds are two to one that, when A and B play tennis, A wins. Suppose that A and B play two matches. What is the probability that A wins at least one match?

Solution:

The experiment consists of observing the winner (A or B) for each of two matches. Let AB denote the event that player A wins the first match and player B wins the second. The sample space for the experiment consists of four sample points:

$$E1 : AA, E2 : AB, E3 : BA, E4 : BB$$

Because A has a better chance of winning any match, it does not seem appropriate to assign equal probabilities to these sample points.

$$P(E1) = 4/9, P(E2) = 2/9, P(E3) = 2/9, P(E4) = 1/9.$$

The event of interest is that A wins at least one game. Thus, if we denote the event of interest as C, it is easily seen that $C = E1 \cup E2 \cup E3$. Finally, $P(C) = P(E1) + P(E2) + P(E3) = 4/9 + 2/9 + 2/9 = 8/9$.

15. A single car is randomly selected from among all of those registered at a local tag agency. What do you think of the following claim? “All cars are either Volkswagens or they are not. Therefore, the probability is 1/2 that the car selected is a Volkswagen.”

Solution:

Unless exactly 1/2 of all cars in the lot are Volkswagens, the claim is not true.

16. Two additional jurors are needed to complete a jury for a criminal trial. There are six prospective jurors, two women and four men. Two jurors are randomly selected from the six available.

a. Define the experiment and describe one sample point. Assume that you need describe only the two jurors chosen and not the order in which they were selected.

b. List the sample space associated with this experiment.

c. What is the probability that both of the jurors selected are women?

Solution:

a. The experiment consists of randomly selecting two jurors from a group of two women and four men.

b. Denoting the women as w1, w2 and the men as m1, m2, m3, m4, the sample space is

w1,m1	w2,m1	m1,m2	m2,m3	m3,m4
w1,m2	w2,m2	m1,m3	m2,m4	
w1,m3	w2,m3	m1,m4		
w1,m4	w2,m4			w1,w2

c. $P(\text{w1,w2}) = 1/15$

17. The Bureau of the Census reports that the median family income for all families in the United States during the year 2003 was $43,318. That is, half of all American families had incomes exceeding this amount, and half had incomes equal to or below this amount. Suppose that four families are surveyed and that each one reveals whether its income exceeded $43,318 in 2003.

a. List the points in the sample space.

b. Identify the simple events in each of the following events:

A: At least two had incomes exceeding $43,318.

B: Exactly two had incomes exceeding $43,318.

C: Exactly one had income less than or equal to $43,318.

c. Make use of the given interpretation for the median to assign probabilities to the simple events and find P(A), P(B), and P(C).

Solution:

a. Define the events: G = family income is greater than \$43,318, N otherwise. The points are E1: GGGG E2: GGGN E3: GGNG E4: GNGG
E5: NGGG E6: GGNN E7: GNGN E8: NGGN
E9: GNNG E10: NGNG E11: NNGG E12: GNNN
E13: NGNN E14: NNGN E15: NNNG E16: NNNN

b. A = {E1, E2, …, E11} B = {E6, E7, …, E11} C = {E2, E3, E4, E5}
c. If $P(\text{E}) = P(\text{N}) = .5$, each element in the sample space has probability 1/16. Thus,
$P(A) = 11/16$, $P(B) = 6/16$, and $P(C) = 4/16$.

18. A boxcar contains six complex electronic systems. Two of the six are to be randomly selected for thorough testing and then classified as defective or not defective.

a. If two of the six systems are actually defective, find the probability that at least one of the two systems tested will be defective. Find the probability that both are defective.

b. If four of the six systems are actually defective, find the probabilities indicated in part (a).

Solution:

a. There are four "good" systems and two "defactive" systems. If two out of the six systems are chosen randomly, there are 15 possible unique pairs. Denoting the systems as g1, g2, g3, g4, d1, d2, the sample space is given by S = {g1g2, g1g3, g1g4, g1d1, g1d2, g2g3, g2g4, g2d1, g2d2, g3g4, g3d1, g3d2, g4g1, g4d1, d1d2}. Thus:
P(at least one defective) = 9/15 P(both defective) = P(d1d2) = 1/15
b. If four are defective, P(at least one defective) = 14/15. P(both defective) = 6/15.

19. The names of 3 employees are to be randomly drawn, without replacement, from a bowl containing the names of 30 employees of a small company. The person whose name is drawn first receives \$100, and the individuals whose names are drawn second and third receive \$50 and \$25, respectively. How many sample points are associated with this experiment?

Solution:

Because the prizes awarded are different, the number of sample points is the number of ordered arrangements of r = 3 out of the possible n = 30 names. Thus, the number of sample points in S is

$$P_3^{30} = \frac{30!}{27!} = (30)(29)(28) = 24{,}360.$$

20. Suppose that an assembly operation in a manufacturing plant involves four steps, which can be performed in any sequence. If the manufacturer wishes to compare the assembly time for each of the sequences, how many different sequences will be involved in the experiment?

Solution:

The total number of sequences equals the number of ways of arranging the $n = 4$ steps taken $r = 4$ at a time, or

$$P_4^4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 24.$$

21. A labor dispute has arisen concerning the distribution of 20 laborers to four different construction jobs. The first job (considered to be very undesirable) required 6 laborers; the second, third, and fourth utilized 4, 5, and 5 laborers, respectively. The dispute arose over an alleged random distribution of the laborers to the jobs that placed all 4 members of a particular ethnic group on job 1. In considering whether the assignment represented injustice, a mediation panel desired the probability of the observed event. Determine the number of sample points in the sample space S for this experiment. That is, determine the number of ways the 20 laborers can be divided into groups of the appropriate sizes to fill all of the jobs. Find the probability of the observed event if it is assumed that the laborers are randomly assigned to jobs.

Solution:

The number ofways of assigning the 20 laborers to the four jobs is equal to the number *of ways of partitioning the 20 into four groups of sizes* $n1 = 6, n2 = 4, n3 = n4 = 5.$ *Then*

$$N = \binom{20}{6\ 4\ 5\ 5} = \frac{20!}{6!\,4!\,5!\,5!}.$$

By a random assignment of laborers to the jobs, we mean that each of the N sample points has probability equal to 1/N. If A denotes the event of interest and n_a the number of sample points in A, the sum of the probabilities of the sample points in A is $P(A) = n_a(1/N) = n_a/N$. The number of sample points in A, n_a, is the number of ways of assigning laborers to the four jobs with the 4 members of the ethnic group all going to job. The remaining 16 laborers need to be assigned to the remaining jobs. Because there remain two openings for job 1, this can be done in

$$n_a = \binom{16}{2\ 4\ 5\ 5} = \frac{16!}{2!\,4!\,5!\,5!}$$

ways. It follows that

$$P(A) = \frac{n_a}{N} = 0.0031.$$

Thus, if laborers are randomly assigned to jobs, the probability that the 4 members of the ethnic group all go to the undesirable job is very small. There is reason to doubt that the jobs were randomly assigned.

22. An airline has six flights from New York to California and seven flights from California to Hawaii per day. If the flights are to be made on separate days, how many different flight arrangements can the airline offer from New York to Hawaii?

<u>Solution:</u>

The total number of flights is 6*7 = 42.

23. A businesswoman in Philadelphia is preparing an itinerary for a visit to six major cities. The distance traveled, and hence the cost of the trip, will depend on the order in which she plans her route.

a. How many different itineraries (and trip costs) are possible?

b. If the businesswoman randomly selects one of the possible itineraries and Denver and San Francisco are two of the cities that she plans to visit, what is the probability that she will visit Denver before San Francisco?

Solution:

a. There are 6! = 720 possible itineraries.
b. In the 720 orderings, exactly 360 have Denver before San Francisco and 360 have San Francisco before Denver. So, the probability is .5.

24. An experiment consists of tossing a pair of dice.

a. Use the combinatorial theorems to determine the number of sample points in the sample space S.

b. Find the probability that the sum of the numbers appearing on the dice is equal to 7.

Solution:

a. By the *mn* rule, there are 6*6 = 36 possible roles.
b. Define the event $A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$. Then, $P(A) = 6/36$.

25. How many different seven-digit telephone numbers can be formed if the first digit cannot be zero?

Solution:

If the first digit cannot be zero, there are 9 possible values. For the remaining six, there are 10 possible values. Thus, the total number is $9*10*10*10*10*10*10 = 9*10^6$.

26. A fleet of nine taxis is to be dispatched to three airports in such a way that three go to airport A, five go to airport B, and one goes to airport C. In how many distinct ways can this be accomplished?

Solution:

$$\binom{9}{3}\binom{6}{5}\binom{1}{1} = 504 \text{ ways.}$$

27. Students attending the University of Florida can select from 130 major areas of study. A student's major is identified in the registrar's records with a two-or three-letter code (for example, statistics majors are identified by STA, math majors by MS). Some students opt for a double major and complete the requirements for both of the major areas before graduation. The registrar was asked to consider assigning these double majors a distinct two- or three-letter code so that they could be identified through the student records' system.

a. What is the maximum number of possible double majors available to University of Florida students?

b. If any two- or three-letter code is available to identify majors or double majors, how many major codes are available?

c. How many major codes are required to identify students who have either a single major or a double major?

d. Are there enough major codes available to identify all single and double majors at the University of Florida?

Solution:

a. $\binom{130}{2} = 8385.$

b. There are 26*26 = 676 two-letter codes and 26*26*26 = 17,576 three-letter codes. Thus, 18,252 total major codes.

c. 8385 + 130 = 8515 required.

d. Yes.

28. A local fraternity is conducting a raffle where 50 tickets are to be sold—one per customer. There are three prizes to be awarded. If the four organizers of the raffle each buy one ticket, what is the probability that the four organizers win

a. all of the prizes?

b. exactly two of the prizes?

c. exactly one of the prizes?

d. none of the prizes?

Solution:

There are $\binom{50}{3} = 19{,}600$ ways to choose the 3 winners. Each of these is equally likely.

a. There are $\binom{4}{3} = 4$ ways for the organizers to win all of the prizes. The probability is 4/19600.

b. There are $\binom{4}{2}\binom{46}{1} = 276$ ways the organizers can win two prizes and one of the other 46 people to win the third prize. So, the probability is 276/19600.

c. $\binom{4}{1}\binom{46}{2} = 4{,}140$. The probability is 4140/19600.

d. $\binom{46}{3} = 15{,}180$. The probability is 15180/19600.

29. Five firms, F1, F2, . . . , F5, each offer bids on three separate contracts, C1,C2, and C3. Any one firm will be awarded at most one contract. The contracts are quite different, so an assignment of C1 to F1, say, is to be distinguished from an assignment of C2 to F1.

a. How many sample points are there altogether in this experiment involving assignment of contracts to the firms? (No need to list them all.)

b. Under the assumption of equally likely sample points, find the probability that F3 is awarded a contract.

Solution:

a. In choosing three of the five firms, order is important. So $P_3^5 = 60$ sample points.

b. If F_3 is awarded a contract, there are $P_2^4 = 12$ ways the other contracts can be assigned. Since there are 3 possible contracts, there are $3*12 = 36$ total number of ways to award F_3 a contract. So, the probability is $36/60 = 0.6$.

30. A study is to be conducted in a hospital to determine the attitudes of nurses toward various administrative procedures. A sample of 10 nurses is to be selected from a total of the 90 nurses employed by the hospital.

a. How many different samples of 10 nurses can be selected?

b. Twenty of the 90 nurses are male. If 10 nurses are randomly selected from those employed by the hospital, what is the probability that the sample of ten will include exactly 4 male (and 6 female) nurses?

Solution:

a. $\binom{90}{10}$ **b.** $\binom{20}{4}\binom{70}{6}\Big/\binom{90}{10} = 0.111$

31. Two cards are drawn from a standard 52-card playing deck. What is the probability that the draw will yield an ace and a face card?

Solution:

There are $\binom{52}{2} = 1326$ ways to draw two cards from the deck. The probability is $4*12/1326 = 0.0362$.

32. Five cards are dealt from a standard 52-card deck. What is the probability that we draw

a. 1 ace, 1 two, 1 three, 1 four, and 1 five (this is one way to get a “straight”)?

b. any straight?

Solution:

There are $\binom{52}{5} = 2{,}598{,}960$ ways to draw five cards from the deck.

a. $\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 4^5 = 1024$. So, the probability is 1024/2598960 = 0.000394.

b. There are 9 different types of "straight" hands. So, the probability is $9*4^5/2598960 =$ 0.00355. Note that this also includes "straight flush" and "royal straight flush" hands.

33. Suppose that we ask *n* randomly selected people whether they share your birthday.

a. Give an expression for the probability that no one shares your birthday (ignore leap years).

b. How many people do we need to select so that the probability is at least .5 that at least one shares your birthday?

Solution:

a. $\dfrac{364(364)(364)\cdots(364)}{365^n} = \dfrac{364^n}{365^n}$. **b.** With $n = 253$, $1-\left(\dfrac{364}{365}\right)^{253} = 0.5005$.

34. The eight-member Human Relations Advisory Board of Gainesville, Florida, considered the complaint of a woman who claimed discrimination, based on sex, on the part of a local company. The board, composed of fivewomen and three men, voted 5–3 in favor of the plaintiff, the five women voting in favor of the plaintiff, the three men against. The attorney representing the company appealed the board's decision by claiming sex bias on the part of the board members. If there was no sex bias among the board members, it might be reasonable to conjecture that any group of five board members would be as likely to vote for the complainant as any other group of five. If this were the case, what is the probability that the vote would split along sex lines (five women for, three men against)?

Solution:

There are $\binom{8}{5} = 56$ sample points in the experiment, and only one of which results in choosing five women. So, the probability is 1/56.

35. Consider the following events in the toss of a single die: A: Observe an odd number, B: Observe an even number, C: Observe a 1 or 2.

a. Are A and B independent events?
b. Are A and C independent events?

<u>Solution</u>

a. To decide whether A and B are independent, we must see whether they satisfy the conditions of Definition 2.10. In this example, P(A) = 1/2, P(B) = 1/2, and P(C) = 1/3. Because A ∩ B = ∅, P(A|B) = 0, and it is clear that P(A|B) = P(A). Events A and B are dependent events.

b. Are A and C independent? Note that P(A|C) = 1/2 and, as before, P(A) = 1/2. Therefore, P(A|C) = P(A), and A and C are independent.

36. Three brands of coffee, X, Y , and Z, are to be ranked according to taste by a judge. Define the following events:

A: Brand X is preferred to Y .
B: Brand X is ranked best.
C: Brand X is ranked second best.
D: Brand X is ranked third best.

If the judge actually has no taste preference and randomly assigns ranks to the brands, is event A independent of events B, C, and D?

<u>Solution:</u>

The six equally likely sample points for this experiment are given by

$$E1 : XY\,Z, E3 : Y\,XZ, E5 : ZXY,$$

$$E2 : XZY, E4 : Y\,ZX, E6 : ZY\,X,$$

where XY Z denotes that X is ranked best, Y is second best, and Z is last. Then $A = \{E1, E2, E5\}, B = \{E1, E2\}, C = \{E3, E5\}, D = \{E4, E6\}$, and it follows that

$$P(A) = 1/2, P(A|B) = P(A \cap B)$$

$$P(B) = 1, P(A|C) = 1/2, P(A|D) = 0.$$

Thus, events A and C are independent, but events A and B are dependent. Events A and D are also dependent.

37. If two events, A and B, are such that $P(A) = .5, P(B) = .3,$ and $P(A \cap B) = .1$, find the following:

a. P(A|B)
b. P(B|A)
c. P(A|A ∪ B)
d. P(A|A ∩ B)
e. P(A ∩ B|A ∪ B)

Solution:

a. $P(A|B) = .1/.3 = 1/3$. **b.** $P(B|A) = .1/.5 = 1/5$.
c. $P(A|A \cup B) = .5/(.5+.3-.1) = 5/7$ **d.** $P(A|A \cap B) = 1$, since A has occurred.
e. $P(A \cap B| A \cup B) = .1(.5+.3-.1) = 1/7$.

38. Gregor Mendel was a monk who, in 1865, suggested a theory of inheritance based on the science of genetics. He identified heterozygous individuals for flower color that had two alleles (one r = recessive white color allele and one R = dominant red color allele). When these individuals were mated, 3/4 of the offspring were observed to have red flowers, and 1/4 had white flowers. The following table summarizes this mating; each parent gives one of its alleles to form the gene of the offspring.

	Sex		
Outcome	Male (M)	Female (F)	*Total*
Pass (A)	24	36	60
Fail ($\bar{A}$)	16	24	40
Total	40	60	100

We assume that each parent is equally likely to give either of the two alleles and that, if either one or two of the alleles in a pair is dominant (R), the offspring will have red flowers. What is the probability that an offspring has

a. at least one dominant allele?

b. at least one recessive allele?

c. one recessive allele, given that the offspring has red flowers?

Solution:

a. P(at least one R) = P(Red) 3/4. **b.** P(at least one r) = 3/4.
c. P(one r | Red) = .5/.75 = 2/3.

39. Cards are dealt, one at a time, from a standard 52-card deck.

a. If the first 2 cards are both spades, what is the probability that the next 3 cards are also spades?

b. If the first 3 cards are all spades, what is the probability that the next 2 cards are also spades?

c. If the first 4 cards are all spades, what is the probability that the next card is also a spade?

Solution:

a. Given the first two cards drawn are spades, there are 11 spades left in the deck. Thus, the probability is $\frac{\binom{11}{3}}{\binom{50}{3}} = 0.0084$. Note: this is also equal to $P(S_3S_4S_5|S_1S_2)$.

b. Given the first three cards drawn are spades, there are 10 spades left in the deck. Thus, the probability is $\frac{\binom{10}{2}}{\binom{49}{2}} = 0.0383$. Note: this is also equal to $P(S_4S_5|S_1S_2S_3)$.

c. Given the first four cards drawn are spades, there are 9 spades left in the deck. Thus, the probability is $\frac{\binom{9}{1}}{\binom{48}{1}} = 0.1875$. Note: this is also equal to $P(S_5|S_1S_2S_3S_4)$

40. A study of the posttreatment behavior of a large number of drug abusers suggests that the likelihood of conviction within a two-year period after treatment may depend upon the offenders education. The proportions of the total number of cases falling in four education–conviction categories are shown in the following table:

	Status within 2 Years after Treatment		
Education	Convicted	Not Convicted	*Total*
10 years or more	.10	.30	.40
9 years or less	.27	.33	.60
Total	.37	.63	1.00

Suppose that a single offender is selected from the treatment program. Define the events: A: The offender has 10 or more years of education. B: The offender is convicted within two years after completion of treatment. Find the following:

a. P(A).

b. P(B).

c. P(A ∩ B).

d. P(A ∪ B).

e. $P(\bar{A})$.

f. $P(\overline{A \cup B})$.

g. $P(\overline{A \cap B})$.

h. P(A|B).

i. P(B|A).

Solution:

a. 0.40 **b.** 0.37 **c.** 0.10 **d.** $0.40 + 0.37 - 0.10 = 0.67$

e. $1 - 0.4 = 0.6$ **f.** $1 - 0.67 = 0.33$ **g.** $1 - 0.10 = 0.90$

h. $.1/.37 = 0.27$ **i.** $1/.4 = 0.25$

41. Suppose that A and B are mutually exclusive events, with P(A) > 0 and P(B) < 1. Are A and B independent? Prove your answer.

Solution:

If A and B are M.E., $P(A \cap B) = 0$. **But,** $P(A)P(B) > 0$. So they are not independent.

42. If A and B are mutually exclusive events and P(B) > 0, show that

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

Solution:

$P(A \mid A \cup B) = P(A)/P(A \cup B) = \dfrac{P(A)}{P(A)+P(B)}$, since A and B are M.E. events.

43. If A and B are independent events, show that A and $\bar{B}$ are also independent. Are $\bar{A}$ and $\bar{B}$ independent?

Solution:

$P(A \mid \bar{B}) = P(A \cap \bar{B})/P(\bar{B}) = \dfrac{P(\bar{B} \mid A)P(A)}{P(\bar{B})} = \dfrac{[1-P(B \mid A)]P(A)}{P(\bar{B})} = \dfrac{[1-P(B)]P(A)}{P(\bar{B})} =$

$\dfrac{P(\bar{B})P(A)}{P(\bar{B})} = P(A)$. So, A and $\bar{B}$ are independent.

$P(\bar{B} \mid \bar{A}) = P(\bar{B} \cap \bar{A})/P(\bar{A}) = \dfrac{P(\bar{A} \mid \bar{B})P(\bar{B})}{P(\bar{A})} = \dfrac{[1-P(A \mid \bar{B})]P(\bar{B})}{P(\bar{A})}$. From the above,

A and $\bar{B}$ are independent. So $P(\bar{B} \mid \bar{A}) = \dfrac{[1-P(A)]P(\bar{B})}{P(\bar{A})} = \dfrac{P(\bar{A})P(\bar{B})}{P(\bar{A})} = P(\bar{B})$. So,

$\bar{A}$ and $\bar{B}$ are independent

44. Suppose that A and B are two events such that P(A) + P(B) > 1.

a. What is the smallest possible value for $P(A \cap B)$?

b. What is the largest possible value for $P(A \cap B)$?

Solution:

a. $P(A) + P(B) - 1$.

b. the smaller of $P(A)$ and $P(B)$.

45. Suppose that A and B are two events such that P(A) + P(B) < 1.

a. What is the smallest possible value for $P(A \cap B)$?

b. What is the largest possible value for $P(A \cap B)$?

Solution:

a. 0, since they could be disjoint.
b. the smaller of $P(A)$ and $P(B)$.

46. Can A and B be mutually exclusive if P(A) = .4 and P(B) = .7? If P(A) = .4 and P(B) = .3? Why?

Solution:

If A and B are M.E., $P(A \cup B) = P(A) + P(B)$. This value is greater than 1 if $P(A) = 0.4$ and $P(B) = 0.7$. So they cannot be M.E. It is possible if $P(A) = 0.4$ and $P(B) = 0.3$.

47. Two events A and B are such that P(A) = .2, P(B) = .3, and P(A ∪ B) = .4. Find the following:

a. P(A ∩ B)

b. $P(\bar{A} \cup \bar{B})$

c. $P(\bar{A} \cap \bar{B})$

d. $P(\bar{A} | B)$

Solution:

Part a is found using the Addition Rule. Parts b and c use DeMorgan's Laws.

a. $0.2 + 0.3 - 0.4 = 0.1$

b. $1 - 0.1 = 0.9$

c. $1 - 0.4 = 0.6$

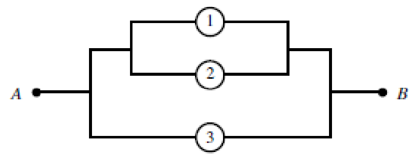
d. $P(\bar{A} \mid B) = \dfrac{P(\bar{A} \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)} = \dfrac{.3 - .1}{.3} = 2/3.$

48. In a game, a participant is given three attempts to hit a ball. On each try, she either scores a hit, H, or a miss, M. The game requires that the player must alternate which hand she uses in successive attempts. That is, if she makes her first attempt with her right hand, she must use her left hand for the second attempt and her right hand for the third. Her chance of scoring a hit with her right hand is .7 and with her left hand is .4. Assume that the results of successive attempts are independent and that she wins the game if she scores at least two hits in a row. If she makes her first attempt with her right hand, what is the probability that she wins the game?

Solution:

Let H denote a hit and let M denote a miss. Then, she wins the game in three trials with the events HHH, HHM, and MHH. If she begins with her right hand, the probability she wins the game, assuming independence, is $(.7)(.4)(.7) + (.7)(.4)(.3) + (.3)(.4)(.7) = 0.364$.

49. Consider the following portion of an electric circuit with three relays. Current will flow from point a to point b if there is at least one closed path when the relays are activated. The relays may malfunction and not close when activated. Suppose that the relays act independently of one another and close properly when activated, with a probability of .9.



a. What is the probability that current will flow when the relays are activated?

b. Given that current flowed when the relays were activated, what is the probability that relay 1 functioned?

Solution:

a. P(current flows) = 1 – P(all three relays are open) $= 1 - (.1)^3 = 0.999$.

b. Let A be the event that current flows and B be the event that relay 1 closed properly. Then, $P(B|A) = P(B \cap A)/P(A) = P(B)/P(A) = .9/.999 = 0.9009$. Note that $B \subset A$.

50. Articles coming through an inspection line are visually inspected by two successive inspectors. When a defective article comes through the inspection line, the probability that it gets by the first inspector is 0.1. The second inspector will "miss" five out of ten of the defective items that get past the first inspector. What is the probability that a defective item gets by both inspectors?

Solution:

Let A be the event the item gets past the first inspector and B the event it gets past the second inspector. Then, $P(A) = 0.1$ and $P(B|A) = 0.5$. Then $P(A \cap B) = .1(.5) = 0.05$.

51. It is known that a patient with a disease will respond to treatment with probability equal to 0.9. If three patients with the disease are treated and respond independently, find the probability that at least one will respond.

<u>Solution:</u>

Define the following events:

A: At least one of the three patients will respond.
B1: The first patient will not respond.
B2: The second patient will not respond.
B3: The third patient will not respond.

Then observe that $\bar{A}$ = B1∩B2∩B3 and $P(A) = 1 - P(\bar{A}) = 1 - P(B1 \cap B2 \cap B3)$. Applying the multiplicative law, we have

$$P(B1 \cap B2 \cap B3)P(B1)P(B2|B1)P(B3|B1\ \cap\ B2),$$

where, because the events are independent,

$$P(B2|B1) = P(B2) = 0.1\ and\ P(B3|B1 \cap B2) = P(B3)\ = 0.1.$$

Substituting $P(Bi\,) = 0.1, i\ =\ 1,2,3,$ we obtain P(A) = 1 − $(.1)^3$ = .999. Notice that we have demonstrated the utility of complementary events. This result is important because frequently it is easier to find the probability of the complement, P(A), than to find P(A) directly.

52. A monkey is to demonstrate that she recognizes colors by tossing one red, one black, and one white ball into boxes of the same respective colors, one ball to a box. If the monkey has not learned the colors and merely tosses one ball into each box at random, find the probabilities of the following results:

a. There are no color matches.

b. There is exactly one color match.

<u>Solution:</u>

This problem can be solved by listing sample points because only three balls are involved, but a more general method will be illustrated. Define the following events: A1: A color match occurs in the red box. A2: A color match occurs in the black box. A3: A color match occurs in the white box. There are 3! = 6 equally likely ways of randomly tossing the balls into the boxes with one ball in each box. Also, there are only 2! = 2 ways of tossing the balls into the boxes if one particular box is required to have a color match. Hence,

$P(A1) = P(A2) = P(A3) = 2/6 = 1/3.$

Similarly, it follows that

$P(A1 \cap A2) = P(A1 \cap A3) = P(A2 \cap A3) = P(A1 \cap A2 \cap A3) = 1/6.$

We can now answer parts (a) and (b) by using the event-composition method.

a. Notice that

P(no color matches) = 1 − P(at least one color match)

$= 1 - P(A1 \cup A2 \cup A3)$

$= 1 - [P(A1) + P(A2) + P(A3) - P(A1 \cap A2) - P(A1 \cap A3) - P(A2 \cap A3) + P(A1 \cap A2 \cap A3)]$

$= 1 - [3(1/3) - 3(1/6) + (1/6)] = 2/6 = 1/3.$

b. We leave it to you to show that

P(exactly one match)

$= P(A1) + P(A2) + P(A3) - 2[P(A1 \cap A2) + P(A1 \cap A3) + P(A2 \cap A3)] + 3[P(A1 \cap A2 \cap A3)]$

$= (3)(1/3) - (2)(3)(1/6) + (3)(1/6) = 1/2.$

53. An advertising agency notices that approximately 1 in 50 potential buyers of a product sees a given magazine ad, and 1 in 5 sees a corresponding ad on television. One in 100 sees both. One in 3 actually purchases the product after seeing the ad, 1 in 10 without seeing it. What is the probability that a randomly selected potential customer will purchase the product?

Solution:

Define the following events:

A: buyer sees the magazine ad

B: buyer sees the TV ad

C: buyer purchases the product

The following are known: $P(A) = .02$, $P(B) = .20$, $P(A \cap B) = .01$. Thus $P(A \cap B) = .21$. Also, $P(\text{buyer sees no ad}) = P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 1 - 0.21 = 0.79$. Finally, it is known that $P(C \mid A \cup B) = 0.1$ and $P(C \mid \overline{A} \cap \overline{B}) = 1/3$. So, we can find $P(C)$ as

$P(C) = P(C \cap (A \cup B)) + P(C \cap (\overline{A} \cap \overline{B})) = (1/3)(.21) + (.1)(.79) = 0.149.$

54. A state auto-inspection station has two inspection teams. Team 1 is lenient and passes all automobiles of a recent vintage; team 2 rejects all autos on a first inspection because their "headlights are not properly adjusted." Four unsuspecting drivers take their autos to the station for inspection on four different days and randomly select one of the two teams.

a. If all four cars are new and in excellent condition, what is the probability that three of the four will be rejected?

b. What is the probability that all four will pass?

Solution:

a. From the description of the problem, there is a 50% chance a car will be rejected. To find the probability that three out of four will be rejected (i.e. the drivers chose team 2), note that there are $\binom{4}{3} = 4$ ways that three of the four cars are evaluated by team 2. Each one has probability $(.5)(.5)(.5)(.5)$ of occurring, so the probability is $4(.5)^4 = 0.25$.

b. The probability that all four pass (i.e. all four are evaluated by team 1) is $(.5)^4 = 1/16$.

55. A football team has a probability of .75 of winning when playing any of the other four teams in its conference. If the games are independent, what is the probability the team wins all its conference games?

Solution:

By independence, $(.75)(.75)(.75)(.75) = (.75)^4$.

56. Suppose that two balanced dice are tossed repeatedly and the sum of the two uppermost faces is determined on each toss. What is the probability that we obtain

a. a sum of 3 before we obtain a sum of 7?

b. a sum of 4 before we obtain a sum of 7?

Solution:

a. Define the events: A: obtain a sum of 3 B: do not obtain a sum of 3 or 7
Since there are 36 possible rolls, $P(A) = 2/36$ and $P(B) = 28/36$. Obtaining a sum of 3 before a sum of 7 can happen on the 1st roll, the 2nd roll, the 3rd roll, etc. Using the events above, we can write these as A, BA, BBA, $BBBA$, etc. The probability of obtaining a sum of 3 before a sum of 7 is given by $P(A) + P(B)P(A) + [P(B)]^2P(A) + [P(B)]^3P(A) + \ldots$. (Here, we are using the fact that the rolls are independent.) This is an infinite sum, and it follows as a geometric series. Thus, $2/36 + (28/36)(2/36) + (28/36)^2(2/26) + \ldots = 1/4$.

b. Similar to part a. Define C: obtain a sum of 4 D: do not obtain a sum of 4 or 7
Then, $P(C) = 3/36$ and $P(D) = 27/36$. The probability of obtaining a 4 before a 7 is 1/3.

57. A new secretary has been given n computer passwords, only one of which will permit access to a computer file. Because the secretary has no idea which password is correct, he chooses one of the passwords at random and tries it. If the password is incorrect, he discards it and randomly selects another password from among those remaining, proceeding in this manner until he finds the correct password.

a. What is the probability that he obtains the correct password on the first try?

b. What is the probability that he obtains the correct password on the second try? The third try?

c. A security system has been set up so that if three incorrect passwords are tried before the correct one, the computer file is locked and access to it denied. If n = 7, what is the probability that the secretary will gain access to the file?

Solution:

a. $1/n$

b. $\frac{n-1}{n} \cdot \frac{1}{n-1} = 1/n$. $\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{1}{n-2} = 1/n$.

c. $P(\text{gain access}) = P(\text{first try}) + P(\text{second try}) + P(\text{third try}) = 3/7$.

58. An electronic fuse is produced by five production lines in a manufacturing operation. The fuses are costly, are quite reliable, and are shipped to suppliers in 100-unit lots. Because testing is destructive, most buyers of the fuses test only a small number of fuses before deciding to accept or reject lots of incoming fuses. All five production lines produce fuses at the same rate and normally produce only 2% defective fuses, which are dispersed randomly in the output. Unfortunately, production line 1 suffered mechanical difficulty and produced 5% defectives during the month of March. This situation became known to the manufacturer after the fuses had been shipped. A customer received a lot produced in March and tested three fuses. One failed.

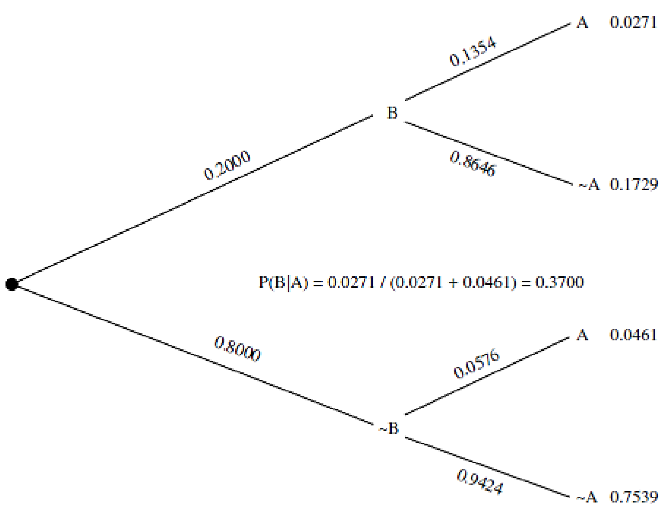
a. What is the probability that the lot was produced on line 1?

b. What is the probability that the lot came from one of the four other lines?

Solution:

Let B denote the event that a fuse was drawn from line 1 and let A denote the event that a fuse was defective. Then it follows directly that

$P(B) = 0.2$ and $P(A|B) = 3(.05)(.95)^2 = .135375$.



Similarly,

$P(\bar{B}) = 0.8$ and $P(A|\bar{B}) = 3(.02)(.98)^2 = .057624$.

Note that these conditional probabilities were very easy to calculate. Using the law of total probability,

$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B}) = (.135375)(.2) + (.057624)(.8) = .0731742$.

$P(B|A) = P(B \cap A)P(A)= P(A|B)P(B)P(A)= (.135375)(.2).0731742= .37$,

$P(\bar{B}|A) = 1 - P(B|A) = 1 - .37 = .63$.

Figure, obtained using the applet Bayes' Rule as a Tree, illustrates the various steps in the computation of P(B|A) .

59. A diagnostic test for a disease is such that it (correctly) detects the disease in 90% of the individuals who actually have the disease. Also, if a person does not have the disease, the test will report that he or she does not have it with probability .9. Only 1% of the population has the disease in question. If a person is chosen at random from the population and the diagnostic test indicates that she has the disease, what is the conditional probability that she does, in fact, have the disease? Are you surprised by the answer? Would you call this diagnostic test reliable?

<u>Solution:</u>

Define the events for the person: D: has the disease H: test indicates the disease

Thus, $P(H|D) = .9$, $P(\overline{H} \mid \overline{D}) = .9$, $P(D) = .01$, and $P(\overline{D}) = .99$. Thus,

$$P(D \mid H) = \frac{P(H \mid D)P(D)}{P(H \mid D)P(D) + P(H \mid \overline{D})P(\overline{D})} = 1/12.$$

60. Males and females are observed to react differently to a given set of circumstances. It has been observed that 70% of the females react positively to these circumstances, whereas only 40% of males react positively. A group of 20 people, 15 female and 5 male, was subjected to these circumstances, and the subjects were asked to describe their reactions on a written questionnaire. A response picked at random from the 20 was negative. What is the probability that it was that of a male?

Solution:

Define the events: P: positive response M: male respondent F: female respondent

$P(P|F) = .7$, $P(P|M) = .4$, $P(M) = .25$. Using Bayes' rule,

$$P(M \mid \overline{P}) = \frac{P(\overline{P} \mid M)P(M)}{P(\overline{P} \mid M)P(M) + P(\overline{P} \mid F)P(F)} = \frac{.6(.25)}{.6(.25) + .3(.75)} = 0.4.$$

61. A student answers a multiple-choice examination question that offers four possible answers. Suppose the probability that the student knows the answer to the question is .8 and the probability that the student will guess is .2. Assume that if the student guesses, the probability of selecting the correct answer is .25. If the student correctly answers a question, what is the probability that the student really knew the correct answer?

Solution:

Define the events: G: student guesses C: student is correct

$$P(\overline{G} \mid C) = \frac{P(C \mid \overline{G})P(\overline{G})}{P(C \mid \overline{G})P(\overline{G}) + P(C \mid G)P(G)} = \frac{1(.8)}{1(.8) + .25(.2)} = 0.9412.$$

62. Of the travelers arriving at a small airport, 60% fly on major airlines, 30% fly on privately owned planes, and the remainder fly on commercially owned planes not belonging to a major airline. Of those traveling on major airlines, 50% are traveling for business reasons, whereas 60% of those arriving on private planes and 90% of those arriving on other commercially owned planes are traveling for business reasons. Suppose that we randomly select one person arriving at this airport. What is the probability that the person

a. is traveling on business?

b. is traveling for business on a privately owned plane?

c. arrived on a privately owned plane, given that the person is traveling for business reasons?

d. is traveling on business, given that the person is flying on a commercially owned plane?

Solution:

Let M = major airline, P = private airline, C = commercial airline, B = travel for business

a. $P(B) = P(B|M)P(M) + P(B|P)P(P) + P(B|C)P(C) = .6(.5) + .3(.6) + .1(.9) = 0.57.$

b. $P(B \cap P) = P(B|P)P(P) = .3(.6) = 0.18.$

c. $P(P|B) = P(B \cap P)/P(B) = .18/.57 = 0.3158.$

d. $P(B|C) = 0.90.$

63. Five identical bowls are labeled 1, 2, 3, 4, and 5. Bowl i contains i white and 5 − i black balls, with i = 1, 2, . . . , 5. A bowl is randomly selected and two balls are randomly selected (without replacement) from the contents of the bowl.

a. What is the probability that both balls selected are white?

b. Given that both balls selected are white, what is the probability that bowl 3 was selected?

Solution:

Let A = {both balls are white}, and for $i = 1, 2, \ldots 5$

A_i = both balls selected from bowl i are white. Then $\bigcup A_i = A$.

B_i = bowl i is selected. Then, $P(B_i) = .2$ for all i.

a. $P(A) = \sum P(A_i \mid B_i)P(B_i) = \frac{1}{5}\left[0 + \frac{2}{5}\left(\frac{1}{4}\right) + \frac{3}{5}\left(\frac{2}{4}\right) + \frac{4}{5}\left(\frac{3}{4}\right) + 1\right] = 2/5.$

b. Using Bayes' rule, $P(B_3|A) = \dfrac{\frac{3}{50}}{\frac{2}{50}} = 3/20.$

64. Define an experiment as tossing two coins and observing the results. Let Y equal the number of heads obtained.

a. Identify the sample points in S, assign a value of Y to each sample point, and identify the sample points associated with each value of the random variable Y .

b. Compute the probabilities for each value of Y.

Solution:

a. Let H and T represent head and tail, respectively; and let an ordered pair of symbols identify the outcome for the first and second coins. (Thus, HT implies a head on the first coin and a tail on the second.) Then the four sample points in S are E1: HH, E2: HT, E3: T H and

E4: TT. The values of Y assigned to the sample points depend on the number of heads associated with each point. For E1 : HH, two heads were observed, and E1 is assigned the value Y = 2. Similarly, we assign the values Y = 1 to E2 and E3 and Y = 0 to E4. Summarizing, the random variable Y can take three values, Y = 0, 1, and 2, which are events defined by specific collections of sample points:

{Y = 0} = {E4}, {Y = 1} = {E2, E3}, {Y = 2} = {E1}.

b. The event {Y = 0} results only from sample point E4. If the coins are balanced, the sample points are equally likely; hence, P(Y = 0) = P(E4) = 1/4. Similarly,

P(Y = 1) = P(E2) + P(E3) = 1/2 and P(Y = 2) = P(E1) = 1/4.

DISCRETE RANDOM VARIABLES AND THEIR PROBABILITY DISTRUBITION

65. A supervisor in a manufacturing plant has three men and three women working for him. He wants to choose two workers for a special job. Not wishing to show any biases in his selection, he decides to select the two workers at random. Let Y denote the number of women in his selection. Find the probability distribution for Y.

Solution:

The supervisor can select two workers from six in $\binom{6}{2} = 15$ ways. Hence, S contains 15 sample points, which we assume to be equally likely because random sampling was employed. Thus, $P(E_i) = 1/15$, for $i = 1, 2, \ldots, 15$. The values for Y that have nonzero probability are 0, 1, and 2. The number of ways of selecting $Y = 0$ women is $\binom{3}{0}\binom{3}{2}$ because the supervisor must select zero workers from the three women and two from the three men. Thus, there are $\binom{3}{0}\binom{3}{2} = 1 \cdot 3 = 3$ sample points in the event $Y = 0$, and

$$p(0) = P(Y = 0) = \frac{\binom{3}{0}\binom{3}{2}}{15} = \frac{3}{15} = \frac{1}{5}.$$

Similarly,

$$p(1) = P(Y = 1) = \frac{\binom{3}{1}\binom{3}{1}}{15} = \frac{9}{15} = \frac{3}{5},$$

$$p(2) = P(Y = 2) = \frac{\binom{3}{2}\binom{3}{0}}{15} = \frac{3}{15} = \frac{1}{5}.$$

Notice that $(Y = 1)$ is by far the most likely outcome. This should seem reasonable since the number of women equals the number of men in the original group.

66. When the health department tested private wells in a county for two impurities commonly found in drinking water, it found that 20% of the wells had neither impurity, 40% had impurity A, and 50% had impurity B. (Obviously, some had both impurities.) If a well is randomly chosen from those in the county, find the probability distribution for Y , the number of impurities found in the well.

Solution:

$P(Y = 0) = P(\text{no impurities}) = .2$, $P(Y = 1) = P(\text{exactly one impurity}) = .7$, $P(Y = 2) = .1$.

67. A group of four components is known to contain two defectives. An inspector tests the components one at a time until the two defectives are located. Once she locates the two defectives, she stops testing, but the second defective is tested to ensure accuracy. Let Y denote the number of the test on which the second defective is found. Find the probability distribution for Y.

Solution:

$p(2) = P(DD) = 1/6$, $p(3) = P(DGD) + P(GDD) = 2(2/4)(2/3)(1/2) = 2/6$, $p(4) = P(GGDD) + P(DGGD) + P(GDGD) = 3(2/4)(1/3)(2/2) = 1/2$.

68. A problem in a test given to small children asks them to match each of three pictures of animals to the word identifying that animal. If a child assigns the three words at random to the three pictures, find the probability distribution for Y , the number of correct matches.

Solution:

There are $3! = 6$ possible ways to assign the words to the pictures. Of these, one is a perfect match, three have one match, and two have zero matches. Thus,

$p(0) = 2/6$, $p(1) = 3/6$, $p(3) = 1/6$.

69. Each of three balls are randomly placed into one of three bowls. Find the probability distribution for Y = the number of empty bowls.

Solution:

There are $3^3 = 27$ ways to place the three balls into the three bowls. Let Y = # of empty bowls. Then:

$p(0) = P(\text{no bowls are empty}) = \frac{3!}{27} = \frac{6}{27}$

$p(2) = \text{P}(2 \text{ bowls are empty}) = \frac{3}{27}$

$p(1) = \text{P}(1 \text{ bowl is empty}) = 1 - \frac{6}{27} - \frac{3}{27} = \frac{18}{27}$.

70. In order to verify the accuracy of their financial accounts, companies use auditors on a regular basis to verify accounting entries. The company's employees make erroneous entries 5% of the time. Suppose that an auditor randomly checks three entries.

a. Find the probability distribution for Y , the number of errors detected by the auditor.

b. Construct a probability histogram for p(y).

c. Find the probability that the auditor will detect more than one error.

Solution:

The random variable Y takes on vales 0, 1, 2, and 3.

a. Let E denote an error on a single entry and let N denote no error. There are 8 sample points: $EEE, EEN, ENE, NEE, ENN, NEN, NNE, NNN$. With $P(E) = .05$ and $P(N) = .95$ and assuming independence:

$P(Y = 3) = (.05)^3 = 0.000125$ $\quad$ $P(Y = 2) = 3(.05)2(.95) = 0.007125$

$P(Y = 1) = 3(.05)^2(.95) = 0.135375$ $\quad$ $P(Y = 0) = (.95)^3 = 0.857375.$

b. The graph is omitted.

c. $P(Y > 1) = P(Y = 2) + P(Y = 3) = 0.00725.$

71. Persons entering a blood bank are such that 1 in 3 have type O+ blood and 1 in 15 have type O− blood. Consider three randomly selected donors for the blood bank. Let X denote the number of donors with type O+ blood and Y denote the number with type O− blood. Find the probability distributions for X and Y . Also find the probability distribution for X + Y , the number of donors who have type O blood.

Solution:

There is a 1/3 chance a person has O^+ blood and 2/3 they do not. Similarly, there is a 1/15 chance a person has O^- blood and 14/15 chance they do not. Assuming the donors are randomly selected, if X = # of O^+ blood donors and Y = # of O^- blood donors, the probability distributions are

	0	1	2	3
$p(x)$	$(2/3)^3 = 8/27$	$3(2/3)^2(1/3) = 12/27$	$3(2/3)(1/3)^2 = 6/27$	$(1/3)^3 = 1/27$
$p(y)$	2744/3375	588/3375	14/3375	1/3375

Note that $Z = X + Y$ = # will type O blood. The probability a donor will have type O blood is $1/3 + 1/15 = 6/15 = 2/5$. The probability distribution for Z is

	0	1	2	3
$p(z)$	$(3/5)^3 = 27/125$	$3(3/5)^2(2/5) = 54/27$	$3(3/5)(2/5)^2 = 36/125$	$(2/5)^3 = 8/125$

72. The probability distribution for a random variable Y is given in table. Find the mean, variance, and standard deviation of Y .

y	$p(y)$
0	1/8
1	1/4
2	3/8
3	1/4

Solution:

$$\mu = E(Y) = \sum_{y=0}^{3} yp(y) = (0)(1/8) + (1)(1/4) + (2)(3/8) + (3)(1/4) = 1.75,$$

$$\sigma^2 = E[(Y-\mu)^2] = \sum_{y=0}^{3}(y-\mu)^2 p(y)$$

$$= (0-1.75)^2(1/8) + (1-1.75)^2(1/4) + (2-1.75)^2(3/8) + (3-1.75)^2(1/4)$$

$$= .9375,$$

$$\sigma = +\sqrt{\sigma^2} = \sqrt{.9375} = .97.$$

73. The manager of an industrial plant is planning to buy a new machine of either type A or type B. If t denotes the number of hours of daily operation, the number of daily repairs Y1 required to maintain a machine of type A is a random variable with mean and variance both equal to .10t. The number of daily repairs Y2 for a machine of type B is a random variable with mean and variance both equal to .12t. The daily cost of operating A is $CA(t) = 10t + 30Y_1^2$; for B it is $CB(t) = 8t + 30Y_2^2$. Assume that the repairs take negligible time and that each night the machines are tuned so that they operate essentially like new machines at the start of the next day. Which machine minimizes the expected daily cost if a workday consists of (a) 10 hours and (b) 20 hours?

Solution:

The expected daily cost for A is

$$E[C_A(t)] = E\left[10t + 30Y_1^2\right] = 10t + 30E\left(Y_1^2\right)$$
$$= 10t + 30\{V(Y_1) + [E(Y_1)]^2\} = 10t + 30[.10t + (.10t)^2]$$
$$= 13t + .3t^2.$$

In this calculation, we used the known values for $V(Y_1)$ and $E(Y_1)$ and the fact that $V(Y_1) = E(Y_1^2) - [E(Y_1)]^2$ to obtain that $E(Y_1^2) = V(Y_1) + [E(Y_1)]^2 = .10t + (.10t)^2$. Similarly,

$$E[C_B(t)] = E\left[8t + 30Y_2^2\right] = 8t + 30E\left(Y_2^2\right)$$
$$= 8t + 30\{V(Y_2) + [E(Y_2)]^2\} = 8t + 30[.12t + (.12t)^2]$$
$$= 11.6t + .432t^2.$$

Thus, for scenario (a) where $t = 10$,

$$E[C_A(10)] = 160 \quad \text{and} \quad E[C_B(10)] = 159.2,$$

which results in the choice of machine B.

For scenario (b), $t = 20$ and

$$E[C_A(20)] = 380 \quad \text{and} \quad E[C_B(20)] = 404.8,$$

resulting in the choice of machine A.

In conclusion, machines of type B are more economical for short time periods because of their smaller hourly operating cost. For long time periods, however, machines of type A are more economical because they tend to be repaired less frequently.

74. An insurance company issues a one-year \$1000 policy insuring against an occurrence A that historically happens to 2 out of every 100 owners of the policy. Administrative fees are \$15 per policy and are not part of the company's "profit." How much should the company charge for the policy if it requires that the expected profit per policy be \$50? [Hint: If C is the premium for the policy, the company's "profit" is C−15 if A does not occur and C−15−1000 if A does occur.]

Solution:

Let P be a random variable that represents the company's profit. Then, $P = C - 15$ with probability 98/100 and $P = C - 15 - 1000$ with probability 2/100. Then, $E(P) = (C - 15)(98/100) + (C - 15 - 1000)(2/100) = 50$. Thus, $C = \$85$.

75. Who is the king of late night TV? An Internet survey estimates that, when given a choice between David Letterman and Jay Leno, 52% of the population prefers to watch Jay Leno. Three late night TV watchers are randomly selected and asked which of the two talk show hosts they prefer.

a. Find the probability distribution for Y, the number of viewers in the sample who prefer Leno.

b. Construct a probability histogram for p(y).

c. What is the probability that exactly one of the three viewers prefers Leno?

d. What are the mean and standard deviation for Y?

e. What is the probability that the number of viewers favoring Leno falls within 2 standard deviations of the mean?

Solution:

a. $p(0) = P(Y = 0) = (.48)^3 = .1106$, $p(1) = P(Y = 1) = 3(.48)^2(.52) = .3594$, $p(2) = P(Y = 2) = 3(.48)(.52)^2 = .3894$, $p(3) = P(Y = 3) = (.52)^3 = .1406$.
b. The graph is omitted.
c. $P(Y = 1) = .3594$.
d. $\mu = E(Y) = 0(.1106) + 1(.3594) + 2(.3894) + 3(.1406) = 1.56$,
$\sigma^2 = V(Y) = E(Y^2) - [E(Y)]^2 = 0^2(.1106) + 1^2(.3594) + 2^2(.3894) + 3^2(.1406) - 1.56^2 = 3.1824 - 2.4336 = .7488$. So, $\sigma = 0.8653$.
e. $(\mu - 2\sigma, \mu + 2\sigma) = (-.1706, 3.2906)$. So, $P(-.1706 < Y < 3.2906) = P(0 \le Y \le 3) = 1$.

76. In a gambling game a person draws a single card from an ordinary 52-card playing deck. A person is paid $15 for drawing a jack or a queen and $5 for drawing a king or an ace. A person who draws any other card pays $4. If a person plays this game, what is the expected gain?

Solution:

Define G to be the gain to a person in drawing one card. The possible values for G are $15, $5, or $–4 with probabilities 3/13, 2/13, and 9/13 respectively. So,
$E(G) = 15(3/13) + 5(2/13) - 4(9/13) = 4/13$ (roughly $.31).

77. Two construction contracts are to be randomly assigned to one or more of three firms: I, II, and III. Any firm may receive both contracts. If each contract will yield a profit of $90,000 for the firm, find the expected profit for firm I. If firms I and II are actually owned by the same individual, what is the owner's expected total profit?

Solution:

Let X_1 = # of contracts assigned to firm 1; X_2 = # of contracts assigned to firm 2. The sample space for the experiment is {(I,I), (I,II), (I,III), (II,I), (II,II), (II,III), (III,I), (III,II), (III,III)}, each with probability 1/9. So, the probability distributions for X_1 and X_2 are:

x_1	0	1	2
$p(x_1)$	4/9	4/9	1/9

x_2	0	1	2
$p(x_2)$	4/9	4/9	1/9

Thus, $E(X_1) = E(X_2) = 2/3$. The expected profit for the owner of both firms is given by $90000(2/3 + 2/3) = \$120,000$.

78. A potential customer for an $85,000 fire insurance policy possesses a home in an area that, according to experience, may sustain a total loss in a given year with probability of .001 and a 50% loss with probability .01. Ignoring all other partial losses, what premium should the insurance company charge for a yearly policy in order to break even on all $85,000 policies in this area?

Solution:

Let Y = the payout on an individual policy. Then, $P(Y = 85,000) = .001$, $P(Y = 42,500) = .01$, and $P(Y = 0) = .989$. Let C represent the premium the insurance company charges. Then, the company's net gain/loss is given by $C - Y$. If $E(C - Y) = 0$, $E(Y) = C$. Thus, $E(Y) = 85000(.001) + 42500(.01) + 0(.989) = 510 = C$.

79. Suppose that Y is a discrete random variable with mean μ and variance σ^2 and let W = 2Y.

a. Do you expect the mean of W to be larger than, smaller than, or equal to μ=E(Y)? Why?

b. Express E(W) = E(2Y) in terms of μ = E(Y). Does this result agree with your answer to part (a)?

c. Recalling that the variance is a measure of spread or dispersion, do you expect the variance of W to be larger than, smaller than, or equal to σ^2 = V(Y)? Why?

d. Use the result in part (b) to show that $V(W) = E\{[W - E(W)]^2\} = E[4(Y - \mu)^2] = 4\sigma^2$; that is, W = 2Y has variance four times that of Y .

Solution:

a. The mean of W will be larger than the mean of Y if $\mu > 0$. If $\mu < 0$, the mean of W will be smaller than μ. If $\mu = 0$, the mean of W will equal μ.

b. $E(W) = E(2Y) = 2E(Y) = 2\mu$.
c. The variance of W will be larger than σ^2, since the spread of values of W has increased.
d. $V(X) = E[(X - E(X))^2] = E[(2Y - 2\mu)^2] = 4E[(Y - \mu)^2] = 4\sigma^2$.

80. Let Y be a discrete random variable with mean μ and variance σ^2. If a and b are constants, prove that,

a. $E(aY + b) = aE(Y) + b = a\mu + b$.

b. $V(aY + b) = a^2V(Y) = a^2\sigma^2$.

Solution:

a. $E(aY + b) = E(aY) + E(b) = aE(Y) + b = a\mu + b$.
b. $V(aY + b) = E[(aY + b - a\mu - b)^2] = E[(aY - a\mu)^2] = a^2E[(Y - \mu)^2] = a^2\sigma^2$.

81. Consider the population of voters described in Example 3.6. Suppose that there are N = 5000 voters in the population, 40% of whom favor Jones. Identify the event favors Jones as a success S. It is evident that the probability of S on trial 1 is .40. Consider the event B that S occurs on the second trial. Then B can occur two ways: The first two trials are both successes or the first trial is a failure and the second is a success. Show that P(B) = .4. What is P(B| the first trial is S)? Does this conditional probability differ markedly from P(B)?

Solution:

With $B = SS \cup FS$, $P(B) = P(SS) + P(FS) = \frac{2000}{5000}\left(\frac{1999}{4999}\right) + \frac{3000}{5000}\left(\frac{2000}{4999}\right) = 0.4$
$P(B|\text{first trial success}) = \frac{1999}{4999} = 0.3999$, which is not very different from the above.

82. In 2003, the average combined SAT score (math and verbal) for college-bound students in the United States was 1026. Suppose that approximately 45% of all high school graduates took this test and that 100 high school graduates are randomly selected from among all high school grads in the United States. Which of the following random variables has a distribution that can be approximated by a binomial distribution? Whenever possible, give the values for n and p.

a. The number of students who took the SAT

b. The scores of the 100 students in the sample

c. The number of students in the sample who scored above average on the SAT

d. The amount of time required by each student to complete the SAT

e. The number of female high school grads in the sample

Solution:

a. Not a binomial random variable.
b. Not a binomial random variable.
c. Binomial with $n = 100$, p = proportion of high school students who scored above 1026.
d. Not a binomial random variable (not discrete).
e. Not binomial, since the sample was not selected among all female HS grads.

83. A complex electronic system is built with a certain number of backup components in its subsystems. One subsystem has four identical components, each with a probability of .2 of failing in less than 1000 hours. The subsystem will operate if any two of the four components are operating. Assume that the components operate independently. Find the probability that

a. exactly two of the four components last longer than 1000 hours.

b. the subsystem operates longer than 1000 hours.

Solution:

Let Y = # of components failing in less than 1000 hours. Then, Y is binomial with $n = 4$ and $p = .2$.

a. $P(Y=2) = \binom{4}{2} .2^2 (.8)^2 = 0.1536.$

b. The system will operate if 0, 1, or 2 components fail in less than 1000 hours. So, P(system operates) = .4096 + .4096 + .1536 = .9728.

84. A multiple-choice examination has 15 questions, each with five possible answers, only one of which is correct. Suppose that one of the students who takes the examination answers each of the questions with an independent random guess. What is the probability that he answers at least ten questions correctly?

Solution: Let Y = # of correct answers. Then, Y is binomial with $n = 15$ and $p = .2$.

$P(Y \geq 10) = 1 - P(Y \leq 9) = 1 - 1.000 = 0.000$ (to three decimal places).

85. Many utility companies promote energy conservation by offering discount rates to consumers who keep their energy usage below certain established subsidy standards. A recent EPA report notes that 70% of the island residents of Puerto Rico have reduced their electricity usage sufficiently to qualify for discounted rates. If five residential subscribers are randomly selected from San Juan, Puerto Rico, find the probability of each of the following events:

a. All five qualify for the favorable rates.

b. At least four qualify for the favorable rates.

Solution:

Let Y = # of qualifying subscribers. Then, Y is binomial with $n = 5$ and $p = .7$.

a. $P(Y = 5) = .7^5 = .1681$

b. $P(Y \geq 4) = P(Y = 4) + P(Y = 5) = 5(.7^4)(.3) + .7^5 = .3601 + .1681 = 0.5282$.

86. A fire-detection device utilizes three temperature-sensitive cells acting independently of each other in such a manner that any one or more may activate the alarm. Each cell possesses a probability of p = .8 of activating the alarm when the temperature reaches 100◦ Celsius or more. Let Y equal the number of cells activating the alarm when the temperature reaches 100◦.

a. Find the probability distribution for Y .

b. Find the probability that the alarm will function when the temperature reaches 100◦.

Solution:

Note that Y is binomial with $n = 3$ and $p = .8$. The alarm will function if $Y = 1$, 2, or 3. Thus, $P(Y \geq 1) = 1 - P(Y = 0) = 1 - .008 = 0.992$.

87. Tay-Sachs disease is a genetic disorder that is usually fatal in young children. If both parents are carriers of the disease, the probability that their offspring will develop the disease is approximately .25. Suppose that a husband and wife are both carriers and that they have three children. If the outcomes of the three pregnancies are mutually independent, what are the probabilities of the following events?

a. All three children develop Tay-Sachs.

b. Only one child develops Tay-Sachs.

c. The third child develops Tay-Sachs, given that the first two did not.

Solution:

There is a 25% chance the offspring of the parents will develop the disease. Then, Y = # of offspring that develop the disease is binomial with $n = 3$ and $p = .25$.

a. $P(Y = 3) = (.25)^3 = 0.015625.$

b. $P(Y = 1) = 3(.25)(.75)^2 = 0.421875$

c. Since the pregnancies are mutually independent, the probability is simply 25%.

88. In the 18th century, the Chevalier de Mere asked Blaise Pascal to compare the probabilities of two events. Below, you will compute the probability of the two events that, prior to contrary gambling experience, were thought by de Mere to be equally likely.

a. What is the probability of obtaining at least one 6 in four rolls of a fair die?

b. If a pair of fair dice is tossed 24 times, what is the probability of at least one double six?

Solution:

a. P(at least one 6 in four rolls) = 1 – P(no 6's in four rolls) = $1 - (5/6)^4 = 0.51775$.

b. Note that in a single toss of two dice, P(double 6) = 1/36. Then:
P(at least one double 6 in twenty–four rolls) = 1 – P(no double 6's in twenty–four rolls) = $= 1 - (35/36)^{24} = 0.4914$.

89. A manufacturer of floor wax has developed two new brands, A and B, which she wishes to subject to homeowners' evaluation to determine which of the two is superior. Both waxes, A and B, are applied to floor surfaces in each of 15 homes.

Assume that there is actually no difference in the quality of the brands. What is the probability that ten or more homeowners would state a preference for

a. brand A?

b. either brand A or brand B?

Solution:

Let Y = # of housewives preferring brand A. Thus, Y is binomial with $n = 15$ and $p = .5$.

a. Using the Appendix, $P(Y \geq 10) = 1 - P(Y \leq 9) = 1 - .849 = 0.151$.

b. $P(10$ or more prefer A or $B) = P(6 \leq Y \leq 9) = 0.302$.

90. Suppose that Y is a binomial random variable with n > 2 trials and success probability p. Use the technique presented in Theorem 3.7 and the fact that

$E\{Y(Y-1)(Y-2)\} = E(Y^3) - 3E(Y^2) + 2E(Y)$ to derive $E(Y^3)$.

Solution:

$$E\{Y(Y-1)Y-2)\} = \sum_{y=0}^{n} \frac{y(y-1)(y-2)n!}{y!(n-y)!} p^y(1-p)^{n-y} = \sum_{y=3}^{n} \frac{n(n-1)(n-2)(n-3)!}{(y-3)!(n-3-(y-3))!} p^y(1-p)^{n-y}$$

$$= n(n-1)(n-2)p^3 \sum_{z=0}^{n-3} \binom{n-3}{z} p^z(1-p)^{n-3-z} = n(n-1)(n-2)p^3 .$$

Equating this to $E(Y^3) - 3E(Y^2) + 2E(Y)$, it is found that

$E(Y^3) = 3n(n-1)p^2 - n(n-1)(n-2)p^3 + np$.

91. Ten motors are packaged for sale in a certain warehouse. The motors sell for \$100 each, but a double-your-money-back guarantee is in effect for any defectives the purchaser may receive. Find the expected net gain for the seller if the probability of any one motor being defective is .08. (Assume that the quality of any one motor is independent of that of the others.)

Solution:

If Y = # of defective motors, then Y is binomial with $n = 10$ and $p = .08$. Then, $E(Y) = .8$. The seller's expected next gain is \$1000 – \$200$E(Y)$ = \$840.

92. Of the volunteers donating blood in a clinic, 80% have the Rhesus (Rh) factor present in their blood.

a. If five volunteers are randomly selected, what is the probability that at least one does not have the Rh factor?

b. If five volunteers are randomly selected, what is the probability that at most four have the Rh factor?

c. What is the smallest number of volunteers who must be selected if we want to be at least 90% certain that we obtain at least five donors with the Rh factor?

<u>Solution:</u>

Let Y = # with Rh^+ blood. Then, Y is binomial with $n = 5$ and $p = .8$

a. $1 - P(Y = 5) = .672$.

b. $P(Y \le 4) = .672$.

c. We need n for which $P(Y \ge 5) = 1 - P(Y \le 4) > .9$. The smallest n is 8.

93. Suppose that the probability of engine malfunction during any one-hour period is p = .02. Find the probability that a given engine will survive two hours.

<u>Solution:</u>

Letting Y denote the number of one-hour intervals until the first malfunction, we have

$$P(\text{survive two hours}) = P(Y \ge 3) = \sum_{y=3}^{\infty} p(y).$$

Because $\sum_{y=1}^{\infty} p(y) = 1$,

$$P(\text{survive two hours}) = 1 - \sum_{y=1}^{2} p(y)$$
$$= 1 - p - qp = 1 - .02 - (.98)(.02) = .9604.$$

94. If the probability of engine malfunction during any one-hour period is p = .02 and Y denotes the number of one-hour intervals until the first malfunction, find the mean and standard deviation of Y .

Solution:

it follows that Y has a geometric distribution with $p = .02$. Thus, $E(Y) = 1/p = 1/(.02) = 50$, and we expect to wait quite a few hours before encountering a malfunction. Further, $V(Y) = .98/.0004 = 2450$, and it follows that the standard deviation of Y is $\sigma = \sqrt{2450} = 49.497$.

95. Suppose that 30% of the applicants for a certain industrial job possess advanced training in computer programming. Applicants are interviewed sequentially and are selected at random from the pool. Find the probability that the first applicant with advanced training in programming is found on the fifth interview.

Solution:

$(.7)^4(.3) = 0.07203.$

96. About six months into GeorgeW. Bush's second term as president, a Gallup poll indicated that a near record (low) level of 41% of adults expressed "a great deal" or "quite a lot" of confidence in the U.S. Supreme Court Suppose that you conducted your own telephone survey at that time and randomly called people and asked them to describe their level of confidence in the Supreme Court. Find the probability distribution for Y , the number of calls until the first person is found who does not express "a great deal" or "quite a lot" of confidence in the U.S. Supreme Court.

Solution:

Y has a geometric distribution with $p = 1 - .41 = .59.$

97. Let Y denote a geometric random variable with probability of success p.

a. Show that for a positive integer a, $P(Y > a) = q^a$.

b. Show that for positive integers a and b, $P(Y > a + b|Y > a) = q^b = P(Y > b)$.

This result implies that, for example, $P(Y > 7|Y > 2) = P(Y > 5)$. Why do you think this property is called the memoryless property of the geometric distribution?

c. In the development of the distribution of the geometric random variable, we assumed that the experiment consisted of conducting identical and independent trials until the first success was observed. In light of these assumptions, why is the result in part (b) "obvious"?

Solution:

a. $P(Y > a) = \sum_{y=a+1}^{\infty} q^{y-1} p = q^a \sum_{x=1}^{\infty} q^{x-1} p = q^a$.

b. From part (a), $P(Y > a + b \mid Y > a) = \frac{P(Y > a + b, Y > a)}{P(Y > a)} = \frac{P(Y > a + b)}{P(Y > a)} = \frac{q^{a+b}}{q^a} = q^b$.

c. The results in the past are not relevant to a future outcome (independent trials).

98. A certified public accountant (CPA) has found that nine of ten company audits contain substantial errors. If the CPA audits a series of company accounts, what is the probability that the first account containing substantial errors

a. is the third one to be audited?

b. will occur on or after the third audited account?

Solution:

Let Y = # of accounts audited until the first with substantial errors is found.

a. $P(Y = 3) = .1^2(.9) = .009$.

b. $P(Y \geq 3) = P(Y > 2) = .1^2 = .01$.

99. The probability of a customer arrival at a grocery service counter in any one second is equal to .1. Assume that customers arrive in a random stream and hence that an arrival in any one second is independent of all others. Find the probability that the first arrival

a. will occur during the third one-second interval.

b. will not occur until at least the third one-second interval.

Solution:

Let Y = # of one second intervals until the first arrival, so that $p = .1$
a. $P(Y = 3) = (.9)2(.1) = .081$.
b. $P(Y \geq 3) = P(Y > 2) = .9^2 = .81$.

100. How many times would you expect to toss a balanced coin in order to obtain the first head?

Solution:

With $p = 1/2$, then $\mu = 1/(1/2) = 2$.

101. In responding to a survey question on a sensitive topic (such as "Have you ever tried marijuana?"), many people prefer not to respond in the affirmative. Suppose that 80% of the population have not tried marijuana and all of those individuals will truthfully answer no to your question. The remaining 20% of the population have tried marijuana and 70% of those individuals will lie. Derive the probability distribution of Y , the number of people you would need to question in order to obtain a single affirmative response.

Solution:

Let Y = # of people questioned before a "yes" answer is given. Then, Y has a geometric distribution with $p = P(\text{yes}) = P(\text{smoker and "yes"}) + P(\text{nonsmoker and "yes"}) = .3(.2) + 0 = .06$. Thus, $p(y) = .06(.94)^{y-1}$. $y = 1, 2, \ldots$.

102. A geological study indicates that an exploratory oil well drilled in a particular region should strike oil with probability .2. Find the probability that the third oil strike comes on the fifth well drilled.

Solution:

Assuming independent drillings and probability .2 of striking oil with any one well, let Y denote the number of the trial on which the third oil strike occurs. Then it is reasonable to assume that Y has a negative binomial distribution with $p = .2$. Because we are interested in $r = 3$ and $y = 5$,

$$P(Y = 5) = p(5) = \binom{4}{2}(.2)^3(.8)^2$$
$$= 6(.008)(.64) = .0307.$$

103. A large stockpile of used pumps contains 20% that are in need of repair. A maintenance worker is sent to the stockpile with three repair kits. She selects pumps at random and tests them one at a time. If the pump works, she sets it aside for future use. However, if the pump does not work, she uses one of her repair kits on it. Suppose that it takes 10 minutes to test a pump that is in working condition and 30 minutes to test and repair a pump that does not work. Find the mean and variance of the total time it takes the maintenance worker to use her three repair kits.

Solution:

Let Y denote the number of the trial on which the third nonfunctioning pump is found. It follows that Y has a negative binomial distribution with $p = .2$. Thus, $E(Y) = 3/(.2) = 15$ and $V(Y) = 3(.8)/(.2)^2 = 60$. Because it takes an additional 20 minutes to repair each defective pump, the total time necessary to use the three kits is

$$T = 10Y + 3(20).$$

Using the result derived in Exercise 3.33, we see that

$$E(T) = 10E(Y) + 60 = 10(15) + 60 = 210$$

and

$$V(T) = 10^2 V(Y) = 100(60) = 6000.$$

Thus, the total time necessary to use all three kits has mean 210 and standard deviation $\sqrt{6000} = 77.46$.

104. A geological study indicates that an exploratory oil well should strike oil with probability .2.

a. What is the probability that the first strike comes on the third well drilled?

b. What is the probability that the third strike comes on the seventh well drilled?

c. What assumptions did you make to obtain the answers to parts (a) and (b)?

d. Find the mean and variance of the number of wells that must be drilled if the company wants to set up three producing wells.

Solution:

a. Geometric probability calculation: $(.8)^2(.2) = .128$.

b. Negative binomial probability calculation: $\binom{6}{2}(.2)^3(.8)^4 = .049$.

c. The trials are independent and the probability of success is the same from trial to trial.

105. The employees of a firm that manufactures insulation are being tested for indications of asbestos in their lungs. The firm is requested to send three employees who have positive indications of asbestos on to a medical center for further testing. If each test costs \$20, find the expected value and variance of the total cost of conducting the tests necessary to locate the three positives.

Solution:

The total cost is given by $20Y$. So, $E(20Y) = 20E(Y) = 20\frac{3}{.4} = \$150$. Similarly, $V(20Y) = 400V(Y) = 4500$.

106. Ten percent of the engines manufactured on an assembly line are defective. What is the probability that the third nondefective engine will be found

a. on the fifth trial?

b. on or before the fifth trial?

c. Given that the first two engines tested were defective, what is the probability that at least two more engines must be tested before the first nondefective is found?

d. Find the mean and variance of the number of the trial on which a the first nondefective engine is found.

e. Find the mean and variance of the number of the trial on which the third nondefective engine is found.

Solution:

a. $P(Y=5) = \binom{4}{2}(.9)^3(.1)^2 = .04374$.

b. $P(Y \le 5) = P(Y=3) + P(Y=4) + P(Y=5) = .729 + .2187 + .04374 = .99144$.

c. $P(Y \ge 4 \mid Y > 2) = P(Y > 3 \mid Y > 2) = P(Y > 1) = 1 - P(Y = 0) = .1$.

_107._ In a sequence of independent identical trials with two possible outcomes on each trial, S and F, and with P(S) = p, what is the probability that exactly y trials will occur before the r th success?

Solution:

Define a random variable $X = y$ trials before the before the first success, $y = r - 1, r, r + 1, \ldots$. Then, $X = Y - 1$, where Y has the negative binomial distribution with parameters r and p. Thus, $p(x) = \frac{y!}{(r-1)!(y-r+1)!} p^r q^{y+1-r}, y = r - 1, r, r + 1, \ldots$.

_108._ An important problem encountered by personnel directors and others faced with the selection of the best in a finite set of elements is exemplified by the following scenario. From a group of 20 Ph.D. engineers, 10 are randomly selected for employment. What is the probability that the 10 selected include all the 5 best engineers in the group of 20?

Solution:

For this example N = 20, n = 10, and r = 5. That is, there are only 5 in the set of 5 best engineers, and we seek the probability that Y = 5, where Y denotes the number of best engineers among the ten selected. Then

$$p(5) = \frac{\binom{5}{5}\binom{15}{5}}{\binom{20}{10}} = \left(\frac{15!}{5!10!}\right)\left(\frac{10!10!}{20!}\right) = \frac{21}{1292} = .0162.$$

_109._ An industrial product is shipped in lots of 20. Testing to determine whether an item is defective is costly, and hence the manufacturer samples his production rather than using a 100% inspection plan. A sampling plan, constructed to minimize the number of defectives shipped to customers, calls for sampling five items from each lot and rejecting the lot if more than one defective is observed. (If the lot is rejected, each item in it is later tested.) If a lot contains four defectives, what is the probability that it will be rejected? What is the expected number of defectives in the sample of size 5? What is the variance of the number of defectives in the sample of size 5?

Solution:

Let Y equal the number of defectives in the sample. Then $N = 20$, $r = 4$, and $n = 5$. The lot will be rejected if $Y = 2, 3,$ or 4. Then

$$\begin{aligned} P(\text{rejecting the lot}) &= P(Y \geq 2) = p(2) + p(3) + p(4) \\ &= 1 - p(0) - p(1) \\ &= 1 - \frac{\binom{4}{0}\binom{16}{5}}{\binom{20}{5}} - \frac{\binom{4}{1}\binom{16}{4}}{\binom{20}{5}} \\ &= 1 - .2817 - .4696 = .2487. \end{aligned}$$

The mean and variance of the number of defectives in the sample of size 5 are

$$\mu = \frac{(5)(4)}{20} = 1 \quad \text{and} \quad \sigma^2 = 5\left(\frac{4}{20}\right)\left(\frac{20-4}{20}\right)\left(\frac{20-5}{20-1}\right) = .632.$$

110. A warehouse contains ten printing machines, four of which are defective. A company selects five of the machines at random, thinking all are in working condition. What is the probability that all five of the machines are nondefective?

Solution:

Use the hypergeometric probability distribution with $N = 10$, $r = 4$, $n = 5$. $P(Y = 0) = \frac{1}{42}$.

111. In southern California, a growing number of individuals pursuing teaching credentials are choosing paid internships over traditional student teaching programs. A group of eight candidates for three local teaching positions consisted of five who had enrolled in paid internships and three who enrolled in traditional student teaching programs. All eight candidates appear to be equally qualified, so three are randomly selected to fill the open positions. Let Y be the number of internship trained candidates who are hired.

a. Does Y have a binomial or hypergeometric distribution? Why?

b. Find the probability that two or more internship trained candidates are hired.

c. What are the mean and standard deviation of Y?

Solution:

a. The random variable Y follows a hypergeometric distribution. The probability of being chosen on a trial is dependent on the outcome of previous trials.

b. $P(Y \geq 2) = P(Y = 2) + P(Y = 3) = \dfrac{\binom{5}{2}\binom{3}{1}}{\binom{8}{3}} + \dfrac{\binom{5}{3}}{\binom{8}{3}} = .5357 + .1786 = 0.7143$.

c. $\mu = 3(5/8) = 1.875$, $\sigma^2 = 3(5/8)(3/8)(5/7) = .5022$, so $\sigma = .7087$.

112. Seed are often treated with fungicides to protect them in poor draining, wet environments. A small-scale trial, involving five treated and five untreated seeds, was conducted prior to a large-scale experiment to explore how much fungicide to apply. The seeds were planted in wet soil, and the number of emerging plants were counted. If the solution was not effective and four plants actually sprouted, what is the probability that

a. all four plants emerged from treated seeds?

b. three or fewer emerged from treated seeds?

c. at least one emerged from untreated seeds?

Solution:

Let Y = # of treated seeds selected.

a. $P(Y = 4) = \dfrac{\binom{5}{4}\binom{5}{0}}{\binom{10}{4}} = .0238$

b. $P(Y \leq 3) = 1 - P(Y = 4) = 1 - \dfrac{\binom{5}{4}\binom{5}{0}}{\binom{10}{4}} = 1 - .0238 = .9762.$

c. same as part b. above.

113. A group of six software packages available to solve a linear programming problem has been ranked from 1 to 6 (best to worst). An engineering firm, unaware of the rankings, randomly selected and then purchased two of the packages. Let Y denote the number of packages purchased by the firm that are ranked 3, 4, 5, or 6. Give the probability distribution for Y.

Solution:

The random variable Y follows a hypergeometric distribution with $N = 6$, $n = 2$, and $r = 4$.

114. Specifications call for a thermistor to test out at between 9000 and 10,000 ohms at 25◦ Celcius. Ten thermistors are available, and three of these are to be selected for use. Let Y denote the number among the three that do not conform to specifications. Find the probability distributions for Y (in tabular form) under the following conditions:

a. Two thermistors do not conform to specifications among the ten that are available.

b. Four thermistors do not conform to specifications among the ten that are available.

Solution:

a. The probability function for Y is $p(y) = \dfrac{\binom{2}{y}\binom{8}{3-y}}{\binom{10}{3}}$, $y = 0, 1, 2$. In tabular form, this is

y	0	1	2
$p(y)$	14/30	14/30	2/30

b. The probability function for Y is $p(y) = \dfrac{\binom{4}{y}\binom{6}{3-y}}{\binom{10}{3}}$, $y = 0, 1, 2, 3$. In tabular form, this is

y	0	1	2	3
$p(y)$	5/30	15/30	9/30	1/30

115. A jury of 6 persons was selected from a group of 20 potential jurors, of whom 8 were African American and 12 were white. The jury was supposedly randomly selected, but it contained only 1 African American member. Do you have any reason to doubt the randomness of the selection?

Solution:

The probability of an event as rare or rarer than one observed can be calculated according to the hypergeometric distribution. Let Y = # of black members. Then, Y is hypergeometric and $P(Y \le 1) = \dfrac{\binom{8}{3}\binom{12}{5}}{\binom{20}{6}} + \dfrac{\binom{8}{0}\binom{12}{6}}{\binom{20}{6}} = .187$. This is nearly 20%, so it is not unlikely.

116. Suppose that a radio contains six transistors, two of which are defective. Three transistors are selected at random, removed from the radio, and inspected. Let Y equal the number of defectives observed, where Y = 0, 1, or 2. Find the probability distribution for Y.

Solution:

The probability distribution for Y is given by

y	0	1	2
$p(y)$	1/5	3/5	1/5

117. In an assembly-line production of industrial robots, gearbox assemblies can be installed in one minute each if holes have been properly drilled in the boxes and in ten minutes if the holes must be redrilled. Twenty gearboxes are in stock, 2 with improperly drilled holes. Five gearboxes must be selected from the 20 that are available for installation in the next five robots.

a. Find the probability that all 5 gearboxes will fit properly.

b. Find the mean, variance, and standard deviation of the time it takes to install these 5 gearboxes.

Solution:

Let Y = # of improperly drilled gearboxes. Then, Y is hypergeometric with $N = 20$, $n = 5$, and $r = 2$.

a. $P(Y = 0) = .553$

b. The random variable T, the total time, is given by $T = 10Y + (5 - Y) = 9Y + 5$. Thus,
$E(T) = 9E(Y) + 5 = 9[5(2/20)] + 5 = 9.5$.
$V(T) = 81V(T) = 81(.355) = 28.755$, $\sigma = 5.362$.

118. Cards are dealt at random and without replacement from a standard 52 card deck. What is the probability that the second king is dealt on the fifth card?

Solution:

Let the event A = 2nd king is dealt on 5th card. The four possible outcomes for this event are $\{KNNNK, NKNNK, NNKNK, NNNKK\}$, where K denotes a king and N denotes a non–king. Each of these outcomes has probability: $\left(\frac{4}{52}\right)\left(\frac{48}{51}\right)\left(\frac{47}{50}\right)\left(\frac{46}{49}\right)\left(\frac{3}{48}\right)$. Then, the desired probability is $P(A) = 4\left(\frac{4}{52}\right)\left(\frac{48}{51}\right)\left(\frac{47}{50}\right)\left(\frac{46}{49}\right)\left(\frac{3}{48}\right) = .016$.

119. Show that the probabilities assigned by the Poisson probability distribution satisfy the requirements that $0 \le p(y) \le 1$ for all y and $\sum p(y) = 1$

<u>Solution:</u>

Because $\lambda > 0$, it is obvious that $p(y) > 0$ for $y = 0, 1, 2, \ldots,$ and that $p(y) = 0$ otherwise. Further,

$$\sum_{y=0}^{\infty} p(y) = \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} e^{-\lambda} = e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{-\lambda} e^{\lambda} = 1$$

because the infinite sum $\sum_{y=0}^{\infty} \lambda^y / y!$ is a series expansion of e^{λ}.

120. Suppose that a random system of police patrol is devised so that a patrol officer may visit a given beat location Y = 0, 1, 2, 3, . . . times per half-hour period, with each location being visited an average of once per time period. Assume that Y possesses, approximately, a Poisson probability distribution. Calculate the probability that the patrol officer will miss a given location during a half-hour period. What is the probability that it will be visited once? Twice? At least once?

<u>Solution:</u>

For this example the time period is a half-hour, and the mean number of visits per half-hour interval is $\lambda = 1$. Then

$$p(y) = \frac{(1)^y e^{-1}}{y!} = \frac{e^{-1}}{y!}, \qquad y = 0, 1, 2, \ldots .$$

The event that a given location is missed in a half-hour period corresponds to $(Y = 0)$, and

$$P(Y = 0) = p(0) = \frac{e^{-1}}{0!} = e^{-1} = .368.$$

Similarly,

$$p(1) = \frac{e^{-1}}{1!} = e^{-1} = .368,$$

and

$$p(2) = \frac{e^{-1}}{2!} = \frac{e^{-1}}{2} = .184.$$

The probability that the location is visited *at least* once is the event $(Y \ge 1)$. Then

$$P(Y \ge 1) = \sum_{y=1}^{\infty} p(y) = 1 - p(0) = 1 - e^{-1} = .632.$$

_121._ A certain type of tree has seedlings randomly dispersed in a large area, with the mean density of seedlings being approximately five per square yard. If a forester randomly locates ten 1-square-yard sampling regions in the area, find the probability that none of the regions will contain seedlings.

Solution:

If the seedlings really are randomly dispersed, the number of seedlings per region, Y, can be modeled as a Poisson random variable with $\lambda = 5$. (The average density is five per square yard.) Thus,

$$P(Y = 0) = p(0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-5} = .006738.$$

The probability that $Y = 0$ on ten independently selected regions is $(e^{-5})^{10}$ because the probability of the intersection of independent events is equal to the product of the respective probabilities. The resulting probability is extremely small. Thus, if this event actually occurred, we would seriously question the assumption of randomness, the stated average density of seedlings, or both.

_122._ Industrial accidents occur according to a Poisson process with an average of three accidents per month. During the last two months, ten accidents occurred. Does this number seem highly improbable if the mean number of accidents per month, μ, is still equal to 3? Does it indicate an increase in the mean number of accidents per month?

Solution:

The number of accidents in *two* months, Y, has a Poisson probability distribution with mean $\lambda^\star = 2(3) = 6$. The probability that Y is as large as 10 is

$$P(Y \geq 10) = \sum_{y=10}^{\infty} \frac{6^y e^{-6}}{y!}.$$

The tedious calculation required to find $P(Y \geq 10)$ can be avoided by using Table 3, Appendix 3, software such as R [`ppois(9,6)` yields $P(Y \leq 9)$]; or the empirical rule. From Theorem 3.11,

$$\mu = \lambda^\star = 6, \qquad \sigma^2 = \lambda^\star = 6, \qquad \sigma = \sqrt{6} = 2.45.$$

The empirical rule tells us that we should expect Y to take values in the interval $\mu \pm 2\sigma$ with a high probability.

Notice that $\mu + 2\sigma = 6 + (2)(2.45) = 10.90$. The observed number of accidents, $Y = 10$, does not lie more than 2σ from μ, but it is close to the boundary. Thus, the observed result is not highly improbable, but it may be sufficiently improbable to warrant an investigation. See Exercise 3.210 for the exact probability $P(|Y - \lambda| \leq 2\sigma)$.

123. Let Y denote a random variable that has a Poisson distribution with mean $\lambda = 2$. Find

a. $P(Y = 4)$.

b. $P(Y \geq 4)$.

c. $P(Y < 4)$.

d. $P(Y \geq 4 | Y \geq 2)$.

Solution:

a. $P(Y = 4) = \frac{2^4}{4!}e^{-2} = .090.$

b. $P(Y \geq 4) = 1 - P(Y \leq 3) = 1 - .857 = .143$ (using

c. $P(Y < 4) = P(Y \leq 3) = .857.$

d. $P(Y \geq 4 \mid Y \geq 2) = \dfrac{P(Y \geq 4)}{P(Y \geq 2)} = .143/.594 = .241$

124. The random variable Y has a Poisson distribution and is such that p(0) = p(1). What is p(2)?

Solution:

If $p(0) = p(1)$, $e^{-\lambda} = \lambda e^{-\lambda}$. Thus, $\lambda = 1$. Therefore, $p(2) = \frac{1^2}{2!}e^{-1} = .1839$.

125. Customers arrive at a checkout counter in a department store according to a Poisson distribution at an average of seven per hour. If it takes approximately ten minutes to serve each customer, find the mean and variance of the total service time for customers arriving during a 1-hour period. (Assume that a sufficient number of servers are available so that no customer must wait for service.) Is it likely that the total service time will exceed 2.5 hours?

Solution:

Let S = total service time = $10Y$. From Ex. 3.122, Y is Poisson with $\lambda = 7$. Therefore, $E(S) = 10E(Y) = 7$ and $V(S) = 100V(Y) = 700$. Also, $P(S > 150) = P(Y > 15) = 1 - P(Y \leq 15) = 1 - .998 = .002$, and unlikely event.

126. The number of typing errors made by a typist has a Poisson distribution with an average of four errors per page. If more than four errors appear on a given page, the typist must retype the whole page. What is the probability that a randomly selected page does not need to be retyped?

Solution:

Let Y = # of typing errors per page. Then, Y is Poisson with $\lambda = 4$ and $P(Y \le 4) = .6288$.

127. The number of knots in a particular type of wood has a Poisson distribution with an average of 1.5 knots in 10 cubic feet of the wood. Find the probability that a 10-cubic-foot block of the wood has at most 1 knot.

Solution:

Let the random variable Y = # of knots in the wood. Then, Y has a Poisson distribution with $\lambda = 1.5$ and $P(Y \le 1) = .5578$.

128. Assume that the tunnel in Exercise 3.132 is observed during ten two-minute intervals, thus giving ten independent observations $Y1, Y2, \ldots, Y10$, on the Poisson random variable. Find the probability that Y > 3 during at least one of the ten two-minute intervals.

Solution:

Let X = # of two–minute intervals with more than three cars. Therefore, X is binomial with $n = 10$ and $p = .01899$ and $P(X \ge 1) = 1 - P(X = 0) = 1 - (1-.01899)^{10} = .1745$.

129. A salesperson has found that the probability of a sale on a single contact is approximately .03. If the salesperson contacts 100 prospects, what is the approximate probability of making at least one sale?

Solution:

Using the Poisson approximation, $\lambda \approx np = 100(.03) = 3$, so $P(Y \ge 1) = 1 - P(Y = 0) = .9524$.

130. The probability that a mouse inoculated with a serum will contract a certain disease is .2. Using the Poisson approximation, find the probability that at most 3 of 30 inoculated mice will contract the disease.

Solution:

Using the Poisson approximation to the binomial with $\lambda \approx np = 30(.2) = 6$.
Then, $P(Y \leq 3) = .1512$.

131. In the daily production of a certain kind of rope, the number of defects per foot Y is assumed to have a Poisson distribution with mean λ = 2. The profit per foot when the rope is sold is given by X, where X = 50 − 2Y − Y 2. Find the expected profit per foot.

Solution:

Note that if Y is Poisson with $\lambda = 2$, $E(Y) = 2$ and $E(Y^2) = V(Y) + [E(Y)]^2 = 2 + 4 = 6$. So,
$E(X) = 50 - 2E(Y) - E(Y^2) = 50 - 2(2) - 6 = 40$.

132. A food manufacturer uses an extruder (a machine that produces bite-size cookies and snack food) that yields revenue for the firm at a rate of $200 per hour when in operation. However, the extruder breaks down an average of two times every day it operates. If Y denotes the number of breakdowns per day, the daily revenue generated by the machine is R = 1600−50Y 2. Find the expected daily revenue for the extruder.

Solution:

Note that if Y is Poisson with $\lambda = 2$, $E(Y) = 2$ and $E(Y^2) = V(Y) + [E(Y)]^2 = 2 + 4 = 6$. So,
$E(X) = 50 - 2E(Y) - E(Y^2) = 50 - 2(2) - 6 = 40$.

133. The number of customers per day at a sales counter, Y , has been observed for a long period of time and found to have mean 20 and standard deviation 2. The probability distribution of Y is not known. What can be said about the probability that, tomorrow, Y will be greater than 16 but less than 24?

Solution:

We want to find $P(16 < Y < 24)$. From Theorem 3.14 we know that, for any $k \geq 0$, $P(|Y - \mu| < k\sigma) \geq 1 - 1/k^2$, or

$$P[(\mu - k\sigma) < Y < (\mu + k\sigma)] \geq 1 - \frac{1}{k^2}.$$

Because $\mu = 20$ and $\sigma = 2$, it follows that $\mu - k\sigma = 16$ and $\mu + k\sigma = 24$ if $k = 2$. Thus,

$$P(16 < Y < 24) = P(\mu - 2\sigma < Y < \mu + 2\sigma) \geq 1 - \frac{1}{(2)^2} = \frac{3}{4}.$$

In other words, tomorrow's customer total will be between 16 and 24 with a fairly high probability (at least 3/4).

Notice that if σ were 1, k would be 4, and

$$P(16 < Y < 24) = P(\mu - 4\sigma < Y < \mu + 4\sigma) \geq 1 - \frac{1}{(4)^2} = \frac{15}{16}.$$

Thus, the value of σ has considerable effect on probabilities associated with intervals.

134. Let Y be a random variable with mean 11 and variance 9. Using Tchebysheff's theorem, find

a. a lower bound for $P(6 < Y < 16)$.

b. the value of C such that $P(|Y - 11| \geq C) \leq .09$.

Solution:

a. The value 6 lies $(11–6)/3 = 5/3$ standard deviations below the mean. Similarly, the value 16 lies $(16–11)/3 = 5/3$ standard deviations above the mean. By Tchebysheff's theorem, at least $1 - 1/(5/3)^2 = 64\%$ of the distribution lies in the interval 6 to 16.

b. By Tchebysheff's theorem, $.09 = 1/k^2$, so $k = 10/3$. Since $\sigma = 3$, $k\sigma = (10/3)^3 = 10 = C$.

135. This exercise demonstrates that, in general, the results provided by Tchebysheff's theorem cannot be improved upon. Let Y be a random variable such that

$$p(-1) = 1/18 \quad ,p(0) = 16/ 18 \quad ,p(1) = 1/18$$

a. Show that E(Y) = 0 and V(Y) = 1/9.

b. Use the probability distribution of Y to calculate P(|Y − μ| ≥ 3σ). Compare this exact probability with the upper bound provided by Tchebysheff's theorem to see that the bound provided by Tchebysheff's theorem is actually attained when k = 3.

<u>Solution:</u>

a. $E(Y) = -1(1/18) + 0(16/18) + 1(1/18) = 0$. $E(Y^2) = 1(1/18) + 0(16/18) + 1(1/18) = 2/18 = 1/9$. Thus, $V(Y) = 1/9$ and $\sigma = 1/3$.

b. $P(|Y - 0| \geq 1) = P(Y = -1) + P(Y = 1) = 1/18 + 1/18 = 2/18 = 1/9$. According to Tchebysheff's theorem, an upper bound for this probability is $1/3^2 = 1/9$.

136. For a certain type of soil the number of wireworms per cubic foot has a mean of 100. Assuming a Poisson distribution of wireworms, give an interval that will include at least 5/9 of the sample values of wireworm counts obtained from a large number of 1-cubic-foot samples.

<u>Solution:</u>

Using Tchebysheff's theorem, $5/9 = 1 - 1/k^2$, so $k = 3/2$. The interval is $100 \pm (3/2)10$, or 85 to 115.

137. A balanced coin is tossed three times. Let Y equal the number of heads observed.

a. Use the formula for the binomial probability distribution to calculate the probabilities associated with Y = 0, 1, 2, and 3.

b. Construct a probability distribution similar to the one in Table 3.1.

c. Find the expected value and standard deviation of Y , using the formulas E(Y) = np and V(Y) = npq.

d. Using the probability distribution from part (b), find the fraction of the population measurements lying within 1 standard deviation of the mean. Repeat for 2 standard deviations.

e. How do your results compare with the results of Tchebysheff's theorem and the empirical rule?

Solution:

a. The binomial probabilities are $p(0) = 1/8$, $p(1) = 3/8$, $p(2) = 3/8$, $p(3) = 1/8$.

b. The graph represents a symmetric distribution.

c. $E(Y) = 3(1/2) = 1.5$, $V(Y) = 3(1/2)(1/2) = .75$. Thus, $\sigma = .866$.

d. For *one* standard deviation about the mean: $1.5 \pm .866$ or $(.634, 2.366)$
This traps the values 1 and 2, which represents 7/8 or 87.5% of the probability. This is consistent with the empirical rule.
For *two* standard deviations about the mean: $1.5 \pm 2(.866)$ or $(-.232, 3.232)$
This traps the values 0, 1, and 2, which represents 100% of the probability. This is consistent with both the empirical rule and Tchebysheff's theorem.

138. In May 2005, Tony Blair was elected to an historic third term as the British prime minister. A Gallop U.K. poll (http://gallup.com/poll/content/default.aspx?ci=1710, June 28, 2005) conducted after Blair's election indicated that only 32% of British adults would like to see their son or daughter grows up to become prime minister. If the same proportion of Americans would prefer that their son or daughter grow up to be president and 120 American adults are interviewed,

a. what is the expected number of Americans who would prefer their child grow up to be president?

b. what is the standard deviation of the number Y who would prefer that their child grow up to be president?

c. is it likely that the number of Americans who prefer that their child grow up to be president exceeds 40?

Solution:

a. The expected value is $120(.32) = 38.4$
b. The standard deviation is $\sqrt{120(.32)(.68)} = 5.11$.
c. It is quite likely, since 40 is close to the mean 38.4 (less than .32 standard deviations away).

139. For a certain section of a pine forest, the number of diseased trees per acre, Y , has a Poisson distribution with mean $\lambda = 10$. The diseased trees are sprayed with an insecticide at a cost of \$3 per tree, plus a fixed overhead cost for equipment rental of \$50. Letting C denote the total spraying cost for a randomly selected acre, find the expected value and standard deviation for C. Within what interval would you expect C to lie with probability at least .75?

Solution:

For $C = 50 + 3Y$, $E(C) = 50 + 3(10) = \$80$ and $V(C) = 9(10) = 90$, so that $\sigma = 9.487$.
Using Tchebysheff's theorem with $k = 2$, we have $P(|Y - 80| < 2(9.487)) \geq .75$, so that the required interval is $(80 - 2(9.487), 80 + 2(9.487))$ or $(61.03, 98.97)$.

CONTINUOUS RANDOM VARIABLES AND THEIR PROBABILITY DISTRUBITION

140. Let Y be a continuous random variable with probability density function given by

$$f(y) = \begin{cases} 3y^2, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find F(y). Graph both f (y) and F(y).

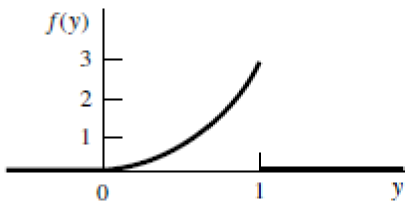
<u>Solution:</u>

$$F(y) = \int_{-\infty}^{y} f(t)\,dt,$$

we have, for this example,

$$F(y) = \begin{cases} \int_{-\infty}^{y} 0\,dt = 0, & \text{for } y < 0, \\ \int_{-\infty}^{0} 0\,dt + \int_{0}^{y} 3t^2\,dt = 0 + t^3\big]_0^y = y^3, & \text{for } 0 \le y \le 1, \\ \int_{-\infty}^{0} 0\,dt + \int_{0}^{1} 3t^2\,dt + \int_{1}^{y} 0\,dt = 0 + t^3\big]_0^1 + 0 = 1, & \text{for } 1 < y. \end{cases}$$

Notice that some of the integrals that we evaluated yield a value of 0. These are included for completeness in this initial example. In future calculations, we will not explicitly display any integral that has value 0. The graph of $F(y)$ is given in Figure 4.7.



141. Given $f(y) = cy^2, 0 \le y \le 2$, and $f(y) = 0$ elsewhere, find the value of c for which $f(y)$ is a valid density function.

Solution:

We require a value for c such that

$$F(\infty) = \int_{-\infty}^{\infty} f(y)\,dy = 1$$

$$= \int_0^2 cy^2\,dy = \frac{cy^3}{3}\bigg]_0^2 = \left(\frac{8}{3}\right)c.$$

Thus, $(8/3)c = 1$, and we find that $c = 3/8$.

142. Let Y be a random variable with *p(y)* given in the table below.

y	1	2	3	4
$p(y)$	.4	.3	.2	.1

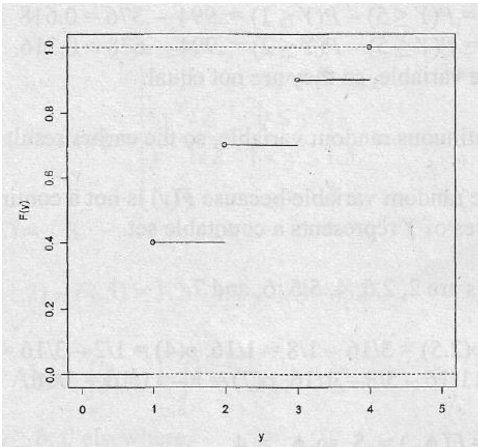
a. Give the distribution function, F(y). Be sure to specify the value of F(y) for all y, $-\infty < y < \infty$.

b. Sketch the distribution function given in part (a).

Solution:

a. $F(y) = P(Y \le y) = \begin{cases} 0 & y < 1 \\ .4 & 1 \le y < 2 \\ .7 & 2 \le y < 3 \\ .9 & 3 \le y < 4 \\ 1 & y \ge 4 \end{cases}$

b.



143. Suppose that Y is a random variable that takes on only integer values 1, 2, . . . and has distribution function F(y). Show that the probability function p(y) = P(Y = y) is given by

$$p(y) = \begin{cases} F(1), & y = 1, \\ F(y) - F(y-1), & y = 2, 3, \ldots . \end{cases}$$

Solution:

For $y = 2, 3, \ldots, F(y) - F(y-1) = P(Y \le y) - P(Y \le y-1) = P(Y = y) = p(y)$. Also, $F(1) = P(Y \le 1) = P(Y = 1) = p(1)$.

144. A random variable Y has the following distribution function:

$$F(y) = P(Y \le y) = \begin{cases} 0, & \text{for } y < 2, \\ 1/8, & \text{for } 2 \le y < 2.5, \\ 3/16, & \text{for } 2.5 \le y < 4, \\ 1/2 & \text{for } 4 \le y < 5.5, \\ 5/8, & \text{for } 5.5 \le y < 6, \\ 11/16, & \text{for } 6 \le y < 7, \\ 1, & \text{for } y \ge 7. \end{cases}$$

a. Is Y a continuous or discrete random variable? Why?

b. What values of Y are assigned positive probabilities?

c. Find the probability function for Y.

d. What is the median, $\varphi_{.5}$, of Y ?

Solution:

a. Y is a discrete random variable because $F(y)$ is not a continuous function. Also, the set of possible values of Y represents a countable set.

b. These values are 2, 2.5, 4, 5.5, 6, and 7.

c. $p(2) = 1/8, p(2.5) = 3/16 - 1/8 = 1/16, p(4) = 1/2 - 3/16 = 5/16, p(5.5) = 5/8 - 1/2 = 1/8, p(6) = 11/16 - 5/8 = 1/16, p(7) = 1 - 11/16 = 5/16.$

d. $P(Y \le \phi_{.5}) = F(\phi_{.5}) = .5$, so $\phi_{.5} = 4$.

145. Suppose that Y possesses the density function

$$f(y) = \begin{cases} cy, & 0 \le y \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find the value of c that makes f (y) a probability density function.

b. Find F(y).

c. Graph f (y) and F(y).

d. Use F(y) to find P(1 ≤ Y ≤ 2).

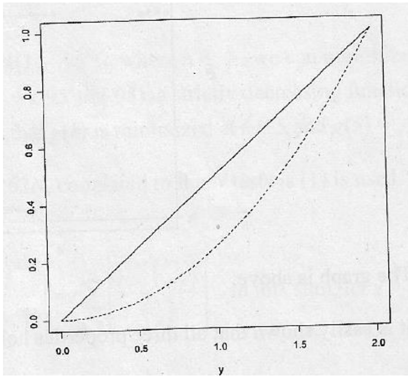
e. Use f (y) and geometry to find P(1 ≤ Y ≤ 2).

<u>Solution:</u>

a. $\int_0^2 cy\,dy = \left[cy^2/2\right]_0^2 = 2c = 1$, so $c = 1/2$.

b. $F(y) = \int_{-\infty}^{y} f(t)dt = \int_0^y \frac{t}{2}dt = \frac{y^2}{4}$, $0 \le y \le 2$.

c.



d. $P(1 \le Y \le 2) = F(2) - F(1) = 1 - .25 = .75$.

e. Note that $P(1 \le Y \le 2) = 1 - P(0 \le Y < 1)$. The region $(0 \le y < 1)$ forms a triangle (in the density graph above) with a base of 1 and a height of .5. So, $P(0 \le Y < 1) = \frac{1}{2}(1)(.5) = .25$ and $P(1 \le Y \le 2) = 1 - .25 = .75$.

146. A supplier of kerosene has a 150-gallon tank that is filled at the beginning of each week. His weekly demand shows a relative frequency behavior that increases steadily up to 100 gallons and then levels off between 100 and 150 gallons. If Y denotes weekly demand in hundreds of gallons, the relative frequency of demand can be modeled by

$$f(y) = \begin{cases} y, & 0 \le y \le 1, \\ 1, & 1 < y \le 1.5, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find F(y).

b. Find P(0 ≤ Y ≤ .5).

c. Find P(.5 ≤ Y ≤ 1.2).

Solution:

a. For $0 \le y \le 1$, $F(y) = \int_0^y t\,dt = y^2/2$. For $1 < y \le 1.5$, $F(y) = \int_0^1 t\,dt + \int_1^y dt = 1/2 + y - 1$ $= y - 1/2$. Hence,

$$F(y) = \begin{cases} 0 & y < 0 \\ y^2/2 & 0 \le y \le 1 \\ y - 1/2 & 1 < y \le 1.5 \\ 1 & y > 1.5 \end{cases}$$

b. $P(0 \le Y \le .5) = F(.5) = 1/8$.

c. $P(.5 \le Y \le 1.2) = F(1.2) - F(.5) = 1.2 - 1/2 - 1/8 = .575$.

147. Let the distribution function of a random variable Y be

$$F(y) = \begin{cases} 0, & y \le 0, \\ \dfrac{y}{8}, & 0 < y < 2, \\ \dfrac{y^2}{16}, & 2 \le y < 4, \\ 1, & y \ge 4. \end{cases}$$

a. Find the density function of Y .

b. Find P(1 ≤ Y ≤ 3).

c. Find P(Y ≥ 1.5).

d. Find P(Y ≥ 1|Y ≤ 3).

Solution:

a. Differentiating $F(y)$ with respect to y, we have

$$f(y) = \begin{cases} 0 & y \le 0 \\ .125 & 0 < y < 2 \\ .125y & 2 \le y < 4 \\ 0 & y \ge 4 \end{cases}$$

b. $F(3) - F(1) = 7/16$

c. $1 - F(1.5) = 13/16$

d. $7/16/(9/16) = 7/9$.

148. Suppose that Y has density function

$$f(y) = \begin{cases} (3/2)y^2 + y, & 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find the mean and variance of Y **.**

Solution:

$$E(Y) = \int_0^1 (1.5y^3 + y^2)dy = \frac{3y^4}{8} + \frac{y^3}{3}\Big]_0^1 = 17/24 = .708.$$

$$E(Y^2) = \int_0^1 (1.5y^4 + y^3)dy = \frac{3y^5}{10} + \frac{y^4}{4}\Big]_0^1 = 3/10 + 1/4 = .55.$$

So, $V(Y) = .55 - (.708)2 = .0487$.

149. If Y has distribution function,

$$F(y) = \begin{cases} 0, & y \le 0, \\ \frac{y}{8}, & 0 < y < 2, \\ \frac{y^2}{16}, & 2 \le y < 4, \\ 1, & y \ge 4. \end{cases}$$

find the mean and variance of Y .

Solution:

$E(Y) = \int_0^2 .125y\,dy + \int_2^4 .125y^2 dy = 31/12$, $E(Y^2) = \int_0^2 .125y^2 dy + \int_2^4 .125y^3 dy = 47/6$.

So, $V(Y) = 47/6 - (31/12)^2 = 1.16$.

150. For certain ore samples, the proportion Y of impurities per sample is a random variable with density function given by,

$$f(y) = \begin{cases} (3/2)y^2 + y, & 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

The dollar value of each sample is $W = 5 - 0.5Y$. Find the mean and variance of W.

Solution:

$$E(Y) = \int_0^1 (1.5y^3 + y^2)dy = \frac{3y^4}{8} + \frac{y^3}{3}\Big]_0^1 = 17/24 = .708.$$

$$E(Y^2) = \int_0^1 (1.5y^4 + y^3)dy = \frac{3y^5}{10} + \frac{y^4}{4}\Big]_0^1 = 3/10 + 1/4 = .55.$$

So, $V(Y) = .55 - (.708)2 = .0487$.

$E(W) = E(5 - .5Y) = 5 - .5E(Y) = 5 - .5(.708) = \$4.65$.

$V(W) = V(5 - .5Y) = .25V(Y) = .25(.0487) = .012$.

151. The temperature Y at which a thermostatically controlled switch turns on has probability density function given by

$$f(y) = \begin{cases} 1/2, & 59 \le y \le 61, \\ 0, & \text{elsewhere.} \end{cases}$$

Find E(Y) and V(Y).

Solution:

$E(Y) = .5\int_{59}^{61} y\,dy = .5\frac{y^2}{2}\Big]_{59}^{61} = 60$, $E(Y^2) = .5\int_{59}^{61} y^2 dy = .5\frac{y^3}{3}\Big]_{59}^{61} = 3600\frac{1}{3}$. Thus,

$V(Y) = 3600\frac{1}{3} - (60)^2 = \frac{1}{3}$.

152. The pH of water samples from a specific lake is a random variable Y with probability density function given by

$$f(y) = \begin{cases} (3/8)(7 - y)^2, & 5 \le y \le 7, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find E(Y) and V(Y).

b. Find an interval shorter than (5, 7) in which at least three-fourths of the pH measurements must lie.

c. Would you expect to see a pH measurement below 5.5 very often? Why?

Solution:

a. $E(Y) = \frac{3}{8}\int_5^7 y(7-y)^2\,dy = \frac{3}{8}\left[\frac{49}{2}y^2 - \frac{14}{3}y^3 + \frac{y^4}{4}\right]_5^7 = 5.5$

$E(Y^2) = \frac{3}{8}\int_5^7 y^2(7-y)^2\,dy = \frac{3}{8}\left[\frac{49}{3}y^3 - \frac{14}{4}y^4 + \frac{y^5}{5}\right]_5^7 = 30.4$, so $V(Y) = .15$.

b. Using Tchebysheff's theorem, a two standard deviation interval about the mean is given by $5.5 \pm 2\sqrt{.15}$ or (4.725, 6.275). Since $Y \geq 5$, the interval is (5, 6.275).

c. $P(Y < 5.5) = \frac{3}{8}\int_5^{5.5}(7-y)^2\,dy = .5781$, or about 58% of the time (quite common).

153. Daily total solar radiation for a specified location in Florida in October has probability density function given by

$$f(y) = \begin{cases} (3/8)(7-y)^2, & 5 \leq y \leq 7, \\ 0, & \text{elsewhere.} \end{cases}$$

with measurements in hundreds of calories. Find the expected daily solar radiation for October.

Solution:

$$E(Y) = \int_2^6 y\left(\tfrac{3}{32}\right)(y-2)(6-y)\,dy = 4.$$

154. If Y is a continuous random variable such that $E[(Y-a)^2] < \infty$ for all a, show that $E[(Y-a)^2]$ is minimized when $a = E(Y)$.

[Hint: $E[(Y-a)^2] = E(\{[Y-E(Y)] + [E(Y)-a]\}^2)$.]

Solution:

Let $\mu = E(Y)$. Then,
$$\begin{aligned} E[(Y-a)^2] &= E[(Y-\mu+\mu-a)^2] \\ &= E[(Y-\mu)^2] - 2E[(Y-\mu)(\mu-a)] + (\mu-a)^2 \\ &= \sigma^2 + (\mu-a)^2. \end{aligned}$$

The above quantity is minimized when $\mu = a$.

155. Arrivals of customers at a checkout counter follow a Poisson distribution. It is known that, during a given 30-minute period, one customer arrived at the counter. Find the probability that the customer arrived during the last 5 minutes of the 30-minute period.

Solution:

As just mentioned, the actual time of arrival follows a uniform distribution over the interval of (0, 30). If Y denotes the arrival time, then

$$P(25 \le Y \le 30) = \int_{25}^{30} \frac{1}{30}\,dy = \frac{30-25}{30} = \frac{5}{30} = \frac{1}{6}.$$

The probability of the arrival occurring in any other 5-minute interval is also 1/6.

156. If a parachutist lands at a random point on a line between markers A and B, find the probability that she is closer to A than to B. Find the probability that her distance to A is more than three times her distance to B.

Solution:

The distance Y is uniformly distributed on the interval A to B, If she is closer to A, she has landed in the interval (A, $\frac{A+B}{2}$). This is one half the total interval length, so the probability is .5. If her distance to A is more than three times her distance to B, she has landed in the interval ($\frac{3B+A}{4}$, B). This is one quarter the total interval length, so the probability is .25.

157. A random variable Y has a uniform distribution over the interval (θ1, θ2). Derive the variance of Y .

Solution:

First find $E(Y^2) = \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} y^2\,dy = \frac{1}{\theta_2-\theta_1}\left[\frac{y^3}{3}\right]_{\theta_1}^{\theta_2} = \frac{\theta_2^3-\theta_1^3}{3(\theta_2-\theta_1)} = \frac{\theta_1^2+\theta_1\theta_2+\theta_2^2}{3}$. Thus,

$$V(Y) = \frac{\theta_1^2+\theta_1\theta_2+\theta_2^2}{3} - \left(\frac{\theta_2+\theta_1}{2}\right)^2 = \frac{(\theta_2-\theta_1)^2}{12}.$$

158. A circle of radius r has area $A = \pi r^2$. If a random circle has a radius that is uniformly distributed on the interval (0, 1), what are the mean and variance of the area of the circle?

Solution:

Let $A = \pi R^2$, where R has a uniform distribution on the interval (0, 1). Then,

$$E(A) = \pi E(R^2) = \pi \int_0^1 r^2\,dr = \frac{\pi}{3}$$

$$V(A) = \pi^2 V(R^2) = \pi^2 [E(R^4) - \left(\frac{1}{3}\right)^2] = \pi^2 \left[\int_0^1 r^4\,dr - \left(\frac{1}{3}\right)^2\right] = \pi^2 \left[\frac{1}{5} - \left(\frac{1}{3}\right)^2\right] = \frac{4\pi^2}{45}$$

159. Upon studying low bids for shipping contracts, a microcomputer manufacturing company finds that intrastate contracts have low bids that are uniformly distributed between 20 and 25, in units of thousands of dollars. Find the probability that the low bid on the next intrastate shipping contract

a. is below $22,000.

b. is in excess of $24,000.

Solution:

Let Y = low bid (in thousands of dollars) on the next intrastate shipping contract. Then, Y is uniform on the interval (20, 25).

a. $P(Y < 22) = 2/5 = .4$

b. $P(Y > 24) = 1/5 = .2$.

160. The failure of a circuit board interrupts work that utilizes a computing system until a new board is delivered. The delivery time, Y , is uniformly distributed on the interval one to five days. The cost of a board failure and interruption includes the fixed cost c0 of a new board and a cost that increases proportionally to Y^2. If C is the cost incurred, $C = c_0 + c_1 Y^2$.

a. Find the probability that the delivery time exceeds two days.

b. In terms of c_0 and c_1, find the expected cost associated with a single failed circuit board.

Solution:

The density for Y = delivery time is $f(y) = \frac{1}{4}$, $1 \le y \le 5$. Also, $E(Y) = 3$, $V(Y) = 4/3$.

a. $P(Y > 2) = 3/4$.

b. $E(C) = E(c_0 + c_1Y^2) = c_0 + c_1E(Y^2) = c_0 + c_1[V(Y) + (E(Y))^2] = c_0 + c_1[4/3 + 9]$

161. A telephone call arrived at a switchboard at random within a one-minute interval. The switch board was fully busy for 15 seconds into this one-minute period. What is the probability that the call arrived when the switchboard was not fully busy?

Solution:

If Y has a uniform distribution on the interval (0, 1), then $P(Y > 1/4) = 3/4$.

162. The cycle time for trucks hauling concrete to a highway construction site is uniformly distributed over the interval 50 to 70 minutes. What is the probability that the cycle time exceeds 65 minutes if it is known that the cycle time exceeds 55 minutes?

Solution:

Let Y = cycle time. Thus, Y has a uniform distribution on the interval (50, 70). Then,
$P(Y > 65 \mid Y > 55) = P(Y > 65)/P(Y > 55) = .25/(.75) = 1/3$.

163. The number of defective circuit boards coming off a soldering machine follows a Poisson distribution. During a specific eight-hour day, one defective circuit board was found.

a. Find the probability that it was produced during the first hour of operation during that day.

b. Find the probability that it was produced during the last hour of operation during that day.

c. Given that no defective circuit boards were produced during the first four hours of operation, find the probability that the defective board was manufactured during the fifth hour.

Solution:

Let Y = time when the defective circuit board was produced. Then, Y has an approximate uniform distribution on the interval (0, 8).

a. $P(0 < Y < 1) = 1/8$.

b. $P(7 < Y < 8) = 1/8$

c. $P(4 < Y < 5 \mid Y > 4) = P(4 < Y < 5)/P(Y > 4) = (1/8)/(1/2) = 1/4$.

164. Let Z denote a normal random variable with mean 0 and standard deviation 1.

a. Find P(Z > 2).

b. Find $P(-2 \le Z \le 2)$.

c. Find $P(0 \le Z \le 1.73)$.

Solution:

a Since $\mu = 0$ and $\sigma = 1$, the value 2 is actually $z = 2$ standard deviations above the mean. Proceed down the first (z) column in Table 4, Appendix 3, and read the area opposite $z = 2.0$. This area, denoted by the symbol $A(z)$, is $A(2.0) = .0228$. Thus, $P(Z > 2) = .0228$.

b Refer to Figure 4.12, where we have shaded the area of interest. In part (a) we determined that $A_1 = A(2.0) = .0228$. Because the density function is symmetric about the mean $\mu = 0$, it follows that $A_2 = A_1 = .0228$ and hence that

$$P(-2 \le Z \le 2) = 1 - A_1 - A_2 = 1 - 2(.0228) = .9544.$$

c Because $P(Z > 0) = A(0) = .5$, we obtain that $P(0 \le Z \le 1.73) = .5 - A(1.73)$, where $A(1.73)$ is obtained by proceeding down the z column in Table 4, Appendix 3, to the entry 1.7 and then across the top of the table to the column labeled .03 to read $A(1.73) = .0418$. Thus,

$$P(0 \le Z \le 1.73) = .5 - .0418 = .4582.$$

165. The achievement scores for a college entrance examination are normally distributed with mean 75 and standard deviation 10. What fraction of the scores lies between 80 and 90?

Solution:

Then the desired fraction of the population is given by the area between

$$z_1 = \frac{80 - 75}{10} = .5 \quad \text{and} \quad z_2 = \frac{90 - 75}{10} = 1.5.$$

$$A = A(.5) - A(1.5) = .3085 - .0668 = .2417.$$

166. If Z is a standard normal random variable, find the value z0 such that

a. $P(Z > z_0) = .5$.

b. $P(Z < z_0) = .8643$.

c. $P(-z_0 < Z < z_0) = .90$.

d. $P(-z_0 < Z < z_0) = .99$.

Solution:

a. $z_0 = 0$.
b. $z_0 = 1.10$
c. $z_0 = 1.645$
d. $z_0 = 2.576$

167. What is the median of a normally distributed random variable with mean μ and standard deviation σ?

Solution:

Since the density function is symmetric about the parameter μ, $P(Y < \mu) = P(Y > \mu) = .5$. Thus, μ is the median of the distribution, regardless of the value of σ.

168. A company that manufactures and bottles apple juice uses a machine that automatically fills 16-ounce bottles. There is some variation, however, in the amounts of liquid dispensed into the bottles that are filled. The amount dispensed has been observed to be approximately normally distributed with mean 16 ounces and standard deviation 1 ounce. determine the proportion of bottles that will have more than 17 ounces dispensed into them.

Solution:

a. Note that the value 17 is $(17 - 16)/1 = 1$ standard deviation above the mean. So, $P(Z > 1) = .1587$.

169. The weekly amount of money spent on maintenance and repairs by a company was observed, over a long period of time, to be approximately normally distributed with mean \$400 and standard deviation \$20. How much should be budgeted for weekly repairs and maintenance to provide that the probability the budgeted amount will be exceeded in a given week is only .1?

Solution:

For the standard normal, $P(Z > z_0) = .1$ if $z_0 = 1.28$. So, $y_0 = 400 + 1.28(20) = \$425.60$.

170. A machining operation produces bearings with diameters that are normally distributed with mean 3.0005 inches and standard deviation .0010 inch. Specifications require the bearing diameters to lie in the interval 3.000±.0020 inches. Those outside the interval are considered scrap and must be remachined. What should the mean diameter be in order that the fraction of bearings scrapped be minimized?

Solution:

In order to minimize the scrap fraction, we need the maximum amount in the specifications interval. Since the normal distribution is symmetric, the mean diameter should be set to be the midpoint of the interval, or $\mu = 3.000$ in.

171. The grade point averages (GPAs) of a large population of college students are approximately normally distributed with mean 2.4 and standard deviation .8. If students possessing a GPA less than 1.9 are dropped from college, what percentage of the students will be dropped?

Solution:

The z–score for 1.9 is $(1.9 - 2.4)/.8 = -.625$. Thus, $P(Z < -.625) = .2660$.

172. Wires manufactured for use in a computer system are specified to have resistances between .12 and .14 ohms. The actual measured resistances of the wires produced by company A have a normal probability distribution with mean .13 ohm and standard deviation .005 ohm.

a. What is the probability that a randomly selected wire from company A's production will meet the specifications?

b. If four of these wires are used in each computer system and all are selected from company A, what is the probability that all four in a randomly selected system will meet the specifications?

Solution:

Let Y = the measured resistance of a randomly selected wire.

a. $P(.12 \le Y \le .14) = P(\frac{.12-.13}{.005} \le Z \le \frac{.14-.13}{.005}) = P(-2 \le Z \le 2) = .9544.$

b. Let X = # of wires that do not meet specifications. Then, X is binomial with $n = 4$ and $p = .9544$. Thus, $P(X = 4) = (.9544)^4 = .8297$.

173. The width of bolts of fabric is normally distributed with mean 950 mm (millimeters) and standard deviation 10 mm.

a. What is the probability that a randomly chosen bolt has a width of between 947 and 958mm?

b. What is the appropriate value for C such that a randomly chosen bolt has a width less than C with probability .8531?

Solution:

Let Y = width of a bolt of fabric, so Y has a normal distribution with $\mu = 950$ mm and $\sigma = 10$ mm.

a. $P(947 \le Y \le 958) = P(\frac{947-950}{10} \le Z \le \frac{958-950}{10}) = P(-.3 \le Z \le .8) = .406$

b. It is necessary that $P(Y \le c) = .8531$. Note that for the standard normal, we find that $P(Z \le z_0) = .8531$ when $z_0 = 1.05$. So, $c = 950 + (1.05)(10) = 960.5$ mm.

174. A soft-drink machine can be regulated so that it discharges an average of μ ounces per cup. If the ounces of fill are normally distributed with standard deviation 0.3 ounce, give the setting for μ so that 8-ounce cups will overflow only 1% of the time.

Solution:

Let Y = volume filled, so that Y is normal with mean μ and $\sigma = .3$ oz. They require that $P(Y > 8) = .01$. For the standard normal, $P(Z > z_0) = .01$ when $z_0 = 2.33$. Therefore, it must hold that $2.33 = (8 - \mu)/.3$, so $\mu = 7.301$.

175. The SAT and ACT college entrance exams are taken by thousands of students each year. The mathematics portions of each of these exams produce scores that are approximately normally distributed. In recent years, SAT mathematics exam scores have averaged 480 with standard deviation 100. The average and standard deviation for ACT mathematics scores are 18 and 6,respectively.

a. An engineering school sets 550 as the minimum SAT math score for new students. What percentage of students will score below 550 in a typical year?

b. What score should the engineering school set as a comparable standard on the ACT math test?

Solution:

a. Let Y = SAT math score. Then, $P(Y < 550) = P(Z < .7) = 0.758$.

b. If we choose the same percentile, $18 + 6(.7) = 22.2$ would be comparable on the ACT math test.

176. Show that the normal density with parameters μ and σ has inflection points at the values μ−σ and μ + σ. (Recall that an inflection point is a point where the curve changes direction from concave up to concave down, or vice versa, and occurs when the second derivative change sign. Such a change in sign may occur when the second derivative equals zero.)

Solution:

The second derivative of $f(y)$ is found to be $f''(y) = \left(\frac{1}{\sigma^3\sqrt{2\pi}}\right)e^{-(y-\mu)^2/2\sigma^2}\left[1-\frac{(\mu-y)^2}{\sigma^2}\right]$. Setting this equal to 0, we must have that $\left[1-\frac{(\mu-y)^2}{\sigma^2}\right] = 0$ (the other quantities are strictly positive). The two solutions are $y = \mu + \sigma$ and $\mu - \sigma$.

177. ***a***. If $\alpha > 0$, $\Gamma(\alpha)$ is defined by $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy$ show that $\Gamma(1) = 1$.

b. If $\alpha > 1$, integrate by parts to prove that $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.

Solution:

a. $\Gamma(1) = \int_0^\infty e^{-y}dy = -e^{-y}\Big]_0^\infty = 1.$

b. $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy = \left[-y^{\alpha-1}e^{-y}\right]_0^\infty + \int_0^\infty(\alpha-1)y^{\alpha-2}e^{-y}dy = (\alpha-1)\Gamma(\alpha-1).$

178. The operator of a pumping station has observed that demand for water during early afternoon hours has an approximately exponential distribution with mean 100 cfs (cubic feet per second).

a. Find the probability that the demand will exceed 200 cfs during the early afternoon on a randomly selected day.

b. What water-pumping capacity should the station maintain during early afternoons so that the probability that demand will exceed capacity on a randomly selected day is only .01?

Solution:

Let Y = water demand in the early afternoon. Then, Y is exponential with $\beta = 100$ cfs.

a. $P(Y > 200) = \int_{200}^\infty \frac{1}{100}e^{-y/100}dy = e^{-2} = .1353.$

b. We require the 99th percentile of the distribution of Y:

$P(Y > \phi_{.99}) = \int_{\phi_{.99}}^\infty \frac{1}{100}e^{-y/100}dy = e^{-\phi_{.99}/100} = .01$. So, $\phi_{.99} = -100\ln(.01) = 460.52$ cfs.

179. If Y has an exponential distribution and P(Y > 2) = .0821, what is

a. $\beta = E(Y)$?

b. $P(Y \le 1.7)$?

Solution:

a. Note that $\int_2^{\infty} \frac{1}{\beta} e^{-y/\beta} dy = e^{-2/\beta} = .0821$, so $\beta = .8$.

b. $P(Y \le 1.7) = 1 - e^{-1.7/.8} = .5075$.

180. Historical evidence indicates that times between fatal accidents on scheduled American domestic passenger flights have an approximately exponential distribution. Assume that the mean time between accidents is 44 days.

a. If one of the accidents occurred on July 1 of a randomly selected year in the study period, what is the probability that another accident occurred that same month?

b. What is the variance of the times between accidents?

Solution:

Let Y = time between fatal airplane accidents. So, Y is exponential with $\beta = 44$ days.

a. $P(Y \le 31) = \int_0^{31} \frac{1}{44} e^{-y/44} dy = 1 - e^{-31/44} = .5057$.

b. $V(Y) = 44^2 = 1936$.

181. A manufacturing plant uses a specific bulk product. The amount of product used in one day can be modeled by an exponential distribution with $\beta = 4$ (measurements in tons). Find the probability that the plant will use more than 4 tons on a given day.

Solution:

$P(Y > 4) = \int_4^{\infty} \frac{1}{4} e^{-y/4} dy = e^{-1} = .3679$.

182. The magnitude of earthquakes recorded in a region of North America can be modeled as having an exponential distribution with mean 2.4, as measured on the Richter scale. Suppose that the magnitude of earthquakes striking the region has a gamma distribution with $\alpha = .8$ and $\beta = 2.4$.

a. What is the mean magnitude of earthquakes striking the region?

b. What is the probability that the magnitude an earthquake striking the region will exceed 3.0 on the Richter scale?

c. Compare your answers to Exercise 4.88(a). Which probability is larger? Explain.

d. What is the probability that an earthquake striking the regions will fall between 2.0 and 3.0 on the Richter scale?

Solution:

Let Y have a gamma distribution with $\alpha = .8$, $\beta = 2.4$.

a. $E(Y) = (.8)(2.4) = 1.92$

b. $P(Y > 3) = .21036$

c. The probability found in Ex. 4.88 (a) is larger. There is greater variability with the exponential distribution.

d. $P(2 \le Y \le 3) = P(Y > 2) - P(Y > 3) = .33979 - .21036 = .12943.$

183. Explosive devices used in mining operations produce nearly circular craters when detonated. The radii of these craters are exponentially distributed with mean 10 feet. Find the mean and variance of the areas produced by these explosive devices.

Solution:

Let R denote the radius of a crater. Therefore, R is exponential w/ $\beta = 10$ and the area is $A = \pi R^2$. Thus,

$$E(A) = E(\pi R^2) = \pi E(R^2) = \pi(100 + 100) = 200\pi.$$

$$V(A) = E(A^2) - [E(A)]^2 = \pi^2[E(R^4) - 200^2] = \pi^2[240{,}000 - 200^2] = 200{,}000\pi^2,$$

where $E(R^4) = \int_0^\infty \frac{1}{10} r^4 e^{-r/10} dr = 10^4\Gamma(5) = 240.000.$

184. Four-week summer rainfall totals in a section of the Midwest United States have approximately a gamma distribution with $\alpha = 1.6$ and $\beta = 2.0$.

a. Find the mean and variance of the four-week rainfall totals.

b. What is the probability that the four-week rainfall total exceeds 4 inches?

Solution:

Let the random variable Y = four–week summer rainfall totals

a. $E(Y) = 1.6(2) = 3.2$, $V(Y) = 1.6(2^2) = 6.4$

b. $P(Y > 4) = .28955$.

185. The response times on an online computer terminal have approximately a gamma distribution with mean four seconds and variance eight seconds^2.

a. Use Tchebysheff's theorem to give an interval that contains at least 75% of the response times.

b. What is the actual probability of observing a response time in the interval you obtained in part (a)?

Solution:

a. Using Tchebysheff's theorem, two standard deviations about the mean is given by

$4 \pm 2\sqrt{8} = 4 \pm 5.657$ or $(-1.657, 9.657)$, or simply $(0, 9.657)$ since Y must be positive.

b. $P(Y < 9.657) = 1 - .04662 = 0.95338$.

186. The weekly amount of downtime Y (in hours) for an industrial machine has approximately a gamma distribution with $\alpha = 3$ and $\beta = 2$. The loss L (in dollars) to the industrial operation as a result of this downtime is given by L = 30Y + 2Y 2.

a. Find the expected value and variance of L.

Solution:

Let Y have a gamma distribution with $\alpha = 3$ and $\beta = 2$. Then, the loss $L = 30Y + 2Y^2$.
Then,

$$E(L) = E(30Y + 2Y^2) = 30E(Y) + 2E(Y^2) = 30(6) + 2(12 + 6^2) = 276,$$
$$V(L) = E(L^2) - [E(L)]^2 = E(900Y^2 + 120Y^3 + 4Y^4) - 276^2.$$

$$E(Y^3) = \int_0^\infty \frac{y^5}{16} e^{-y/2} = 480 \qquad E(Y^4) = \int_0^\infty \frac{y^6}{16} e^{-y/2} = 5760$$

Thus, $V(L) = 900(48) + 120(480) + 4(5760) - 276^2 = 47{,}664$.

187. Suppose that Y has a gamma distribution with parameters α and β.

a. If a is any positive or negative value such that $\alpha + a > 0$, show that $E(Y^a) = \beta^a(\alpha + a)/(\alpha)$

b. Why did your answer in part (a) require that $\alpha + a > 0$?

c. Show that, with a = 1, the result in part (a) gives $E(Y) = \alpha\beta$.

d. Use the result in part (a) to give an expression for $E(\sqrt{Y})$. What do you need to assume about α?

e. Use the result in part (a) to give an expression for $E(1/Y)$, $E(1/\sqrt{Y})$, and $E(1/Y^2)$. What do you need to assume about α in each case?

<u>Solution:</u>

a.

$$E(Y^a) = \int_0^\infty y^a \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} dy = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty y^{a+\alpha-1} e^{-y/\beta} dy = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{\Gamma(a+\alpha)\beta^{a+\alpha}}{1} = \frac{\beta^a \Gamma(a+\alpha)}{\Gamma(\alpha)}$$

b. For the gamma function $\Gamma(t)$, we require $t > 0$.

c. $E(Y^1) = \frac{\beta^1\Gamma(1+\alpha)}{\Gamma(\alpha)} = \frac{\beta\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta.$

d. $E(\sqrt{Y}) = E(Y^{.5}) = \frac{\beta^{.5}\Gamma(.5+\alpha)}{\Gamma(\alpha)}$, $\alpha > 0$.

e. $E(1/Y) = E(Y^{-1}) = \frac{\beta^{-1}\Gamma(-1+\alpha)}{\Gamma(\alpha)} = \frac{1}{\beta(\alpha-1)}$, $\alpha > 1$.

$E(1/\sqrt{Y}) = E(Y^{-.5}) = \frac{\beta^{-.5}\Gamma(-.5+\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha-.5)}{\sqrt{\beta}\Gamma(\alpha)}$, $\alpha > .5$.

188. A gasoline wholesale distributor has bulk storage tanks that hold fixed supplies and are filled every Monday. Of interest to the wholesaler is the proportion of this supply that is sold during the week. Over many weeks of observation, the distributor found that this proportion could be modeled by a beta distribution with $\alpha = 4$ and $\beta = 2$. Find the probability that the wholesaler will sell at least 90% of her stock in a given week.

<u>Solution:</u>

If Y denotes the proportion sold during the week, then

$$f(y) = \begin{cases} \dfrac{\Gamma(4+2)}{\Gamma(4)\Gamma(2)} y^3(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$P(Y > .9) = \int_{.9}^{\infty} f(y)\,dy = \int_{.9}^{1} 20(y^3 - y^4)\,dy$$

$$= 20\left\{ \frac{y^4}{4}\bigg]_{.9}^{1} - \frac{y^5}{5}\bigg]_{.9}^{1} \right\} = 20(.004) = .08.$$

It is *not* very likely that 90% of the stock will be sold in a given week.

189. The relative humidity Y , when measured at a location, has a probability density function given following. Find the value of k that makes f (y) a density function.

$$f(y) = \begin{cases} ky^3(1-y)^2, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Solution:

a. The random variable Y follows the beta distribution with $\alpha = 4$ and $\beta = 3$, so the constant $k = \frac{\Gamma(4+3)}{\Gamma(4)\Gamma(3)} = \frac{6!}{3!2!} = 60$.

190. The percentage of impurities per batch in a chemical product is a random variable Y with density function

$$f(y) = \begin{cases} 12y^2(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean and variance of the percentage of impurities in a randomly selected batch of the chemical.

Solution:

$E(Y) = 3/5$ and $V(Y) = 1/25$.

191. Verify that if Y has a beta distribution with α = β = 1, then Y has a uniform distribution over (0, 1). That is, the uniform distribution over the interval (0, 1) is a special case of a beta distribution.

Solution:

For $\alpha = \beta = 1$, $f(y) = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} y^{1-1}(1-y)^{1-1} = 1,\ 0 \le y \le 1$, which is the uniform distribution.

192. During an eight-hour shift, the proportion of time Y that a sheet-metal stamping machine is down for maintenance or repairs has a beta distribution with α = 1 and β = 2. That is,

$$f(y) = \begin{cases} 2(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a. The cost (in hundreds of dollars) of this downtime, due to lost production and cost of maintenance and repair, is given by $C = 10 + 20Y + 4Y^2$. Find the mean and variance of C.

b. Find an interval for which the probability that C will lie within it is at least .75.

Solution:

a.

$$E(C) = 10 + 20E(Y) + 4E(Y^2) = 10 + 20\left(\tfrac{1}{3}\right) + 4\left(\tfrac{2}{9*4} + \tfrac{1}{9}\right) = \tfrac{52}{3}$$

$$V(C) = E(C^2) - [E(C)]^2 = E[(10 + 20Y + 4Y^2)^2] - \left(\tfrac{52}{3}\right)^2$$

$$E[(10 + 20Y + 4Y^2)^2] = 100 + 400E(Y) + 480E(Y^2) + 160E(Y^3) + 16E(Y^4)$$

Using mathematical expectation, $E(Y^3) = \frac{1}{10}$ and $E(Y^4) = \frac{1}{15}$. So,

$$V(C) = E(C^2) - [E(C)]^2 = (100 + 400/3 + 480/6 + 160/10 + 16/15) - (52/3)^2 = 29.96.$$

b.

it is shown that $E(C) = \frac{52}{3}$ and $V(C) = 29.96$, so, the standard deviation is $\sqrt{29.96} = 5.474$. Thus, using Tchebysheff's theorem with $k = 2$, the interval is

$$|Y - \tfrac{52}{3}| \le 2(5.474) \text{ or } (6.38, 28.28)$$

193. Errors in measuring the time of arrival of a wave front from an acoustic source sometimes have an approximate beta distribution. Suppose that these errors, measured in microseconds, have approximately a beta distribution with $\alpha = 1$ and $\beta = 2$.

a. What is the probability that the measurement error in a randomly selected instance is less than .5 μs?

b. Give the mean and standard deviation of the measurement errors.

<u>Solution:</u>

a. $P(Y < .5) = \int_0^{.5} 2(1-y)dy = 2y - y^2\Big]_0^{.5} = .75$

b. $E(Y) = 1/3$, $V(Y) = 1/18$, so $\sigma = 1/\sqrt{18} = .2357$.

194. The proportion of time per day that all checkout counters in a supermarket are busy is a random variable Y with a density function given by

a. Find the value of c that makes f (y) a probability density function.

b. Find E(Y). (Use what you have learned about the beta-type distribution.)

c. Calculate the standard deviation of Y .

<u>Solution:</u>

The random variable Y has a beta distribution with $\alpha = 3$, $\beta = 5$.

a. The constant $c = \frac{\Gamma(3+5)}{\Gamma(3)\Gamma(5)} = \frac{7!}{2!4!} = 105$.

b. $E(Y) = 3/8$.

c. $V(Y) = 15/(64*9) = 5/192$, so $\sigma = .1614$.

195. Find μ_k for the uniform random variable with $\theta_1 = 0$ and $\theta_2 = \theta$ for $k=1,2,3$.

<u>Solution:</u>

$$\mu'_k = E(Y^k) = \int_{-\infty}^{\infty} y^k f(y)\,dy = \int_0^{\theta} y^k \left(\frac{1}{\theta}\right) dy = \frac{y^{k+1}}{\theta(k+1)}\Big]_0^{\theta} = \frac{\theta^k}{k+1}.$$

Thus,

$$\mu'_1 = \mu = \frac{\theta}{2}, \qquad \mu'_2 = \frac{\theta^2}{3}, \qquad \mu'_3 = \frac{\theta^3}{4},$$

196. A machine used to fill cereal boxes dispenses, on the average, μ ounces per box. The manufacturer wants the actual ounces dispensed Y to be within 1 ounce of μ at least 75% of the time. What is the largest value of σ, the standard deviation of Y , that can be tolerated if the manufacturer's objectives are to be met?

Solution:

We require $P(|Y-\mu| \le .1) \ge .75 = 1 - 1/k^2$. Thus, $k = 2$. Using Tchebysheff's inequality, $1 = k\sigma$ and so $\sigma = 1/2$.

197. Find $P(|Y - \mu| \le 2\sigma)$ for the uniform random variable. Compare with the corresponding probabilistic statements given by Tchebysheff's theorem and the empirical rule.

Solution:

For the uniform distribution on (θ_1, θ_2), $\mu = \frac{\theta_1+\theta_2}{2}$ and $\sigma^2 = \frac{(\theta_2-\theta_1)^2}{12}$. Thus,

$$2\sigma = \frac{(\theta_2-\theta_1)}{\sqrt{3}}.$$

The probability of interest is

$$P(|Y-\mu| \le 2\sigma) = P(\mu - 2\sigma \le Y \le \mu + 2\sigma) = P\left(\frac{\theta_1+\theta_2}{2} - \frac{(\theta_2-\theta_1)}{\sqrt{3}} \le Y \le \frac{\theta_1+\theta_2}{2} + \frac{(\theta_2-\theta_1)}{\sqrt{3}}\right)$$

It is not difficult to show that the range in the last probability statement is greater than the actual interval that Y is restricted to, so

$$P\left(\frac{\theta_1+\theta_2}{2} - \frac{(\theta_2-\theta_1)}{\sqrt{3}} \le Y \le \frac{\theta_1+\theta_2}{2} + \frac{(\theta_2-\theta_1)}{\sqrt{3}}\right) = P(\theta_1 \le Y \le \theta_2) = 1.$$

Note that Tchebysheff's theorem is satisfied, but the probability is greater than what is given by the empirical rule. The uniform is not a mound–shaped distribution.