Linear Programming
Problem Set 3 – CS 6515/4540 (Fall 2025)

This problem set is due on **Tuesday October 1st**. Submission is via Gradescope. Your solution must be a typed pdf (e.g. via LaTeX) – no handwritten solutions.

9 Convex Sets

We will study properties of convex sets in the Euclidean space.

1. Does every convex body $K \subseteq \mathbb{R}^n$ have a finite number of vertices[1]? Give a proof or a counterexample.

   No, consider a circle. Every point on the circumference is a vertex.

2. Is the intersection of two convex sets also a convex set? What about union? For each, prove it or give a counterexample.

   Yes, the intersection of two convex sets is a convex set. For two points x and y both in $K_1 \cap K_2$, we have for all $0 \leq \lambda \leq 1$, $\lambda x + (1-\lambda)y \in K_1$ by convexity of K_1, and $\lambda x + (1-\lambda)y \in K_2$ by convexity of K_2, hence $\lambda x + (1-\lambda)y \in K_1 \cap K_2$, so $K_1 \cap K_2$ is convex.

   For union, no. Just consider K_1 to be a single point and K_2 another point.

3. Is a convex set in n dimensions always bounded (finite volume) or could be unbounded? Give a proof that it is bounded or a counterexample.

   Consider all points $x \in \mathbb{R}^n$ where all coordinates of x are nonnegative, this is an unbounded set and it is convex.

10 LPs

Alice is trying to get enough oranges and bananas to host a fruit party. To successfully host a party she needs **at least 8 oranges** and **at least 20 bananas**. Unfortunately, her local grocery story only sells fruit in bundles. Bundle A costs 3 dollars and contains one orange and two bananas. Bundle B costs 2 dollars and contains three oranges and a banana. Fortunately, the grocery story will allow Alice to buy fractions of bundles (i.e. she can buy 2.5 bundle As to get 2.5 oranges and 5 bananas). They will not allow Alice to buy negative bundles (i.e. she cannot buy -1 bundle As and 3 bundle Bs to get 5 oranges and a banana).

Alice would like to buy x_A bundle As and x_B bundle Bs to guarantee she has at least 8 oranges and at least 20 bananas. Moreover, she would like to minimize her dollars spent.

1. Write a linear program whose solution is the optimal choice of x_A, x_B for Alice's problem and briefly justify why this is the correct LP.

[1] Points $u \in K$ is called a vertex if cannot be expressed as a convex combination of some finite number of other points in K

We have the following LP:

$$\begin{aligned}
\min \quad & 3x_A + 2x_B \\
s.t.\ & x_A + 3x_B \geq 8 \\
& 2x_A + x_B \geq 20 \\
& x_A, x_B \geq 0
\end{aligned}$$

In the above, x_A and x_B represent the number of bundle As and bundle Bs to be bought respectively. The objective is to minimize the cost, and since one bundle of A and B cost 3 and 2 respectively, we minimize $3x_A + 2x_B$. The first and second non-trivial constraints represent the constraints that Alice needs to satisfy to host the party. Finally, the trivial constraints ensure that Alice cannot buy negative bundles.

2. Show that there exists a solution to this linear program with objective value 30 (and hence Alice has to spend at most 30 dollars).

We can verify that $x_A = 10$ and $x_B = 0$ is indeed a feasible solution and leads to objective value 30.

$$\begin{aligned}
& 3x_a + 2x_b = 30 + 0 = 30 \\
& x_A + 3x_B = 10 \geq 8 \\
& 2x_A + x_B = 20 \geq 20 \\
& x_A = 10 \geq 0, x_B = 0 \geq 0
\end{aligned}$$

3. Prove that every feasible solution to this linear program has value at least 30 (prove this by taking a non-negative linear combination of the constraints), so the solution in the previous part is optimal.

We can express the objective as a nonnegative linear combination of constraints and hence obtain a lower bound as follows:

$$\underbrace{3x_A + 2x_B}_{cost} = (3/2) \cdot \underbrace{(2x_A + x_B)}_{\text{constraint } 2x_A + x_B \geq 20} + (1/2) \cdot \underbrace{(x_B)}_{\text{constraint } x_B \geq 0} \geq (3/2) \cdot 20 + (1/2) \cdot 0 = 30$$

11 LP Equivalent Formulation

1. Show that the class of left LPs (LP1) can efficiently represent any LP on the right (LP2). In other words, if we have a polynomial time algorithm to solve any LP1 on the left, then we can solve in polynomial time any LP2 on the right.

$$\max \sum_{i=1}^{n} c_i x_i \qquad \text{(LP1)}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} A_{ji} x_i = b_j, \quad \forall j \in \{1, \ldots, m\}$$

$$x_i \geq 0, \quad \forall i \in \{1, \ldots, n\}.$$

$$\max \sum_{i=1}^{n} c_i x_i \qquad \text{(LP2)}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} A_{ji} x_i \leq b_j, \quad \forall j \in \{1, \ldots, m\}$$

(Hint: You need two ideas: (1) We can replace unconstrained x by $x = x^+ - x^-$ where $x^+ \geq 0$ and $x^- \geq 0$. (2) We can replace $Ax \leq b$ by $Ax + Z = b$ for some vector $Z \geq 0$.)

Given LP2, we construct LP1 as

$$\max\ (c^\top, -c^\top, 0)\begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix}$$
$$\left[\mathbf{A}|-\mathbf{A}|\mathbf{I}\right]\begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} = b$$
$$x^+, x^-, z \geq 0$$
$$x^+, x^- \in \mathbb{R}^n, z \in \mathbb{R}^m$$

Then solve LP1 and return $x = x^+ - x^-$.

Correctness We now argue that the two LPs are equivalent:
1) For the optimal solution x^+, x^-, z of LP1, define $x = x^+ - x^-$. Then we have $\mathbf{A}x = \mathbf{A}x^+ - \mathbf{A}x^- \leq \mathbf{A}x^+ - \mathbf{A}x^- + z = b$ (where the inequality uses $z \geq 0$). So this is a feasible solution of LP2. It has the same cost as the solution for LP1 because $c^\top x = c^\top x^+ - c^\top x^-$. We next need to argue that no better solution for LP2 exists.
2) Observe that for the optimal solution x of LP2, we can define $x^+ = \max(0, x)$, $x^- = -\min(x, 0)$, $z = b - \mathbf{A}x$. These satisfy $\mathbf{A}x^+ - \mathbf{A}x^- + z = b$ and $c^\top x^+ - c^\top x^- = c^\top x$. Further, $x^+, x^- \geq 0$ by definition and $z \geq 0$ since $\mathbf{A}x \leq b$. So they are a feasible solution for LP1 with the same cost. Thus LP2 cannot have a solution with better cost than the optimal solution of LP1.

2. Now use a similar idea to show that Farkas' Lemma A below implies Farkas' Lemma B.

- Farkas' Lemma A says that the system of inequalities $\sum_{i=1}^n A_{ji}x_i = b_j$ for $j \in \{1, \ldots, m\}$ and $x_i \geq 0$ for $i \in \{1, \ldots, n\}$ are infeasible iff there exist λ_j for $j \in \{1, \ldots, m\}$ such that

$$\sum_{j=1}^m \lambda_j b_j < 0 \qquad \text{and} \qquad \sum_{j=1}^m \lambda_j A_{ji} \geq 0 \quad \forall i \in \{1, \ldots, n\}.$$

- Farkas' Lemma B says that the system of inequalities $\sum_{i=1}^n A_{ji}x_i \leq b_j$ for $j \in \{1, \ldots, m\}$ are infeasible iff there exist *non-negative* $\lambda_j \geq 0$ for $j \in \{1, \ldots, m\}$ such that

$$\sum_{j=1}^m \lambda_j b_j < 0 \qquad \text{and} \qquad 1q\sum_{j=1}^m \lambda_j A_{ji} = 0 \quad \forall i \in \{1, \ldots, n\}.$$

Note that $[k]$ is shorthand for the set $\{1, \ldots, k\}$.
We must show both directions of Farkas' Lemma B. We first start with the simpler reverse direction ($\Leftarrow$). So, assume that there exist $\lambda_j \geq 0$ such that:

$$\sum_{j=1}^{m} \lambda_j b_j < 0 \qquad \text{and} \qquad \sum_{j=1}^{m} \lambda_j A_{ji} = 0 \quad \forall i \in \{1, \ldots, n\}.$$

For the sake of contradiction, assume that there exists a feasible solution to the system of inequalities $\sum_{i=1}^{n} A_{ji} x_i \leq b_j$ for every $j \in [m]$, say $\{x^*\}_{i \in [n]}$. Then, for any $j \in [m]$, we have $\sum_{i=1}^{n} A_{ji} x_i^* \leq b_j$. Multiply this inequality by λ_j; notice that we can safely do this while **preserving the direction of the inequality** because $\lambda_j \geq 0$, and so we get $\sum_{i=1}^{n} \lambda_j A_{ji} x_i^* \leq \lambda_j b_j$. Summing all such inequalities for every $j \in [m]$, we get:

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j A_{ji} x_i^* \leq \sum_{j=1}^{m} \lambda_j b_j \Rightarrow \sum_{j=1}^{m} \lambda_j b_j \geq \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_j A_{ji} x_i^* \Rightarrow \sum_{j=1}^{m} \lambda_j b_j \geq 0$$

In the above, we use the fact that for every $i \in [n]$, we have $\sum_{j=1}^{m} \lambda_j A_{ji} = 0$ which implies $\sum_{j=1}^{m} \lambda_j A_{ji} x_i^* = 0$. However, the implication $\sum_{j=1}^{m} \lambda_j b_j \geq 0$ is a contradiction to $\sum_{j=1}^{m} \lambda_j b_j < 0$. Hence, we have the reverse implication.

Now, we prove the forward implication ($\Rightarrow$). We assume that the system of inequalities $\sum_{i=1}^{n} A_{ji} x_i \leq b_j$ is infeasible. Using the trick from Part 1, we can transform this LP to the form of the LP in Farkas Lemma A:

$$\sum_{i=1}^{n} A_{ji} x_i^+ + \sum_{i=1}^{n} (-A_{ji}) x_i^- + z_j = b_j \ \forall\, j \in [m]$$

Now, we have reached the form in Farkas Lemma A, with the matrix "A", the input "x" and vector "b" in the statement of Farkas Lemma A being given by the matrix A', input x' and b' defined as follows (using matrix and vector notation):

$$A' = \begin{pmatrix} \mathbf{A} & -\mathbf{A} & \mathbf{I}_m \end{pmatrix}; x' = \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{z} \end{bmatrix}; b' = \mathbf{b}$$

Notice that the number of variables (length of x'), and hence the number of columns of A' is equal to $2n + m$, while the number of constraints remains m. Applying Farkas Lemma A, we obtain that there exist $\lambda_j \in \mathbb{R} \ \forall\, j \in [m]$ such that, we have:

$$\sum_{j=1}^{m} \lambda_j A'_{ji} \geq 0 \ \forall\, i \in [2n+m]; \sum_{j=1}^{m} \lambda_j b_j < 0$$

Expanding the condition on the matrix A' above for the last m columns, we effectively take a dot product of each of the last m columns of A, which are given by the identity matrix $\mathbf{I}_m$ with the vector $\{\lambda_j\}_{j\ in[m]}$ giving us $\lambda_j \geq 0$ for every $j \in [m]$. This illustrates that these $\lambda_j \in \mathbb{R}$ are in fact nonnegative, and we also have the condition $\sum_{j=1}^{m} \lambda_j b_j < 0$, and so we have proven one part of Farkas Lemma B.
It remains to show the other part of Farkas Lemma B. Applying the condition on matrix A' to columns i and $i + n$ for every $i \in [n]$, we get:

$$\sum_{j=1}^{m} \lambda_j A_{ji} \geq 0 \text{ and } \sum_{j=1}^{m} \lambda_j (-A_{ji}) \geq 0 \text{ respectively } \Rightarrow \sum_{j=1}^{m} \lambda_j A_{ji} = 0$$

Hence, we have shown that the other condition is also true. This completes the proof.

12 LP for Regression

Suppose we are given n points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n) \in \mathbb{R} \times \mathbb{R}$. The goal of this problem is to find a line $y = ax + b$, where $a, b \in \mathbb{R}$ that fits these points as closely as possible, where closeness is defined according to some objective function. Write a polynomial-sized LP for the following settings:

1. ℓ_1 regression: Objective is to minimize $\sum_{i=1}^{n} |y_i - ax_i - b|$.

   We introduce variables z_i:

$$
\begin{aligned}
&\min \sum_{i=1}^{n} z_i \\
&s.t.\ z_i \geq y_i - ax_i - b \\
&\qquad z_i \geq b - y_i + ax_i \\
&\qquad z_i \geq 0 \ \forall\ i \in [n], a, b \in \mathbb{R}(\text{unconstrained})
\end{aligned}
$$

2. ℓ_∞ regression: Objective is to minimize $\max_{i=1}^{n} |y_i - ax_i - b|$

   We introduce variable Z:

$$
\begin{aligned}
&\min Z \\
&s.t.\ Z \geq y_i - ax_i - b \\
&\qquad Z \geq b - y_i + ax_i \\
&\qquad Z \geq 0, a, b \in \mathbb{R}(\text{unconstrained})
\end{aligned}
$$

3. Write the dual LPs for both the above LPs.

You may directly write the dual LPs using the general transformation at the start of Lec 11 (note: dual of dual is the primal LP, so we have to transform the min LP to its dual here). For clarity, below we will derive them from first principles.

$L1$-regression: we first apply the trick from Question 11, to get the modified LP with nonnegative variables:

$$
\begin{aligned}
&\min \sum_{i=1}^{n} z_i \\
&s.t.\ (a^+ - a^-)x_i + (b^+ - b^-) + z_i \geq y_i \\
&\quad - (a^+ - a^-)x_i - (b^+ - b^-) + z_i \geq -y_i \\
&\quad z_i \geq 0\ \forall\ i \in [n], a^+, a^-, b^+, b^- \geq 0
\end{aligned}
$$

Now, we have an LP of the form:

$$
\begin{aligned}
&\min c'^T x' \\
&s.t.\ A'x' \geq b' \\
&\quad x' \geq 0
\end{aligned}
$$

where:

$$
A' = \begin{pmatrix} \mathbf{I}_n & \mathbf{x} & -\mathbf{x} & \mathbf{1} & -\mathbf{1} \\ \mathbf{I}_n & -\mathbf{x} & \mathbf{x} & -\mathbf{1} & \mathbf{1} \end{pmatrix}; x' = \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ a^+ \\ a^- \\ b^+ \\ b^- \end{bmatrix}; b' = \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ -y_1 \\ \vdots \\ -y_n \end{bmatrix}
$$

The corresponding dual would be (in general):

$$
\begin{aligned}
&\max b'^T y' \\
&s.t.\ A'^T y' \leq c' \\
&\quad y' \geq 0
\end{aligned}
$$

So, writing the dual (with variables $\{p_i\}_{i \in [n]}$ corresponding to the first set of n constraints in the primal, and $\{q_i\}_{i \in [n]}$ corresponding to the second set of n constraints in the primal):

$$
\begin{gathered}
\max \sum_{i=1}^{n} y_i(p_i - q_i) \\
s.t.\ p_i + q_i \leq 1\ \forall\ i \in [n] \\
\sum_{i=1}^{n} x_i(q_i - p_i) \leq 0 \text{ and } \sum_{i=1}^{n} x_i(p_i - q_i) \leq 0 \Rightarrow \sum_{i=1}^{n} x_i(p_i - q_i) = 0\ \forall\ i \in [n] \\
\sum_{i=1}^{n}(q_i - p_i) \leq 0 \text{ and } \sum_{i=1}^{n}(p_i - q_i) \leq 0 \Rightarrow \sum_{i=1}^{n}(p_i - q_i) = 0\ \forall\ i \in [n] \\
p_i, q_i \geq 0\ \forall\ i \in [n]
\end{gathered}
$$

L_∞**-regression**: Again, applying the trick from Question 11:

$$\begin{aligned}
&\min Z \\
&s.t.\ (a^+ - a^-)x_i + (b^+ - b^-) + Z \geq y_i \\
&\quad -(a^+ - a^-)x_i - (b^+ - b^-) + Z \geq -y_i \\
&\quad Z, a^+, a^-, b^+, b^- \geq 0
\end{aligned}$$

Now, we have again reached the standard LP form described above with the values of A', x' and b':

$$A' = \begin{pmatrix} \mathbf{1} & \mathbf{x} & -\mathbf{x} & \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & -\mathbf{x} & \mathbf{x} & -\mathbf{1} & \mathbf{1} \end{pmatrix}; x' = \begin{bmatrix} Z \\ a^+ \\ a^- \\ b^+ \\ b^- \end{bmatrix}; b' = \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ -y_1 \\ \vdots \\ -y_n \end{bmatrix}$$

Writing out the corresponding duals (again, define p_i, q_i corresponding to the first n and second n constraints respectively):

$$\begin{gathered}
\max \sum_{i=1}^{n} y_i(p_i - q_i) \\
s.t. \sum_{i=1}^{n} (p_i + q_i) \leq 1 \\
\sum_{i=1}^{n} x_i(q_i - p_i) \leq 0 \text{ and } \sum_{i=1}^{n} x_i(p_i - q_i) \leq 0 \Rightarrow \sum_{i=1}^{n} x_i(p_i - q_i) = 0\ \forall\ i \in [n] \\
\sum_{i=1}^{n} (q_i - p_i) \leq 0 \text{ and } \sum_{i=1}^{n} (p_i - q_i) \leq 0 \Rightarrow \sum_{i=1}^{n} (p_i - q_i) = 0\ \forall\ i \in [n] \\
p_i, q_i \geq 0\ \forall\ i \in [n]
\end{gathered}$$