Linear Programming
Problem Set 4 – CS 6515/4540 (Fall 2025)

Solution to problem set 4, question 15

15 Facility Location

We analyze the same LP-rounding algorithm from class, except that now each ball B_j has radius

$$r_j = \tfrac{4}{3} \cdot L_j, \qquad L_j = \sum_i d(i,j) y^*_{ij},$$

instead of radius $2L_j$. As defined y^*_{ij} are the fractional assignment variables, that consumer j is assigned to facility i, in the LP relaxation.
As in the 6-approximation analysis, the algorithm processes facilities in nondecreasing order of L_j, and selects a ball B_j if it does not intersect any previously chosen ball. Thus, the chosen balls are disjoint.
For finding the appropriate LP-relaxation for the assignment cost. Suppose we don't open a facility in B_j, then by the greedy selection process, B_j intersects with some ball B_k with $L_k \leq L_j$. Hence, the distance between j and a facility in B_k is at most

$$d(j,k) \;\leq\; \tfrac{4}{3} L_j + \tfrac{8}{3} L_k \;\leq\; 4L_j.$$

Therefore, after our transformation the total assignment cost $\leq 4L_j$.
Consider a ball B_j where the algorithm opens a facility. Let p be the fraction of j's fractional mass that lies *outside* B_j. Then, every such point has distance at least $\frac{4}{3} L_j$ from j, so the expected distance satisfies

$$\mathbb{E}[\text{distance from } j] \;\geq\; p \cdot \tfrac{4}{3} L_j.$$

But by our definition L_j is the average fractional connection cost, so this expectation is at most L_j. Hence

$$p \cdot \tfrac{4}{3} L_j \;\leq\; L_j \quad \Rightarrow \quad p \leq \tfrac{3}{4}.$$

Therefore, at least $1 - p \geq \frac{1}{4}$ of the mass must lie inside B_j. In other words,

$$\sum_{i \in B_j} y^*_{ij} \;\geq\; \tfrac{1}{4}.$$

and by our constraints, we also know

$$y^*_{ij} \leq x^*_i$$

Since we choose to open the cheapest facility in B_j, its cost can be charged to this fractional opening mass. This implies that the total facility opening cost is at most

$$\leq 4 \cdot \sum_i f_i x^*_i,$$

where x^*_i are the LP facility opening variables. Thus the facility cost is bounded by 4−times the optimal cost.
Therefore, By modifying the radius of each ball to $\frac{4}{3} L_j$, the same greedy LP-rounding algorithm achieves a 4-approximation for the facility location problem, improving over the 6-approximation bound from class.