Convex Programming
Problem Set 5 – CS 6515/4540 (Fall 2025)

Solutions to Problem Set 5, Q18.

18 Convex Functions

1. Yes, $f+g$ is also convex. Consider a point $z = \lambda x + (1-\lambda)y$ where $\lambda \in [0,1]$. Since f and g are convex functions, we have

$$f(z) \leq \lambda f(x) + (1-\lambda) f(y),$$
$$g(z) \leq \lambda g(x) + (1-\lambda) g(y).$$

   Then,

$$\begin{aligned}(f+g)(z) &= f(z) + g(z) \\ &\leq \lambda f(x) + (1-\lambda) f(y) + \lambda g(x) + (1-\lambda) g(y) \\ &= \lambda (f+g)(x) + (1-\lambda)(f+g)(y).\end{aligned}$$

   Hence, $f+g$ is convex.

2. The product fg need not be convex. Consider the counterexample:

$$f(x) = x, \quad g(x) = -x.$$

   Both f and g are linear, hence convex. Their product is

$$(fg)(x) = x \cdot (-x) = -x^2,$$

   which is concave (since $(fg)''(x) = -2 < 0$). Therefore, the product of two convex functions need not be convex.

3. Yes, the function $h(x) = \max(f(x), g(x))$ is always convex. For any $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, define $z = \lambda x + (1-\lambda)y$. Since f and g are convex,

$$f(z) \leq \lambda f(x) + (1-\lambda) f(y),$$
$$g(z) \leq \lambda g(x) + (1-\lambda) g(y).$$

   Taking the maximum on both sides gives

$$\max(f(z), g(z)) \leq \max\left(\lambda f(x) + (1-\lambda) f(y),\ \lambda g(x) + (1-\lambda) g(y)\right).$$

$$h(z) = \max(f(z), g(z)) \leq \lambda \max(f(x), g(x)) + (1-\lambda) \max(f(y), g(y)).$$

   Hence, h is convex.

4. If f is α-strongly convex, then for all $x, y \in \mathbb{R}$,

$$f(y) \geq f(x) + f'(x)(y-x) + \frac{\alpha}{2}(y-x)^2.$$

   Let $x^\star$ be the unique minimizer of f, where $f'(x^\star) = 0$. Then for any $x \in \mathbb{R}$,

$$f(x) \geq f(x^\star) + \frac{\alpha}{2}(x - x^\star)^2.$$

For large $|x|$, say $|x| \geq R \geq 2|x^\star|$, we have

$$(x - x^\star)^2 \geq (|x| - |x^\star|)^2 \geq \left(\frac{|x|}{2}\right)^2 = \frac{x^2}{4}.$$

Hence,

$$f(x) \geq f(x^\star) + \frac{\alpha}{8}x^2.$$

If $f(x^\star) \geq 0$, we immediately have $f(x) \geq \frac{\alpha}{8}x^2$. Otherwise, choose R large enough (e.g., $R \geq \sqrt{\frac{16|f(x^\star)|}{\alpha}}$) so that

$$f(x) \geq \frac{\alpha}{16}x^2 \quad \text{for all } |x| \geq R.$$

Therefore, there exist constants $C = \frac{\alpha}{16} > 0$ and $R > 0$ such that

$$f(x) \geq Cx^2 \quad \text{for all } |x| \geq R,$$

i.e.,

$$f(x) = \Omega(x^2).$$