DP and Graph Algorithms
Problem Set 2 – CS 6515/4540 (Fall 2025)

Answers to problem set 2, question 8.

8 Bipartite Matching: Forbidden Paths

We'll prove the result in 2 parts, 1^{st} we'll prove the left => right condition.
Suppose there exists a directed $v_s \to v_t$ path in G avoiding F:

$$P: \quad v_s = i_0 \to i_1 \to \cdots \to i_l = v_t.$$

We construct a perfect matching M in $\widehat{G}$ such that:

- For each edge (i_j, i_{j+1}) on the path, match

$$i_j \in L \quad \text{with} \quad i'_{j+1} \in R.$$

- For every vertex $r \in V$ not on the path and not in F, match $r \in L$ with $r' \in R$ (the identity edge).

Now we'll prove how the above construction is a valid perfect matching, i.e. covers all vertices $\in V$ and every vertex is incident to exactly 1 edge in G. Another observation is that $L = R$.

- Every left vertex i_j on the path: P with $i < l$ is matched with i'_{j+1}. The left vertex $v_t = i_l \in L$ was deleted, so it requires no matching. Every left vertex not on the path: P is matched with its own right copy. There are no other vertices remaining in L that are unmatched.
- On the right, each i'_{j+1} is matched to i_j on the path. In particular, v'_t is matched to i_{l-1}. The right vertex $v'_s \in R$ was deleted, so it requires no matching. There are no other vertices remaining in R that are unmatched.

Thus, M is a perfect matching in $\widehat{G}$.

Now to prove the converse, given a perfect matching M in $\widehat{G}$ we will try and construct a directed path from v_s to v_t in G.

Start at the left vertex v_s. Since M is perfect, v_s is matched to exactly one right vertex in M; call it i'_0. Because v'_s was deleted, $i'_0 \neq v'_s$. Therefore, the edge $\{v_s, i'_0\}$ in $\widehat{G}$ must correspond to a genuine directed edge $v_s \to i_0$ in G. Thus, the edge gives the first step of a directed walk starting at v_s in G.
Now, consider the right vertex i'_0. Because of the identity edges in $\widehat{G}$, there must be an edge from $i_0 \in L$ to $i'_0 \in R$. To note, this edge will not be a part of M, as we've already taken the edge $\{v_s, i'_0\}$ that's incident on i'_0. Since we have a perfect matching $i_0 \in L$ must be matched via some other edge to some vertex in R, say i'_1. The edge $\{i_0, i'_1\}$ must exist and be in the perfect matching. Which then implies $\{i_0, i_1\}$ must be an actual edge in G.

If we keep continuing this process, we end up with a directed path $\{v_s \to i_0 \to i_1 \to \cdots \to i_l\}$ where each connection either follows from a directed edge or an identity edge in $\widehat{G}$. All encountered vertices avoid F, since forbidden vertices were removed from $\widehat{G}$. Because the vertex set is finite and $|L| = |R|$, two outcomes are possible:

- We are able to construct a path till $i_l = v'_t$ for some l which implies a directed path in G from v_s to v_t.

- We end up in a cycle wherein some left vertex, i_k repeats before reaching v_t'. So, say we have i_k link back to some i_j for $j < k$ or there exist and edge from i_k to i_j'. This edge must be in M as otherwise i_k will not be matched to any edge, but we already have an edge incident to i_j' in M. Therefore, we have some vertex i_j' matched to two different edges in M, contradicting the definition of a perfect matching. Hence, this case is impossible.

Thus, there exists a directed path from v_s to v_t in G that avoids all forbidden vertices.