Linear Programming
Problem Set 3 – CS 6515/4540 (Fall 2025)

Answers to problem set 3, question 9.

9 Convex Sets

1. By definition a convex body is a non-empty and closed bounded convex set with non-empty interior. Therefore, this can be a solid sphere or a solid desk which have infinite number of points on their boundary or infinite vertices. Therefore, a convex body need not have a finite number of vertices.

2. Say we have two convex sets, C_1 and C_2. Let's consider the union and intersection as separate scenarios:

   Intersection:

   - If $C_1 \cap C_2 = \varnothing$, then the intersection is empty, and therefore an empty (valid) convex set.
   - If $C_1 \cap C_2 \neq \varnothing$, let $x, y \in C_1 \cap C_2$. Since $x, y \in C_1$ and $x, y \in C_2$, and both C_1 and C_2 are convex, the line segment $L = \{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$ lies in C_1 and in C_2, hence also in $C_1 \cap C_2$. Thus, $C_1 \cap C_2$ is a convex set.
   - If $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$, then $C_1 \cap C_2 = \min\{C_1, C_2\}$ which is also a convex set.

   Union:

   - If $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$, then $C_1 \cup C_2 = \max\{C_1, C_2\}$ and the union is again a convex set.
   - Counterexample (disjoint case): Consider two disks of radius 1 in $\mathbb{R}^2$ centered at $(0, 0)$ and $(4, 0)$. Choosing x in the first disk and y in the second disk, the line segment $[x, y]$ passes through points that belong to neither disk.
   - Counterexample (overlap case): Consider two disks of radius 1 centered at $(0, 0)$ and $(1, 0)$. Taking $x = (0, 1)$ on boundary of the first disk and $y = (1, 1)$ on the boundary of the second disk, the line segment $[x, y]$ passes through points outside both disks.

3. A convex set in n dimensions can be unbounded. Consider the set $Q = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ i.e. the first quadrant of the plane. For any two points $(x_1, y_1), (x_2, y_2) \in Q$, the line segment $L = \{\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) : 0 \leq \lambda \leq 1\}$ lies entirely in Q. Hence, Q is convex and Q is clearly unbounded.