

- * HW 3 out, due on Thurs next week
- * Exam 1 to be graded by early next week ← Any feedback?
- * Recap lec 9

LP Duality

Primal LP

$$\max_{\mathbf{x}} \sum_{i=1}^n c_i x_i \quad \Leftrightarrow \quad \max_{\mathbf{x}} \mathbf{c}^T \mathbf{x}$$

s.t.

$$\sum_j A_{ji} x_i \leq b_j \quad \forall j \in \{1, \dots, m\}$$

$$y_j \quad x_i \geq 0 \quad \forall i \in \{1, \dots, n\}$$

standard form of LP

s.t. $A \mathbf{x} \leq \mathbf{b}$

$\mathbf{x} \geq 0$

Dual LP

$$\min_{\mathbf{y}} \sum_j y_j b_j \quad \Leftrightarrow \quad \min_{\mathbf{y}} \mathbf{y}^T \mathbf{b}$$

s.t.

$$\sum_{j=1}^m y_j A_{ji} \geq c_i \quad \forall i \in \{1, \dots, n\}$$

$$y_j \geq 0 \quad \forall j \in \{1, \dots, m\}$$

$\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$

$\mathbf{y} \geq 0$

Thm (Weak LP Duality) Any feasible dual LP soln \mathbf{y} gives an upper bound, i.e., $\text{opt}_{\text{Primal LP}} \leq \mathbf{y}^T \mathbf{b}$

Pf: After multiplying j -th constraint by y_j and summing gives

$$\begin{aligned} & \sum_{j=1}^m y_j \left[\sum_{i=1}^n A_{ji} x_i \right] \leq \sum_{j=1}^m y_j b_j \\ \Leftrightarrow & \sum_{i=1}^n x_i \left[\sum_{j=1}^m y_j A_{ji} \right] \leq \sum_{j=1}^m y_j b_j \\ & \geq c_i \text{ since } \mathbf{y} \text{ is dual LP feasible} \end{aligned}$$

Since $x_i \geq 0$

$$\Rightarrow \sum_{i=1}^n x_i c_i \leq \sum_{j=1}^m y_j b_j \Rightarrow \text{opt}_{\text{Primal LP}} \leq \sum_j y_j b_j$$

■

$$\begin{array}{l|l} \max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} & \min_{\mathbf{y}} \mathbf{y}^T \mathbf{b} \\ \text{s.t. } A \mathbf{x} \leq \mathbf{b} & \text{s.t. } \mathbf{A}^T \mathbf{y} \geq \mathbf{c} \\ \mathbf{x} \geq 0 & \mathbf{y} \geq 0 \end{array}$$

Example

$$\begin{array}{l|l} \max_{\mathbf{x}} 5x_1 + 4x_2 + 6x_3 & \min_{\mathbf{y}} 3y_1 + 10y_2 + 6y_3 \\ \text{s.t. } x_1 + 2x_3 \leq 3 \leftarrow y_1 & \text{s.t. } y_1 + 2y_2 \geq 5 \\ 2x_1 + x_2 + x_3 \leq 10 \leftarrow y_2 & y_1 + 7y_2 \geq 4 \end{array}$$

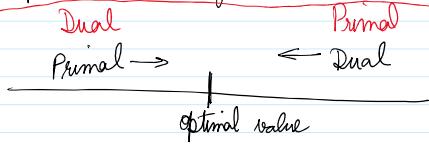
$$\begin{array}{l}
 \begin{array}{l}
 \begin{array}{l}
 x_1 + x_3 = 0 \leq 0 \\
 2x_1 + x_2 + x_3 \leq 10 \leftarrow y_2 \\
 2x_2 + x_3 \leq 6 \leftarrow y_3 \\
 x_i \geq 0 \quad \text{for } i \in \{1, 2, 3\}
 \end{array}
 \end{array}
 \end{array}
 \quad \left| \quad \begin{array}{l}
 \text{s.t.} \quad y_1 + 2y_2 \geq 5 \\
 y_2 + 2y_3 \geq 6 \\
 y_1 + y_2 + y_3 \geq 6 \\
 y_j \geq 0 \quad \forall j \in \{1, 2, 3\}
 \end{array}
 \right.$$

Thm (Strong LP duality)

If optimal primal LP value is finite then optimal dual LP value is the same.

Also, if $\text{opt}_{\text{primal}} \rightarrow \infty$ then dual LP is infeasible

and if primal LP is infeasible then $\text{opt}_{\text{dual}} \rightarrow -\infty$



Wrong, instead

Correct

if $\text{opt}_{\text{dual}} \rightarrow -\infty$ then primal LP is infeasible

Before the proof, let us see an application.

Max-Flow Min-Cut using LP Duality

Write max st flow as a different LP.

Recall : directed graph G with edge capacities c_e .

Let P denote all $s-t$ simple ^{directed} paths.

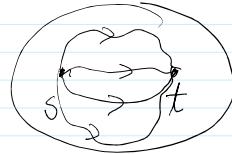
$$\begin{array}{l}
 \text{Max-flow-LP} \quad \max \sum_{p \in P} x_p
 \end{array}$$

$$\begin{array}{l}
 \text{s.t.} \\
 \forall e \in E \quad \sum_{\substack{p \ni e \\ p \in P}} x_p \leq c_e \\
 \forall p \in P \quad x_p \geq 0
 \end{array}$$

$$\min \sum_{e \in E} y_e c_e$$

$$\text{s.t.} \quad \forall p \in P: \sum_{e \in p} y_e \geq 1$$

$$\forall e \in E: \quad y_e \geq 0$$



$$\begin{array}{l}
 \text{By strong LP duality:} \quad \sum_{p \in P} x_p^* = \sum_{e \in E} y_e^* c_e \\
 \uparrow \quad \text{opt primal} \quad \quad \quad \text{opt dual}
 \end{array}$$

The dual LP may return a fractional optimal soln but it looks like a 'min cut' LP.

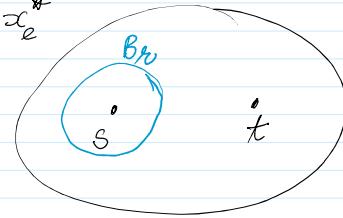
Obs: Any integral dual soln is a valid $s-t$ cut since all $s-t$ paths are disconnected.

Obs: Any integral dual soln is a valid s - t cut since all s - t paths are disconnected.

Lemma: The dual LP always has an optimal integral soln, and hence max-flow = min-cut.

Pf: Consider directed graph G with edge weights $w_e = x_e^*$

For $r \geq 0$, let B_r denote all vertices at distance at most r from s .



Note that for $r \in (0, 1)$, $B_r \not\ni t$.

Claim: $\min_{r \in (0, 1)} \sum_{e \in E(B_r, V \setminus B_r)} c_e \leq \sum_e y_e^* c_e$ \Rightarrow there is an integral optimal soln.

Pf: Suppose we choose a random $r \sim \text{Unif}[0, 1]$

$$\Pr[\text{edge } e = (u, v) \text{ is cut}] \leq |\text{dist}(u, s) - \text{dist}(v, s)| \leq y_e^*$$

$$\Rightarrow \mathbb{E}[\text{Cut value}] = \sum_{e \in E} c_e \cdot \Pr[\text{edge } (u, v) \text{ is cut}]$$

$$\mathbb{E}[X + Y + \dots] = \mathbb{E}[X] + \mathbb{E}[Y] + \dots \leq \sum_e c_e y_e^*$$

Linearity of Expectation

Since best r could only be better than a random r , this completes the proof. \blacksquare