Convex Programming
Problem Set 5 – CS 6515/4540 (Fall 2025)

Solutions to problem set 5, Q21

21 Non-Convex Function

1. Apply the L-smoothness inequality with $x = x_t$ and $y = x_{t+1} = x_t - \eta\nabla f(x_t)$:

$$\begin{aligned} f(x_{t+1}) &\leq f(x_t) + \langle \nabla f(x_t),\, x_{t+1} - x_t\rangle + \frac{L}{2}\|x_{t+1} - x_t\|^2 \\ &= f(x_t) + \langle \nabla f(x_t),\, -\eta\nabla f(x_t)\rangle + \frac{L}{2}\| - \eta\nabla f(x_t)\|^2 \\ &= f(x_t) - \eta\|\nabla f(x_t)\|^2 + \frac{L\eta^2}{2}\|\nabla f(x_t)\|^2 \\ &= f(x_t) - \eta\Big(1 - \frac{L\eta}{2}\Big)\|\nabla f(x_t)\|^2. \end{aligned}$$

   Since $\eta \leq \frac{1}{L}$, it follows that $1 - \frac{L\eta}{2} \geq \frac{1}{2}$, hence:

$$f(x_{t+1}) \leq f(x_t) - \frac{\eta}{2}\|\nabla f(x_t)\|^2.$$

2. From (1), we have for each iteration k:

$$\frac{\eta}{2}\|\nabla f(x_k)\|^2 \leq f(x_k) - f(x_{k+1}).$$

   Summing from $k = 0$ to $t - 1$, the telescoping sum gives:

$$\frac{\eta}{2}\sum_{k=0}^{t-1}\|\nabla f(x_k)\|^2 \leq f(x_0) - f(x_t).$$

   Using $f(x_t) \geq f(x^*)$, we obtain:

$$\frac{\eta}{2}\sum_{k=0}^{t-1}\|\nabla f(x_k)\|^2 \leq f(x_0) - f(x^*),$$

   which implies:

$$\sum_{k=0}^{t-1}\|\nabla f(x_k)\|^2 \leq \frac{2}{\eta}\big(f(x_0) - f(x^*)\big).$$

3. From the inequality in (2), we have:

$$\frac{1}{t}\sum_{k=0}^{t-1}\|\nabla f(x_k)\|^2 \leq \frac{2}{\eta t}(f(x_0) - f(x^*)).$$

   Since the minimum is less than or equal to the average:

$$\min_{0\leq k\leq t-1}\|\nabla f(x_k)\|^2 \leq \frac{1}{t}\sum_{k=0}^{t-1}\|\nabla f(x_k)\|^2.$$

Taking square roots on both sides yields:

$$\min_{0 \le k \le t-1} \|\nabla f(x_k)\| \le \sqrt{\frac{2}{\eta t}\big(f(x_0) - f(x^*)\big)}.$$

This completes the proof.