

- * HW 3 out, due on Thurs next week
- * Exam 1 to be graded by early next week ← Any feedback?
- * Recap lec 9

LP Duality

Primal LP

$$\begin{aligned} \max_x \quad & \sum_{i=1}^n c_i x_i \\ \text{s.t.} \quad & \sum_j A_{ji} x_i \leq b_j \quad \forall j \in \{1, \dots, m\} \\ & x_i \geq 0 \quad \forall i \in \{1, \dots, n\} \end{aligned}$$

Standard form of LP

$$\begin{aligned} \max_x \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min_y \quad & \sum_j y_j b_j \\ \text{s.t.} \quad & \sum_{j=1}^m y_j A_{ji} \geq c_i \quad \forall i \in \{1, \dots, n\} \\ & y_j \geq 0 \quad \forall j \in \{1, \dots, m\} \end{aligned}$$

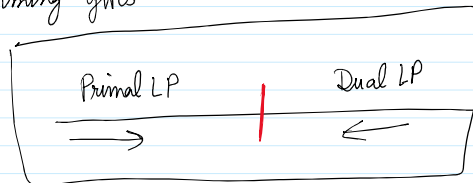
Standard form of Dual LP

$$\begin{aligned} \min_y \quad & y^T b \\ \text{s.t.} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Thm (Weak LP Duality) Any feasible dual LP soln y gives an upper bound, i.e., $\text{opt}_{\text{Primal LP}} \leq y^T b$

Pf: After multiplying j -th constraint by y_j and summing gives

$$\begin{aligned} \sum_{j=1}^m y_j \left[\sum_{i=1}^n A_{ji} x_i \right] &\leq \sum_{j=1}^m y_j b_j \\ \Leftrightarrow \sum_{i=1}^n x_i \left[\sum_{j=1}^m y_j A_{ji} \right] &\leq \sum_{j=1}^m y_j b_j \\ &\geq c_i \text{ since } y \text{ is dual LP feasible} \end{aligned}$$



Since $x_i \geq 0$

$$\Rightarrow \sum_{i=1}^n x_i c_i \leq \sum_{j=1}^m y_j b_j \Rightarrow \text{opt}_{\text{Primal LP}} \leq \sum_{j=1}^m y_j b_j$$

$$\begin{array}{l|l} \max_x \quad c^T x & \min_y \quad y^T b \\ \text{s.t.} \quad Ax \leq b & \text{s.t.} \quad A^T y \geq c \\ x \geq 0 & y \geq 0 \end{array}$$

Example

$$\begin{array}{l|l} \max & 5x_1 + 4x_2 + 6x_3 \\ \text{s.t.} & 1x_1 + 10x_3 \leq 3 \leftarrow y_1 \\ & 2x_1 + x_2 + x_3 \leq 10 \leftarrow y_2 \end{array} \quad \begin{array}{l} \min \quad 3y_1 + 10y_2 + 6y_3 \\ \text{s.t.} \quad y_1 + 2y_2 \geq 5 \\ \quad \quad 4 + 10y_2 \geq 4 \end{array}$$

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 5 \leftarrow y_1 \\ 2x_1 + x_2 + x_3 &\leq 10 \leftarrow y_2 \\ 2x_2 + x_3 &\leq 6 \leftarrow y_3 \\ x_i &\geq 0 \text{ for } i \in \{1, 2, 3\} \end{aligned}$$

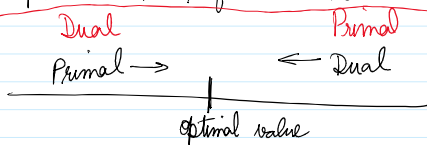
$$\begin{aligned} \text{s.t. } y_1 + 2y_2 &\geq 5 \\ y_2 + 2y_3 &\geq 4 \\ y_1 + y_2 + y_3 &\geq 6 \\ y_j &\geq 0 \quad \forall j \in \{1, 2, 3\} \end{aligned}$$

Thm (Strong LP Duality)

If optimal primal LP value is finite then optimal dual LP value is the same.

Also, if $\text{opt}_{\text{primal}} \rightarrow \infty$ then dual LP is infeasible

and if primal LP is infeasible then $\text{opt}_{\text{dual}} \rightarrow -\infty$



Wrong, instead

Correct

if $\text{opt}_{\text{dual}} \rightarrow -\infty$ then primal LP is infeasible

Before the proof, let us see an application.

Max-Flow Min-Cut using LP Duality

Write max s-t flow as a different LP.

Recall: directed graph G with edge capacities c_e .

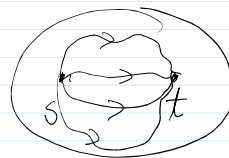
Let \mathcal{P} denote all s-t simple ^{directed} paths.

Max-flow-LP

$$\begin{aligned} \text{max } \sum_{p \in \mathcal{P}} x_p \\ \text{s.t. } \forall e \in E: \sum_{\substack{p \ni e \\ p \in \mathcal{P}}} x_p &\leq c_e \\ \forall p \in \mathcal{P}: x_p &\geq 0 \end{aligned}$$

$$\text{min } \sum_{e \in E} y_e c_e$$

$$\begin{aligned} \text{s.t. } \forall p \in \mathcal{P}: \sum_{e \in p} y_e &\geq 1 \\ \forall e \in E: y_e &\geq 0 \end{aligned}$$



By strong LP duality: $\sum_{p \in \mathcal{P}} x_p^* = \sum_{e \in E} y_e^* c_e$

\uparrow \leftarrow \leftarrow
 opt primal opt dual

The dual LP may return a fractional optimal soln but it looks like a 'min cut' LP.

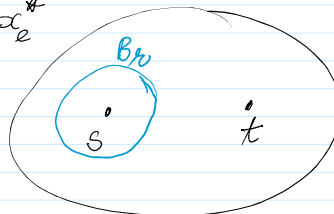
Obs: Any integral dual soln is a valid s-t cut since all s-t paths are disconnected.

Obs: Any integral dual soln is a valid s-t cut since all s-t paths are disconnected.

Lemma: The dual LP always has an optimal integral soln, and hence max-flow = min-cut.

Pf: Consider directed graph G with edge weights $w_e = x_e^*$

For $r \geq 0$, let B_r denote all vertices at distance at most r from s .



Note that for $r \in (0, 1)$, $B_r \not\ni t$.

Claim:
$$\min_{r \in (0, 1)} \sum_{e \in E(B_r, V \setminus B_r)} c_e \leq \sum_e y_e^* c_e$$

\Rightarrow there is an integral optimal soln.

Pf: Suppose we choose a random $r \sim \text{Unif}[0, 1]$

$$\Pr[\text{edge } e = (u, v) \text{ is cut}] \leq |\text{dist}(u, s) - \text{dist}(v, s)| \leq y_e^*$$

$$\Rightarrow \mathbb{E}[\text{Cut value}] = \sum_{e \in E} c_e \cdot \Pr[\text{edge } (u, v) \text{ is cut}]$$

Linearity of Expectation

$$\mathbb{E}[X + Y + \dots] = \mathbb{E}[X] + \mathbb{E}[Y] + \dots \leq \sum_e c_e y_e^*$$

Since best r could only be better than a random r , this completes the proof. \square