* HW 2 - due today
* Exam1 on Sep 18 : 4 problems : 1st pb short answer $\approx 35\%$

Recap : Matchings

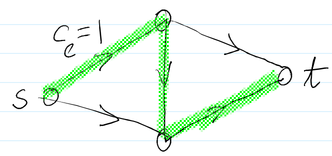
Today : Max Flow

## Maximum Flow

Directed Graph $G = (V, E)$

Edges have capacities $c_e$ ← non-neg integers

Source $s \in V$ and Sink $t \in V$ for $s \neq t$



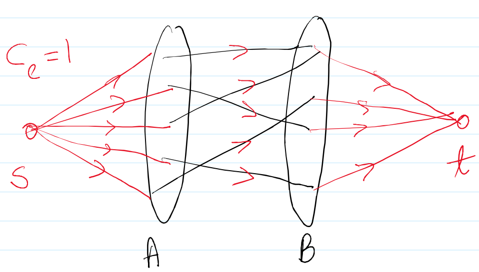

Defn (Flow): Non-neg real number $f_e$ for all $e \in E$

s.t. $0 \leq f_e \leq c_e \quad \forall e \in E$

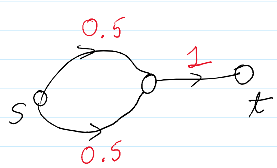
and $\forall u \notin \{s,t\}$ : $\underbrace{\sum_{v} f_{vu}}_{\text{Incoming flow}} = \underbrace{\sum_{v} f_{uv}}_{\text{Outgoing flow}}$

Goal: Find a flow that maximizes $\sum_{u \in V} f_{su}$

Obs : This problem generalizes finding max matching in bipartite graphs.




max-flow in this graph = max matching




## Ford-Fulkerson Algo

Idea 1: Start with some flow $f$

Gradually improve it : By 1 in each iteration

We will construct a graph, $G^f$ on $V$ and find an

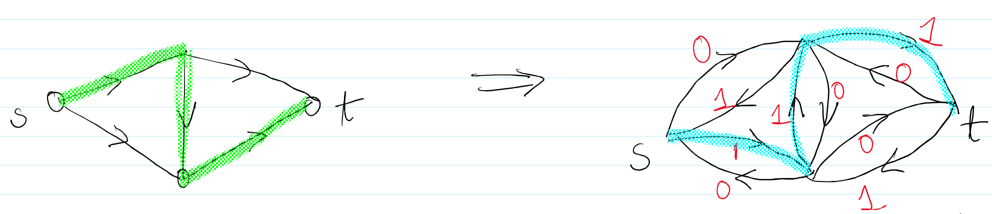
We will construct a $\underbrace{\text{graph}}_{\text{residual graph}}$ $G^f$ on $V$ and find an $s-t$ path.

Obs: Given flow $f$:

1) Increase $f_e$ to $c_e$
2) Decrease $f_e$ until $0$

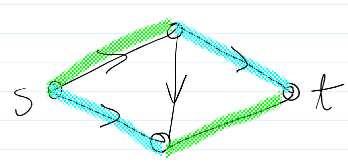
Residual Graph $G^f$: ← Directed

1) Same vertices $V$
2) Edge $e \in E$ has capacity of $c_e - f_e$
3) Add Edges $-e$ for $e \in E$ with capacity $f_e$




Find a $s$-$t$ path $p$ in $G^f$

Update flow $f$ to $f+p$ which increases total flow value by 1.




Alg:

1) Start $f$ = all 0s
2) Create $G^f$
3) Find $s-t$ path $p$ in $G^f$ ← augmenting path
   (Break if no such path exists)
4) $f = f + p$.
5) Go back to Step 2.

Time: $O(m * \text{max-flow})$ ← Pseudo polytime algo

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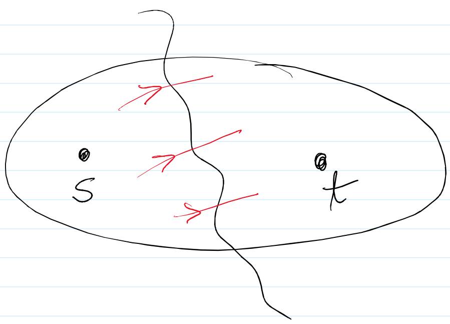
Remark: Variants of this alg are polytime

Thm: If there is no s-t path in $G^f$ then $f$ is max-flow.

## Min s-t Cut

Same input as max-flow: Directed $G$
Capacities $c_e$
$s, t \in V$




Goal: Find a cut $(X, V \setminus X)$
where $X \ni s$
$V \setminus X \ni t$

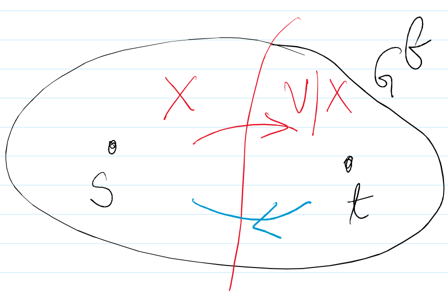
to min $\displaystyle\sum_{e \in (X, V \setminus X)} c_e$

Obs: Max s-t flow $\leq$ Min s-t Cut

Thm: Max s-t flow = Min s-t Cut

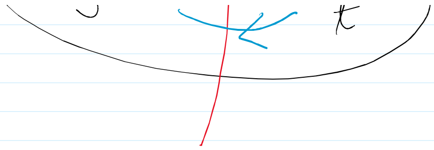
Pf: WTS if $G^f$ has no s-t path then
$\exists$ a cut $(X, V \setminus X)$ with cut value $\overset{①}{=}$ flow value.

Let $X$ denote all vertices reachable from $s$ in $G^f$.




Claim: $(X, V \setminus X)$ satisfies ①

Pf: $e \in (X, V \setminus X)$




If, $e \in (X, V \setminus X)$

$$f_e = c_e$$

$$e \in (V \setminus X, X)$$

$$f_e = 0$$

This completes the proof $\because$ total s-t flow

= Total cut capacity

$\blacksquare$