Lec 24

Tuesday, November 18, 2025 1:24 PM

* Today's lecture is optional (not on exams)

* Exam 3 on Nov 20 (Thurs)

Experts Problem:

- n experts
- Each round $t \in \{1, \dots, T\}$

(1) Alg chooses an expert/action $a_t \in \{1, .., n\}$ random choice is allowed

(2) Cost of each expert $c_t(i) \in [-1, 1]$ revealed & Alg gets $c_t(a_t)$

Objective: Min total Alg cost $\sum_{t=1}^{T} c_t(a_t)$

compared to best fixed expert $\min_i \sum_t c_t(i)$

__Remark:__ This problem is useful in Forecasting (weather, stocks), Spam detection, LP solving

Thm (Lec21): OGD implies a random alg with expected avg regret

$$\underbrace{\frac{1}{T}\left[\mathbb{E}\left[\sum_t c_t(a_t)\right] - \sum_t c_t(i)\right]}_{\text{Avg Regret}} \leq \frac{2\sqrt{n}}{\sqrt{T}}$$

Thm (today): Hedge alg has $\mathbb{E}[\text{Avg Regret}] \leq O\left(\sqrt{\frac{\log n}{T}}\right)$

Obs: Randomization is necessary to obtain $o(1)$ avg regret.

$n = 2$ experts $\qquad c_t(i)$

Cost $\quad c_1(1) = 1 \qquad c_2(a_2) = 1 \qquad c_3(a_3) = 1$

$\qquad c_1(2) = 0 \qquad c_2(\bar{a}_2) = 0 \qquad c_3(\bar{a}_3) = 0 \quad \dots$

For det alg, there is sequence where Alg incurs 1 cost per step

$\implies T$ cost overall

Benchmark $\min_i \sum_t c_t(i) \leq T/2$

$\Rightarrow$ Total Regret $\geq T - T/2 = T/2$

$\Rightarrow$ Avg regret $\geq 1/2$

Thm (today): Hedge alg has $\mathbb{E}[\text{Avg Regret}] \leq O\left(\sqrt{\frac{\log n}{T}}\right)$

Alg (1) Maintain wts $w_t(i) \geq 0 \leftarrow$ expert i

Initialized to 1 at $t=1$

Hedge Alg

(2) In each round t:

choose expert i with prob $\dfrac{w_t(i)}{\sum_{i'=1}^{n} w_t(i')}$

(3) $w_{t+1}(i) = w_t(i) \cdot \exp(-\eta \, c_t(i))$

η Step size $\approx \sqrt{\frac{\log n}{T}}$

$= \underbrace{w_1(i)}_{=1} \cdot \exp\left(-\eta \sum_{s \leq t} c_s(i)\right)$

Pf: Define potential $\phi_t = \sum_{i=1}^{n} w_t(i) \rightarrow \phi_1 = n$

Lemma: $\phi_{t+1} \leq \phi_t \exp(-\eta \, Alg_t + \eta^2)$

$\mathbb{E}[\text{Cost of alg in } t\text{-th step}]$

Assuming lemma,

$w_{T+1}(i) \leq \phi_{T+1} \leq \overset{=n}{\phi_1} \exp\left(-\eta \underbrace{\boxed{\sum_{t=1}^{T} Alg_t}}_{= Alg} + \eta^2 T\right)$

$\Rightarrow \log(w_{T+1}(i)) = \log\left(\exp\left(-\sum_{t=1}^{T} \eta \, c_t(i)\right)\right) \leq \log\left(n \cdot \exp(-\eta Alg + \eta^2 T)\right)$

$\Rightarrow -\eta \sum_t c_t(i) \leq \log n + (-\eta Alg + \eta^2 T)$

$\Rightarrow Alg \leq \sum_t c_t(i) + \frac{\log n}{\eta} + \eta T$

$$\underbrace{\frac{\log n}{\eta} + \eta T}_{\le\, 2\sqrt{T \log n} \text{ for } \eta = \sqrt{\frac{\log n}{T}}}$$

Lemma: $$\Phi_{t+1} \le \Phi_t \exp\left(-\eta\, Alg_t + \eta^2\right)$$

Pf: $e^{-x} \le 1 - x + x^2$ for $|x| \le 1$

$e^{-x} \ge 1 - x \qquad e^{-x} \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \dots$

$$\Phi_{t+1} = \sum_i w_{t+1}(i) = \Phi_t \sum_i \frac{w_t(i)}{\Phi_t} \exp\left(-\eta\, c_t(i)\right)$$

$$\le \Phi_t \sum_i \frac{w_t(i)}{\Phi_t}\left(1 - \eta\, c_t(i) + \eta^2 \overbrace{c_t^2(i)}^{\le 1}\right)$$

$$\le \Phi_t\left(1 - \eta\, Alg_t + \eta^2\right)$$

$$\le \Phi_t \exp\left(-\eta\, Alg_t + \eta^2\right) \qquad \blacksquare$$

Remark: Hedge alg can be captured as a generalization of OGD called Online Mirror Descent