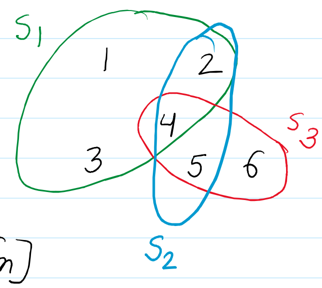
* HW-4 due Today
* Exam 2 on Oct 21 : LPs, Approx, Rand Alg
* Recap : Randomization helps with fast runtime and also with Approx Alg.

Set Cover



- Universe of elements $[n] = \{1, 2, \dots, n\}$
- m subsets $\mathcal{S} = \{S_1, \dots, S_m\}$
  where $S_i \subseteq [n]$ and $\bigcup_{i \in [m]} S_i = [n]$
- Goal: min # subsets s.t. their union is $[n]$

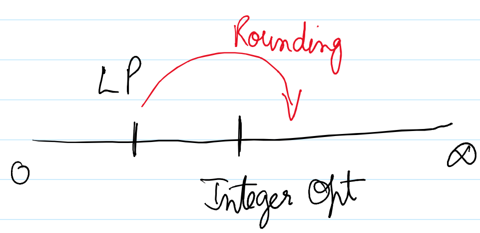
Remark: This pb generalizes vertex cover (when elems are edges) and in general is $\Omega(\log n)$-approx hard assuming $P \neq NP$.

Thm: There is a randomized alg with $\mathbb{E}[\text{Size}] = O(\log n) \cdot \text{Opt}$.

Randomized LP Rounding

Plan :

1) Integer Program for Set Cover
2) IP $\longrightarrow$ LP relaxation
3) Solve LP
4) Rounding



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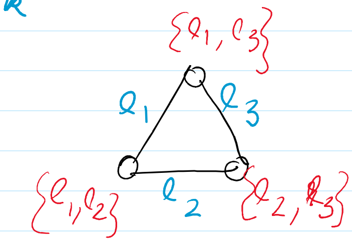
$$\text{s.t.} \quad x_k \in \{0,1\} \xrightarrow{\text{LP relaxation}} 0 \le x_k \le 1$$

$$\forall e \in [n]: \sum_{k : S_k \ni e} x_k \ge 1$$



(1) Obs: $IP_{opt} = \min$ Set Cover

(2) Write LP relax

(3) Solve LP to x_k^* : We know $\sum_{k \in [m]} x_k^* \le Opt_{\text{Set Cover}}$

(4) Rounding $x_k^* \longrightarrow X_k \in \{0,1\}$ which are feasible

Thm: Rand Rounding returns feasible $X_k \in \{0,1\}$

s.t. $\mathbb{E}\left[\sum_{k \in [m]} X_k\right] \le O(\log n) \cdot \sum_{k \in [m]} x_k^*$

Alg: (1) For each set k,

indep take it w.p. $\min\{1, 10 \cdot \log n \cdot x_k^*\}$

(2) If this soln is infeasible, then take arbitrary set per element.

Obs: $\mathbb{E}\left[\text{Cost of Alg in Step 1}\right] \le 10 \log n \cdot \sum_k x_k^*$

Lemma 1: Step 1 of alg is feasible w.p. $1 - \frac{1}{n^9}$.

The lemma implies theorem :

$\mathbb{E}[\text{Cost of Alg}] = \mathbb{E}[\text{Cost of Step 1}] + \mathbb{E}[\text{Cost of Step 2}]$

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$$\mathbb{E}[\text{Cost Alg}] = \mathbb{E}[\text{Cost of Step 1}] + \mathbb{E}[\text{Cost of Step 2}]$$

$$\leq \left(10 \cdot \log n \sum_k x_k^*\right) + \underbrace{P[\text{Reach Step 2}]}_{\text{lem 1} \atop \leq 1/n^9} \cdot n$$

Pf of lemma 1: Two types of $e \in [n]$:

Case 1: Sps $\exists S_k \ni e$ with $x_k^* \geq \frac{1}{10 \cdot \log n}$

$\Rightarrow$ Elem e is covered w.p. 1 in Step 1.

Case 2: Now sps all sets $S_k \ni e$ have $x_k^* < \frac{1}{10 \log n}$

Recall: $\sum_{k \in [m]} x_k^* \geq 1$

$P[e \text{ is not covered in Step 1}]$

$$= \prod_{k : S_k \ni e} \left(1 - 10 \cdot \log n \cdot x_k^*\right)$$

Fact:
$1 - x \leq e^{-x} \approx 1 - x + \frac{x^2}{2} - \cdots$
for $x \in [0,1]$

$$\leq \prod_{k : S_k \ni e} e^{-10 \log n \cdot x_k^*}$$

$$= e^{\sum_{S_k \ni e} x_k^* (-10 \log n)} \leq e^{-10 \log n} = \frac{1}{n^{10}}$$

$$P[\text{Step 1 fails}] = P\left[\bigcup_e \{e \text{ is not covered in Step 1}\}\right]$$

Union $\leq \sum$ $P[e \text{ is not covered in Step 1}]$

Union Bound

$$\leq \sum_{e \in [n]} P[e \text{ is not covered in Step 1}]$$

$$\leq \sum_{e} \frac{1}{n^{10}} = \frac{1}{n^{9}}.$$

$\blacksquare$