Online Convex Optimization & Complexity
Problem Set 6 – CS 6515/4540 (Fall 2025)

This problem set is due on **Monday November 17th**. Submission is via Gradescope. Your solution must be a typed pdf (e.g. via LaTeX) – no handwritten solutions.

22 Online Gradient Descent

Online gradient descent is frequently utilized with stochastic gradient descent. Suppose that we want to run online gradient descent but cannot query the gradient of our convex function f_t, however we can construct a random vector variable G_t that estimates ∇f_t at any x_t:

$$\nabla f_t(x_t) = \mathbb{E}[G_t]$$

Show that if we run Online Gradient Descent using G_t as our gradient of f_t at time t, then we have

$$\mathbb{E}\left[\sum_{t=0}^{T} f_t(x_t) - f_t(x^*)\right] \leq \frac{DG}{\sqrt{T}}$$

where G in this case will satisfy $\mathbb{E}[||G_t||^2] \leq G^2$ for all t.

23 Missing Proof Details

1. In Lecture 20 we assumed that the following convex body K has diameter $O(\sqrt{n})$. Prove that.

$$K = \{x \in \mathbb{R}_{\geq 0}^{n \times n} \mid \sum_i x_{ij} = 1 \quad \forall j \quad \text{and} \quad \sum_j x_{ij} = 1 \quad \forall i\}.$$

2. In Lecture 21 while proving the minimax theorem, by applying OGD and taking T large enough, we obtained

$$\frac{1}{T}\sum_t M(x_t, a_t) \geq \frac{1}{T}\max_i \sum_t M(i, a_t) - \epsilon.$$

   Prove why the LHS is at most expression B and why the LHS is at least expression A$-\epsilon$ to complete the proof of the minimax theorem.

3. Suppose you run online gradient descent in n-dimensions where $f(x) = \sum_{i=1}^n a_i x_i$ for some non-negative reals $a_i \in [0, A]$ and where convex body $K = \{x \in \mathbb{R}_{\geq 0}^n \mid \sum_i x_i = B\}$ for some positive B. After T iterations of online gradient descent, what is the best possible upper bound on the regret of the algorithm? (Give the answer in Big-O notation in terms of parameters A, B, and n.)

24 Zero Sum Games

Consider a zero-sum game between players A and B with A having actions $\{A_1, A_2\}$ and B having actions $\{B_1, B_2\}$. The entries in this table correspond to the rewards of player A (i.e., the costs of player B):

	B_1	B_2
A_1	2	-1
A_2	-1	1

1. If player A has to commit a strategy before player B (i.e., B will choose their best strategy/action knowing A's randomized strategy), what strategy would they adopt and how much expected reward would it get?

   (Hint: First solve for a general randomized strategy playing A_1 with probability p and A_2 with probability $1 - p$, and then optimize p.)

2. Now answer the above question when player B has to first commit a strategy (and then player A chooses their best strategy/action): what strategy would B adopt and how much would be the expected cost?

3. Are the answers to both the above parts the same? Why does it agree with the minimax theorem?

25 Reductions