Linear Programming
Problem Set 3 – CS 6515/4540 (Fall 2025)

Answers to problem set 3, question 11.

11 LP Equivalent Formulation

1. Consider the LP2 instance

$$\begin{aligned} &\max\ c^T x \\ &\text{s.t. } Ax \le b, \\ &x \in \mathbb{R}^n \ \text{ (some } x_i \text{ may be free variables).} \end{aligned}$$

   For each unrestricted variable x_i, introduce

$$x_i = x_i^+ - x_i^-, \qquad x_i^+, x_i^- \ge 0.$$

   If $x_i \ge 0$ already, we can keep x_i as x_i^+ and set $x_i^- = 0$.
   Next replace each inequality constraint

$$(Ax)_j \le b_j$$

   by

$$(Ax)_j + z_j = b_j, \qquad z_j \ge 0.$$

   Let

$$\overline{x} := \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} \ge 0, \qquad \overline{c} := \begin{pmatrix} c \\ -c \\ 0 \end{pmatrix}, \qquad \overline{A} := \begin{pmatrix} A & -A & I \end{pmatrix}.$$

   With the above transforms, LP2 can now be converted to look like LP1

$$\begin{aligned} &\max\ \overline{c}^T \overline{x} \\ &\text{s.t. } \overline{A}\overline{x} = b, \\ &\overline{x} \ge 0. \end{aligned}$$

   The new LP is equivalent to the original as any feasible solution $\overline{x}$ for the transformed LP maps to a feasible x for LP2 with the same objective value, and vice versa. The transformation adds at most n extra variables (for x^-, x^+) and m extra variables for the identity, zero matrix and z, so the size grows only linearly. Thus, if we can solve the transformed LP (similar to LP1) in polynomial time, we can also solve LP2 in polynomial time.

2. It's easy to prove the reverse condition. If there exists a *non-negative* $\lambda_j \ge 0$ for $j \in \{1, \dots, m\}$ such that

$$\sum_{j=1}^{m} \lambda_j b_j < 0 \qquad \text{and} \qquad \sum_{j=1}^{m} \lambda_j A_{ji} = 0 \quad \forall i \in \{1, \dots, n\}.$$

   then the system of inequalities will be infeasible as the LHS will be $= 0$ and the RHS is strictly < 0.

   To prove the forward condition, we'll show a conversion from the system of equations with inequality to a form of the Farkas' Lemma A.

We replace each inequality constraint by

$$\sum A_{ji}x_i < b_j => \sum A_{ji}x_i + z_j = b_j \quad z \geq 0$$

For each unrestricted variable x_i, introduce

$$x_i = x_i^+ - x_i^-, \qquad x_i^+, x_i^- \geq 0.$$

If $x_i \geq 0$ already, we can keep x_i as x_i^+ and set $x_i^- = 0$.
Let

$$\overline{x} := \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} \geq 0, \qquad \overline{A} := \begin{pmatrix} A & -A & I \end{pmatrix}.$$

With the above transforms, we can re-write our system of inequality to:

$$\overline{A}\overline{x} = b,$$
$$\overline{x} \geq 0$$

By assumption the original $\leq$-system is infeasible, so the equality system $\overline{A}\overline{x} = b$, $\overline{x} \geq 0$ is infeasible. Applying Farkas' Lemma A to this equality system yields a vector $\lambda \in \mathbb{R}^m$ such that

$$\lambda^T b < 0 \qquad \text{and} \qquad \lambda^T \overline{A} \geq 0 \quad \text{(componentwise).}$$

Examine the componentwise inequalities $\lambda^T \overline{A} \geq 0$ column block by block:

- Columns corresponding to x^+ (these columns are the columns of A) give

$$\sum_{j=1}^{m} \lambda_j A_{ji} \geq 0 \qquad \text{for each } i.$$

- Columns corresponding to x^- (these columns are the columns of $-A$) give

$$\sum_{j=1}^{m} \lambda_j(-A_{ji}) \geq 0 \quad \Longrightarrow \quad -\sum_{j=1}^{m} \lambda_j A_{ji} \geq 0 \quad \Longrightarrow \quad \sum_{j=1}^{m} \lambda_j A_{ji} \leq 0.$$

Combining the two displayed inequalities for each fixed i yields

$$\sum_{j=1}^{m} \lambda_j A_{ji} = 0 \qquad \text{for every } i = 1, \ldots, n,$$

i.e. $\lambda^T A = 0$.

Finally, the columns corresponding to variables z_j from the identity matrix I_m. The component wise inequalities for those columns give

$$\lambda_j \geq 0 \qquad \text{for each } j = 1, \ldots, m,$$

so $\lambda \geq 0$.

Thus the λ produced by Farkas' Lemma A satisfies

$$\lambda \geq 0, \qquad \lambda^T b < 0, \qquad \lambda^T A = 0,$$

which is exactly the requirements in Farkas' Lemma B. Hence Farkas' Lemma A => Farkas' Lemma B.