Online Convex Optimization & Complexity
Problem Set 6 – CS 6515/4540 (Fall 2025)

Solutions for HW6, Q24

24 Zero Sum Games

1. Let A play A_1 with probability p and A_2 with probability $1-p$. If B plays B_1 then A's expected payoff is
$$u(B_1) = 2 \cdot p + (-1) \cdot (1-p) = 3p - 1.$$
If B plays B_2 then A's expected payoff is
$$u(B_2) = (-1) \cdot p + 1 \cdot (1-p) = 1 - 2p.$$
Since B sees p and then chooses the action minimizing A's payoff (or maximizing B's payoff), A's guarantee when choosing p is
$$\min\{\, 3p-1,\ 1-2p \,\}.$$
A should choose p to maximize this worst-case value. Set the two expressions equal to find the crossover:
$$3p - 1 = 1 - 2p \quad \Longrightarrow \quad 5p = 2 \quad \Longrightarrow \quad p = \tfrac{2}{5}.$$
At $p = \frac{2}{5}$ the common value is
$$3 \cdot \frac{2}{5} - 1 = \frac{6}{5} - 1 = \frac{1}{5}.$$
Thus A's optimal committed strategy is $\big(p(A_1) = \frac{2}{5},\ p(A_2) = \frac{3}{5}\big)$, and the expected reward A guarantees is $\frac{1}{5}$.

2. Let B play B_1 with probability q and B_2 with probability $1-q$. If A plays A_1 the expected payoff is
$$u(A_1) = 2q + (-1)(1-q) = 3q - 1,$$
and if A plays A_2 it is
$$u(A_2) = -1 \cdot q + 1 \cdot (1-q) = 1 - 2q.$$
Given q, A will choose the action giving the larger payoff, so the payoff resulting from B's choice q is
$$\max\{\, 3q-1,\ 1-2q \,\}.$$
B wants to choose q to minimize this maximum. Again equate the two branches:
$$3q - 1 = 1 - 2q \quad \Longrightarrow \quad q = \tfrac{2}{5},$$
and the resulting value is $\frac{1}{5}$. So B's optimal committed strategy is $q(B_1) = \frac{2}{5}, q(B_2) = \frac{3}{5}$, and the expected cost (payoff to A) is $\frac{1}{5}$.

3. Yes, both (1) and (2) gave the same expected values: $\frac{1}{5}$. This is precisely the statement of the minimax theorem for zero-sum games: the value of the game
$$\min_x \max_y M(x,y) \;=\; \max_y \min_x M(x,y)$$
and optimal mixed strategies exist for both players producing the same value. Intuitively, when each player can randomize, committing first or second does not change the guaranteed payoff in a zero-sum game.