Linear Programming
Problem Set 4 – CS 6515/4540 (Fall 2025)

Answers to problem set 4, Q17.

17 Max 2-SAT

1. We take any boolean assignment $x_i \in \{0,1\}$ and define the ILP variables as follows:
   - For each literal that is the variable x_i, use $y = x_i$; if the literal is $\neg x_i$, use $y = 1 - x_i$.
   - For each clause j, set $z_j = 1$ if the clause is satisfied by the assignment, and $z_j = 0$ otherwise.

   This satisfies the LP constraints:

$$\begin{aligned} 1 \geq x_i \geq 0 &\quad \text{holds because } x_i \in \{0,1\}, \\ z_j \leq 1 &\quad \text{holds because } z_j \in \{0,1\}, \\ y_{j1} \geq z_j &\quad \forall j \in J_1,\ \text{since the literal must be true if } z_j = 1, \\ y_{j1} + y_{j2} \geq z_j &\quad \forall j \in J_2,\ \text{since at least one literal is true if } z_j = 1. \end{aligned}$$

   Therefore, the LP optimum (which is the maximum of the objective over all feasible fractional solutions) is at least the maximum number of clauses that can be satisfied by any boolean assignment. This also holds from the fact that with the ILP (any boolean assignment) we're constraining the feasible set of solutions.

2. Let (x_i, y_j, z_j) be the optimal, possibly fractional, LP solution. Now we're constructing a random boolean assignment by setting each variable x_i to true independently with probability x_i, and false otherwise.

   Let's define X_j as the indicator random variable for the event that clause j is satisfied by this random assignment. We will then show that $\mathbb{E}[X_j] \geq \frac{3}{4} z_j$ for all j.

   Scenario 1: $j \in J_1$

   The probability that the literal is true equals the corresponding y_{j1} (either x_i or $1 - x_i$). The LP constraint $y_{j1} \geq z_j$ implies:

$$\mathbb{E}[X_j] = \Pr[\text{literal is true}] = y_{j1} \geq z_j \geq \frac{3}{4} z_j.$$

   Scenario 2: $j \in J_2$

   Let $p_1 = y_{j1}$ and $p_2 = y_{j2}$ be the probabilities that the two literals are true. The clause, j, is satisfied unless both literals are false, so:

$$\mathbb{E}[X_j] = 1 - (1 - p_1)(1 - p_2) = p_1 + p_2 - p_1 p_2.$$

   From the LP constraint, $p_1 + p_2 \geq z_j$. We know that for a fixed sum $S = p_1 + p_2$, the product $p_1 p_2$ is maximized when $p_1 = p_2 = S/2$, giving:

$$\mathbb{E}[X_j] \geq S - \frac{S^2}{4}.$$

   Since $z_j \leq 1$, $S \in [0,1]$ and $S \geq z_j$, we have:

$$S - \frac{S^2}{4} = S\left(1 - \frac{S}{4}\right) \geq \frac{3}{4} S \geq \frac{3}{4} z_j.$$

Thus, for any clause j,

$$\mathbb{E}[X_j] \geq \frac{3}{4} z_j.$$

By linearity of expectation,

$$\mathbb{E}\left[\sum_j X_j\right] = \sum_j \mathbb{E}[X_j] \geq \frac{3}{4} \sum_j z_j.$$

Hence, the randomized rounding yields an expected number of satisfied clauses at least $\frac{3}{4}$ times the LP optimum. Therefore, this gives a $\frac{3}{4}$-approximation algorithm for Max 2-SAT in expectation.