Linear Programming
Problem Set 4 – CS 6515/4540 (Fall 2025)

Solutions to problem set 4, Q14.

14 LP-Based Set Cover

Introduce a variable $x_i \in \{0,1\}$ for each set S_i, where $x_i = 1$ indicates that S_i is chosen or $i \in T$.

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} x_i \\ \text{subject to} \quad & \sum_{i: e \in S_i} x_i \geq 1, \quad \forall e \in [n], \\ & x_i \in \{0,1\}, \quad \forall i \in [m]. \end{aligned}$$

We relax the integrality constraints to obtain the LP relaxation:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} x_i \\ \text{subject to} \quad & \sum_{i: e \in S_i} x_i \geq 1, \quad \forall e \in [n], \\ & x_i \geq 0, \quad \forall i \in [m]. \end{aligned}$$

Let $x^* = (x_1^*, x_2^*, \ldots, x_m^*)$ be an optimal LP solution, and let $C^* = \sum_i x_i^*$ denote its objective value. Clearly, C^* is a lower bound on the optimal ILP as we've increased the feasible region.
We define the rounded set

$$T := \{i \in [m] : x_i^* \geq 1/f\}.$$

We will show that T is a valid set cover and that $|T| \leq f \cdot C^*$.

- Fix any element $j \in [n]$. From the LP constraints, we have

  $$\sum_{i: j \in S_i} x_i^* \geq 1.$$

  By definition of f, the left-hand side includes at most f terms. If all of these terms were less than $1/f$, we would have

  $$\sum_{i: j \in S_i} x_i^* < f \cdot \frac{1}{f} = 1,$$

  contradicting the constraint above. Thus, there must exist at least one i with $j \in S_i$ and $x_i^* \geq 1/f$, implying $i \in T$.

  Hence, for every $j \in [n]$, there exists some S_i with $i \in T$ that covers j, so

  $$\bigcup_{i \in T} S_i = [n].$$

  Therefore, T is a feasible cover.

- Each chosen set $i \in T$ satisfies $x_i^* \geq 1/f$. Therefore,

$$|T| \leq \sum_{i \in T} \frac{x_i^*}{1/f} \leq f \sum_{i=1}^{m} x_i^* = f \cdot C^*.$$

  Thus, by choosing all i such that $x_i^* \geq 1/f$ we've increased the size of our optimal set T by a max of f times.

Therefore, we have an f-approximation algorithm for the set cover problem.