

Online Convex Optimization & Complexity

Problem Set 6 – CS 6515/4540 (Fall 2025)

Solutions to HW6, Q25

25 Reductions: Dominating Set

- Given a graph $G = (V, E)$ we form a set-cover instance with universe $U = V$ and family of sets

$$\mathcal{F} = \{N[v] : v \in V\},$$

i.e. one set for each vertex consisting of its closed neighborhood. A collection of vertices $D \subseteq V$ is a dominating set of G iff the corresponding family $\{N[v] : v \in D\}$ covers U . Hence the minimum dominating set size equals the size of the minimum subcollection of \mathcal{F} that covers V .

Therefore any α -approximation algorithm for Set Cover, when run on the instance (U, \mathcal{F}) , returns a collection of closed neighborhoods of size at most α times optimum. Interpreting those chosen sets as vertices yields an α -approximate dominating set for G .

- Given universe $U = \{u_1, \dots, u_n\}$ and family $\mathcal{S} = \{S_1, \dots, S_m\}$, construct graph $G = (V, E)$ as follows:

$$V = U \cup \{s_1, \dots, s_m\},$$

i.e. create one vertex u_j for every universe element and one vertex s_i for every set S_i . Put edges

$$(s_i, u_j) \in E \text{ iff } u_j \in S_i,$$

and make the set-vertices $\{s_1, \dots, s_m\}$ into a clique (i.e. add all edges $(s_i, s_{i'})$ for $i \neq i'$). There are no edges among element-vertices u_j themselves.

(\Rightarrow) Any set cover of size k yields a dominating set of size k . If $\mathcal{C} \subseteq \{S_1, \dots, S_m\}$ is a cover (every u_j lies in some S_i with $S_i \in \mathcal{C}$), take $D = \{s_i : S_i \in \mathcal{C}\}$. Every element-vertex u_j is adjacent to some chosen s_i (since \mathcal{C} covers), so each u_j is dominated. Also all s_i -vertices are in the clique, so every s -vertex is dominated (a chosen s dominates all other s -vertices). Thus D is a dominating set of size k .

(\Leftarrow) Conversely, any dominating set D can be converted into a set-vertex-only dominating set D' with $|D'| \leq |D|$. Let $D \subseteq V$ be any dominating set. Construct D' as follows:

- include every set-vertex $s_i \in D$ into D' ;
- for every element-vertex $u_j \in D$, choose (arbitrarily) one set S_i with $u_j \in S_i$ (such an S_i exists because otherwise u_j would be isolated in the reduction), and include s_i in D' .

If several element-vertices of D select the same s_i we only add s_i once; therefore $|D'| \leq |D|$.

We must check D' dominates all vertices:

- Any element-vertex u was dominated by D . If it was dominated by a set-vertex $s \in D$, then that s remains in D' and so u is still dominated. If u itself belonged to D , then we mapped u to some s_i with $u \in S_i$ and put that s_i into D' , so u is dominated.
- Any set-vertex s was dominated by D . If $s \in D$ then $s \in D'$ and is dominated. Otherwise s had a neighbor in D . Such a neighbor can only be either a set-vertex (which remains in D') or an element-vertex $u \in D$ adjacent to s . In the latter case we replaced that element u by some set-vertex $s_u \in D'$, and because all set-vertices form a clique s is adjacent to s_u , hence s is dominated by D' . Thus every set-vertex is also dominated.

So D' is a dominating set consisting only of set-vertices, with $|D'| \leq |D|$.

Finally, the set of set-vertices in any dominating set D' corresponds exactly to a collection of sets that covers U : each element u must be adjacent to some $s_i \in D'$, which means $u \in S_i$. Hence the chosen set-vertices form a set-cover.

Thus an α -approximation algorithm for Dominating Set yields an α -approximation for Set Cover.