Linear Programming
Problem Set 3 – CS 6515/4540 (Fall 2025)

Answers to problem set 3, question 12.

12 LP for Regression

1. For the ℓ_1 regression problem:

$$\min_{a,b} \sum_{i=1}^{n} \left| y_i - (ax_i + b) \right|.$$

Introducing variables $t_i \geq 0$ to represent absolute values, s.t $|y_i - ax_i - b| = t_i$. We get the following primal LP:

$$\begin{aligned}
\min_{a,b,t} \quad & \sum_{i=1}^{n} t_i \\
\text{s.t.} \quad & y_i - ax_i - b \leq t_i, \quad i = 1, \dots, n, \\
& y_i - ax_i - b \geq -t_i, \quad i = 1, \dots, n, \\
& t_i \geq 0, \quad i = 1, \dots, n.
\end{aligned}$$

This is a polynomial sized LP as we've added n+2 variables and 3n constraints. To note the variables of our primal LP are, a, b, t_i. Equivalently,

$$\boxed{\begin{aligned}
\min_{a,b,t} \quad & \sum_{i=1}^{n} t_i \\
\text{s.t.} \quad & ax_i + b + t_i \geq y_i, \quad i = 1, \dots, n, \\
& -ax_i - b + t_i \geq -y_i, \quad i = 1, \dots, n, \\
& t_i \geq 0, \quad i = 1, \dots, n.
\end{aligned}}$$

Our A (constraint) matrix can be thought of as

$$A = \begin{bmatrix}
x_1 & 1 & 1 & 0 & \cdots & 0 \\
x_2 & 1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & & & \ddots & \vdots \\
x_n & 1 & 0 & 0 & \cdots & 1 \\
-x_1 & -1 & 1 & 0 & \cdots & 0 \\
-x_2 & -1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & & & \ddots & \vdots \\
-x_n & -1 & 0 & 0 & \cdots & 1
\end{bmatrix} \in \mathbb{R}^{2n \times (n+2)}.$$

and the variable vector as

$$v = \begin{bmatrix} a \\ b \\ t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} \in \mathbb{R}^{n+2}.$$

Our primal LP can be written as,

$$\begin{aligned} \min_{v} \quad & c^\top v \\ \text{s.t.} \quad & Av \le b, \end{aligned}$$

where

$$v = \begin{bmatrix} a \\ b \\ t_1 \\ \vdots \\ t_n \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{n+2}, \quad b = \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ -y_1 \\ \vdots \\ -y_n \end{bmatrix} \in \mathbb{R}^{2n}.$$

2. For the ℓ_∞ regression problem:

$$\min_{a,b} \max_{i=1}^{n} \left| y_i - (ax_i + b) \right|.$$

Introducing a single variable $T \ge 0$ to represent the maximum absolute error, s.t. $|y_i - ax_i - b| \le T$ for all i. We get the following primal LP:

$$\begin{aligned} \min_{a,b,T} \quad & T \\ \text{s.t.} \quad & y_i - ax_i - b \le T, \quad i = 1, \dots, n, \\ & -(y_i - ax_i - b) \le T, \quad i = 1, \dots, n, \\ & T \ge 0. \end{aligned}$$

This is a polynomial sized LP as we've 3 variables and 2n+1 constraints. To note the variables of our primal LP are, a, b, T. Equivalently,

$$\boxed{\begin{aligned} \min_{a,b,T} \quad & T \\ \text{s.t.} \quad & -ax_i - b + T \ge -y_i, \quad i = 1, \dots, n, \\ & ax_i + b + T \ge y_i, \quad i = 1, \dots, n, \\ & T \ge 0. \end{aligned}}$$

Our A (constraint) matrix can be thought of as

$$A = \begin{bmatrix} -x_1 & -1 & 1 \\ -x_2 & -1 & 1 \\ \vdots & \vdots & \vdots \\ -x_n & -1 & 1 \\ x_1 & 1 & 1 \\ x_2 & 1 & 1 \\ \vdots & \vdots & \vdots \\ x_n & 1 & 1 \end{bmatrix} \in \mathbb{R}^{2n \times 3}.$$

and the variable vector as

$$v = \begin{bmatrix} a \\ b \\ T \end{bmatrix} \in \mathbb{R}^3.$$

Our primal LP can be written as,

$$\begin{aligned} \min_{v} \quad & c^\top v \\ \text{s.t.} \quad & Av \ge b, \end{aligned}$$

where

$$v = \begin{bmatrix} a \\ b \\ T \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3, \quad b = \begin{bmatrix} -y_1 \\ \vdots \\ -y_n \\ y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^{2n}.$$

3. - For the ℓ_1 regression problem:
   To convert to dual form we introduce dual variables $\alpha_i, \beta_i \geq 0$ The dual LP:

   $$\max_{u \geq 0} b^\top u \quad \text{s.t.} \quad A^\top u \leq c,$$

   where

   $$u = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}^n$$

   To note, since a and b are free variable in the primal LP, they will correspond to equality constraints in the dual.
   The dual objection function becomes,

   $$b^\top u = \sum_{i=1}^{n} -y_i\alpha_i + \sum_{i=1}^{n} y_i\beta_i = \sum_{i=1}^{n}(\beta_i - \alpha_i)y_i.$$

   And the new constraints are,

   $$\sum_{i=1}^{n} x_i(\alpha_i - \beta_i) = 0, \quad a \text{ is a free variable in the primal}$$

   $$\sum_{i=1}^{n}(\alpha_i - \beta_i) = 0, \quad b \text{ is a free variable in the primal}$$

   $$\alpha_i + \beta_i \leq 1, \quad i = 1, \ldots, n.$$

   Therefore, the dual problem is

   $$\boxed{\begin{aligned} \max_{\alpha,\beta} \quad & \sum_{i=1}^{n}(\beta_i - \alpha_i)y_i \\ \text{s.t.} \quad & \alpha_i + \beta_i \leq 1, \quad i = 1, \ldots, n, \\ & \alpha_i, \beta_i \geq 0, \quad i = 1, \ldots, n, \\ & \sum_{i=1}^{n}(\alpha_i - \beta_i)x_i = 0, \\ & \sum_{i=1}^{n}(\alpha_i - \beta_i) = 0. \end{aligned}}$$

   - For the ℓ_∞ regression problem: To convert to dual form we introduce dual variables $\alpha_i, \beta_i \geq 0$ corresponding to the first n and second n constraints, respectively. The dual LP:

   $$\max_{u \geq 0} b^\top u \quad \text{s.t.} \quad A^\top u \leq c,$$

where

$$u = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}^n.$$

To note, since a and b are free variable in the primal LP, they will correspond to equality constraints in the dual.

The dual objective function becomes,

$$b^\top u = \sum_{i=1}^{n} (-y_i)\alpha_i + \sum_{i=1}^{n} (y_i)\beta_i = \sum_{i=1}^{n} (\beta_i - \alpha_i) y_i.$$

And the new constraints are,

$$\begin{aligned}
\sum_{i=1}^{n} x_i(\beta_i - \alpha_i) = 0, &\quad a \text{ is a free variable in the primal} \\
\sum_{i=1}^{n} (\beta_i - \alpha_i) = 0, &\quad b \text{ is a free variable in the primal} \\
\sum_{i=1}^{n} (\alpha_i + \beta_i) \le 1, &\quad T \text{ is nonnegative in the primal.}
\end{aligned}$$

Therefore, the dual problem is

$$\boxed{\begin{aligned}
\max_{\alpha,\beta} \quad & \sum_{i=1}^{n} (\beta_i - \alpha_i) y_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} (\alpha_i + \beta_i) \le 1, \\
& \alpha_i, \beta_i \ge 0, \quad i = 1, \ldots, n, \\
& \sum_{i=1}^{n} (\beta_i - \alpha_i) x_i = 0, \\
& \sum_{i=1}^{n} (\beta_i - \alpha_i) = 0.
\end{aligned}}$$