Part of Exam 2

Linear Programming

Continuous n variables $x_1, x_2, \dots, x_n$

Objective : $\max \sum_{i=1}^{n} c_i x_i$ where $c_i \in \mathbb{R}$ $\qquad \max\ c^T x$

Subject to $\sum_{i=1}^{n} a_i^{(j)} x_i \le b^{(j)}$ for $j \in \{1, \dots, m\}$ $\longleftarrow Ax \le b$
$\ge$

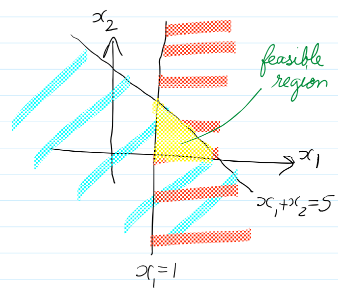
where $a_i^{(j)}$ & $b^{(j)} \in \mathbb{R}$

$$j \underbrace{\left[\quad a_{(i)}^{j} \quad \right]}_{m \times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \le \begin{bmatrix} b^{(1)} \\ \vdots \\ b^{(m)} \end{bmatrix}$$

Eg. $\max\ 2x_1 + 3x_2$

s.t. $x_1 + x_2 \le 5$

$x_1 \ge 1 \iff -x_1 \le -1$



Feasible region: Set of $\vec{x} \in \mathbb{R}^n$ that satisfy all constraints

Smoothie Example

Sps 3 oranges, 10 Bananas, 6 Strawberries

You can make 3 kinds of smoothies :

(1) 1 Or + 2 Ban : $\$5/l$

(2) 2 Str + 1 Ban : $\$4/l$

(3) 1 Str + 1 Or + 1 Ban : $\$6/l$

Goal : max the money made

Let x_i for $i \in \{1,2,3\}$ denote amount of smoothie (i) in litres

Obj : $\max\ 5x_1 + 4x_2 + 6x_3$

s.t. $1.x_1 + 1.x_3 \le 3$ — Orange

$2x_1 + x_2 + x_3 \le 10$ — Bananas

$2x_2 + x_3 \le 6$ — Straw

$x_i \ge 0$ for $i \in \{1,2,3\}$

Eg 2 Max-flow

Let f_e for $e \in E$

$\max \sum_{v \in V} f_{sv}$

$$\text{s.t. } \forall u \neq \{s,t\}: \quad \sum_{vu} f_{su} = \sum_{uv} f_{uv}$$

$$\forall (u,v) \in E \quad 0 \le f_{uv} \le c_{uv}$$

Thm: Any LP can be solved in poly(# input bits) time $\approx n^3$

Remarks:

(1) Equalities are allowed in constraints

$$a_1x_1 + a_2x_2 = 5$$

$$\Updownarrow \quad a_1x_1 + a_2x_2 \le 5$$

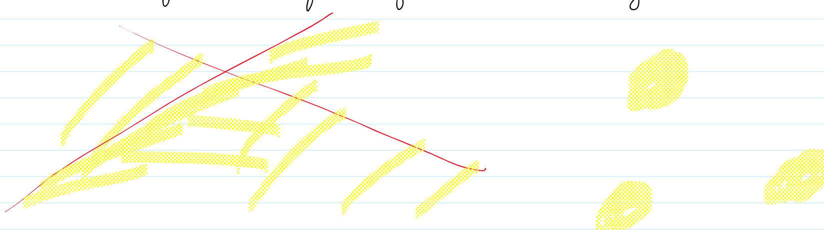
$$-a_1x_1 - a_2x_2 \le -5$$

(2) Strict inequalities are not allowed $\quad a_1x_1 + a_2x_2 < 5$

E.g. max x_1

s.t. $x_1 < 5$

Ques: Is the feasible region of an LP always connected?

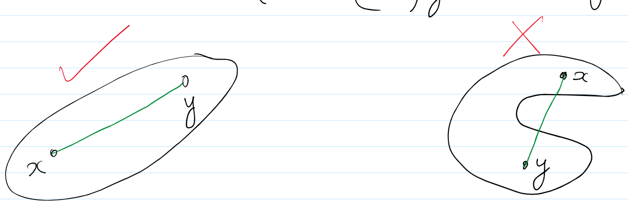


Ans: Yes

In fact, feasible is always convex.

Defn (Convex) A set $S \subseteq \mathbb{R}^n$ if for any $x \in S$ and $y \in S$ we have

$$\alpha x + (1-\alpha) y \in S \quad \text{for } \alpha \in [0,1]$$



Thm: Feasible region of LP is convex.

Pf: Note that if x satisfies $\sum_i a_i x_i \le b$ — α

and y 〃 $\sum_i a_i y_i \le b$ — $(1-\alpha)$

and y . $\sum_i a_i y_i \le b$ ( )

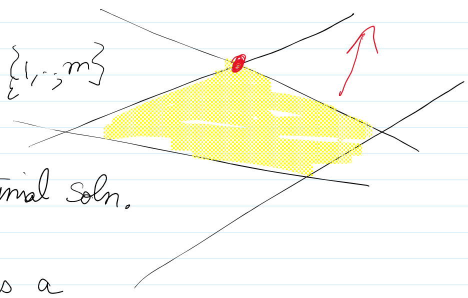
$$\Rightarrow \sum_i a_i \left(\alpha x_i + (1-\alpha) y_i \right) \le b$$

$\Rightarrow \alpha x + (1-\alpha) y$ is also feasible. $\blacksquare$

Thm: LPs can be solved in polytime

Ques: Given an LP, why does it even have a poly(#input bits) size soln?

$$\max \sum_i c_i x_i$$
$$\text{s.t. } \sum_i a_i^{(j)} x_i \le b^{(j)} \text{ for } j \in \{1, \dots, m\}$$



Obs: There always exists a vertex optimal soln.

Defn (Vertex) In n dim, vertex is a soln to n linearly indep linear-equalities.

Now any vertex is a soln to $A' x^* = b'$

$\Rightarrow x^* = (A')^{-1} b'$ Subset of n constraints

This has poly sized representation

Proof of LP optimality

Suppose we are given an LP : $\max c^T x$ s.t. $Ax \le b$

A friend claims that x^* is the optimal soln to LP.

Q: How can your friend convince you that x^* is optimal?

Eg. $\max 5x_1 + 4x_2 + 6x_3$

s.t. $1 \cdot x_1 + 1 \cdot x_3 \le 3$ — Orange (a)

$2x_1 + x_2 + x_3 \le 10$ — Bananas (b)

$2x + x_3 \le 6$ — Straw (c)

$$2x_1 + x_2 + x_3 \leq 10 \quad \text{— Bananas} \qquad (b)$$

$$2x_2 + x_3 \leq 6 \quad \text{— Straw} \qquad (c)$$

$$x_i \geq 0 \quad \text{for } i \in \{1,2,3\}$$

Friend says $x_1 = 3 = x_2$, $x_3 = 0$ is optimal

Objective value $= 27$

Pf: Consider $5 * (a) + 2 * (c) + \{-x_3 \leq 0\}$

$$\Rightarrow 5(x_1 + x_3) + 2(2x_2 + x_3) + (-x_3) \leq 5 * 3 + 2 \times 6 + 0$$

$$\Rightarrow 5x_1 + 4x_2 + 6x_3 \leq 27$$

$\Rightarrow$ Every soln has objective ≤ 27

Thm: For any LP, there exists a non-neg linear combination of constraints such that LHS = Objective

RHS = Optimal value.