Linear Programming
Problem Set 3 – CS 6515/4540 (Fall 2025)

This problem set is due on **Thursday October 2nd**. Submission is via Gradescope. Your solution must be a typed pdf (e.g. via LaTeX) – no handwritten solutions.

9 Convex Sets

We will study properties of convex sets in the Euclidean space.

1. Does every convex body $K \subseteq \mathbb{R}^n$ have a finite number of vertices[1]? Give a proof or a counterexample.

2. Is the intersection of two convex sets also a convex set? What about union? For each, prove it or give a counterexample.

3. Is a convex set in n dimensions always bounded (finite volume) or could be unbounded? Give a proof that it is bounded or a counterexample.

10 LPs

Alice is trying to get enough oranges and bananas to host a fruit party. To successfully host a party she needs **at least 8 oranges** and **at least 20 bananas**. Unfortunately, her local grocery story only sells fruit in bundles. Bundle A costs 3 dollars and contains one orange and two bananas. Bundle B costs 2 dollars and contains three oranges and a banana. Fortunately, the grocery story will allow Alice to buy fractions of bundles (i.e. she can buy 2.5 bundle As to get 2.5 oranges and 5 bananas). They will not allow Alice to buy negative bundles (i.e. she cannot buy -1 bundle As and 3 bundle Bs to get 5 oranges and a banana).

Alice would like to buy x_A bundle As and x_B bundle Bs to guarantee she has at least 8 oranges and at least 20 bananas. Moreover, she would like to minimize her dollars spent.

1. Write a linear program whose solution is the optimal choice of x_A, x_B for Alice's problem and briefly justify why this is the correct LP.

2. Show that there exists a solution to this linear program with objective value 30 (and hence Alice has to spend at most 30 dollars).

3. Prove that every feasible solution to this linear program has value at least 30 (prove this by taking a non-negative linear combination of the constraints), so the solution in the previous part is optimal.

11 LP Equivalent Formulation

1. Show that the class of left LPs (LP1) can efficiently represent any LP on the right (LP2). In other words, if we have a polynomial time algorithm to solve any LP1 on the left, then we can solve in polynomial time any LP2 on the right.

[1] Points $u \in K$ is called a vertex if cannot be expressed as a convex combination of some finite number of other points in K

$$\max \sum_{i=1}^{n} c_i x_i \qquad \text{(LP1)}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} A_{ji} x_i = b_j, \quad \forall j \in \{1, \ldots, m\}$$

$$x_i \geq 0, \quad \forall i \in \{1, \ldots, n\}.$$

$$\max \sum_{i=1}^{n} c_i x_i \qquad \text{(LP2)}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} A_{ji} x_i \leq b_j, \quad \forall j \in \{1, \ldots, m\}$$

(Hint: You need two ideas: (1) We can replace unconstrained x by $x = x^+ - x^-$ where $x^+ \geq 0$ and $x^- \geq 0$. (2) We can replace $Ax \leq b$ by $Ax + Z = b$ for some vector $Z \geq 0$.)

2. Now use a similar idea to show that Farkas' Lemma A below implies Farkas' Lemma B.

   - Farkas' Lemma A says that the system of inequalities $\sum_{i=1}^{n} A_{ji} x_i = b_j$ for $j \in \{1, \ldots, m\}$ and $x_i \geq 0$ for $i \in \{1, \ldots, n\}$ are infeasible iff there exist λ_j for $j \in \{1, \ldots, m\}$ such that

   $$\sum_{j=1}^{m} \lambda_j b_j < 0 \qquad \text{and} \qquad \sum_{j=1}^{m} \lambda_j A_{ji} \geq 0 \quad \forall i \in \{1, \ldots, n\}.$$

   - Farkas' Lemma B says that the system of inequalities $\sum_{i=1}^{n} A_{ji} x_i \leq b_j$ for $j \in \{1, \ldots, m\}$ are infeasible iff there exist *non-negative* $\lambda_j \geq 0$ for $j \in \{1, \ldots, m\}$ such that

   $$\sum_{j=1}^{m} \lambda_j b_j < 0 \qquad \text{and} \qquad \sum_{j=1}^{m} \lambda_j A_{ji} = 0 \quad \forall i \in \{1, \ldots, n\}.$$

12 LP for Regression

Suppose we are given n points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n) \in \mathbb{R} \times \mathbb{R}$. The goal of this problem is to find a line $y = ax + b$, where $a, b \in \mathbb{R}$ that fits these points as closely as possible, where closeness is defined according to some objective function. Write a polynomial-sized LP for the following settings:

1. ℓ_1 regression: Objective is to minimize $\sum_{i=1}^{n} |y_i - ax_i - b|$.
2. ℓ_∞ regression: Objective is to minimize $\max_{i=1}^{n} |y_i - ax_i - b|$
3. Write the dual LPs for both the above LPs.