* HW 5 due today
* HW 6 to be released soon

Online Convex Optimization / Online Learning

Example (Regression): Given T labeled samples $(a_1, b_1) \dots (a_T, b_T)$ where $a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$

Find x s.t. $a_t x \approx b_t$ $\leftarrow$ offline optim / Learning

Formally, $\min_x \sum_{t=1}^{T} (a_t x - b_t)^2 = \min_x \underbrace{\|Ax - b\|^2}_{f(x)}$

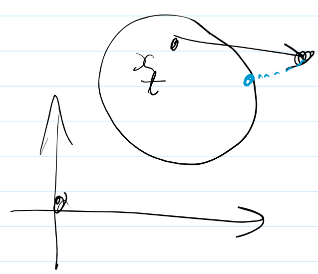
Gradient descent will find x s.t. $f(x) - f(x^*) \leq \epsilon$.

What if the samples not given upfront and arrive over time?

Problem Model (Online Convex Opt)

1) We are given a convex body $K \subseteq \mathbb{R}^n$
2) T rounds:
   (a) Alg plays $x_t \in K$
   (b) Convex cost function $f_t : \mathbb{R}^n \to \mathbb{R}$ is revealed

E.g. $f_t(x) = (a_t x - b_t)^2$



Goal: $\min \sum_{t=0}^{T-1} f_t(x_t)$

Benchmark: $\min_{x^* \in K} \sum_{t=0}^{T-1} f_t(x^*) \neq \sum_{t=0}^{T-1} \min_{y_t^* \in K} f_t(y_t^*)$

Thm: Online gradient descent guarantees

$x_{t+1} = \Pi_K \left[x_t - \eta \nabla f_t(x_t) \right]$

$$\underbrace{\frac{1}{T} \sum_{t=0}^{T-1} \left[f_t(x_t) - f_t(x^*) \right]}_{\text{Average regret}} \leq \frac{DG}{\sqrt{T}}$$

where $D = \text{Diameter}(K)$ & $G = \max_t \|\nabla f_t(x)\|$ for $x \in K$

Pf: (Same as offline grad descent)

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Define a potential function $\phi_t = \|x_t - x^*\|^2 \cdot \frac{1}{2\eta}$

We will show that

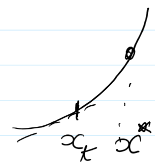
$$\left[f_t(x_t) - f_t(x^*)\right] \leq \left(\phi_t - \phi_{t+1}\right) + \frac{\eta G^2}{2} \qquad \text{—— (1)}$$

This suffices since

$$\underbrace{\sum_{t=0}^{T-1}\left[f_t(x_t) - f_t(x^*)\right]}_{\text{Regret}} \leq \underbrace{\phi_0}_{=\frac{D^2}{2\eta}} - \underbrace{\phi_T}_{\geq 0} + \frac{\eta G^2 T}{2}$$

$$\leq \frac{D^2}{2\eta} + \frac{\eta G^2 T}{2} = DG\sqrt{T}. \qquad \boxed{\eta = \frac{D}{G\sqrt{T}}}$$

Finally, to prove (1),



$$2\eta \cdot \phi_{t+1} = \|x_{t+1} - x^*\|^2$$

$$= \|x_t - \eta \nabla f_t(x_t) - x^*\|^2$$

$$= \underbrace{\|x_t - x^*\|^2}_{=\phi_t \cdot 2\eta} + \eta^2 \underbrace{\|\nabla f_t(x_t)\|^2}_{\leq G^2} + 2\eta \underbrace{\langle \nabla f_t(x_t), x^* - x_t\rangle}_{\leq f_t(x^*) - f_t(x_t)}$$

$$\Rightarrow f_t(x_t) - f_t(x^*) \leq \phi_t - \phi_{t+1} + \frac{\eta G^2}{2}.$$

$\blacksquare$

Applications

(a) Experts Problem:

- n experts
- Each round $t \in \{1, \dots, T\}$

(1) Alg chooses an expert/action $a_t \in \{1, \dots, n\}$ — random choice is allowed

(2) Cost of each expert $c_t(i) \in [-1, 1]$ revealed & Alg gets $c_t(a_t)$

Objective: Min total Alg cost $\sum_t c_t(a_t)$

compared to best fixed expert $\min_i \sum_t c_t(i)$

Remark: This problem is useful in Forecasting (weather, stocks), Spam detection, LP solving

Thm: OGD implies avg regret

$$\frac{1}{T}\left[\sum_t c_t(a_t) - \sum_t c_t(i)\right] \le \frac{\sqrt{n}}{\sqrt{T}}$$

Pf: Let x_t denote Alg's t-th distrib over n experts

Let $K = \{ x \mid \sum_i x(i) = 1, \; x(i) \ge 0 \}$

$\Rightarrow$ diameter$(K) \le 1$

$$f_t(x) = \sum_i c_t(i) \cdot x(i) \quad \leftarrow \text{linear fn}$$

$$\Rightarrow \|\nabla f_t\| = \|c_t\| \le \sqrt{n}$$

Hence, OGD implies

$$\frac{1}{T} \underbrace{\sum_t f_t(x_t)}_{= \sum_i x_t(i) \cdot c_t(i) \,=\, \mathbb{E}[\text{Alg's cost at step } t]} - \min_{x^* \in K} \underbrace{\sum_t f_t(x^*)}_{= \sum_i x^*(i) \sum_t c_t(i) \,=\, \min_i \sum_t c_t(i)} \le \frac{\overset{\le 1}{D}\;\overset{\le \sqrt{n}}{G}}{\sqrt{T}}$$

$\blacksquare$

Remark: There is a different alg, called Multiplicative Weights, with average regret $\frac{\sqrt{\log n}}{\sqrt{T}}$ for Experts problem

(b) Min-cost Perfect Matching

* n people and n tasks

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* On each day $t \in \{1, \dots, T\}$

(1) Play a perfect matching M_t

(2) Cost of each edge revealed $c_t(i,j) \in [0,1]$ for $i,j \in \{1, \dots n\}$

Goal: $\min \sum_{t=1}^{T} \sum_{i=1}^{n} c_t(i, M_t(i))$

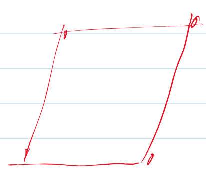
Benchmark: Best-fixed matching M^*

$$\sum_t \sum_i c_t(i, M^*(i))$$

Thm: We can use OGD to play (randomized) matchings with average regret $\leq \frac{n^{1.5}}{\sqrt{T}}$.

Pf: Let x_t be t-th fractional matching

$$K = \left\{ x \in \mathbb{R}_{\geq 0}^{n \times n} \text{ s.t. } \forall i \ \sum_j x_{ij} = 1, \ \forall j \ \sum_i x_{ij} = 1 \right\} \leftarrow \text{Polytope of bipartite matchings}$$



$\mathrm{Diam}(K) = \sqrt{n} \leftarrow$ Exercise

$$f_t(x) = \sum_{ij} x_{ij} \cdot c_t(i,j) \leftarrow \text{linear}$$

$$\Longrightarrow G \leq \| \vec{C}_t \| \leq \sqrt{n^2} = n$$

Step t: Alg plays a random perfect matching where edge (i,j) appears with prob $x_t(i,j)$ } possible because K is exactly bipartite matching polytope

OGD gives $\underline{DG} = \underline{\sqrt{n} \cdot n}$.

OGD gives $\frac{DG}{\sqrt{T}} = \frac{\sqrt{n} \cdot n}{\sqrt{T}}$.

polytope

$\blacksquare$