Online Convex Optimization & Complexity
Problem Set 6 – CS 6515/4540 (Fall 2025)

Solutions to HW6, Q23

23 Missing Proof Details

1. For any $x \in K$, we have $0 \le x_{ij} \le 1$ and
$$\sum_{i,j} x_{ij} = n.$$
Thus
$$\|x\|_F^2 = \sum_{i,j} x_{ij}^2 \le \sum_{i,j} x_{ij} = n, \qquad \Rightarrow \qquad \|x\|_F \le \sqrt{n}.$$
Hence for any $x, y \in K$,
$$\|x-y\|_F \le \|x\|_F + \|x\|_F \le 2\sqrt{n}.$$
Therefore $\mathrm{diam}(K) = O(\sqrt{n})$.

2. We know,
$$\frac{1}{T}\sum_{t=1}^{T} M(x_t, a_t) \ge \max_i \frac{1}{T}\sum_{t=1}^{T} M(i, a_t) \; - \; \varepsilon. \tag{R2}$$
Now define the time-averaged mixed strategies
$$\overline{x} \; := \; \frac{1}{T}\sum_{t=1}^{T} x_t, \qquad \overline{a} \; := \; \frac{1}{T}\sum_{t=1}^{T} a_t.$$
Observe that M is linear in each argument, so for any fixed x,
$$\frac{1}{T}\sum_{t=1}^{T} M(x, a_t) = M\big(x, \overline{a}\big).$$
Therefore
$$\max_x \frac{1}{T}\sum_{t=1}^{T} M(x, a_t) = \max_x M(x, \overline{a}).$$
By definition of $B = \max_x \min_a M(x,a)$ we have $\max_x M(x, \overline{a}) \le B$ (since B is the maximum, over all single-column strategies a, of the corresponding minima).
$$\frac{1}{T}\sum_{t=1}^{T} M(x_t, a_t) \le \max_x M(x, \overline{a}) + \varepsilon \le B + \varepsilon.$$
Similarly, linearity in the first argument gives for any fixed a:
$$\frac{1}{T}\sum_{t=1}^{T} M(x_t, a) = M\big(\overline{x}, a\big).$$

Hence

$$\min_a \frac{1}{T} \sum_{t=1}^{T} M(x_t, a) = \min_a M(\overline{x}, a).$$

By definition of $A = \min_a \max_x M(x, a)$ we have $\min_a M(\overline{x}, a) \geq A$ (since A is the minimum, over all mixed row strategies x, of the corresponding worst-case value $\min_a M(x, a)$).

$$\frac{1}{T} \sum_{t=1}^{T} M(x_t, a_t) \geq \min_a M(\overline{x}, a) - \varepsilon \geq A - \varepsilon.$$

Combining the two bounds we obtain

$$A - \varepsilon \ \leq \ \frac{1}{T} \sum_{t=1}^{T} M(x_t, a_t) \ \leq \ B + \varepsilon.$$

Letting $T \to \infty$ (so $\varepsilon \to 0$) gives $A \leq B$. Since $B \leq A$ holds by definition, we get $A = B$, which proves the minimax theorem.

3. $K = \{x \in \mathbb{R}^n_{\geq 0} : \sum_i x_i = B\}$

   Each loss is

$$f_t(x) = \sum_{i=1}^{n} a_t^i x_i, \qquad a_t^i \in [0, A].$$

Diameter of K is:

$$\|Be_i - Be_j\|_2 = \sqrt{B^2 + B^2} = \sqrt{2}\, B.$$

$$D = O(B).$$

$$\nabla f_t(x) = (a_t^1, \ldots, a_t^n), \qquad \|\nabla f_t(x)\|_2 \leq A\sqrt{n}.$$

$$G = O(A\sqrt{n}).$$

Euclidean OGD satisfies

$$\mathrm{Regret}_T \leq D\, G\, \sqrt{T}.$$

Hence

$$\mathrm{Regret}_T = O\Big(B \cdot A\sqrt{n} \cdot \sqrt{T}\Big) = O\Big(AB\sqrt{nT}\Big).$$

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