# Practice Problems (Exam 3) – CS 6515/4540 (Fall 2025)

Collection of problems for students to practice

# 1 Short Answer Questions

Give the following prompt to an LLM (like chatGPT, Gemini, CoPilot) to test your basic understanding of class topics such as Convex body, Convex function, strong convexity, smoothness, gradient descent, experts problem, minimax theorem, orthogonal vectors problem:

"Can you give me simple short-answer/multiple-choice questions to test my knowledge of ..."

# 2 Convex Optimization

1. Hazan's book [Lemma 2.4 in 2.2.2]
   Assume function $f$ is $\alpha$-strongly convex and $\beta$-smooth. Prove the following bounds on $||f(x_t) - f(x^*)||$ where $f(x^*) = \min_{x \in \mathcal{R}^d} f(x)$ (Note that $K = \mathcal{R}^d$ here, hence $\nabla f(x^*) = \mathbf{0}$.)
   (a) $||f(x_t) - f(x^*)|| \leq \frac{\beta}{2}||x_t - x^*||^2$
   (b) $||f(x_t) - f(x^*)|| \geq \frac{\alpha}{2}||x_t - x^*||^2$
   (c) $||f(x_t) - f(x^*)|| \geq \frac{1}{2\beta}||\nabla f(x_t)||^2$
   (d) $||f(x_t) - f(x^*)|| \leq \frac{1}{2\alpha}||\nabla f(x_t)||^2$
2. Hazan's book Chp 2 problems 3, 5, 7

# 3 Online Learning

1. Orabana's book Problem 2.2: Prove that for online gradient descent with changing step size $\frac{D}{G\sqrt{t}}$ in the $t$-th iteration (instead of $\frac{D}{G\sqrt{T}}$) gives $O(DG\sqrt{T})$ total regret (i.e., is suboptimal only by a constant factor, and now you don't need to know $T$ upfront).
2. Hazan's book Chp 3 problems 2, 3.

# 4 Reductions

## 4.1 LHI and OV.

You are given the following two problems:

1. **Orthogonal Vectors (OV):**
   **Input:** Two sets $A$ and $B$, each containing $n$ binary vectors of dimension $d$.
   **Question:** Does there exist a pair $(a, b)$ with $a \in A$ and $b \in B$ such that their dot product $a \cdot b = 0$?
2. **Low-Hamming-Intersection (LHI):**
   **Input:** Two sets $X$ and $Y$, each containing $n$ binary vectors of dimension $d$, and an integer threshold $t$.
   **Question:** Does there exist a pair $(x, y)$ such that the number of positions where both $x$ and $y$ have a 1 is at most $t$?

1. Show that OV is a special case of LHI.

2. Given a black box method $LHI(X, Y, t)$ that computes the answer for the LHI problem, construct an algorithm for the OV problem. The reduction should be $O(1)$ time.

3. Show that, if there exists an algorithm that computes the answer for LHI problem in $O(n^{2-\epsilon}d)$ time, then there is an algorithm for the OV problem that computes its answer in $O(n^{2-\epsilon}d)$ time.

## 4.2 Triangle Detection via Boolean Matrix Multiplication

A Boolean matrix product of two $n \times n$ Boolean matrices $A, B \in \{0,1\}^{n\times n}$ is defined as the matrix

$$C = A \times B, \qquad C[i,j] = \bigvee_{k=1}^{n} \big(A[i,k] \wedge B[k,j]\big).$$

You are given an algorithm that computes Boolean matrix multiplication in time $T(n)$.

Let $G = (V, E)$ be a directed graph on $n$ vertices, and let $M$ be its adjacency matrix, so $M[u,v] = 1$ iff $(u \to v) \in E$. A *directed triangle* is a triple $(u, v, w)$ with edges $u \to v$, $v \to w$, and $w \to u$.

Answer the following:

1. Define Boolean matrices $A$ and $B$ (in terms of $M$) such that the Boolean product $C = A \times B$ satisfies:
$$C[u,w] = 1 \iff \exists v \,\big(u \to v \text{ and } v \to w \text{ in } G\big).$$
2. Explain how to use $C$ and $M$ to decide whether $G$ contains a directed triangle.
3. Conclude that triangle detection can be solved in time $O(T(n))$.

## 4.3

**Hardness via Reduction from Orthogonal Vectors** Let $G = (V, E)$ be a bipartite graph with partition $V = A \cup B$, where all edges go between $A$ and $B$. For two vertices $x, y \in A$, let $d(x, y)$ denote their graph distance in $G$.

Your task is to analyze the following computational problem:

FIND-DISTANT-PAIR: Given $G = (A \cup B, E)$, output two *distinct* vertices $u, v \in A$ such that $d(u, v) > 2$.

Recall the ORTHOGONAL VECTORS problem:

Given two sets of Boolean vectors $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$, each in $\{0,1\}^d$ with $d = O(\log m)$, determine whether there exist $x_i \in X$ and $y_j \in Y$ such that their inner product satisfies $x_i \cdot y_j = 0$.

The Orthogonal Vectors Conjecture states that this problem cannot be solved in $O(m^{2-\varepsilon})$ time for any $\varepsilon > 0$.

Construct a reduction from ORTHOGONAL VECTORS to FIND-DISTANT-PAIR:

1. Describe how to build a bipartite graph $G = (A \cup B, E)$ from a given OV instance $(X, Y)$.
2. Show that if there exists an orthogonal pair $(x_i, y_j)$, then there exist $u, v \in A$ with $d(u, v) > 2$.
3. Show the converse: if every pair of vertices in $A$ has distance at most 2, then no orthogonal vector pair exists in $(X, Y)$.

**(Hardness consequence).** Conclude that an $O(|A|^{2-\varepsilon}|B|)$-time algorithm for FIND-DISTANT-PAIR would violate the Orthogonal Vectors Conjecture.

## 5 Solutions (partial)

### 5.1 LHI and OV.

You are given the following two problems:

1. **Orthogonal Vectors (OV):**
   **Input:** Two sets $A$ and $B$, each containing $n$ binary vectors of dimension $d$.
   **Question:** Does there exist a pair $(a, b)$ with $a \in A$ and $b \in B$ such that their dot product $a \cdot b = 0$?

2. **Low-Hamming-Intersection (LHI):**
   **Input:** Two sets $X$ and $Y$, each containing $n$ binary vectors of dimension $d$, and an integer threshold $t$.
   **Question:** Does there exist a pair $(x, y)$ such that the number of positions where both $x$ and $y$ have a 1 is at most $t$?

1. Show that OV is a special case of LHI.

> **Solution Sketch:** OV is just LHI when $t = 0$.

2. Given a black box method $LHI(X, Y, t)$ that computes the answer for the LHI problem, construct an algorithm for the OV problem. The reduction should be $O(1)$ time.

> **Solution Sketch:** Assuming the function is $LHI(X, Y, t)$ and we want to write the function $OV(A, B)$, then the algorithm reduces to:
>
> $OV(A, B): \ return\ LHI(A, B, 0)$

3. Show that, if there exists an algorithm that computes the answer for LHI problem in $O(n^{2-\epsilon}d)$ time, then there is an algorithm for the OV problem that computes its answer in $O(n^{2-\epsilon}d)$ time.

> **Solution Sketch:** The previous answer showed that we can reduce the OV problem to the LHI problem in $O(1)$ time. Hence, the time to run the OV problem depends on the time to run the LHI problem here. If we use the $O(n^{2-\epsilon}d)$ time algorithm for LHI, we have an $O(n^{2-\epsilon}d)$ time algorithm for the OV problem.

### 5.2 Triangle Detection via Boolean Matrix Multiplication

A Boolean matrix product of two $n \times n$ Boolean matrices $A, B \in \{0, 1\}^{n \times n}$ is defined as the matrix

$$C = A \times B, \qquad C[i, j] = \bigvee_{k=1}^{n} \big(A[i, k] \wedge B[k, j]\big).$$

You are given an algorithm that computes Boolean matrix multiplication in time $T(n)$.

Let $G = (V, E)$ be a directed graph on $n$ vertices, and let $M$ be its adjacency matrix, so $M[u, v] = 1$ iff $(u \to v) \in E$. A *directed triangle* is a triple $(u, v, w)$ with edges $u \to v$, $v \to w$, and $w \to u$.

Answer the following:

1. Define Boolean matrices $A$ and $B$ (in terms of $M$) such that the Boolean product $C = A \times B$ satisfies:

$$C[u, w] = 1 \iff \exists v \, \big(u \to v \text{ and } v \to w \text{ in } G\big).$$

2. Explain how to use $C$ and $M$ to decide whether $G$ contains a directed triangle.

3. Conclude that triangle detection can be solved in time $O(T(n))$.

**(a)** Take

$$A = M, \qquad B = M.$$

Then for any vertices $u, w$,

$$C[u,w] = 1 \iff \exists v \left( M[u,v] = 1 \wedge M[v,w] = 1 \right),$$

which means exactly that there is a length-2 path $u \to v \to w$ in $G$.

**(b)** A directed triangle $u \to v \to w \to u$ exists iff

$$C[u,w] = 1 \quad \text{and} \quad M[w,u] = 1.$$

Thus we scan all $(u,w)$ pairs; if both conditions hold, $G$ has a triangle.

**(c)** Computing $C = A \times B$ takes $T(n)$ time. Checking all $n^2$ entries of $C$ and $M$ takes $O(n^2)$ time, which is dominated by $T(n)$. Hence triangle detection runs in $O(T(n))$ time.