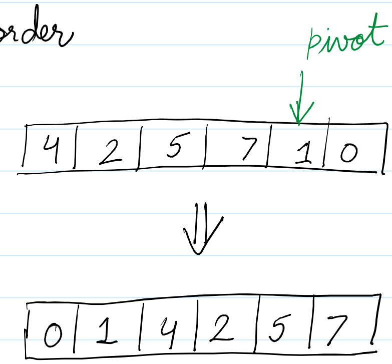
* HW-4 due Oct 16
* HW-3 graded
* Exam 2 on Oct 21 : LPs, Approx, Rand Alg
* Recap : Linearity of Expectation, Markov's ineq, Median finding, Algorithm has access to random bits

Quick Sort

n distinct numbers in an array A. Sort in Ascending order



function QSort (A, st, end)

If $st = end$ then Halt

Else

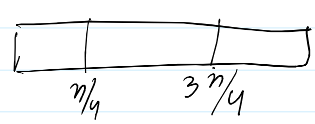
1) Uniformly randomly choose pivot $\in [st, end]$
2) Rearrange s.t. elems $< A[\text{pivot}]$ go to left and larger go to right
3) QSort $(A, st, st + \text{rank } A[\text{pivot}])$
   QSort $(A, st + \text{rank } A[\text{pivot}], end)$

$X_{2j} = 1$

How to analyze as a random recursion:

$$T(n) = 2T(n/2) + O(n)$$

$$T(n) = T(n-1) + O(n)$$



Hope: $T(n) \approx T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + O(n)$

$\approx O(n \log n)$

Time is a r.v.

(A) What is its expectation?

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(B) How likely is the time to be close to expectation.

Thm: $\mathbb{E}[\text{Time of QSort}] = O(n \log n)$

Plan: 1) Write $\text{Time} \leq \underbrace{X_1 + X_2 + X_3 + \ldots}_{\text{Simple random vars}}$

2) Apply linearity of expectation

QSort in English: 1) Choose a random pivot.

2) Compare all elems to pivot and rearrange

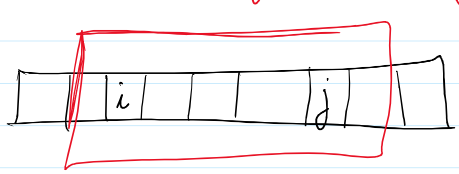
Define X_{ij}: R.V. $X_{ij} = \begin{cases} 1 & \text{if } i\text{-th smallest and } j\text{-th smallest ever compared} \\ 0 & \text{otherwise} \end{cases}$

Claim 1: Run-time of QSort $= O\left(\sum_i \sum_{j>i} X_{ij}\right)$

True because the runtime of alg in any iteration (excluding the recursion cost) equals length of subarray = Number of X_{ij}'s that convert from 0 to 1.

Claim 2: $\mathbb{E}\left[\sum_i \sum_{j>i} X_{ij}\right] = O(n \log n)$.

Think of Sorted Array



$j > i$

Pf: $\mathbb{E}[X_{ij}] = \dfrac{2}{|j-i|+1}$

For $i \neq j$: $P[X_{ij} = 1] =$

$$\sum_i \sum_{j>i} \mathbb{E}[X_{ij}] = \sum_i \sum_{j>i} \frac{2}{j-i+1}$$

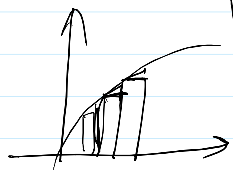
$$= \sum_i \sum_{k=1}^{n-i} \frac{2}{k+1} \leq \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2}{k+1}$$

$$\leq \sum_{i=1}^{n} 2(\log n + 1)$$

$$\leq O(n \log n)$$

Fact: $\underbrace{\sum_{k=1}^{n} \frac{1}{k}}$ $\leq (\log n)+1$

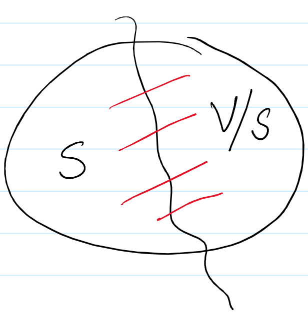
$$\approx \int_{x=1}^{n} \frac{1}{x}\,dx = \log n$$



Approx Algs using Randomization

Max-Cut Pb: Undirected unwtd $G=(V,E)$

$$\max_{S \subseteq V} |E(S, V \setminus S)|$$



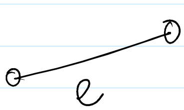
NP-Hard

Thm: A simple rand alg with

$$\mathbb{E}[\text{Cut Size}] = \frac{1}{2}|E| \geq \frac{1}{2}|\text{Opt MaxCut}|$$

Alg: Choose each vertex $v \in V$ into S indep w.p. $1/2$

$$\text{Cut-Size} = \sum_{e \in E} X_e \quad \text{where } X_e = \begin{cases} 1 & \text{if } e \in E(S, V \setminus S) \\ 0 & \text{o.w.} \end{cases}$$



Note: $\mathbb{E}[X_e] = P[e \text{ is Cut}]$

$$= 1/2$$

$$\Rightarrow \mathbb{E}[\text{Cut Size}] = \sum_e 1/2 = \frac{|E|}{2} \qquad \blacksquare$$