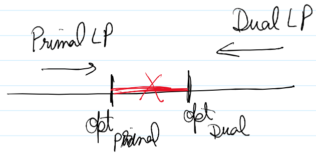
* HW 3 due Oct 2nd

* Recap LP Strong Duality

$\max\ c^T x$ s.t. $A_{m\times n} x \le b$ $x \ge 0$	$\min\ y^T b$ s.t. $A^T y \ge c$ $y \ge 0$



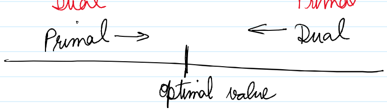
Thm (Strong LP duality)

If optimal primal LP value is finite then optimal dual LP value is the same.

Also, if $opt_{Primal} \longrightarrow \infty$ then dual LP is infeasible

and if ~~primal~~ Dual LP is infeasible then $opt_{\text{~~Dual~~ Primal}} \longrightarrow -\infty$ Wrong, instead

Correct: if $opt_{Dual} \longrightarrow -\infty$ then primal LP is infeasible



E.g. $\max\ 2x_1 - x_2$ s.t. $x_1 - x_2 \le 1$ $-x_1 + x_2 \le -2$ $x_1, x_2 \ge 0$	$\min\ y_1 - 2y_2$ s.t. $y_1 - y_2 \ge 2$ $-y_1 + y_2 \ge -1$ $y_1, y_2 \ge 0$

both are infeasible

Before the proof, let us see an application.

Max - Flow Min - Cut using LP Duality

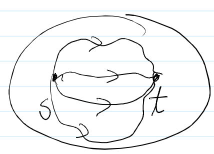
Write max s-t flow as a different LP.

Recall : directed graph G with edge capacities c_e.

Let $\mathcal{P}$ denote all s-t simple directed paths.

Max-flow-LP

$\max \sum_{p\in\mathcal{P}} x_p$ s.t. $y_e \rightarrow \forall e\in E \quad \sum_{p\ni e,\ p\in\mathcal{P}} x_p \le c_e$ $\forall p\in\mathcal{P} \quad x_p \ge 0$	$\min \sum_{e\in E} y_e c_e$ s.t. $\forall p\in\mathcal{P}: \sum_{e\in p} y_e \ge 1$ $\forall e\in E: \quad y_e \ge 0$



By strong LP duality : $$\sum_{p\in\mathcal{P}} x_p^* = \sum_{e\in E} y_e^* c_e$$
opt primal (x_p^*), opt dual ($y_e^* c_e$)

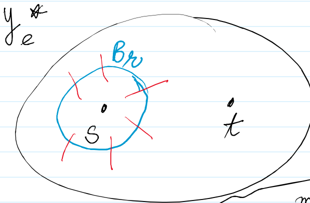
The dual LP may return a fractional optimal soln but it looks like a 'min cut' LP.

Obs : Any integral dual soln is a valid s-t cut since all s-t paths are disconnected.

Lemma: The dual LP always has an optimal integral soln, and hence max-flow = min-cut.

Pf: Consider directed graph G with edge weights $w_e = y_e^*$

For $r \geq 0$, let B_r denote all vertices at distance at most r from s.



Note that for $r \in (0,1)$, $B_r \not\ni t$.

$$\min \sum_{e \in E} y_e c_e$$

s.t. $\forall p \in P: \sum_{e \in p} y_e \geq 1$

$\forall e \in E: \quad y_e \geq 0$

Claim: $$\min_{r \in (0,1)} \sum_{e \in E(B_r, V \setminus B_r)} c_e \leq \sum_e y_e^* c_e$$

$\Rightarrow$ there is an integral optimal soln.

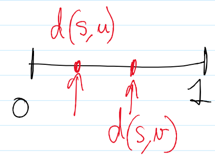
Pf: Suppose we choose a random $r \sim Unif[0,1]$

$$\Pr[\text{directed edge } e=(u,v) \text{ is cut}] = \Pr[u \in B_r \text{ and } v \notin B_r] = \max\{dist(v,s) - dist(u,s), 0\} \leq y_e^*$$

since (u,v) has length y_e^*

$$\Rightarrow \mathbb{E}[\text{Cut value}] = \sum_{e \in E} c_e \cdot \Pr[\text{edge } (u,v) \text{ is cut}]$$

Linearity of Expectation: $\mathbb{E}[X+Y+\dots] = \mathbb{E}[X] + \mathbb{E}[Y] + \dots$



$$\leq \sum_e c_e y_e^*$$

Since best r could only be better than a random r, this completes the proof. ∎

LP duality (General form):

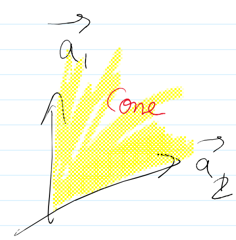
$$\max \; c_1^T x_1 + c_2^T x_2 + c_3^T x_3 + c_4^T x_4$$

s.t.
- $A_1 x_1 \leq b_1$ — y_1
- $x_1 \geq 0$
- $A_2 x_2 \leq b_2$ — y_2
- $A_3 x_3 = b_3$ — y_3
- $x_3 \geq 0$
- $A_4 x_4 = b_4$ — y_4

Here x_1, x_2, x_3, x_4 are vectors of variables

$$\min \; b_1^T y_1 + b_2^T y_2 + b_3^T y_3 + b_4^T y_4$$

s.t.
- $A_1^T y_1 \geq c_1, \; y_1 \geq 0$
- $A_2^T y_2 = c_2, \; y_2 \geq 0$
- $A_3^T y_3 \geq c_3$
- $A_4^T y_4 = c_4$



Farkas' lemma

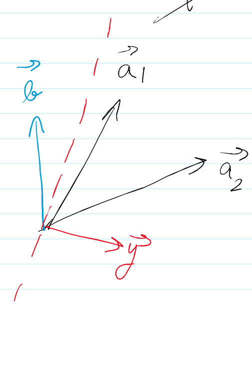
Given 3 vectors $\vec{a}_1$, $\vec{a}_2$ and $\vec{b}$, does there exist $x_1 \geq 0$, $x_2 \geq 0$ s.t.

$$\vec{a}_1 x_1 + \vec{a}_2 x_2 = \vec{b} \; ?$$

How to prove no solution exists?

— Find a separating plane $\vec{y}$ s.t.

then

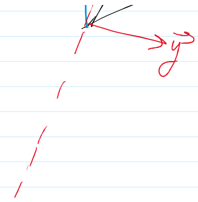


How to prove no solution exists?

— Find a separating plane $\vec{y}$ s.t.

$\vec{y} \cdot \vec{b} < 0$, $\quad \vec{a}_1 \cdot \vec{y} \geq 0$

$\vec{a}_2 \cdot \vec{y} \geq 0$



Farkas' lemma: $\quad \sum_i \vec{a}_i x_i = \vec{b}$

If $Ax = b$ and $x \geq 0$ is infeasible then

$\exists y$ s.t. $y^T A \geq 0$ and $y^T b < 0$.

$\vec{y} \cdot \vec{a}_i \geq 0$

→ We will not formally prove it is basically the picture

Proof of Strong Duality via Farkas' Lemma

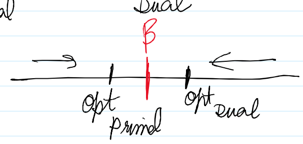
$\max\ c^T x$	$\min\ y^T b$
s.t. $Ax \leq b$	s.t. $A^T y \geq c$
$x \geq 0$	$y \geq 0$

Instead of the standard form of LP, consider

$\max\ c^T x$	$\min\ b^T y$
s.t. $Ax = b$	s.t. $A^T y \geq c$
$x \geq 0$	

We will prove $\text{Opt}_{\text{Primal}} = \text{Opt}_{\text{Dual}}$.

Suppose not.



We will show contradiction by showing a feasible dual with value $< \text{Opt}_{\text{Dual}}$.

Take any β such that $\text{Opt}_{\text{Primal}} < \beta < \text{Opt}_{\text{dual}}$

Now $\quad c^T x \geq \beta$, $\quad Ax = b$, $\quad x \geq 0$ is infeasible $\xRightarrow{\text{Add a slack variable } t}$ $\quad c^T x - t = \beta$, $\quad Ax = b$, $\quad x \geq 0,\ t \geq 0$ is infeasible

$$\Rightarrow \begin{bmatrix} c^T & -1 \\ & 0 \\ A & 0 \\ & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ t \end{bmatrix} = \begin{bmatrix} \beta \\ \\ b \\ \\ \end{bmatrix} \text{ is infeasible}$$

$x \geq 0,\ t \geq 0$

By Farkas' lemma, $\exists \begin{bmatrix} \lambda \\ \vec{y} \end{bmatrix} \in \mathbb{R}^{m+1}$ s.t. $\lambda \beta + \vec{y} \cdot \vec{b} < 0$,

$$-\lambda \geq 0$$

and $\forall i: \lambda c_i + \vec{y} \cdot \vec{a}_i \geq 0$

$$\Longrightarrow \underbrace{\frac{\vec{y}}{-\lambda} \cdot \vec{b} < \beta}_{\text{lower objective}} \quad \text{and} \quad \underbrace{\frac{\vec{y}}{-\lambda} \cdot \vec{a}_i \geq c_i}_{\text{feasible dual}} \qquad \blacksquare$$