⁎ Exam1 graded : Median 74 + 13

⁎ HW 3 due on Thursday

⁎ Recap : LPs and LP duality

Approximation Algorithms

While designing an algorithm, we usually want :

① Fast

② Optimal

③ General

But if these are not simultaneously achievable, we relax

e.g. ① Polytime → Pseudo-polytime

→ ② Optimal → Approx

③ General graphs → Trees

Focus in this part of course

Approx Alg : For any optimization pb where the goal is
to max/min an objective s.t. constraints,
(e.g., MST, Knapsack, indep set,..)

we say that an algo is α-approx for $\alpha \geq 1$ if

Ⓐ For a max pb, alg value (objective) $\geq \frac{1}{\alpha}$ opt value

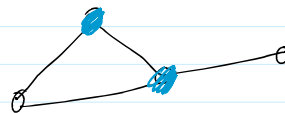Ⓑ For a min pb, alg cost (objective) $\leq \alpha \cdot$ opt cost

If someone says β-approx for $\beta < 1$, then take $\alpha = 1/\beta$.

Vertex Cover

Given undirected $G = (V, E)$

Find smallest $S \subseteq V$ s.t.

$\forall (u, v) \in E$ at least one of u or v is in S
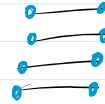
Q1: Is there a polytime algo to find optimal soln?

— No, unless $P = NP$

Q2: Can we get α-approx in polytime? What is the smallest α?

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Thm: There is a 2-approx polytime algo. ← It is unknown whether <2 is possible in polytime.

maximal greedy

Pf: Find a (max-size) matching M and return all matched vertices, i.e., size $= 2|M|$

Claim 1: This is a valid vertex cover

If an edge is not covered, M is not max-matching.

Claim 2: Optimum vertex cover has size $\geq |M|$
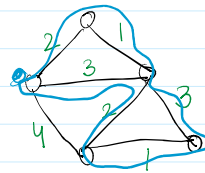
Any soln needs to cover the M-edges.
Since these M-edges don't share any vertex,
opt $\geq |M|$.

Hence, we have a 2-approx alg.

Traveling Salesman Problem

Given an undirected graph $G = (V, E)$
weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

Goal: Find a tour π (cycle with vertex repeats allowed) that visits all vertices and has shortest length.
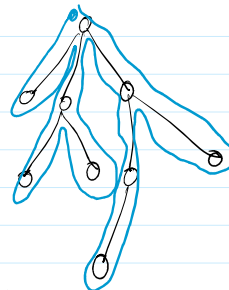
Remark: Known to be NP-Hard.

we even know $(1.5-\delta)$ approx algo

Thm: There is a 2-approx algo for this problem.

Pf: Find MST of G.

To get tour π_{Alg}, take 2 copies of every MST edge.

$\Rightarrow \quad \pi_{Alg} \leq 2 \cdot MST$

Note that optimum tour connects all vertices

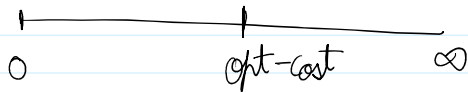$\Rightarrow \quad \Pi_{Alg} \leq 2 \cdot MST$

Note that optimum tour connects all vertices

$$\Rightarrow \quad \Pi_{OPT} \geq MST$$

Hence, a 2-approx.

LP - Based Approx Algo

Idea:

① Capture the pb 'roughly' using an LP. Solve the LP.

② Convert the optimal fractional soln into an integ soln, while losing approx ratio.

Vertex Cover Pb

Given $G = (V, E)$

Find $S \subseteq V$ s.t. every edge is covered.

Define LP vars x_u for $u \in V$

$$x_u = \begin{cases} 1 & \text{if } v \in S^* \\ 0 & o.w. \end{cases}$$
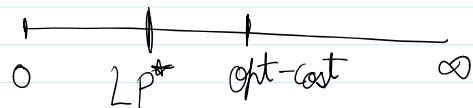
$$\min \sum_u x_u$$

$$s.t. \quad x_u \geq 0$$

$$\forall (a, b) \in E:$$
$$x_a + x_b \geq 1$$

Observ: Any $\{0, 1\}$ integ soln to this LP is a valid vertex cover.

Claim 1: Optimal LP value $= \sum_u x_u^* \leq$ min-vertex-cover.

Pf: Suppose S^* is the opt integ soln.

$$x_u = \begin{cases} 1 & \text{if } u \in S^* \\ 0 & o.w. \end{cases}$$

$u \quad \{ 0 \quad \text{o.w.}$

This is feasible soln and has objective value $\lceil S^* \rceil$

Alg: Return all u s.t. $x_u^* \geq 1/2$.

Claim 2: This is a valid vertex cover.
$\forall (a,b): x_a^* + x_b^* \geq 1 \Rightarrow$ at least one of $\underset{\underset{\downarrow}{claim 1}}{a \text{ or } b \text{ in vertex cover}}$

Claim 3: Its cost $\leq 2 \cdot \sum_u x_u^* \underset{claim 1}{\leq} 2 \cdot \text{opt}$

$\therefore$ I am at most doubling x_u^*.