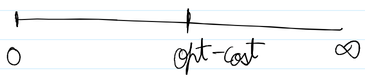
* HW 3 due today
* HW 4 to be released soon (due in 2 weeks from today)
* No lecture on Tues (Fall break)

## LP-Based Approx Algo




Idea:

1. Capture the pb 'roughly' using an LP. Solve the LP.
2. Convert the optimal fractional solns into an integ soln, while losing approx ratio.

## Vertex Cover Revisited

Given $G=(V,E)$

Find $S \subseteq V$ s.t. every edge is covered.

Define LP vars $x_u$ for $u \in V$

Intuitively, $x_u = \begin{cases} 1 & \text{if } v \in S^* \\ 0 & \text{o.w.} \end{cases}$

$$\min \sum_u x_u$$

s.t. $x_u \geq 0 \quad \forall u \in V$

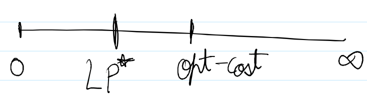
$\forall (a,b) \in E$:

$x_a + x_b \geq 1$

~~$x_u \in \{0,1\}$~~

Observe: Any $\{0,1\}$ integ soln to this LP is a valid vertex cover.

Claim 1: Optimal LP value $= \sum_u x_u^* \leq$ min-vertex-cover.




Pf: Suppose $S^*$ is the opt integ soln.

$x_u = \begin{cases} 1 & \text{if } u \in S^* \\ 0 & \text{o.w.} \end{cases}$

This is feasible soln and has objective value $|S^*|$

$$\boxed{\text{Alg: Return all } u \text{ s.t. } x_u^* \geq 1/2.}$$

Claim 2: This is a valid vertex cover:

$\forall (a,b)$: $x_a^* + x_b^* \geq 1 \Rightarrow$ at least one of a or b in vertex cover

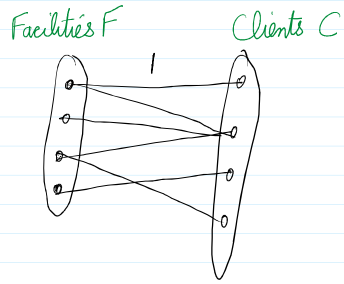
Claim 3: Alg's cost $\leq 2 \cdot \sum_u x_u^* \overset{\text{Claim 1}}{\leq} 2 \cdot \text{opt}$

∴ Taking at most doubling $x_u^*$.

Claim 3: Alg's cost $\leq 2 \cdot \sum_u x_u \leq 2 \cdot \text{opt}$

$\therefore$ I am at most doubling $x_u^*$. $\blacksquare$

## Facility Location




Consider a graph with vertices $F \cup C$ and a metric $d$:

edge lengths that satisfy triangle ineq $d_{ab} \leq d_{ac} + d_{cb}$

Each facility $i \in F$ has an opening cost $f_i \in \mathbb{R}_{\geq 0}$

Goal: Decide a subset $S \subseteq F$ of facilities to open to

$$\min \underbrace{\sum_{i \in S} f_i}_{\text{Opening costs}} + \underbrace{\sum_{j \in C} \min_{i \in S} d_{ij}}_{\text{Assignment costs}}$$

Remark: This problem is NP-Complete. We also know PTAS is impossible, (APX-Hard)
i.e., there exists a constant $c > 0$ s.t. better than $(1+c)$ approx means $P = NP$.

Thm: There exists a 6-approx polytime alg for facility location.

Plan:
1. Write an Integer Program (IP) — generalizes LP since we can also write integ constraints like $x_i \in \{0,1\}$
2. LP relaxation → Convert IP to LP
3. Solve LP to get optimal fractional soln $x^*$
4. LP rounding → Convert $x^*$ to integral feasible soln while increasing objective by at most 6-factor

Step 1: Let $x_i \in \{0,1\}$ denote whether facility $i \in F$ is opened

Let $y_{ij} \in \{0,1\}$ denote whether client $j \in C$ assigned to $i \in F$

$$\min \sum_i f_i x_i + \sum_{ij} d_{ij} y_{ij}$$

$$\text{s.t.} \quad y_{ij} \le x_i \quad \forall i \in F, \forall j \in C$$

$$\sum_{i \in F} y_{ij} = 1 \quad \forall j \in C$$

$$x_i \in \{0,1\}, \quad y_{ij} \in \{0,1\}$$

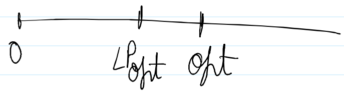
Obs: This IP exactly captures facility location.

Step ② : Change $x_i, y_{ij} \in \{0,1\}$ to $[0,1]$

Step ③ : Solve LP to get $x_i^*$ and $y_{ij}^*$.

Obs: $\sum_i f_i x_i^* + \sum_{ij} d_{ij} y_{ij}^* = LP_{opt} \le OPT$ = optimal integ soln cost

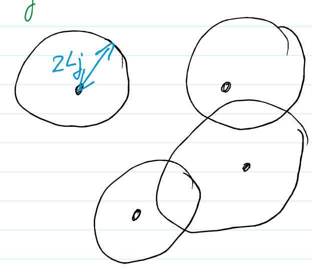
because the optimum integ soln is feasible




Step ④: Theorem: We can convert $x_i^*, y_{ij}^*$ into integral feasible soln $X_i \in \{0,1\}$, $Y_{ij} \in \{0,1\}$ where

$$\sum_i f_i X_i + \sum_{ij} d_{ij} Y_{ij} \le 6 \cdot \Big( \sum_i f_i x_i^* + \sum_j \underbrace{\sum_i d_{ij} y_{ij}^*}_{L_j} \Big)$$

Pf: Let $L_j = \sum_i d_{ij} y_{ij}^*$. ← avg distance of client $j$ to a facility in LP soln.




Draw $2 \cdot L_j$ radius balls $B_j$ around all $j \in C$.

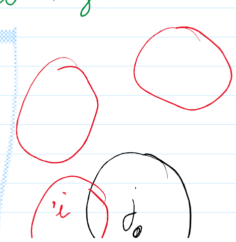
Note: we are "happy" for client $j$ if she is assigned any facility in $B_j$

Ques: Can we open one facility from each $B_j$? No, ∵ opening costs could be too high

Alg: Sort $L_1 \le L_2 \le \ldots \le L_{|C|}$

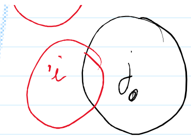
For $j$ in 1 to $|C|$ :

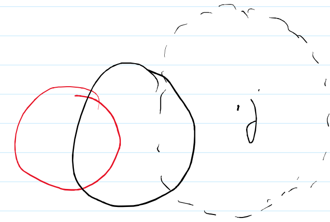
If $B_j$ is untouched, open cheapest facility in $B_j$



For $j$ in 1 to $|C|$ :

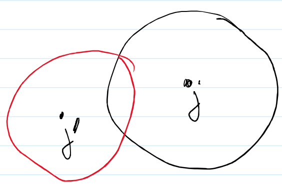
If $B_j$ is untouched, open cheapest facility in $B_j$.
Anyways, assign $j$ to the new facility
or the facility for the overlapping ball




Claim 1: Assignment cost for $j \in C$ is $\leq 6 L_j$

Pf: Either $B_j$ opens a facility $\Rightarrow \leq 2 L_j$ cost

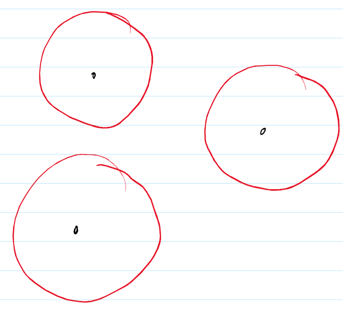



Or $B_j$ overlaps with $B_{j'}$ that opens facility

$$\leq \text{Radius}(B_j) + 2 \cdot \text{Radius}(B_{j'}) \leq 2L_j + 2 \cdot 2 L_{j'}$$
$$\leq 6 L_j$$

Claim 2: Total facility opening cost $\leq 2 \cdot \sum_i f_i x_i^*$

Pf. We only open facilities in disjoint balls of radius $2 L_j$.



Consider any $B_j$ where we open a facility.

Obs: $\sum_{i \in B_j} x_i^* \geq \sum_{i \in B_j} y_{ij}^* \geq 1/2$

Otherwise, if $> 1/2$ fraction of $j$ goes outside $2 L_j$, avg length $> L_j$
Recall $L_j = \sum_i d_{ij} \cdot y_{ij}^*$ and $\sum_i y_{ij}^* = 1$.

Given this, since we open the cheapest facility in $B_j$, our opening cost is at most doubling each $x_i^*$.