Maxwell's equations in a Dielectric medium

Maxwell summarized the existing ideas of electric and magnetic fields and their inter-related phenomena into four equations (in 1860) which are known as the Maxwell's equation. This also paved the way for describing radiation as an electromagnetic wave.

The first two equations are the Gauss's law for electric and magnetic fields.

- Divergence of the electric field is given by the charge density divided by $\boldsymbol{\varepsilon_o}$

  $$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_o}$$

- Divergence of the magnetic field is uniformly zero

  $$\vec{\nabla}.\vec{B} = 0$$

- The curl of the electric field is equal to the rate of change of the magnetic field which is the standard Faraday's law of electromagnetic induction.

  $$\vec{\nabla}\mathrm{x}\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

- The curl of the magnetic field is given by the current density through the closed loop and the displacement current

  $$\vec{\nabla}\mathrm{x}\vec{B} = \mu_o\vec{j} + \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

  This equation is an extension of the Ampere's law with the addition of the displacement current when there was a component of the time varying electric fields

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[A review of the operations with the ∇ operator (nabla operator)

The $\vec{\nabla}$ operator is a partial differential mathematical operator and is given by $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ *where* $\hat{\imath}$, $\hat{\jmath}$, *and* $\hat{k}$ *are the unit vectors in the three orthogonal directions.*

The operator $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ *when operates on a scalar quantity gives rise to a vector. For example the $\vec{\nabla}$ operator operating on an electric potential gives the electric field at the point*

$$\vec{\nabla}\boldsymbol{V} = \hat{\boldsymbol{\imath}}\frac{\partial \boldsymbol{V_x}}{\partial \boldsymbol{x}} + \hat{\boldsymbol{\jmath}}\frac{\partial \boldsymbol{V_y}}{\partial \boldsymbol{y}} + \hat{\boldsymbol{k}}\frac{\partial \boldsymbol{V_z}}{\partial \boldsymbol{z}} = \hat{\boldsymbol{\imath}}\,\boldsymbol{E}_x + \hat{\boldsymbol{\jmath}}\boldsymbol{E_y} + \hat{\boldsymbol{k}}\boldsymbol{E_z} = \vec{\boldsymbol{E}}$$

The dot product of the $\vec{\nabla}$ operator with another vector field gives rise to the divergence of the defined field or the rate of change of the field in the three orthogonal directions.

The cross product of $\vec{\nabla}$ operator with a vector field gives the curl of the field and results in a vector which is perpendicular to both $\vec{\nabla}$ and the given vector.

Another important identity with the $\vec{\nabla}$ operator is the curl of the curl of a vector ie.,

$$\vec{\nabla}\mathrm{x}\vec{\nabla}\mathrm{x}\vec{E} = \vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2\vec{E}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ *is the scalar Laplacian operator.]*

Wave equation from Maxwell's equations

In the case of free space (which does not have sources of charges and currents) then the Maxwell's equations reduce to

$$\vec{\nabla}.\vec{E} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\vec{\nabla}.\vec{B} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\vec{\nabla}\mathbf{x}\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$\vec{\nabla}\mathbf{x}\vec{B} = +\mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Taking the curl of curl of the electric field the equation can be written as

$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$ this reduces to

$$\vec{\nabla}(\vec{\nabla}.\vec{\mathrm{E}}) - \nabla^2\vec{\mathrm{E}} = \left(-\frac{\partial\vec{\nabla}\times\vec{B}}{\partial t}\right)$$

Since $\vec{\nabla}.\vec{E} = 0$ this reduces to $-\nabla^2\vec{\mathrm{E}} = \left(-\frac{\partial\vec{\nabla}\times\vec{B}}{\partial t}\right)$

Substituting for curl of B the above equation simplifies to $\nabla^2\vec{\mathrm{E}} = \left(\mu_o\varepsilon_o\frac{\partial^2\mathrm{E}^2}{\partial t^2}\right)$.

But we know that $\mu_o\varepsilon_o = \frac{1}{c^2}$ and the equation reduces to

$\nabla^2\vec{\mathrm{E}} = \left(\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}\right)$ which is the general form of a wave equation. Maxwell concluded that this should be an electric vector in free space travelling at the speed of light $c = \sqrt{\frac{1}{\mu_o\varepsilon_o}}$

In a very similar way we could starting from the curl of the curl of the magnetic field show that

$\nabla^2\vec{\mathrm{B}} = \left(\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}\right)$. This describes a transverse magnetic field vector travelling at the speed of light.

The electric and magnetic waves must therefore be representing light and hence Maxwell proposed that light could be treated as electromagnetic waves, where the electric and magnetic vectors are mutually perpendicular and perpendicular to the direction of propagation of the radiation.

Consider a 1D electric wave E_x associated with radiation propagating in the Z direction which can be represented as

$$E_x = A\cos(wt + kz)$$

This implies that the electric field vector has only an x component and the other two components E_y and E_z are zero.

Hence the associated magnetic component of the EM wave can be evaluated using the Maxwell's third equation namely $\vec{\nabla}\mathbf{x}\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

Evaluating the curl of the electric field

$$\vec{\nabla}\mathrm{x}\vec{E} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{bmatrix} = \hat{\imath}\times 0 + \hat{\jmath} * \frac{\partial E_x}{\partial z} + \hat{k} * 0 = \hat{\jmath}\ \frac{\partial}{\partial z}[A\cos(wt+kz)] = \hat{\jmath} * k * A\sin(wt+kz)$$

this implies that $-\frac{\partial \vec{B}}{\partial t} = \hat{\jmath} * k * A\sin(wt+kz)$

Integrating the above equation with respect to time t we get $-B = \hat{\jmath} * k * A\cos(wt+kz) * \left(\frac{1}{w}\right) = \hat{\jmath} * A\cos(wt+kz) * \left(\frac{1}{\frac{w}{k}}\right) = \hat{\jmath}.E_x * \frac{1}{c}$ since c=w/k is the velocity of the radiations. We note that the magnetic component of the EM wave has only the Y component and the magnitude of the wave is $\frac{1}{c}$ times the magnitude of the electric component of the wave. Thus we conclude that the EM waves have a coupled electric and magnetic field components which are mutually perpendicular to each other and both are perpendicular to the direction of propagation of radiation.

<u>Energy of EM waves</u>

Classically the energy of waves is equivalent to it's intensity which is square of the amplitude of the waves.

The energy associated with an Electric field per unit volume of free space is $E_n = \frac{1}{2}\varepsilon_o E^2$ where E is the electric field.

The energy content of the electric component of the wave $= \frac{1}{2}\varepsilon_o {E_x}^2 = \frac{1}{2}\varepsilon_o {E_{ox}}^2\cos^2(wt+kz)$

The energy content of the magnetic component of the wave $= \frac{1}{2}\frac{{B_y}^2}{\mu_o} = \frac{1}{2}\frac{{E_x}^2}{c^2\mu_o} = \frac{1}{2}\varepsilon_o {E_x}^2$

Hence the total energy content of the wave is the sum of the two components $= \varepsilon_o {E_x}^2$.

Poynting Vector

E_x is however a time varying component and hence to determine the average energy of the wave transmitted per unit time through unit area can be found out as

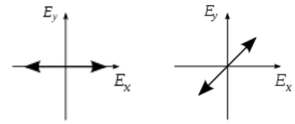
$$\text{Average Energy} = c\varepsilon_o \int_0^{2\pi} {E_x}^2\,dt = \int_0^{2\pi} c\varepsilon_o {E_{ox}}^2\cos^2(wt+kz)\,dt$$

$$= \frac{1}{2}\varepsilon_o c{E_{ox}}^2 = \frac{1}{2}c\frac{{B_{oy}}^2}{\mu_o} = \frac{1}{2}\frac{E_{ox}B_{oy}}{\mu_o}$$

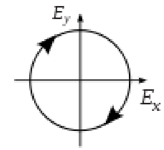
This implies that the average energy content of EM waves to be proportional to the square of the amplitude of the electric or magnetic vector and is independent of the frequency of the waves. Thus the classical picture of the EM waves as carriers of energy gives a picture of frequency independence. For this reason some of the observed phenomena of interaction of light with matter could not be consistently explained in spite of the fact that all other observed phenomena of radiation could be explained by the Maxwell's EM wave theory.

Polarisation states of EM waves

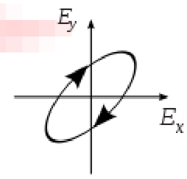
Light is a transverse electromagnetic wave, but natural light is generally unpolarized, all planes of propagation being equally probable.



Light in the form of a plane wave in space is said to be linearly polarized. The addition of a horizontally linearly polarized wave and a vertically polarized wave of the same amplitude in the same phase result in a linearly polarized at a 45^{o} angle



If light is composed of two plane waves of equal amplitude but differing in phase by 90°, then the light is said to be circularly polarized.



If two plane waves of differing amplitude are related in phase by 90°, or if the relative phase is other than 90° then the light is said to be elliptically polarized.