UNIT III

FOURIER SERIES REPRESENTATION OF PERIODIC SIGNALS

1. Response of LTI Systems to complex exponentials.
2. Trigonometric Fourier series.
3. Fourier series representation of continuous time periodic signals.
4. Convergence of Fourier series.
5. Properties of continuous time Fourier series.
6. Introduction to Fourier series representation of Discrete time periodic signals.

Joseph Fourier

The Response of LTI Systems to Complex Exponentials

- In previous chapter we represented signal as weighted superposition of delayed impulse functions.

- In this chapter we represent signal as a <u>weighted superposition of complex sinusoids</u>

Response of LTI systems to complex exponentials

$$\text{continuous time}: \; e^{st} \;\rightarrow\; H(s)\,e^{st}$$

$$\text{discrete time}: \; z^{n} \;\rightarrow\; H(z)\,z^{n}$$

S = σ + j ω

Z = r e $^{j\,\Omega}$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau
\end{aligned}
$$

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y(t) &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \\
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&= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau \\
&= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau
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$$

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&= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau \\
&= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \\
&= H(s)\, e^{st}
\end{aligned}
$$

For Discrete time signals $\quad Let\ x[n] = z^n$

We know for LTI System Output $\quad y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$

$$= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

$$y[n] = H(z)x[n] = H(z)z^n$$

Eigenvalue

Eigenfunction

$$where\ \ H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

Similarly for discrete time

$$\text{discrete time} : z^n \quad \rightarrow \quad H(z)\, z^n$$

In both the cases the output is the scaled version of input

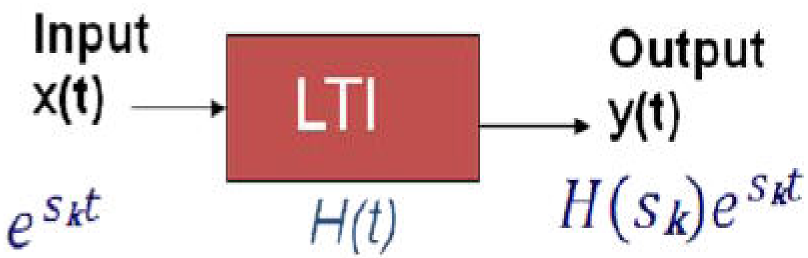
e^{st} or $z^{\,n}$ are called Eigen function

H(s) AND H(z) are called Eigen value

Key Properties: for Input to LTI System

1. To represent signals as linear combinations of basic signals.
2. Set of basic signals used to construct a broad class of signals.
3. The response of an LTI system to each signal should be simple enough in structure.
4. It then provides us with a convenient representation for the response of the system.
5. Response is then a linear combination of basic signal.

Eigenfunction and Superposition Properties



$$a_1 e^{s_1 t} \rightarrow a_1 H(s_1) e^{s_1 t}$$

$$a_2 e^{s_2 t} \rightarrow a_2 H(s_2) e^{s_2 t}$$

$$a_3 e^{s_3 t} \rightarrow a_3 H(s_3) e^{s_3 t}$$

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

$$\rightarrow x(t) = \sum_k a_k e^{s_k t} \rightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

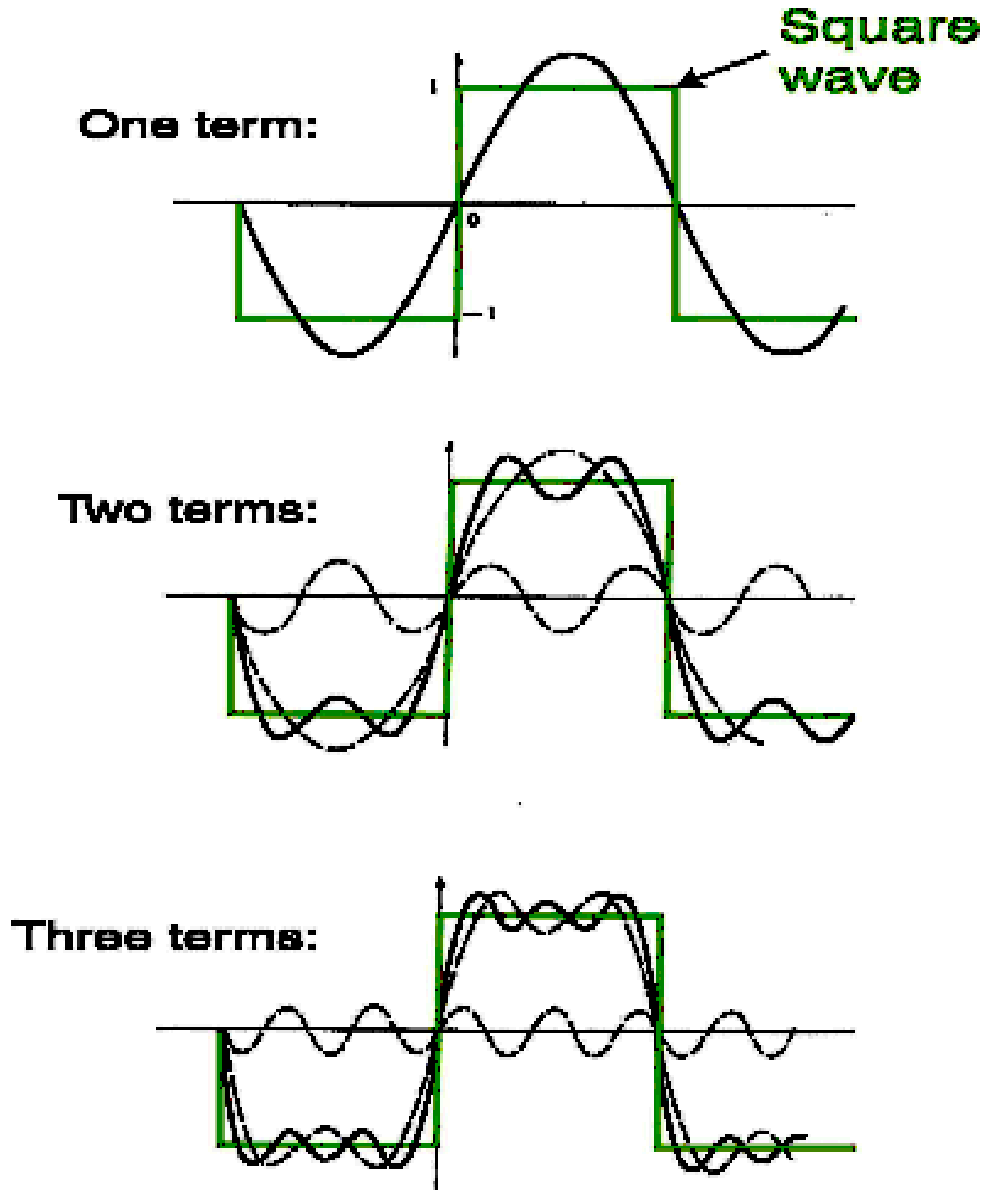
Similarly for DT Signals

$$\rightarrow x[n] = \sum_k a_k z_k^{\,n} \rightarrow y[n] = \sum_k a_k H(z_k) z_k^{\,n}$$

Hence for both continuous time and discrete time, each coefficient in this representation of the output is obtained as the product of the corresponding coefficient a_k of the input and the system's eigenvalue or $H(z_k)$ associated with the eigenfunction $e^{s_k t}$ or $z_k^{\,n}$ respectively.

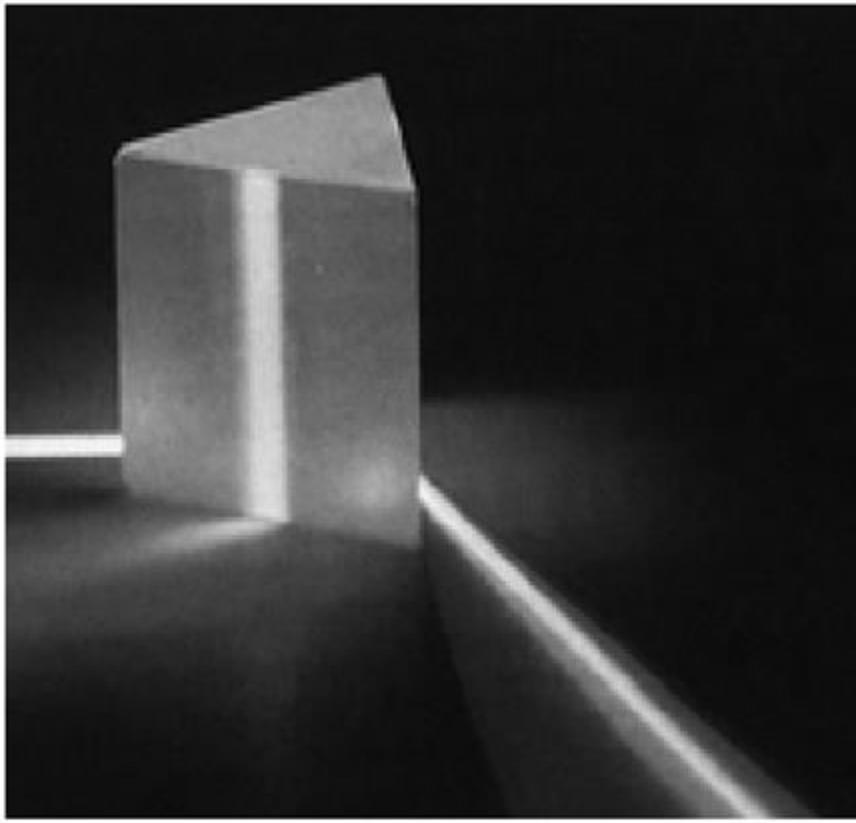
ADVANTAGE

- The operation of convolution has become multiplication
- This is the powerful motivation for representing signals as weighted super positions of complex sinusoids.
- Rather than describing the signal’s behavior as a function of time, the weights describe it as function of frequency.


Square wave
One term:
1
0
-1
Two terms:
Three terms:

Frequency Domain

<u>Analogy: Prism</u>



- Physical device that separates white light into its constituent colours.
- Each colour depends on it wavelength or frequency

YOUNG LADY?

OR

OLD WOMAN?

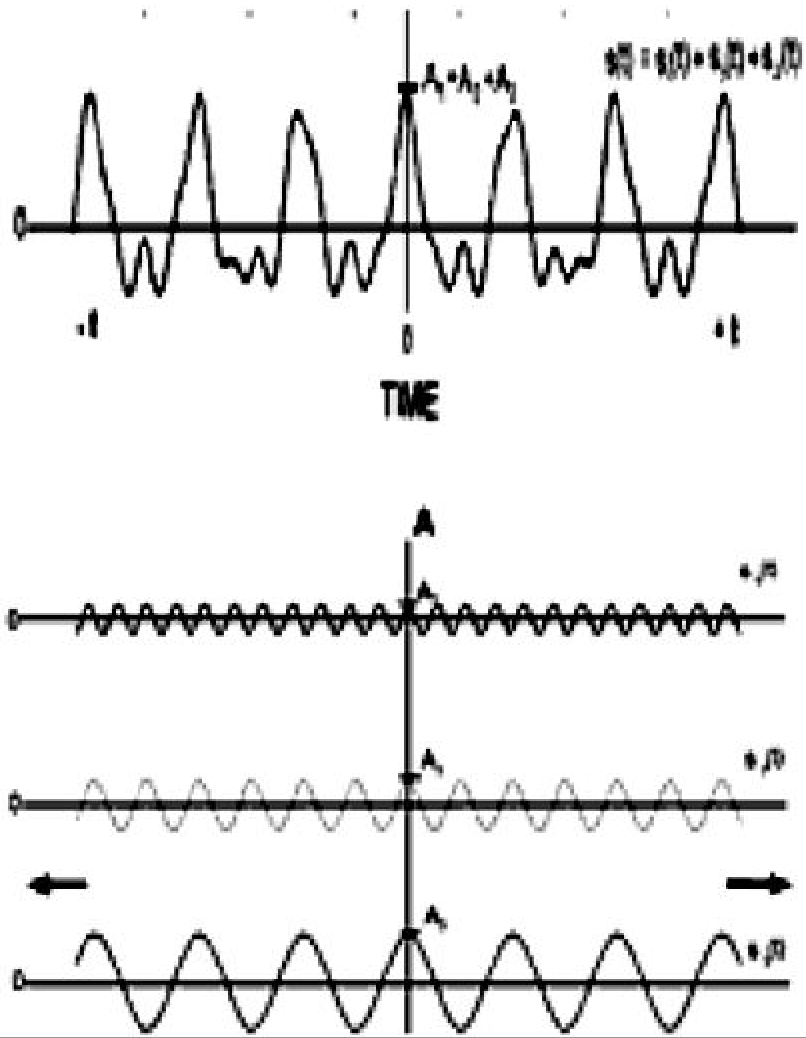
WHAT U SEE?

Fourier transform of x [n]

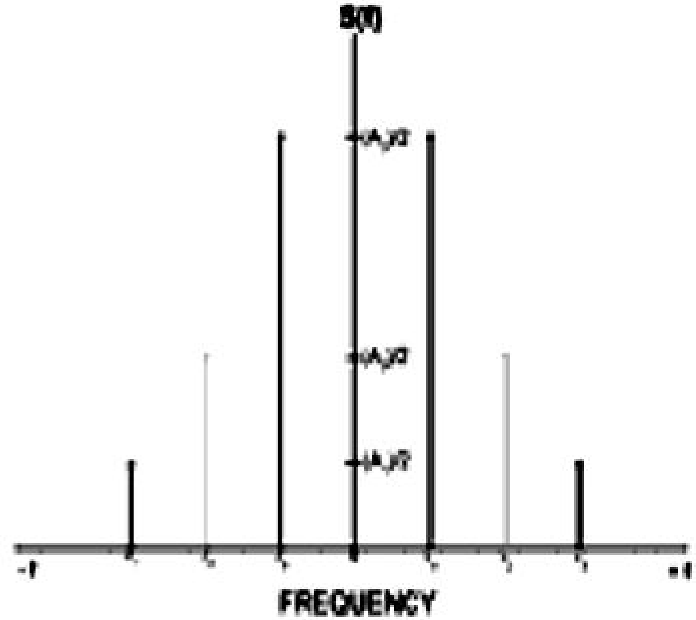
(time domain signal) is

$X(e^{j\omega})$ or X(W)
(frequency domain signal)

Time Domain



Fourier Domain?



Although s & z may be arbitrary complex numbers, Fourier analysis involves restricting our attention to purely imaginary number (sigma = 0).

Similarly for z, we restrict the range of values of z to unit magnitude (r=1).

$$S = \sigma + j\,\omega \longrightarrow S = 0 + j\,\omega$$

$$Z = r\,e^{j\,\Omega} \longrightarrow Z = 1\,e^{j\,\Omega}$$

$$e^{j\Omega n} \longrightarrow \boxed{h[n]} \longrightarrow H(e^{j\Omega})\, e^{j\Omega n}$$

FREQUENCY RESPONSE OF THE SYSTEM

$$e^{j\omega t} \longrightarrow \boxed{h[n]} \longrightarrow H(e^{j\omega})\, e^{j\omega t}$$

Continuous-Time LTI System

$$y(t)=\int_{-\infty}^{\infty}h(\tau)e^{j\omega(t-\tau)}d\tau=e^{j\omega t}\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau$$

$$=H(j\omega)e^{j\omega t}$$

Frequency response:

$$H(j\omega)=\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau$$

♣ **Polar form complex number $c = a + jb$:**

$$c=|c|e^{j\arg\{c\}} \quad \textbf{where} \quad |c|=\sqrt{a^2+b^2} \quad \text{and} \quad \arg\{c\}=\tan^{-1}\left(\frac{b}{a}\right)$$

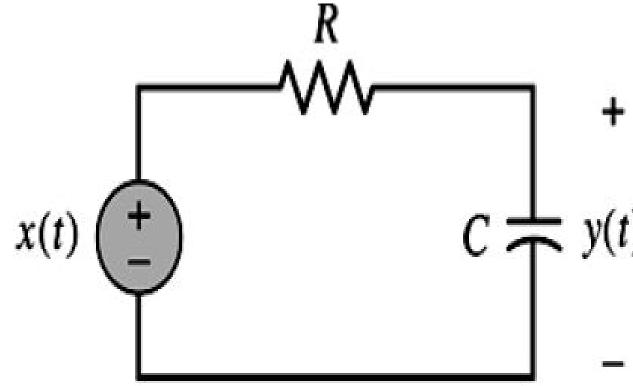
♣ **Polar form for $H(j\omega)$:**

$$H(j\omega)=|H(j\omega)|e^{j\arg\{H(j\omega)\}}$$

where $|H(j\omega)| = \text{Magnitude response}$ and $\arg\{H(j\omega)\} = \text{Phase respnse}$

$$\Rightarrow \quad y(t)=|H(j\omega)|e^{j(\omega t+\arg\{H(j\omega)\})}$$

RC Circuit: Frequency response



$$h(t)=\frac{1}{RC}e^{-t/RC}u(t)$$

Frequency response:

$$H(j\omega)=\frac{1}{RC}\int_{-\infty}^{\infty}e^{-\frac{\tau}{RC}}u(\tau)e^{-j\omega\tau}d\tau=\frac{1}{RC}\int_{0}^{\infty}e^{-\left(j\omega+\frac{1}{RC}\right)\tau}d\tau$$

$$=\frac{1}{RC}\frac{-1}{\left(j\omega+\frac{1}{RC}\right)}e^{-\left(j\omega+\frac{1}{RC}\right)\tau}\Bigg|_{0}^{\infty}$$

$$=\frac{1}{RC}\frac{-1}{\left(j\omega+\frac{1}{RC}\right)}(0-1)$$

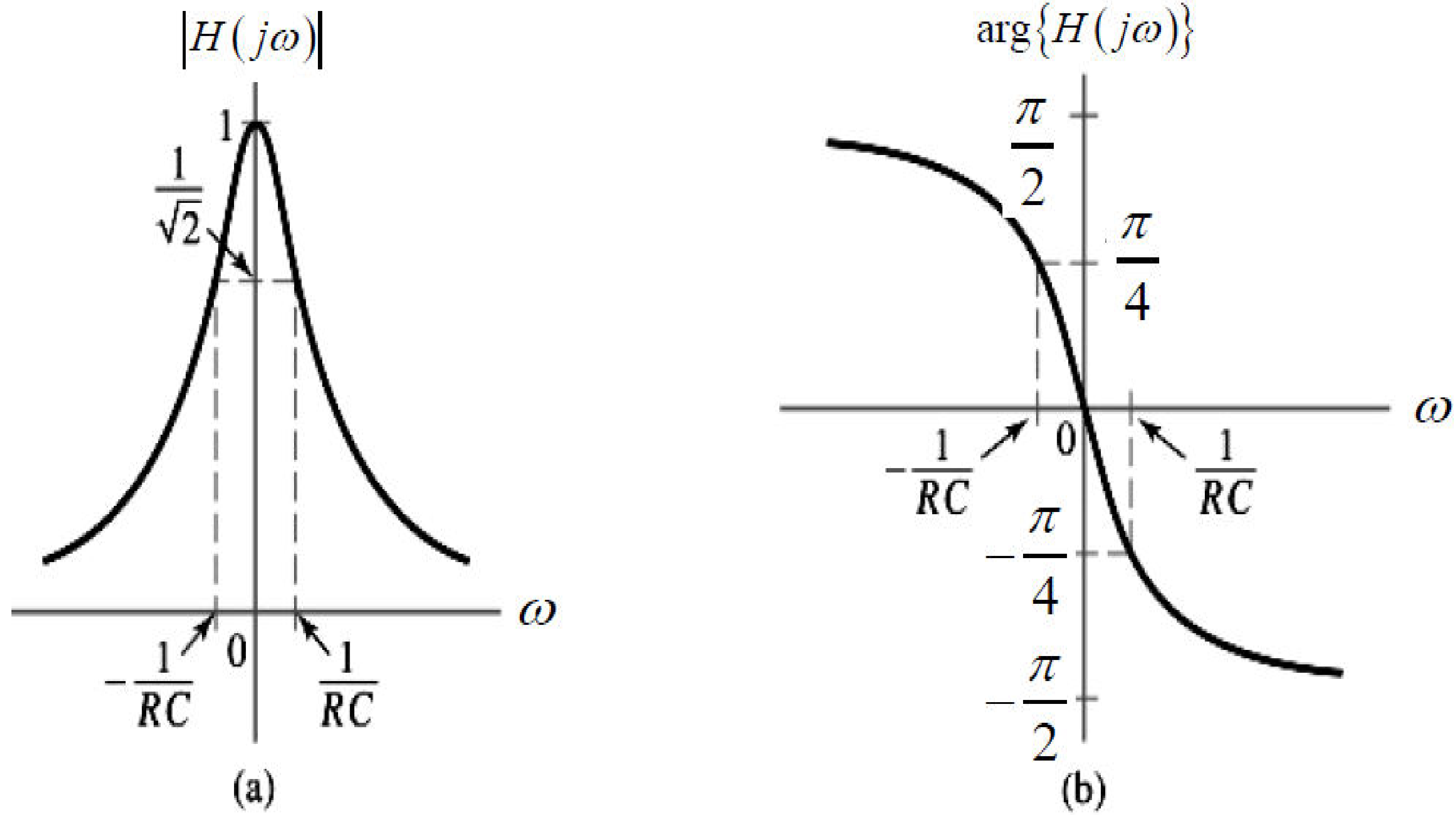
$$=\frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}$$

Magnitude response:

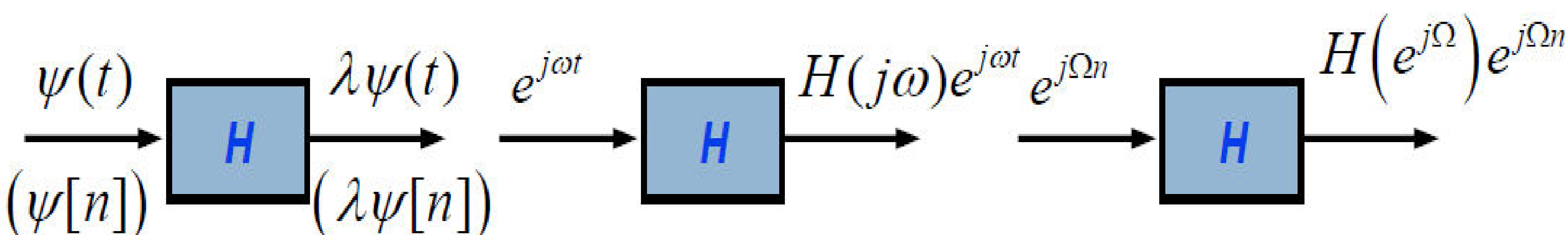
$$|H(j\omega)|=\frac{\frac{1}{RC}}{\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}}$$

Phase response:

$$\arg\{H(j\omega)\}=-\arctan(\omega RC)$$



$|H(j\omega)|$
1
$\frac{1}{\sqrt{2}}$
$-\frac{1}{RC}$
0
$\frac{1}{RC}$
ω
(a)
$\arg\{H(j\omega)\}$
$\frac{\pi}{2}$
$\frac{\pi}{4}$
ω
$-\frac{1}{RC}$
0
$\frac{1}{RC}$
$-\frac{\pi}{4}$
$-\frac{\pi}{2}$
(b)

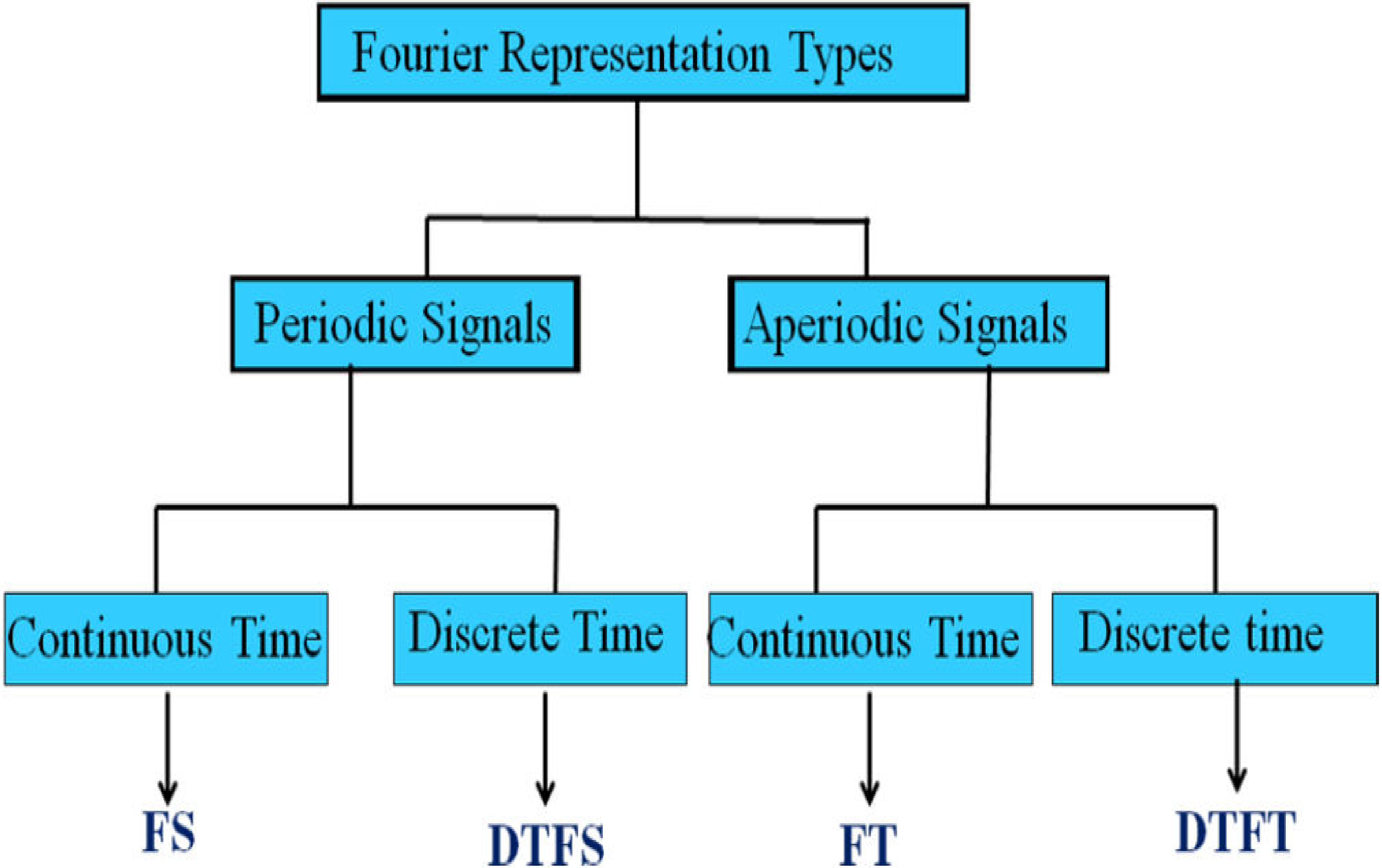


$\psi(t)$
$(\psi[n])$
H
$\lambda\psi(t)$
$(\lambda\psi[n])$
$e^{j\omega t}$
H
$H(j\omega)e^{j\omega t}$
$e^{j\Omega n}$
H
$H\left(e^{j\Omega}\right)e^{j\Omega n}$

APPLICATION

- HEAT TRANSFER
- AC CURRENT AND VOLTAGE SINUSOIDAL (ENABLES TO ANALYSE CIRCUIT)
- SIGNALS TRANSMITTED BY RADIO, TV ARE SINUSOIDAL.
- AUDIO SIGNALS AND IMAGE CAN ALSO BE REPRESENTED USING FOURIER REPRESENTATIONS.

Fourier Representation for four Signal Classes



FOURIER ANALYSIS (DISCRETE TIME)

- PERFECTLY SUITED FOR DIGITAL IMPLEMENTATION

EXERCISE

- Consider an LTI System for which

  Y(t)=x(t-3), if input to the system is complex exponential e^(j2t) find the o/p of system and frequency response of the system.
- Consider an LTI System

x(t)=cos 4t + cos 7t

Find y(t) if impulse response of the system is delta(t-3)

Fourier series of periodic continuous time signals

Any periodic continuous time signal with period, $T = 2\pi / \omega_0$, can be written as a sum of complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t}$$

where

$$s_k = jk\omega_0 = jk\frac{2\pi}{T}$$

How to determine a_k?

Determination of *spectral coefficients*, a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Determination of *spectral coefficients*, a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Multiplying both sides by $e^{-jn\omega_0 t}$ and integrating from 0 to T, we get

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt$$

Determination of *spectral coefficients*, a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Multiplying both sides by $e^{-jn\omega_0 t}$ and integrating from 0 to T, we get

$$\begin{aligned} \int_0^T x(t)\, e^{-jn\omega_0 t} dt &= \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt \\ &= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt \end{aligned}$$

Determination of *spectral coefficients*, a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Multiplying both sides by $e^{-jn\omega_0 t}$ and integrating from 0 to T, we get

$$\begin{aligned}
\int_0^T x(t) e^{-jn\omega_0 t} dt &= \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k T \delta[k-n] = T a_n
\end{aligned}$$

$$a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

Fourier Representation for Continuous Time Signals

PERIODIC SIGNALS

– The Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

– The Analysis Equation

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \tag{3.20}$$

is also periodic with period of T.

- $k = 0$, $x(t)$ is a constant.
- $k = +1$ and $k = -1$, both have fundamental frequency equal to ω_0 and are collectively referred to as ***the fundamental components*** or ***the first harmonic components***.
- $k = +2$ and $k = -2$, the components are referred to as ***the second harmonic components***.
- $k = +N$ and $k = -N$, the components are referred to as ***the Nth harmonic components***.

Eq. (3.20) can also be expressed as

$$x(t) = x^*(t) = \sum_{k=-\infty}^{+\infty} a^*_k \, e^{-jk\omega_0 t} , \tag{3.21}$$

where we assume that $x(t)$ is real, that is, $x(t) = x^*(t)$.

Replacing k by $-k$ in the summation, we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a^*_{-k} \, e^{jk\omega_0 t} , \tag{3.22}$$

which , by comparison with Eq. (3.20), requires that $a_k = a^*_{-k}$, or equivalently

$$a^*_k = a_{-k} . \tag{3.23}$$

To derive the alternative forms of the Fourier series, we rewrite the summation in Eq.

$$x(t) = a_0 + \sum_{k=1}^{+\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk(2\pi/T)t} \right].$$

Substituting a^*_k for a_{-k}, we have

$$x(t) = a_0 + \sum_{k=1}^{+\infty} \left[a_k e^{jk\omega_0 t} + a^*_k \, e^{-jk(2\pi/T)t} \right].$$

Since the two terms inside the summation are complex conjugate of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\}. \tag{3.26}$$

If a_k is expressed in polar from as

$$a_k = A_k e^{j\theta_k},$$

then Eq. (3.26) becomes

$$x(t) = a_0 + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\left\{A_k e^{j(k\omega_0 t + \theta_k)}\right\}.$$

That is

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(k\omega_0 t + \theta_k).$$

Another form is obtained by writing a_k in rectangular form as

$$a_k = B_k + jC_k$$

then Eq. (3.26) becomes

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty}\left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right].$$

For real periodic functions, the Fourier series in terms of complex exponential has the following *three equivalent forms:*

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$
$$x(t) = a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(k\omega_0 t + \theta_k)$$
$$x(t) = a_0 + 2\sum_{k=1}^{+\infty} \left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right]$$


THANK
YOU!