Network Analysis & Systems

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UE18EC201: Network Analysis & Systems

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Network Analysis and Synthesis

Part V: Network Synthesis

Network Functions (1)

- Let $H(s)$ be a network function:

$$H(s) = \frac{M_1(s) + N_1(s)}{M_2(s) + N_2(s)}$$

where M_1 and M_2 are the even parts, and N_1 and N_2 are the odd parts.

- Also,

$$\begin{aligned} H(s) &= \frac{M_1(s) + N_1(s)}{M_2(s) + N_2(s)} \frac{M_2(s) - N_2(s)}{M_2(s) - N_2(s)} \\ &= \frac{M_1 M_2 - N_1 N_2}{M_2^2 - N_2^2} + \frac{N_1 M_2 - M_1 N_2}{M_2^2 - N_2^2} \\ &\triangleq \text{Even}\,[H(s)] + \text{Odd}\,[H(s)] \end{aligned}$$

Network Functions (2)

- On the imaginary axis,

$$H(j\omega) = \text{Re}\ [H(j\omega)] + j\text{Im}\ [H(j\omega)]$$

- Clearly,

$$\text{Re}\ [H(j\omega)] = \text{Even}\ [H(s)]|_{s=j\omega}$$

and

$$j\text{Im}\ [H(j\omega)] = \text{Odd}\ [H(s)]|_{s=j\omega}$$

That is, the even part is real and the odd part is imaginary.

LC Immittance Functions (1)

- Let $Z(s)$ be the impedance of a passive one-port.

LC Immittance Functions (1)

- Let $Z(s)$ be the impedance of a passive one-port.
- For a pure reactive network, the average power dissipated is zero.
- Therefore,

$$\text{Re}\,[Z(j\omega)] = \text{Even}\,[Z(j\omega)] = 0$$

where

$$\text{Even}\,[Z(s)] = \frac{M_1(s)M_2(s) - N_1(s)N_2(s)}{M_2^2(s) - N_2^2(s)}$$

- Clearly, Even $[Z(j\omega)] = 0$ implies,

$$M_1(j\omega)M_2(j\omega) - N_1(j\omega)N_2(j\omega) = 0$$

LC Immittance Functions (2)

$$M_1(j\omega)M_2(j\omega) - N_1(j\omega)N_2(j\omega) = 0$$

implies

- $Z(s) = \frac{M_1 M_2 - N_1 N_2}{M_2^2 - N_2^2} + \frac{N_1 M_2 - M_1 N_2}{M_2^2 - N_2^2} = \frac{N_1 M_2 - M_1 N_2}{M_2^2 - N_2^2}$.
- Either $M_1 = 0 = N_2$ or $M_2 = 0 = N_1$.

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- Either $M_1 = 0 = N_2$ or $M_2 = 0 = N_1$.

Thus,

- If $M_1 = 0 = N_2$, then $Z(s) = \frac{N_1(s)}{M_2(s)}$.

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Thus,

- If $M_1 = 0 = N_2$, then $Z(s) = \frac{N_1(s)}{M_2(s)}$.
- If $M_2 = 0 = N_1$, then $Z(s) = \frac{M_1(s)}{N_2(s)}$.

LC Immittance Functions (2)

$$M_1(j\omega)M_2(j\omega) - N_1(j\omega)N_2(j\omega) = 0$$

implies

- $Z(s) = \frac{M_1M_2 - N_1N_2}{M_2^2 - N_2^2} + \frac{N_1M_2 - M_1N_2}{M_2^2 - N_2^2} = \frac{N_1M_2 - M_1N_2}{M_2^2 - N_2^2}$.
- Either $M_1 = 0 = N_2$ or $M_2 = 0 = N_1$.

Thus,

- If $M_1 = 0 = N_2$, then $Z(s) = \frac{N_1(s)}{M_2(s)}$.
- If $M_2 = 0 = N_1$, then $Z(s) = \frac{M_1(s)}{N_2(s)}$.
- Thus LC immittance functions are a ratio of odd to even polynomials or even to odd polynomials.
- All poles and zeros lie on the imaginary axis.

LC Immittance Functions (3)

Consider an LC immittance function

$$H(s) = \frac{a_4 s^4 + a_2 s^2 + a_0}{b_5 s^5 + b_3 s^3 + b_1 s}$$

- All coefficients are real and positive.
- All poles and zeros lie on the imaginary axis.
- There are no multiple poles or zeros on the imaginary axis.

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- All coefficients are real and positive.
- All poles and zeros lie on the imaginary axis.
- There are no multiple poles or zeros on the imaginary axis.
- The highest powers must differ by unity: If $2n$ is the degree of the numerator, then the denominator has degree either $2n-1$ or $2n+1$.
- If the degree of denominator is $2n-1$, then $H(s)$ has a simple pole at $s=\infty$.
- If the degree of denominator is $2n+1$, then $H(s)$ has a simple zero at $s=\infty$.

LC Immittance Functions (4)

$$H(s) = \frac{a_4 s^4 + a_2 s^2 + a_0}{b_5 s^5 + b_3 s^3 + b_1 s}$$

- The lowest terms of numerator and denominator must differ by one; otherwise, there would be multiple poles or zeros at $s = 0$, a contradiction.

LC Immittance Functions (4)

$$H(s) = \frac{a_4 s^4 + a_2 s^2 + a_0}{b_5 s^5 + b_3 s^3 + b_1 s}$$

- The lowest terms of numerator and denominator must differ by one; otherwise, there would be multiple poles or zeros at $s = 0$, a contradiction.
- If the polynomial is even, it should contain all even powers.
- If the polynomial is odd, it should contain all odd powers.
- Otherwise, there would be poles or zeros not on the imaginary axis, a contradiction.

LC Immittance Functions (5)

Therefore,

$$\begin{aligned} H(s) &= \frac{K(s^2+\omega_1^2)(s^2+\omega_3^2)\cdots(s^2+\omega_i^2)\cdots}{s(s^2+\omega_2^2)(s^2+\omega_4^2)\cdots(s^2+\omega_j^2)\cdots} \\ &= \frac{k_0}{s} + \frac{2k_2 s}{s^2+\omega_2^2} + \frac{2k_4 s}{s^2+\omega_4^2} + \cdots + k_\infty s \end{aligned}$$

- The residues must be real and positive.
- Clearly, $H(j\omega)$ is a pure reactance $jX(\omega)$, where

$$X(\omega) = -\frac{k_0}{\omega} + \frac{2k_2\omega}{-\omega^2+\omega_2^2} + \frac{2k_4\omega}{-\omega^2+\omega_4^2} + \cdots + k_\infty \omega$$

- Moreover,

$$\frac{dX(\omega)}{d\omega} = \frac{k_0}{\omega^2} + k_\infty + \frac{2k_2(\omega^2+\omega_2^2)}{(\omega^2-\omega_2^2)^2} + \cdots \geq 0$$

LC Immittance Functions (6)

Let

$$H(s) = \frac{Ks(s^2 + \omega_3^2)}{(s^2 + \omega_2^2)(s^2 + \omega_4^2)}$$

Therefore,

$$H(j\omega) = jX(\omega) =$$

LC Immittance Functions (6)

Let

$$H(s) = \frac{Ks(s^2+\omega_3^2)}{(s^2+\omega_2^2)(s^2+\omega_4^2)}$$

Therefore,

$$H(j\omega) = jX(\omega) = j\frac{K\omega(-\omega^2+\omega_3^2)}{(-\omega^2+\omega_2^2)(-\omega^2+\omega_4^2)}$$

- What is the plot of $X(\omega)$ vs. ω?

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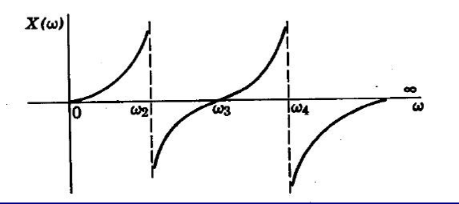
- What is the plot of $X(\omega)$ vs. ω?
- Recall that $\frac{dX(\omega)}{d\omega} \geq 0$.
- ω_2, ω_3, and ω_4 are called internal critical frequencies.
- $s = 0$ and $s = \infty$ are called external critical frequencies.

LC Immittance Functions (7)

$$H(s) = \frac{Ks(s^2+\omega_3^2)}{(s^2+\omega_2^2)(s^2+\omega_4^2)}$$



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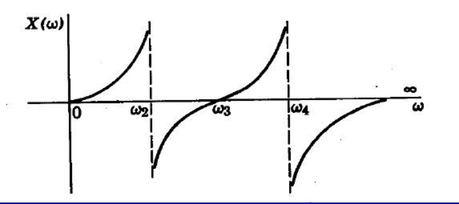

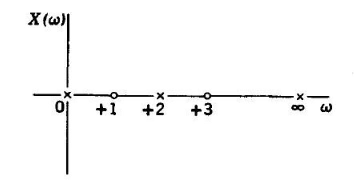

LC Immittance Functions (8)

Summary of Properties:

- It is the ratio of odd to even polynomials or even to odd polynomials.
- The poles and zeros are simple and lie on the imaginary axis.
- The poles and zeros interlace on the imaginary axis.
- The highest powers of numerator and denominator must differ by unity; the lowest powers also differ by unity.
- There must be a zero or a pole at the origin and infinity.

LC Immittance Functions (9)

- $$H(s) = \frac{Ks(s^2+4)}{(s^2+1)(s^2+3)}$$

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- $$H(s) = \frac{s^5+4s^3+5s}{3s^4+6s^2}$$

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- $$H(s) = \frac{Ks(s^2+4)}{(s^2+1)(s^2+3)}$$

- $$H(s) = \frac{s^5+4s^3+5s}{3s^4+6s^2}$$

- $$H(s) = \frac{K(s^2+1)(s^2+9)}{(s^2+2)(s^2+10)}$$

LC Immittance Functions (10)

$$H(s) = \frac{k_0}{s} + \frac{2k_2 s}{s^2 + \omega_2^2} + \frac{2k_4 s}{s^2 + \omega_4^2} + \cdots + k_\infty s$$

Let $H(s)$ be an impedance.

- The first term is a $1/k_0$ F capacitor and the last term is a k_∞ H inductor.

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- The first term is a $1/k_0$ F capacitor and the last term is a k_∞ H inductor.
- The term $\frac{2k_i s}{s^2+\omega_i^2}$ is a parallel tank circuit with $1/2k_i$ F capacitor in parallel with a $2k_i/\omega_i^2$ H inductor.
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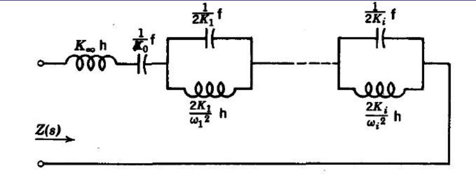

LC Immittance Functions (11)

$$Z(s) = \frac{2(s^2+1)(s^2+9)}{s(s^2+4)} =$$

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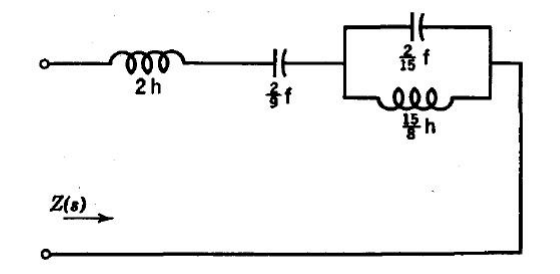

LC Immittance Functions (12)

$$H(s) = \frac{k_0}{s} + \frac{2k_2 s}{s^2 + \omega_2^2} + \frac{2k_4 s}{s^2 + \omega_4^2} + \cdots + k_\infty s$$

Let $H(s)$ be an admittance.

- The first term is a $1/k_0$ H inductor and the last term is a k_∞ F capacitor.

LC Immittance Functions (12)

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Let $H(s)$ be an admittance.

- The first term is a $1/k_0$ H inductor and the last term is a k_∞ F capacitor.
- The term $\frac{2k_i s}{s+\omega_i^2}$ is a series circuit with $1/2k_i$ H inductor in series with a $2k_i/\omega_i^2$ F capacitor.
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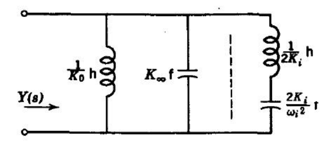

LC Immittance Functions (13)

$$Y(s) = \frac{s(s^2+2)(s^2+4)}{s(s^2+1)(s^2+3)} =$$

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$$Y(s) = \frac{s(s^2+2)(s^2+4)}{s(s^2+1)(s^2+3)} = s + \frac{3s/2}{s^2+1} + \frac{s/2}{s^2+3}$$

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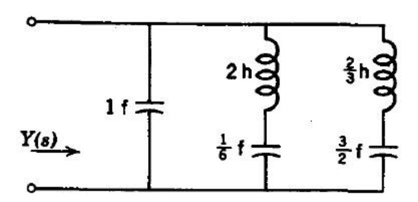

LC Immittance Functions (14)

- Recall: the degrees of the numerator and denominator always differ by unity.
- Therefore, there is always a pole or zero at $s = \infty$.

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- Therefore, there is always a pole or zero at $s = \infty$.
- Thus, for an impedance function $Z(s)$, the degree of the numerator is $2n$ and the degree of the denominator is $2n - 1$, implying that there is a pole at $s = \infty$.
- Clearly, we can remove an impedance $L_1 s$ so that $Z_2(s) \triangleq Z(s) - L_1 s$ is still an LC impedance function.

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- Clearly, we can remove an impedance $L_1 s$ so that $Z_2(s) \triangleq Z(s) - L_1 s$ is still an LC impedance function.
- The degree of the denominator of $Z_2(s)$ is $2n - 1$ making the degree of the numerator $2n - 2$!

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- Therefore, there is always a pole or zero at $s = \infty$.
- Thus, for an impedance function $Z(s)$, the degree of the numerator is $2n$ and the degree of the denominator is $2n - 1$, implying that there is a pole at $s = \infty$.
- Clearly, we can remove an impedance $L_1 s$ so that $Z_2(s) \triangleq Z(s) - L_1 s$ is still an LC impedance function.
- The degree of the denominator of $Z_2(s)$ is $2n - 1$ making the degree of the numerator $2n - 2$!
- Clearly $Z_2(s)$ has a zero at $s = \infty$; or $1/Z_2(s)$ has a pole at $s = \infty$.
- This process can be continued, leading to a CFE with a ladder network.

LC Immittance Functions (15)

$$Z(s) = \frac{2s^5 + 12s^3 + 16s}{s^4 + 4s^2 + 3}$$

Therefore,

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Therefore,

$$
\begin{array}{l}
s^4 + 4s^2 + 3\,\big)\,2s^5 + 12s^3 + 16s\,\big(\,2s \leftrightarrow Z \\
\qquad\qquad\quad 2s^5 + 8s^3 + 6s \\
\qquad\qquad\quad \overline{\qquad\quad 4s^3 + 10s\,\big)\,s^4 + 4s^2 + 3\,\big(\,\tfrac{1}{4}s \leftrightarrow Y} \\
\qquad\qquad\qquad\qquad\qquad s^4 + \tfrac{5}{2}s^2 \\
\qquad\qquad\qquad\qquad\qquad \overline{\quad \tfrac{3}{2}s^2 + 3\,\big)\,4s^3 + 10s\,\big(\,\tfrac{8}{3}s \leftrightarrow Z} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 4s^3 + 8s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{\quad 2s\,\big)\,\tfrac{3}{2}s^2 + 3\,\big(\,\tfrac{3}{4}s \leftrightarrow Y} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \tfrac{3}{2}s^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\qquad 3\,\big)\,2s\,\big(\,\tfrac{2}{3}s \leftrightarrow Z} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =
\end{array}
$$

LC Immittance Functions (16)

$$Z(s) = 2s + \cfrac{1}{\cfrac{s}{4} + \cfrac{1}{\cfrac{8s}{3} + \cfrac{1}{\cfrac{3s}{4} + \cfrac{1}{2s/3}}}}$$

LC Immittance Functions (16)

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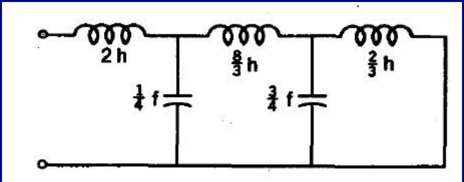


LC Immittance Functions (17)

- If the initial function is an impedance, the first quotient is an impedance, the second is an admittance (due to the expansion of the reciprocal), and so on.
- This is a Cauer ladder network.

LC Immittance Functions (17)

- If the initial function is an impedance, the first quotient is an impedance, the second is an admittance (due to the expansion of the reciprocal), and so on.
- This is a Cauer ladder network.
- In contrast, if $H(s)$ has a pole at $s = 0$, the aforementioned procedure can be used to successively remove the poles at $s = 0$.
- For this the polynomials are arranged in ascending order.

LC Immittance Functions (18)

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)}$$

Therefore,

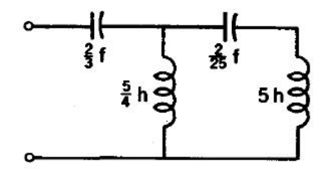
LC Immittance Functions (18)

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)}$$

Therefore,

$$2s + s^3 \overline{)3 + 4s^2 + s^4}(3/2s \leftrightarrow Z$$
$$3 + \tfrac{3}{2}s^2$$
$$\tfrac{5}{2}s^2 + s^4)2s + s^3(4/5s \leftrightarrow Y$$
$$2s + \tfrac{4}{5}s^3$$
$$\tfrac{1}{5}s^3)\tfrac{5}{2}s^2 + s^4(25/2s \leftrightarrow Z$$
$$\tfrac{5}{2}s^2$$
$$s^4)\tfrac{1}{5}s^3(1/5s \leftrightarrow Y$$
$$\tfrac{1}{5}s^3$$

LC Immittance Functions (19)



- This is also a Cauer ladder network.

LC Immittance Functions (20)

Canonical Forms

- Foster Series Network or Foster I form: PFE of an impedance function.
- Foster Parallel Network or Foster II form: PFE of an admittance function.
- Cauer I form: CFE when the polynomials are arranged in descending order.
- Cauer II form: CFE when the polynomials are arranged in ascending order.
- Cauer I and II forms result in ladder networks and are applicable for an immittance function.