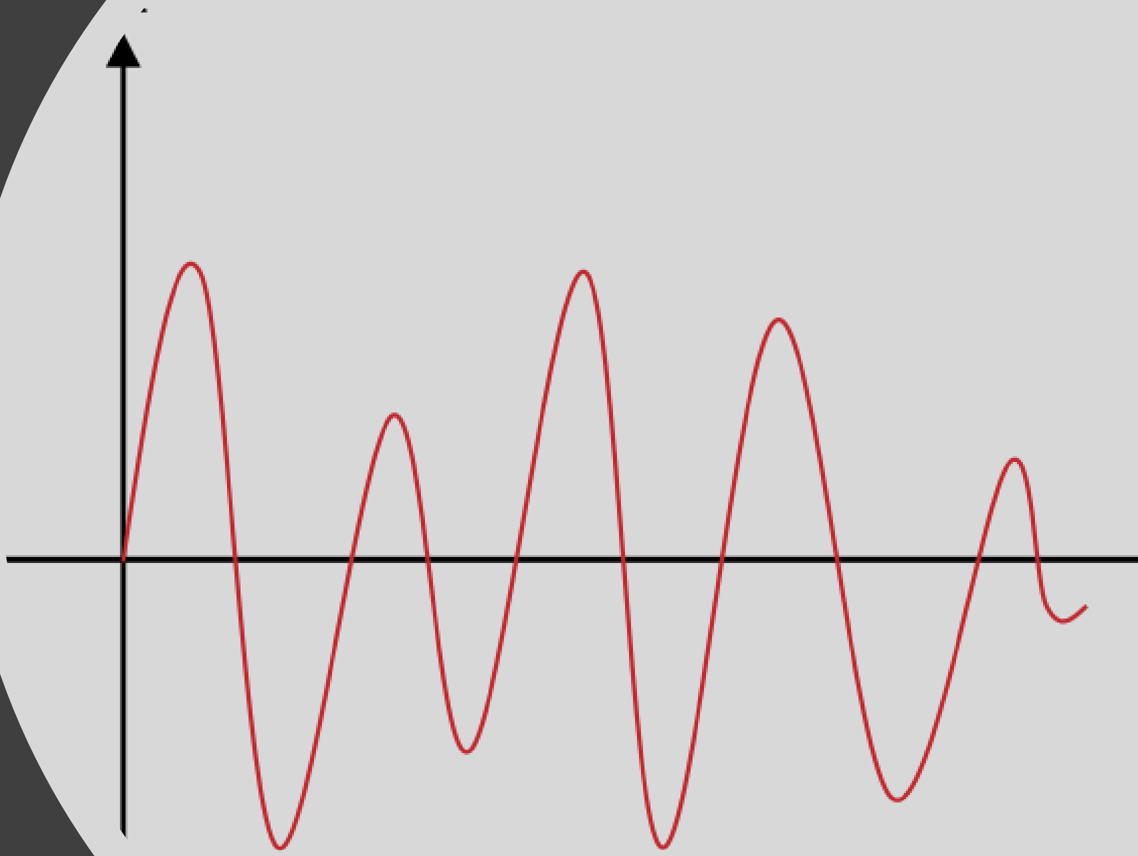
SIGNALS AND SYSTEMS UNIT 1

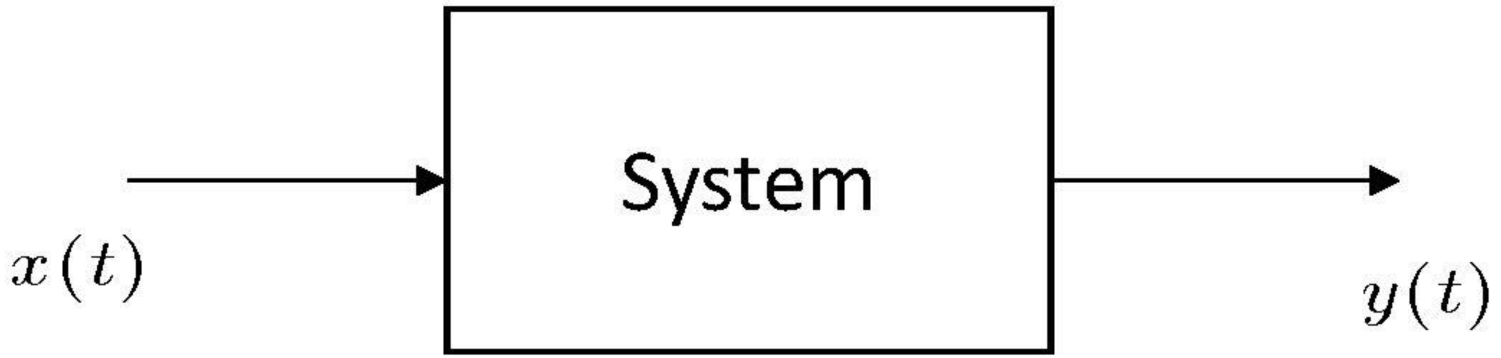
REFERENCE BOOK -SIMON HAYKIN

What is a Signal?

- A signal is formally defined as a function of one or more variables that conveys information on the nature of a physical phenomenon.
- A signal which is a function of one independent variable is said to 1D . When it is a function of two or more variables then it is said to be multi-dimensional.
- Eg: Speech(1D) , Image(2D)



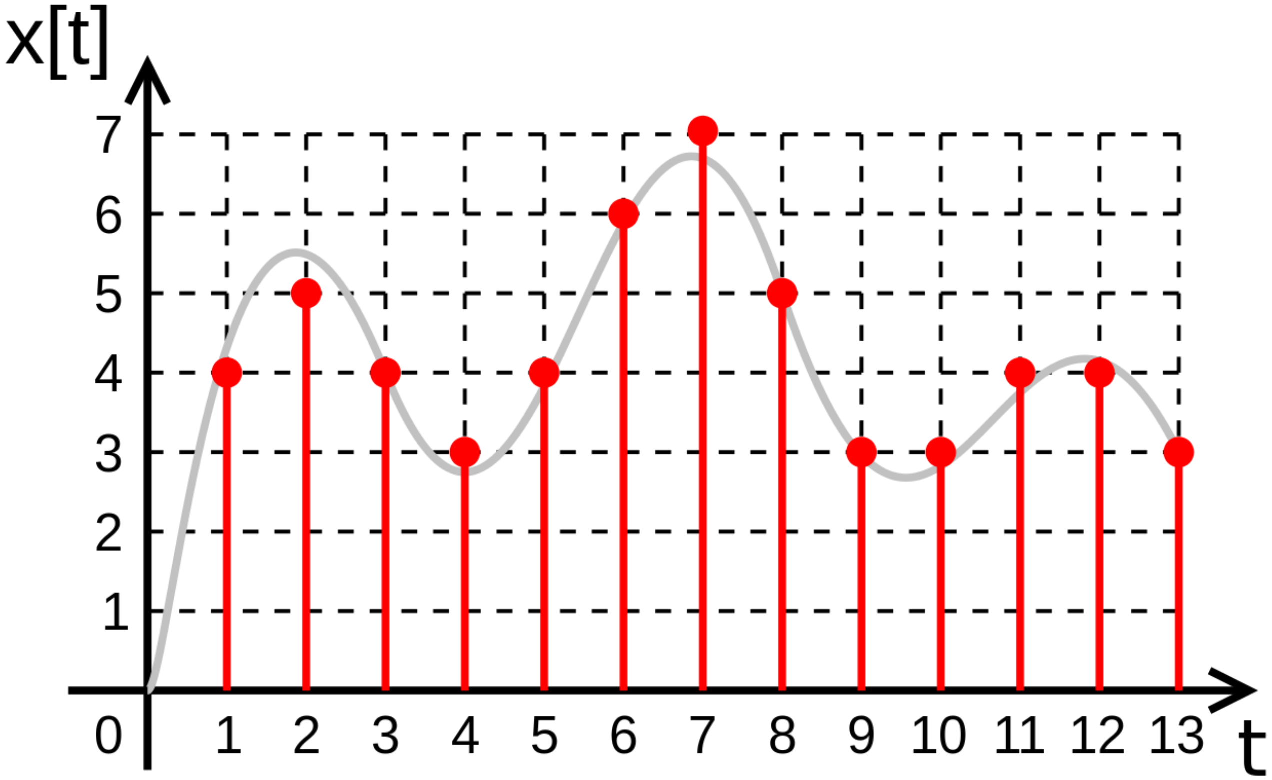
What is a System?



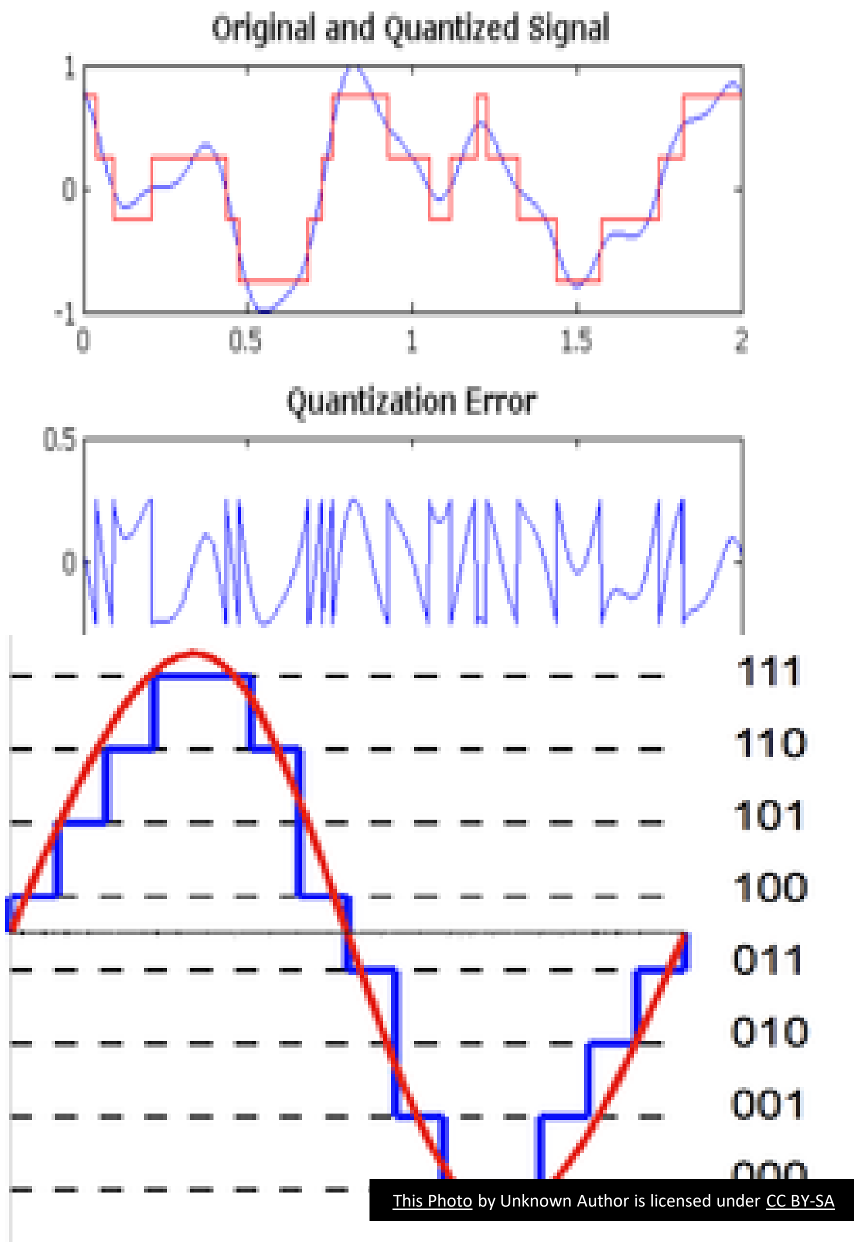
- A system is formally defined as an entity that manipulates one or more signals to accomplish a function, thereby yielding new signals.

Important terms:

- **Sampling**: which converts the signal into a sequence of numbers, with each number representing the amplitude of the signal at a particular instant of time.



- **Quantization**: which involves representing each number produced by the sampler to the nearest level selected from a finite number of discrete amplitude levels.
- After the combination of sampling and quantization, the representation of the signal will be discrete in both amplitude and time.



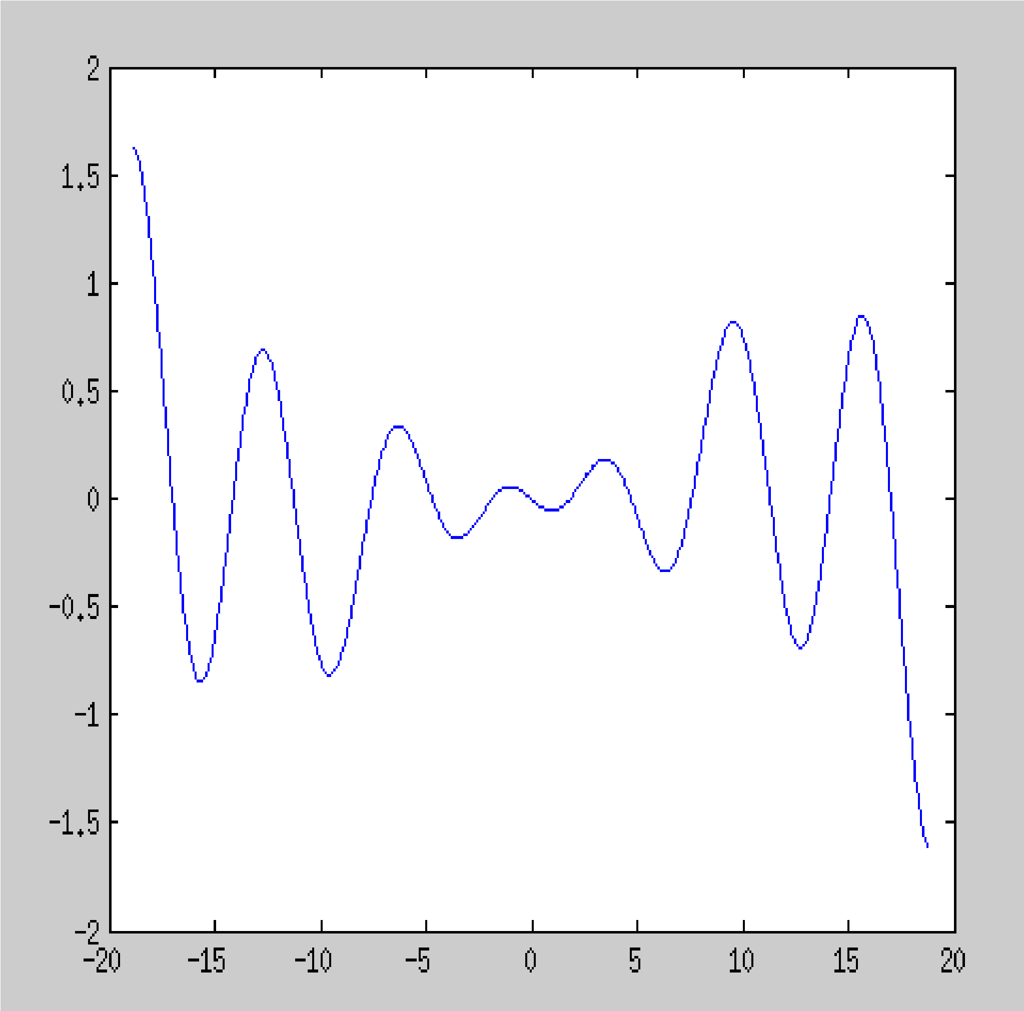
Classification of signals

Continuous and Discrete time signals :

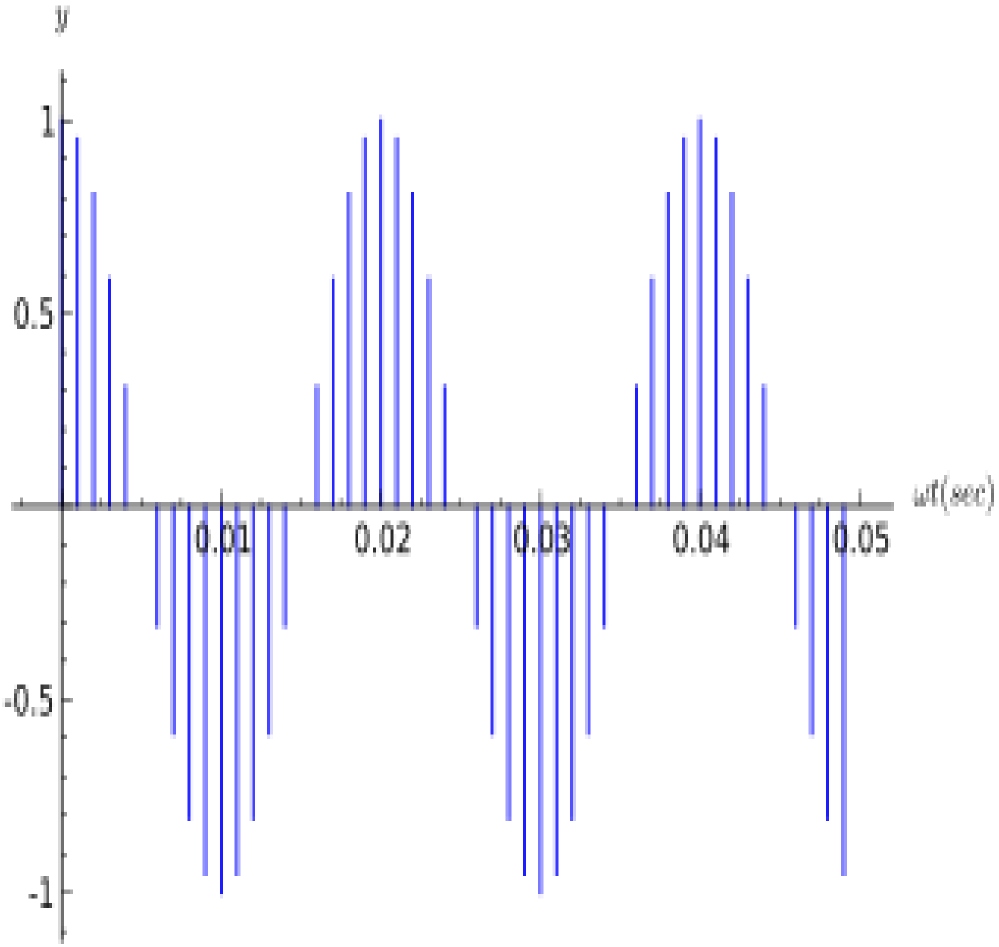
Continuous time :a signal x(t) is said to be a continuous-time signal if it is defined for all time t. Represented as x(t).

Discrete time : is defined only at discrete instants of time. Represented as x[n].

- **Continuous time signal**

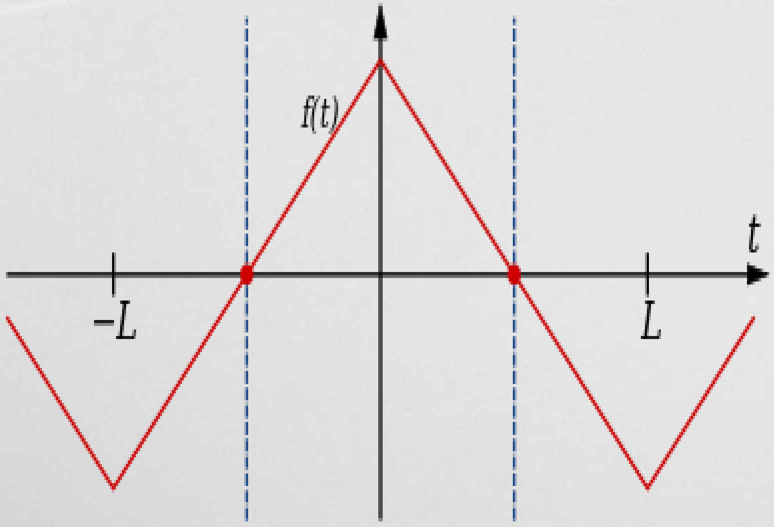


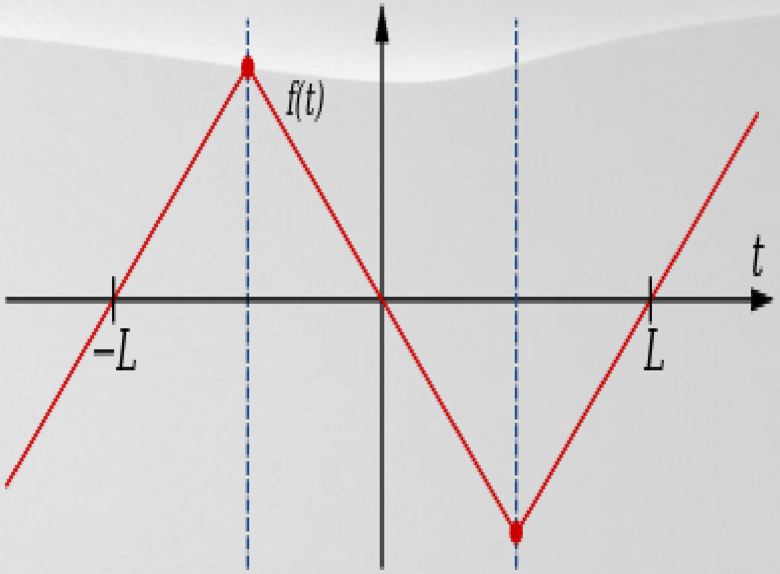
- **Discrete time signal**



- **EVEN signal**
- A continuous-time signal x(t) is said to be an even signal if x(—t) = x(t) for all t.
- In other words, even signals are symmetric about the vertical axis, or time origin.

- **ODD Signal**
- The signal x{t) is said to be an odd signal if x(—t) = —x(t) for all t.
- odd signals are antisymmetric about the time origin.





- **Example 1:**
- Consider the signal x(t)={ $\sin\left(\frac{\pi t}{t}\right)$, $\quad -T \leq t \leq T$

  0 , otherwise}

Is the signal x(t) an even or an odd function of time t?

Solution: Replacing t with –t yields

x(t)= {$\sin\left(-\frac{\pi t}{t}\right), -T \leq t \leq T$

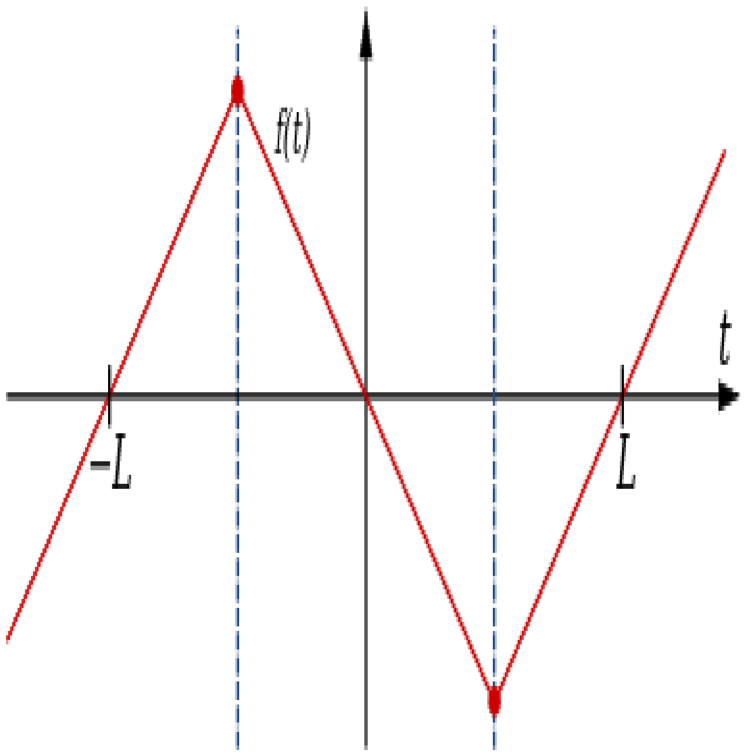
0 , otherwise}

{-$\sin\left(\frac{\pi t}{t}\right), -T \leq t \leq T$

0 , otherwise}

= -x(t) for all t,

Hence , x(t) is an odd signal.



- If a signal can neither be classified as even nor odd then we can represent it as a sum of even and odd signals.

$x(t)= x_e(t) + x_o(t)$

----- $x_o(t)= \frac{1}{2}\ (x(t)-x(-t)\ )$ (odd part of the signal)

-----$x_e(t)= \frac{1}{2}\ (x(t)+x(-t)\ )$ (even part of the signal)

--**Odd signals will have the value zero at the origin.**

- **Example 2**: Find the even and odd parts of the signals.
- x(t) = e^(-2t) cos t.
- **Solution**:
  Replacing t with -t in the expression for x(t) yields
- x(—t) = e^(2t) cos(—t)=e^(2t) cost.
- xe(t) = $\frac{1}{2}$ (e^(-2t) cos t + e^(2t) cos(t))

= cosh(2t)cos(t)

xo(t)=$\frac{1}{2}$ (e^(−2t) cos t − e^(2t) cos(t))

= -sinh(2t) cos t,

▶ **Problem 1.1** Find the even and odd components of each of the following signals:

(a) $x(t) = \cos(t) + \sin(t) + \sin(t)\cos(t)$

(b) $x(t) = 1 + t + 3t^2 + 5t^3 + 9t^4$

(c) $x(t) = 1 + t\cos(t) + t^2\sin(t) + t^3\sin(t)\cos(t)$

(d) $x(t) = (1 + t^3)\cos^3(10t)$

Answers:

(a) Even: $\cos(t)$

Odd: $\sin(t)(1 + \cos(t))$

(b) Even: $1 + 3t^2 + 9t^4$

Odd: $t + 5t^3$

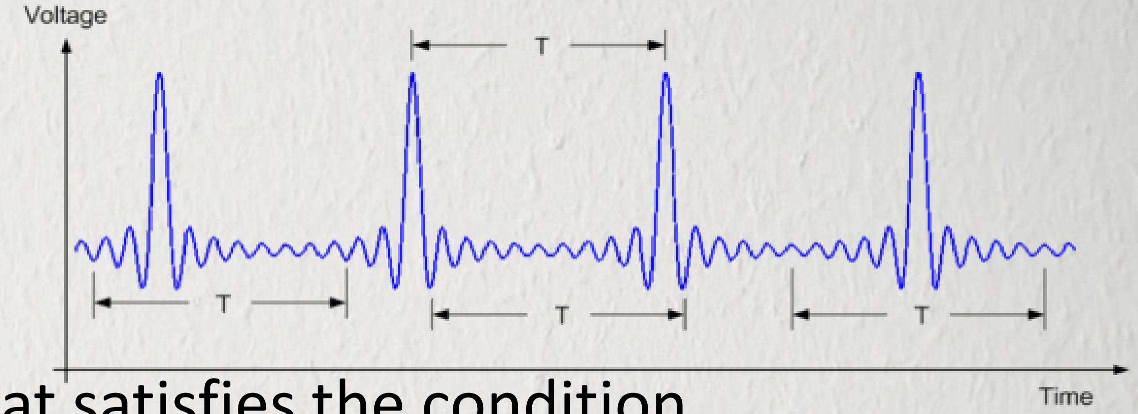
(c) Even: $1 + t^3\sin(t)\cos(t)$

Odd: $t\cos(t) + t^2\sin(t)$

(d) Even: $\cos^3(10t)$

Odd: $t^3\cos^3(10t)$ ◀

Periodic and Non-periodic Signals



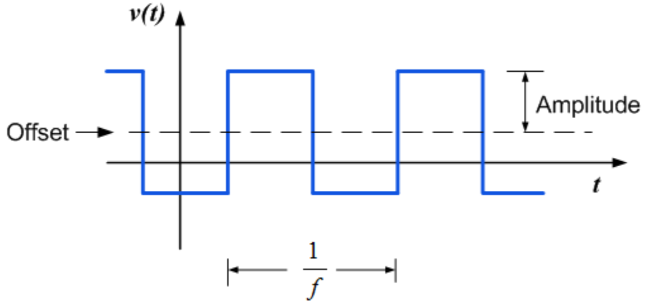
1) **<u>Periodic Signal (CT)</u>** –

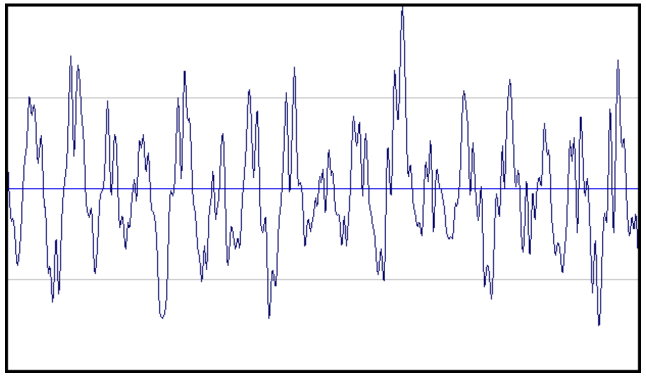
- A periodic signal x(t) is a function of time that satisfies the condition

  $x(t) = x(t + T)$ for all t

- where **T is a positive constant**. If this condition is satisfied for $T = T_0$, say, then it is also satisfied for $T = 2T_0 , 3T_0 , 4T_0$. The smallest value of T that satisfies is called the fundamental period of x(t). Accordingly, the fundamental period T defines the duration of one complete cycle of x(t).

- The reciprocal of the fundamental period T is called the fundamental frequency of the periodic signal x(t); it describes how frequently the periodic signal x(t) repeats itself. We thus formally write $f = \frac{1}{T} \ Hz$





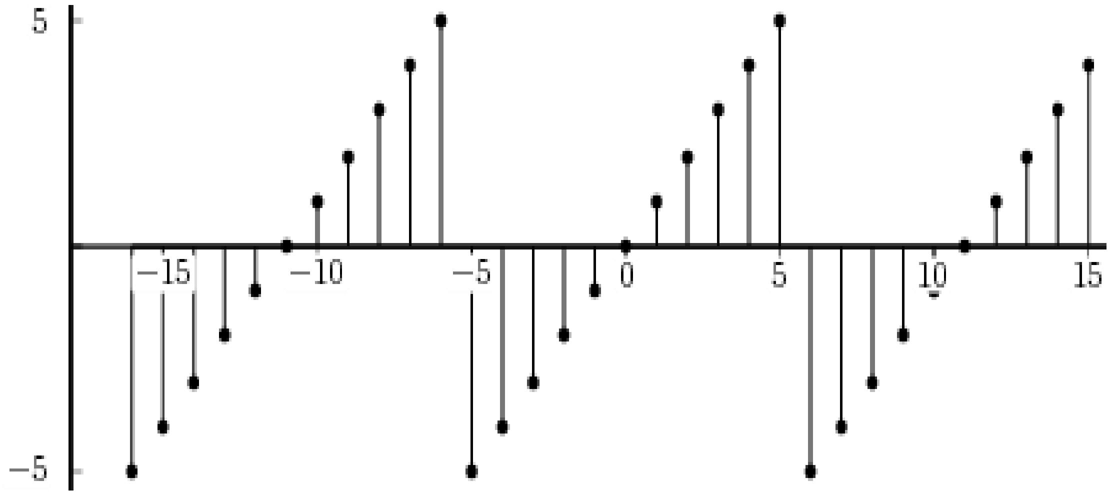
- Angular frequency measured in radians/seconds.
- $\omega = \frac{2\pi}{T}$ radians/sec.

2) **<u>Non-Periodic signal (CT)</u>**

- Any signal x(t) for which no value of T satisfies the condition of Eq. (1.7) is called an aperiodic or nonperiodic signal.

Periodic and Non-Periodic signals(DT)
Periodic DT signals

- A discrete-time signal x[n] is said to be periodic if x[n] = x[n + N] for integer n, where **N is a positive integer**.
- The smallest integer N for which equation is satisfied is called the fundamental period of the discrete-time signal x[n].
- The fundamental frequency of x[n] is defined by $\omega = \frac{2\pi m}{n}$



Non-periodic DT signals

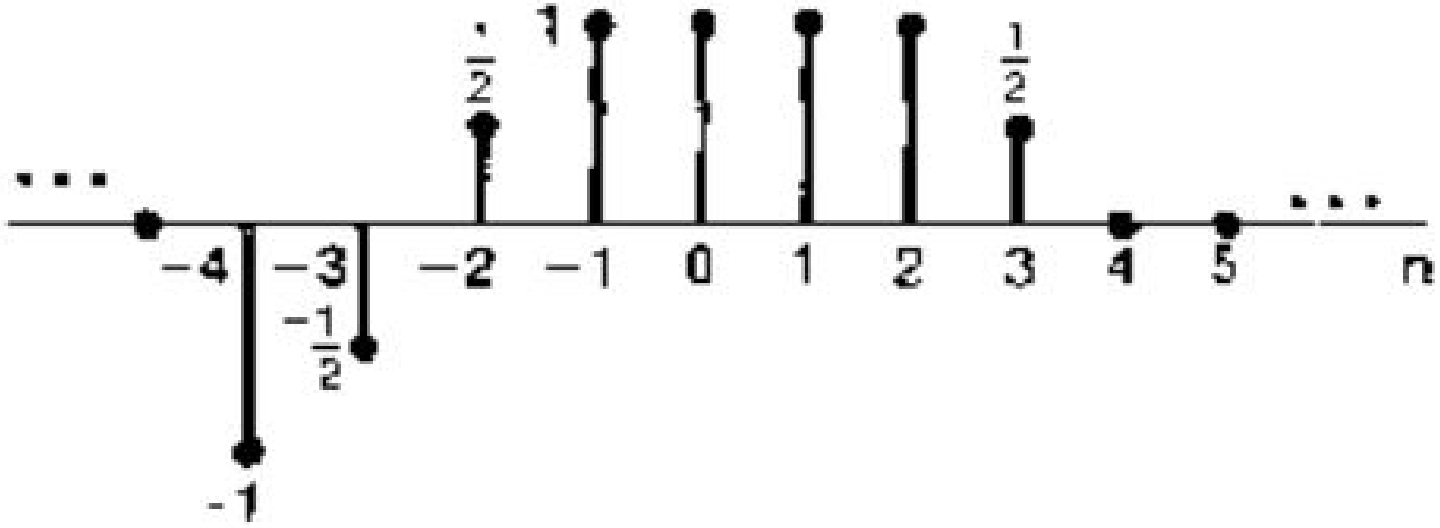


Figure P1.22

Example: For each of the following signals, determine whether it is periodic, and if it is, find the fundamental period:

(a) $x(t) = \cos^2(2\pi t)$

(b) $x(t) = \sin^3(2t)$

(c) $x(t) = e^{-2t}\cos(2\pi t)$

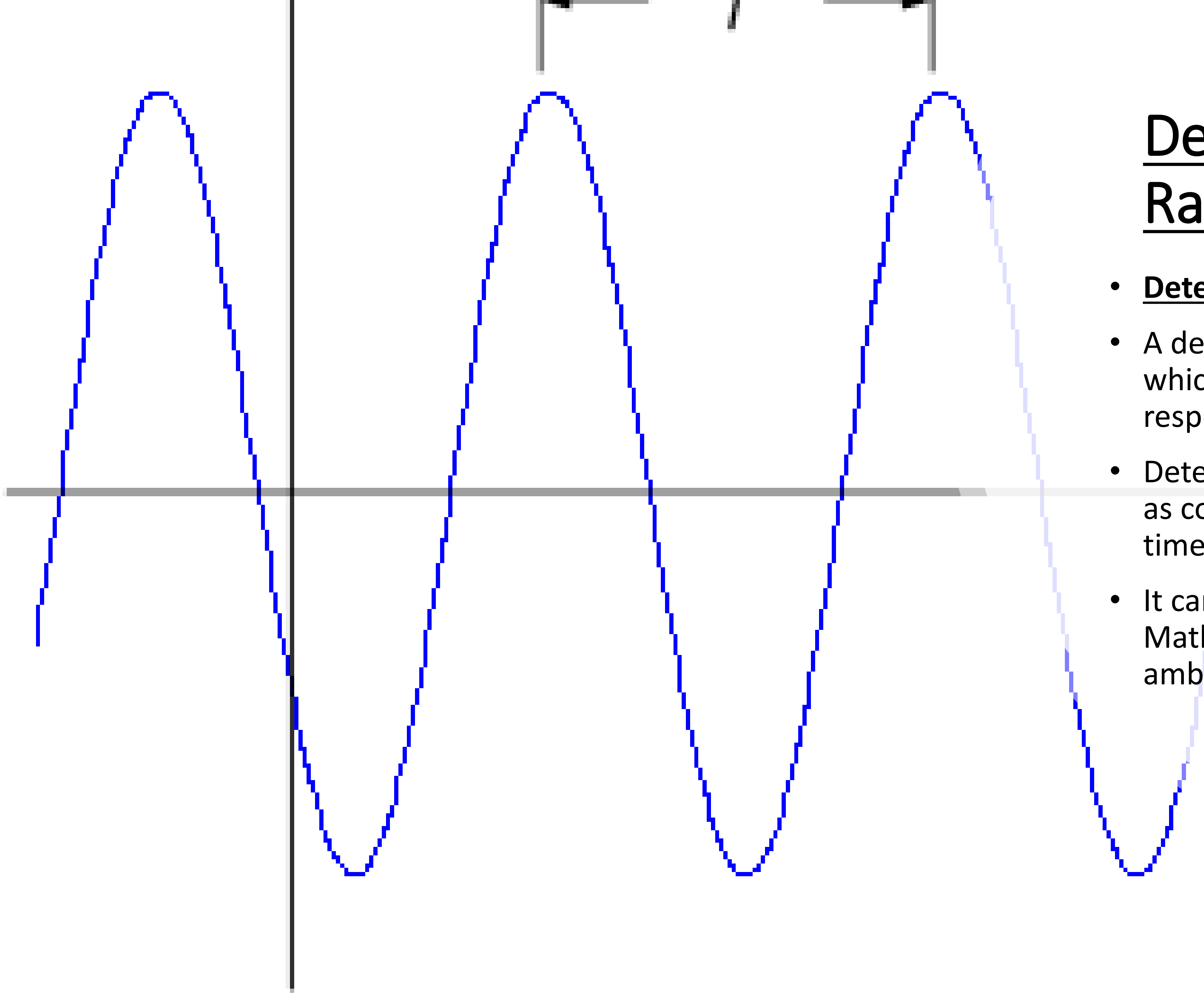
(d) $x[n] = (-1)^n$

(e) $x[n] = (-1)^{n^2}$

(f) $x[n] = \cos(2n)$

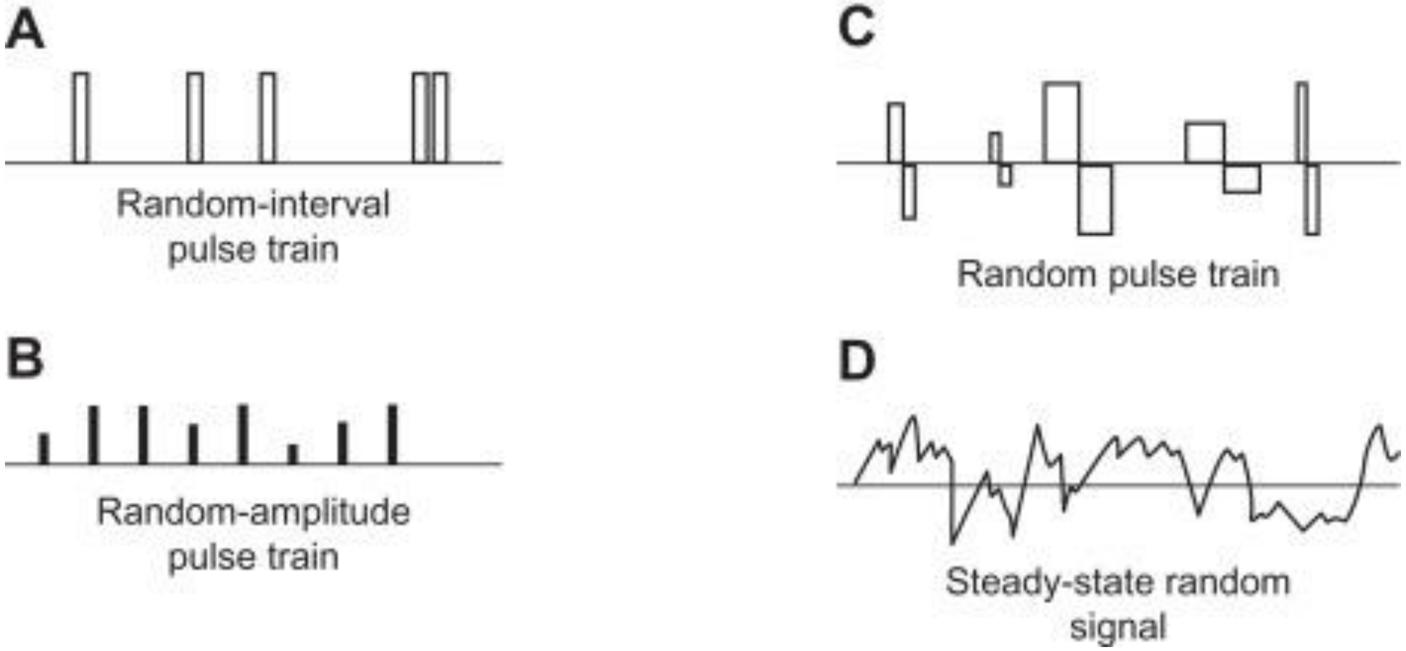
(g) $x[n] = \cos(2\pi n)$

(a) Periodic, with a fundamental period of 0.5 s
(b) Periodic, with a fundamental period of $(1/\pi)$ s
(c) Nonperiodic
(d) Periodic, with a fundamental period of 2 samples
(e) Periodic, with a fundamental period of 2 samples
(f) Nonperiodic
(g) Periodic, with a fundamental period of 1 sample



Deterministic and Random signals

- **Deterministic signal**
- A deterministic signal is a signal about which there is no uncertainty with respect to its value at any time.
- Deterministic signals may be modeled as completely specified functions of time.
- It can be expressed in the form of Mathematical expression without any ambiguity.



- **Random Signal**
- a random signal is a signal about which there is uncertainty before it occurs.

Energy and Power Signals

- **Energy Signals**:
- **CT:**
- Finite energy and zero power.
- p(t) = x^2(t)
- Energy of a CT signal - $\lim_{T \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}}$ x^2(t) dt
- $=\int_{-\infty}^{\infty} x\hat{}2(t)dt$
- Power of a CT signal- P= $\lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}}$ x^2(t) dt
- Power of a **periodic signal with fundamental time period** T ,
- P=$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}}$ x^2(t) dt

- **DT**:
- Energy E is defined as $= \sum_{n=-\infty}^{\infty} x^{\wedge}2[n]$,
- Average power is defined as : $\lim_{n\to\infty} \frac{1}{2N} \sum_{n=-\infty}^{\infty} x^{\wedge}2[n]$
- Average power in a periodic signal is defined by: $\frac{1}{N} \sum_{n=0}^{N-1} x^{\wedge}2[n]$
- A signal is referred to as an energy signal if and only if the total energy of the signal satisfies the condition
- 0 < E < ∞.
- The signal is referred to as a power signal if and only if the average power of the signal satisfies the condition
- 0 < P < ∞.

The energy and power classifications of signals are mutually exclusive.

In particular an energy signal has zero time-averaged power, whereas a power signal has infinite energy.

Periodic signals and random signals are power signals, whereas signals that are both deterministic and nonperiodic are usually viewed as energy signals.

► **Problem 1.9** Categorize each of the following signals as an energy signal or a power signal, and find the energy or time-averaged power of the signal:

(a) $x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2 - t, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$

(b) $x[n] = \begin{cases} n, & 0 \le n < 5 \\ 10 - n, & 5 \le n \le 10 \\ 0, & \text{otherwise} \end{cases}$

(c) $x(t) = 5\cos(\pi t) + \sin(5\pi t), -\infty < t < \infty$

(d) $x(t) = \begin{cases} 5\cos(\pi t), & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$

(e) $x(t) = \begin{cases} 5\cos(\pi t), & -0.5 \le t \le 0.5 \\ 0, & \text{otherwise} \end{cases}$

(f) $x[n] = \begin{cases} \sin(\pi n), & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

(g) $x[n] = \begin{cases} \cos(\pi n), & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

(h) $x[n] = \begin{cases} \cos(\pi n), & n \ge 0 \\ 0, & \text{otherwise} \end{cases}$

Answers:

(a) Energy signal, energy $= \frac{2}{3}$

(b) Energy signal, energy $= 85$

(c) Power signal, power $= 13$

(d) Energy signal, energy $= 25$

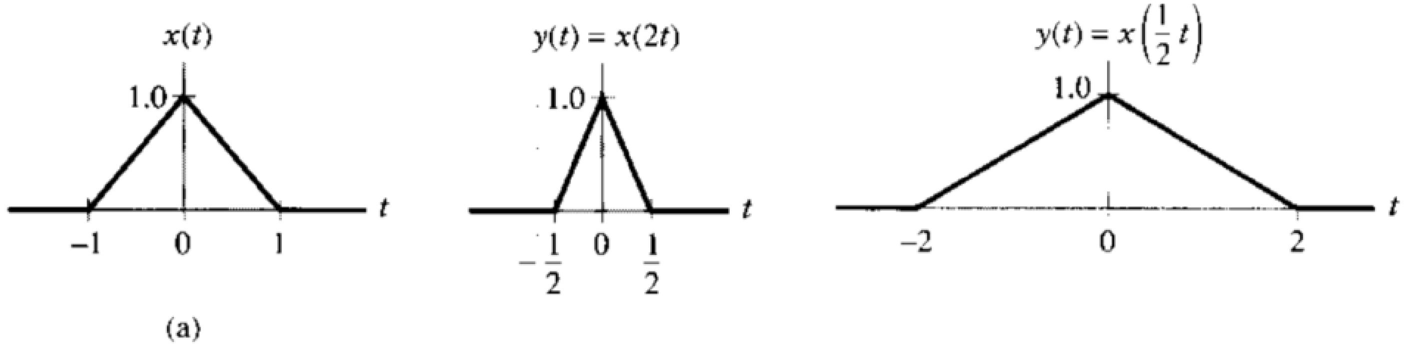
(e) Energy signal, energy $= 12.5$

(f) Zero signal

(g) Energy signal, energy $= 9$

(h) Power signal, power $= \frac{1}{2}$

Operations performed on independent variable

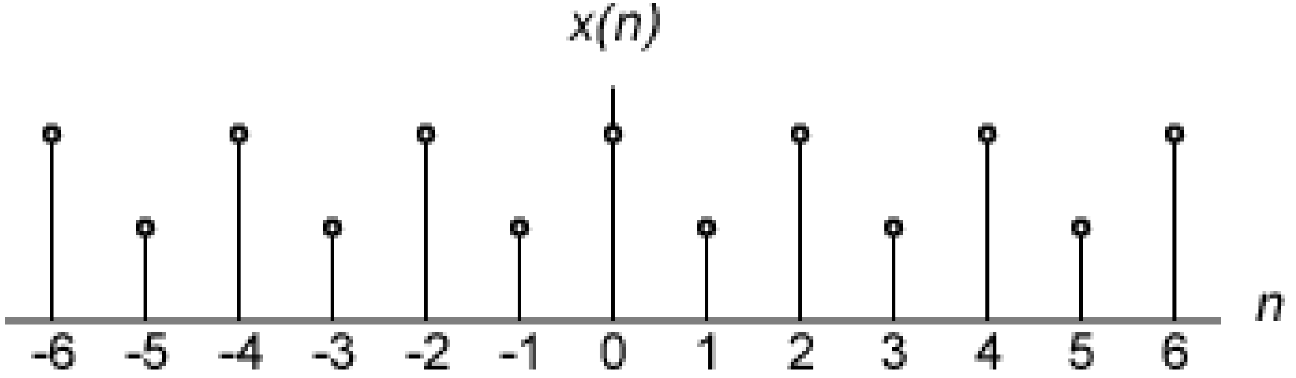


- **Time scaling** : The signal y(t) obtained by scaling the independent variable, time t, by a factor a is defined by y(t) = x(at).
- If a > 1 , the signal y(t) is a compressed version of x(t).
- If 0 < a < 1, the signal y(t) is an expanded (stretched) version of x(t).

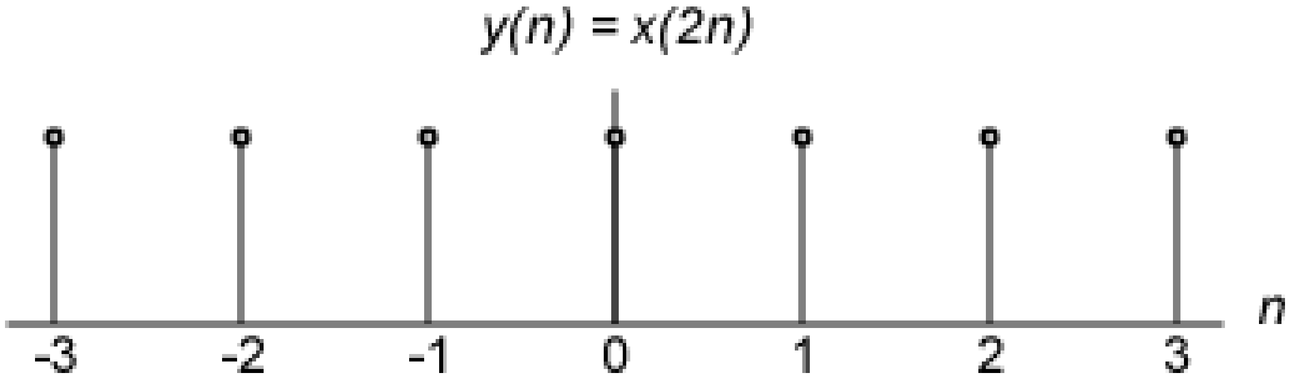
- In DT case , y[n] = x[kn],

k > 0

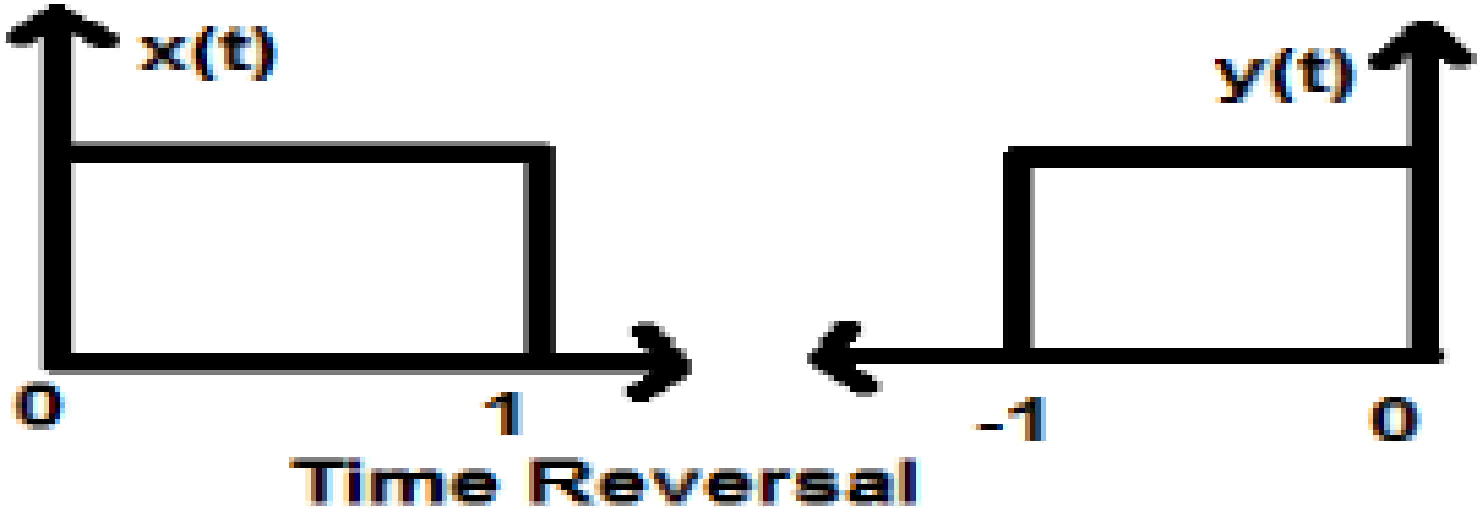
- The samples x[kn] for n = ± 1, ±3, . . . are lost because putting it = 2 in x[kn] causes these samples to be skipped.



(a)



(b)



- **<u>Time Reversal</u>**:
- Let x(t) denote a continuous-time signal. Let y(t) denote the signal obtained by replacing time t with –t ; that is, y(t) = x(-t)

EXAMPLE 1.3 REFLECTION Consider the triangular pulse $x(t)$ shown in Fig. 1.22(a). Find the reflected version of $x(t)$ about the amplitude axis (i.e., the origin).

Solution: Replacing the independent variable t in $x(t)$ with $-t$, we get $y(t) = x(-t)$, as shown in the figure.

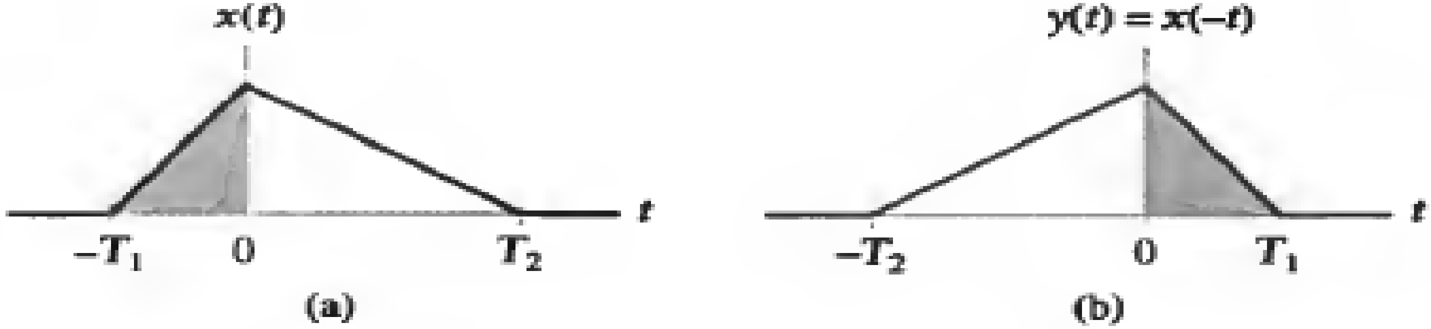


FIGURE 1.22 Operation of reflection: (a) continuous-time signal $x(t)$ and (b) reflected version of $x(t)$ about the origin.

Note that for this example, we have

$$x(t) = 0 \quad \text{for } t < -T_1 \text{ and } t > T_2.$$

Correspondingly, we find that

$$y(t) = 0 \quad \text{for } t > T_1 \text{ and } t < -T_2.$$

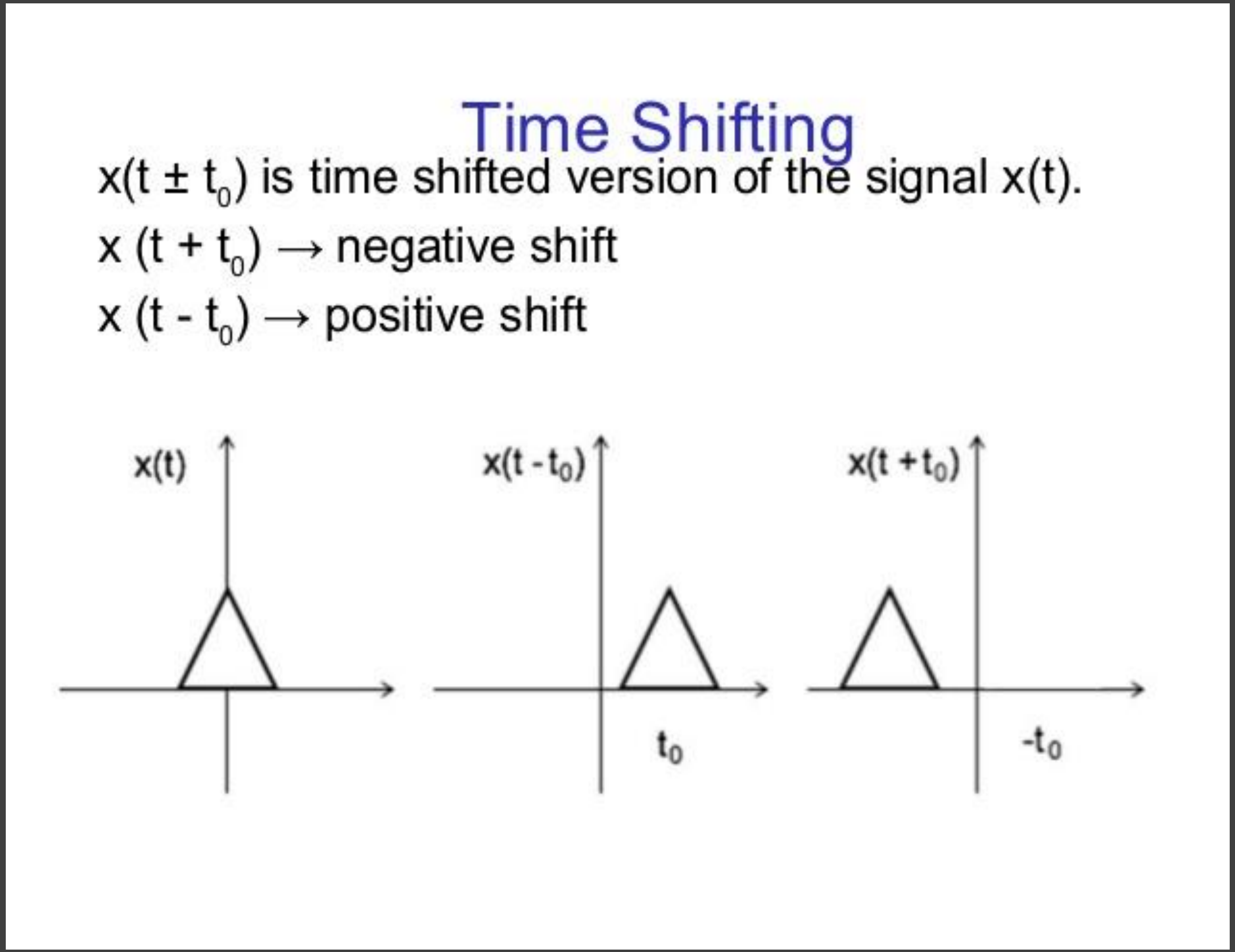
▶ **Problem 1.10** Let

$$x[n] = \begin{cases} n & \text{for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}.$$

Determine $y[n] = x[2n]$.

Answer: $y[n] = 0 \quad \text{for all } n.$

- **Time shifting** :Let x(t) denote a continuous-time signal. Then the time-shifted version of x(t) is defined by
- y(t) = x(t - to), where to is the time shift.
- If to > 0, the waveform of y(t) is obtained by shifting x(t) toward the right, relative to the time axis.
- If to < 0, x(t) is shifted to the left.



▶ **Problem 1.11** The discrete-time signal

$$x[n] = \begin{cases} 1, & n = 1 \\ -1, & n = -1 \\ 0, & n = 0 \text{ and } |n| > 1 \end{cases}.$$

Find the composite signal

$$y[n] = x[n] + x[-n].$$

Answer: $y[n] = 0$ for all integer values of n.

▶ **Problem 1.12** Repeat Problem 1.11 for

$$x[n] = \begin{cases} 1, & n = -1 \text{ and } n = 1 \\ 0, & n = 0 \text{ and } |n| > 1 \end{cases}.$$

Answer: $y[n] = \begin{cases} 2, & n = -1 \text{ and } n = 1 \\ 0, & n = 0 \text{ and } |n| > 1 \end{cases}.$

▶ **Problem 1.13** The discrete-time signal

$$x[n] = \begin{cases} 1, & n = 1, 2 \\ -1, & n = -1, -2 \\ 0, & n = 0 \text{ and } |n| > 2 \end{cases}.$$

Find the time-shifted signal $y[n] = x[n + 3]$.

Answer:
$$y[n] = \begin{cases} 1, & n = -1, -2 \\ -1, & n = -4, -5 \\ 0, & n = -3, n < -5, \text{ and } n > -1 \end{cases}.$$

Precedence rule:

Let $y(t)$ denote a continuous-time signal that is derived from another continuous-time signal $x(t)$ through a combination of time shifting and time scaling; that is , $y(t) = x(at - b)$

1) Time shifting operation

2) Time scaling

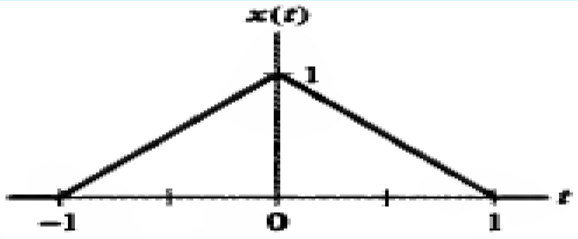
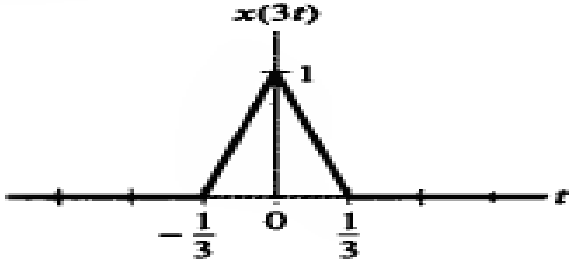


FIGURE 1.26 Triangular pulse for Problem 1.14.

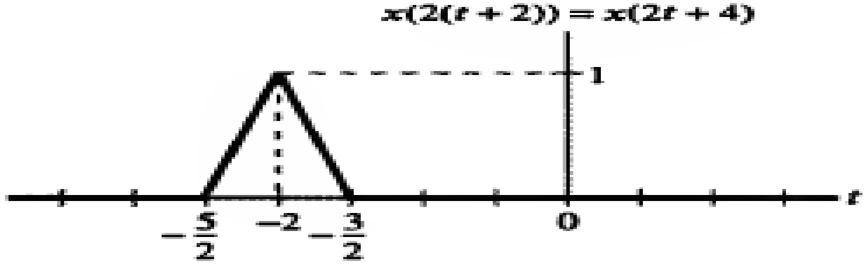
► **Problem 1.14** A triangular pulse signal $x(t)$ is depicted in Fig. 1.26. Sketch each of the following signals derived from $x(t)$:

(a) $x(3t)$
(b) $x(3t + 2)$
(c) $x(-2t - 1)$
(d) $x(2(t + 2))$
(e) $x(2(t - 2))$
(f) $x(3t) + x(3t + 2)$

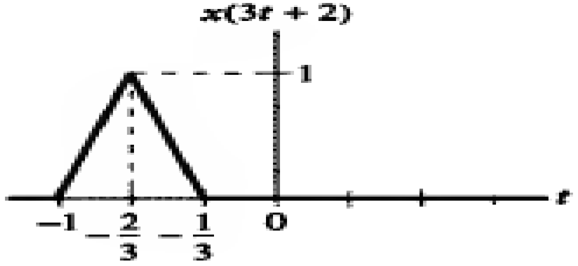
Answers:



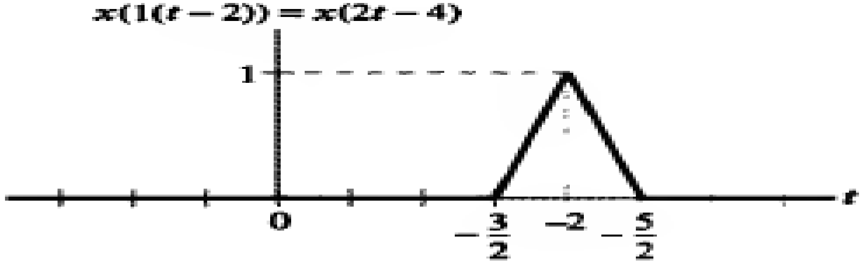
(a)



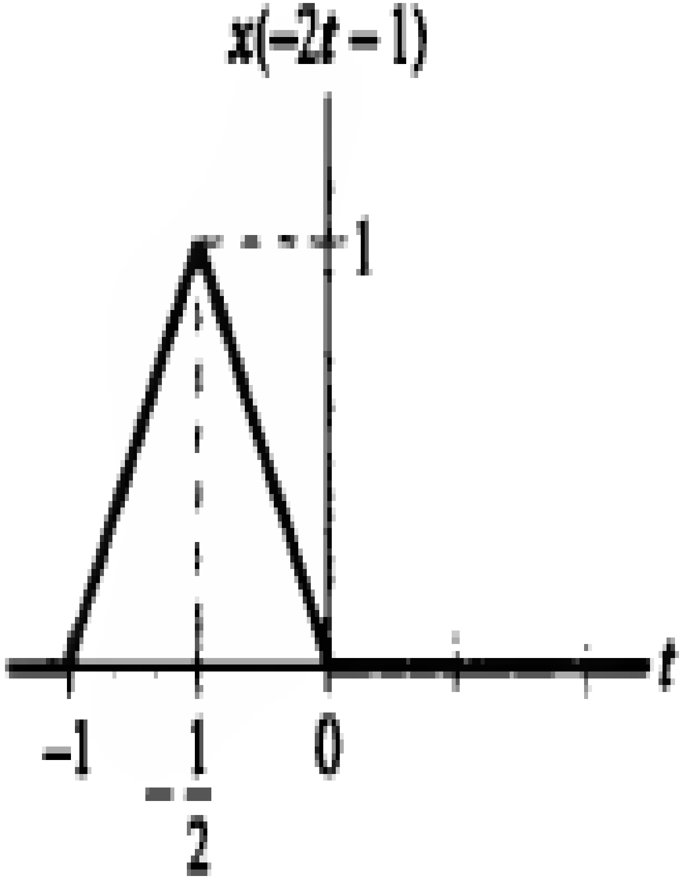
(d)



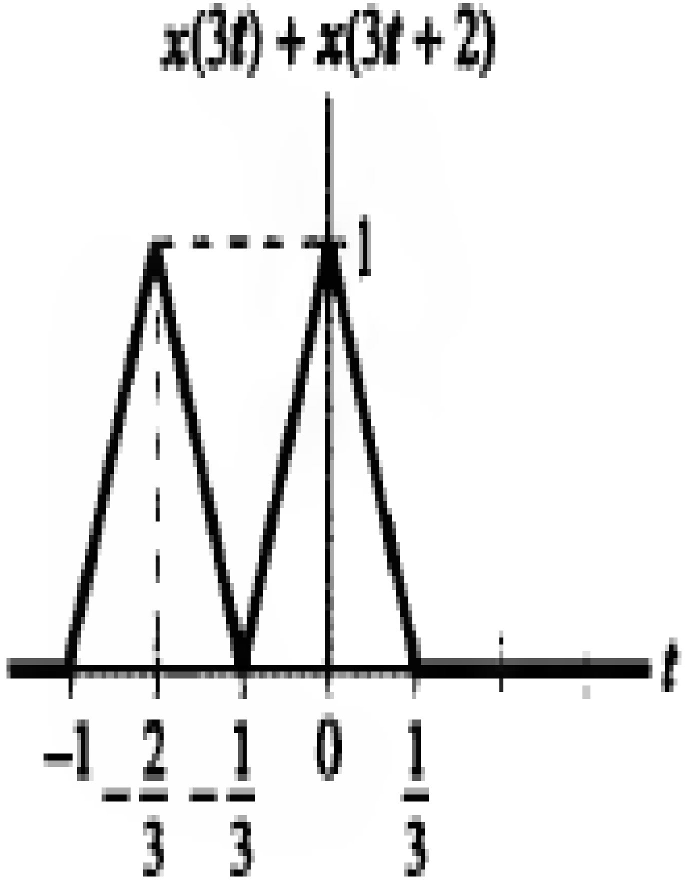
(b)



(e)



(c)



(f)

► **Problem 1.15** Consider a discrete-time signal

$$x[n] = \begin{cases} 1, & -2 \le n \le 2 \\ 0, & |n| > 2 \end{cases}.$$

Find $y[n] = x[3n - 2]$.

Answer: $y[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases}$. ◄

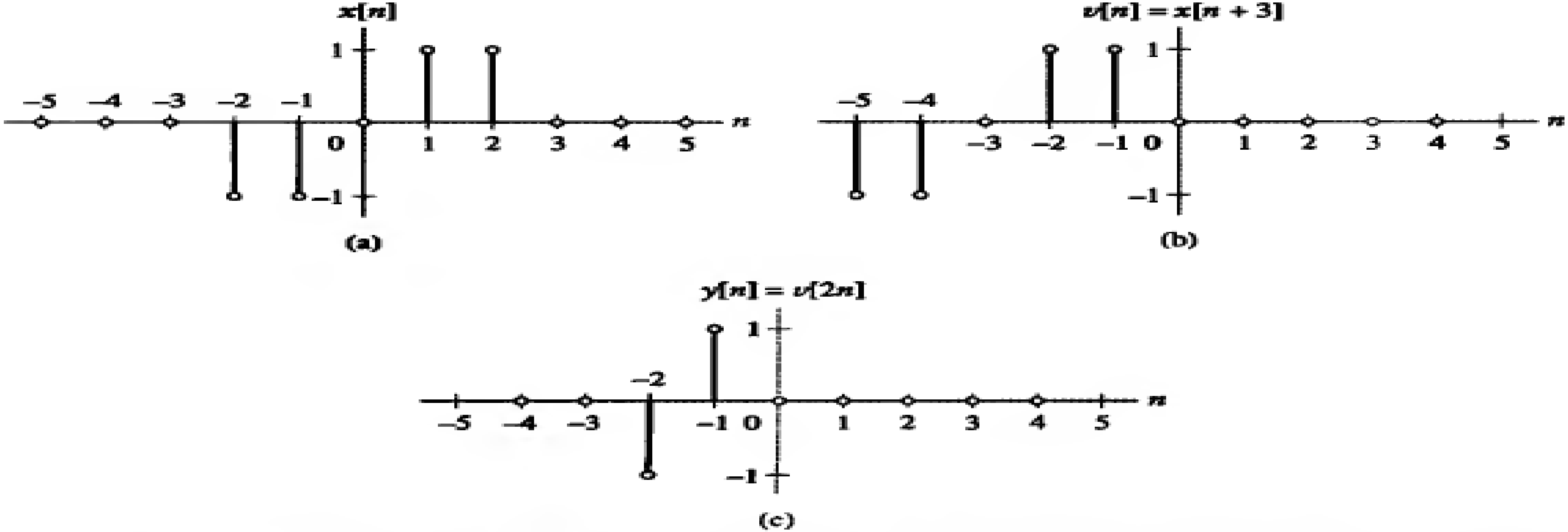
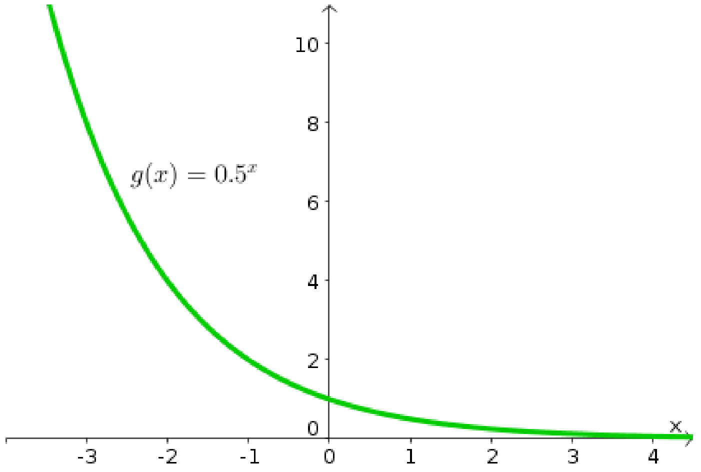


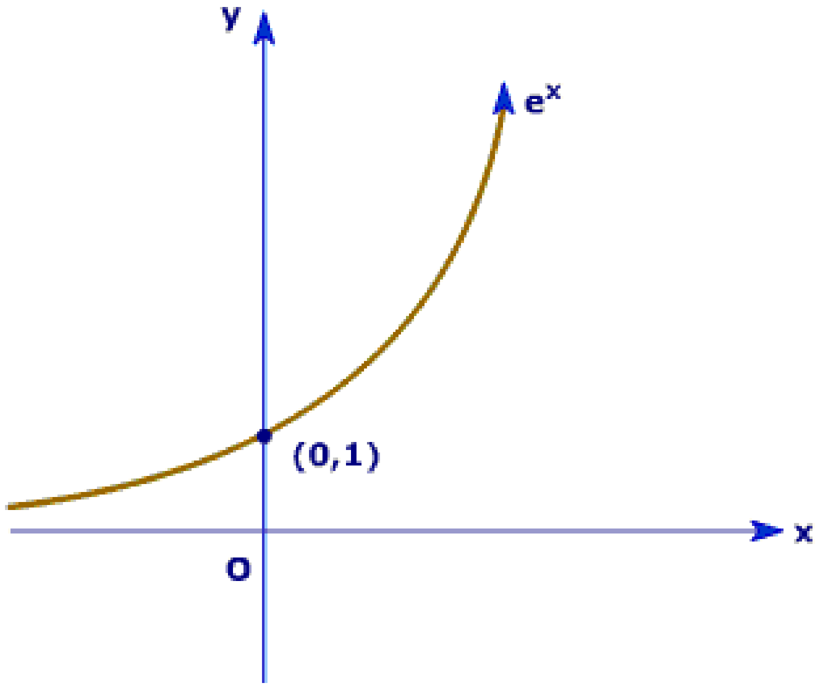
FIGURE 1.27 The proper order of applying the operations of time scaling and time shifting for the case of a discrete-time signal. (a) Discrete-time signal $x[n]$, antisymmetric about the origin. (b) Intermediate signal $v(n)$ obtained by shifting $x[n]$ to the left by 3 samples. (c) Discrete-time signal $y[n]$ resulting from the compression of $v[n]$ by a factor of 2, as a result of which two samples of the original $x[n]$, located at $n = -2, +2$, are lost.

- **Elementary Signals**:
- **Exponential Signals**:
- A real exponential signal, in its most general form, is written as x(t) = Be^(at)

where both B and a are real parameters.

- B is the amplitude of the exponential signal measured at time t = 0.
- we may identify two special cases:
- Decaying exponential, for which a < 0
- Growing exponential, for which a > 0





▶ **Problem 1.16** Determine the fundamental period of the sinusoidal signal

$$x[n] = 10\cos\left(\frac{4\pi}{31}n + \frac{\pi}{5}\right).$$

Answer: $N = 31$ samples. ◀

▶ **Problem 1.17** Consider the following sinusoidal signals:

(a) $x[n] = 5\sin[2n]$
(b) $x[n] = 5\cos[0.2\pi n]$
(c) $x[n] = 5\cos[6\pi n]$
(d) $x[n] = 5\sin[6\pi n/35]$

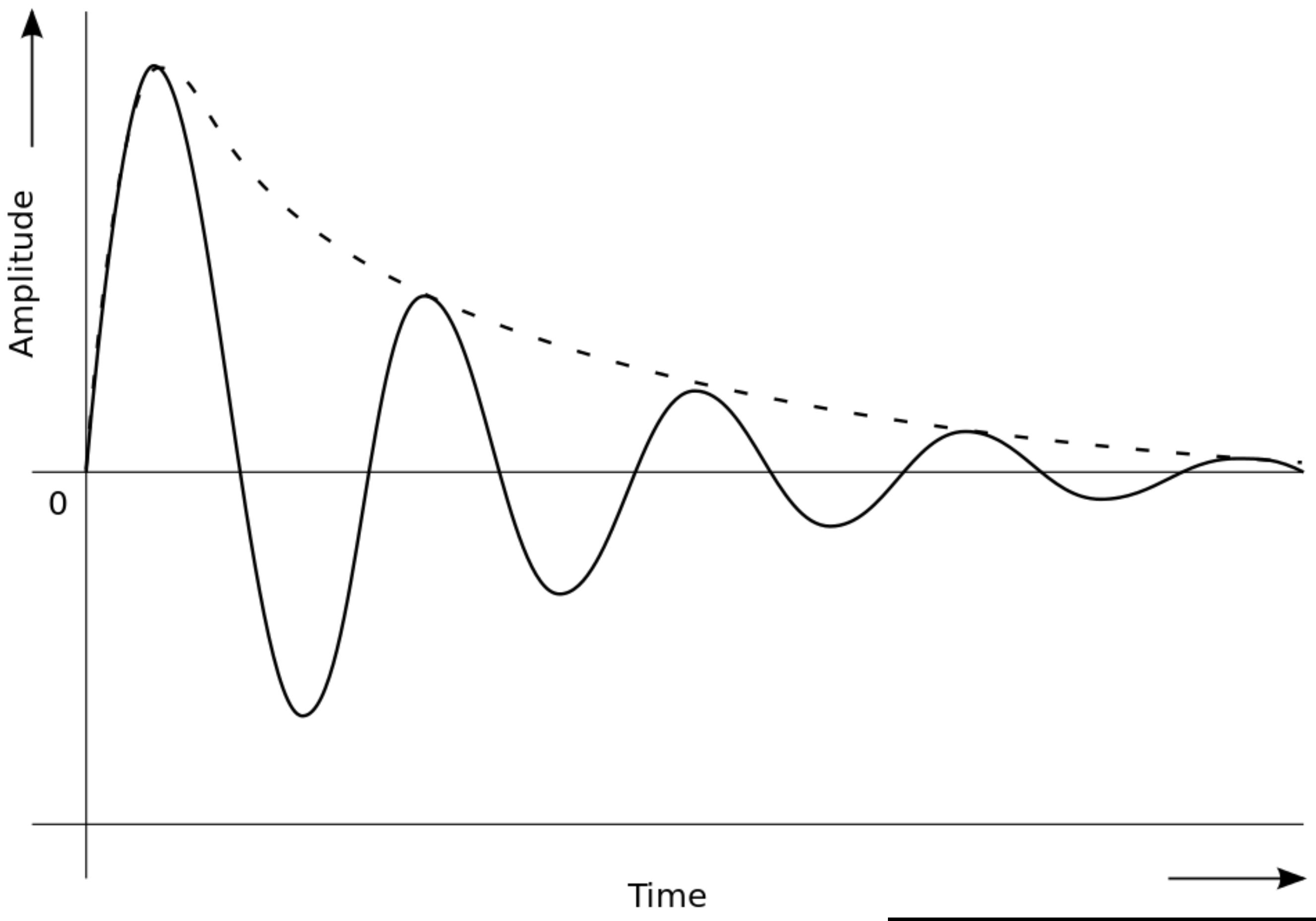
Determine whether each $x(n)$ is periodic, and if it is, find its fundamental period.

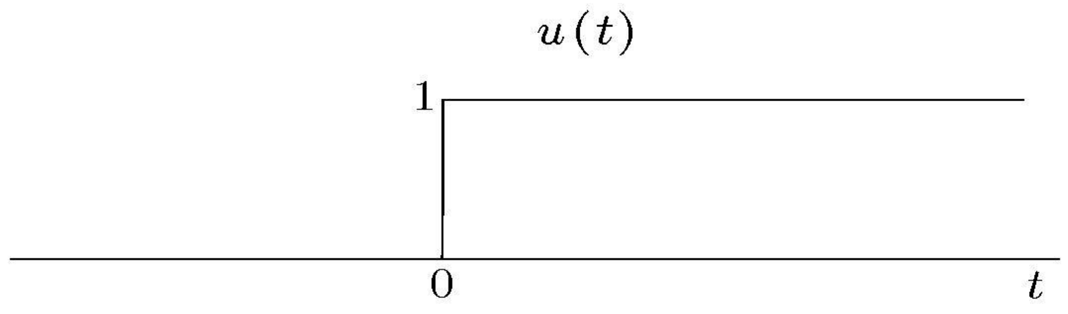
Answers: (a) Nonperiodic. (b) Periodic, fundamental period = 10. (c) Periodic, fundamental period = 1. (d) Periodic, fundamental period = 35. ◀

▶ **Problem 1.18** Find the smallest angular frequencies for which discrete-time sinusoidal signals with the following fundamental periods would be periodic: (a) $N = 8$, (b) $N = 32$, (c) $N = 64$, and (d) $N = 128$.

Answers: (a) $\Omega = \pi/4$. (b) $\Omega = \pi/16$. (c) $\Omega = \pi/32$. (d) $\Omega = \pi/64$. ◀

- **Exponentially damped sinusoidal signal**:
- The multiplication of a sinusoidal signal by a real-valued decaying exponential signal results in a new signal referred to as an exponentially damped sinusoidal signal.
- x(t) = Ae^ (-at) sin(ωt + ∅), a > 0.
- **Complex valued exponential signal**:
- x(t) = Ae^(at+jwt) , a>0





Step function :

- The continuous-time version of the unit-step function is defined by
- u(t)= {1 , t ≥ 0

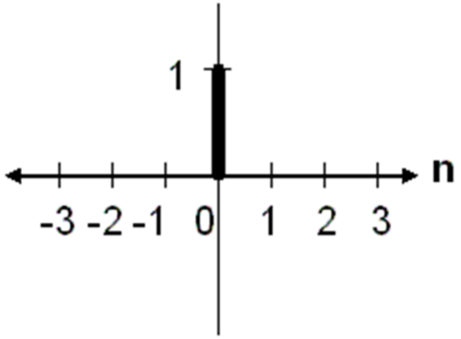
  0 , t< 0 }

For DT

- u[n]= {1 , n ≥ 0

  0 , n < 0 }

- **<u>Impulse function</u>** :
- The discrete-time version of the unit impulse is defined by

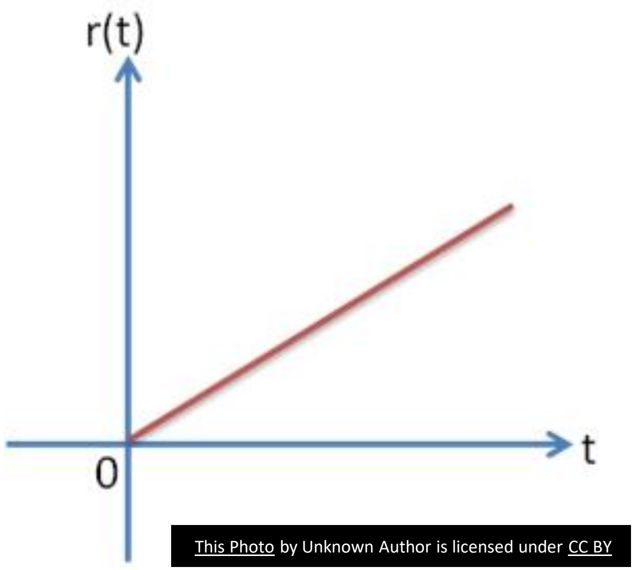


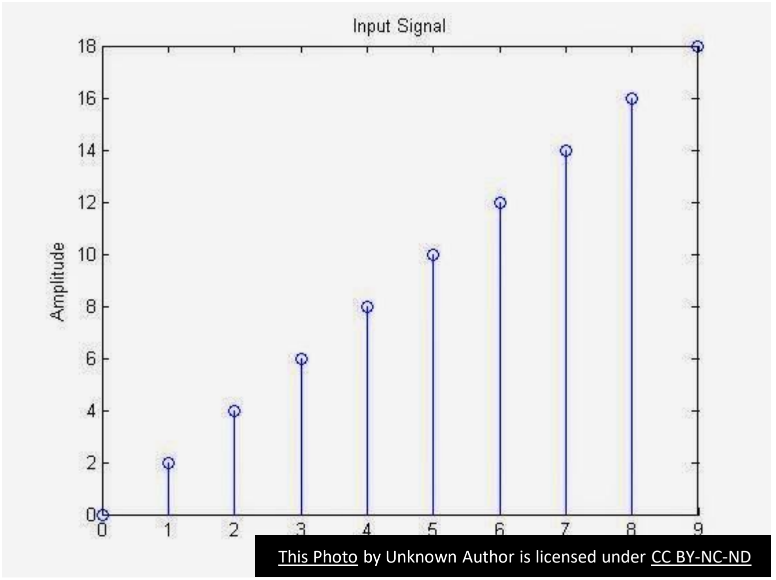
$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0. \end{cases}$$

- **the total area under the unit impulse is unity**. The impulse ∂(t) is also referred to as the Dirac delta function.
- $\int_{-\infty}^{\infty} \boldsymbol{\partial}\textbf{(t)}\, \boldsymbol{dt} = \mathbf{1}$
- Differentitation of unit step function u(t) gives impulse function $\boldsymbol{\partial}$**(t) =**$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\boldsymbol{u}(\boldsymbol{t})$
- Step function u(t) is the integral of the impulse function ∂(t) with respect to time t **: u(t) =** $\int_{-\infty}^{\boldsymbol{t}} \boldsymbol{\partial}\textbf{(t)}\, \boldsymbol{dt}$
- the unit impulse ∂(t) is an even function of time f **$\boldsymbol{\partial}$(−t) = $\boldsymbol{\partial}$(t)**
- Let x(t) be such a function, and consider the product of x(t) and the time-shifted delta function $\partial(t - t_0)$. We can express the integral of this product as

$\int_{-\infty}^{\infty} \boldsymbol{x}(\boldsymbol{t})\boldsymbol{\delta}(\boldsymbol{t} - \mathbf{t_0})dt = x(t_0)$

Another important property is **time scaling property $\boldsymbol{\partial}$(at) $= \frac{\mathbf{1}}{\boldsymbol{a}}$ $\boldsymbol{\partial}$(t) , a>0**





- **Ramp function** : integral of the step function u(t) is a ramp function of unit slope.

For CT : r(t) = { t , t$\geq$ 0

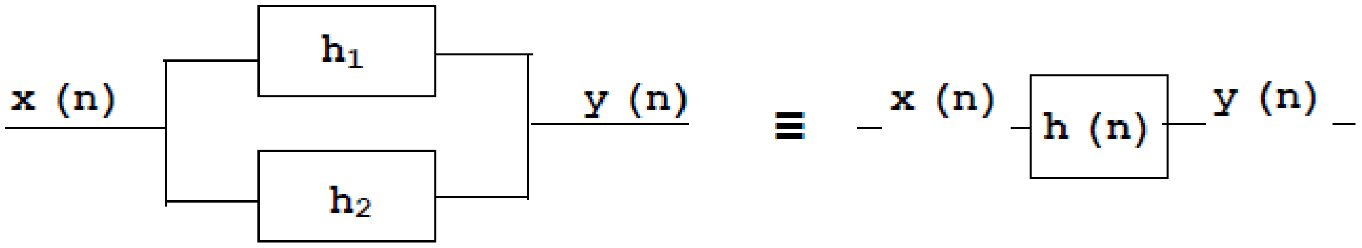
0 , t$<$ 0}

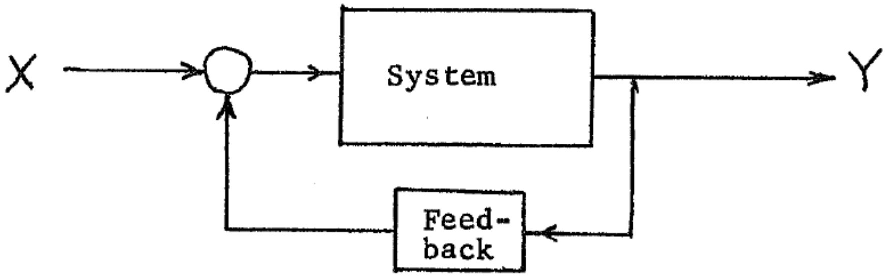
r(t) = tu(t) (Equivalent expression)

For DT: r[n] = { n , n $\geq$ 0

0 , n $<$ 0 }

r(n) = nu(n) (Equivalent expression)





- **Interconnections of systems** :
- When input is given to the system then the o/p can be written as y(t) = H{x(t)}.

 where H denotes the operation of the system on the input signal.

- System can be connected in parallel , cascade and feedback connections

- **Properties of the system** :

1. **Stability** : A system is said to be bounded-input, bounded-output (BIBO) stable if and only if every bounded input results in a bounded output. The output of such a system does not diverge if the input does not diverge.

- Both output and input should follow the condition $|y(t)| < \infty$, $|x(t)| < \infty$
- If the system doesn't follow this condition then it is **unstable system**.

2. **Memory** : A system is said to possess memory if its output signal depends on past or future values of the input signal.

eg : $y(t) = x(t) + x(t-2) + x(t+4)$

Memoryless : a system is said to be memoryless if its output signal depends only on the present value of the input signal.

Eg: $y(t) = x(t)$

EXAMPLE 1.13 MOVING-AVERAGE SYSTEM (CONTINUED) Show that the moving-average system described in Example 1.12 is BIBO stable.

Solution: Assume that

$$|x[n]| < M_x < \infty \quad \text{for all } n.$$

Using the given input–output relation

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]),$$

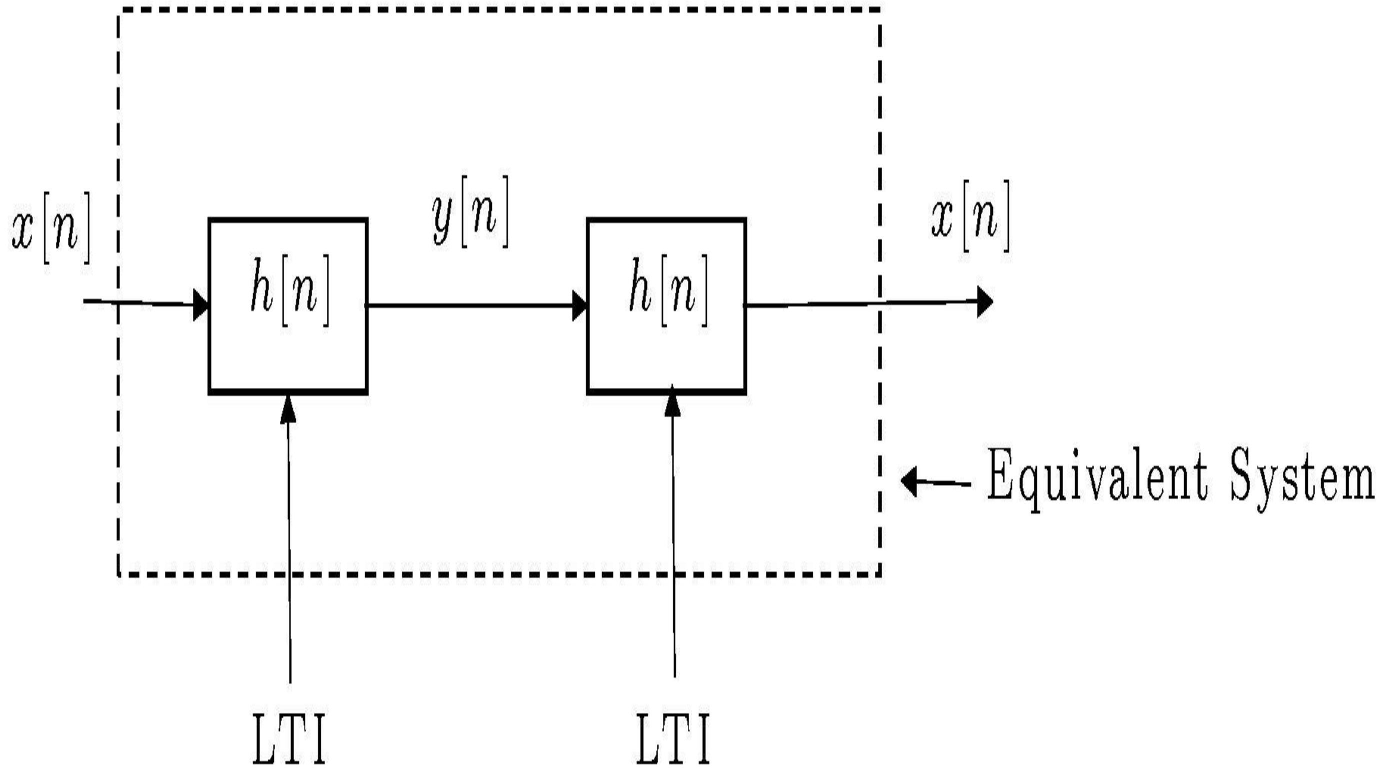
we may write

$$\begin{aligned} |y[n]| &= \frac{1}{3}|x[n] + x[n-1] + x[n-2]| \\ &\le \frac{1}{3}(|x[n]| + |x[n-1]| + |x[n-2]|) \\ &\le \frac{1}{3}(M_x + M_x + M_x) \\ &= M_x. \end{aligned}$$

Hence, the absolute value of the output signal $y[n]$ is always less than the maximum absolute value of the input signal $x[n]$ for all n, which shows that the moving-average system is stable. ■

- **3. <u>Causal</u>:** A system is said to be causal if the present value of the output signal depends only on the present or past values of the input signal.
- Eg: $y(t) = x(t) + x(t-2) + x(t-4)$
- **<u>Non Casual</u>**: the output signal of a noncausal system depends on one or more future values of the input signal.
- Eg : $y(t) = x[n] + x[n+1] + x[n-2]$
- **4. <u>Invertibility</u>**:
- A system is said to be invertible if the input of the system can be recovered from the output. We may view the set of operations needed to recover the input as a second system connected in cascade with the given system, such that the output signal of the second system is equal to the input signal applied to the given system.
- the operator H represent a continuous-time system, with the input signal x(t) producing the output signal y(t). Let the output signal y(t) be applied to a second continuous-time system represented by the operator Hinv
- HinvH = I (where I is the identity operator)

- Hinv is the inverse operator and the system associated with it is called the inverse system.



- **Time invariance** : A system is said to be time invariant if a time delay or time advance of the input signal leads to an identical time shift in the output signal.
- Otherwise, it is time variant.

- **Eg 1** : $y(t) = tx(t)$. Check for time invariance.

Solution : 1) Delay the input by m

$y(t)= tx(t-m)$

2)Delay the output by m

$y(t-m)= (t-m)x(t-m)$

Both are not equal hence it is time variant.

Eg2: $y(t)=x(t)$

Solution : $y(t) = x(t-m)$

$y(t-m) = x(t-m)$

Both are equal hence it is time invariant.

Linearity : A system is said to be linear interms of i/p and o/p it it satisfies the following 2 conditions:

Superposition : if i/p is x1(t) then o/p is y1(t)

if i/p is x2(t) then o/p is y2(t)

If i/p x(t) = x1(t)+ x2(t) then o/p y(t) = y1(t) + y2(t) should be satisfied.

Homogenity : if i/p is x(t) then o/p is y(t)

If the i/p is scaled to x(at) then o/p is y(at) should be satisfied.

When both the above mentioned conditions are not satisfied then the system is said to be non_linear.

- **Eg 1** : Consider a discrete-time system described by the input-output relation
- y[n] = nx[n]. Show that the system is linear.
- **Solution**: y1[n] = nx1[n]

  y2[n] = nx2[n]

y3[n] = nx1[n] + nx2[n] = n(x1[n]+x2[n]) ----1

If i/p x3[n] = x1[n]+x2[n]

y3[n] = n(x1[n]+x2[n])-----2

Equation 1 and 2 are same

Hence the system is **linear**