Module -7
Properties of Fourier Series and Complex Fourier Spectrum.

Objective: To understand the change in Fourier series coefficients due to different signal operations and to plot complex Fourier spectrum.

Introduction:

The Continuous Time Fourier Series is a good analysis tool for systems with periodicexcitation. Understanding properties of Fourier series makes the work simple in calculating the Fourier series coefficients in the case when signals modified by some basic operations.

Graphical representation of a periodic signal in frequency domain represents Complex Fourier Spectrum.

Description:

Properties of continuous- time Fourier series

The Fourier series representation possesses a number of important properties that are useful for various purposes during the transformation of signals from one form to other . Some of the properties are listed below.

[$x_1(t)$ and $x_2(t)$] are two periodic signals with period T and with Fourier series coefficients C_n and D_n respectively.

1) **Linearity property**

The linearity property states that, if

$$x_1(t)\overset{Fs}{\leftrightarrow}C_n \text{ and } x_2(t)\overset{Fs}{\leftrightarrow}D_n$$

then $$Ax_1(t)+Bx_2(t)\overset{Fs}{\leftrightarrow}AC_n+BD_n$$

proof: From the definition of Fourier series, we have

$$FS[Ax_1(t)+Bx_2(t)]=\frac{1}{T}\int_{t_0}^{t_0+T}[Ax_1(t)+Bx_2(t)]e^{-jn\omega_0 t}dt$$

$$=A\left(\frac{1}{T}\int_{t_0}^{t_0+T}x_1(t)e^{-jn\omega_0 t}dt\right)+B\left(\frac{1}{T}\int_{t_0}^{t_0+T}x_2(t)e^{-jn\omega_0 t}dt\right)$$

$$=AC_n+BD_n$$

or $$\boxed{Ax_1(t)+Bx_2(t)\overset{Fs}{\leftrightarrow}AC_n+BD_n}$$ proved

2) **Time shifting property**

The time shifting property states that,if

$$x(t)\overset{Fs}{\leftrightarrow}C_n$$

then $$x(t-t_0)\overset{Fs}{\leftrightarrow}e^{-jn\omega_0 t_0}C_n$$

proof: From the definition of Fourier series, we have

$$x(t)=\sum_{n=-\infty}^{\infty}C_n\,e^{jn\omega_0 t}$$

$$x(t-t_0)=\sum_{n=-\infty}^{\infty}C_n\,e^{jn\omega_0 (t-t_0)}$$

$$=\sum_{n=-\infty}^{\infty}[C_n\,e^{-jn\omega_0 t}]e^{jn\omega_0 t}$$

$$=FS^{-1}[C_n e^{-jn\omega_0 t}]$$

$$\boxed{x(t-t_0)\overset{Fs}{\leftrightarrow}e^{-jn\omega_0 t_0}C_n}$$

Or proved

3) Time reversal property

The time reversal property states that,if

$$x(t)\overset{Fs}{\leftrightarrow}C_n$$

then $$x(-t)\overset{Fs}{\leftrightarrow}C_{-n}$$

proof: From the definition of Fourier series, we have

$$x(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\,\omega_0 t}$$

$$x(-t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\,\omega_0 (-t)}$$

substituting n = -p in the right hand side, we get

$$x(-t)=\sum_{p=\infty}^{-\infty} C_{-p}\, e^{j\,(-p)\omega_0 (-t)} = \sum_{p=-\infty}^{\infty} C_{-p}\, e^{jp\,\omega_0 t}$$

substituting p = n , we get

$$x(-t)=\sum_{n=-\infty}^{\infty} C_{-n}\, e^{jn\,\omega_0 t} = FS^{-1}[C_{-n}]$$

$$\boxed{x(-t)\overset{Fs}{\leftrightarrow}C_{-n}}$$ proved

4) Time scaling property

The time scaling property states that , if

$$x(t)\overset{Fs}{\leftrightarrow}C_n$$

then $$x(\alpha t)\overset{Fs}{\leftrightarrow}C_n \text{ with } \omega_0\rightarrow\alpha\omega_0$$

proof: From the definition of Fourier series, we have

$$x(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\,\omega_0 t}$$

$$x(\alpha t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\,\omega_0 \alpha t} = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\,(\omega_0 \alpha) t} = FS^{-1}[C_n]$$

where $\omega_0\rightarrow\alpha\omega_0$.

$$\boxed{x(\alpha t)\overset{Fs}{\leftrightarrow}C_n \text{ with fundamental frequency of } \alpha\omega_0}$$ proved.

5) Time differential property

The time differential property states that, if

$$x(t)\overset{Fs}{\leftrightarrow}C_n$$

then $$\frac{dx(t)}{dt}\overset{Fs}{\leftrightarrow}jn\omega_0 Cn$$

proof: From the definition of Fourier series, we have

$$x(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\,\omega_0 t}$$

Differentiating both sides with respect to t, we get

$$\frac{dx(t)}{dt}=\sum_{n=-\infty}^{\infty} C_n\, \frac{d(e^{jn\,\omega_0 t})}{dt} = \sum_{n=-\infty}^{\infty} C_n\,(jn\omega_0)\, e^{jn\,\omega_0 t}$$

$$=\sum_{n=-\infty}^{\infty} C_n\,(jn\omega_0)\, e^{jn\,\omega_0 t} = FS^{-1}[jn\omega_0 C_n]$$

$$\boxed{\frac{dx(t)}{dt}\overset{Fs}{\leftrightarrow}jn\omega_0 Cn}$$ Proved.

6) Time integration property

The time integration property states that,

if $\quad x(t)\overset{Fs}{\leftrightarrow}C_n$

then $\quad \int_{-\infty}^{t} x(\tau)d\tau \overset{FS}{\leftrightarrow} \frac{C_n}{jn\,\omega_0} \quad$ (if $C_0 = 0$)

proof: From the definition of Fourier series, we have

$$x(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\,\omega_0 t}$$

$$\int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{t}\left[\sum_{n=-\infty}^{\infty} C_n\, e^{jn\,\omega_0 \tau}\right]d\tau$$

Interchanging the order of integration and summation, we get

$$\int_{-\infty}^{t} x(\tau)d\tau = \sum_{n=-\infty}^{\infty} C_n \int_{-\infty}^{t} e^{jn\,\omega_0 \tau} d\tau$$

$$=\sum_{n=-\infty}^{\infty} C_n \left[\frac{e^{jn\,\omega_0 \tau}}{jn\,\omega_0}\right]_{-\infty}^{t}$$

$$=\sum_{n=-\infty}^{\infty} \left(\frac{C_n}{jn\,\omega_0}\right) e^{jn\,\omega_0 t} = FS^{-1}\left[\frac{C_n}{jn\,\omega_0}\right]$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \overset{FS}{\leftrightarrow} \frac{C_n}{jn\,\omega_0}, \quad C_0 = 0}$$

Proved.

7) Convolution theorem or property

The convolution theorem or property states that , The Fourier series of the convolution of two time domain functions $x_1(t)$ and $x_2(t)$ is equal to the multiplication of their Fourier series coefficients, i.e.”Convolution of two functions in time domain is equivalent to multiplication of their Fourier coefficients in frequency domain”.

The convolution property states that,if

$$x_1(t)\overset{Fs}{\leftrightarrow}C_n \text{ and } x_2(t)\overset{Fs}{\leftrightarrow}D_n$$

then $\quad x_1(t)*x_2(t)\overset{Fs}{\leftrightarrow}TC_nD_n$

proof: From the definition of Fourier series, we have

$$FS[x_1(t)*x_2(t)]=\frac{1}{T}\int_{t_0}^{t_0+T}[x_1(t)*x_2(t)]\,e^{-jn\,\omega_0 t}dt = \frac{1}{T}\int_{0}^{T}[x_1(t)*x_2(t)]\,e^{-jn\,\omega_0 t}dt$$

But from the convolution integral for a periodic signal, we have

$$x_1(t)*x_2(t)=\int_0^T x_1(\tau)*x_2(t-\tau)\,d\tau \quad \text{or} \quad x_1(t)*x_2(t)=\int_0^T x_1(t-\tau)*x_2(\tau)\,d\tau$$

Substituting back, we get

$$FS[x_1(t)*x_2(t)] = \frac{1}{T}\int_0^T\left(\int_0^T x_1(\tau)*x_2(t-\tau)\,d\tau\right)e^{-jn\,\omega_0 t}dt$$

Interchanging the order of integration, we get

$$FS[x_1(t)*x_2(t)] = \frac{1}{T}\int_0^T x_1(\tau)\left[\int_0^T x_2(t-\tau)\,e^{-jn\,\omega_0 t}dt\right]d\tau$$

Substituting t-**τ** =t' in RHS, we have dt=dt'. Substituting back, we get

$$FS[x_1(t)*x_2(t)] = \frac{1}{T}\int_0^T x_1(\tau)\left[\int_{-\tau}^{T-\tau} x_2(t')\,e^{-jn\,\omega_0 (t'+\tau)}dt'\right]d\tau$$

$$=T\left(\frac{1}{T}\int_0^T x_1(\tau)e^{-jn\,\omega_0 t}d\boldsymbol{\tau}\right)\left(\frac{1}{T}\int_0^T x_1(t')e^{-jn\,\omega_0 t'}dt'\right)$$

$$=T(C_n)(D_n)$$

0 to T or –**τ** to T-**τ** or t_0 to t_0+T will have the same period.

$$\boxed{x_1(t)*x_2(t)\overset{Fs}{\leftrightarrow}TC_nD_n}$$

Proved.

8) Modulation or Multiplication property

The Modulation or Multiplication property states that,if

$$x_1(t) \overset{Fs}{\leftrightarrow} C_n \text{ and } x_2(t) \overset{Fs}{\leftrightarrow} D_n$$

then

$$x_1(t)x_2(t) \overset{Fs}{\leftrightarrow} \sum_{l=-\infty}^{\infty} C_l D_{n-l}$$

proof: From the definition of Fourier series, we have

$$FS[x_1(t)x_2(t)] = \frac{1}{T}\int_{t_0}^{t_0+T} [x_1(t)x_2(t)]\, e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\left(\sum_{l=-\infty}^{\infty} C_l e^{jn\omega_0 t}\right) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\left(\sum_{l=-\infty}^{\infty} C_l e^{-j(n-l)\omega_0 t}\right) dt$$

Interchanging the order of integration and summation, we have

$$FS[x_1(t)x_2(t)] = \sum_{l=-\infty}^{\infty} C_l\left(\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t) e^{-j(n-l)\omega_0 t}\, dt\right) = \sum_{l=-\infty}^{\infty} C_l D_{n-l}$$

$$\boxed{x_1(t)x_2(t) \overset{Fs}{\leftrightarrow} \sum_{l=-\infty}^{\infty} C_l D_{n-l}}$$ Proved.

9) Conjugation and Conjugate symmetry property

If $x(t) \overset{Fs}{\leftrightarrow} C_n$

Then the Conjugation property states that

$$x^*(t) \overset{Fs}{\leftrightarrow} C_{-n}^* \quad \text{[for complex x(t)]}$$

proof: Conjugation property

From the definition of Fourier series, we have

$$FS[x^*(t)] = \frac{1}{T}\int_{t_0}^{t_0+T} x*(t)\, e^{-jn\omega_0 t} dt = \left(\frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{jn\omega_0 t} dt\right)^*$$

$$= \left(\frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-j(-n)\omega_0 t} dt\right)^* = [C_{-n}]^*$$

$$\boxed{x^*(t) \overset{Fs}{\leftrightarrow} C_{-n}^*}$$ Proved.

10) Parseval's Relation or Theorem or property

If $x_1(t) \overset{Fs}{\leftrightarrow} C_n$ [for complex $x_1(t)$]

And $x_2(t) \overset{Fs}{\leftrightarrow} D_n$ [for complex $x_2(t)$]

Then, the parsevals relation states that

$$\frac{1}{T}\int_{t_0}^{t_0+T} [x_1(t)x_2^*(t)]\, dt = \sum_{n=-\infty}^{\infty} C_n D_n^* \quad \text{[for complex } x_1(t) \text{ and } x_2(t)]$$

And parseval's identity states that

$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 \qquad \text{if } x_1(t) = x_2(t) = x(t)$$

Proof: parsevals relation

$$LHS = \frac{1}{T}\int_{t_0}^{t_0+T} [x_1(t)x_2^*(t)]\, dt = \frac{1}{T}\int_{t_0}^{t_0+T} \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right) x_2^* dt$$

Interchanging the order of integration and summation in the RHS , we have

$$\frac{1}{T}\int_{t_0}^{t_0+T} [x_1(t)x_2^*(t)]\, dt = \sum_{n=-\infty}^{\infty} C_n\left(\frac{1}{T}\int_{t_0}^{t_0+T} x_2^*(t) e^{jn\omega_0 t}\, dt\right)$$

$$= \sum_{n=-\infty}^{\infty} C_n\left(\frac{1}{T}\int_{t_0}^{t_0+T} x_2(t) e^{-jn\omega_0 t}\, dt\right)^* = \sum_{n=-\infty}^{\infty} C_n\, [D_n]^*$$

Proved

Parseval's identity

$$\boxed{\frac{1}{T}\int_{t_0}^{t_0+T}[x_1(t)x_2^*(t)]\,dt=\sum_{n=-\infty}^{\infty} C_n D_n^*}$$

If $x_1(t) = x_2(t) = x(t)$, then

$$\frac{1}{T}\int_{t_0}^{t_0+T}[x(t)x^*(t)]\,dt=\sum_{n=-\infty}^{\infty} C_n C_n^*$$

Since $|x(t)|^2 = x(t)x^*(t)$ and $|C_n|^2 = C_n C_n^*$, substituting these values in above equation, we get

$$\boxed{\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2\,dt =\sum_{n=-\infty}^{\infty} |C_n|^2}$$

Proved.

Complex Fourier Spectrum

The Fourier spectrum of a periodic signal x(t) is a plot of its Fourier coefficients versus frequency ω. It is in two parts : (a) Amplitude spectrum and (b) phase spectrum. The plot of the amplitude of Fourier coefficients verses frequency is known as the amplitude spectra, and the plot of the phase of Fourier coefficients verses frequency is known as phase spectra. The two plots together are known as Fourier frequency spectra of x(t).This type of representation is also called frequency domain representation. The Fourier spectrum exists only at discrete frequencies $n\omega_o$, where n=0,1,2,….. Hence it is known as discrete spectrum or line spectrum. The envelope of the spectrum depends only upon the pulse shape, but not upon the period of repetition.

The trigonometric representation of a periodic signal x(t) contains both sine and cosine terms with positive and negative amplitude coefficients (a_n and b_n) but with no phase angles.

The cosine representation of a periodic signal contains only positive amplitude coefficients with phase angle θ_n. Therefore, we can plot amplitude spectra(A_n versus ω)and phase spectra (θ_n versus ω). Since, in this representation, Fourier coefficients exist only for positive frequencies, this spectra is called single-sided spectra.

The exponential representation of a periodic signal x(t) contains amplitude coefficients C_n Which are complex. Hence, they can be represented by magnitude and phase. Therefore, we can plot two spectra, the magnitude spectrum($|C_n|$ versus ω) and phase spectrum ($\lfloor C_n$ versus ω). The spectra can be plotted for both positive and negative frequencies.Henceit is called two-sided spectra.

The below figure (a) represents the spectrum of a trigonometric Fourier series extending from 0 to∞,producing a one-sided spectrum as no negative frequencies exist here. The figure (b) represents the spectrum of a complex exponential Fourier series extending from -∞to∞,producing a two-sided spectrum.

The amplitude spectrum of the exponential Fourier series is symmetrical Fourier series is symmetrical about the vertical axis. This is true for all periodic functions.

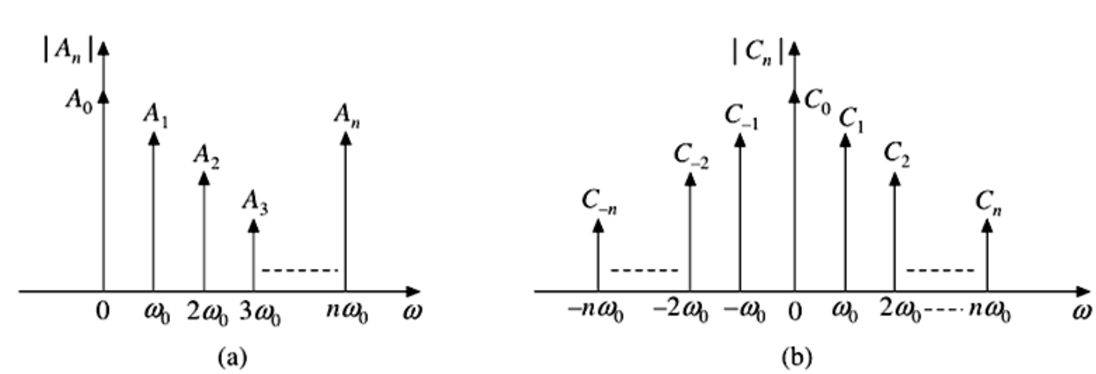


Fig: Complex frequency spectrum for (a) Trigonometric Fourier series and (b) complex exponential Fourier series.

If C_n is a general complex number, then

$$C_n = |C_n|\, e^{j\theta_n}$$

$$C_{-n} = |C_n|\, e^{-j\theta_n}$$

$$C_n = |C_{-n}|$$

The magnitude spectrum is symmetrical about the vertical axis passing through the origin, and the phase spectrum is antisymmetrical about the vertical axis passing through the origin. So the magnitude spectrum exhibits even symmetry and phase spectrum exhibits odd symmetry.

When x(t) is real , then $C_{-n}=C_n^*$, the complex conjugate of C_n.

Solved Problems:

Problem 1: Show that the magnitude spectrum of every periodic function is symmetrical about the vertical axis passing through the origin and the phase spectrum is antisymmetrical about the vertical axis passing through the origin.

Solution: The coefficient C_n of exponential Fourier series is given by

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega_0 t}dt)$$

And

$$C_{-n} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{jn\omega_0 t}dt)$$

It is evident from these equations that the coefficients C_n and C_{-n} are complex conjugate of each other , that is

$$C_n = C_n^*$$

Hence $|C_n| = |C_{-n}|$

It, therefore, follows that the magnitude spectrum is symmetrical about the vertical axis passing through the origin, and hence is an even function of ω.

If C_n is real, then C_{-n} is also real and C_n is equal to C_{-n}. is complex, let

$C_n = |C_n|\ e^{j\theta_n}$

then $C_{-n} = |C_n|\ e^{-j\theta_n}$

The phase of C_n is θ_n;however , the phase of C_{-n} is $-\theta_n$.Hence , it is obvious that the phase spectrum is antisymmetrical, and the magnitude spectrum is symmetrical about the vertical axis passing through the origin.

Problem 2: With regard to Fourier series representation, justify the following statement: odd functions have only sine terms.

Solution: We know that the trigonometric Fourier series of a periodic function x(t) in any interval t_0 to t_0+T or 0 to T or $-\frac{T}{2}$ to $\frac{T}{2}$ is given by

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where $$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\ dt$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos n\omega_0 t\ \ dt$$

and $$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\omega_0 t\, dt$$

For an odd function

$$x(t) = -x(-t)$$

Also even part is zero, i.e. $x_e(t) = 0$ and $x(t) = x_0(t)$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\ dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_0(t)\ dt = 0$$

Since the integration of an odd function,over one cycle is always zero.

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos n\omega_0 t\ \ dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_0(t) \cos n\omega_0 t\ \ dt$$

Here $x_0(t)$ is odd and $\cos n\omega_0 t$ is even. So $x_0(t) \cos n\omega_0 t$, i.e. the product if an odd function and even function is odd. So integration over one cycle is zero. Therefore,

$a_n = 0$

Now, $$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\omega_0 t\, dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_0(t) \sin n\omega_0 t\, dt$$

Now, $x_0(t)$ is odd $\sin n\omega_0 t$ is also odd. Therefore, the product $x_0(t)\sin n\omega_0 t$ is even. So we have to evaluate the integral:

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\omega_0 t\, dt = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t) \sin n\omega_0 t\, dt$$

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Thus, the Fourier series of odd functions contains only sine terms.

Problem3: With regard to Fourier series representation, justify the following statement: Even functions have no sine terms.

Solution: We know that the trigonometric Fourier series of a periodic function x(t) in any interval t_0 to t_0+T or 0 to T or $-\frac{T}{2}$ to $\frac{T}{2}$ is given by

$$x(t)= a_0+\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where $$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos n\omega_0 t \; dt$$

and $$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\omega_0 t\, dt$$

For an even function

$$x(t) = x(-t)$$

Also odd part is zero, i.e. $x(t) = x_e(t)$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_e(t)\, dt = \frac{2}{T}\int_{0}^{\frac{T}{2}} x(t)\, dt$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos n\omega_0 t \; dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_e(t) \cos n\omega_0 t \; dt$$

x_e(t) is even and also $\cos n\omega_0 t$ is even. So, the product x_e(t)$\cos n\omega_0 t$ is even. So we have to evaluate the integral:

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_e(t) \cos n\omega_0 t \; dt = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t) \cos n\omega_0 t \; dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\omega_0 t\, dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_e(t) \sin n\omega_0 t\, dt$$

Here x_e(t) is even $\sin n\omega_0$tand is odd. So, the product $x_e(t) \sin n\omega_0 t$ is odd. Thus, the integral over one complete cycle is zero.

$b_n = 0$

$$x(t)= a_0+\sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

Thus, the Fourier series of even functions contains no sine terms.

Problem 4: With regard to Fourier series representation, justify the following statement: Functions with half wave symmetry have only odd harmonics.

Solution: We know that the trigonometric Fourier series of a periodic function x(t) in any interval t_0 to t_0+T or 0 to T or $-\frac{T}{2}$ to $\frac{T}{2}$ is given by

$$x(t)= a_0+\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where $a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt$

$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos n\omega_0 t\; dt$

and $b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin n\omega_0 \mathrm{t}\, dt$

For a half wave symmetric function,

$$x(t) = -x(t \pm \frac{T}{2})$$

Therefore, $a_0 = 0$

$a_n = \frac{2}{T}\int_0^{T} x(t)\cos n\omega_0 t\; dt$

$$= \frac{2}{T}\left[\int_0^{\frac{T}{2}} x(t)\cos n\omega_0 t\; dt + \int_{\frac{T}{2}}^{T} x(t)\cos n\omega_0 t\; dt\right]$$

To have the limits of second integration also from 0 to T/2 , change the variable t by t + (T/2) in the second integration.

$$a_n = \frac{2}{T}\left[\int_0^{\frac{T}{2}} x(t)\cos n\omega_0 t\; dt + \int_0^{\frac{T}{2}} x\left(t+\frac{T}{2}\right)\cos n\omega_0\left(t+\frac{T}{2}\right)\; dt\right]$$

But $x(t) = -x(t \pm \frac{T}{2})$

$$a_n = \frac{2}{T}\left[\int_0^{\frac{T}{2}} x(t)\cos n\omega_0 t\; dt + \int_0^{\frac{T}{2}} -x(t)\cos(n\omega_0 t + n\pi)\, dt\right]$$

$$= \frac{2}{T}\left[\int_0^{\frac{T}{2}} x(t)\cos n\omega_0 t - x(t)\cos(n\omega_0 t + n\pi)\right] dt$$

$$a_n = \begin{cases} 0 & for\ even\ n \\ \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos n\omega_0 t & for\ odd\ n \end{cases}$$

Now,

$b_n = \frac{2}{T}\int_0^{\frac{T}{2}} x(t)\sin n\omega_0 \mathrm{t}\, dt$

$$= \frac{2}{T}\left[\int_0^{\frac{T}{2}} x(t)\sin n\omega_0 \mathrm{t}\, dt + \int_{\frac{T}{2}}^{T} x(t)\sin n\omega_0 \mathrm{t}\, dt\right]$$

To have the limits of second integration also from 0 to T/2 change the variable t by t + (T/2) in the second integration.

$$b_n = \frac{2}{T}\left[\int_0^{\frac{T}{2}} x(t)\sin n\omega_0 \mathrm{t}\, dt + \int_0^{\frac{T}{2}} x\left(t+\frac{T}{2}\right)\sin n\omega_0\left(\mathrm{t}+\frac{T}{2}\right) dt\right]$$

But $x(t) = -x(t \pm \frac{T}{2})$

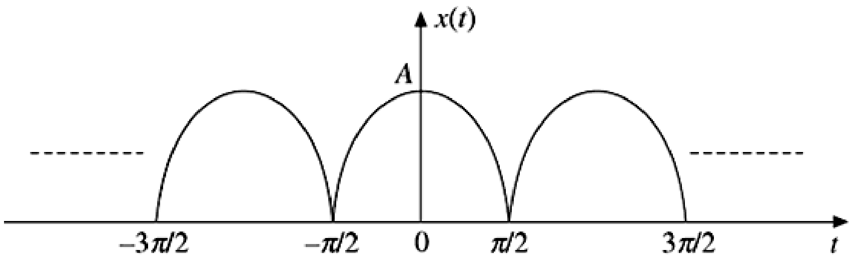
$b_n \; = \frac{2}{T}\,[\int_0^{\frac{T}{2}} x(t)\sin n\omega_0 \mathrm{t}\, dt + \int_0^{\frac{T}{2}} -x(t)\sin(\, n\omega_0 \mathrm{t} + \mathrm{n\pi})\, dt]$

$$= \frac{2}{T}\,[\int_0^{\frac{T}{2}} x(t)\sin n\omega_0 \mathrm{t}\, dt \text{ - } \int_0^{\frac{T}{2}} x(t)\sin n\omega_0 \mathrm{t}\cos \mathrm{n\pi})\, dt]$$

$$b_n = \begin{cases} 0 & for\ even\ n \\ \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin n\omega_0 \mathrm{t} & for\ odd\ n \end{cases}$$

Thus, the Fourier series of half wave symmetric function consists of only odd harmonics.

Problem 5: Draw the complex fourier spectrum of given figure.



Solution: It is a plot between C_n and $n\omega_0 = \omega$. As C_n is real, only amplitude plot is sufficient. The amplitude versus frequency plot, called the amplitude spectrum.

To draw the frequency spectrum,

$$\mathrm{x(t)} = \frac{2A}{\pi} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\mathrm{A}}{\pi}\,[\frac{(-1)^{\mathrm{n}}}{2\mathrm{n}+1} + \frac{(-1)^{\mathrm{n}+1}}{2\mathrm{n}-1}]\mathrm{e}^{\mathrm{j2nt}}$$

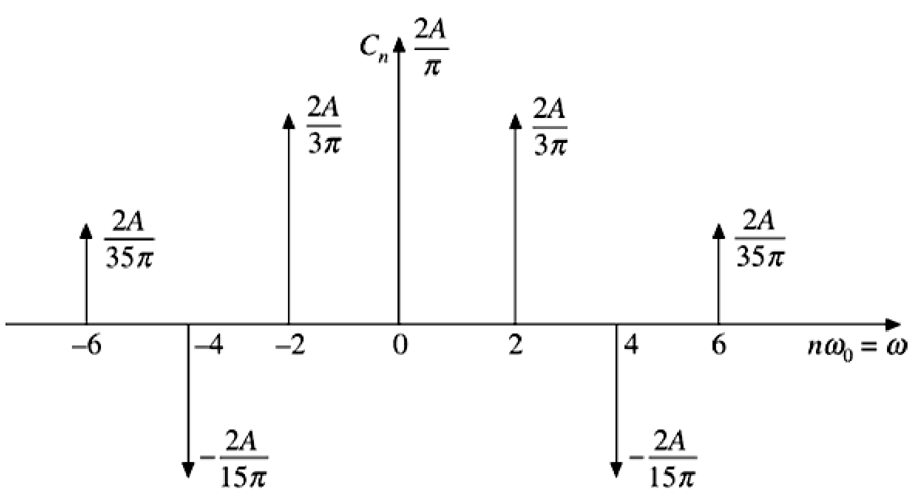
exponential Fourier series coefficients

$C_0 = \frac{2A}{\pi}$

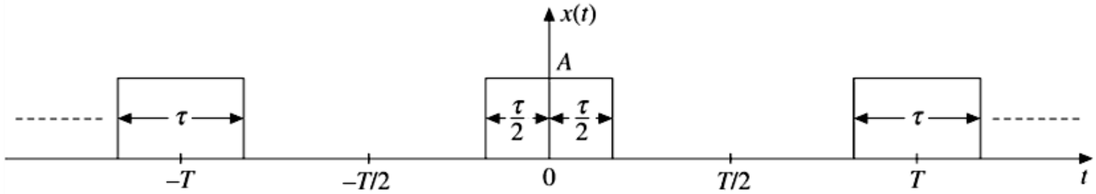
$C_1 = \frac{2A}{3\pi}$

$C_2 = \frac{2A}{15\pi}$

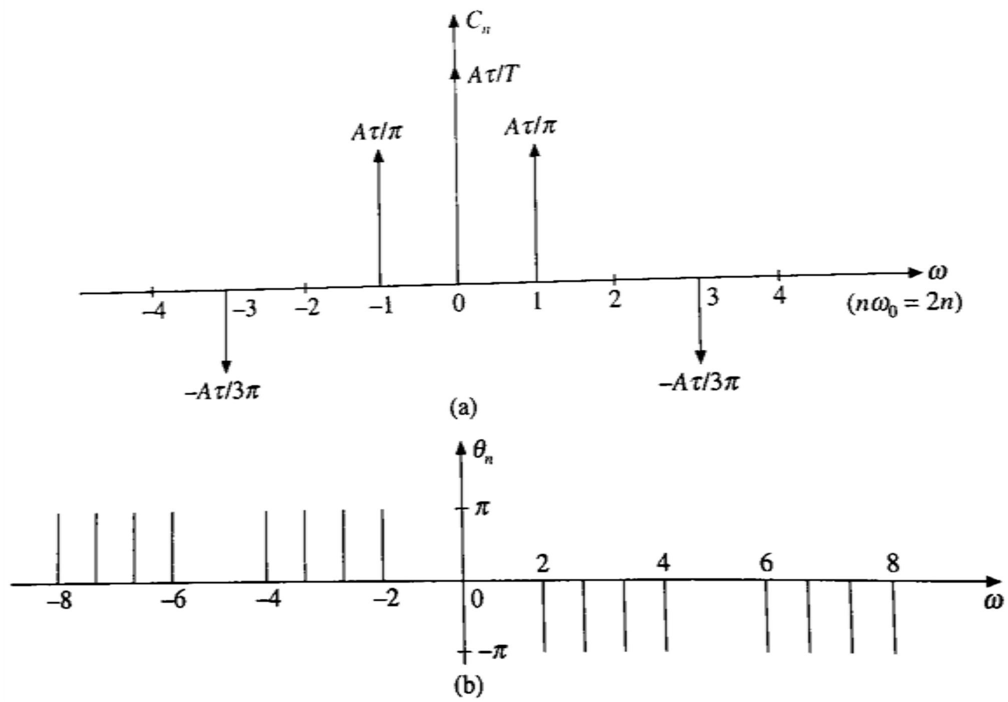
$C_3 = \frac{2A}{35\pi}$



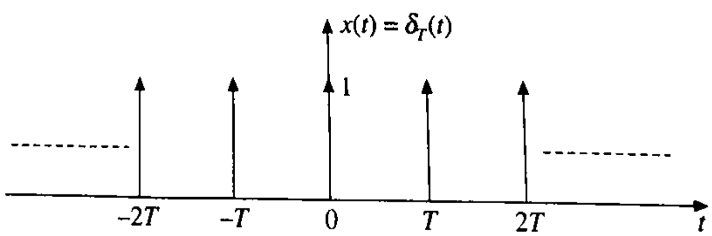
Problem 6: For the periodic gate function shown in below figure, plot the magnitude and phase spectra.



Solution: Amplitude and phase spectra are shown in below figures respectively.



Example 7: Find the complex exponential Fourier series and the trigonometric Fourier series of unit impulse train $\delta_T(t)$ shown in bellow figure.



Solution: The periodic waveform shown in above figure with period T can be expressed as

$$x(t) = \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Let $\quad t_0 = -\frac{T}{2}$

So $\quad t_0 + T = -\frac{T}{2}$

In one period, only $\delta(t)$ exists.

Fundamental frequency $\omega_0 = (2\pi/T)$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\, e^{-jn\omega_0 t}\,dt$$

$\delta(t) = 1$ at $t = 0$ $= 0$ elsewhere

$$= \frac{1}{T}e^0 = \frac{1}{T}$$

$C_n = \frac{1}{T}$

The exponential Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\left(\frac{2\pi}{T}\right)t} = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\left(\frac{2\pi}{T}\right)t}$$

To get the trigonometric Fourier coeifficients , we have

$a_0 = C_0 = \frac{1}{T}$

$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\ dt = \frac{2}{T}\int_0^T \delta(t)\cos n\omega_0 t\ dt = \frac{2}{T}$

$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\, dt = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\, dt = 0$

Therefore , the trigonometric Fourier series is

$$x(t) + a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = \frac{1}{T} + \frac{2}{T}\sum_{n=1}^{\infty} \cos n\omega_0 t$$

The complex Fourier series coefficients are

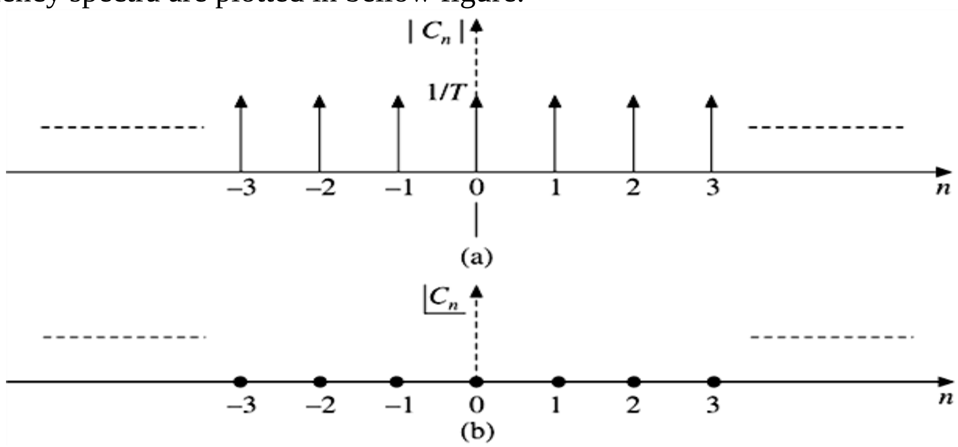
$C_n = \frac{1}{T}$ for all n

The magnitude and phase spectrum are

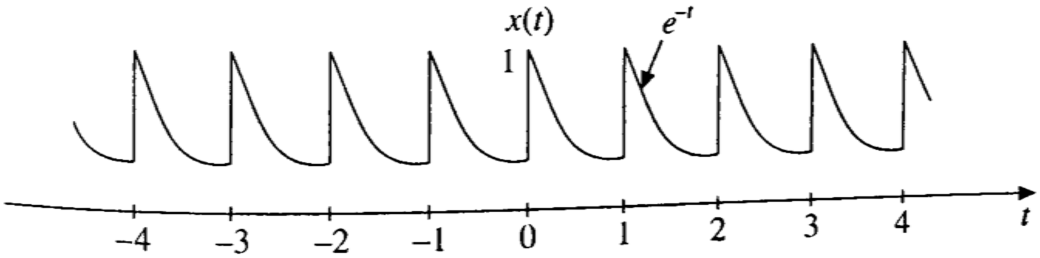
$|C_n| = \left|\frac{1}{T}\right|$ for all n

$\angle C_n = 0^o$ for all n

The frequency spectra are plotted in bellow figure.



Problem 8: Find the Fourier series of the signal $x(t)=e^{-t}$ with T = 1 sec as shown in bellow figure. Draw its magnitude and phase spectra.



Solution: Given signal $x(t)=e^{-t}$

Period T = 1 sec

Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$

The exponential Fourier series is

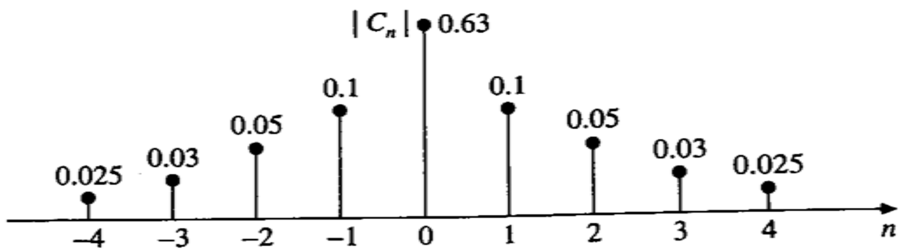
$$C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\, dt = \frac{1}{1}\int_0^1 e^{-t}e^{-jn2\pi t}\, dt$$

$$= \int_0^1 e^{-(1+jn2\pi)t}\, dt = \left[\frac{e^{-(1+jn2\pi)t}}{-(1+jn2\pi)}\right]_0^1$$

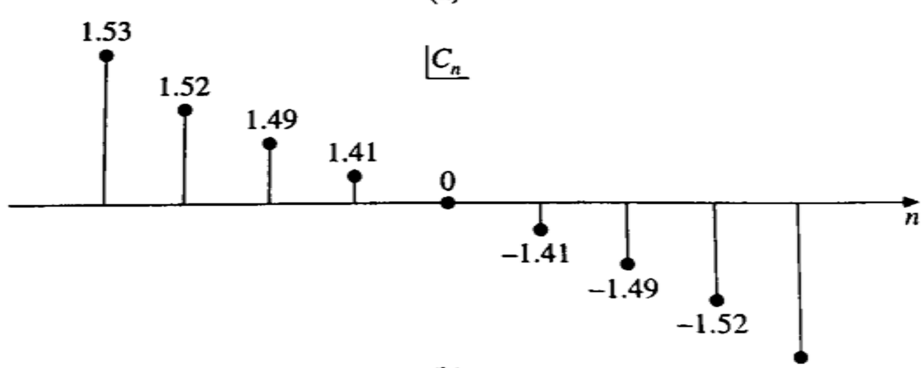
$$= \frac{e^{-(1+jn2\pi)}-1}{-(1+jn2\pi)} = \frac{1-e^{-1}e^{-jn2\pi}}{(1+jn2\pi)} = \frac{1-e^{-1}}{(1+jn2\pi)}$$

$$x(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty}\left(\frac{1-e^{-1}}{1+jn2\pi}\right) e^{jn2\pi t}$$

The spectra are shown in bellow figure.

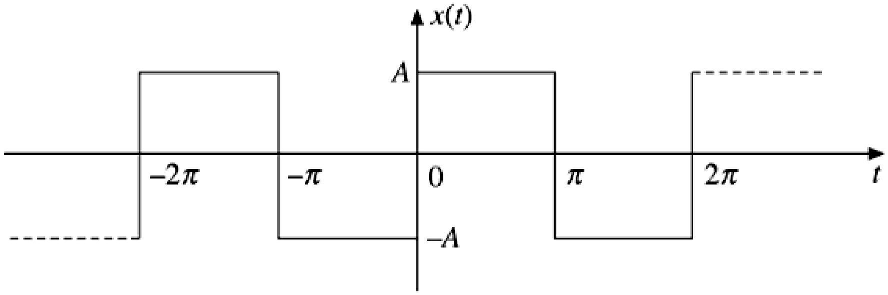


(a)



(b)

Problem 9: Obtain the exponential Fourier Series for the wave form shown in below figure. Also draw the frequency spectrum.



Solution: The periodic waveform shown in fig with a period T= 2π can be expressed as:

$$x(t) = \begin{cases} A & 0 \le t \le \pi \\ -A & \pi \le t \le 2\pi \end{cases}$$

Let

$t_0 = 0$, $t_0 + T = 2\pi$

and Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

Exponential Fourier series

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt$$
$$= \frac{1}{2\pi}\int_0^{\pi} A\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} -A\,dt = 0$$

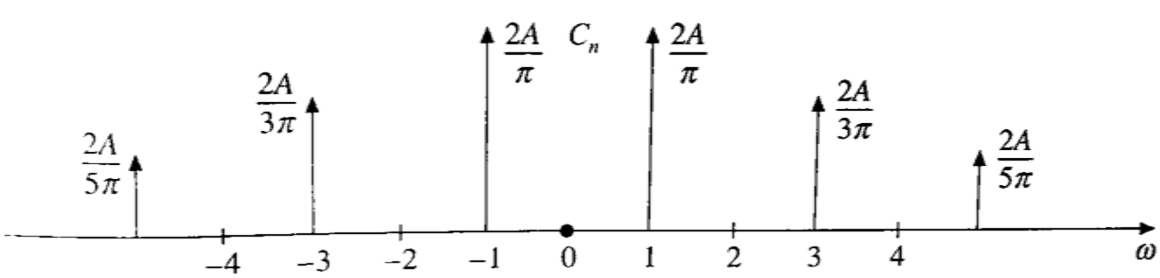
$$C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$
$$= \frac{1}{2\pi}\int_0^{\pi} Ae^{-jnt}\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} -Ae^{-jnt}\,dt$$
$$= -\frac{A}{j2n\pi}\left[(-1)^n - 1\right] - \left[1 - (-1)^n\right] = -j\frac{A}{2n\pi}$$

$$C_n = \begin{cases} \left(-j\frac{2A}{\pi n}\right) & \text{for odd n} \\ 0 & \text{for even n} \end{cases}$$

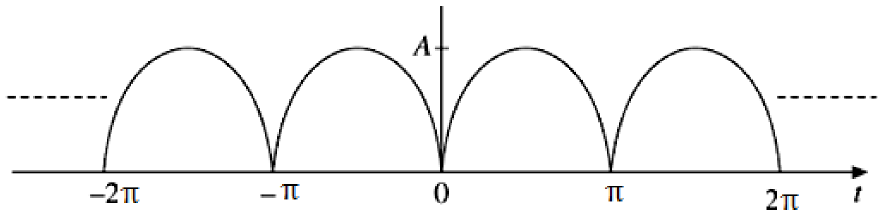
$$\therefore \quad x(t) = C_0 + \sum_{\substack{n=-\infty \\ n \ne 0}}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} -j\frac{A}{2n\pi}e^{jnt}$$

$C_0 = 0$, $C_{-1} = C_1 = \frac{2A}{\pi}$, $C_{-3} = C_3 = \frac{2A}{3\pi}$, $C_{-5} = C_5 = \frac{2A}{5\pi}$

The frequency spectrum is shown in the below figure.



Problem10: Find the exponential Fourier series and plot the frequency spectrum for the full wave rectified sine wave given in below figure



Solution: The waveform shown in fig can be expressed over one period(0 to π) as:

$$x(t) = A \sin \omega t \text{ where } \omega = \frac{2\pi}{2\pi} = 1$$

because it is part of a sine wave with period = 2π

$$x(t) = A \sin \omega t \quad 0 \le t \le \pi$$

The full wave rectified sine wave is periodic with period T = π

Let

$$t_0 = 0,\ t_0 + T = 0 + \pi = \pi$$

and Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$

The exponential Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j2nt}$$

where $C_n = \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t}\, dt$

$$= \frac{1}{\pi}\int_0^\pi A \sin t\, e^{-j2nt}\, dt = \frac{A}{\pi}\int_0^\pi \sin t\, e^{-j2nt}\, dt$$

$$= \frac{A}{j2\pi}\left[\frac{e^{j(1-2n)t} - e^0}{j(1-2n)} - \frac{e^{-j(1-2n)t} - e^0}{-j(1-2n)}\right]$$

$$\therefore C_n = \frac{2A}{\pi(1-4n^2)}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

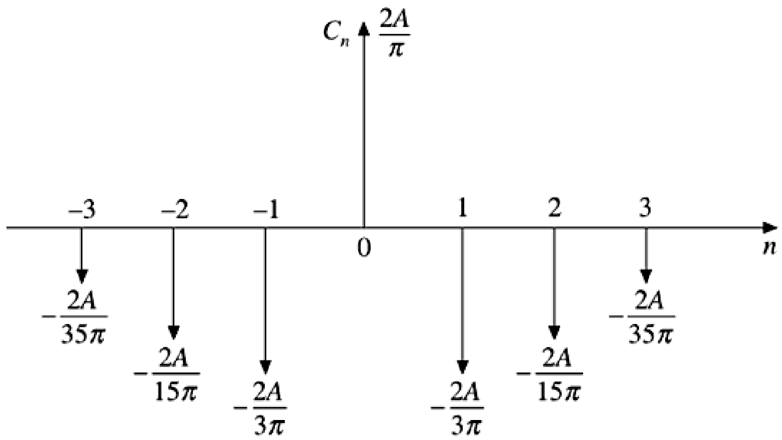
$$= \frac{1}{\pi}\int_0^\pi A \sin t\, dt = \frac{A}{\pi}[-\cos t]_0^\pi = \frac{2A}{\pi}$$

The exponential Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{2A}{\pi(1-4n^2)} e^{j2nt} = \frac{2A}{\pi} + \frac{2A}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{e^{j2nt}}{1-4n^2}\right)$$

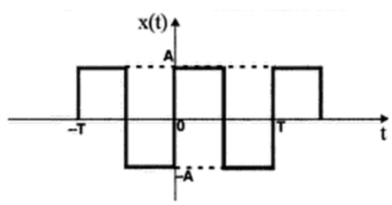
$$C_0 = \frac{2A}{\pi},\quad C_{-1} = C_1 = \frac{2A}{3\pi},\ C_{-2} = C_2 = -\frac{2A}{15\pi},\ C_{-3} = C_3 = -\frac{2A}{35\pi}$$

The frequency spectrum is shown in the below figure.

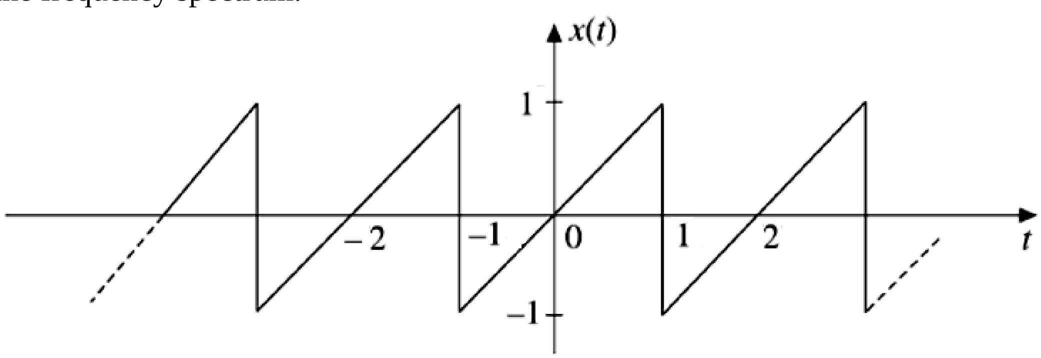


Assignment Problems:

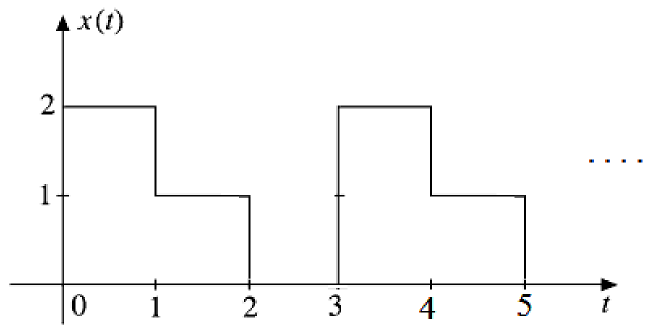
1. Find the trigonometric Fourier series for the periodic signal x(t) shown in figure below. Also draw the frequency spectrum.



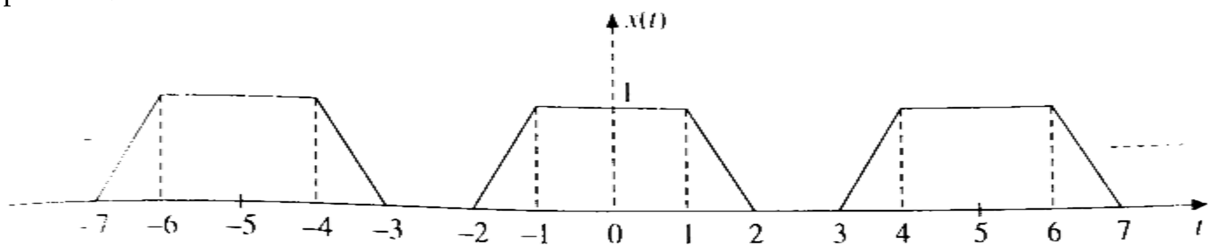
2. Find the trigonometric Fourier series for the periodic signal x(t) shown in figure below. Also draw the frequency spectrum.



3. Compute the exponential Fourier series of the signal shown in figure below. Also draw the frequency spectrum.



4. Find the exponential series of the signal shown in the figure below. Also draw the frequency spectrum.



5. Find the trigonometric Fourier series of x(t) $=t^2$ over the interval (-1, 1).
6. Find the Fourier series for the periodic signal x(t) $=t^2$ for $0 \le t \le 1$, so that it repeats every 1 sec.
7. Determine the time signal corresponding to the magnitude and phase spectra shown in below figure with $\omega_o = \pi$.

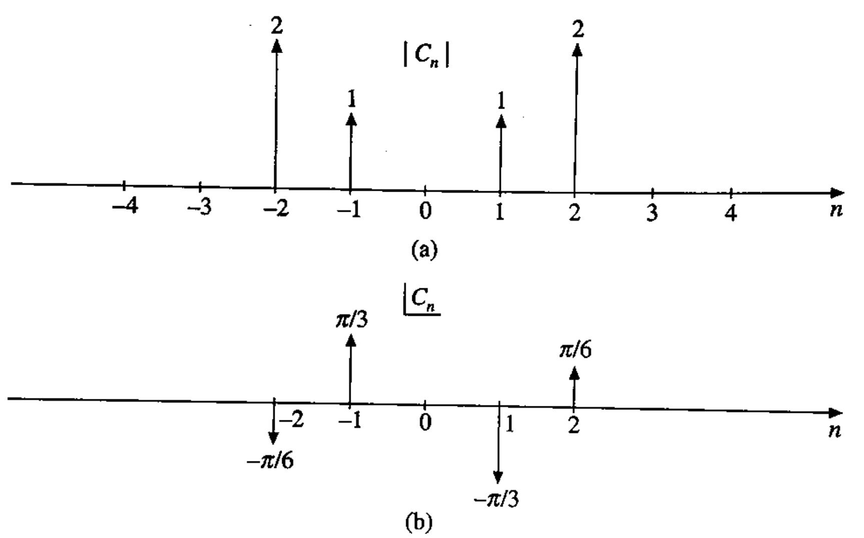


Figure (a) Magnitude and (b) Phase spectra

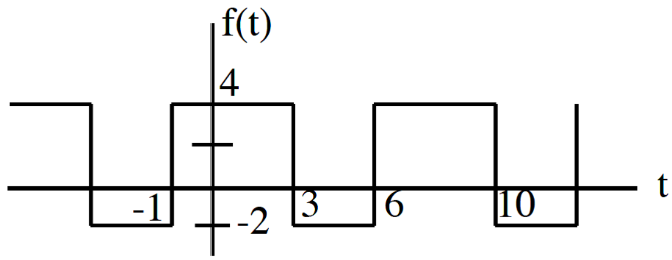
8. Determine the Fourier series representation for x(t) = 2 $\sin(2\pi t - 3)$+$\sin 6\pi t$.
9. Find the complex exponential Fourier series coefficient of the signal x(t) = 3cos $4\omega_0$t. Also draw the frequency spectrum.
10. Find the complex exponential Fourier series coefficient of the signal x(t) = 2cos $3\omega_0$t. Also draw the frequency spectrum.

Simulation:

MATLAB can be used to find and plot the exponential Fourier series of a periodic function. Using the symbolic math toolbox, we can perform the integrations necessary to find the coefficients Fn.

Exponential Fourier Series

For the waveform, *f(t)*, shown below:



we can write the exponential Fourier Series expansion as:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t}$$

Where

$$F_n = \frac{1}{T}\int_T f(t)e^{-jn\omega_0 t}\,dt = \begin{cases} \dfrac{10}{7}, \quad n = 0 \\ \dfrac{1}{jn2\pi}\left[4e^{jn2\pi/7} + 2e^{-jn12\pi/7} - 6e^{-jn6\pi/7}\right], \quad n \neq 0 \end{cases}$$

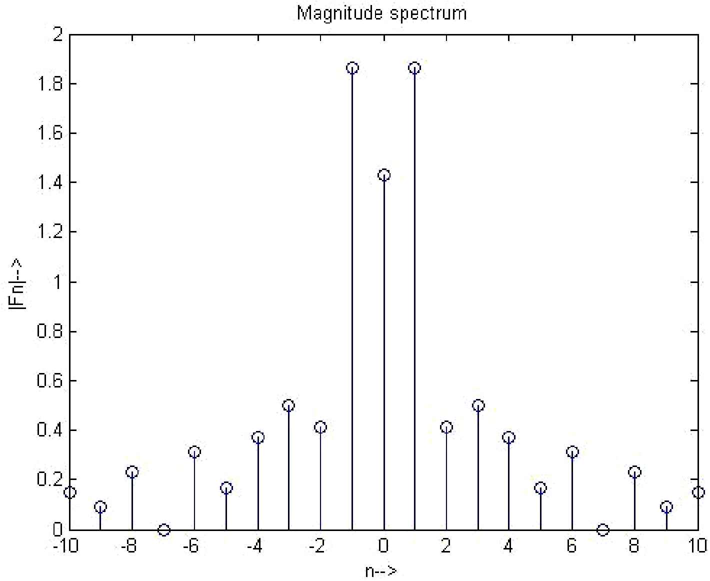
The Fn are complex coefficients. Thus, when we plot them we need to plot the magnitude andphase separately. In Matlab we use the “abs” function to find the magnitude and “angle” to findthe phase angle. The following Matlab statements show one method for obtaining these plots:

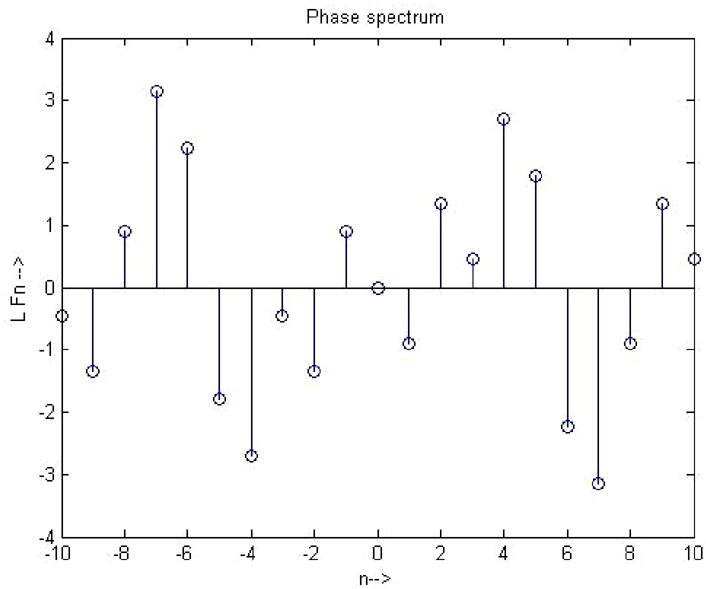
Program

```
clc;clear all; close all;
n=-10:10;
Fn=(1./(j*n*2*pi)).*(4*exp(j*n*2*pi/7)+ 2*exp(-j*n*12*pi/7)-6*exp(-
j*n*6*pi/7));
Fn(find(n==0))=10/7;
stem(n,abs(Fn))
xlabel('n-->');
ylabel('|Fn|-->');
title('Magnitude spectrum');
figure,
stem(n,angle(Fn))
xlabel('n-->');
```

```
ylabel('L Fn -->');
title('Phase spectrum');
```

Since the value of Fn is different at n = 0, we assign that value using the *find* command. Thestatement: find(n==0) will locate the index of the n vector that contains a zero. In this case thatindex is 6, since the zero is found in the sixth position in n. That value is then used to index Fn,so the value of Fn(6) is set to 10/7. This corresponds to the value for n = 0. These commandsresult in the following magnitude and phase plots:





Note that the magnitude spectrum is even and the phase spectrum is odd as expected.

References:

[1] Alan V.Oppenheim, Alan S.Willsky and S.Hamind Nawab, “Signals & Systems”, Second edition, Pearson Education, 8th Indian Reprint, 2005.

[2] M.J.Roberts, “Signals and Systems, Analysis using Transform methods and MATLAB”, Second edition,McGraw-Hill Education,2011

[3] John R Buck, Michael M Daniel and Andrew C.Singer, “Computer explorations in Signals and Systems using MATLAB”,Prentice Hall Signal Processing Series

[4] P Ramakrishna rao, “Signals and Systems”, Tata McGraw-Hill, 2008

[5] Tarun Kumar Rawat, “Signals and Systems”, Oxford University Press,2011

[6] A.Anand Kumar, “Signals and Systems” , PHI Learning Private Limited ,2011