Z Transforms

Reference book-Simon Haykin

- Representation interms of complex exponential signals is termed as z-transform, the discrete time counterpart of laplace transform.
- Effect of applying a complex exponential input to an LTI system, let $z = re^{j\Omega}$ be a complex number with magnitude r and angle Ω. The signal $x[n] = z^n$ is a complex exponential signal. We use $z = re^{j\Omega}$ to write

$$x[n] = r^n \cos(\Omega n) + jr^n \sin(\Omega n). \tag{7.1}$$

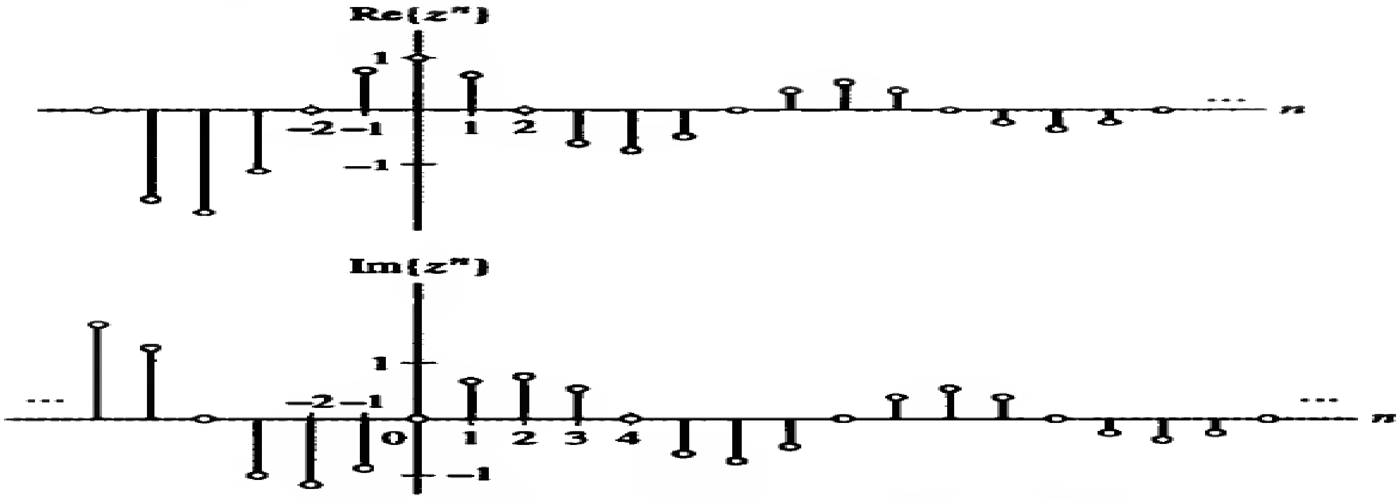


FIGURE 7.1 Real and imaginary parts of the signal z^n.

The real part of x[n] is an exponentially damped cosine and the imaginary part is an exponentially damped sine.

- The positive number r determines the damping factor and Ω is the sinusoidal frequency. x[n] is a complex sinusoid if r=1.

$$\begin{aligned} y[n] &= H\{x[n]\} \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k]. \end{aligned}$$

- We use x[n]= z^n to obtain

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} \\ &= z^n\left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right). \end{aligned}$$

- We define the transfer function $$\boxed{H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}}$$

- so that we can write it as $H\{z^n\} = H(z)z^n.$
- Z transform of an arbitrary signal is defined as $\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},}$
- Inverse Z transform $\boxed{x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}\,dz.}$
- Relationship between x[n] and X(z) is expressed as $x[n] \xleftrightarrow{z} X(z).$
- **Convergence of z-transform**
- The z-transform should be absolutely summable $\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty.$

  The range of r for which this condition is satisfied is termed the **region of convergence (ROC)** of the z-transform.
- The z transform exists for signals that do not have DTFT.

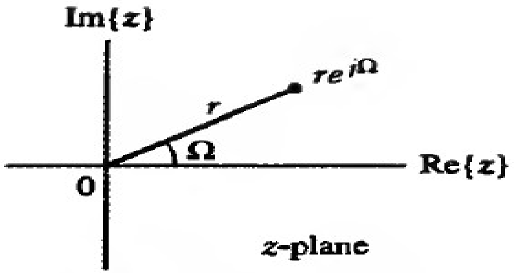
- The existence of DTFT requires absolute summability of x[n]. By restricting the values of r, we ensure that $x[n]r^{-n}$ is absolutely summable, even though x[n] is not.
- 

FIGURE 7.3 The z-plane. A point $z = re^{j\Omega}$ is located at a distance r from the origin and an angle Ω relative to the real axis.

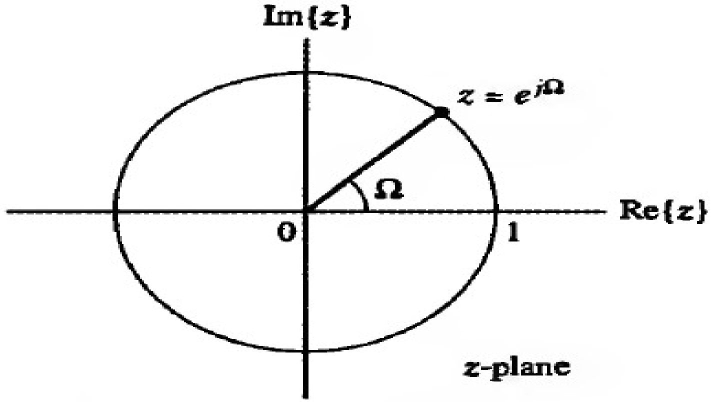


FIGURE 7.4 The unit circle, $z = e^{j\Omega}$, in the z-plane.

- The equation $z = e^{j\Omega}$ describes a circle of unit radius centered at origin in the z-plane.

EXAMPLE 7.1 THE Z-TRANSFORM AND THE DTFT Determine the z-transform of the signal

$$x[n] = \begin{cases} 1, & n = -1 \\ 2, & n = 0 \\ -1, & n = 1 \\ 1, & n = 2 \\ 0, & \text{otherwise} \end{cases}.$$

Use the z-transform to determine the DTFT of $x[n]$.

Solution: We substitute the prescribed $x[n]$ into Eq. (7.4) to obtain

$$X(z) = z + 2 - z^{-1} + z^{-2}.$$

We obtain the DTFT from $X(z)$ by substituting $z = e^{j\Omega}$:

$$X(e^{j\Omega}) = e^{j\Omega} + 2 - e^{-j\Omega} + e^{-j2\Omega}.$$

- **Poles and zeros**

  The most commonly encountered form of the z-transform is the ratio of the two polynomials in z^{-1},

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}.$$

- The roots of the numerator polynomial are the **zeros** of X(z) and the roots of the denominator polynomial are the **poles** of X(z).
- Location of zeros are denoted with the "○" symbol and location of poles with the "×" symbol in the z plane.

Example 7.2 z-Transform of a Causal Exponential Signal Determine the z-transform of the signal

$$x[n] = \alpha^n u[n].$$

Depict the ROC and the locations of poles and zeros of $X(z)$ in the z-plane.

Solution: Substituting $x[n] = \alpha^n u[n]$ into Eq. (7.4) yields

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n. \end{aligned}$$

This is a geometric series of infinite length in the ratio α/z; the sum converges, provided that $|\alpha/z| < 1$, or $|z| > |\alpha|$. Hence,

$$\begin{aligned} X(z) &= \frac{1}{1 - \alpha z^{-1}}, \qquad |z| > |\alpha| \\ &= \frac{z}{z - \alpha}, \qquad |z| > |\alpha|. \end{aligned} \tag{7.7}$$

There is thus a pole at $z = \alpha$ and a zero at $z = 0$, as illustrated in Fig. 7.5. The ROC is depicted as the shaded region of the z-plane. ■

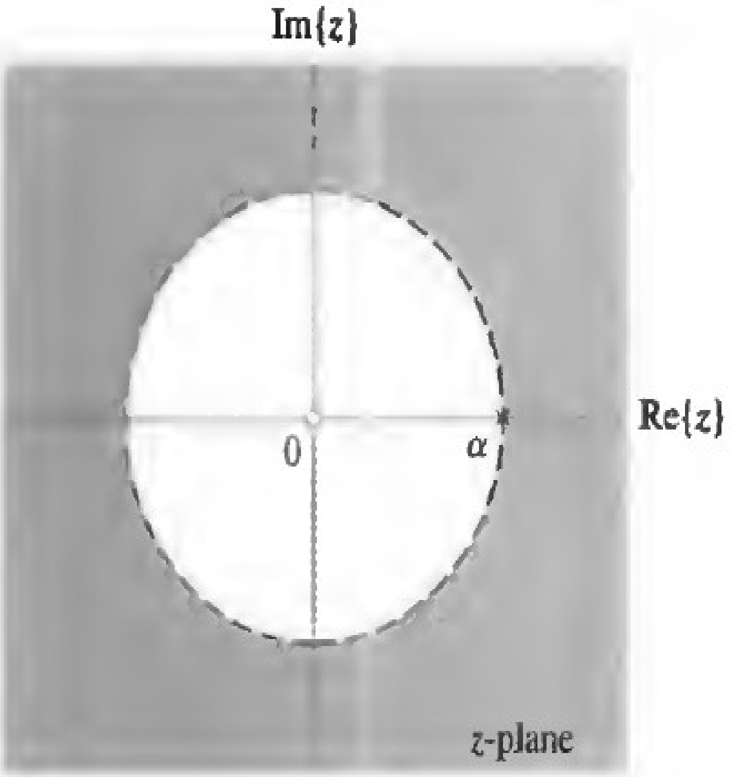


FIGURE 7.5 Locations of poles and zeros of $x[n] = \alpha^n u[n]$ in the z-plane. The ROC is the shaded area.

- Two different time signal may have identical z transforms, but different ROC's.

EXAMPLE 7.3 z-TRANSFORM OF AN ANTICAUSAL EXPONENTIAL SIGNAL Determine the z-transform of the signal

$$y[n] = -\alpha^n u[-n-1].$$

Depict the ROC and the locations of poles and zeros of $X(z)$ in the z-plane.

Solution: We substitute $y[n] = -\alpha^n u[-n-1]$ into Eq. (7.4) and write

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1] z^{-n} \\ &= -\sum_{n=-\infty}^{-1} \left(\frac{\alpha}{z}\right)^n \\ &= -\sum_{k=1}^{\infty} \left(\frac{z}{\alpha}\right)^k \\ &= 1 - \sum_{k=0}^{\infty} \left(\frac{z}{\alpha}\right)^k. \end{aligned}$$

The sum converges, provided that $|z/\alpha| < 1$, or $|z| < |\alpha|$. Hence,

$$\begin{aligned} Y(z) &= 1 - \frac{1}{1 - z\alpha^{-1}}, \qquad |z| < |\alpha|, \\ &= \frac{z}{z-\alpha}, \qquad |z| < |\alpha|. \end{aligned} \tag{7.8}$$

The ROC and the locations of poles and zeros are depicted in Fig. 7.6. ■

Note that Y(z) in eqn (7.8) is identical to eqn (7.7), even though the time signals are quite different. Only the ROC differentiate the two transforms.

- We must know the ROC to determine the correct inverse z transform.

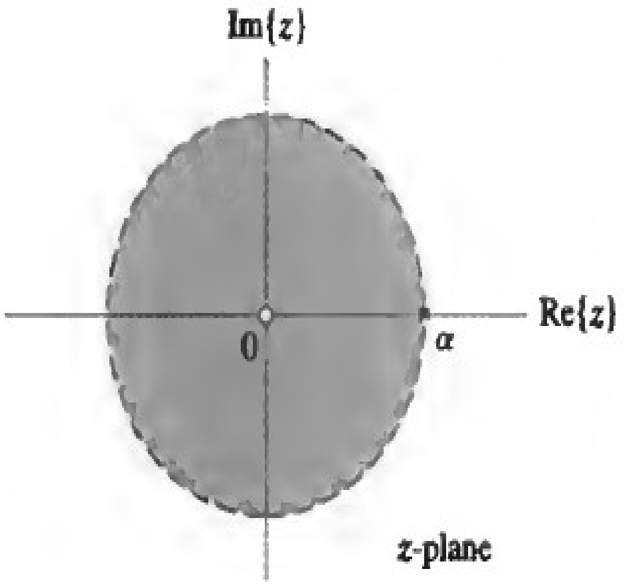


FIGURE 7.6 ROC and locations of poles and zeros of $x[n] = -\alpha^n u[-n-1]$ in the z-plane.

Example 7.4 z-Transform of a Two-Sided Signal Determine the z-transform of

$$x[n] = -u[-n-1] + \left(\frac{1}{2}\right)^n u[n].$$

Depict the ROC and the locations of poles and zeros of $X(z)$ in the z-plane.

Solution: Substituting for $x[n]$ in Eq. (7.4), we obtain

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} - u[-n-1] z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n - \sum_{n=-\infty}^{-1} \left(\frac{1}{z}\right)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n + 1 - \sum_{k=0}^{\infty} z^k.
\end{aligned}$$

Both sums must converge in order for $X(z)$ to converge. This implies that we must have $|z| > 1/2$ and $|z| < 1$. Hence,

$$\begin{aligned}
X(z) &= \frac{1}{1 - \frac{1}{2}z^{-1}} + 1 - \frac{1}{1-z}, \qquad 1/2 < |z| < 1 \\
&= \frac{z\left(2z - \frac{3}{2}\right)}{\left(z - \frac{1}{2}\right)(z-1)}, \qquad 1/2 < |z| < 1.
\end{aligned}$$

The ROC and the locations of the poles and zeros are depicted in Fig. 7.7. In this case, the ROC is a ring in the z-plane. ∎

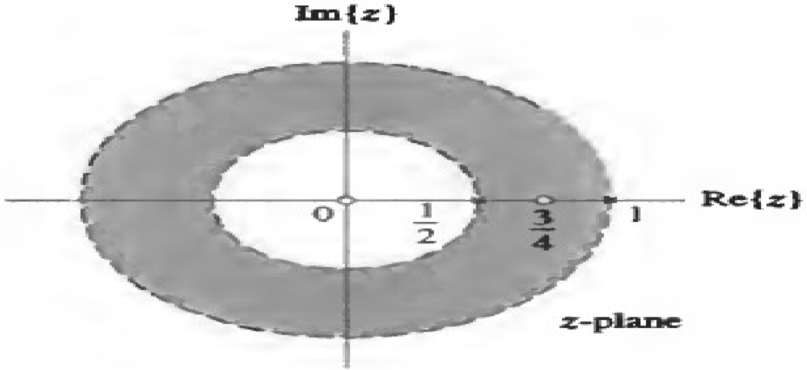


FIGURE 7.7 ROC and locations of poles and zeros in the z-plane for Example 7.4.

► **Problem 7.1** Determine the z-transform, the ROC, and the locations of poles and zeros of $X(z)$ for the following signals:

(a)

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{-1}{3}\right)^n u[n]$$

(b)

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{-1}{3}\right)^n u[-n-1]$$

(c)

$$x[n] = -\left(\frac{3}{4}\right)^n u[-n-1] + \left(\frac{-1}{3}\right)^n u[n]$$

(d)

$$x[n] = e^{j\Omega_o n} u[n]$$

Answers:

(a)

$$X(z) = \frac{z\left(2z - \frac{1}{6}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}, \qquad |z| > 1/2$$

Poles are at $z = 1/2$ and $z = -1/3$, and zeros are at $z = 0$ and $z = 1/12$

(b)

$$X(z) = \frac{z\left(2z - \frac{1}{6}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}, \qquad |z| < 1/3$$

Poles are at $z = 1/2$ and $z = -1/3$, and zeros are at $z = 0$ and $z = 1/12$

(c)

$$X(z) = \frac{z(2z - 5/12)}{(z - 3/4)(z + 1/3)}, \qquad 1/3 < |z| < 3/4$$

Poles are at $z = 3/4$ and $z = -1/3$, and zeros are at $z = 0$ and $z = 5/24$

(d)

$$X(z) = \frac{z}{z - e^{j\Omega_o}}, \qquad |z| > 1$$

A pole is at $z = e^{j\Omega_o}$, and a zero is at $z = 0$ ◀

Properties of Region of Convergence(ROC)

1) ROC cannot contain any poles. This is because the ROC is defined as the set of all the z for which the z-transform converges. Hence X(z) must be finite for all values of z in the ROC.

- If d is a pole then $|X(d)| = \infty$, and the z transform does not converge at the pole.

Thus, the pole cannot lie in the ROC.

2) The ROC for finite duration signal includes the entire z plane, except possibly at z=0 or $|z| = \infty$ (or both).

- The ROC of a right-sided signal is of the form $|z| > r_+$.
- The ROC of a left-sided signal is of the form $|z| < r_-$.
- The ROC of a two-sided signal is of the form $r_+ < |z| < r_-$.

- If a signal consists of a sum of exponentials, then the ROC is the intersection of the ROCs associated with each term; it will have a radius

  greater than that of the pole of largest radius associated with the right-sided terms and a radius less than that of the pole of smallest radius associated with the left-handed terms.

EXAMPLE 7.5 ROCs OF TWO-SIDED SIGNALS Identify the ROC associated with the z-transform for each of the following signals:

$$x[n] = (-1/2)^n u[-n] + 2(1/4)^n u[n];$$

$$y[n] = (-1/2)^n u[n] + 2(1/4)^n u[n];$$

$$w[n] = (-1/2)^n u[-n] + 2(1/4)^n u[-n].$$

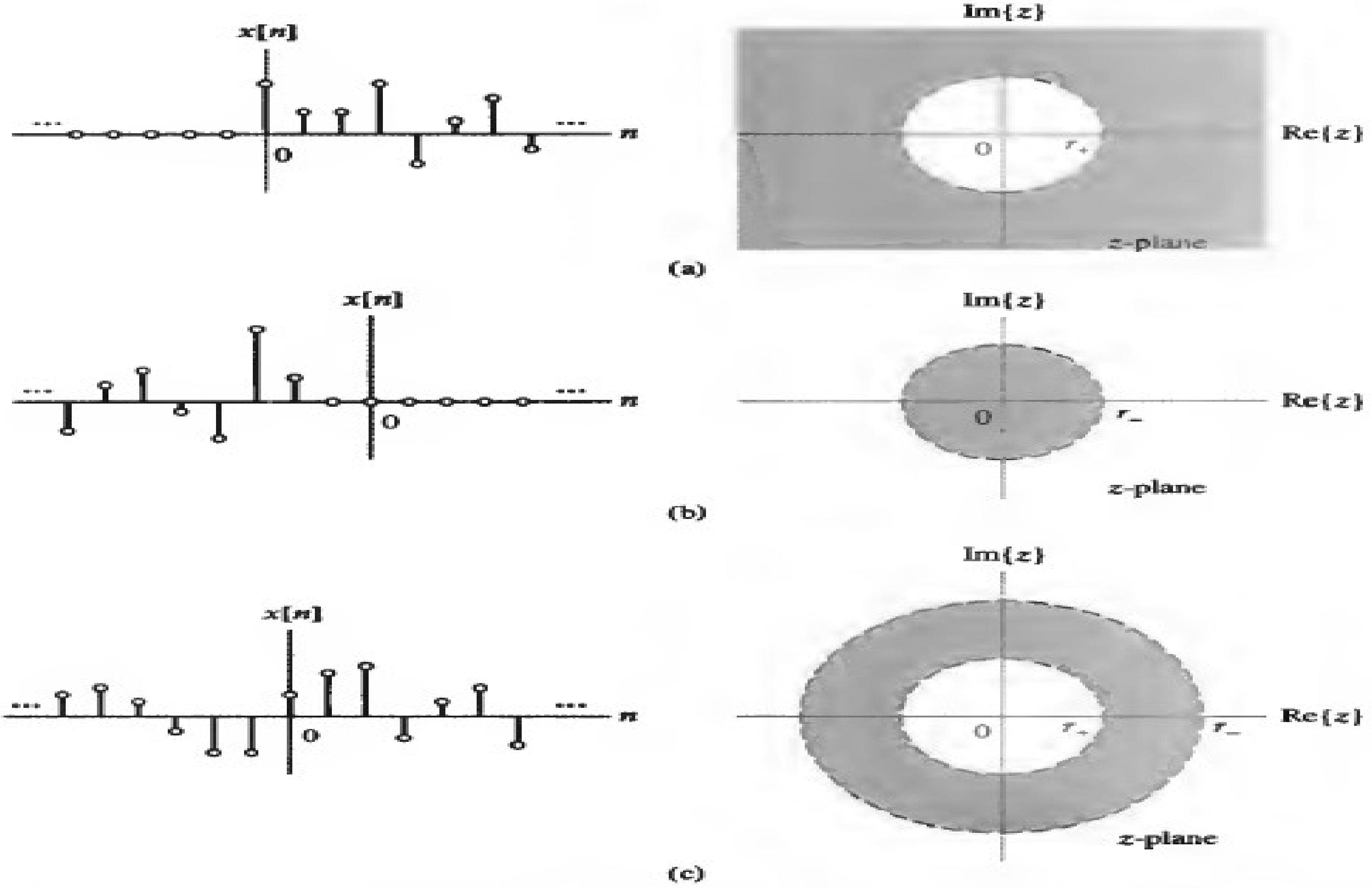


FIGURE 7.8 The relationship between the ROC and the time extent of a signal. (a) A right-sided signal has an ROC of the form $|z| > r_+$. (b) A left-sided signal has an ROC of the form $|z| < r_-$. (c) A two-sided signal has an ROC of the form $r_+ < |z| < r_-$.

Solution: Beginning with $x[n]$, we use Eq. (7.4) to write

$$X(z) = \sum_{n=-\infty}^{0}\left(\frac{-1}{2z}\right)^{n} + 2\sum_{n=0}^{\infty}\left(\frac{1}{4z}\right)^{n}$$

$$= \sum_{k=0}^{\infty}(-2z)^{k} + 2\sum_{n=0}^{\infty}\left(\frac{1}{4z}\right)^{n}.$$

The first series converges for $|z| < \frac{1}{2}$, while the second converges for $|z| > \frac{1}{4}$. Both series must converge for $X(z)$ to converge, so the ROC is $\frac{1}{4} < |z| < \frac{1}{2}$. The ROC of this two-sided signal is depicted in Fig. 7.9(a). Summing the two geometric series, we obtain

$$X(z) = \frac{1}{1+2z} + \frac{2z}{z-\frac{1}{4}},$$

which has poles at $z = -1/2$ and $z = 1/4$. Note that the ROC is the ring-shaped region located between the poles.

Next, $y[n]$ is a right-sided signal, and again using the definition of the z-transform given by Eq. (7.4), we write

$$Y(z) = \sum_{n=0}^{\infty}\left(\frac{-1}{2z}\right)^n + 2\sum_{n=0}^{\infty}\left(\frac{1}{4z}\right)^n.$$

The first series converges for $|z| > 1/2$, while the second converges for $|z| > 1/4$. Hence, the combined ROC is $|z| > 1/2$, as depicted in Fig. 7.9(b). In this case, we write

$$Y(z) = \frac{z}{z+\frac{1}{2}} + \frac{2z}{z-\frac{1}{4}},$$

for which the poles are again at $z = -1/2$ and $z = 1/4$. The ROC is outside a circle containing the pole of largest radius, $z = -1/2$.

The last signal, $w[n]$, is left sided and has z-transform given by

$$\begin{aligned} W(z) &= \sum_{n=-\infty}^{0}\left(\frac{-1}{2z}\right)^n + 2\sum_{n=-\infty}^{0}\left(\frac{1}{4z}\right)^n \\ &= \sum_{k=0}^{\infty}(-2z)^k + 2\sum_{k=0}^{\infty}(4z)^k. \end{aligned}$$

Here, the first series converges for $|z| < 1/2$, while the second series converges for $|z| < 1/4$, giving a combined ROC of $|z| < 1/4$, as depicted in Fig. 7.9(c). In this case, we have

$$W(z) = \frac{1}{1 + 2z} + \frac{2}{1 - 4z},$$

where the poles are at $z = -1/2$ and $z = 1/4$. The ROC is inside a circle containing the pole of smallest radius, $z = 1/4$.

This example illustrates that the ROC of a two-sided signal is a ring, the ROC of a right-sided signal is the exterior of a circle, and the ROC of a left-sided signal is the interior of a circle. In each case, the poles define the boundaries of the ROC. ■

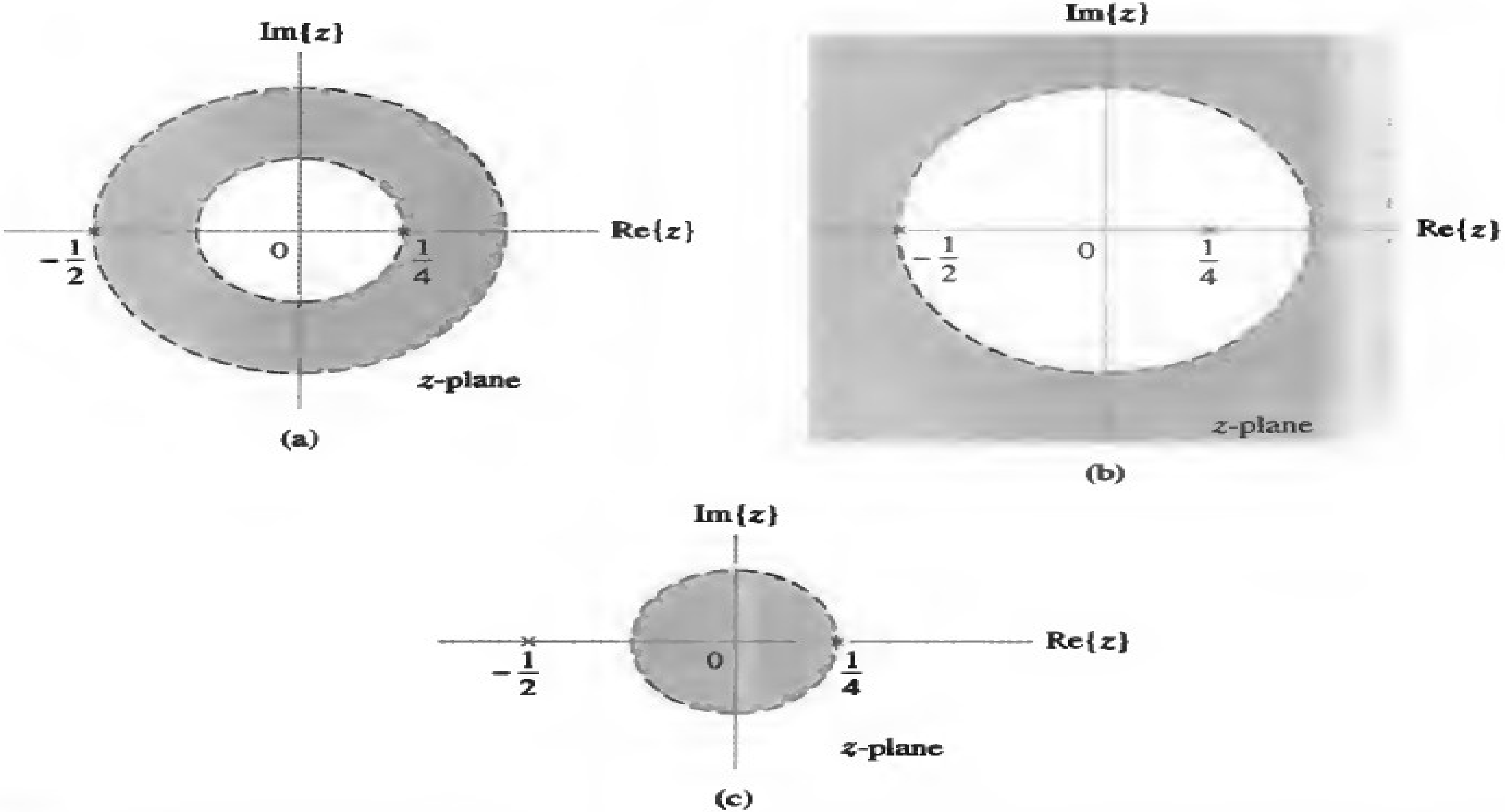


FIGURE 7.9 ROCs for Example 7.5. (a) Two-sided signal $x[n]$ has ROC in between the poles. (b) Right-sided signal $y[n]$ has ROC outside of the circle containing the pole of largest magnitude. (c) Left-sided signal $w[n]$ has ROC inside the circle containing the pole of smallest magnitude.

▶ **Problem 7.2** Determine the z-transform and ROC for the two-sided signal

$$x[n] = \alpha^{|n|},$$

assuming that $|\alpha| < 1$. Repeat for $|\alpha| > 1$.

Answer: For $|\alpha| < 1$,

$$X(z) = \frac{z}{z - \alpha} - \frac{z}{z - 1/\alpha}, \qquad |\alpha| < |z| < 1/|\alpha|$$

For $|\alpha| > 1$, the ROC is the empty set

- **Properties of Z transform**
- We assume that

$$x[n] \xleftrightarrow{z} X(z), \quad \text{with ROC } R_x$$

and

$$y[n] \xleftrightarrow{z} Y(z), \quad \text{with ROC } R_y.$$

- **1) Linearity**: The linearity property states that the z-transform of sum of signals is just the sum of individual z-transform.

$$\boxed{ax[n] + by[n] \xleftrightarrow{z} aX(z) + bY(z), \quad \text{with ROC at least } R_x \cap R_y.}$$

- The z transform of the sum is valid only whereever both X(z) and Y(z) converge.

EXAMPLE 7.6 POLE–ZERO CANCELLATION Suppose

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{3}{2}\right)^n u[-n-1] \xleftrightarrow{z} X(z) = \frac{-z}{\left(z - \frac{1}{2}\right)\left(z - \frac{3}{2}\right)},$$

$$\text{with ROC } \frac{1}{2} < |z| < \frac{3}{2}$$

and

$$y[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} Y(z) = \frac{-\frac{1}{4}z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}, \quad \text{with ROC } |z| > \frac{1}{2}.$$

Evaluate the z-transform of $ax[n] + by[n]$.

Solution: The pole–zero plots and ROCs for $x[n]$ and $y[n]$ are depicted in Figs. 7.10(a) and (b), respectively. The linearity property given by Eq. (7.11) indicates that

$$ax[n] + by[n] \xleftrightarrow{z} a\frac{-z}{\left(z - \frac{1}{2}\right)\left(z - \frac{3}{2}\right)} + b\frac{-\frac{1}{4}z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}.$$

In general, the ROC is the intersection of individual ROCs, or $\frac{1}{2} < |z| < \frac{3}{2}$ in this example, which corresponds to the ROC depicted in Fig. 7.10(a). Note, however, what happens when $a = b$: We have

$$ax[n] + ay[n] = a\left(-\left(\frac{3}{2}\right)^n u[-n-1] + \left(\frac{1}{4}\right)^n u[n]\right),$$

and we see that the term $\left(\frac{1}{2}\right)^n u[n]$ has been canceled in the time-domain signal. The ROC is now easily verified to be $\frac{1}{4} < |z| < \frac{3}{2}$, as shown in Fig. 7.10(c). This ROC is larger than the intersection of the individual ROCs, because the term $\left(\frac{1}{2}\right)^n u[n]$ is no longer present. Combining z-transforms and using the linearity property gives

$$\begin{aligned} aX(z) + aY(z) &= a\left(\frac{-z}{(z-\frac{1}{2})(z-\frac{3}{2})} + \frac{-\frac{1}{4}z}{(z-\frac{1}{4})(z-\frac{1}{2})}\right) \\ &= a\frac{-\frac{1}{4}z(z-\frac{3}{2}) - z(z-\frac{1}{4})}{(z-\frac{1}{4})(z-\frac{1}{2})(z-\frac{3}{2})} \\ &= a\frac{-\frac{5}{4}z(z-\frac{1}{2})}{(z-\frac{1}{4})(z-\frac{1}{2})(z-\frac{3}{2})}. \end{aligned}$$

The zero at $z = \frac{1}{2}$ cancels the pole at $z = \frac{1}{2}$, so we have

$$aX(z) + aY(z) = a\frac{-\frac{5}{4}z}{(z-\frac{1}{4})(z-\frac{3}{2})}.$$

Hence, cancellation of the $\left(\frac{1}{2}\right)^n u[n]$ term in the time domain corresponds to cancellation of the pole at $z = \frac{1}{2}$ by a zero in the z-domain. This pole defined the ROC boundary, so the ROC enlarges when the pole is removed. ■

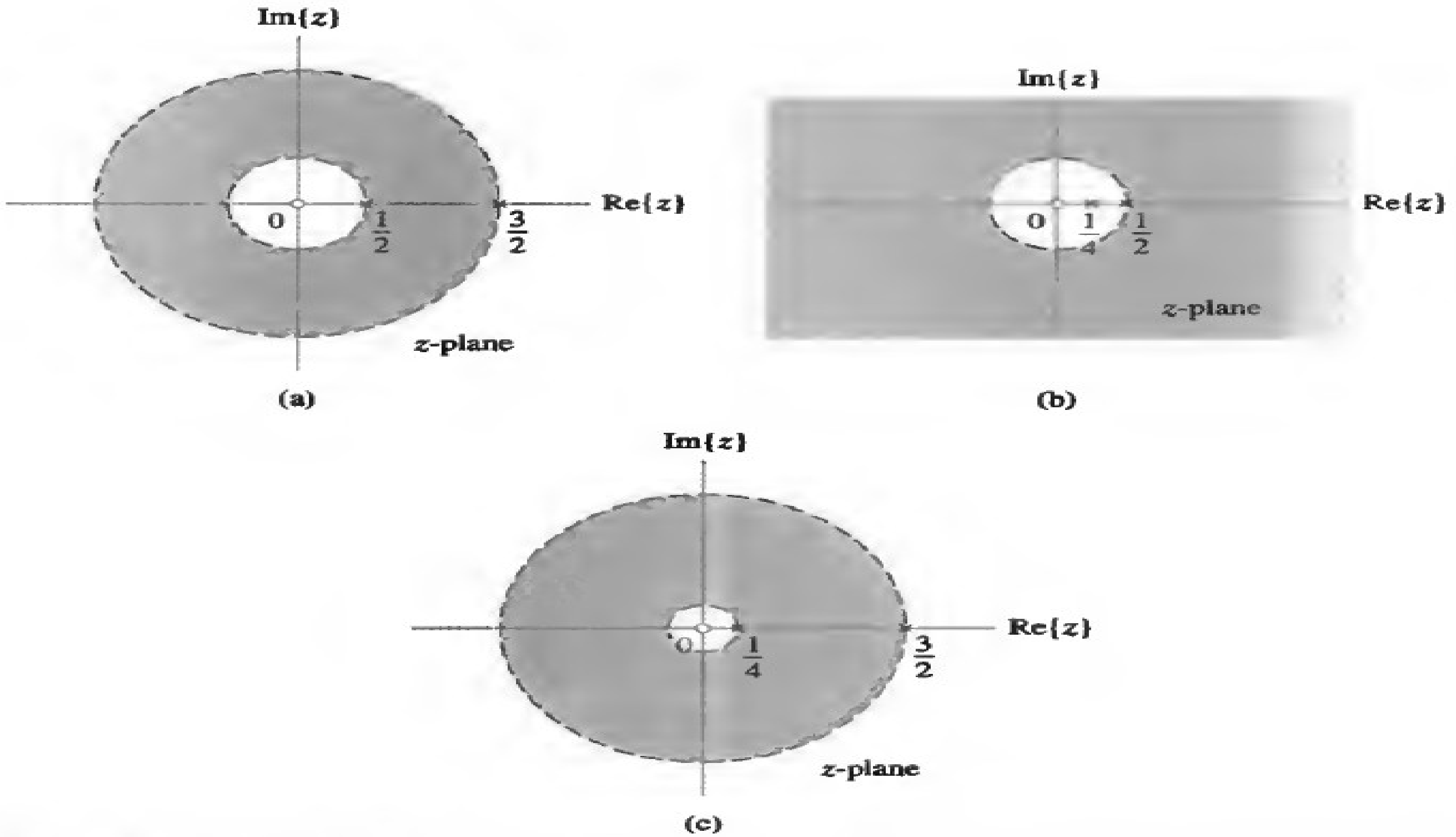


FIGURE 7.10 ROCs for Example 7.6. (a) ROC and pole–zero plot for $X(z)$. (b) ROC and pole–zero plot for $Y(z)$. (c) ROC and pole–zero plot for $a(X(z) + Y(z))$.

- **2)Time Reversal**

$$x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right), \quad \text{with ROC} \frac{1}{R_x}.$$

- Time reversal, or reflection, corresponds to replacing z by z^{-1}. Hence, if R_x is of the form $a < |z| < b$, the ROC of the reflected signal is $a < 1/|z| < b$, or $1/b < |z| < 1/a$.

- **3)Time Shift**

$$x[n - n_o] \xleftrightarrow{z} z^{-n_o}X(z), \quad \text{with ROC } R_x, \text{ except possibly } z = 0 \text{ or } |z| = \infty.$$

- **4) Multiplication**

$$\alpha^n x[n] \xleftrightarrow{z} X\left(\frac{z}{\alpha}\right), \quad \text{with ROC } |\alpha| R_x.$$

- 5) **Differentiation in the z-domain**

$$nx[n] \xleftrightarrow{z} -z\frac{d}{dz}X(z), \quad \text{with ROC } R_x.$$

Example 7.7 Applying Multiple Properties Find the z-transform of the signal

$$x[n] = \left(n\left(\frac{-1}{2}\right)^n u[n]\right) * \left(\frac{1}{4}\right)^{-n} u[-n].$$

Solution: First we find the z-transform of $w[n] = n(\frac{-1}{2})^n u[n]$. We know from Example 7.2 that

$$\left(\frac{-1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{z}{z+\frac{1}{2}}, \quad \text{with ROC } |z| > \frac{1}{2}.$$

Thus, the z-domain differentiation property of Eq. (7.16) implies that

$$\begin{aligned} w[n] = n\left(\frac{-1}{2}\right)^n u[n] \xleftrightarrow{z} W(z) &= -z\frac{d}{dz}\left(\frac{z}{z+\frac{1}{2}}\right), \quad \text{with ROC } |z| > \frac{1}{2} \\ &= -z\left(\frac{z+\frac{1}{2}-z}{(z+\frac{1}{2})^2}\right) \\ &= \frac{-\frac{1}{2}z}{(z+\frac{1}{2})^2}, \quad \text{with ROC } |z| > \frac{1}{2}. \end{aligned}$$

Next, we find the z-transform of $y[n] = \left(\frac{1}{4}\right)^{-n} u[-n]$. We do this by applying the time-reversal property given in Eq. (7.12) to the result of Example 7.2. Noting that

$$\left(\frac{1}{4}\right)^n u[n] \xleftrightarrow{z} \frac{z}{z - \frac{1}{4}}, \quad \text{with ROC } |z| > \frac{1}{4},$$

we see that Eq. (7.12) implies that

$$\begin{aligned} y[n] \xleftrightarrow{z} Y(z) &= \frac{\frac{1}{z}}{\frac{1}{z} - \frac{1}{4}}, \quad \text{with ROC } \frac{1}{|z|} > \frac{1}{4} \\ &= \frac{-4}{z - 4}, \quad \text{with ROC } |z| < 4. \end{aligned}$$

Last, we apply the convolution property given in Eq. (7.15) to obtain $X(z)$ and thus write

$$\begin{aligned} x[n] = w[n] * y[n] \xleftrightarrow{z} X(z) &= W(z)Y(z), \quad \text{with ROC } R_w \cap R_y \\ &= \frac{2z}{(z-4)\left(z + \frac{1}{2}\right)^2}, \quad \text{with ROC } \frac{1}{2} < |z| < 4. \end{aligned}$$

■

EXAMPLE 7.8 z-TRANSFORM OF AN EXPONENTIALLY DAMPED COSINE Use the properties of linearity and multiplication by a complex exponential to find the z-transform of

$$x[n] = a^n \cos(\Omega_o n)u[n],$$

where a is real and positive.

Solution: First we note from Example 7.2 that $y[n] = a^n u[n]$ has the z-transform

$$Y(z) = \frac{1}{1 - az^{-1}}, \quad \text{with ROC } |z| > a.$$

Now we rewrite $x[n]$ as the sum

$$x[n] = \frac{1}{2}e^{j\Omega_o n}y[n] + \frac{1}{2}e^{-j\Omega_o n}y[n]$$

and apply the property of multiplication by a complex exponential given in Eq. (7.14) to each term, obtaining

$$\begin{aligned}
X(z) &= \frac{1}{2}Y(e^{-j\Omega_o}z) + \frac{1}{2}Y(e^{j\Omega_o}z), \quad \text{with ROC } |z| > a \\
&= \frac{1}{2}\frac{1}{1 - ae^{j\Omega_o}z^{-1}} + \frac{1}{2}\frac{1}{1 - ae^{-j\Omega_o}z^{-1}} \\
&= \frac{1}{2}\left(\frac{1 - ae^{-j\Omega_o}z^{-1} + 1 - ae^{j\Omega_o}z^{-1}}{(1 - ae^{j\Omega_o}z^{-1})(1 - ae^{-j\Omega_o}z^{-1})}\right) \\
&= \frac{1 - a\cos(\Omega_o)z^{-1}}{1 - 2a\cos(\Omega_o)z^{-1} + a^2z^{-2}}, \quad \text{with ROC } |z| > a.
\end{aligned}$$

■

► **Problem 7.3** Find the z-transform of the following signals:

(a)

$$x[n] = u[n-2] * (2/3)^n u[n]$$

(b)

$$x[n] = \sin(\pi n/8 - \pi/4)u[n-2]$$

(c)

$$x[n] = (n-1)(1/2)^n u[n-1] * (1/3)^n u[n+1]$$

(d)

$$x[n] = (2)^n u[-n-3]$$

Answers:

(a)

$$X(z) = \frac{1}{(z-1)(z-2/3)}, \quad \text{with ROC } |z| > 1$$

(b)

$$X(z) = \frac{z^{-1}\sin(\pi/8)}{z^2 - 2z\cos(\pi/8) + 1}, \quad \text{with ROC } |z| > 1$$

(c)

$$X(z) = \frac{3/4\, z^2}{(z-1/3)(z-1/2)^2}, \quad \text{with ROC } |z| > 1/2$$

(d)

$$X(z) = \frac{-z^3}{4(z-2)}, \quad \text{with ROC } |z| < 2$$

- **Inversion of Z-transform**

  We encounter z-transforms that are a rational function of z^{-1}.

$$X(z) = \frac{B(z)}{A(z)}$$
$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}},$$

- Depending on the ROC, the inverse z transform associated is given by

$$\boxed{A_k(d_k)^n u[n] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}}, \quad \text{with ROC } |z| > d_k}$$

or

$$\boxed{-A_k(d_k)^n u[-n-1] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}}, \quad \text{with ROC } |z| < d_k.}$$

- The relationship between ROC associated with X(z) and each pole determines whether the right-sided or left-sided z transform is chosen.

- If the pole di is repeated r times, then there are r terms in the partial-fraction expansion associated with the pole:

$$\frac{A_{i_1}}{1 - d_i z^{-1}}, \frac{A_{i_2}}{(1 - d_i z^{-1})^2}, \ldots, \frac{A_{i_r}}{(1 - d_i z^{-1})^r}.$$

- ROC of X(z) determines whether the right-or left handed inverse transform is chosen. If the ROC is of the form |z|>di, the right sided inverse z transform is chosen.

- $$A\frac{(n+1)\cdots(n+m-1)}{(m-1)!}(d_i)^n u[n] \xleftrightarrow{z} \frac{A}{(1 - d_i z^{-1})^m}, \quad \text{with ROC } |z| > d_i.$$

- If ROC is of the form |z|<di, then the left-sided inverse z transform is chosen.

$$-A\frac{(n+1)\cdots(n+m-1)}{(m-1)!}(d_i)^n u[-n-1] \xleftrightarrow{z} \frac{A}{(1 - d_i z^{-1})^m}, \quad \text{with ROC } |z| < d_i.$$

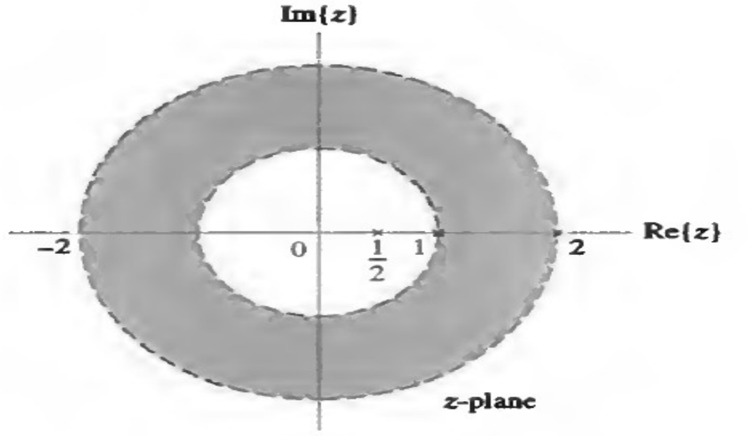
- ROC of X(z) is the intersection of ROCs associated with the individual terms in the partial fraction expansion. To choose the correct inverse transform, we must infer the ROC of each term from the ROC of X(z).
- If ROC of X(z) has radius greater than that of the pole associated with a given term, we choose the right-sided inverse transform. If the ROC of X(z) has a radius less than that of the pole, we choose the left-sided inverse transform for that term.
- 

FIGURE 7.12 Locations of poles and ROC for Example 7.9.

Example 7.9 Inversion by Partial-Fraction Expansion Find the inverse z-transform of

$$X(z) = \frac{1 - z^{-1} + z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})(1 - z^{-1})}, \quad \text{with ROC } 1 < |z| < 2.$$

Solution: We use a partial-fraction expansion to write

$$X(z) = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - 2z^{-1}} + \frac{A_3}{1 - z^{-1}}.$$

Solving for A_1, A_2, and A_3 gives

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 2z^{-1}} - \frac{2}{1 - z^{-1}}.$$

Now we find the inverse z-transform of each term, using the relationship between the locations of the poles and the ROC of $X(z)$, each of which is depicted in Fig. 7.12. The figure shows that the ROC has a radius greater than the pole at $z = \frac{1}{2}$, so this term has the right-sided inverse transform

$$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1 - \frac{1}{2}z^{-1}}.$$

The ROC also has a radius less than the pole at $z = 2$, so this term has the left-sided inverse transform

$$-2(2)^n u[-n-1] \xleftrightarrow{z} \frac{2}{1 - 2z^{-1}}.$$

Finally, the ROC has a radius greater than the pole at $z = 1$, so this term has the right-sided inverse z-transform

$$-2u[n] \xleftrightarrow{z} -\frac{2}{1 - z^{-1}}.$$

Combining the individual terms gives

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2(2)^n u[-n-1] - 2u[n].$$

▶ **Problem 7.4** **Repeat Example** (7.9) for the following ROCs:

(a) $\frac{1}{2} < |z| < 1$

(b) $|z| < 1/2$

Answers:

(a)

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2(2)^n u[-n-1] + 2u[-n-1]$$

(b)

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - 2(2)^n u[-n-1] + 2u[-n-1]$$

EXAMPLE 7.10 INVERSION OF AN IMPROPER RATIONAL FUNCTION Find the inverse z-transform of

$$X(z) = \frac{z^3 - 10z^2 - 4z + 4}{2z^2 - 2z - 4}, \quad \text{with ROC } |z| < 1.$$

Solution: The poles at $z = -1$ and $z = 2$ are found by determining the roots of the denominator polynomial. The ROC and pole locations in the z-plane are depicted in Fig. 7.13. We convert $X(z)$ into a ratio of polynomials in z^{-1} in accordance with Eq. (7.17). We do this by factoring z^3 from the numerator and $2z^2$ from the denominator, yielding

$$X(z) = \frac{1}{2}z\left(\frac{1 - 10z^{-1} - 4z^{-2} + 4z^{-3}}{1 - z^{-1} - 2z^{-2}}\right).$$

The factor $\frac{1}{2}z$ is easily incorporated later by using the time-shift property, so we focus on the ratio of polynomials in parentheses. Using long division to reduce the order of the numerator polynomial, we have

$$\begin{array}{r|l}
 & -2z^{-1} + 3 \\
\hline
-2z^{-2} - z^{-1} + 1 & 4z^{-3} - 4z^{-2} - 10z^{-1} + 1 \\
 & 4z^{-3} + 2z^{-2} - 2z^{-1} \\
\cline{2-2}
 & \quad -6z^{-2} - 8z^{-1} + 1 \\
 & \quad -6z^{-2} - 3z^{-1} + 3 \\
\cline{2-2}
 & \qquad - 5z^{-1} - 2
\end{array}.$$

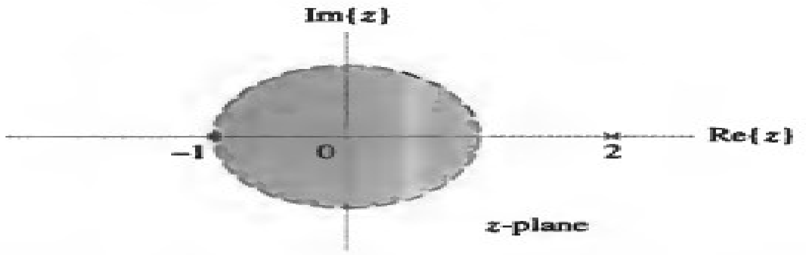


FIGURE 7.13 Locations of poles and ROC for Example 7.10.

Thus, we may write

$$\frac{1 - 10z^{-1} - 4z^{-2} + 4z^{-3}}{1 - z^{-1} - 2z^{-2}} = -2z^{-1} + 3 + \frac{-5z^{-1} - 2}{1 - z^{-1} - 2z^{-2}}$$

$$= -2z^{-1} + 3 + \frac{-5z^{-1} - 2}{(1 + z^{-1})(1 - 2z^{-1})}.$$

Next, using a partial-fraction expansion, we have

$$\frac{-5z^{-1} - 2}{(1 + z^{-1})(1 - 2z^{-1})} = \frac{1}{1 + z^{-1}} - \frac{3}{1 - 2z^{-1}}$$

and thus define

$$X(z) = \frac{1}{2} z W(z), \tag{7.18}$$

where

$$W(z) = -2z^{-1} + 3 + \frac{1}{1 + z^{-1}} - \frac{3}{1 - 2z^{-1}}, \quad \text{with ROC } |z| < 1.$$

The ROC has a smaller radius than either pole, as shown in Fig. 7.13, so the inverse z-transform of $W(z)$ *is*

$$w[n] = -2\delta[n - 1] + 3\delta[n] - (-1)^n u[-n - 1] + 3(2)^n u[-n - 1].$$

Finally, we apply the time-shift property (Eq. (7.13)) to Eq. (7.18) to obtain

$$x[n] = \frac{1}{2} w[n + 1]$$

and thus

$$x[n] = -\delta[n] + \frac{3}{2}\delta[n + 1] - \frac{1}{2}(-1)^{n+1} u[-n - 2] + 3(2)^n u[-n - 2].$$ ■

► **Problem 7.5** Find the time-domain signals corresponding to the following z-transforms:

(a)

$$X(z) = \frac{(1/4)z^{-1}}{(1 - (1/2)z^{-1})(1 - (1/4)z^{-1})}, \quad \text{with ROC } 1/4 < |z| < 1/2$$

(b)

$$X(z) = \frac{16z^2 - 2z + 1}{8z^2 + 2z - 1}, \quad \text{with ROC } |z| > \frac{1}{2}$$

(c)

$$X(z) = \frac{2z^3 + 2z^2 + 3z + 1}{2z^4 + 3z^3 + z^2}, \quad \text{with ROC } |z| > 1$$

Answers:

(a)

$$x[n] = -(1/4)^n u[n] - (1/2)^n u[-n - 1]$$

(b)

$$x[n] = -\delta[n] + \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{-1}{2}\right)^n u[n]$$

(c)

$$x[n] = \delta[n - 2] + 2(-1)^{n-1}u[n - 1] - (-1/2)^{n-1}u[n - 1]$$

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