Signals And Systems (UE17EC204)

Dr. T. S. Chandar

Department of ECE
PES University,
Bengaluru

Unit III
Representation of Periodic (Continuous-time & Discrete-time) signals using Fourier series.

(Chapter 3 of prescribed Textbook – Sections 3.1 to 3.7)

Introduction

- Representation of signals as the weighted superposition of delayed impulse responses studied in previous chapters.
- Representation of signals as the weighted superposition of complex sinusoids studied in this chapter.
- Frequency response of LTI systems studied in detail.

Frequency response of an LTI system

x^2

- Considering the output $\mathrm{y}[n]$ of a Discrete time LTI system with impulse response $h[n]$ and input $x[n] = e^{j\Omega n}$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{j\Omega(n-k)}$$

$$= e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k} = H(e^{j\Omega})\, e^{j\Omega n}$$

- Here, $H\left(e^{j\Omega}\right)$ is a complex scaling factor

x^2

Frequency response of an LTI system

- Analogously in continuous time,

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau \\
&= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau} d\tau = H(j\omega)\, e^{j\omega t}
\end{aligned}
$$

x^2

Frequency response of an LTI system

- The output of a complex sinusoidal input to an LTI system is a complex sinusoid of the same frequency as the input, multiplied by the frequency response of the system

$$H(j\omega) = |H(j\omega)|e^{j\,arg(H(j\omega))}$$

$$\therefore y(t) = |H(j\omega)|e^{j(\omega t + arg(H(j\omega)))}$$

- The system modifies the input's amplitude by $|H(j\omega)|$ and phase by $arg\big(H(j\omega)\big)$

x^2

Frequency response of an LTI system

- The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude
- In CT, $e^{st} \rightarrow H(s)e^{st}$
- In DT, $z^{n} \rightarrow H(n)z^{n}$
- Signal for which the system output is a constant (mostly complex) times the input is called an **Eigenfunction** of the system [e^{st} or z^{n}].
- Amplitude factor is referred to as the system's **Eigenvalue** [$H(s)$ or $H(z)$]

Frequency response of an LTI system

x^2

- Decomposition of general signals in terms of eigenfunction is useful:

If $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$, by eigenfunction property,

$$a_1 e^{s_1 t} \rightarrow a_1 H(s_1) e^{s_1 t}$$
$$a_2 e^{s_2 t} \rightarrow a_2 H(s_2) e^{s_2 t}$$
$$a_3 e^{s_3 t} \rightarrow a_3 H(s_3) e^{s_3 t}$$

- From superposition property, response to the sum is the sum of responses:

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

x^2

Frequency response of an LTI system

- Representation of signals as a linear combination of complex exponentials leads to a convenient expression for the response of an LTI system.
- If input $x(t) = \sum_k a_k e^{s_k t}$

output $y(t) = \sum_k a_k H(s_k) e^{s_k t}$

- Analogously for DT,

$$x[n] = \sum_k a_k {z_k}^n \,;\; y[n] = \sum_k a_k H(z_k) {z_k}^n$$

- For both CT and DT, the output can be represented as a linear combination of complex exponentials if input is represented as a linear combination of the same complex exponentials.

Fourier Series Representation of CT periodic signals

- Two basic periodic signals exist:

Sinusoidal $x(t) = \cos(\omega_0 t)$

Complex exponential $x(t) = e^{j\omega_0 t}$

Both signals are periodic with fundamental frequency ω_0 and fundamental period $T = \frac{2\pi}{\omega_0}$

- Set of harmonically related complex exponentials associated with $e^{j\omega_0 t}$ is denoted by $\Phi_k(t) \; = \; e^{jk\omega_0 t} \; = \; e^{jk\left(\frac{2\pi}{T}\right)t}, K = 0, \pm 1, \pm 2, etc.$

- For $|k| \geq 2$, fundamental period of $\Phi_k(t)$ is a fraction of T

Fourier Series Representation of CT periodic signals

- Therefore, linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

Is also periodic with period T

- For $k = 0$, $x(t)$ is constant
- For $k = \pm 1$, terms have fundamental frequency equal to ω_0 (fundamental components or first harmonic components)
- For $k = \pm 2$, terms periodic with half the period (twice the frequency; second harmonic components)
- For $k = \pm N$, Nth frequency components

Example 1:

Considering $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$

With fundamental frequency 2π

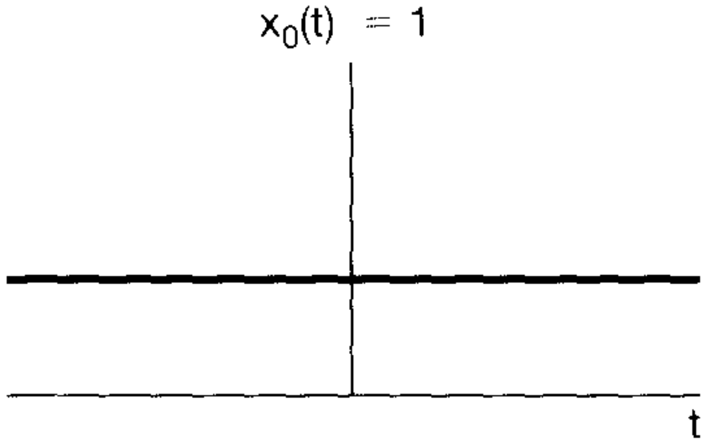
$a_0 = 1; a_1 = a_{-1} = \frac{1}{4}; a_2 = a_{-2} = \frac{1}{2}; a_3 = a_{-3} = \frac{1}{3}$

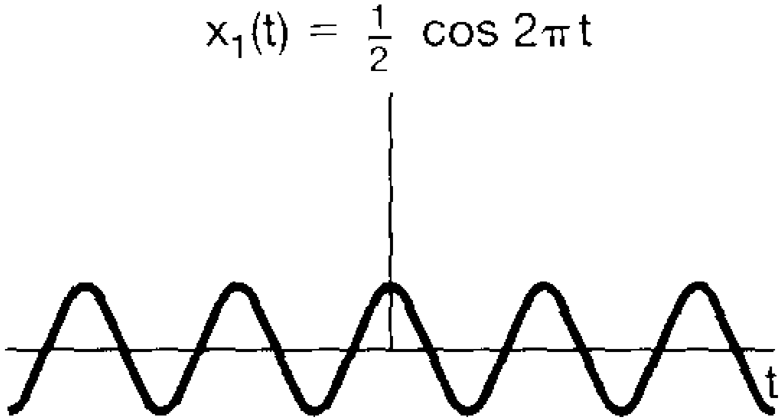
Expanding, we get

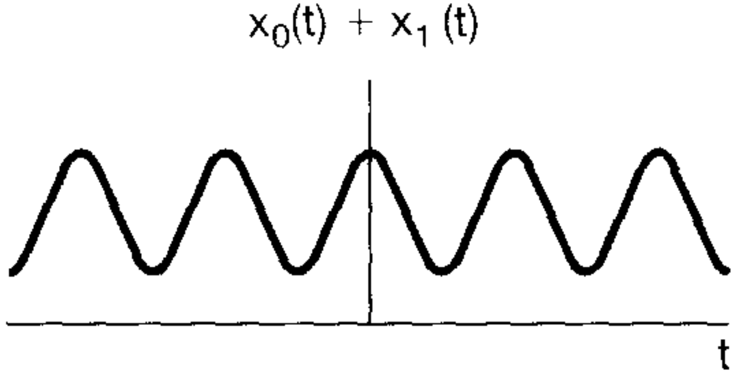
$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

Using Euler's relation,

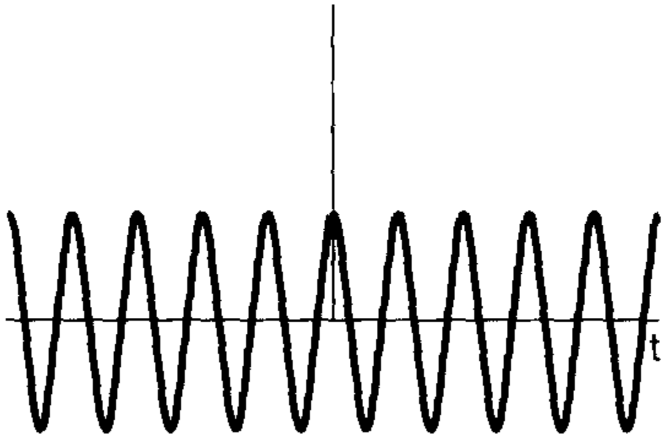
$$x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$



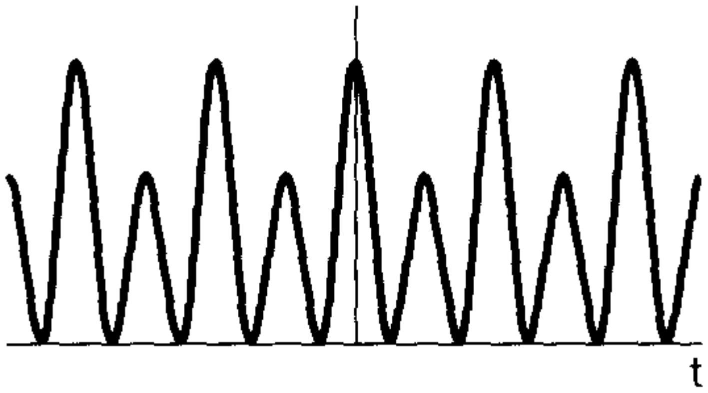




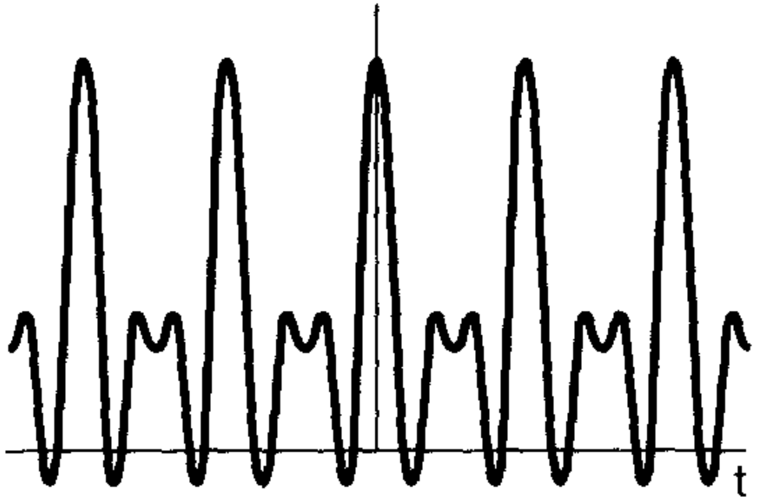
$x_2(t) = \cos 4\pi t$



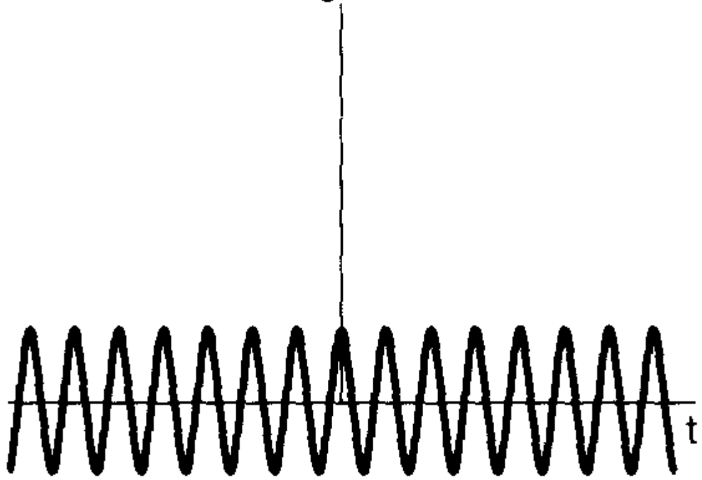
$x_0(t) + x_1(t) + x_2(t)$



$x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t)$



$x_3(t) = \frac{2}{3} \cos 6\pi t$



Fourier Series Representation of CT periodic signals

- Considering real $x(t), x(t) = x^*(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k{}^* e^{-jk\omega_0 t}$$

$$k \to -k, x(t) = \sum_{k=-\infty}^{\infty} a_{-k}{}^* e^{jk\omega_0 t}$$

Comparing with $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$,

$a_k = a_{-k}{}^*$ and $a_{-k} = a_k{}^*$

Fourier Series Representation of CT periodic signals

Alternate forms of Fourier Series,

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right] = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + {a_k}^{*} e^{-jk\omega_0 t}\right]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2Re\left\{a_k e^{jk\omega_0 t}\right\}$$

$a_k = A_k e^{j\theta}$ then,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

If $a_k = B_k + jC_k$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t)$$

Derivation of FS representation of CT periodic signals

We know that $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Multiply both sides by $e^{-jn\omega_0 t}$,

$$x(t)e^{-j\omega_0 nt} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-j\omega_0 nt}$$

Integrating both sides from $0\ to\ T = \frac{2\pi}{\omega_0}$,

$$\int_0^T x(t)e^{-j\omega_0 nt}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\, dt$$

Using Euler's formula,

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$

Derivation of FS representation of CT periodic signals

Therefore, $\int_0^T x(t)e^{-j\omega_0 nt}\,dt = Ta_n$

$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$, this holds true over any interval T

Therefore, $a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}\,dt$

“If $x(t)$ has a **Fourier Series representation** (an expression as a linear combination of harmonically related complex exponentials), the **Fourier Series coefficients** are given by the above equation”

Derivation of FS representation of CT periodic signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \rightarrow Equation\ 1$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t} = \frac{1}{T}\int_T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t} \qquad \rightarrow Equation\ 2$$

Equation 1:Synthesis equation

Equation 2:Analysis equation

Average power over one period of $x(t)$ is given by

$a_0 = \frac{1}{T}\int_T x(t)\,dt$ when k=0

Example 2:

Considering $x(t) = \sin(\omega_0 t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$

Comparing with synthesis equation,

$a_1 = -a_{-1} = \frac{1}{2j}; a_k = 0 \; for \; k \neq \pm 1$

*Example 3:

* - important

Periodic square wave $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T \end{cases}$

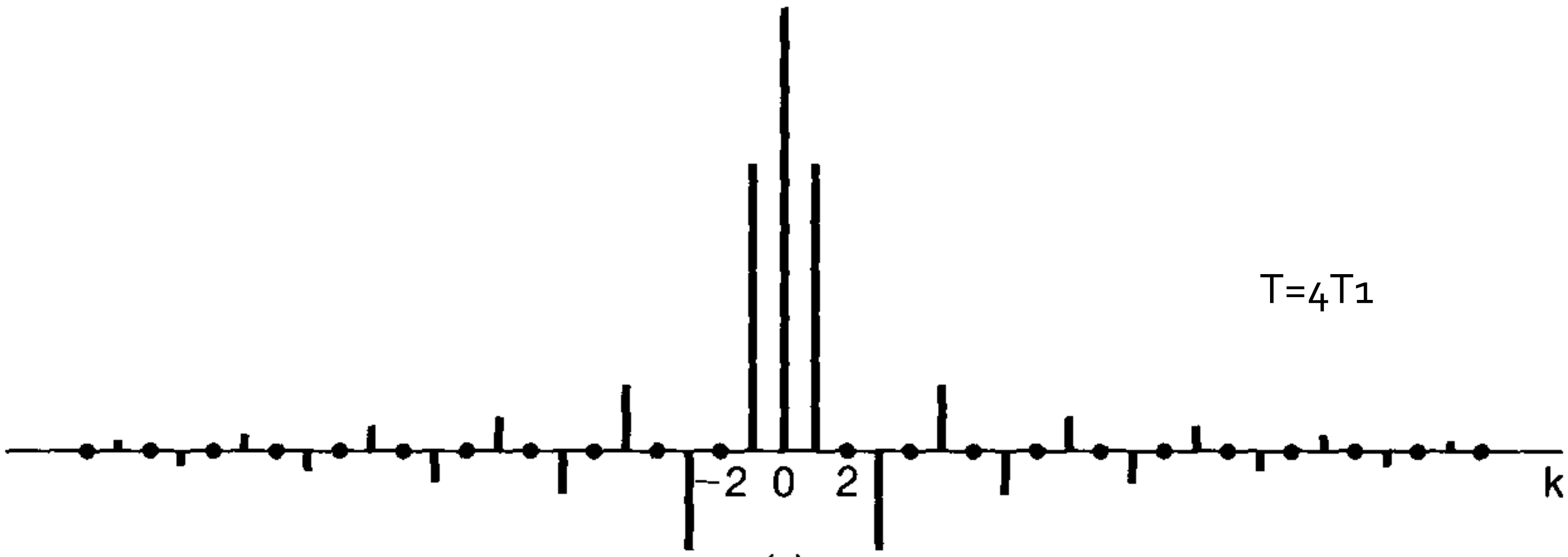
$$k = 0, a_0 = \frac{1}{T}\int_{-T1}^{T1} 1.dt = \frac{2T_1}{T}$$

$$k \neq 0, a_k = \frac{1}{T}\int_{-T1}^{T1} e^{-jk\omega_0 t}\, dt = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T1)}{k\pi}$$

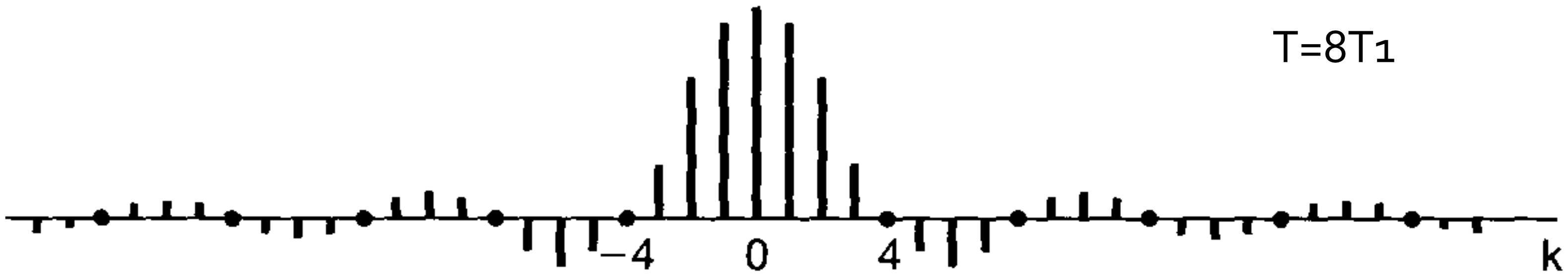
If $T = 4T_1$, $x(t)$ is a square wave which is unity for half period and zero for half period

$$\omega_0 T_1 = \frac{\pi}{2};\ a_k = \frac{\sin\left(\frac{\pi k}{2}\right)}{k\pi}$$

$$a_0 = \frac{1}{2};\ a_1 = a_{-1} = \frac{1}{\pi};\ a_3 = a_{-3} = -\frac{1}{3\pi};\ a_5 = a_{-5} = \frac{1}{5\pi}$$



(a)



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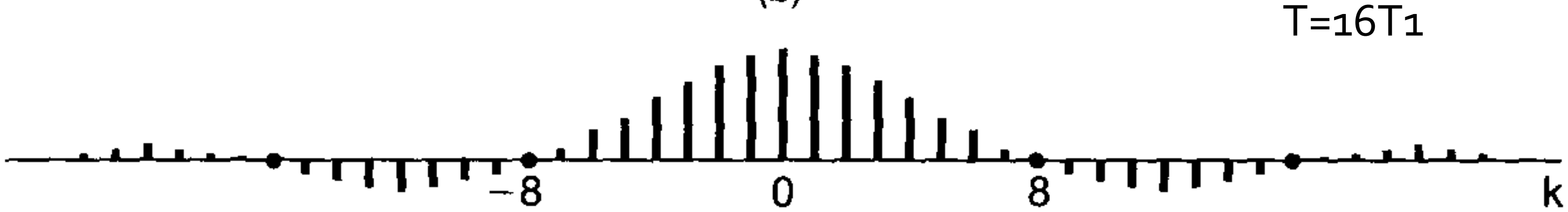

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Convergence of the Fourier Series

Approximating a given signal $x(t)$ by a linear combination of finite number of harmonically related complex exponentials,

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

Approximation error $e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$

Energy in error, $E_N = \int_T |e_N(t)|^2 dt$

to minimise error, $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$

The best approximation using only a finite number of harmonically related complex exponentials is by truncating the Fourier series to a desired number of terms: as $N \to \infty, E_N \to 0$

As N increases, new terms are added, energy in error decreases.

Zero energy does not imply that $x(t)$ and its Fourier Series Representation are equal at every t

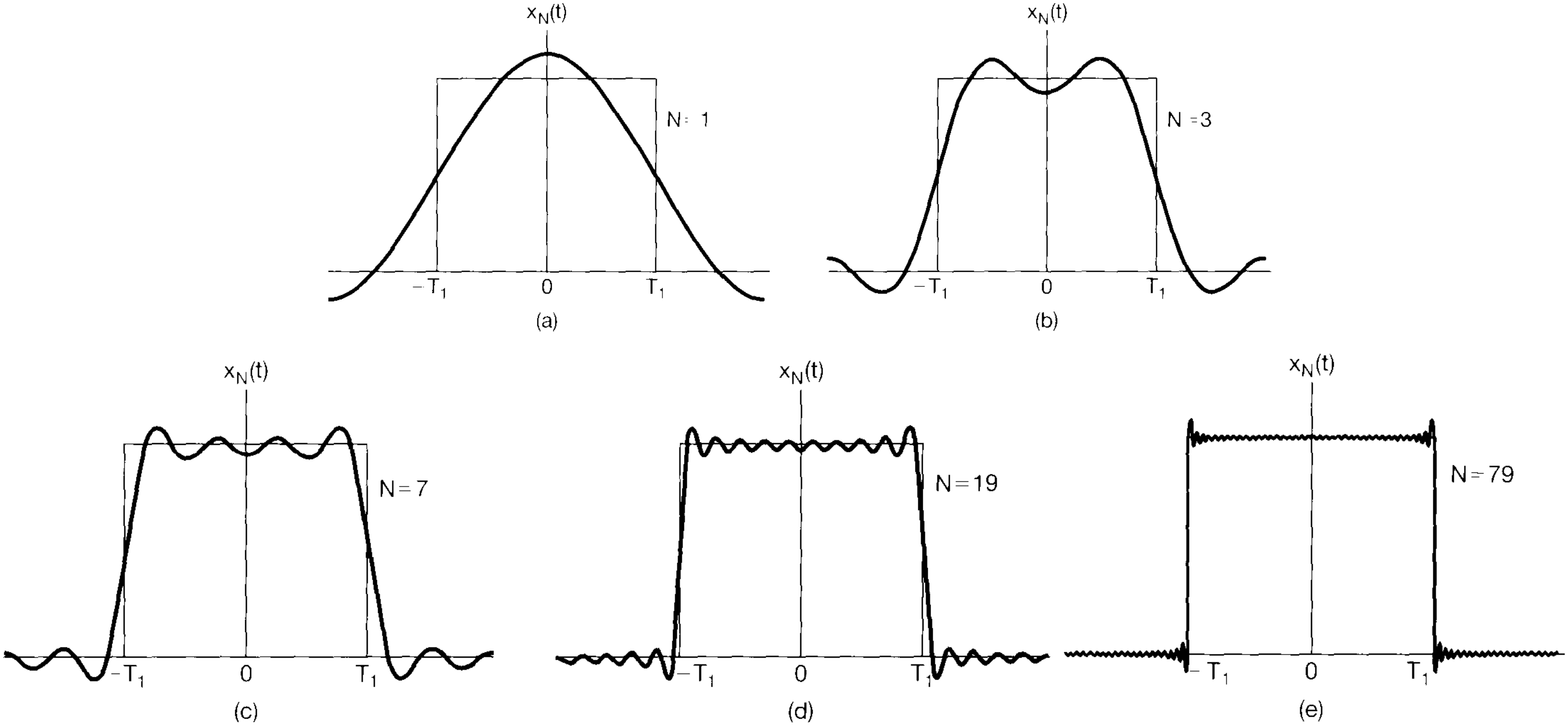
Dirichlet Conditions

- Over any period, $x(t)$ must be absolutely integrable: $\int_T |x(t)| dt < \infty$
- In any finite time interval, $x(t)$ is of bounded variation (there are no more than a finite number of maxima and minima during any single period of the signal
- In any finite time interval, only finite number of discontinuities exist
- $x(t)$ must be a single valued function.

Gibbs Phenomenon

- As N increases, ripples in the partial sums become compressed towards the discontinuities, but for any finite value of N, the peak amplitude of the ripples remains constant.
- The truncated Fourier Series approximation of $x(t)$, which is discontinuous, will show high frequency ripples and overshoots $x(t)$ near the discontinuities.
- N is large so that total energy is insignificant

Convergence of Fourier Series and Gibbs Phenomenon



Properties of CT Fourier Series

Notation used: $x(t) \overset{FS}{\leftrightarrow} a_k\ ;\ period\ T, \omega_0 = \frac{2\pi}{T}$

- **Linearity**: if $x(t) \overset{FS}{\leftrightarrow} a_k$ and $y(t) \overset{FS}{\leftrightarrow} b_k$ then linear combination is also periodic with period T

$$z(t) = Ax(t) + By(t) \overset{FS}{\leftrightarrow} Aa_k + Bb_k\ =\ c_k$$

- **Time shifting**: application of time shift to periodic $x(t)$ preserves period T

$$x(t - t_0) \overset{FS}{\leftrightarrow} e^{-jk\omega_0 t_0} a_k$$

- **Time scaling**: application of scaling changes the period of a signal

$$x(\alpha t) \overset{FS}{\leftrightarrow} a_k\ if\ \alpha > 0$$

Properties of CT Fourier Series

- **Time reversal**: period remains unchanged

$$x(-t) \overset{FS}{\leftrightarrow} a_{-k}$$

- **Multiplication**: $x(t)y(t) \overset{FS}{\leftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$
- **Time differentiation**: $\frac{d(x(t))}{dt} \overset{FS}{\leftrightarrow} j\omega_0 k a_k = \frac{jk2\pi a_k}{T}$
- **Time integration**: $\int_{-\infty}^{t} x(t)dt \overset{FS}{\leftrightarrow} a_k \frac{1}{jk\omega_0}$
- **Parseval's relation**: total average power in a periodic signal is the sum of average powers in all its components(sum of square values of the Fourier Series Coefficients)

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Properties of CT Fourier Series

- **Conjugate symmetry and conjugation**:

$$x^*(t) \overset{FS}{\leftrightarrow} {a^*}_k$$

For real $x(t)$, $x^*(t) = x(t)$

1. $(a_k)^* = a_{-k}$
2. $|a_k| = |a_{-k}|$
3. $\angle a_k = -\angle a_k$

For real and even $x(t)$, real and even a_k

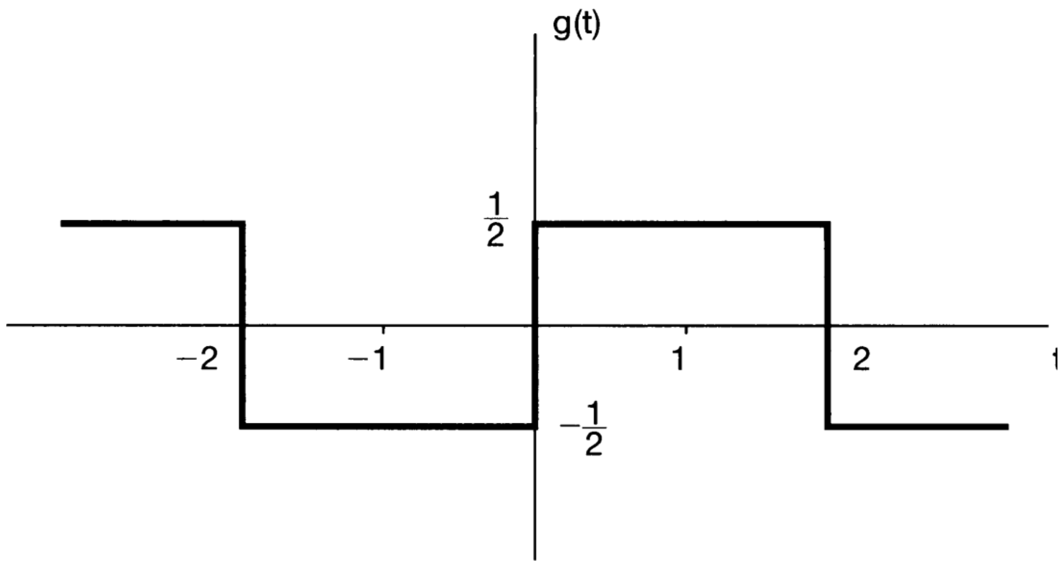
For real and odd $x(t)$, purely imaginary and odd a_k

Example 4:

Considering $g(t) = x(t-1) - \frac{1}{2}$; $g(t) \overset{FS}{\leftrightarrow} d_k$

$$x(t-1) \overset{FS}{\leftrightarrow} a_k e^{-\frac{jk\pi}{2}}$$

$$-\frac{1}{2} \overset{FS}{\leftrightarrow} \begin{cases} 0 & k \neq 0 \\ -\frac{1}{2} & k = 0 \end{cases}$$



$$\therefore d_k = \begin{cases} a_k e^{-\frac{jk\pi}{2}} & k \neq 0 \\ a_0 - \frac{1}{2} & k = 0 \end{cases} = \begin{cases} \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} & k \neq 0 \\ 0 & k = 0 \end{cases}$$

Fourier Series Representation of DT periodic signals

- Different from that of CT in the sense that the Fourier Series representation of a DT periodic signal is a finite series, whereas it is infinite for CT periodic signals.
- Signal $x[n]$ periodic when $x[n]$=$x[n+N]$ with period N
- Set of all DT periodic exponential signals is also periodic with period N and is given by $\Phi_k[n] = e^{jk\omega_0 n} = e^{jk\left(\frac{2\pi}{T}\right)n}, K = 0, \pm 1, \pm 2, etc$
- The signals above have fundamental frequencies which are multiples of $\frac{2\pi}{N}$ and are harmonically related

Fourier Series Representation of DT periodic signals

- Since DT complex exponentials which differ in frequency by a multiple of 2π are identical, only N distinct signals in $\Phi_k[n]$ exist: $\Phi_0[n]$=$\Phi_N[n]$; $\Phi_1[n] = \Phi_{N+1}[n]$, in general, $\Phi_k[n]$=$\Phi_{k+rN}[n]$
- Periodic sequences in terms of linear combinations:

$$x[n] = \sum_k a_k \Phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- Inclusion over N successive values of k is enough

$$x[n] = \sum_{k=<N>} a_k \Phi_k[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

Fourier Series Representation of DT periodic signals

- The exact same set of complex exponential sequences occur for any N successive values of k (be it $k: 0\ to\ N - 1\ (or)\ 3\ to\ N + 2$) in the summation in the previous slide, referred to as the DT Fourier Series
- For N successive values of n corresponding to one period, $x[n]$ is known as the Fourier Series representation.
- Set of n linear equations for N unknown a_k is as given:

$$x[n-1] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)(n-1)}$$

Fourier Series Representation of DT periodic signals

- We know that the sum over one period of the values of a periodic complex exponential is zero, unless the complex exponential is a constant.

$$\sum_{n=<N>} e^{jk\left(\frac{2\pi}{N}\right)n} = \begin{cases} N & k = 0, \pm N, \pm 2N \ldots \\ 0 & elsewhere \end{cases}$$

- Multiplying Fourier Series representation by $e^{-jr\left(\frac{2\pi}{N}\right)n}$, and summing over N terms,

$$\sum_{n=<N>} x[n]e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=<N>} \sum_{k=<N>} a_k\, e^{\frac{j(k-r)\left(\frac{2\pi}{N}\right)}{n}}$$

- Interchange order of summation,
- $\sum_{n=<N>} x[n]e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{k=<N>} a_k \sum_{n=<N>} e^{\frac{j(k-r)\left(\frac{2\pi}{N}\right)}{n}}$

Fourier Series Representation of DT periodic signals

- Innermost sum on RHS = $N \ if \ k = r; 0 \ if \ k \neq r$

Therefore, $Na_r = \sum_{n=<N>} x[n]e^{-jr\left(\frac{2\pi}{N}\right)n}$

And $a_r = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jr\left(\frac{2\pi}{N}\right)n}$

- DT Fourier series pair:

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \rightarrow Equation\ 1$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n} \rightarrow Equation\ 2$$

Equation 1:Synthesis equation

Equation 2:Analysis equation

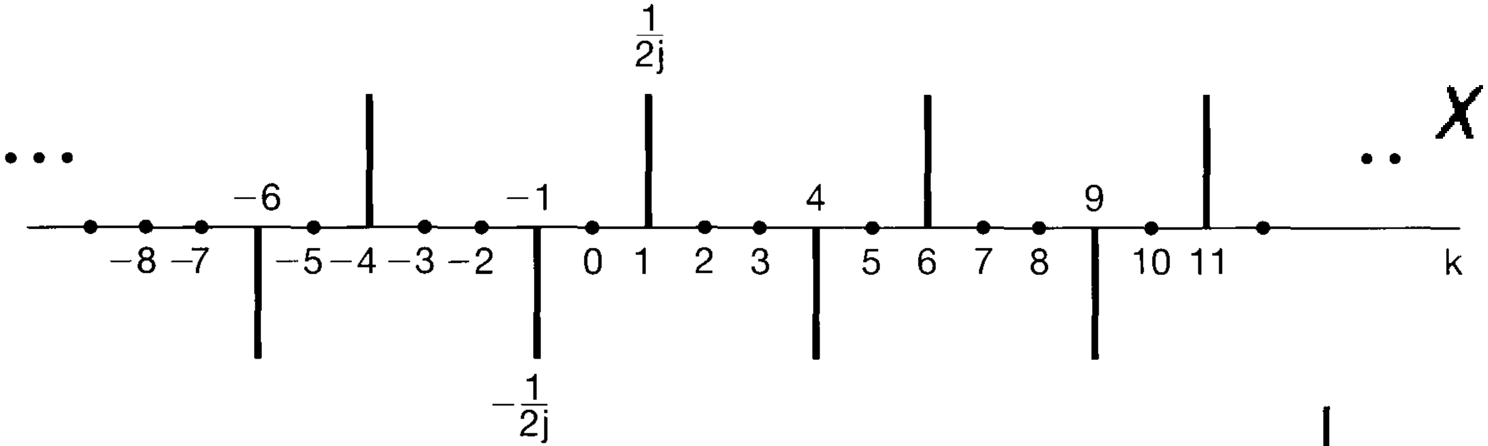
Fourier Series Representation of DT periodic signals

- a_k often referred to as spectral coefficients of x[n] specifying a decomposition of x[n] into a sum of N harmonically related complex exponentials
- Considering more than N sequential values of k, values of a_k repeat periodically with period N

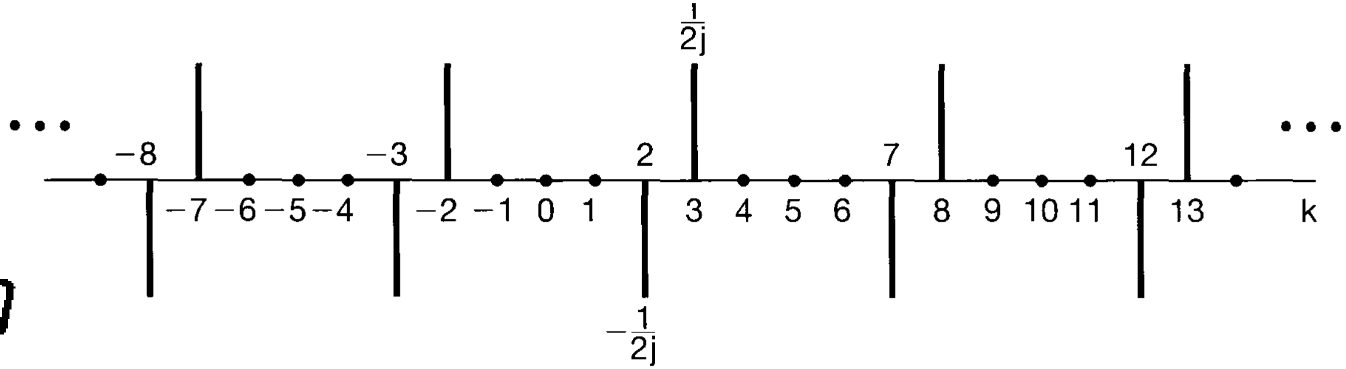
Example 5:

Consider $x[n] = \sin(\omega_0 n)$ *periodic when* $\frac{2\pi}{\omega_0} = N$; $\omega_0 = \frac{2\pi}{N}$

$x[n] = \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right]$; *by inspection,* $a_1 = -a_{-1} = \frac{1}{2j}$; rest all coefficients are zero.

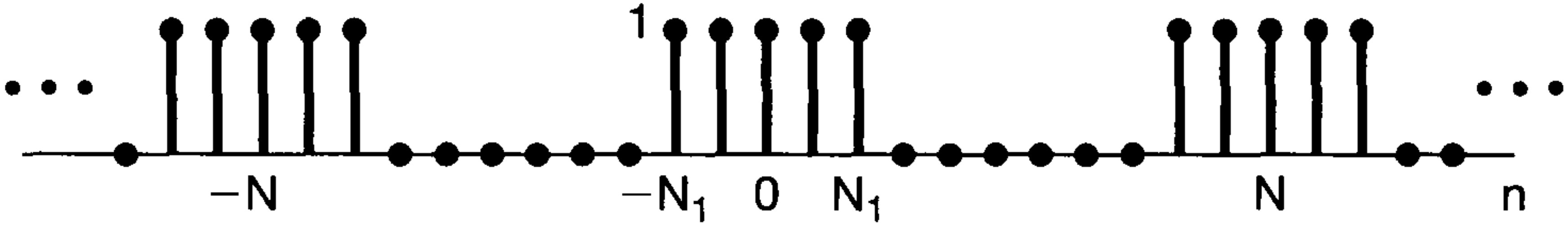


$x[n] = \sin(2\pi/5)n$



$x[n] = \sin 3(2\pi/5)n$

Example 6:



The above waveform has the following Fourier series coefficients:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\left(\frac{2\pi}{N}\right)n}$$

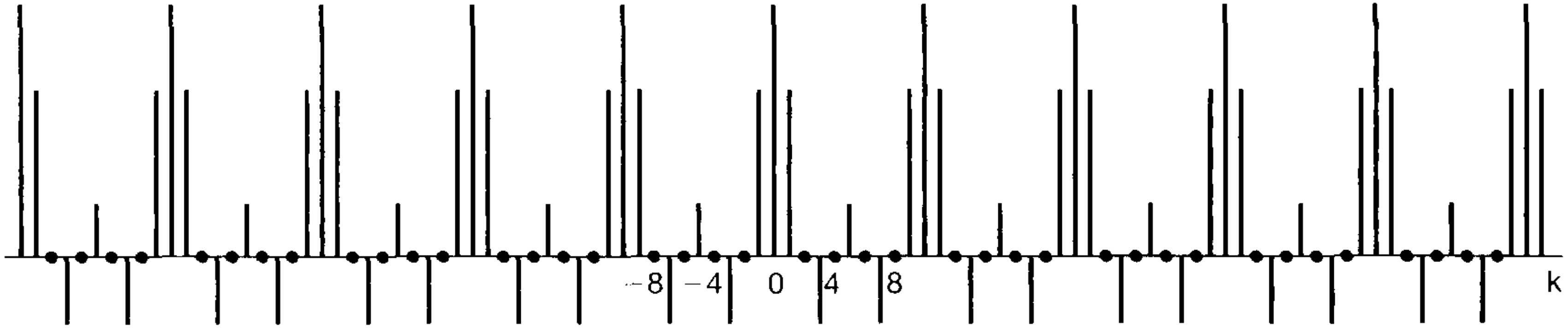
Considering $m = n + N_1$

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)} = \frac{1}{N}e^{jk\left(\frac{2\pi}{N}\right)N_1}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m)}$$

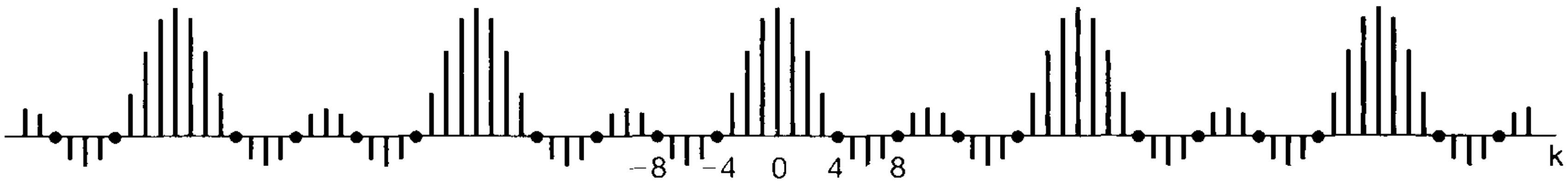
Summation consists of sum of first $2N_1 + 1$ terms in a geometric series

$$a_k = \frac{1}{N}e^{jk\left(\frac{2\pi}{N}\right)N_1}\left(\frac{1-e^{-\frac{jk2\pi(2N_1+1)}{N}}}{1-e^{-jk\left(\frac{2\pi}{N}\right)}}\right)$$

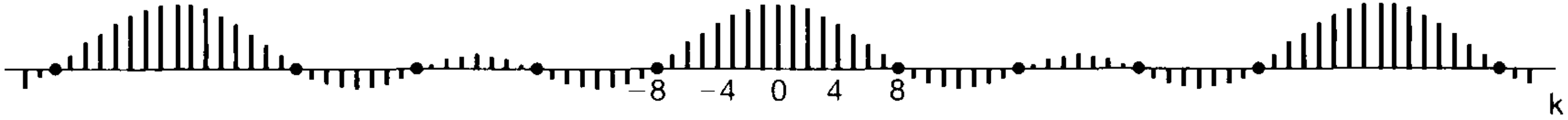
$$a_k = \frac{1}{N}\left(\frac{\sin\left(\frac{2\pi k(N_1+0.5)}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}\right) = \frac{2N_1+1}{N}, K = 0, \pm N, \pm 2N$$



(a)



(b)



(c)

$$2N_1 + 1 = 5$$

$$(a) N = 10$$

$$(b) N = 20$$

$$(c) N = 40$$

Properties of DT Fourier Series

- Notation used: $x[n] \overset{FS}{\leftrightarrow} a_k;\ y[n] \overset{FS}{\leftrightarrow} b_k$
- **Linearity:**$Ax[n] + By[n] \overset{FS}{\leftrightarrow} Aa_k + Bb_k$
- **Time shifting:**$x[n - n_0] \overset{FS}{\leftrightarrow} a_k e^{-jk\left(\frac{2\pi}{N}\right)n_0}$
- **Frequency shifting:**$e^{jM\left(\frac{2\pi}{N}\right)n} x[n] \overset{FS}{\leftrightarrow} a_{k-m}$
- **Conjugation:**$x^*[n] \overset{FS}{\leftrightarrow} a^*{}_{-k}$
- **Time reversal:**$x[-n] \overset{FS}{\leftrightarrow} a_{-k}$
- **Time scaling:**$x_{(m)}[n] = \begin{cases} x\left[\frac{n}{m}\right] & n = im_i \in \mathbb{I} \\ 0 & n \neq im \end{cases} \overset{FS}{\leftrightarrow} \frac{1}{m} a_k$

Properties of DT Fourier Series

- **Periodic convolution:** $\sum_{r=<N>} x[r]y[n-r] \overset{FS}{\leftrightarrow} N a_k b_k$
- **Multiplication:** $x[n]y[n] \overset{FS}{\leftrightarrow} \sum_{l=<N>} a_l b_{k-l}$
- **First difference:** $x[n]-x[n-1] \overset{FS}{\leftrightarrow} \left[1-e^{-jk\left(\frac{2\pi}{N}\right)}\right] a_k$
- **Running sum:** $\sum_{k=-\infty}^{\infty} x[k] \overset{FS}{\leftrightarrow} \left(\frac{1}{1-e^{-jk\left(\frac{2\pi}{N}\right)}}\right) a_k$
- **Parseval's relation:** $\frac{1}{N}\sum_{n=<N>}|x[n]|^2 = \sum_{k=<N>}|a_k|^2$
- **Conjugate symmetry and conjugation:**

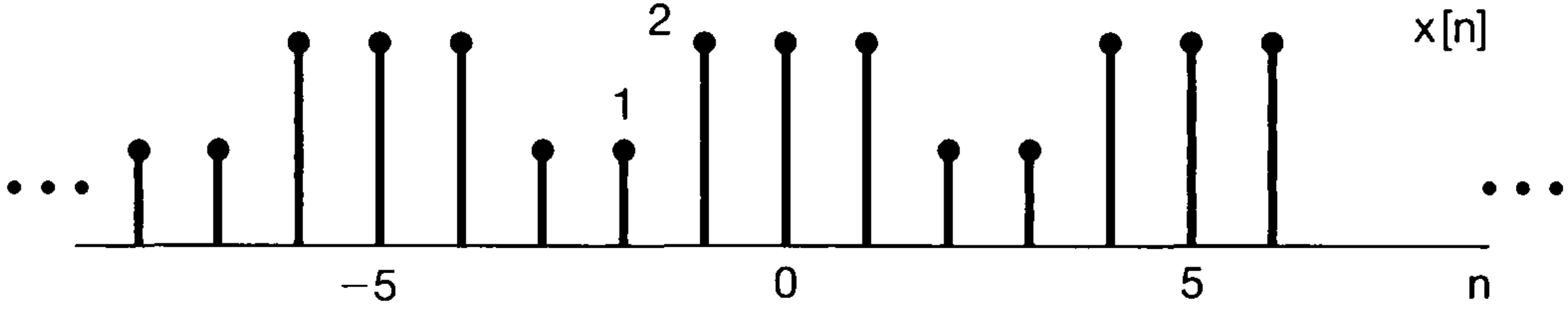
For real $x[n]$, $x^*[n] = x[n]$

1. $(a_k)^* = a_{-k}$
2. $|a_k| = |a_{-k}|$
3. $\angle a_k = -\angle a_k$

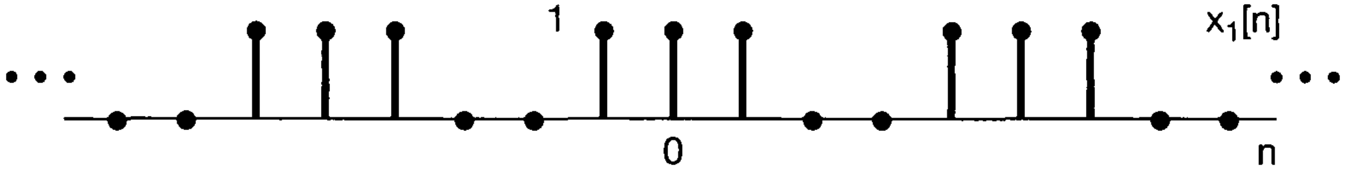
For real and even $x[n]$, real and even a_k

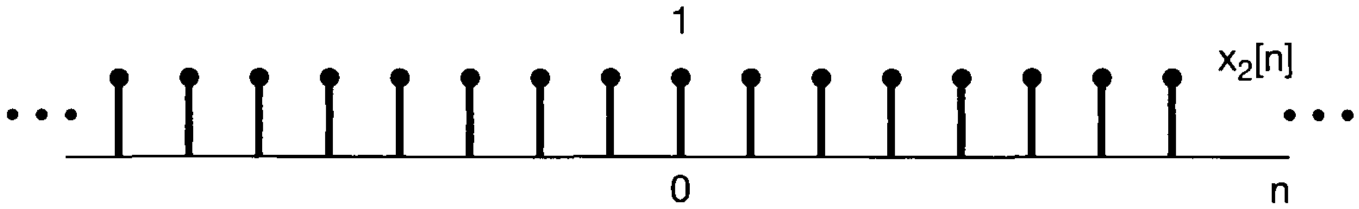
For real and odd $x[n]$, purely imaginary and odd a_k

Example 7:



$x[n]$ can be expressed as a sum of a square wave and a dc sequence.





Square wave $x_1[n]$, dc sequence $x_2[n]$

By linearity, $a_k = b_k + c_k;\ for\ x_1[n], b_k = \begin{cases} \frac{1}{5}\left(\frac{\sin\left(\frac{3\pi k}{5}\right)}{\sin\left(\frac{\pi k}{5}\right)}\right) & k \neq 0, \pm 5, \pm 10 \ldots \\ \frac{3}{5} & k = 0, \pm 5, \pm 10 \ldots \end{cases}, N_1 = 1, N = 5$

$for\ x_2[n]$, only DC value present,

$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1$

$c_k = 1\ when\ k = 5i, i \in \mathbb{I}$

$$\therefore a_k = \begin{cases} \frac{1}{5}\left(\frac{\sin\left(\frac{3\pi k}{5}\right)}{\sin\left(\frac{\pi k}{5}\right)}\right) & k \neq 0, \pm 5, \pm 10\ldots \\ \frac{8}{5} & k = 0, \pm 5, \pm 10 \end{cases}$$