Module - 3

Fourier Series — Representation of Periodic signals

$\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 8 \end{pmatrix} \begin{pmatrix} 0 \\ 7 \\ 9 \end{pmatrix} \right] \rightarrow \text{BASIS } (\hat{i}, \hat{j}, \hat{k})$$

→ Not unique
→ Linearly independent
→ Spanning entire domain (can find other from these)

BASIS

$(\hat{i} \quad \hat{j} \quad \hat{k})$ → This is basis which is orthonormal.

Continous - Periodic ⟶ Fourier series.
Continous - Aperiodic ⟶ Fourier Transform.
Discrete - periodic ⟶ Discrete time fourier series.
Discrete - Aperiodic ⟶ Discrete time fourier Transform.

The response of LTI system to complex exponentials:

→ The study of LTI systems to represent signals as linear combinations of basic signals should ~~sta~~ satisfy the following two properties.

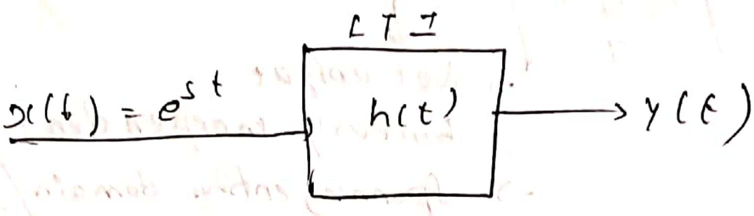
① The set of basic signals can be used to construct a board & usefull class of signals

② The response of an LTI system each signal ~~is~~ should be simple enough in structure to provide us with a convivent representation for the response of the system to any singal constructed as a linear combination of the basic signals

To show that complex exponentials are eigen functions of LTI system;

Consider a system with impulse response $h(t)$

let $x(t) = e^{st}$, $s \rightarrow$ complex number



$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$y(t) = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau$$

$$y(t) = e^{st} H(s)$$

$\Rightarrow$ e^{st} is the eigen function

$H(s)$ is the eigen value

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \leftarrow \text{same as this}$$

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt \rightarrow \text{Laplace function}$$

Hence, we have that complex exponentials of eigen functions of LTI system & the constant $H(s)$ for a specific value of s is then a eigen value associated with the eigen function.

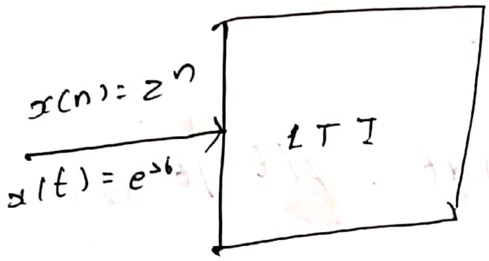
(*) Consider an LTI system with I/p $x[n] = z^n$, let $h(n)$ be the I/p response, then $y(n) = h(n) * x(n)$.

$$= \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k) \cdot z^{n-k}$$

$$y(n) = z^n \sum_{-\infty}^{\infty} h(k) z^{-k}$$

$$y(n) = z^n \cdot H(z), \quad \text{where } H(z) = \sum_{-\infty}^{\infty} h(k) z^{-k}.$$



$$Y(n) = z^n H(z) \rightarrow H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$$

$$Y(t) = e^{st} H(s) \rightarrow H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

Q:- Given I/p $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$
find $y(t)$

$x(t)$ is a linear combination of complex exponentials

$$x(t) = \begin{cases} a_1 e^{s_1 t} \longrightarrow a_1 H(s_1) e^{s_1 t} \\ a_2 e^{s_2 t} \longrightarrow a_2 \cdot H(s_2) e^{s_2 t} \\ a_3 e^{s_3 t} \longrightarrow a_3 H(s_3) e^{s_3 t} \end{cases}$$

we can do this because the system is linear

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

In general, if $x(t) = \sum_k a_k e^{s_k t} \rightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$

$||^{ly}$ if $x(n) = \sum_k a_k z_k^n$

then, $y(n) = \sum_k a_k H(z_k) \cdot z_k^n$

For Both CT & DT systems if the I/p to an LTI system is represented as a linear combination of complex exponentials, then the o/p can also be represent as a linear combination of same complex exponential.

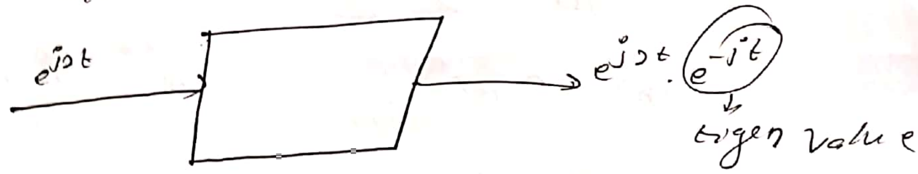
Each co-efficient in the representation of O/p is obtained as a product of corresponding coeff. of I/p & systems Eigen value.

Ex 3.1:

$y(t) = x(t-3)$

if $x(t) = e^{j2t}$

$x(t-3) = e^{j2(t-3)} = e^{2jt} \cdot e^{-j6}$



$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$H(s) = \int_{-\infty}^{\infty} \phi(t)\,\delta(t-t_0)\,dt = \phi(t)\Big|_{t=t_0}$$

$$H(s) = e^{-3s}$$

$$e^{j2t} \longrightarrow e^{j2t} \cdot H(2j)$$

$$e^{j2t} \cdot e^{-6j}$$

$$x(t) = \cos 4t + \cos 7t$$

$$x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-j4t} + \frac{1}{2} e^{j7t} + \frac{1}{2} e^{-j7t}$$

$$\frac{1}{2} e^{j4t} \longrightarrow \frac{1}{2} e^{j4t} H(j4)$$

$$\longrightarrow \frac{1}{2} e^{j4t} e^{-12j} - 1$$

Fourier Series representation of continuous time periodic signals:

$x(t) = x(t+T) \quad \forall t$

$x(t) = \cos \omega_0 t$, $x(t) = e^{j\omega_0 t}$, $T = \frac{2\pi}{\omega_0}$

$e^{jk\omega_0 t}$, $k \to$ integer

Set of harmonically related complex exponentials is given by $\phi_k(t) = e^{jk\omega_0 t}$, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_{-2} e^{-j2\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t}$$

A linear combination of harmonically related complex exponentials of the form,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \longrightarrow (1)$$

(Harmonically related complex exponentials have same periodicity)

$k=0 \Rightarrow$ constant

$k=1 \Rightarrow$ First harmonic signals

$k=2 \Rightarrow$ second — " ——

⋮ ⋮

The representation of a periodic signal in the form of eqn (1), is referred to as fourier series representation.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$a_k \rightarrow$ fourier co-efficient

Ex 3.2: Consider a periodic signal $x(t)$ with fundamental ~~fo~~ frequency 2π & is ~~kep~~ expressed in the form

$x(t) = \sum_{k=-3}^{+3} a_k e^{jk\omega_0 t}$. Given, $a_0 = 1$, $a_1 = a_{-1} = 1/4$, $a_2 = a_{-2} = 1/2$

$a_3 = a_{-3} = 1/3$.

Sol:- $x(t) = a_{-3} e^{-j3\omega_0 t} + a_{-2} e^{-j2\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_0 +$

$a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_3 e^{j3\omega_0 t}$

$x(t) = \frac{1}{3} e^{-j6\pi t} + \frac{1}{2} e^{-j4\pi t} + \frac{1}{4} e^{-j2\pi t} + 1 + \frac{1}{4} e^{j2\pi t} +$

$\frac{1}{2} e^{j4\pi t} + \frac{1}{3} e^{+j6\pi t}$

$= \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) +$

$\frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right) + 1$

$= \frac{1}{4} \times 2 \cos(2\pi t) + \frac{1}{2} \times 2 \cos(4\pi t) + \frac{1}{3} \times 2 \cos(6\pi t) + 1$

$= \frac{1}{2} \cos(2\pi t) + \cos(4\pi t) + \frac{2}{3} \cos(6\pi t) + 1$

If $x(t)$ is real, $\Rightarrow a_k = a_{-k}$

$$x^*(t) = x(t)$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = x(t) \quad (\because x(t) \text{ is real})$$

Replacing k with -k

$$x(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} \text{ --- } (2)$$

Comparing (1) & (2),

$$\Rightarrow \boxed{a_k = a_{-k}^*}$$

If a_k is real, then $\boxed{a_k = a_{-k}}$

Alternate forms of Fourier series :-

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{jk\omega_0 t} \right]$$

$$= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right]$$

$$= a_0 + 2\,Re\left[\sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} \right]$$

i) If $a_k = A_k e^{-j\theta k}$

$$x(t) = a_0 + 2 \sum_{k=-1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

ii) If $a_k = B_k + j C_k$

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} \left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t \right]$$

Fourier series representation :-

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow (1)$$

$a_k \rightarrow$ fourier co-efficients

$a_k \rightarrow$ complex

$a_k = A_k e^{j\theta_k}$

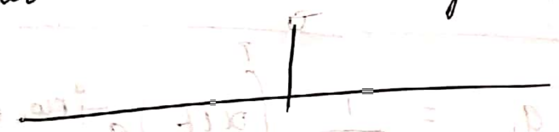
A_k vs ω — Magnitude spectrum

θ_k v/s ω — Phase spectrum

a_k, $k = 0, \pm 1, \pm 2, \ldots$

discrete

$e^{jn\omega_0 t} \rightarrow 1$ single ~~tone~~



To find a_k :-

$\boxed{\text{Inner product : } f(x), g(x) \quad \langle f(x) \cdot g(x) \rangle \triangleq \int f(x)\, g^*(x)\, dx}$

Multiplying both side of eqn (1) by $e^{-jn\omega_0 t}$, we obtain

$$x(t) e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

Integrating both sides from 0 to $T = 2\pi/\omega_0$, we get:

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt \rightarrow (2)$$

Case 1 : for $k \neq n$

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos(k-n)\omega_0 t\,dt + \int_0^T \sin(k-n)\omega_0 t\,dt$$

$$= 0$$

Case 2 : for $k = n$

$$\int_0^T e^0\,dt = T$$

$$\therefore (2) \Rightarrow \int_0^T x(t)\,e^{-jn\omega_0 t} = a_n T$$

$$\Rightarrow a_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt$$

$$\boxed{a_k = \frac{1}{T}\int_0^T x(t)\,e^{-jk\omega_0 t}\,dt}$$

Generalised form

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)\,e^{-jk\omega_0 t}\,dt$$ Analysis eqn

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\,e^{jk\omega_0 t}$$ Synthesis eqn

The set of co-efficients $[a_k] \Rightarrow$ Fourier series Co-efficients or spectral coefficient of $x(t)$

$$\boxed{a_0 = \frac{1}{T}\int_{\langle T\rangle} x(t)\,dt}$$

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(1) Calculate the fourier series coefficients for the following signals

Ex 3.3 $x(t) = \sin \omega_0 t$

Sol:-

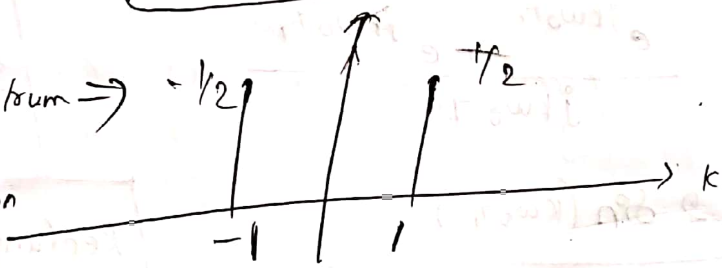
$$x(t) = \frac{e^{j\omega_0 t}}{2j} - \frac{e^{-j\omega_0 t}}{2j}$$

$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{j\omega_0 t}$$

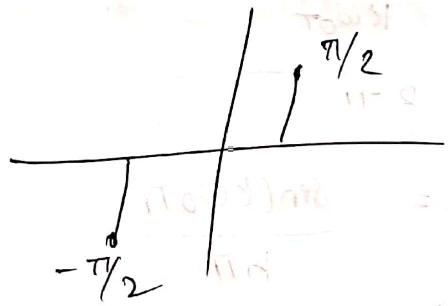
By comparing with $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, we get

$$\boxed{a_1 = \frac{1}{2j} \; ; \; a_{-1} = \frac{-1}{2j}}$$

Magnitude spectrum →
↓
even function



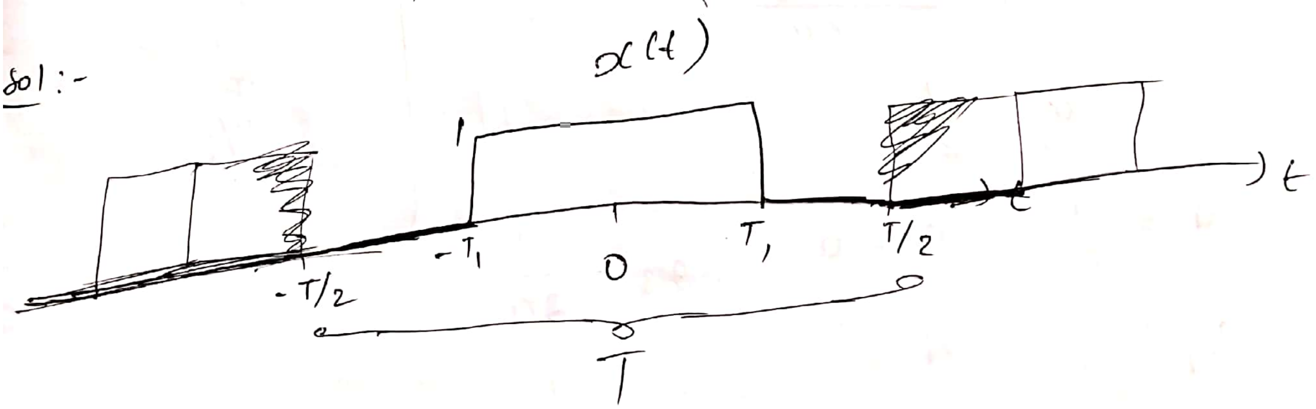
phase spectrum ⇒
↓
odd function



Ex 3.4 $x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$

Ex 3.5
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

Sol:-



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} 1\; e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1}$$

$$= \frac{1}{T}\left[\frac{e^{-jk\omega_0 T_1}}{-jk\omega_0} + \frac{e^{jk\omega_0 T_1}}{jk\omega_0}\right]$$

$$= \frac{e^{jk\omega_0 T_1} \cancel{+} - e^{-jk\omega_0 T_1}}{jk\omega_0 T}$$

$$\boxed{\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta}$$

$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$\boxed{\text{Rectangular} \leftrightarrows \text{sinc (Fourier pair)}}$

Since, $\omega_0 T = 2\pi$

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

WKT, $a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$

Consider $T = 4T_1$,

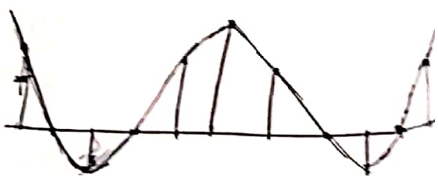
$$a_k = \frac{\sin(k\pi/2)}{k\pi}$$

$$\omega_0 T_1 = \frac{2\pi}{T} \times T_1 = \frac{2\pi}{\cancel{4T_1}} \times T_1 = \pi/2$$

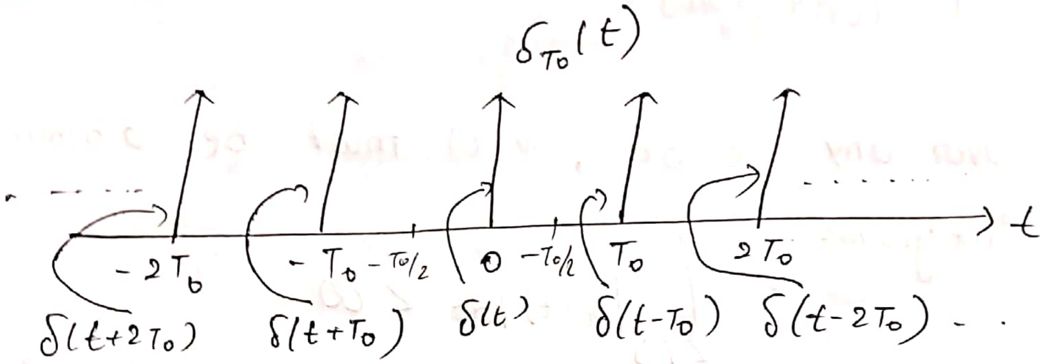
$$a_0 = \frac{1}{2}$$

$a_1 = \frac{1}{\pi}$ $\quad a_2 = 0$ $\quad a_3 = \frac{1}{3\pi}$

$a_{-1} = \frac{1}{\pi}$ $\quad a_{-2} = 0$ $\quad a_{-3} = -\frac{1}{3\pi}$



Consider the periodic Impulse train [$\delta_{T_0}(t)$] determine the complex fourier series representation :-



$$\boxed{\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_0) \qquad k = 0, \pm 1, \pm 2, \pm 3, \ldots}$$

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

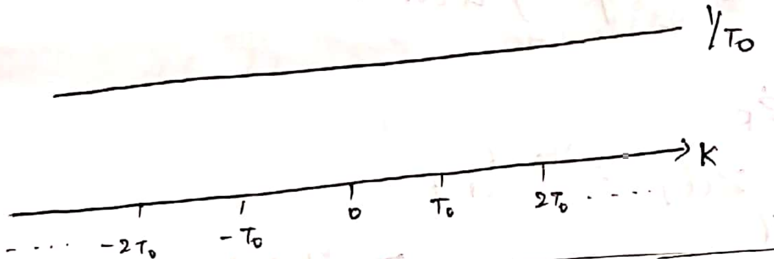
$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{T_0} \left. e^{-jk\omega_0 t} \right|_{t=0}$$

$$a_k = \frac{1}{T_0}$$

Sampling property

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt = \phi(t)\Big|_{t=0}$$



$$\Rightarrow \boxed{\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0}\, e^{jk\omega_0 t}}$$

Convergence of fourier series :-

Dirichlet conditions :-

Condition 1: Over any period, $x(t)$ must be absolutely integrable.

i.e, $\int_{\langle T \rangle} |x(t)|\,dt < \infty$

Condition 2: In any finite interval of time $x(t)$ is of bounded variation i.e., there are no more than a finite number of maxima & minima during any single period of the signal.

Condition 3: In any finite interval of time there are only a finite number of discontinuties

Properties of fourier series :-

$x(t) \xrightarrow{F.S} a_k$

This means $x(t)$ has a_k as fourier coefficie-nts

① Linearity :-

Let $x(t) \xrightarrow{FS} a_k$, $y(t) \xrightarrow{FS} b_k$

$z(t) = a\,x(t) + b\,y(t) \xrightarrow{FS} ?$

Proof: We have $a_k = \frac{1}{T}\int_T x(t)\, e^{-jw_0 kt}\,dt$

Let $z(t) \xleftrightarrow{FS} z_k$

then $z_k = \frac{1}{T}\int_T z(t)\, e^{-jkw_0 t}\,dt$

$z_k = \frac{1}{T}\int_{\langle T \rangle} [a\,x(t) + b\,y(t)]\, e^{-jkw_0 t}\,dt$

$z_k = a\frac{1}{T}\int_T x(t)\, e^{-jkw_0 t}\,dt + b\frac{1}{T}\int_T y(t)\, e^{-jw_0 t}\,dt$

$z_k = a\,a_k + b\,b_k$

$a\,x(t) + b\,y(t) \xleftrightarrow{FS} a\,a_k + b\,b_k$

② Time-shift :

If $x(t) \xleftrightarrow{F.S} a_k$

then $y(t) = x(t-t_0) \xleftrightarrow{F.S} b_k = e^{-jk\omega_0 t_0} a_k$

Proof : Let $y(t) \xleftrightarrow{F.S} b_k$

then $b_k = \frac{1}{T}\int_{\langle T\rangle} y(t)\, e^{-jk\omega_0 t}\, dt$

$$b_k = \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

$t - t_0 = \lambda \Rightarrow dt = d\lambda$

$t = t_0 + \lambda$

$$b_k = \frac{1}{T}\int_T x(\lambda)\, e^{-jk\omega_0 \lambda}\, e^{-jk\omega_0 t_0}\, d\lambda$$

$$b_k = e^{-jk\omega_0 t_0} a_k$$

③ Time reversal : If $x(t) \xleftrightarrow{F.S} a_k$

then $y(t) = x(-t) \xleftrightarrow{F.S} b_k = a_{-k}$

Proof : $x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{+jk\omega_0 t}$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k\, e^{-jk\omega_0 t}$$

Let $k = -m$

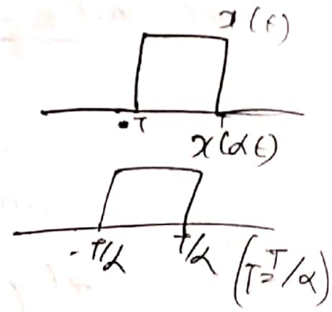
$$\Rightarrow x(-t) = \sum_{m=-\infty}^{\infty} a_{-m}\, e^{jm\omega_0 t}$$

$\Rightarrow x(-t) \xleftrightarrow{F.S} a_{-k}$

④ Time-Scaling:

If $x(t) \xleftrightarrow{F.S} a_k$

then $y(t) = x(\alpha t) \xleftrightarrow{F.S} b_k = a_k$



Proof: $a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$

$$b_k = \frac{1}{T/\alpha}\int_{T/\alpha} x(\alpha t)\, e^{-jk\alpha\omega_0 t}\, dt$$

$$\alpha t = \lambda \Rightarrow dt = d\lambda/\alpha$$

$$b_k = \frac{\cancel{\alpha}}{\cancel{T}}\int_{T/\alpha} x(\lambda)\, e^{-jk\omega_0 \lambda}\, d\lambda/\cancel{\alpha}$$

$$b_k = \frac{1}{T}\int_T x(\lambda)\, e^{-jk\omega_0 \lambda}\, d\lambda$$

$$\boxed{b_k = a_k}$$

⑤ Time-Differentiation:

If $x(t) \xleftrightarrow{F.S} a_k$

then $y(t) = \frac{dx(t)}{dt} \xleftrightarrow{F.S} b_k = jk\omega_0 a_k$

Proof: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} a_k jk\omega_0\, e^{jk\omega_0 t}$$

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{F.S} jk\omega_0 a_k}$$

⑥ Multiplication :

If $x(t) \xleftrightarrow{F.S} a_k$

$y(t) \xleftrightarrow{F.S} b_k$

then $z(t) = x(t) \cdot y(t) \xleftrightarrow{F.S} c_k = a_k * b_k$

Proof :

$$c_k = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt$$

$$c_k = \frac{1}{T}\int_T x(t)\, y(t)\, e^{-jk\omega_0 t}\, dt$$

$$c_k = \frac{1}{T}\int_T \left[\sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t}\right] y(t)\, e^{-jk\omega_0 t}\, dt$$

$$c_k = \frac{1}{T}\sum_{l=-\infty}^{\infty} a_l \int_T y(t)\, e^{-j(k-l)\omega_0 t}\, dt$$

$$c_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}$$

$$\boxed{c_k = a_k * b_k}$$

⑦ conjugation :

if $x(t) \xleftrightarrow{F.S} a_k$

then $x^*(t) \xleftrightarrow{F.S} a^*_{-k}$

Proof:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a^*_k\, e^{-jk\omega_0 t}$$

Replacing k with $-k$,

$$x^*(t) = \sum_{k=-\infty}^{\infty} a^*_{-k}\, e^{jk\omega_0 t}$$

$\Rightarrow x^*(t) \xleftrightarrow{F.S} a^*_{-k}$

further, If $x(t)$ is real,

then $x^*(t) = x(t)$

$\Rightarrow a_k = a^*_{-k}$

(or) $\boxed{a_k^* = a_{-k}}$

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8. Parseval's Relation:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Proof: LHS: $\frac{1}{T}\int_T x(t)\, x^*(t)\, dt$

$$= \frac{1}{T}\int_T x(t)\left[\sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}\right] dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^* \int_T x(t) e^{-jk\omega_0 t}\, dt$$

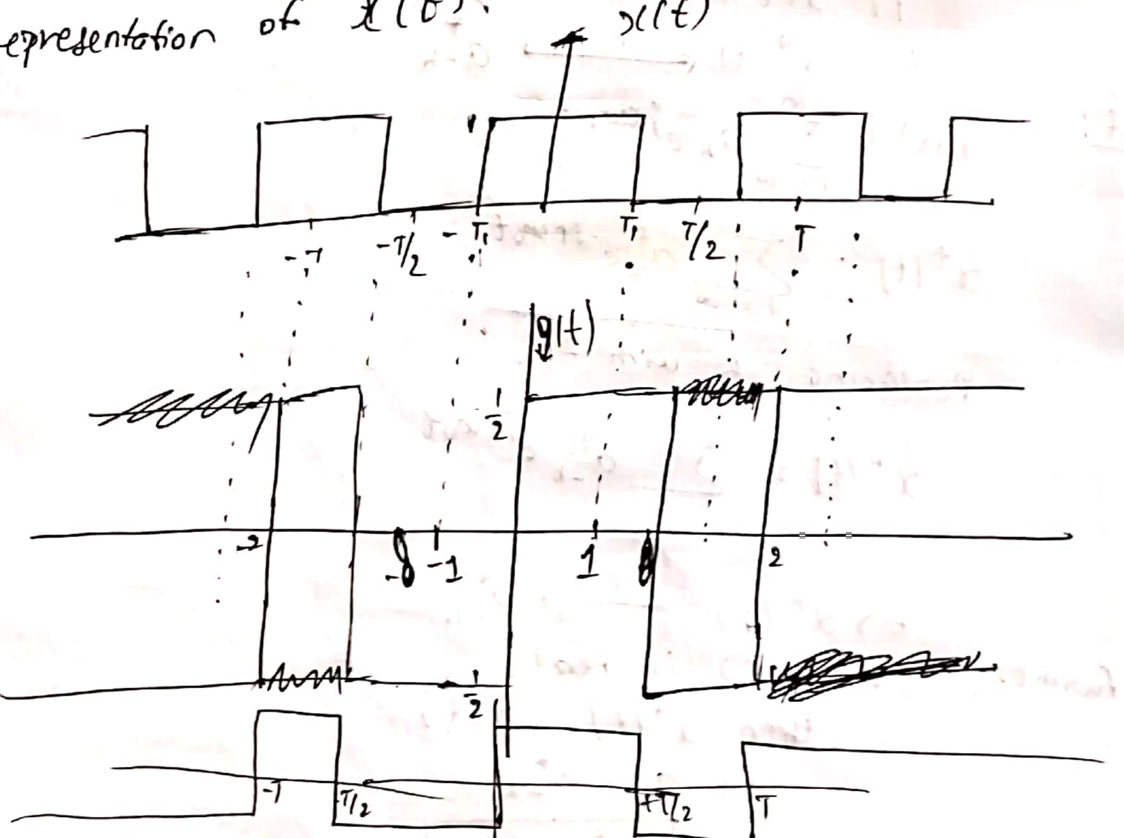
$$= \sum_{k=-\infty}^{\infty} a_k^* a_k$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2$$

= RHS

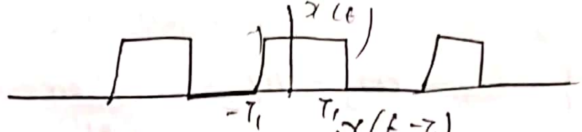
Problems:-

Ex 3.6 Find the F.S representation of $g(t)$ knowing F.S representation of $x(t)$.

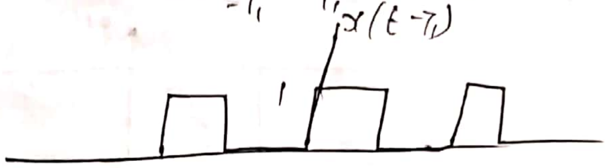


Sol :-

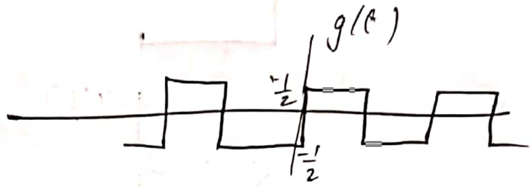
$x(t)$ $\Rightarrow$



$\Rightarrow x(t-T_1)$ $\Rightarrow$



$\Rightarrow x(t-T_1)-\frac{1}{2}$ $\Rightarrow$



$g(t) = x(t-T_1) - \frac{1}{2}$

Refering to $x(t)$, $T=4$ & $T_1 = 1$

$$x(t) \longrightarrow x(t-T_0) - \frac{1}{2} \longrightarrow g(t)$$

$$\Rightarrow g(t) = x(t-T_0) - \frac{1}{2}$$

$$x(t) \xleftrightarrow{F.S} a_k$$

$$x(t-t_0) \xleftrightarrow{F.S} e^{-jk\omega_0 t_0} a_k$$

$$x(t-1) \xleftrightarrow{F.S} e^{-jk\omega_0} a_k = b_k$$

$$+\frac{1}{2} \xleftrightarrow{F.S} c_k = +1/2 \quad , \quad k=0$$

Let $g(t) \xleftrightarrow{F.S} d_k$

$$d_k = \begin{cases} e^{-jk\omega_0} a_k , & k \neq 0 \\ a_0 - 1/2 , & k = 0 \end{cases}$$

c_0

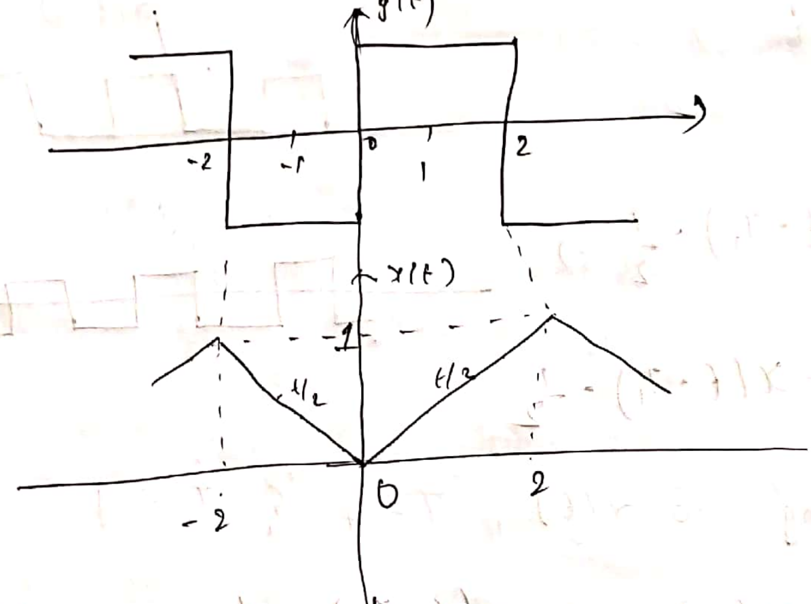
$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$, $k \neq 0$

$a_0 = \frac{1}{2}$, $k=0$

$$d_k = \begin{cases} e^{-jk\pi/2} \dfrac{\sin k\pi/2}{k\pi} , & k \neq 0 \\ 0 , & k=0 \end{cases}$$

Q. Determine the F.S representation $x(t)$

$T = 4$ { ~~fund~~ $\omega_0 = \pi/2$. Given the function $g(t)$



sol:-

$$g(t) = \frac{dx(t)}{dt}$$

$$x(t) \xleftrightarrow{F.S} e_k$$

$$g(t) = \frac{dx(t)}{dt} \xleftrightarrow{F.S} jk\omega_0 \, e_k = d_k \text{ (Previous solution)}$$

$$d_k = jk\omega_0 e_k$$

$$e_k = \frac{d_k}{jk\omega_0}$$

$$e_k = \frac{2d_k}{jk\pi}$$

$$e_k = \frac{2\,\frac{\sin(k\pi/2)}{\cancel{k\pi}}}{j\cancel{k\pi}}$$

$$e_k = \frac{2\sin(k\pi/2)}{j}$$

$$\boxed{e_k = \begin{cases} \frac{2}{j}\sin\left(\frac{k\pi}{2}\right), & k \neq 0 \\ 0, & k = 0 \end{cases}}$$

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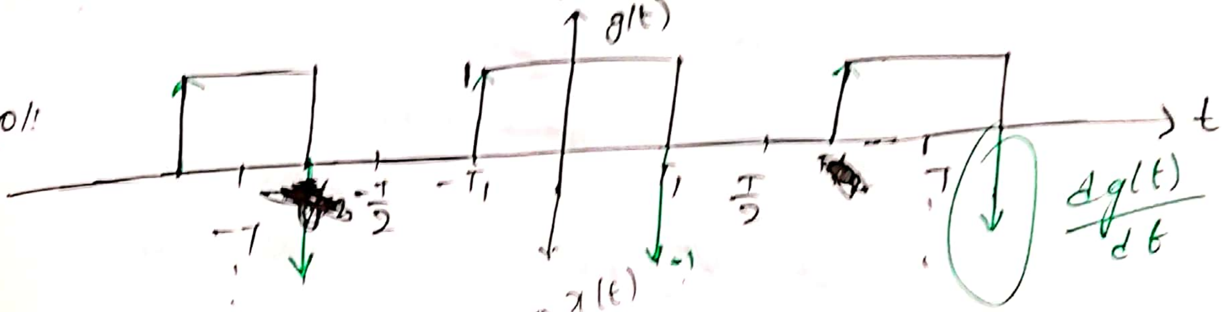
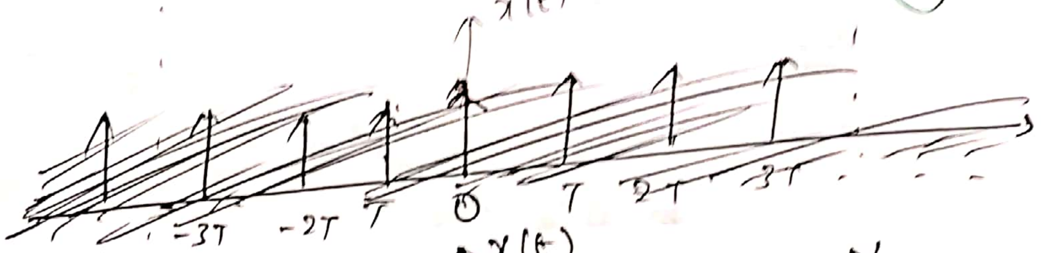
Q 28) Determine the fourier series representation of the signal $g(t)$ Given $x(t)$.

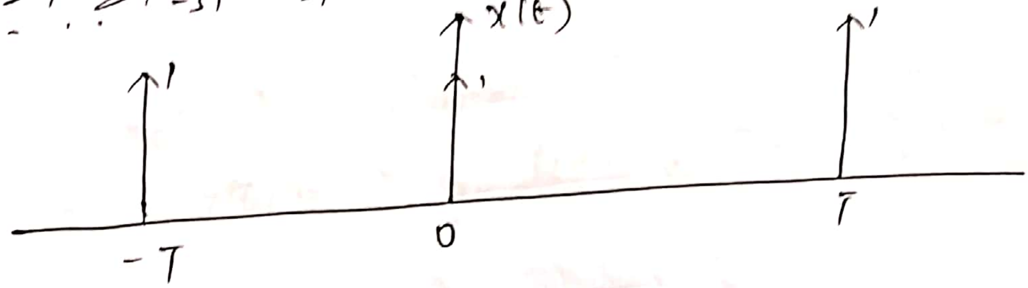
$$g(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Sol:







$$\frac{dg(t)}{dt} = x(t+T_1) - x(t-T_1)$$

$$x(t) \xleftrightarrow{F.S} a_k = \frac{1}{T}$$

$$x(t+T_1) \xleftrightarrow{F.S} e^{jk\omega_0 T_1} \frac{1}{T}$$

$$-x(t-T_1) \xleftrightarrow{F.S} -e^{-jk\omega_0 T_1} \cdot \frac{1}{T}$$

$$\frac{dg(t)}{dt} \xleftrightarrow{F.S} \frac{e^{jk\omega_0 T_1}}{T} - \frac{e^{-jk\omega_0 T_1}}{T}$$

$$= \frac{2\sin(k\omega_0 T_1)}{jT} = b_k$$

Let $q(t) = \frac{dg(t)}{dt} = x(t+T_1) - x(t-T_1)$

Use the property of differentiation to find the F.S of $g(t)$

(Ex 3.9) Suppose, we are given the following facts about the signal $x(t)$

(i) $x(t)$ is a real signal.

(ii) $x(t)$ is periodic with $T=4$ & it has Fourier coefficient a_k

(iii) $a_k = 0$ for $|k| > 1$

(iv) The signal with Fourier coefficients $b_k = e^{-jk\pi/2} a_{-k}$ is odd

(v) $\frac{1}{4} \int_4 |x(t)|^2 dt = \frac{1}{2}$

Find $x(t)$.

Sol:-

using (ii) $\Rightarrow$ $x(t) \xleftrightarrow{F.S} a_k$ $\qquad T = 4$

$$\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$$

using (iii) $\Rightarrow$ $a_k = 0, |t| > 1$

$$\Rightarrow a_0 \neq a_{-1} \neq a_1 \neq 0$$

$$x(t) = \sum a_k e^{jk\omega_0 t}$$

$$\Rightarrow x(t) = \sum_{k=-1}^{1} a_k e^{jk\omega_0 t} = a_0 + a_{-1} e^{-j\pi/2 t} + a_1 e^{+j\pi/2 t}$$

using (i) $\Rightarrow$ $x(t) = x^*(t)$

$$\begin{array}{ccc} x(t) & = & x^*(t) \\ \updownarrow F.S & & \updownarrow F.S \\ a_k & = & a^*_{-k} \end{array} \Rightarrow a^*_k = a_{-k}$$

$$x(t) = a_0 + a^*_1 e^{-j\pi/2 t} + a_1 e^{j\pi/2 t}$$

$$x(t) = a_0 + \left(a_1 e^{+j\pi/2 t}\right)^* + a_1 e^{j\pi/2 t}$$

$$x(t) = a_0 + \underline{2 Re\left(a_1 e^{j\pi/2 t}\right)} \Rightarrow \text{exponential multiple cosine}$$

using (iv) $\Rightarrow$

$$a_k \xleftrightarrow{F.S} x(t) \qquad e^{-jk\pi/2} = e^{-jk\omega_0 t_0}$$

$$a_{-k} \xleftrightarrow{F.S} x(-t)$$

$$x[-(t-t_0)] \xleftrightarrow{F.S} e^{-jk\pi/2} a_{-k}$$

$$x(-t+1) \xleftrightarrow{F.S} b_k$$

Property: If $x(t)$ is real & even $\overset{F.S}{\longleftrightarrow}$ a_k is real & even | real & odd $\overset{F.S}{\longleftrightarrow}$ Imaginary & odd

using (iv) $\Rightarrow$ $x(-t+1)$ is real & b_k is odd $\boxed{b_k = -b_{-k}}$

$\Rightarrow b_0 = 0$ & $b_{-1} = -b_1$

using (v) $\Rightarrow$ using Parseval Relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\frac{1}{4}\int_{<4>} [x(-t+1)]^2 dt = \frac{1}{2}$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |b_k|^2 = |b_{-1}|^2 + |b_1|^2$$

$$\Rightarrow |b_{-1}|^2 + |b_1|^2 = \frac{1}{2}$$

$$|-b_1|^2 + |b_1|^2 = \frac{1}{2}$$

$$|b_1|^2 + |b_1|^2 = \frac{1}{2}$$

$$|b_1|^2 = \frac{1}{4}$$

$x(t) \rightarrow$ real & odd $\overset{F.S}{\longleftrightarrow}$ Img & odd

$$b_1 = \frac{j}{2} \quad \& \quad b_1 = \frac{-j}{2}$$

using (i) $\Rightarrow$

$$a_k = b_k e^{jk\frac{\pi}{2}}$$

$$a_{+k} = b_k e^{-jk\pi/2}$$

$$a_{+1} = \frac{j}{2} e^{-j\pi/2} = \frac{1}{2}$$

$$a_{-1} = \frac{j}{2} e^{j\pi/2} = \frac{1}{2}$$

Discrete - Time Fourier Series :-

$x(n) = x(n+N) \quad , \forall n$

$\phi_k[n] = e^{jk\omega_0 n}$ → set of harmonically related signals

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n}$$

→ for Discrete time periodic signals, the summation varies as N,

$$\Rightarrow \boxed{x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}}$$

$k = 0, 1, 2, \ldots N-1$

(or)

$k = 1, 2, \ldots N$

→ Fourier series representation of D.T periodic signals.

a_k = Fourier co-efficients

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n}$$

Check for the periodicity the F.S- coefficients :-

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n}$$

$$a_{k+N} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-j(k+N)\omega_0 n}$$

$$a_{k+N} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n} e^{-jN\omega_0 n}$$

$$a_{k+N} = \frac{1}{N} \sum_{<N>} x(n) e^{-jk\omega_0 n} \cancel{e^{-jN\frac{2\pi}{N}n}1}$$

$$a_{k+N} = \frac{1}{N} \sum_{<N>} x[n] e^{-jk\omega_0 n}$$

$$\boxed{a_{k+N} = a_k}$$

Problems:-

(1) Determine the discrete time fourier series ~~rep~~ representa-tion for the sequence $x[n] = \cos\left(\frac{\pi}{4}n\right)$

Sol: $x[n] = \cos\left(\frac{\pi}{4}n\right)$

$$\frac{\omega_0}{2\pi} = \frac{\frac{\pi}{4}}{2\pi} = \frac{1}{8} \Rightarrow \frac{\omega_0}{2\pi} = \frac{m}{N}$$

$$\boxed{N=8}$$

$$x[n] = \cos\left(\frac{\pi}{4}n\right) = \boxed{\frac{e^{j\frac{\pi}{4}n}}{2} + \frac{e^{-j\frac{\pi}{4}n}}{2}}$$

$$a_1 e^{j\frac{\pi}{4}n} + a_{-1} e^{-j\frac{\pi}{4}n}$$

$k=1$ $\quad$ $k=-1$

$$\Rightarrow \boxed{a_1 = 1/2 \; ; \; a_{-1} = +1/2}$$

(2) Evaluate the DT. F.S coefficient for the signal $x[n] = \sin\left(\frac{4\pi}{21}\right)n + \cos\left(\frac{10\pi}{21}\right)n + 1$ & sketch the magnitude & phase spectrum.

Sol:- $\frac{\omega_1}{2\pi} = \frac{\frac{4\pi}{21}}{2\pi} = \frac{2}{21} = \frac{m_1}{N_1} \Rightarrow \frac{N_1}{N_2} = \frac{21}{21} = 1$

$$\frac{\omega_2}{2\pi} = \frac{\frac{10\pi}{21}}{2\pi} = \frac{5}{21} = \frac{m_1}{N_2}$$

$$N = LCM(N_1, N_2) = 21$$

$$\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{21}$$

~~$x[n] = \frac{1}{2} e^{\frac{4\pi}{21}n} + \frac{1}{2} e^{-\frac{4\pi}{21}n} + \frac{1}{2}$~~

$$x[n] = \frac{1}{2j} e^{j\frac{4\pi}{21}n} + \frac{1}{2j} e^{-j\frac{4\pi}{21}n} + \frac{1}{2} e^{j\frac{10\pi}{21}n} +$$

$$\frac{1}{2} e^{-j\frac{10\pi}{21}n} + 1$$

—

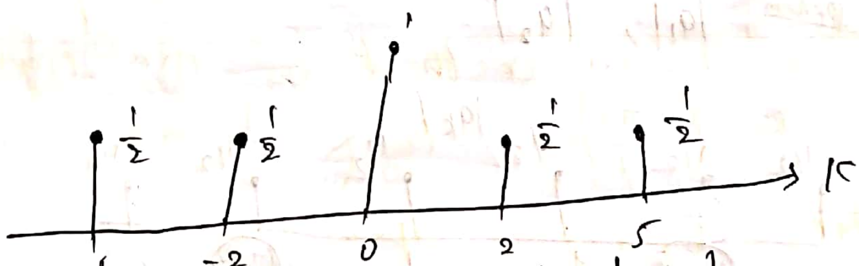
$$x[n] = \frac{1}{2j} e^{j2\frac{2\pi}{21}n} - \frac{1}{2j} e^{-j2\frac{2\pi}{21}n} + \frac{1}{2} e^{j5\frac{2\pi}{21}n} +$$

$$\frac{1}{2} e^{-j5\frac{2\pi}{21}n} + 1$$

$$x(n) = \underset{k=2}{\frac{1}{2j} e^{j2\omega_0 n}} - \underset{k=-2}{\frac{1}{2j} e^{-j2\omega_0 n}} + \underset{k=5}{\frac{1}{2} e^{j5\omega_0 n}} +$$

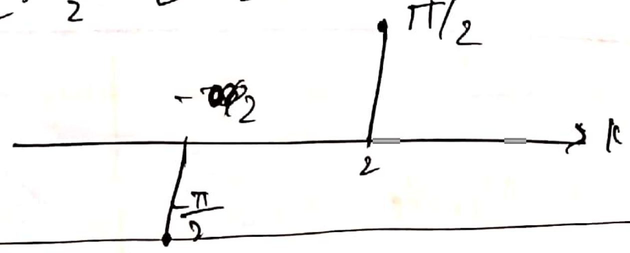
$$\underset{k=-5}{\frac{1}{2} e^{-j5\omega_0 n}} + \underset{k=0}{1}$$

$$\boxed{\begin{aligned} &a_2 = \frac{1}{2j} \;; \quad a_{-2} = -\frac{1}{2j} \\ &a_5 = \frac{1}{2} \;; \quad a_{-5} = \frac{1}{2} \;; \quad a_0 = 1 \end{aligned}}$$



$|a_{-5}| = 0$ $\quad |a_{-2}|$ $\quad |a_0|$ $\quad |a_2|$ $\quad |a_5|$

$\angle a_{-5} = \frac{\pi}{2}$ $\quad \angle a_{-2} = \frac{\pi}{2}$ $\quad \angle a_0 = 0$ $\quad \angle a_2 = 0$ $\quad \angle a_5 = 0$



(3) $x[n] = \cos^2\left(\frac{\pi}{8} n\right)$ $\qquad \frac{\omega_0}{2\pi} = \frac{\frac{\pi}{8}}{2\pi} = \frac{1}{8}$

Sol:- $x[n] = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{4}n$

$$x[n] = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2}\left(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{4}e^{j\frac{\pi}{4}n} + \frac{1}{4}e^{-j\frac{\pi}{4}n}$$

$$\boxed{a_0 = \frac{1}{2} \; ; \; a_1 = \frac{1}{4} \; , \; a_{-1} = \frac{1}{4}}$$

~~Discrete time fourier series~~ :-

(3) : Determine the ~~fourier series~~ cofficients

$$x[n] = \sin(\omega_0 n)$$

Sol: $\frac{\omega_0}{2\pi} = \frac{\omega_0}{2\pi} = \frac{m}{N}$

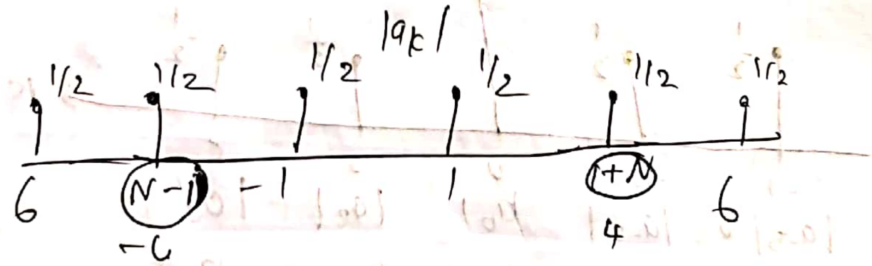
$$\Rightarrow N = \frac{2\pi}{\omega_0}$$

$$x[n] = \frac{1}{2j}e^{j\omega_0 n} - \frac{1}{2j}e^{-j\omega_0 n}$$

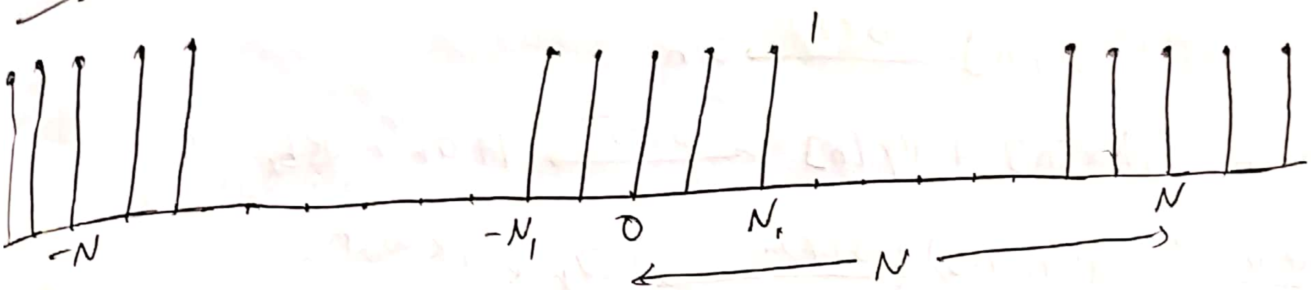
$$\therefore a_1 = \frac{1}{2j} \; ; \; a_{-1} = \frac{-1}{2j}$$

Magnitude spectrum: $|a_1|, |a_2|$

Assume $N=5$



Exr 3 : Q :



Sol: $$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-jk\omega_0 n}$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1\, e^{-jk\frac{2\pi}{N}n}$$

$$\boxed{\sum_{m=0}^{n} a^m = \frac{1-a^{n+1}}{1-a}}$$

Let $m = n + N_1$

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)}$$

$$a_k = \frac{1}{N}\, e^{jk\frac{2\pi}{N}n}\left[\sum_{m=0}^{2N_1} e^{-jk\frac{2\pi}{N}m}\right]$$

$$a_k = \frac{1}{N}\, e^{jk\frac{2\pi}{N}n}\left[\frac{1-e^{jk\frac{2\pi}{N}(2N_1+1)}}{1-e^{-jk\frac{2\pi}{N}}}\right]$$

$$a_k = \frac{1}{N}\left[\frac{\sin\left[2\pi k\,(N_1+1/2)/N\right]}{\sin\left(\frac{\pi k}{N}\right)}\right], \quad k = 0, \pm N,$$

for $k=0$,

$$a_0 = \frac{1}{N}\sum_{m=0}^{2N_1} 1$$

$$\boxed{a_0 = \frac{2N_1+1}{N}} \qquad k = 0, \pm N, \ldots$$

Properties of DTFS :

$$x[n] \xleftrightarrow{DTFS} a_n$$

<u>Linear:</u> $A x[n] + B y[n] \xleftrightarrow{DTFS} A a_k + B b_k$

<u>Time-shift:</u> $x(n-n_0) \xleftrightarrow{DTFS} a_k e^{-jk\omega_0 n_0}$

<u>Conjugate</u> : $x^*[n] \xleftrightarrow{DTFS} a^*_{-k}$

<u>Convolution</u> : $x(n) * y(n) \xleftrightarrow{DTFS} N a_k b_k$

<u>Multiplication</u> : $x(n)\, y(n) \longleftrightarrow \sum_{l=\langle N \rangle} a_l b_{k-l}$

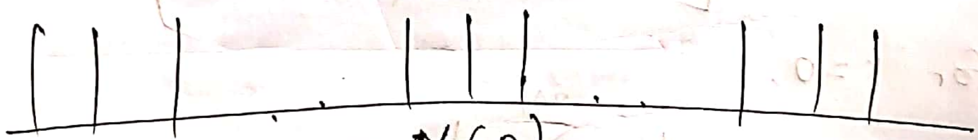
<u>first difference</u> : $x(n) - x(n-1) \longleftrightarrow \left(1 - e^{jk\left(\frac{2\pi}{N}\right)}\right) a_k$

<u>Parseval's Relation</u> : $\frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{k=\langle N \rangle} |a_k|^2$

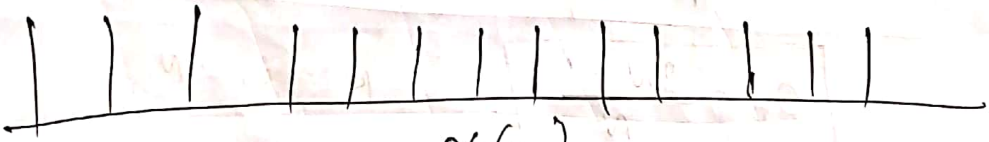
<u>Ex : 3.10</u> :

$x(n)$

<u>Sol</u>



$y(n)$



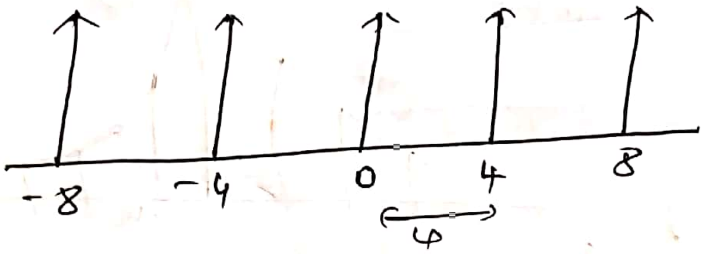
$x(n)$

(8) consider $x[n] = \sum_{m=-\infty}^{\infty} \delta(n-4m)$

i) Sketch $x[n]$

ii) find a_k

Sol:-



$x[n] = \ldots \delta(n+4) + \delta(0) + \delta(n-4) + \ldots$

$N = 4$

$$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

$$a_k = \frac{1}{4} \sum_{k=0}^{3} 1 \cdot e^{-jk\frac{2\pi}{4}n}$$

$$a_k = \frac{1}{4}\left[1 + e^{-j\frac{\pi}{2}n} + e^{-j\pi n} + e^{-j\frac{3\pi}{2}n}\right]$$

$$\boxed{a_0 = \frac{1}{4},\ a_1 = \frac{1}{4};\ a_2 = \frac{1}{4};\ a_3 = \frac{1}{4}}$$