Module 1 : Signals in Natural Domain
Lecture 4 : Properties of Systems

Objectives

In this lecture you will learn the following

- We shall look at different system properties like
- memory
- linearityshift-invariance
- stability
- causality
- We shall even deduce some theorems based on the above properties.

Memory:

Memory is a property relevant only to systems whose input and output signals have the *same independent variable*. **A system is said to be memoryless if its output for each value of the independent variable is dependent only on the input signal at that value of independent variable.** For example the system with description : **y(t) = 5x(t)** (y(t) is the output signal corresponding to input signal x(t)) is memoryless. In the physical world a resistor can be considered to be a memoryless system (with voltage considered to be the input signal, current the output signal).

By definition, a system that does not have this property is said to have memory.

How can we identify if a system has memory?

For a memoryless system, changing the input at any instant can change the output *only at that instant*. If, in some case, a change in input signal at some instant changes the output at some other instant, we can be sure that the system has memory.

Note: Consider a system whose output **Y(t)** depends on input **X(t)** as: **Y(t) = X(t-5) + { X(t) - X(t-5) }**

While at first glance, the system might appear to have memory, it does not. This brings us to the idea that given the description of a system, it need not at all be the most economical one. The same system may have more than one description.

Examples:

Assume **y[n]** and **y(t)** are respectively outputs corresponding to input signals **x[n]** and **x(t)**

1. The identity system **y(t) = x(t)** is of-course **Memoryless**
2. System with description **y[n] = x[n-5] has memory**. The input at any "instant" depends on the input 5 "instants" earlier.
3. System with description

$$y(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda$$

also **has memory**. The output at any instant depends on all past and present inputs.

Linearity:

Now we come to one of the most important and revealing properties systems may have - Linearity. Basically, the principle of linearity is equivalent to the principle of *superposition*, i.e. a system can be said to be linear if, for any two input signals, their linear combination yields as output the same linear combination of the corresponding output signals.

Definition:

> Given any two inputs, acted on by the system as follows:
> $$X_1(t) - > \mathbb{S} - > Y_1(s) \quad ; \quad X_2(t) - > \mathbb{S} - > Y_2(s)$$
> The system is said to be linear if, for any two constants α & β,
> and the signal: $X(t) = \alpha X_1(t) + \beta X_2(t)$,
> The system acts on it as follows:
> $$X(t) - > \mathbb{S} - > Y(s)\text{, where } Y(s) = \alpha Y_1(s) + \beta Y_2(s)$$

(It is not necessary for the input and output signals to have the same independent variable for linearity to make sense. The definition for systems with input and/or output signal being discrete-time is similar.)

Example of linearity

A capacitor, an inductor, a resistor or any combination of these are all linear systems, if we consider the voltage applied across them as an input signal, and the current through them as an output signal. This is because these simple passive circuit components follow the principle of superposition within their ranges of operation.
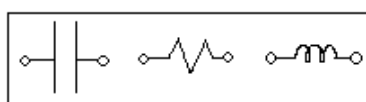
Additivity and Homogeneity:

Linearity can be thought of as consisting of two properties:

- Additivity

  A system is said to be additive if for any two input signals $x_1(t)$ and $x_2(t)$,

  $$X_1(t) \to S \to Y_1(s) \; ; \; X_2(t) \to S \to Y_2(s)$$
  $$\Rightarrow [X_1(t) + X_2(t)] \to S \to [Y_1(s) + Y_2(s)]$$

  i.e. the output corresponding to the sum of any two inputs is the sum of the two outputs.

- Homogeneity (Scaling)

  A system is said to be homogenous if, for any input signal $X(t)$,

  $$X(t) \to S \to Y(s) \Rightarrow \alpha X(t) \to S \to \alpha Y(s) \; \forall \, \alpha$$

  i.e. scaling any input signal scales the output signal by the same factor.

To say a system is linear is equivalent to saying the system obeys both additivity and homogeneity.

    a) We shall first prove homogeneity and additivity imply linearity.

Consider two input signals $x_1(t)$ and $x_2(t)$, with corresponding output signals $y_1(t)$ and $y_2(t)$ respectively. S represents the system transformation.

By Homogeneity,

$\alpha x_1(t) \to S \to \alpha y_1(t)$ and $\beta x_2(t) \to S \to \beta y_2(t)$, where α and β are any two constants.

By Additivity,

$\alpha x_1(t) + \beta x_2(t) \to S \to \alpha y_1(t) + \beta y_2(t)$

Hence proved.

    b) To prove linearity implies homogeneity and additivity.

This is easy; put both constants equal to 1 in the definition to get additivity; one of them to 0 to get homogeneity.

Additivity and homogeneity are independent properties.

We can prove this by finding examples of systems which are additive but not homogeneous, and vice versa.

Again, **y(t)** is the response of the system to the input **x(t)**.

Example of a system which is additive but not homogeneous: $Y(t) = \overline{X(t)}$

[It is homogeneous for real constants but not complex ones - consider $\alpha = i = \sqrt{-1}$]

Example of a system which is homogeneous but not additive: $Y(t) = \dfrac{\{X(t)\}^2}{X(t-1)}$

[*From this example can you generalize to a class of such systems?*]

Examples of Linearity:

Assume **y[n]** and **y(t)** are respectively outputs corresponding to input signals **x[n]** and **x(t)**

1) System with description **y(t) = t . x(t)** is linear.

    Consider any two input signals, $x_1(t)$ and $x_2(t)$, with corresponding outputs $y_1(t)$ and $y_2(t)$.

    a and **b** are arbitrary constants. The output corresponding to $a.x_1(t) + b.x_2(t)$ is

    $= t \, (a.x_1(t) + b.x_2(t))$

    $= t.a.x_1(t) + t.b.x_2(t)$, which is the same linear combination of $y_1(t)$ and $y_2(t)$.

    Hence proved.

2) The system with description $y(t) = \{x(t)\}^2$ is not linear.

    See for yourself that the system is neither additive, nor homogenous.

    Show for yourself that systems with the following descriptions are linear:

$$y(t) = \int_{-\infty}^{t} x(\lambda) \, d\lambda$$

$$y(t) = \frac{d\ x(t)}{dt}$$

Shift Invariance

This is another important property applicable to systems with the same independent variable for the input and output signal. We shall first define the property for continuous time systems and the definition for discrete time systems will follow naturally.

Definition: Say, for a system, the input signal **x(t)** gives rise to an output signal **y(t)**. If the input signal **x(t - t$_0$)** gives rise to output **y(t - t$_0$)**, for every **t$_0$**, and every possible input signal, we say the system is shift invariant.

i.e. for every permissible **x(t)** and every **t$_0$** $x(t) \rightarrow y(t) \Rightarrow x(t - t_0) \rightarrow y(t - t_0)$

In other words, for a shift invariant system, shifting the input signal shifts the output signal by the same offset.
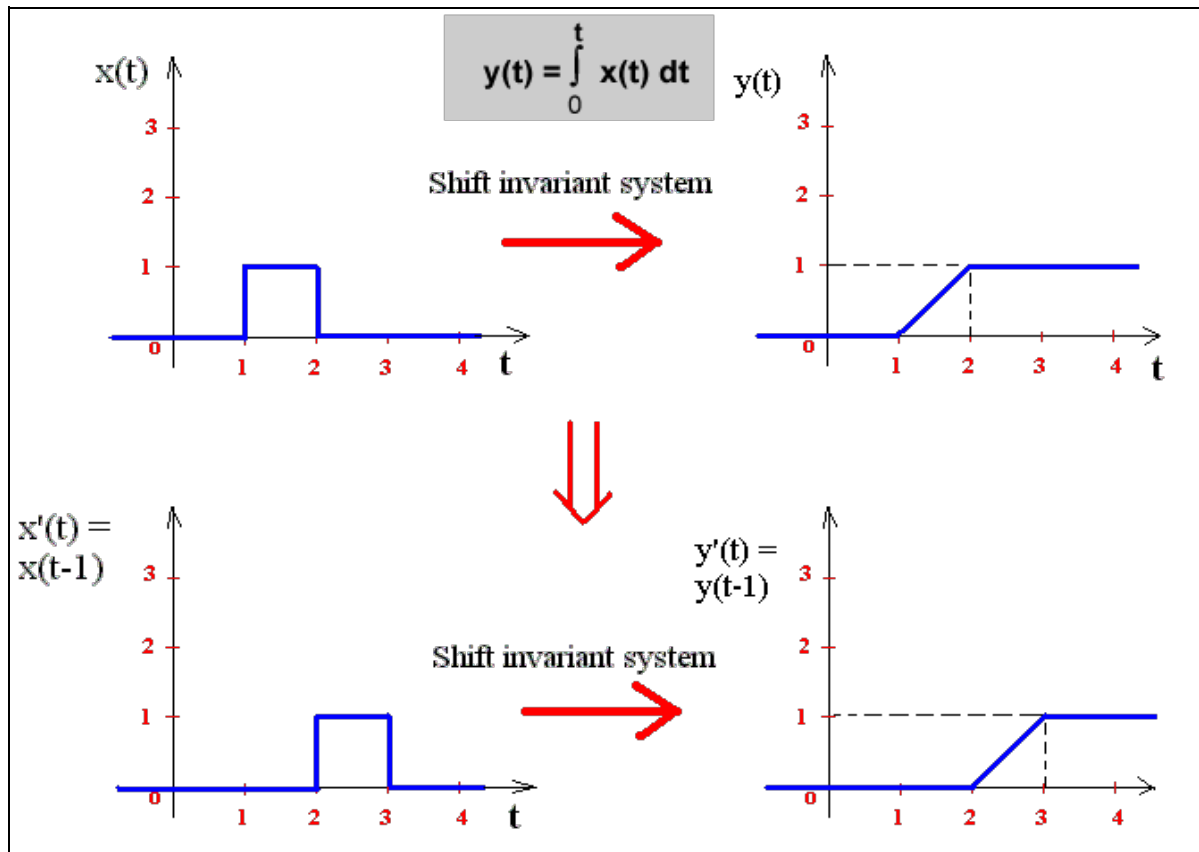
Note this is not to be expected from every system. **x(t)** and **x(t - t$_0$)** are different (related by a shift, but different) input signals and a system, which simply maps one set of signals to another, need not at all map **x(t)** and **x(t - t$_0$)** to output signal also shift by **t$_0$**

As you might have guessed, a discrete time system is said to be shift invariant if for

every permissible input signal x[n] and every integer n₀,

$x[n] \rightarrow y[n] \Rightarrow x[n\text{-}n_0] \rightarrow y[n\text{-}n_0]$

A system that does not satisfy this property is said to be **shift variant.**

Examples of Shift Invariance:

Assume **y[n]** and **y(t)** are respectively outputs corresponding to input signals **x[n]** and **x(t)**

1. The system with description

$y[n] = x[n] + x[n\text{-}5]$

is shift invariant as input x[n-n₀] gives rise to output x[n-n₀] + x[n-5- n₀] which equals y[n-n₀].

2. The system with description

$y(t) = x(2t)$

is shift variant.

The output corresponding to input signal $x(t-t_o)$ is $x(2t-t_o)$.

If this is confusing, let us say the input signal is $g(t) = x(t-t_o)$. The output signal is clearly

$g(2t)$, which is the same as $x(2t-t_o)$.

If the system were shift-invariant, the output "should" have been $x(2(t-t_o))$!

Similarly, you can verify that systems with descriptions $y(t) = x(t^2)$, or $y[n] = n.x[n]$

are also examples of shift variant systems.

Stability

Let us learn about one more important system property known as stability. Most of us are familiar with the word stability, which intuitively means resistance to change or displacement. Broadly speaking a stable system is a one in which small inputs lead to predictable responses that do not diverge, i.e. are bounded. To get the qualitative idea let us consider the following physical example.

Example

Consider an ideal mechanical spring (elongation proportional to tension). If we consider tension in the spring as a function of time as the input signal and elongation as a function of time to be the output signal, it would appear intuitively that the system is stable. A small tension leads only to a finite elongation.

There are various ideas/notions about stability not all of which are equivalent. We shall now introduce the notion of **BIBO** Stability, i.e. **BOUNDED INPUT-BOUNDED OUTPUT STABILITY.**

Statement:

A system is said to be BIBO stable if for any bounded input signal ,

the output signal is bounded.

Given a bounded input , **i.e.** $\exists \ M_x \geq 0 \ni |X(t)| \leq M_x \ \forall t,$

$$\text{then } \exists \ M_y \geq 0 \ni |y(t)| \leq M_y \ \forall t.$$

Note: This should be true for all bound inputs **x(t)**

It is not necessary for the input and output signal to have the same independent variable for this property to make sense. It is valid for continuous time, discrete time and hybrid systems.

Examples

Consider systems with the following descriptions. **y(t)** is the output signal corresponding to the input signal **x(t)**.

$1. y(t) = \{x(t)\}^2$

If $x(t)$ is bounded, then $\exists\, M_x \geq 0 \ni \forall t \; |x(t)| \leq M_x$

$\Rightarrow |y(t)| \leq M_x^2$ which further implies that the output is bounded $\forall t$.

$\therefore$ The system is BIBO stable.

$2. y(t) = \dfrac{1}{x(t)}$, for $t > 0$.

if $x(t) = (1 - e^{-t})u(t)$.

where $u(t)$ is unit step function defined as $u(t) = 1$ for $t \geq 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad u(t) = 0$ for $t < 0$

then the input is bounded by 1 and clearly the output shoots up as $x \to 0$,

hence is unbounded. $\therefore$ The system is BIBO unstable.

$3. y(t) = tx(t)$

Take $x(t) = u(t)$

then the input is bounded by 1 and the output is unbounded,

hence BIBO unstable.

CONCLUSION

BIBO Stable system : In a BIBO stable system, every bounded input is assured to give a bounded output. An unbounded input can give us either a bounded or an unbounded output, i.e. nothing can be said for sure.

BIBO Unstable system: In a BIBO unstable system, there exists at least one bounded input for which output is unbounded. Again, nothing can be said about the system's response to an unbounded input.

Causality

Causality refers to cause and effect relationship (the effect follows the cause). In a causal system, the value of the output signal at any instant depends only on "past" and "present" values of the input signal (i.e. only on values of the input signal at "instants" less than or equal to that "instant"). Such a system is often referred to as being **non-anticipative**, as the system output does not anticipate future values of the input (remember again the reference to time is merely symbolic). As you might have realized, causality as a property is relevant only for systems whose input and output signals have the **same independent variable**. Further, this independent variable must be **ordered** (it makes no sense to talk of "past" and "future" when the independent variable is not ordered).

What this means mathematically is that If two inputs to a causal (continuous-time) system are identical up to some time to, the corresponding outputs must also be equal up to this same time (we'll define the property for continuous-time systems; the definition for discrete-time systems will then be obvious).

Definition

Let $x_1(t)$ and $x_2(t)$ be two input signals to a system and $y_1(t)$ and $y_2(t)$ be their respective outputs.

The system is said to be causal if and only if:

$$
\begin{aligned}
&x_1(t) = x_2(t) && \forall && t \leq t_0 \\
&\Rightarrow y_1(t) = y_2(t) && \forall && t \leq t_0 \quad \text{for every } t_0 \text{ and every } x_1(t) \text{ and } x_2(t)
\end{aligned}
$$

This of course is only another way of stating what we said before: for **any t_0 : $y(t_0)$** depends only on values of **$x(t)$** for **$t <= t_0$**

As an example of the behavior of causal systems, consider the figure below:
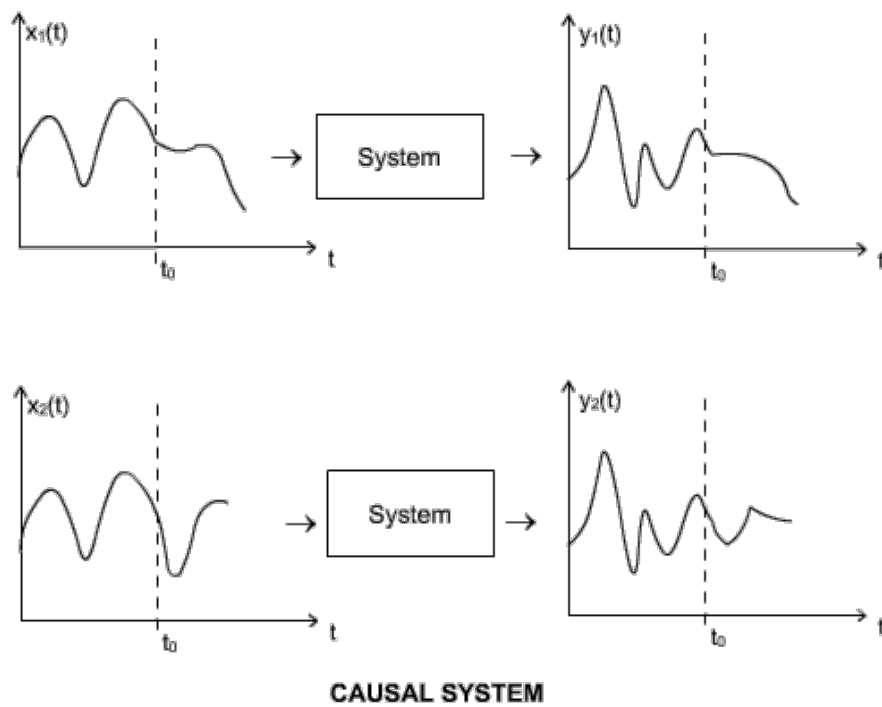
CAUSAL SYSTEM

The two input signals in the figure above are identical to the point $t = t_0$, and the system being a causal system, their corresponding outputs are also identical till the point $t = t_0$.

Examples of Causal systems

Assume y[n] and y(t) are respectively the outputs corresponding to input signals x[n] and x(t)

1. System with description **y[n] = x[n-1] + x[n]** is clearly causal, as output "at" n depends on only values of the input "at instants" less than or equal to **n** (in this case n and n-1).

2. Similarly, the continuous-time system with description $y(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda$ is causal, as value of output at any time t_0 depends on only value of the input at t_0 and before.

3. But system with description **y[n] = x[n+1]** is not causal as output at **n** depends on input one instant later.

Note:

If you think the idea of non-causal systems is counter intuitive, i.e: if you think no system can "anticipate the future", remember the independent variable need not be time. Visualizing non-causal systems with , say one-dimensional space as the independent variable is not difficult at all ! Even if the independent variable is time, we need not always be dealing with real-time, i.e. with the time axes of the input and output signals synchronized. The input signal may be a recorded audio signal and the output may be the same signal played backwards. This is clearly not causal !

Deductions from System Properties

Now that we have defined a few system properties, let us see how powerful inferences can be drawn about systems having one or more of these properties.

Theorem Statement: If a system is additive or homogeneous, then **x(t)=0** implies **y(t)=0**.

Proof:

Case1: The system is additive

$$0 \cdots\cdots > \boxed{\text{system}} \cdots\cdots > y(t)$$

As the system is additive

$$0 + 0 \cdots\cdots > \boxed{\text{system}} \cdots\cdots > y(t) + y(t) = 2y(t)$$

therefore y(t)=2y(t) which implies that y(t) = 0 $\forall t$.

$$0 \cdots\cdots > \boxed{\text{system}} \cdots\cdots > y_1(t)$$

$$5*0 = 0 \cdots\cdots > \boxed{\text{system}} \cdots\cdots > y_2(t)$$

homogeneity implies that $y_2(t) = 5y_1(t)$

$$\therefore y_1(t) = 5y_1(t) \Rightarrow y_1(t) = 0$$

This completes the proof.

Theorem:

Statement : If a causal system is either additive or homogeneous ,then **y(t)** can not be non zero before **x(t)** is non-zero .

Proo f:

Say **x(t) = 0** for all **t** less than or equal to **t$_0$**.

We have to show that the system response **y(t) = 0** for all **t** less than or equal to **t$_0$**.

Since the system is either additive or homogeneous the response to the zero input signal is the zero output signal. The zero input signal and **x(t)** are identical for all **t** less than or equal to **t$_0$**.

Hence, from causality, their output signals are identical for all **t** less than or equal to **t$_0$**.

We conclude the discussion on system properties by noting that this is not an end, but merely a beginning! Through much of our further discussions, we will be looking at an important class of systems - **Linear Shift-Invariant (LSI) Systems.**

Conclusion:

In this lecture you have learnt:

- A system is said to be memoryless, if its output for each value of the independent variable is dependent only on the value of the input signal at that value of the independent variable.

- The principle of linearity is equivalent to the principle of superposition, i.e. a system can be said to be linear if, for any two input signals, their linear combination yields as output the same linear combination of the corresponding output signals.

- To say a system is linear is equivalent to saying that the system obeys both additivity and homogeneity.

- Say, for a system, the input signal x(t) gives rise to an output signal y(t), and it is said to be shift invariant if the input signal x(t - t 0) gives rise to the output y(t - t 0), for every t 0, and every possible input signal.

- A system in which a bounded input leads to a bounded output is said to be BIBO stable.

- In a causal system, the value of the output signal at any instant depends only on the "past" and "present" values of the input signal and/or "past" values of the output signal.

- If a system is additive or homogeneous, then x(t)=0 implies that y(t)=0.

- If a causal system is either additive or homogeneous ,then y(t) can not be non zero before x(t) is non-zero.

Congratulations, you have finished Lecture 4.