Network Analysis & Systems

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UE18EC201: Network Analysis & Systems

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Network Analysis and Synthesis

Unit II: Transient Behaviour — Boundary Conditions

Boundary Conditions

- Boundary: Initial and Final.
- Initial conditions (initial state) are required to determine the arbitrary constants arising whilst solving differential equations.
- Behaviour of the system at the instant of switching.
- Serves as a check on the veracity of the solution.
- Assumption: Switching takes place in zero time.
- Recall: Values of system functions are designated with a $0-$ before switching, and $0+$ after switching; e.g., $i(0-)$ and $v(0+)$.
- Initial conditions depend on the past history and manifests as capacitor voltages and inductor currents in circuit theory.

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- If there is a prior current, then it acts as a current source.

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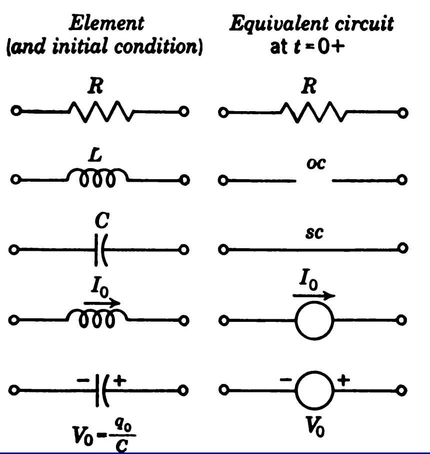
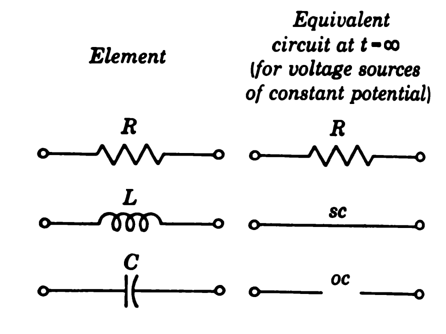
- If there is a prior current, then it acts as a current source.
- The voltage across a capacitor cannot change instantaneously: $v_C(0-) = v_C(0+)$.
- Therefore, closing a switch to include a capacitor with no prior charge across it is equivalent to replacing the capacitor by a short circuit.
- If there is a prior charge, then it acts as a voltage source.

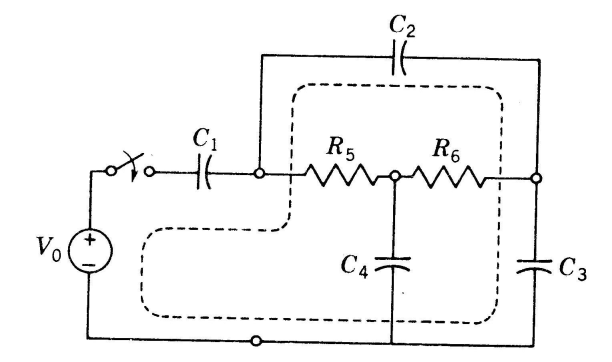
Initial Conditions (2)



Final Conditions



Special Cases (1)



- Capacitors are not initially charged and the switch is closed at $t = 0$.

Special Cases (2)

- Capacitors act like short circuits.
- This results is in the voltage source shorting.
- Therefore, the voltage source produces an infinite current over an infinitesimally short interval of time.
- The charge transferred

$$q = \int_{0-}^{0+} i(\tau) d\tau$$

is however finite and is such that the following must be satisfied:

$$V_0 = v_{C_1}(0+) + v_{C_2}(0+) + v_{C_3}(0+)$$

Special Cases (3)

- That is,

$$V_0 = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3}$$

- Thus the infinite current in the interval $[0-, 0+]$ deposits sufficient charge on the capacitors so that KVL is satisfied.
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- Under this condition, the voltage across a capacitor changes instantaneously.
- A similar analysis holds even if the capacitors are charged prior to switching.

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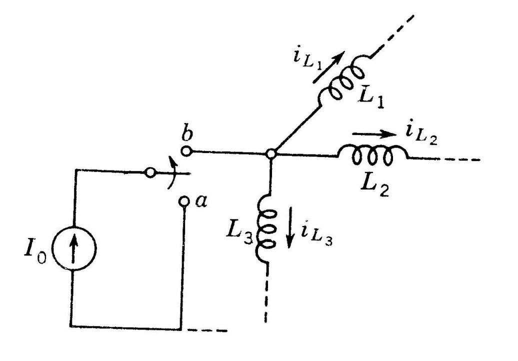
- A similar analysis holds even if the capacitors are charged prior to switching.
- Note that ideal model of capacitor is assumed.
- In practice, there is always some stray resistance which prevents infinite current flowing.

Special Cases (4)



- There are no currents flowing through the inductors initially, and the switch is closed at $t = 0$.

Special Cases (5)

- This is the dual scenario.
- Prior to switching, at $t = 0-$,

$$i_{L_1}(0-) + i_{L_2}(0-) + i_{L_3}(0-) = 0$$

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- Immediately after switching, at $t = 0+$,

$$i_{L_1}(0+) + i_{L_2}(0+) + i_{L_3}(0+) = I_0$$

- Therefore, currents in the inductors must change instantaneously to satisfy both equations.
- An infinite voltage is generated due to the switching action that produces sufficient finite flux for the above to be satisfied:

$$\phi = \int_{0-}^{0+} v(t)dt \Longrightarrow \frac{\phi}{L_1} + \frac{\phi}{L_2} + \frac{\phi}{L_3} = I_0$$

Special Cases (6)

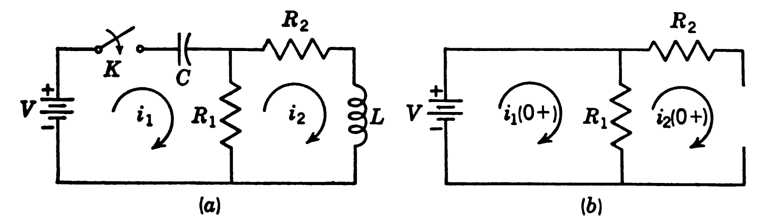
- The infinite voltage in the interval $[0-, 0+]$ is mathematically modelled using the Dirac Delta impulse.
- Under this condition, the current through an inductor changes instantaneously.
- A similar analysis holds even if the inductors are carrying currents prior to switching.
- Note that ideal model of inductor is assumed.
- In practice, there is always some stray resistance which prevents infinite voltage to be generated.

Initial Conditions (3)

Suggested procedure for evaluating initial conditions of currents and voltages:

- Replace all inductors with o.c. or with current sources.
- Replace all capacitors with s.c. or with voltage sources.

Examples (1)



- Assume zero voltage on capacitor and zero current through inductor prior to switching.

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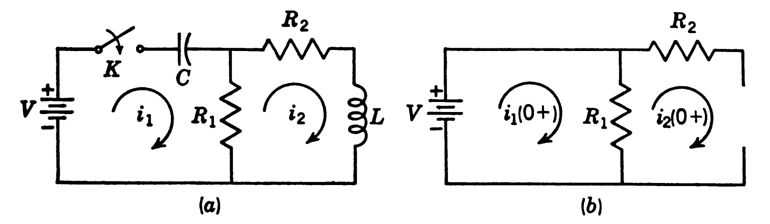

- Assume zero voltage on capacitor and zero current through inductor prior to switching.
- Clearly,

$$i_1(0+) = \frac{V}{R_1}, \quad i_2(0+) = 0$$

Examples (2)

To compute the initial values of the derivatives of the variables:

- Applying KVL to both loops:

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$$\begin{aligned} \frac{1}{C}\int_{-\infty}^{t} i_1(\tau)d\tau + R_i(i_1 - i_2) &= 0 \\ R_1(i_2 - i_1) + R_2 i_2 + L\frac{di_2}{dt} &= 0 \end{aligned}$$

- These equations hold for all t. In particular, it holds for $t = 0+$.

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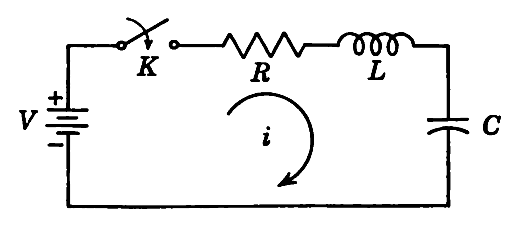
- These equations hold for all t. In particular, it holds for $t = 0+$.
- Thus,

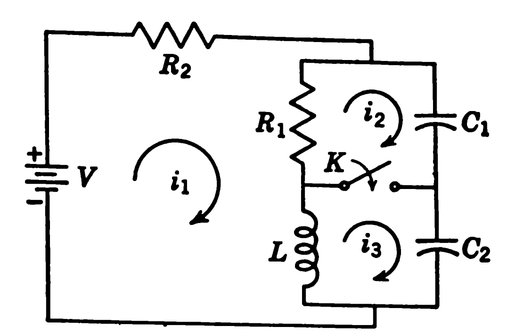
$$\left.\frac{di_2}{dt}\right|_{t=0+} = \frac{V}{L}, \quad \left.\frac{di_1}{dt}\right|_{t=0+} = \frac{V}{L} - \frac{V}{R_1^2 C}$$

Examples (3)



Source: Van Valkenburg, 1975.

Examples (4)



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Second Order Systems (1)

Consider

$$a_0 \frac{d^2 i}{dt^2} + a_1 \frac{di}{dt} + a_2 i = 0$$

Observe that if

$$i(t) = ke^{st}$$

then

$$(a_0 s^2 + a_1 s + a_2) ke^{st} = 0$$

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is the characteristic or auxiliary polynomial. Evidently, the roots are

$$s_1, s_2 = -\frac{a_1}{2a_0} \pm \frac{1}{2a_0}\sqrt{a_1^2 - 4a_0 a_2}$$

Second Order Systems (2)

$$a_0 \frac{d^2 i}{dt^2} + a_1 \frac{di}{dt} + a_2 i = 0$$

With s_1 and s_2 the roots of the characteristic polynomial,

$$i_1(t) = ke^{s_1 t}, \quad i_2(t) = ke^{s_2 t}$$

results in

$$(a_0 s_i^2 + a_1 s_i + a_2) ke^{s_i t} = 0$$

and hence are solutions of the second-order differential equations.

- Since i_1 and i_2 are solutions, then so is any linear combination αi_1 and βi_2, for all α, $\beta \in \mathbb{R}$.
- Hence, we deduce that the ODE corresponds to a linear system.

Second Order Systems (3)

$$a_0 \frac{d^2 i}{dt^2} + a_1 \frac{di}{dt} + a_2 i = 0$$

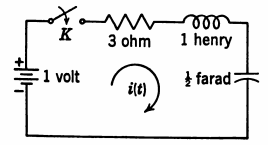
The general solution is

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

The roots s_1 and s_2 may be

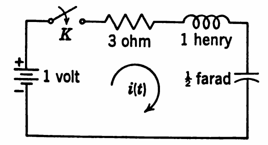
- real and different,
- real and equal, or
- complex conjugate pair.

Second Order Systems (4)



Applying KVL,

Second Order Systems (4)



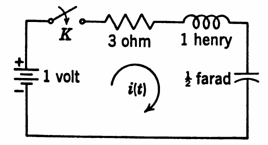
Applying KVL,

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = V$$

Equivalently,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

Second Order Systems (4)



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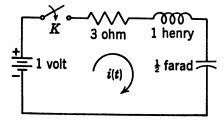
Thus, the auxiliary equation is

$$Ls^2 + Rs + \frac{1}{C} = 0$$

Second Order Systems (5)



That is,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Clearly,

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

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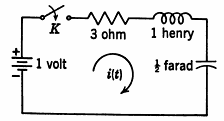

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For the example,

$$s^2 + 3s + 2 = (s+1)(s+2) = 0$$

Second Order Systems (6)

The solution to

$$\frac{d^2 i}{dt^2} + 3\frac{di}{dt} + 2i = 0$$

is

$$i(t) = k_1 e^{-t} + k_2 e^{-2t}$$

- The arbitrary constants k_1 and k_2 are determined from the initial conditions.
- If the switch is closed at $t = 0$, then $i(0+) = 0$.

Second Order Systems (7)

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = V$$

- Therefore, at the instant of switching, the second and third terms are zero:

$$Ri(0+) = 0, \quad \int_{-\infty}^{0+} i(\tau)d\tau = 0$$

- Thus,

$$L\left.\frac{di}{dt}\right|_{t=0+} = V$$

- Hence, the arbitrary constants k_1 and k_2 are determined from the initial conditions $i(0+) = 0$ and $\left.\frac{di}{dt}\right|_{t=0+} = \frac{V}{L} = 1\text{A/sec}.$

Second Order Systems (8)

Thus,

$$k_1 + k_2 = 0, \quad -k_1 - 2k_2 = 1$$

Hence,

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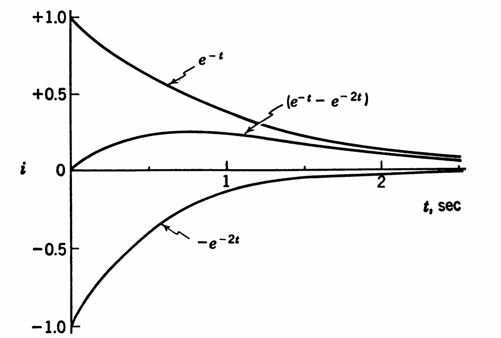

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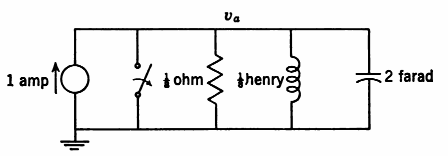


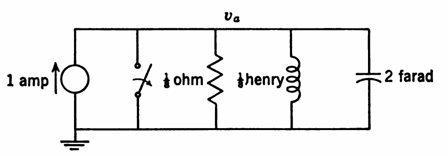
Applying KCL,

$$C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau = I$$

Equivalently,

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

Second Order Systems (9)



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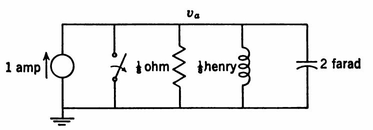
$$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0$$

Second Order Systems (10)



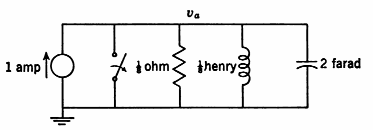
That is,

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Clearly,

$$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Second Order Systems (10)



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For the example,

$$s^2 + 4s + 4 = (s+2)^2 = 0$$

Second Order Systems (11)

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Let the solution be of the form

$$v(t) = y(t)e^{-2t}$$

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This results in

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Integrating twice,

$$y(t) = k_1 + k_2 t$$

Thus,

$$v(t) = k_1 e^{-2t} + k_2 t e^{-2t}$$

Second Order Systems (12)

$$C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau = I$$

- At the instant of switching, $v(0+) = 0$.
- Therefore, at the instant of switching, the second and third terms are zero: $\frac{1}{R}v(0+) = 0$ and $\int_{-\infty}^{0+} v(\tau)d\tau = 0$.
- Thus,

$$C\frac{dv}{dt}\bigg|_{t=0+} = I$$

- Hence, the arbitrary constants k_1 and k_2 are determined from the initial conditions $v(0+) = 0$ and $\frac{dv}{dt}\Big|_{t=0+} = \frac{I}{C} = 0.5\text{V/sec}$.

Second Order Systems (13)

Thus,

$$k_1 = 0, \quad k_2 = 0.5$$

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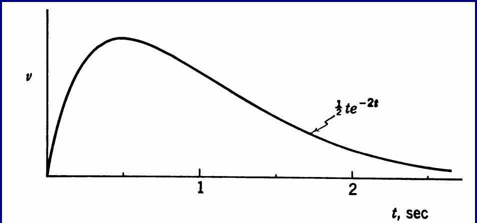
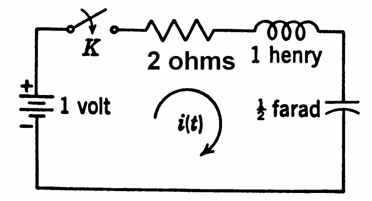


Second Order Systems (14)



Clearly,

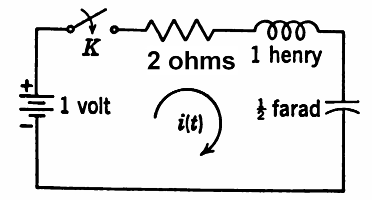
$$s^2 + 2s + 2 = 0 \implies s_1, s_2 = -1 \pm j$$

Therefore,

$$i(t) = k_1 e^{(-1+j)t} + k_2 e^{(-1-j)t} = e^{-t}\left(k_1 e^{jt} + k_2 e^{-jt}\right)$$

Using Euler's identity,

Second Order Systems (14)



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Using Euler's identity,

$$i(t) = e^{-t}\left(k_3 \cos t + k_4 \sin t\right)$$

Second Order Systems (15)

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = V$$

- The initial conditions are the same as before:

$$i(0+) = 0, \quad \left.\frac{di}{dt}\right|_{t=0+} = \frac{V}{L} = 1\text{A}/\text{sec}$$

- Therefore,

Second Order Systems (15)

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- The initial conditions are the same as before:

$$i(0+) = 0, \quad \left.\frac{di}{dt}\right|_{t=0+} = \frac{V}{L} = 1\text{A/sec}$$

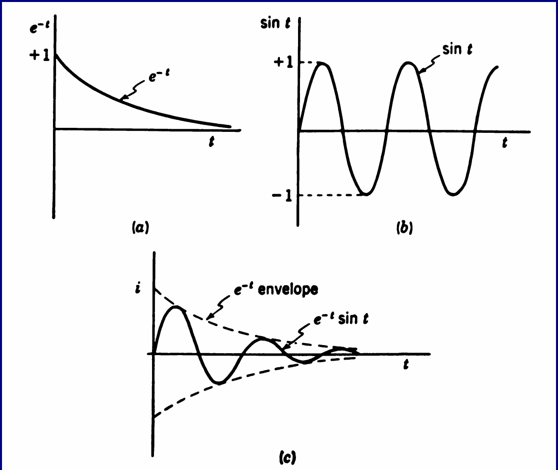
- Therefore,

$$k_3 = 0, \quad k_4 = 1$$

and hence,

$$i(t) = e^{-t}\sin t$$

Second Order Systems (16)



Second Order Systems (17)

In general,

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2 y = 0, \quad a_i > 0$$

Thus,

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In general,

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2y = 0, \quad a_i > 0$$

Thus,

$$s^2 + a_1s + a_2 = 0$$

Canonical form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Thus,

$$\begin{aligned} s_1, s_2 &= -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \end{aligned}$$

Second Order Systems (18)

$$
\begin{aligned}
s_1, s_2 &= -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2} \\
&= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\
&= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}
\end{aligned}
$$

Thus,

$$
\omega_n^2 = a_2 = \frac{1}{LC}, \quad \zeta = \frac{1}{2}\frac{a_1}{\sqrt{a_2}} = \frac{R}{2}\sqrt{\frac{C}{L}}
$$

- ω_n is called the natural frequency.
- ζ is called the damping ratio (dimensionless).

Second Order Systems (19)

Case I: $a_1^2 > 4a_2$ or $\zeta > 1$.

- Therefore,

$$-\zeta\omega_n < \omega_n\sqrt{\zeta^2 - 1}$$

- Accordingly, the roots are both real, negative and unequal.
- The general solution is of the form

$$y(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

  with both terms decaying to zero exponentially (monotonically decreasing).
- This is called an overdamped system.

Second Order Systems (20)

Case II: $a_1^2 = 4a_2$ or $\zeta = 1$ or $R_{cr} = 2\sqrt{\frac{L}{C}}$. (Therefore, $\zeta = \frac{R}{R_{cr}}$.)

- Therefore, both roots are co-located at $-a_1/2$ or $-\omega_n$.
- Accordingly, the roots are both real, negative and equal.
- The general solution is of the form

$$y(t) = k_1 e^{s_1 t} + k_2 t e^{s_2 t}$$

  with both terms decaying to zero eventually.
- This is called a critically damped system.

Note: Often a quantity Q is also defined for circuits as $Q = \frac{1}{2\zeta}$.

Second Order Systems (21)

Case III: $a_1^2 < 4a_2$ or $\zeta < 1$.

$$\begin{aligned} s_1, s_2 &= -\frac{a_1}{2} \pm j\sqrt{a_2 - \left(\frac{a_1}{2}\right)^2} \\ &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \end{aligned}$$

- Therefore, the roots form a complex conjugate pair.
- The general solution is of the form

$$y(t) = e^{-\zeta\omega_n t}\left(k_1 \sin\omega_n t + k_2 \cos\omega_n t\right)$$

with both terms oscillatory but decaying to zero eventually.

- This is called a underdamped damped system.

Second Order Systems (22)

Case IV: $a_1 = 0$, $a_2 \neq 0$ or $\zeta = 0$.

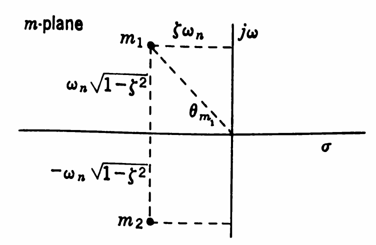
$$\begin{aligned} s_1, s_2 &= \pm j\sqrt{a_2} \\ &= \pm j\omega_n \end{aligned}$$

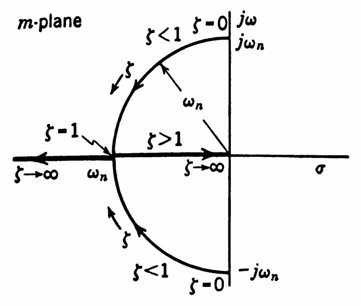
- Therefore, the roots form a conjugate imaginary pair.
- The general solution is of the form

$$y(t) = k_1 \sin \omega_n t + k_2 \cos \omega_n t$$

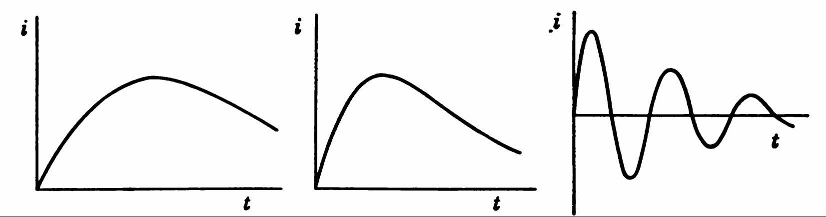
  with both terms oscillatory.
- This is called an oscillatory system.

Second Order Systems (23)





Second Order Systems (24)



Summary:

- Overdamped: $\zeta > 1$ or $Q < 0.5$ or $R > R_{cr}$.
- Critically damped: $\zeta = 1$ or $Q = 0.5$ or $R = R_{cr}$.
- Underdamped: $\zeta < 1$ or $Q > 0.5$ or $R < R_{cr}$.

Second Order Systems (25)

- There is some forcing function on the right-hand-side.
- Complementary solution $y_c(t)$: homogeneous differential equation.
- Particular solution $y_p(t)$: non-homogeneous differential equation.
- Total solution:

$$y(t) = y_c(t) + y_p(t)$$

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- Total solution:

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- In electrical networks the forcing functions are a constant, ramp, sinusoidal, exponential, linear combinations of these or products of these.
- The method of undetermined coefficients is particularly useful here.

Examples (1)

A series RL circuit with the driving force voltage $v(t) = Ve^{-at}$:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}e^{-at}$$

Since the characteristic equation is $s + \frac{R}{L} = 0$, the complementary solution is

$$i_c(t) = ke^{-Rt/L}$$

Case (i): $a \neq R/L$. Let the particular solution be

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Case (i): $a \neq R/L$. Let the particular solution be

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Substituting this in the differential equation yields $A = \frac{V}{R-aL}$. Thus, the total solution is

$$i(t) = ke^{-Rt/L} + \frac{V}{R - aL}e^{-at}$$

Examples (2)

Case (ii): $a = R/L$. Let the particular solution be

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$$i(t) = ke^{-Rt/L} + \frac{V}{L}te^{-at}$$

- The arbitrary constant k is then determined from the initial condition in each case.

Examples (3)

A series RC circuit with the driving force voltage $v(t) = V \sin \omega t$:

$$Ri + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = V \sin \omega t$$

Differentiating and re-arranging,

Examples (3)

A series RC circuit with the driving force voltage $v(t) = V \sin \omega t$:

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Differentiating and re-arranging,

$$\frac{di}{dt} + \frac{1}{RC} i = \frac{\omega V}{R} \sin \omega t$$

Since the characteristic equation is $s + \frac{1}{RC} = 0$, the complementary solution is

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Examples (4)

Let the particular solution be of the form $i_p(t) = A\cos\omega t + B\sin\omega t$. Substituting this in the differential equation, and rearranging

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$i_p(t) = A\cos\omega t + B\sin\omega t$. Substituting this in the differential equation, and rearranging

$$\frac{A}{RC} + \omega B = \frac{\omega V}{R}, \quad \frac{B}{RC} - \omega A = 0$$

Solving for A and B yields

$$A = \frac{\omega CV}{1+\omega^2 R^2 C^2}, \quad B = \frac{\omega^2 RC^2 V}{1+\omega^2 R^2 C^2}$$

Therefore,

$$\begin{aligned} i_p(t) &= \frac{V}{R^2 + \frac{1}{\omega^2 C^2}} \left(\frac{1}{\omega C}\cos\omega t + R\sin\omega t \right) \\ &= \frac{V}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \cos\left(\omega t - \tan^{-1}\omega RC\right) \end{aligned}$$

Examples (5)

Consider an RL network with the excitation $V\sin(\omega t + \theta)$:

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L}\sin(\omega t + \theta)$$

Repeating the earlier procedure,

$$i_p(t) = \frac{V}{\sqrt{R^2 + \omega^2 L^2}} \sin\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

Therefore, the total solution,

$$i(t) = \frac{V}{\sqrt{R^2 + \omega^2 L^2}} \sin\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right) + ke^{-Rt/L}$$

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Therefore, the total solution,

$$i(t)=\frac{V}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t+\theta-\tan^{-1}\frac{\omega L}{R}\right)+ke^{-Rt/L}$$

If the switch is closed at $t=0$, $i(0+)=0$. Therefore,

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Therefore, the total solution,

$$i(t)=\frac{V}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t+\theta-\tan^{-1}\frac{\omega L}{R}\right)+ke^{-Rt/L}$$

If the switch is closed at $t=0$, $i(0+)=0$. Therefore,

$$k=-\frac{V}{\sqrt{R^2+\omega^2L^2}}\sin\left(\theta-\tan^{-1}\frac{\omega L}{R}\right)$$

Examples (6)

$$k = -\frac{V}{\sqrt{R^2 + \omega^2 L^2}} \sin\left(\theta - \tan^{-1}\frac{\omega L}{R}\right)$$

If θ is chosen s.t.

$$\theta = \tan^{-1}\frac{\omega L}{R}$$

then $k = 0$.

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- There is no transient response!

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then $k = 0$.

- There is no transient response!
- A similar phenomenon can be shown to exist for RC networks as well.

Second Order Systems (26)

Factor in $v(t)^*$	Necessary choice for the particular integral†
1. V (a constant)	A
2. $a_1 t^n$	$B_0 t^n + B_1 t^{n-1} + \ldots + B_{n-1} t + B_n$
3. $a_2 e^{rt}$	Ce^{rt}
4. $a_3 \cos \omega t$	$D \cos \omega t + E \sin \omega t$
5. $a_4 \sin \omega t$	
6. $a_5 t^n e^{rt} \cos \omega t$	$(F_1 t^n + \ldots + F_{n-1} t + F_n) e^{rt} \cos \omega t$
7. $a_6 t^n e^{rt} \sin \omega t$	$+ (G_1 t^n + \ldots + G_n) e^{rt} \sin \omega t$

Network Analysis and Synthesis

Unit II: Transient Behaviour — Waveform Synthesis

Waveform Synthesis (1)

Unit step function $1(t)$ (Or, the Heaviside[1] step function.)

$$1(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0 \end{cases}$$

- The step function can be used to synthesise other waveforms.

[1] Oliver Heaviside (1850–1925) was a British self-taught electrical engineer, mathematician and physicist. He adapted complex numbers to the study of electric circuits, invented operational calculus (equivalent to Laplace transform) to solve differential equations, reformulated Maxwell's equations in terms of electric and magnetic forces and independently formulated vector analysis.

Waveform Synthesis (2)

- Rectangular Pulse:

$$p_{a,b}(t) = \begin{cases} 1 & a \leq t \leq b, \\ 0 & \text{otherwise} \end{cases}$$

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- Half-cycle of a sine wave:

$$f(t) = \begin{cases} \sin \pi t & 0 \leq t \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

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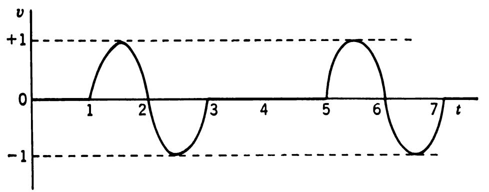
- Half-cycle of a sine wave:

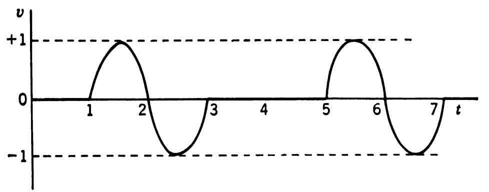
$$f(t) = \begin{cases} \sin \pi t & 0 \leq t \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = \sin \pi t \, 1(t) + \sin \pi (t-1) \, 1(t-1)$$

Waveform Synthesis (3)



Waveform Synthesis (3)

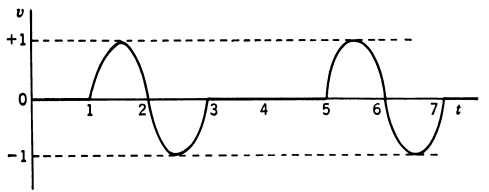


Solutions:

- With step function:

$$f(t) = \sin \pi(t-1)\, 1(t-1)$$

Waveform Synthesis (3)

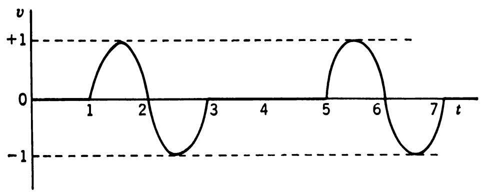


Solutions:

- With step function:

$$f(t) = \sin \pi(t-1)\, 1(t-1) - \sin \pi(t-3)\, 1(t-3)$$

Waveform Synthesis (3)

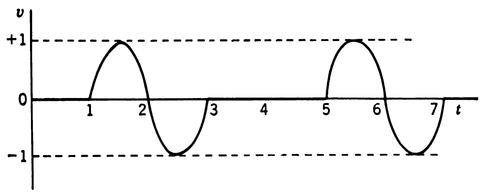


Solutions:

- With step function:

$$\begin{aligned} f(t) &= \sin\pi(t-1)\,1(t-1) - \sin\pi(t-3)\,1(t-3) \\ &+ \sin\pi(t-5)\,1(t-5) \end{aligned}$$

Waveform Synthesis (3)

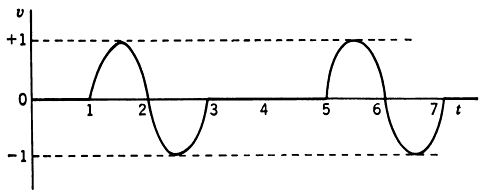


Solutions:

- With step function:

$$
\begin{aligned}
f(t) \;&=\; \sin\pi(t-1)\,1(t-1) - \sin\pi(t-3)\,1(t-3) \\
&\quad + \sin\pi(t-5)\,1(t-5) - \sin\pi(t-7)\,1(t-7)
\end{aligned}
$$

Waveform Synthesis (3)



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- With pulse function:

$$f(t) = -\sin\pi t\,\left(p_{1,3}(t) + p_{5,7}(t)\right)$$

Waveform Synthesis (4)

Unit ramp:

$$r(t) = \begin{cases} t & t \geq 0, \\ 0 & t < 0 \end{cases}$$

Waveform Synthesis (4)

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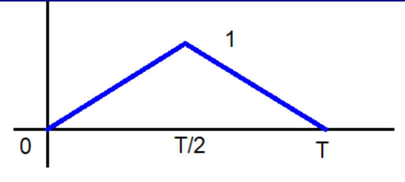
Synthesise the triangular function:



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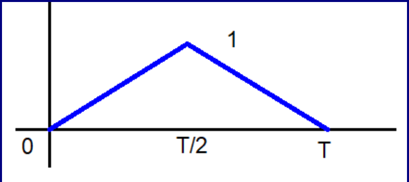

$$f(t) = \frac{2}{T} r(t)$$

Waveform Synthesis (4)

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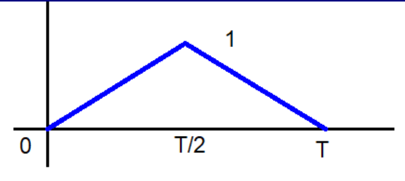

$$f(t) = \frac{2}{T} r(t) - \frac{4}{T} r\left(t - \frac{T}{2}\right)$$

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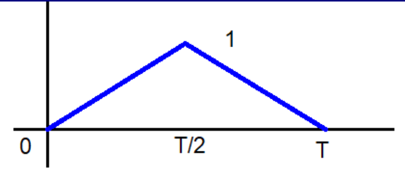

$$f(t) = \frac{2}{T}r(t) - \frac{4}{T}r\left(t - \frac{T}{2}\right) + \frac{2}{T}r(t - T)$$

Network Analysis and Synthesis

Unit II: Transient Behaviour — Network Functions

Network Functions (1)

- First order circuit excited by a DC source:

$$Y(s) = \frac{1}{s + 1/T} y(0-) + \frac{1}{s + 1/T} E(s)$$

where $E(s)$ is the excitation and $Y(s)$ is the response.

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$$Y(s) = \frac{1}{s + 1/T} E(s) = F(s)E(s)$$

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where $E(s)$ is the excitation and $Y(s)$ is the response.

- Under zero initial condition

$$Y(s) = \frac{1}{s + 1/T} E(s) = F(s)E(s)$$

- Similarly, for a second order circuit, under zero initial conditions,

$$Y(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2} E(s) = F(s)E(s)$$

Network Functions (2)

These can be represented as follows:

$$E(s) \longrightarrow \boxed{F(s)} \longrightarrow Y(s)$$

- $F(s)$ is called a network function:

$$F(s) = \frac{Y(s)}{E(s)}$$

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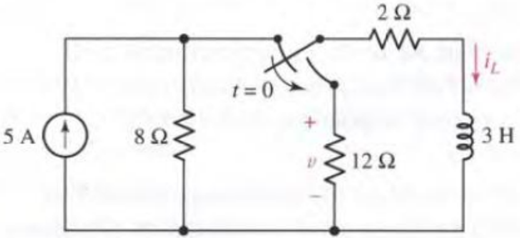
- If the excitation is a current and the response is a voltage, then $F(s)$ is an impedance.
- Together, they are referred to as immitance.

Network Analysis and Synthesis

Unit II: Transient Behaviour — Additional Examples

Examples (1)

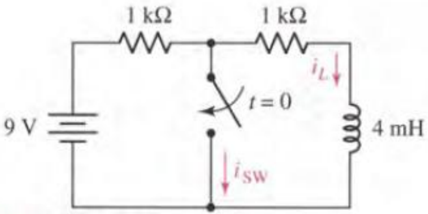
The switch in the circuit is a single-pole, double-throw switch that is drawn to indicate that it closes one circuit before opening the other; this type of switch is often referred to as a "make before break" switch. Assuming the switch has been in the position drawn in the figure for a long time, determine the value of v and i_L (*a*) the instant just *prior* to the switch changing; (*b*) the instant just *after* the switch changes.



Source: Hayt, Kemmerly and Durbin, 2007.

Examples (2)

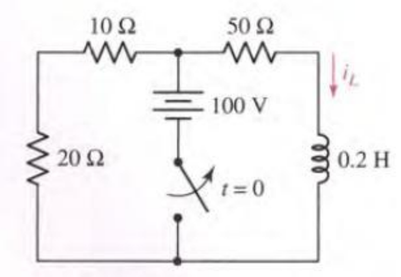
After being in the configuration shown for hours, the switch in the circuit is closed at $t = 0$. At $t = 5\ \mu s$, calculate: (*a*) i_L; (*b*) i_{SW}.



Source: Hayt, Kemmerly and Durbin, 2007.

Examples (3)

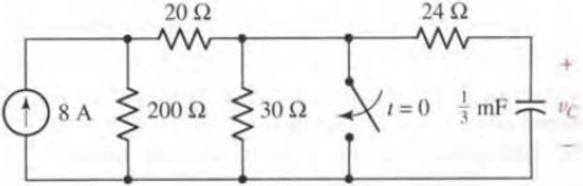
After having been closed for a long time, the switch in the circuit is opened at $t = 0$. (*a*) Find $i_L(t)$ for $t > 0$. (*b*) Evaluate i_L. (*c*) Find t_1 if $i_L(t_1) = 0.5 i_L(0)$.



Source: Hayt, Kemmerly and Durbin, 2007.

Examples (4)

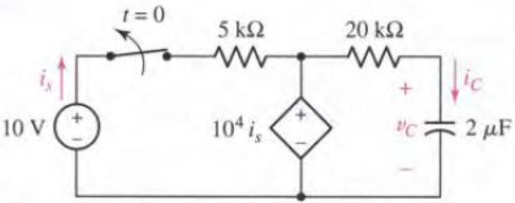
(a) Find $v_C(t)$ for all time in the circuit (b) At what time is $v_C = 0.1 v_C(0)$?



Source: Hayt, Kemmerly and Durbin, 2007.

Examples (5)

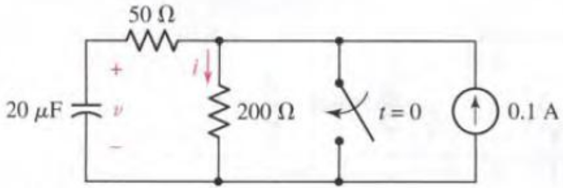
Determine $v_C(t)$ and $i_C(t)$ for the circuit and sketch both curves on the same time axis, $-0.1 < t < 0.1$ s.



Source: Hayt, Kemmerly and Durbin, 2007.

Examples (6)

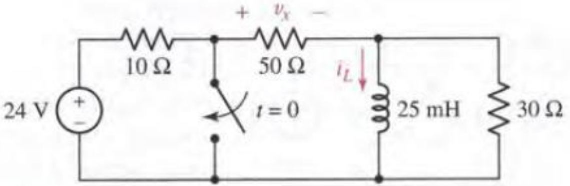
For the circuit determine the value of the current labeled i and the voltage labeled v at $t = 0^+$, $t = 1.5$ ms, and $t = 3.0$ ms.



Source: Hayt, Kemmerly and Durbin, 2007.

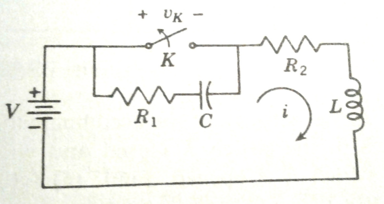
Examples (7)

The switch in the circuit has been open for a long time before it closes at $t = 0$. (*a*) Find $i_L(t)$ for $t > 0$. (*b*) Sketch $v_x(t)$ for $-4 < t < 4$ ms.



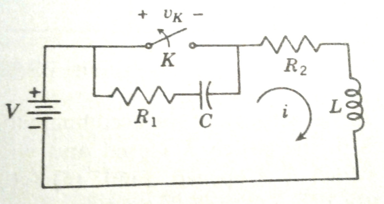
Source: Hayt, Kemmerly and Durbin, 2007.

Examples (8)

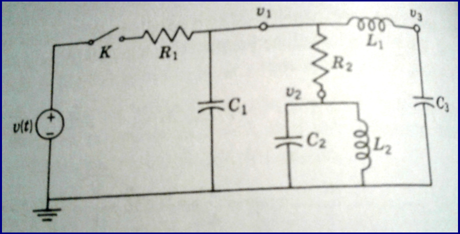


Source: Valkenburg, 1975; Problem 5.25.

Examples (8)

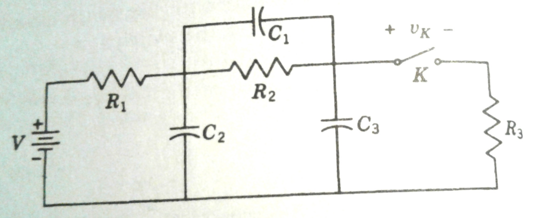


Source: Valkenburg, 1975; Problem 5.25.



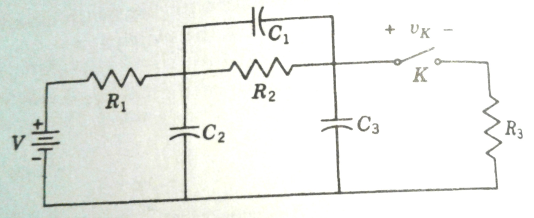
Source: Valkenburg, 1975; Problem 5.27.

Examples (9)



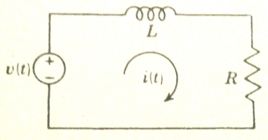
Source: Valkenburg, 1975; Problem 5.28.

Examples (9)



Source: Valkenburg, 1975; Problem 5.28.

6-23. A bolt of lightning having a waveform which is approximated as $v(t) = te^{-t}$ strikes a transmission line having resistance $R = 0.1\ \Omega$ and inductance $L = 0.1$ H (the line-to-line capacitance is assumed negligible). An equivalent network is shown in the accompanying diagram. What is the form of the current as a function of time? (This current will be in amperes per unit volt of the lightning; likewise the time base is normalized.)



Source: Valkenburg, 1975; Problem 6.23.

Examples (10)

6-24. In the network of the figure, the switch K is closed at $t = 0$ with the capacitor initially unenergized. For the numerical values given, find $i(t)$.

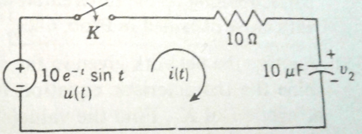


Fig. P6-24.

Source: Valkenburg, 1975; Problem 6.24.

Examples (10)

6-24. In the network of the figure, the switch K is closed at $t = 0$ with the capacitor initially unenergized. For the numerical values given, find $i(t)$.

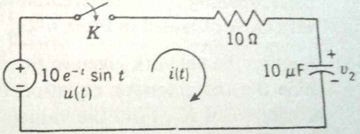


Fig. P6-24.

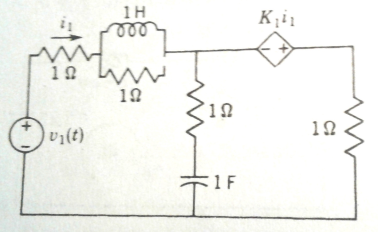
Source: Valkenburg, 1975; Problem 6.24.

6-29. Consider a series RLC network which is excited by a voltage source. (a) Determine the characteristic equation corresponding to the differential equation for $i(t)$. (b) Suppose that L and C are fixed in value but that R varies from 0 to ∞. What will be the locus of the roots of the characteristic equation? (c) Plot the roots of the characteristic equation in the s plane if $L = 1$ H, $C = 1$ μF, and R has the following values: 500 Ω, 1000 Ω, 3000 Ω, 5000 Ω.

Source: Valkenburg, 1975; Problem 6.29.

Examples (11)

-31. Analyze the network given in the figure on the loop basis, and determine the characteristic equation for the currents in the network as a function of K_1. Find the value(s) of K_1 for which the roots of the characteristic equation are on the imaginary axis of the s plane. Find the range of values of K_1 for which the roots of the characteristic equation have positive real parts.



Source: Valkenburg, 1975; Problem 6.31.

Examples (12)

6-33. A switch is closed at $t = 0$ connecting a battery of voltage V with a series RL circuit. (a) Show that the energy in the resistor as a function of time is

$$w_R = \frac{V^2}{R}\left(t + \frac{2L}{R}e^{-Rt/L} - \frac{L}{2R}e^{-2Rt/L} - \frac{3L}{2R}\right) \text{ joules}$$

(b) Find an expression for the energy in the magnetic field as a function of time. (c) Sketch w_R and w_L as a function of time. Show the steady-state asymptotes, that is, the values that w_R and w_L approach as $t \longrightarrow \infty$. (d) Find the total energy supplied by the voltage source in the steady state.

Source: Valkenburg, 1975; Problem 6.33.