Network Analysis & Systems

Koshy George

Dept. of Elect. & Comm. Engineering,

PES University;

PES Centre for Intelligent Systems

UE18EC201: Network Analysis & Systems

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Network Analysis and Synthesis

Part V: Network Synthesis

Overview of Syllabus

Unit V (8+2 hours) Network Synthesis:

- Hurwitz polynomials.
- Positive real functions.
- Elementary synthesis procedures.
- Properties of R-C impedance, R-L admittance, L-C immittance functions.
- Foster forms I and II.
- Cauer forms I and II.

Ref. C: Chapters 10 & 11.

Reference Books

C: F. F. Kuo, *Network Analysis and Synthesis*, 2nd edn., John Wiley, 1966.

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C: F. F. Kuo, *Network Analysis and Synthesis*, 2nd edn., John Wiley, 1966.

I W. Cauer, *Synthesis of Linear Communication Networks, Vol I & II*, 2nd edn., McGraw-Hill, 1958 (Original German edition, 1941).

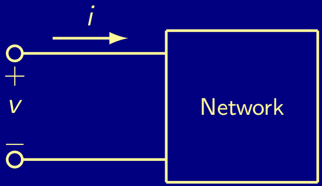
II M. A. Van Valkenburg, *Introduction to Modern Network Synthesis*, John Wiley, 1960.

III L. Weinberg, *Network Analysis and Synthesis*, McGraw-Hill, 1962.

Introduction

- Realisability Conditions.
- Mathematics of Synthesis.
- Our discussion is limited to certain aspects of synthesis of LC, RL, RL and RLC networks.

Network Functions — One-Ports (1)



- The transform impedance at a port is defined as follows:

$$Z(s) = \frac{V(s)}{I(s)}$$

- The transform admittance at a port is defined as follows:

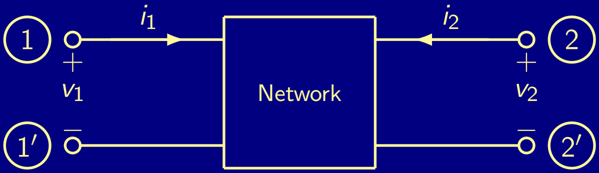
$$Y(s) = \frac{I(s)}{V(s)}$$

- These are called driving-point impedance or admittance, or together referred to as driving-point immittance.

Network Functions — Two-Ports (2)

- A transfer function is used to describe the relationship between the Laplace transform of a quantity at one port to the Laplace transform of a quantity at another port.
- There are three possible forms:
  - The ratio of one voltage to another voltage: voltage transfer ratio.
  - The ratio of one current to another current: current transfer ratio.
  - The ratio of one voltage to another current or one current to another voltage.

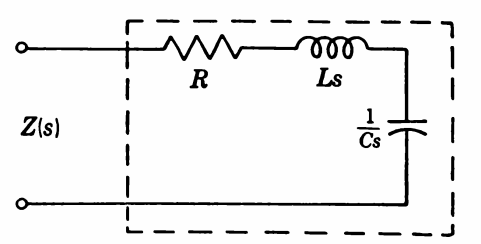
Network Functions — Two-Ports (3)



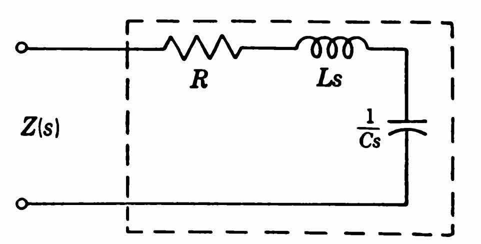
- It is a convention to define a transfer function as the ratio of an output quantity to an input quantity.
- Accordingly, only the following transfer functions are defined:

$$\frac{V_2(s)}{V_1(s)}, \quad \frac{I_2(s)}{I_1(s)}, \quad \frac{V_2(s)}{I_1(s)}, \quad \frac{I_2(s)}{V_1(s)}$$

Examples (1)

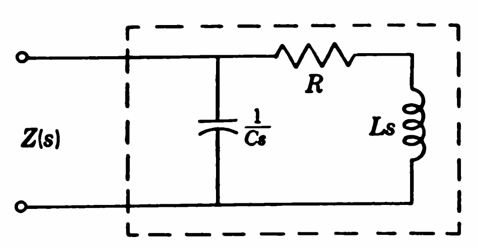


Examples (1)

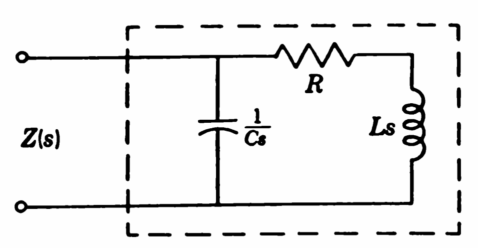


$$Z(s) = L\frac{s^2 + (R/L)s + 1/LC}{s}$$

Examples (2)

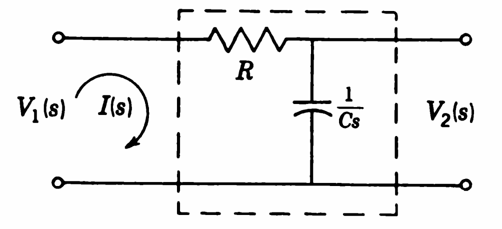


Examples (2)



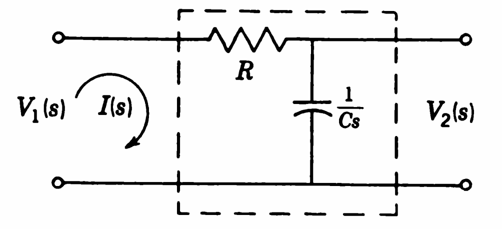
$$Z(s) = \frac{1}{C}\frac{s + R/L}{s^2 + (R/L)s + 1/LC}$$

Examples (3)



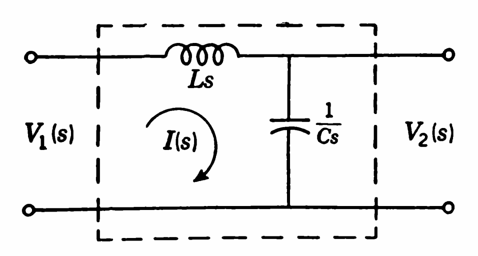
$$\frac{V_2(s)}{V_1(s)} =$$

Examples (3)



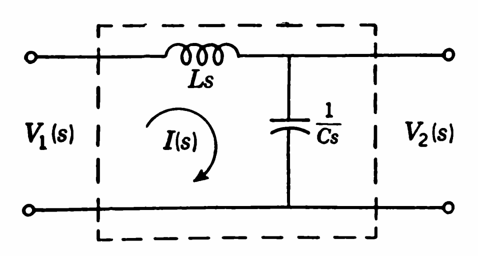
$$\frac{V_2(s)}{V_1(s)} = \frac{1/RC}{s + 1/RC}$$

Examples (4)



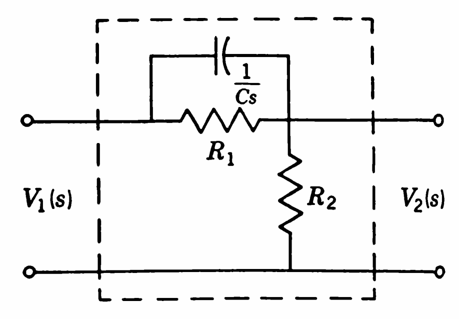
$$\frac{V_2(s)}{V_1(s)} =$$

Examples (4)



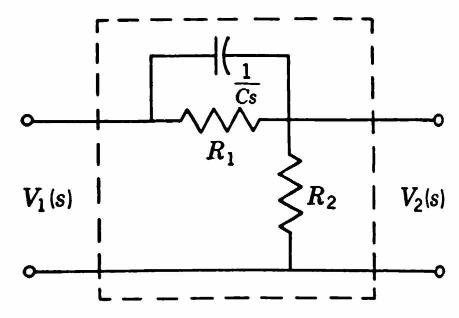
$$\frac{V_2(s)}{V_1(s)} = \frac{1/LC}{s^2 + 1/LC}$$

Examples (5)



$$\frac{V_2(s)}{V_1(s)} =$$

Examples (5)



$$\frac{V_2(s)}{V_1(s)} = \frac{s + 1/R_1 C}{s + (R_1 + R_2)/R_1 R_2 C}$$

Network Functions (4)

All network functions can be written in the following form:

$$
\begin{aligned}
H(s) &= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)} \\
&= k \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}
\end{aligned}
$$

- $A(s) = 0$ has n roots; these are called poles.
- $B(s) = 0$ has m roots; these are called zeros.
- k is called the scale factor.
- A network function is completely specified by its poles, zeros, and the scale factor.

Network Functions (5)

$$
\begin{aligned}
H(s) &= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)} \\
&= k \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}
\end{aligned}
$$

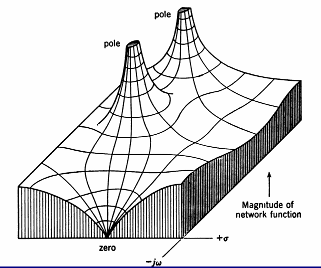
- When $n > m$, there are $n - m$ zeros at infinity.
- When $n < m$, there are $m - n$ poles at infinity.

Network Functions (5)

$$\begin{aligned} H(s) &= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)} \\ &= k \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \end{aligned}$$

- When $n > m$, there are $n - m$ zeros at infinity.
- When $n < m$, there are $m - n$ poles at infinity.
- Thus, counting the number of poles and zeros at infinity, the total number of poles is equal to the total number of zeros.
- The poles or zeros may be simple (or distinct) or multiple. For example, a pole-factor $(s - p_j)^{r_j}$ is said to have multiplicity r_j.

Network Functions (6)



- The poles and zeros are called critical frequencies.
- At the poles the network function becomes infinite, and at the zeros the network function becomes zero.
- The network function has a finite non-zero value at other complex frequencies.
- Without poles and zeros, the network function is an expanse of mathematical desert.

Network Functions (7)

$$
\begin{aligned}
H(s) &= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{B(s)}{A(s)} \\
&= k\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}
\end{aligned}
$$

- The coefficients a_i and b_j are real for network functions.
- Accordingly, such functions are called real-rational functions.

Network Functions (7)

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\end{aligned}
$$

- The coefficients a_i and b_j are real for network functions.
- Accordingly, such functions are called real-rational functions.
- When $n \geq m$, $H(s)$ is said to be proper rational function.
- When $n > m$, $H(s)$ is said to be strictly proper rational function.
- When $n < m$, $H(s)$ is said to be improper rational function.

Network Functions (8)

$$U(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

- Suppose that $H(s)$ is a transfer function. Then, $Y(s)$ is the transform of the output variable of interest and $U(s)$ is the input variable of interest.
- Clearly,

$$Y(s) = H(s)U(s)$$

- Thus, if $u(t)$ is specified, the transfer function enables the determination of the response $y(t)$ of the network.

Network Functions (9)

- Suppose that the partial fraction expansion[1] is as follows:

$$H(s)U(s) = \sum_{i=1}^{n} \frac{c_i}{s - p_i} + \sum_{j=1}^{n'} \frac{c'_j}{s - p'_j}$$

  assuming
  - all the poles are simple,
  - no pole of $H(s)$ cancels a zero of $U(s)$, no pole of $U(s)$ cancels a zero of $H(s)$,
  - and that p_i corresponds to the poles of $H(s)$ and p'_j corresponds to the poles of $U(s)$.
- Then,

$$y(t) = \sum_{i=1}^{n} c_i e^{p_i t} + \sum_{j=1}^{n'} c'_j e^{p'_j t}$$

[1] To be considered in some detail later.

Network Functions (10)

$$y(t) = \sum_{i=1}^{n} c_i e^{p_i t} + \sum_{j=1}^{n'} c'_j e^{p'_j t}$$

- The frequencies corresponding to p_j are called natural oscillations or free oscillations.
- The frequencies corresponding to p'_j are called forced oscillations.

Network Functions (10)

$$y(t) = \sum_{i=1}^{n} c_i e^{p_i t} + \sum_{j=1}^{n'} c_j' e^{p_j' t}$$

- The frequencies corresponding to p_j are called natural oscillations or free oscillations.
- The frequencies corresponding to p_j' are called forced oscillations.
- The poles determine the nature of the waveform of the output.
- The zeros determine the strength (i.e., c_i and c_j') of each part of the response.

Network Functions — One-port (11)

- Consider the driving-point impedance $1/Cs$. There is a pole at $s = 0$ and a zero at $s = \infty$.

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- Similarly, for the driving-point impedance Ls, it behaves as an open circuit at the pole frequency $(s = \infty)$ and a short circuit at the zero frequency $(s = 0)$.

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- At the pole frequency ($s = 0$) it behaves as an open circuit, and at the zero frequency ($s = \infty$) it behaves as a short circuit.
- Similarly, for the driving-point impedance Ls, it behaves as an open circuit at the pole frequency ($s = \infty$) and a short circuit at the zero frequency ($s = 0$).
- More generally, a driving-point impedance $H(s)$ behaves as an open-circuit at a pole frequency and it behaves as a short-circuit at a zero frequency.

Network Functions — One-port (12)

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)}$$

- Since the coefficients are all real, both $B(s)$ and $A(s)$ are real for real s.

Network Functions — One-port (12)

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- Since the coefficients are all real, both $B(s)$ and $A(s)$ are real for real s.
- Further, the roots are either real or complex; if the latter, they must occur as conjugate pairs.

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- Since the coefficients are all real, both $B(s)$ and $A(s)$ are real for real s.
- Further, the roots are either real or complex; if the latter, they must occur as conjugate pairs.
- In a one-port network consisting of only passive elements, the response is bounded if the input is bounded.
- That is, the voltage is always bounded when a current source is applied, and the current is always bounded when a voltage source is applied.

Network Functions — One-port (13)

- Therefore, in the partial fraction expansion,

$$y(t) = \sum_{i=1}^{n} c_i e^{p_i t} + \sum_{j=1}^{n'} c_j' e^{p_j' t}$$

the real part of p_i, Re $p_i \leq 0$.

Network Functions — One-port (13)

- Therefore, in the partial fraction expansion,

$$y(t) = \sum_{i=1}^{n} c_i e^{p_i t} + \sum_{j=1}^{n'} c_j' e^{p_j' t}$$

  the real part of p_i, Re $p_i \leq 0$.
- Hence, the poles and zeros of a driving-point immittance cannot lie in the right-half of s-plane.
- Those poles that lie on the imaginary axis must be simple.
- This implies that all coefficients of $B(s)$ or $A(s)$ are non-zero except if they are even (contains only even powers) or odd (contains only odd powers).

Network Functions — One-port (14)

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)}$$

- In a one-port the driving-point immittance Ls or Cs dominates at high-frequencies.
- Otherwise, it may act as a short-circuit in the case of $1/Ls$ or $1/Cs$, or, in their absence, the resistor dominates.

Network Functions — One-port (14)

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- In a one-port the driving-point immittance Ls or Cs dominates at high-frequencies.
- Otherwise, it may act as a short-circuit in the case of $1/Ls$ or $1/Cs$, or, in their absence, the resistor dominates.
- More generally,

$$\lim_{s \longrightarrow \infty} H(s) = \lim_{s \longrightarrow \infty} s^{m-n}$$

- Therefore, for a passive one-port network, $n - m$ is either -1, 0, or 1.
- The term $n - m$ is called relative degree. Thus, for a passive one-port the relative degree is -1, 0, or 1.

Network Functions — One-port (15)

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{B(s)}{A(s)}$$

- At low frequencies $s \longrightarrow 0$. Thus,

$$H(s) \approx \frac{b_{m-1}s + b_m}{a_{n-1}s + a_n}, \quad b_{m-1} \neq 0,\ a_{n-1} \neq 0$$

and this should approach the behaviour of s, $1/s$, or a constant.

Network Functions — One-port (15)

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  and this should approach the behaviour of s, $1/s$, or a constant.
- For an impedance, $b_m = 0$ and $a_n \neq 0$ implies an inductor, and $b_m \neq 0$ and $a_n = 0$ implies a capacitor.

Network Functions — One-port (15)

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{B(s)}{A(s)}$$

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- For an impedance, $b_m = 0$ and $a_n \neq 0$ implies an inductor, and $b_m \neq 0$ and $a_n = 0$ implies a capacitor.
- The opposite holds for an admittance.

Network Functions — One-port (15)

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- For an impedance, $b_m = 0$ and $a_n \neq 0$ implies an inductor, and $b_m \neq 0$ and $a_n = 0$ implies a capacitor.
- The opposite holds for an admittance.
- If $b_m \neq 0$ and $a_n \neq 0$, the behaviour is that of a resistor.

Network Functions — One-port (15)

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{B(s)}{A(s)}$$

- At low frequencies $s \longrightarrow 0$. Thus,

$$H(s) \approx \frac{b_{m-1}s + b_m}{a_{n-1}s + a_n}, \quad b_{m-1} \neq 0,\ a_{n-1} \neq 0$$

  and this should approach the behaviour of s, $1/s$, or a constant.
- For an impedance, $b_m = 0$ and $a_n \neq 0$ implies an inductor, and $b_m \neq 0$ and $a_n = 0$ implies a capacitor.
- The opposite holds for an admittance.
- If $b_m \neq 0$ and $a_n \neq 0$, the behaviour is that of a resistor.
- To summarise, the terms of the lowest degree may differ at most by one.

Network Functions: Necessary Conditions (One-port) (15)

The necessary conditions for $H(s)$ to be a driving-point function:

- All coefficients must be real and positive.
- There cannot be missing terms, unless all even or all odd terms are missing.
- The real part of all poles and zeros must be negative or zero; if the real part is zero, then that pole or zero must be simple.
- The relative degree may be -1, 0, or 1.
- The terms of lowest degree may differ at most by one.

Network Functions — Two-port (16)

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)}$$

- Recall: These are transfer functions.
- Note: $1/H(s)$ does not make sense!

Network Functions — Two-port (16)

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- Recall: These are transfer functions.
- Note: $1/H(s)$ does not make sense!
- Since the coefficients are all real, both $B(s)$ and $A(s)$ are real for real s.

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- Recall: These are transfer functions.
- Note: $1/H(s)$ does not make sense!
- Since the coefficients are all real, both $B(s)$ and $A(s)$ are real for real s.
- Further, the roots are either real or complex; if the latter, they must occur as conjugate pairs.

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- Recall: These are transfer functions.
- Note: $1/H(s)$ does not make sense!
- Since the coefficients are all real, both $B(s)$ and $A(s)$ are real for real s.
- Further, the roots are either real or complex; if the latter, they must occur as conjugate pairs.
- In a two-port consisting of only passive elements, the response is bounded if the input is bounded.

Network Functions — Two-port (17)

- Therefore, the poles of a transfer function cannot lie in the right-half of s-plane.
- Those poles that lie on the imaginary axis must be simple.
- This implies that all coefficients of $A(s)$ are non-zero except if they are even (contains only even powers) or odd (contains only odd powers).

Network Functions — Two-port (17)

- Therefore, the poles of a transfer function cannot lie in the right-half of s-plane.
- Those poles that lie on the imaginary axis must be simple.
- This implies that all coefficients of $A(s)$ are non-zero except if they are even (contains only even powers) or odd (contains only odd powers).
- There are no restrictions on the zero-locations.
- If the zeros lie in the left-half of s-plane, the network function is called minimum-phase; otherwise, they are called non-minimum-phase.

Network Functions — Two-port (18)

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{B(s)}{A(s)}$$

- For voltage-to-voltage and current-to-current transfer functions,

$$H(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \text{ or } H(s) = \frac{-Y_2(s)}{Y_1(s) + Y_2(s)}$$

and hence,

$$\deg B(s) \leq \deg A(s)$$

- For a transfer function that is an immittance at high frequencies, $H(s) = Ls$ or $H(s) = Cs$, and hence,

$$\deg B(s) \leq \deg A(s) + 1$$

Network Functions: Necessary Conditions (Two-port) (19)

The necessary conditions for $H(s)$ to be a transfer function:

- All coefficients must be real and positive.
- For the denominator polynomial, there cannot be missing terms, unless all even or all odd terms are missing.
- The real part of all poles must be negative or zero; if the real part is zero, then that pole or zero must be simple.
- There are no restrictions on the zero locations.
- For an immitance transfer function, the maximum degree of $B(s)$ is the degree of $A(s)$ plus one.
- For voltage or current ratios, the maximum degree of $B(s)$ is the degree of $A(s)$.