$$\alpha = \frac{1}{2RC} \qquad [11]$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad [10]$$

and A_1 and A_2 must be found by applying the given initial conditions.

We note two basic scenarios possible with Eqs. [12] and [13] depending on the relative sizes of α and ω_0 (dictated by the values of R, L, and C). If $\alpha > \omega_0$, s_1 and s_2 will both be real numbers, leading to what is referred to as an ***overdamped response***. In the opposite case, where $\alpha < \omega_0$, both s_1 and s_2 will have nonzero imaginary components, leading to what is known as an ***underdamped response***. Both of these situations are considered separately in the following sections, along with the special case of $\alpha = \omega_0$, which leads to what is called a ***critically damped response***. We should also note that the general response comprised by Eqs. [9] through [13] describes not only the voltage but all three branch currents in the parallel RLC circuit; the constants A_1 and A_2 will be different for each, of course.

The ratio of α to ω_0 is called the *damping ratio* by control system engineers and is designated by ζ (zeta).

Overdamped:	$\alpha > \omega_0$
Critically damped:	$\alpha = \omega_0$
Underdamped:	$\alpha < \omega_0$

EXAMPLE 9.1

Consider a parallel *RLC* circuit having an inductance of 10 mH and a capacitance of 100 μF. Determine the resistor values that would lead to overdamped and underdamped responses.

We first calculate the resonant frequency of the circuit:

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(10 \times 10^{-3})(100 \times 10^{-6})}} = 10^3 \text{ rad/s}$$

An *overdamped* response will result if $\alpha > \omega_0$; an *underdamped* response will result if $\alpha < \omega_0$. Thus,

$$\frac{1}{2RC} > 10^3$$

and so

$$R < \frac{1}{(2000)(100 \times 10^{-6})}$$

or

$$R < 5\ \Omega$$

leads to an overdamped response; $R > 5\ \Omega$ leads to an underdamped response.

PRACTICE

9.1 A parallel RLC circuit contains a $100\ \Omega$ resistor and has the parameter values $\alpha = 1000\ \text{s}^{-1}$ and $\omega_0 = 800$ rad/s. Find (*a*) C; (*b*) L; (*c*) s_1; (*d*) s_2.

Ans: 5 μF; 312.5 mH; $-400\ \text{s}^{-1}$; $-1600\ \text{s}^{-1}$.

In Fig. 9.4*b*, we draw the circuit at $t = 0^+$, representing the inductor current and capacitor voltage by ideal sources for simplicity. Since neither can change in zero time, we know that $v_C(0^+) = 60$ V.

▶ *Determine if additional information is required.*

We have an equation for the capacitor voltage: $v_C(t) = A_1 e^{-50{,}000t} + A_2 e^{-200{,}000t}$. We now know $v_C(0) = 60$ V, but a third equation is still required. Differentiating our capacitor voltage equation, we find

$$\frac{dv_C}{dt} = -50{,}000 A_1 e^{-50{,}000t} - 200{,}000 A_2 e^{-200{,}000t}$$

which can be related to the capacitor current as $i_C = C(dv_C/dt)$. Returning to Fig. 9.4*b*, KCL yields

$$i_C(0^+) = -i_L(0^+) - i_R(0^+) = 0.3 - [v_C(0^+)/200] = 0$$

▶ *Attempt a solution.*

Application of our first initial condition yields

$$v_C(0) = A_1 + A_2 = 60$$

and application of our second initial condition yields

$$i_C(0) = -20 \times 10^{-9}(50{,}000 A_1 + 200{,}000 A_2) = 0$$

Solving, $A_1 = 80$ V and $A_2 = -20$ V, so that

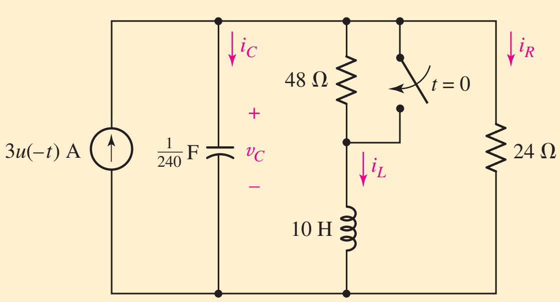
$$v_C(t) = 80e^{-50{,}000t} - 20e^{-200{,}000t} \text{ V}, \qquad t > 0$$

▶ *Verify the solution. Is it reasonable or expected?*

At the very least, we can check our solution at $t = 0$, verifying that $v_C(0) = 60$ V. Differentiating and multiplying by 20×10^{-9}, we can also verify that $i_C(0) = 0$. Also, since we have a source-free circuit for $t > 0$, we expect that $v_C(t)$ must eventually decay to zero as t approaches ∞, which our solution does.

PRACTICE

9.2 After being open for a long time, the switch in Fig. 9.5 closes at $t = 0$. Find (*a*) $i_L(0^-)$; (*b*) $v_C(0^-)$; (*c*) $i_R(0^+)$; (*d*) $i_C(0^+)$; (*e*) $v_C(0.2)$.



■ FIGURE 9.5

Ans: 1 A; 48 V; 2 A; −3 A; −17.54 V.

How do we obtain a *second* initial condition? If we multiply Eq. [18] by 30×10^3, we obtain an expression for $v_C(t)$. Taking the derivative and multiplying by 2 pF yield an expression for $i_C(t)$:

$$i_C = C\frac{dv_C}{dt} = (2 \times 10^{-12})(30 \times 10^3)(A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t})$$

By KCL,

$$i_C(0^+) = i_L(0^+) - i_R(0^+) = 0$$

Thus,

$$-(2 \times 10^{-12})(30 \times 10^3)(3.063 \times 10^6 A_1 + 13.60 \times 10^6 A_2) = 0 \qquad [20]$$

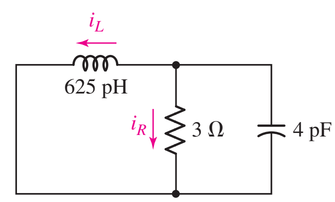
Solving Eqs. [19] and [20], we find that $A_1 = 161.3\ \mu$A and $A_2 = -36.34\ \mu$A. Thus,

$$i_R = \begin{cases} 125\ \mu\text{A} & t < 0 \\ 161.3e^{-3.063\times10^6 t} - 36.34e^{-13.6\times10^6 t}\ \mu\text{A} & t > 0 \end{cases}$$

PRACTICE

9.3 Determine the current i_R through the resistor of Fig. 9.7 for $t > 0$ if $i_L(0^-) = 6$ A and $v_C(0^+) = 0$ V. The configuration of the circuit prior to $t = 0$ is not known.

Ans: $i_R(t) = 2.437(e^{-7.823\times10^{10}t} - e^{-0.511\times10^{10}t})$ A.



■ FIGURE 9.7 Circuit for Practice Problem 9.3.

Graphical Representation of the Overdamped Response

Now let us return to Eq. [17] and see what additional information we can determine about this circuit. We may interpret the first exponential term as having a time constant of 1 s and the other exponential, a time constant of $\frac{1}{6}$ s. Each starts with unity amplitude, but the latter decays more rapidly; $v(t)$ is never negative. As time becomes infinite, each term approaches zero, and the response itself dies out as it should. We therefore have a response curve which is zero at $t = 0$, is zero at $t = \infty$, and is never negative; since it is not everywhere zero, it must possess at least one maximum, and this is not a difficult point to determine exactly. We differentiate the response

$$\frac{dv}{dt} = 84(-e^{-t} + 6e^{-6t})$$

set the derivative equal to zero to determine the time t_m at which the voltage becomes maximum,

$$0 = -e^{-t_m} + 6e^{-6t_m}$$

manipulate once,

$$e^{5t_m} = 6$$

and obtain

$$t_m = 0.358 \text{ s}$$

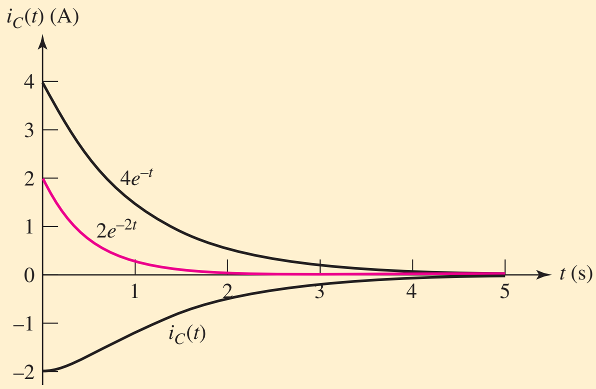


FIGURE 9.9 The current response $i_C(t) = 2e^{-2t} - 4e^{-t}$ A, sketched alongside its two components.

This equation can be solved using an iterative solver routine on a scientific calculator, which returns the solution $t_s = 5.296$ s. If such an option is not available, however, we can approximate Eq. [21] for $t \geq t_s$ as

$$-4e^{-t_s} = -0.02 \qquad [22]$$

Solving,

$$t_s = -\ln\left(\frac{0.02}{4}\right) = 5.298 \text{ s} \qquad [23]$$

which is reasonably close (better than 0.1% accuracy) to the exact solution.

PRACTICE

9.4 (*a*) Sketch the voltage $v_R(t) = 2e^{-t} - 4e^{-3t}$ V in the range $0 < t < 5$ s. (*b*) Estimate the settling time. (*c*) Calculate the maximum positive value and the time at which it occurs.

Ans: See Fig. 9.10; 5.9 s; 544 mV, 896 ms.

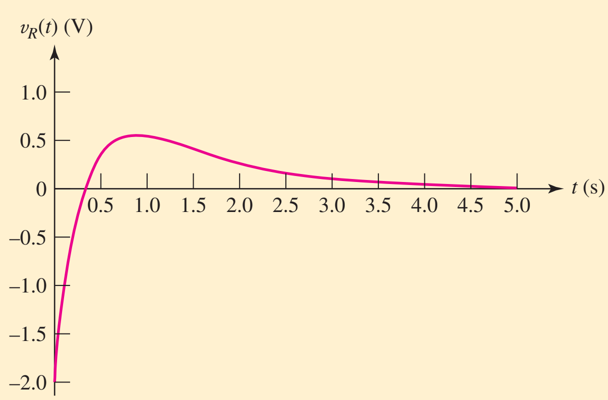
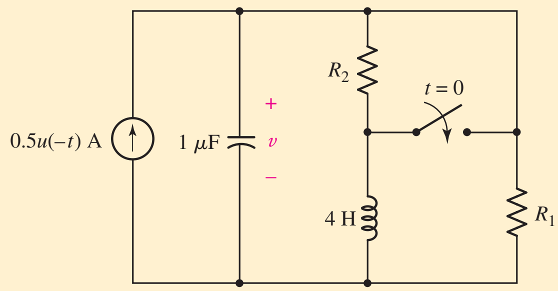


FIGURE 9.10 Response sketched for Practice Problem 9.4*a*.

PRACTICE

9.5 (*a*) Choose R_1 in the circuit of Fig. 9.13 so that the response after $t = 0$ will be critically damped. (*b*) Now select R_2 to obtain $v(0) = 100$ V. (*c*) Find $v(t)$ at $t = 1$ ms.



■ FIGURE 9.13

Ans: 1 kΩ; 250 Ω; −212 V.

9.4 THE UNDERDAMPED PARALLEL *RLC* CIRCUIT

Let us continue the process begun in Sec. 9.3 by increasing R once more to obtain what we will refer to as an ***underdamped*** response. Thus, the damping coefficient α decreases while ω_0 remains constant, α^2 becomes smaller than ω_0^2, and the radicand appearing in the expressions for s_1 and s_2 becomes negative. This causes the response to take on a much different character, but it is fortunately not necessary to return to the basic differential equation again. By using complex numbers, the exponential response turns into a *damped sinusoidal response;* this response is composed entirely of real quantities, the complex quantities being necessary only for the derivation.[1]

The Form of the Underdamped Response

We begin with the exponential form

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

and then let

$$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\sqrt{\omega_0^2 - \alpha^2}$$

where $j \equiv \sqrt{-1}$.

Electrical engineers use "j" instead of "i" to represent $\sqrt{-1}$ to avoid confusion with currents.

We now take the new radical, which is real for the underdamped case, and call it ω_d, the ***natural resonant frequency:***

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

The response may now be written as

$$v(t) = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}) \tag{26}$$

(1) A review of complex numbers is presented in Appendix 5.

PRACTICE

9.6 The switch in the circuit of Fig. 9.19 has been in the left position for a long time; it is moved to the right at $t = 0$. Find (*a*) dv/dt at $t = 0^+$; (*b*) v at $t = 1$ ms; (*c*) t_0, the first value of t greater than zero at which $v = 0$.

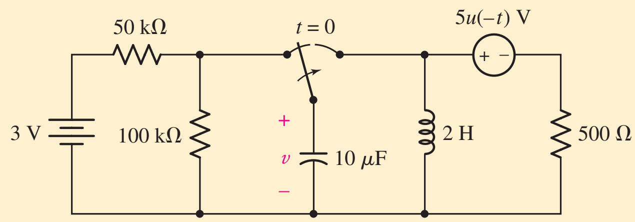


FIGURE 9.19

Ans: −1400 V/s; 0.695 V; 1.609 ms.

COMPUTER-AIDED ANALYSIS

One useful feature in Probe is the ability to perform mathematical operations on the voltages and currents that result from a simulation. In this example, we will make use of that ability to show the transfer of energy in a parallel *RLC* circuit from a capacitor that initially stores a specific amount of energy (1.25 μJ) to an inductor that initially stores no energy.

We choose a 100 nF capacitor and a 7 μH inductor, which immediately enables us to calculate $\omega_0 = 1.195 \times 10^6$ s^{-1}. In order to consider overdamped, critically damped, and underdamped cases, we need to select the parallel resistance in such a way as to obtain $\alpha > \omega_0$ (*overdamped*), $\alpha = \omega_0$ (*critically damped*), and $\alpha < \omega_0$ (*underdamped*). From our previous discussions, we know that for a parallel *RLC* circuit $\alpha = (2RC)^{-1}$. We select $R = 4.1833\ \Omega$ as a close approximation to the critically damped case; obtaining α precisely equal to ω_0 is effectively impossible. If we increase the resistance, the energy stored in the other two elements is dissipated more slowly, resulting in an underdamped response. We select $R = 100\ \Omega$ so that we are well into this regime, and use $R = 1\ \Omega$ (a very small resistance) to obtain an overdamped response.

We therefore plan to run three separate simulations, varying only the resistance R between them. The 1.25 μJ of energy initially stored in the capacitor equates to an initial voltage of 5 V, and so we set the initial condition of our capacitor accordingly.

Once Probe is launched, we select **Add** under the **Trace** menu. We wish to plot the energy stored in both the inductor and the capacitor as a function of time. For the capacitor, $w = \frac{1}{2}Cv^2$, so we click in the **Trace Expression** window, type in "0.5*100E-9*" (without the quotes), click on V(C1:1), return to the **Trace Expression** window and enter "*", click on V(C1:1) once again, and then select **Ok.** We repeat the sequence to obtain the energy stored in the inductor, using 7E-6 instead of 100E-9, and clicking on I(L1:1) instead of V(C1:1).

PRACTICE

9.7 With reference to the circuit shown in Fig. 9.24, find (*a*) α; (*b*) ω_0; (*c*) $i(0^+)$; (*d*) $di/dt|_{t=0^+}$; (*e*) $i(12\text{ ms})$.

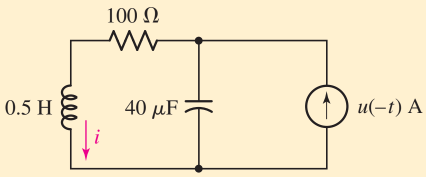


FIGURE 9.24

Ans: 100 s^{-1}; 224 rad/s; 1 A; 0; −0.1204 A.

As a final example, we pause to consider situations where the circuit includes a dependent source. If no controlling current or voltage associated with the dependent source is of interest, we may simply find the Thévenin equivalent connected to the inductor and capacitor. Otherwise, we are likely faced with having to write an appropriate integrodifferential equation, take the indicated derivative, and solve the resulting differential equation as best we can.

EXAMPLE 9.8

Find an expression for $v_C(t)$ in the circuit of Fig. 9.25*a*, valid for $t > 0$.

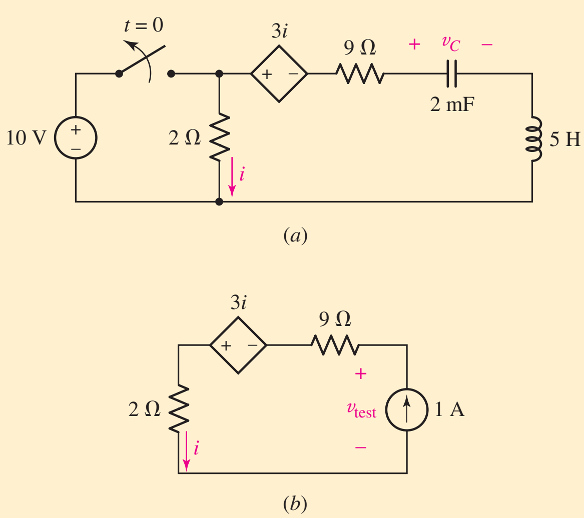


FIGURE 9.25 (*a*) An *RLC* circuit containing a dependent source. (*b*) Circuit for finding R_{eq}.

As we are interested only in $v_C(t)$, it is perfectly acceptable to begin by finding the Thévenin equivalent resistance connected in series with the

(Continued on next page)

PRACTICE

9.8 Find an expression for $i_L(t)$ in the circuit of Fig. 9.27, valid for $t > 0$, if $v_C(0^-) = 10$ V and $i_L(0^-) = 0$. *Note that although it is not helpful to apply Thévenin techniques in this instance, the action of the dependent source links v_C and i_L such that a first-order linear differential equation results.*

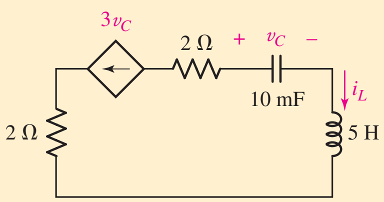


FIGURE 9.27 Circuit for Practice Problem 9.8.

Ans: $i_L(t) = -30e^{-300t}$ A, $t > 0$.

9.6 THE COMPLETE RESPONSE OF THE *RLC* CIRCUIT

We now consider those *RLC* circuits in which dc sources are switched into the network and produce forced responses that do not necessarily vanish as time becomes infinite.

The general solution is obtained by the same procedure that was followed for *RL* and *RC* circuits. The basic steps are (not necessarily in this order) as follows:

1. Determine the initial conditions.
2. Obtain a numerical value for the forced response.
3. Write the appropriate form of the natural response with the necessary number of arbitrary constants.
4. Add the forced response and natural response to form the complete response.
5. Evaluate the response and its derivative at $t = 0$, and employ the initial conditions to solve for the values of the unknown constants.

We note that *it is generally this last step that causes the most trouble for students*, as the circuit must be carefully evaluated at $t = 0$ to make full use of the initial conditions. Consequently, although the determination of the initial conditions is basically no different for a circuit containing dc sources from what it is for the source-free circuits that we have already covered in some detail, this topic will receive particular emphasis in the examples that follow.

Most of the confusion in determining and applying the initial conditions arises for the simple reason that we do not have a rigorous set of rules laid down for us to follow. At some point in each analysis, a situation usually arises in which some thinking is involved that is more or less unique to that particular problem. This is almost always the source of the difficulty.

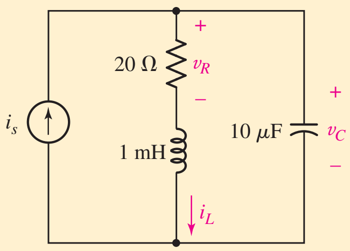
The Easy Part

The *complete* response (arbitrarily assumed to be a voltage response) of a second-order system consists of a *forced* response,

$$v_f(t) = V_f$$

PRACTICE

9.9 Let $i_s = 10u(-t) - 20u(t)$ A in Fig. 9.30. Find (*a*) $i_L(0^-)$; (*b*) $v_C(0^+)$; (*c*) $v_R(0^+)$; (*d*) $i_L(\infty)$; (*e*) $i_L(0.1\text{ ms})$.

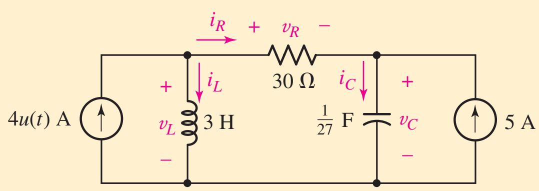


■ FIGURE 9.30

Ans: 10 A; 200 V; 200 V; −20 A; 2.07 A.

EXAMPLE 9.10

Complete the determination of the initial conditions in the circuit of Fig. 9.28, repeated in Fig. 9.31, by finding values at $t = 0^+$ for the first derivatives of the three voltage and three current variables defined on the circuit diagram.



■ FIGURE 9.31 Circuit of Fig. 9.28, repeated for Example 9.10.

We begin with the two energy storage elements. For the inductor,

$$v_L = L\frac{di_L}{dt}$$

and, specifically,

$$v_L(0^+) = L\frac{di_L}{dt}\bigg|_{t=0^+}$$

Thus,

$$\frac{di_L}{dt}\bigg|_{t=0^+} = \frac{v_L(0^+)}{L} = \frac{120}{3} = 40 \text{ A/s}$$

Similarly,

$$\frac{dv_C}{dt}\bigg|_{t=0^+} = \frac{i_C(0^+)}{C} = \frac{4}{1/27} = 108 \text{ V/s}$$

(Continued on next page)

PRACTICE

9.10 Let $v_s = 10 + 20u(t)$ V in the circuit of Fig. 9.33. Find (*a*) $i_L(0)$; (*b*) $v_C(0)$; (*c*) $i_{L,f}$; (*d*) $i_L(0.1\text{ s})$.

Ans: 0.2 A; 10 V; 0.6 A; 0.319 A.

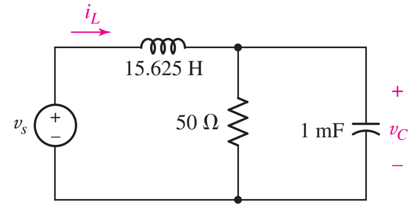


FIGURE 9.33

9.7 THE LOSSLESS *LC* CIRCUIT

When we considered the source-free *RLC* circuit, it became apparent that the resistor served to dissipate any initial energy stored in the circuit. At some point it might occur to us to ask what would happen if we could remove the resistor. If the value of the resistance in a parallel *RLC* circuit becomes infinite, or zero in the case of a series *RLC* circuit, we have a simple *LC* loop in which an oscillatory response can be maintained forever. Let us look briefly at an example of such a circuit, and then discuss another means of obtaining an identical response without the need of supplying any inductance.

Consider the source-free circuit of Fig. 9.34, in which the large values $L = 4$ H and $C = \frac{1}{36}$ F are used so that the calculations will be simple. We let $i(0) = -\frac{1}{6}$ A and $v(0) = 0$. We find that $\alpha = 0$ and $\omega_0^2 = 9\text{ s}^{-2}$, so that $\omega_d = 3$ rad/s. In the absence of exponential damping, the voltage v is simply

$$v = A\cos 3t + B\sin 3t$$

Since $v(0) = 0$, we see that $A = 0$. Next,

$$\left.\frac{dv}{dt}\right|_{t=0} = 3B = -\frac{i(0)}{1/36}$$

But $i(0) = -\frac{1}{6}$ ampere, and therefore $dv/dt = 6$ V/s at $t = 0$. We must have $B = 2$ V and so

$$v = 2\sin 3t \qquad \text{V}$$

which is an undamped sinusoidal response; in other words, our voltage response does not decay.

Now let us see how we might obtain this voltage without using an *LC* circuit. Our intentions are to write the differential equation that v satisfies and then to develop a configuration of op amps that will yield the solution of the equation. Although we are working with a specific example, the technique is a general one that can be used to solve any linear homogeneous differential equation.

For the *LC* circuit of Fig. 9.34, we select v as our variable and set the sum of the downward inductor and capacitor currents equal to zero:

$$\frac{1}{4}\int_{t_0}^{t} v\, dt' - \frac{1}{6} + \frac{1}{36}\frac{dv}{dt} = 0$$

Differentiating once, we have

$$\frac{1}{4}v + \frac{1}{36}\frac{d^2v}{dt^2} = 0$$

or

$$\frac{d^2v}{dt^2} = -9v$$

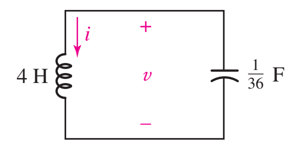


FIGURE 9.34 This circuit is lossless, and it provides the undamped response $v = 2\sin 3t$ V, if $v(0) = 0$ and $i(0) = -\frac{1}{6}$ A.