UNIT IV

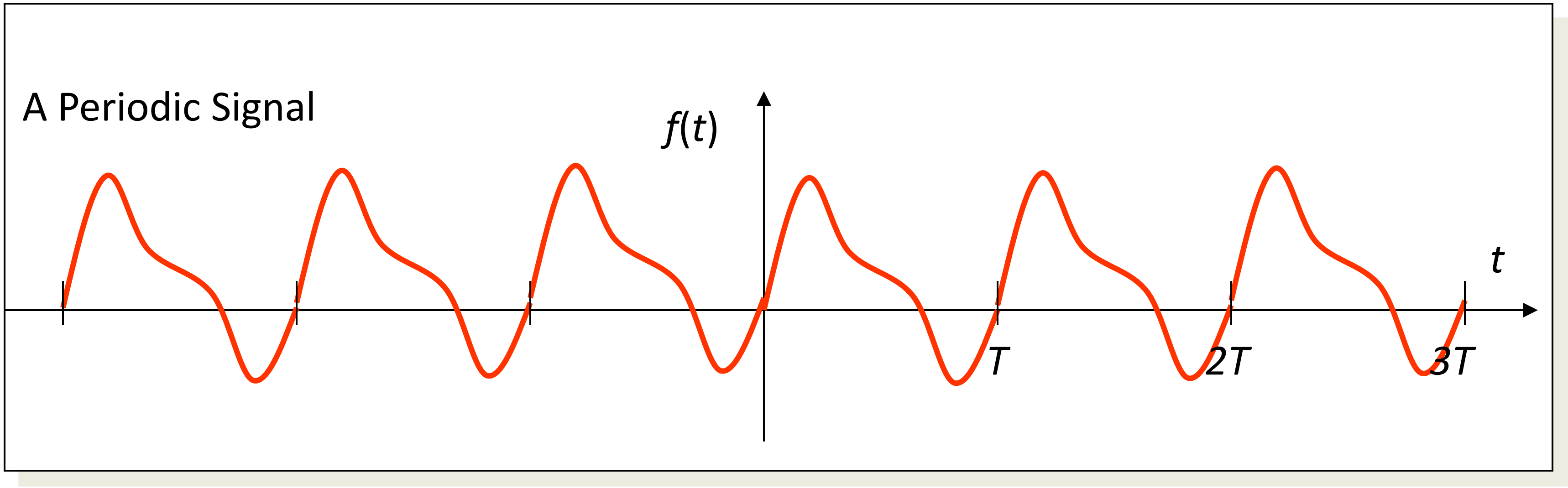
Continuous Time Fourier Transform (CTFT)

Joseph Fourier

	Continuous Time	Discrete Time
Periodic	Continuous Time Fourier Series	Discrete Time Fourier Series
Aperiodic	Continuous Time Fourier Transform	Discrete Time Fourier Transform

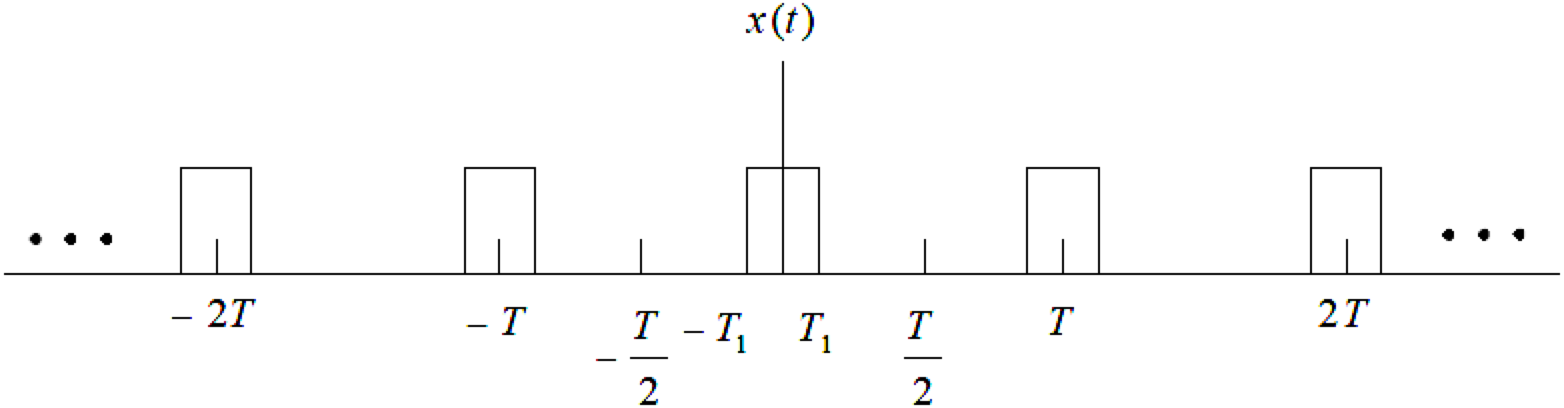
Review of Fourier Series

- Deals with continuous-time periodic signals.
- Discrete frequency spectra.

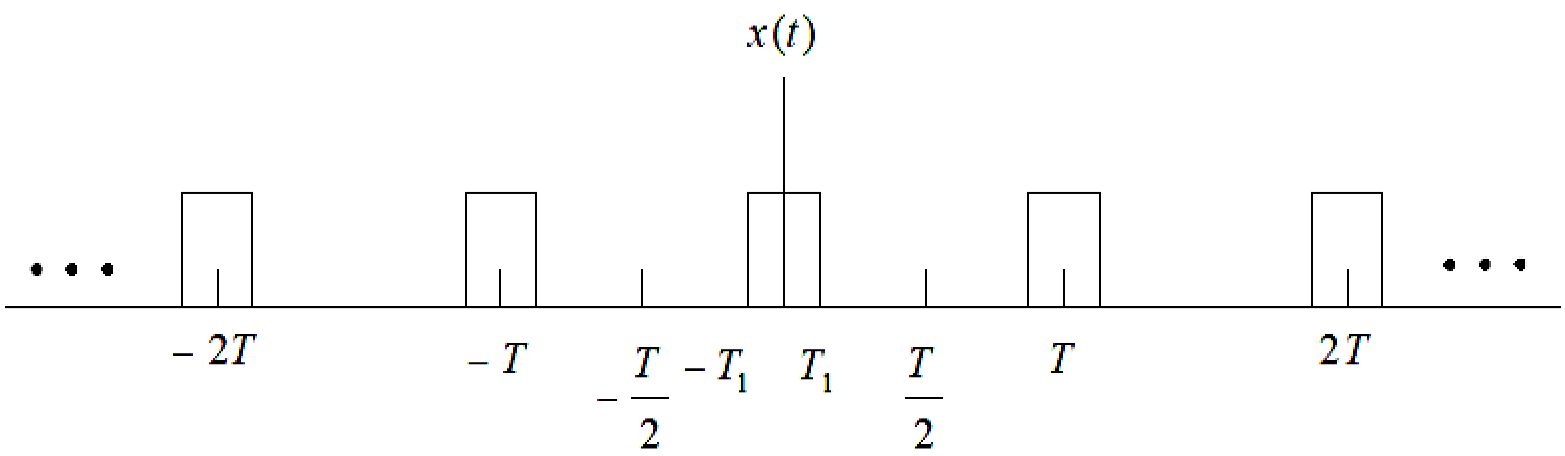


Generalization of Fourier series to aperiodic signals

- How do we get aperiodic signals by adding complex exponentials?



How to Deal with Aperiodic Signal?



If $T \rightarrow \infty$, what happens?

T increases

$\omega 0$ decreases (becomes very very small).

- · A periodic signal can be represented as linear combination of complex exponentials which are harmonically related.

- · An aperiodic signal can be represented as linear combination of complex exponentials, which are infinitesimally close in frequency. So the representation take the form of an integral rather than a sum

Fourier series synthesis equation takes Integral form

- In the Fourier representation, as the period increases the fundamental frequency decreases and the harmonically related components become closer in frequency. As the period becomes infinite, the components form a continuum and the Fourier series becomes an integral.
- Fourier series synthesis equation.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t) = \begin{cases} 1 & , |t| < T_1 \\ 0 & , T_1 < |t| < \infty \end{cases}$$

$$\tilde{x}(t) = \begin{cases} 1 & , |t| < T_1 \\ 0 & , T_1 < |t| < T/2 \end{cases}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

- As $T \to \infty,\ \tilde{x}(t) = x(t)$
- In addition, $\omega_0 \to 0$ as $T \to \infty$

$$k\omega_0 \to \omega$$

$$
\begin{aligned}
\tilde{x}(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\
a_k &= \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt \\
&= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt
\end{aligned}
$$

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$T\, a_k = \int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$X(jk\omega_0) = \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t}\, dt$$

we have for the coefficients a_k,

As $T \to \infty$

$k\omega_0 \to \omega$

$$a_k = \frac{1}{T} X(jk\omega_0)$$

$X(j\omega)$ is the envelope of Ta_k

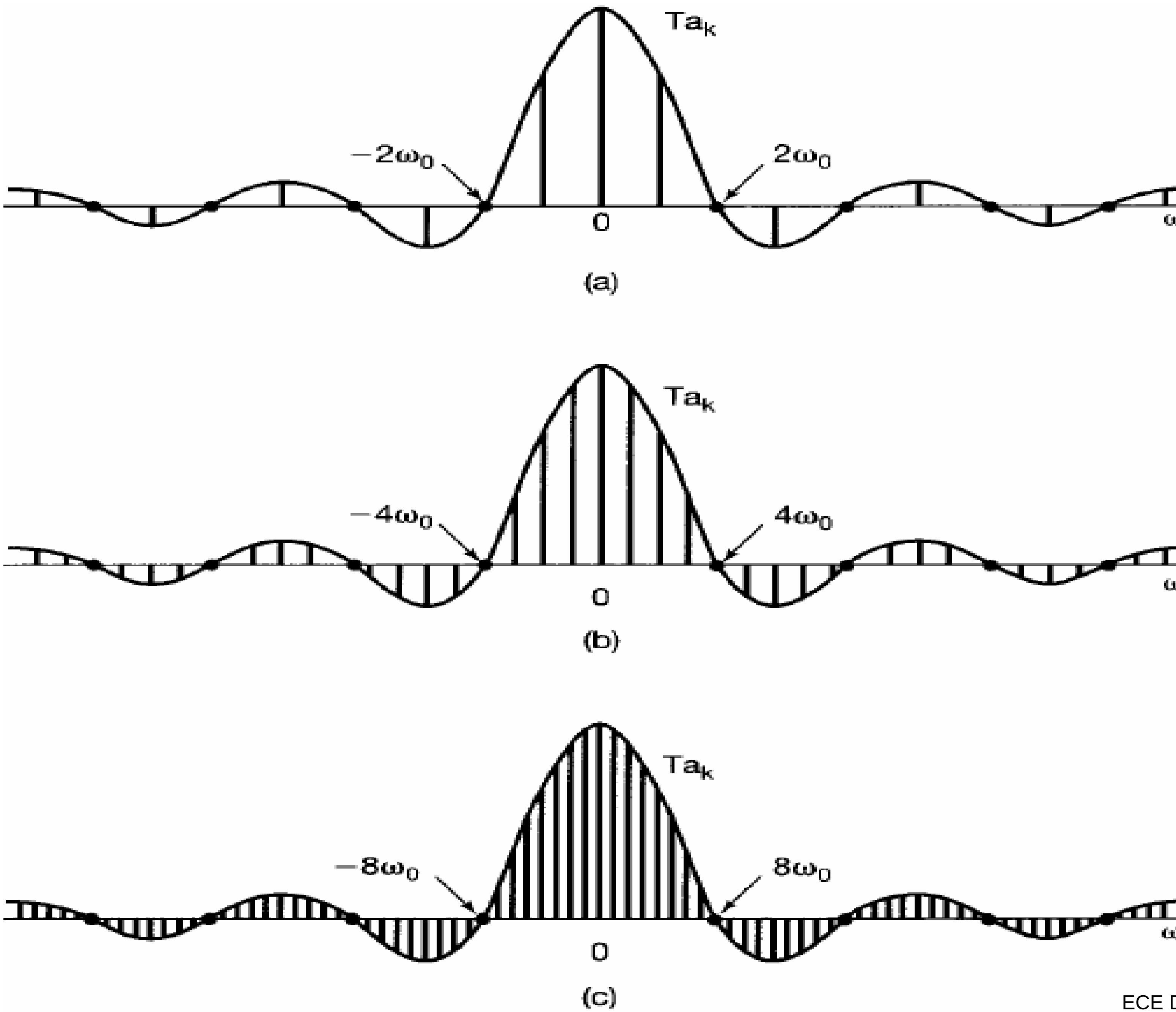
The Fourier coefficients a_k for this square wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}. \qquad Ta_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega = k\omega_0},$$

where $2\sin(\omega T_1)/\omega$ represent the envelope of Ta_k

- When T *increases or the fundamental frequency* $\omega_0 = 2\pi / T$ decreases

- *the envelope is* sampled with a closer and closer spacing. As T *becomes arbitrarily large, the original* periodic square wave approaches a rectangular pulse.
- Ta_k *becomes more and more closely spaced samples of the envelope,* as $T \to \infty$, *the Fourier* series coefficients approaches the envelope function.



Ta_k
$-2\omega_0$
$2\omega_0$
0
ω
(a)
Ta_k
$-4\omega_0$
$4\omega_0$
0
ω
(b)
Ta_k
$-8\omega_0$
$8\omega_0$
0
ω
(c)

$$X(jk\omega_0) = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\, dt$$

we have for the coefficients a_k,

$$a_k = \frac{1}{T} X(jk\omega_0)$$

$$Ta_k = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\, dt$$

Fourier Transform

Analysis equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

Synthesis equation

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} X(jk\omega_0)$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

- As $T \to \infty$, $\tilde{x}(t) = x(t)$
- In addition, $\omega_0 \to 0$ as $T \to \infty$

$d\omega$ $\quad$ $k\omega_0 \to \omega$

Summation becomes integral

Synthesis equation

Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{FT}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{Inverse FT}$$

Fourier Series vs. Fourier Integral

Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Period Function

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-jk\omega_0 t}\, dt$$

Discrete Spectra

Fourier Integral:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

Non-Period Function

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

Continuous Spectra

Existence of the Fourier Transform

Dirichlets Conditions

For existence of FT:

1. $x(t)$ is absolutely integrable, i.e., each coefficient $X(\omega)$ to be finite

$$\boxed{\int_{-\infty}^{\infty} |x(t)|dt < \infty}$$

Condition 2: In any finite interval of time,

$x(t)$ have a finite number of maxima and mi nima.

Condition 3: In any finite interval of time, there

are only a finite number of discontinuities.

Furthermore, each of these discontinuities is finite.

Fourier Transform : Example 1

$$x(t) = e^{-at}u(t) \qquad a > 0$$

Fourier Transform : Example 1

$$x(t) = e^{-at}u(t) \qquad a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Transform : Example 1

$$x(t) = e^{-at}u(t) \qquad a > 0$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt \end{aligned}$$

Fourier Transform : Example 1

$$x(t) = e^{-at}u(t) \qquad a > 0$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \end{aligned}$$

Fourier Transform : Example 1

$$x(t) = e^{-at}u(t) \qquad a > 0$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-(a+j\omega)t}dt \end{aligned}$$

Fourier Transform : Example 1

$$x(t) = e^{-at}u(t) \qquad a > 0$$

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt \\
&= \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\
&= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt \\
&= \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_{0}^{\infty}
\end{aligned}$$

Fourier Transform : Example 1

$$x(t) = e^{-at}u(t) \qquad a > 0$$

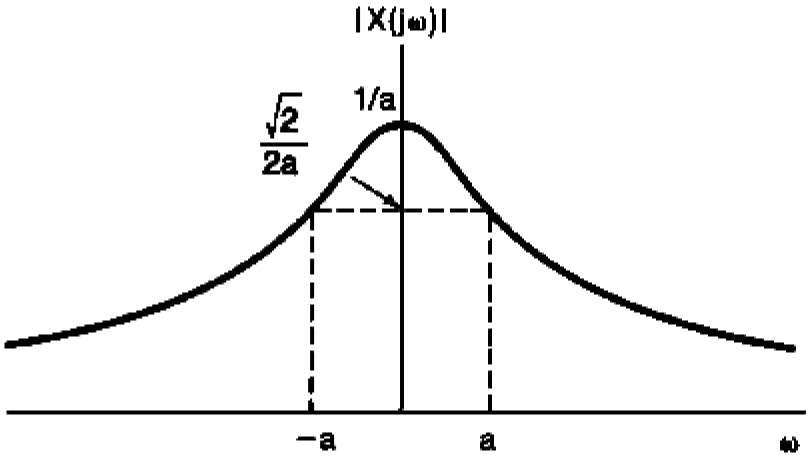
$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt \\
&= \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\
&= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt \\
&= \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty} \\
&= \frac{1}{a+j\omega}
\end{aligned}$$

Fourier Transform : Example 1

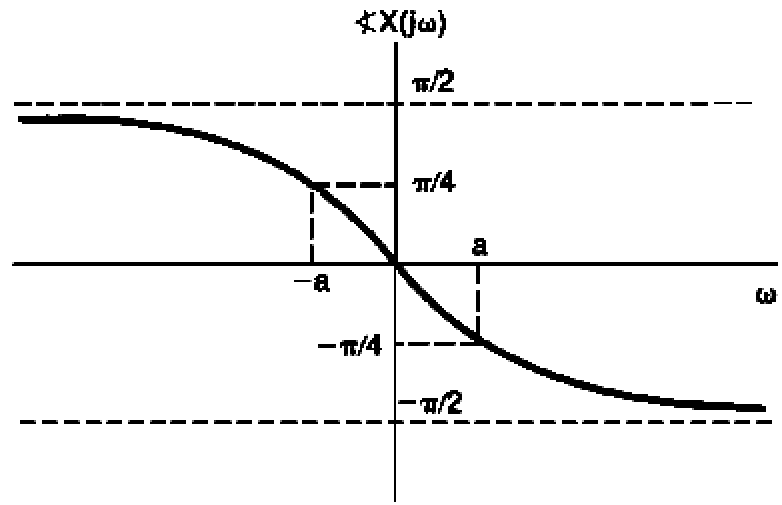
$$x(t) = e^{-at}u(t) \qquad a > 0$$

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt \\
&= \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\
&= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt \\
&= \frac{-1}{a+j\omega} e^{-(a+j\omega)t}\Big|_{0}^{\infty} \\
&= \frac{1}{a+j\omega}
\end{aligned}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \measuredangle X(j\omega) = -\tan^{-1}\frac{\omega}{a}$$



(a)



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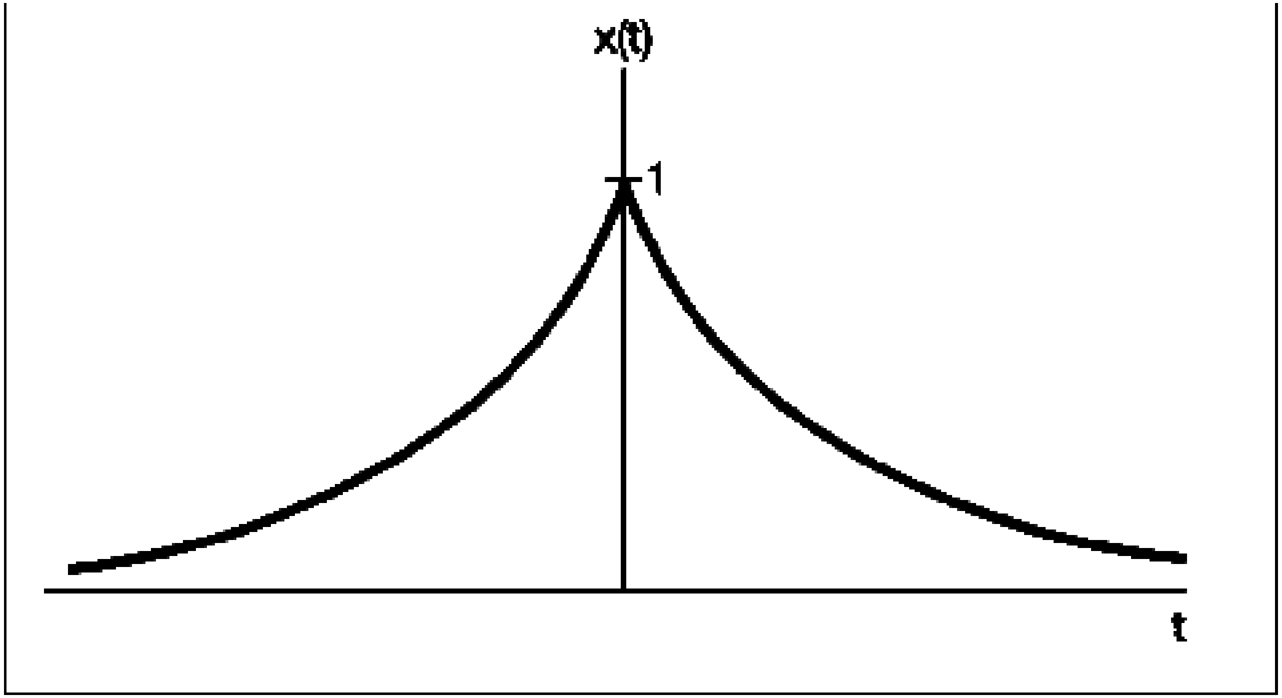
Fourier Transform : Example 2

$$x(t) = e^{-a|t|} \qquad a > 0$$



Fourier Transform : Example 2

$$x(t) = e^{-a|t|} \qquad a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Transform : Example 2

$$x(t) = e^{-a|t|} \qquad a > 0$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt \end{aligned}$$

Fourier Transform : Example 2

$$x(t) = e^{-a|t|} \qquad a > 0$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \end{aligned}$$

Fourier Transform : Example 2

$$x(t) = e^{-a|t|} \qquad a > 0$$

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt \\
&= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{(-a-j\omega)t} dt
\end{aligned}$$

Fourier Transform : Example 2

$$x(t) = e^{-a|t|} \qquad a > 0$$

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&= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{(-a-j\omega)t}\, dt \\
&= \frac{1}{a-j\omega} e^{(a-j\omega)t} \Big|_{-\infty}^{0} + \frac{1}{-a-j\omega} e^{(-a-j\omega)t} \Big|_{0}^{\infty}
\end{aligned}$$

Fourier Transform : Example 2

$$x(t) = e^{-a|t|} \qquad a > 0$$

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{(-a-j\omega)t}\, dt \\
&= \frac{1}{a-j\omega} e^{(a-j\omega)t} \Big|_{-\infty}^{0} + \frac{1}{-a-j\omega} e^{(-a-j\omega)t} \Big|_{0}^{\infty} \\
&= \frac{1}{a-j\omega}(1-0) + \frac{1}{-a-j\omega}(0-1)
\end{aligned}$$

Fourier Transform : Example 2

$$x(t) = e^{-a|t|} \qquad a > 0$$

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{(-a-j\omega)t}\, dt \\
&= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}
\end{aligned}$$

Fourier Transform : Example 2

$$x(t) = e^{-a|t|} \qquad a > 0$$

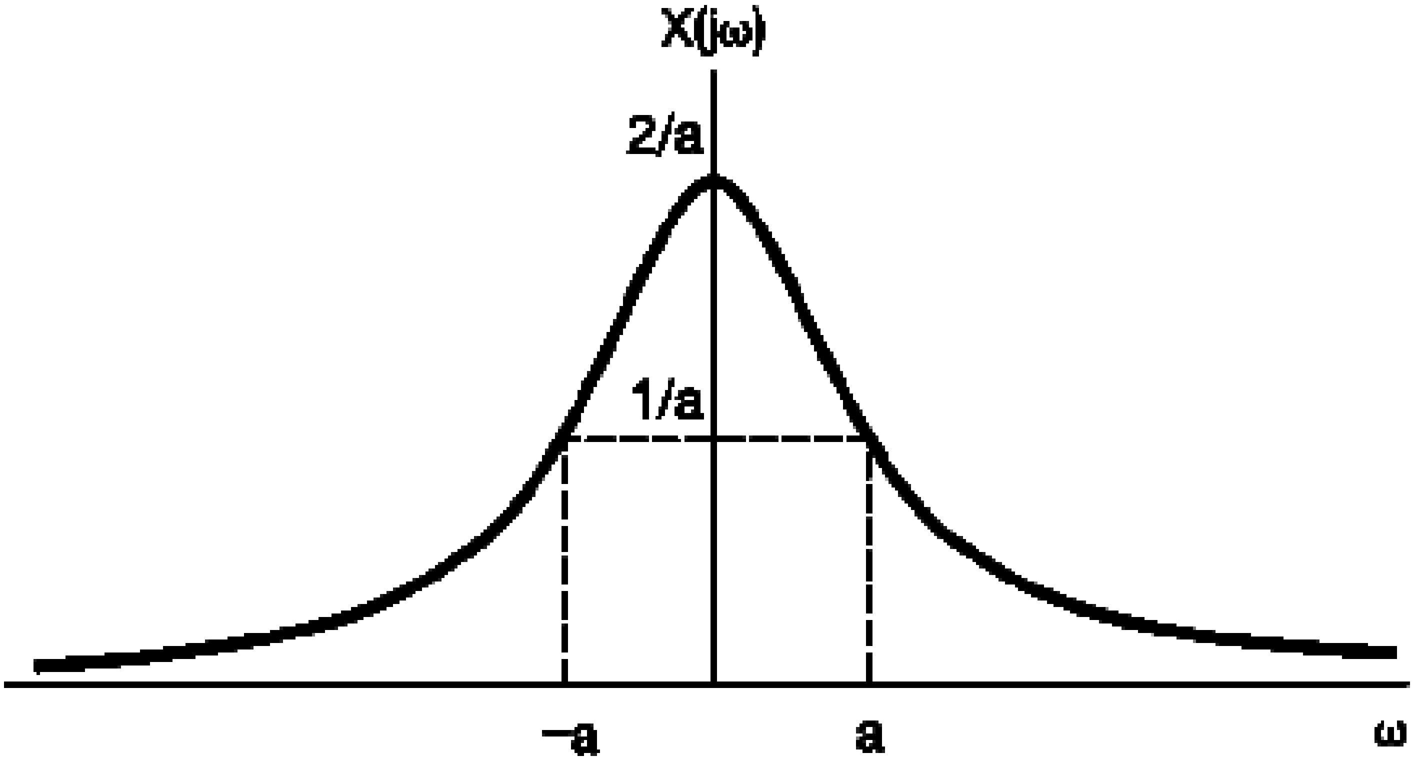
$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{(-a-j\omega)t}\, dt \\
&= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\
&= \frac{2a}{a^2+\omega^2}
\end{aligned}$$

Fourier Transform : Example 2

$$x(t) = e^{-a|t|} \qquad a > 0$$

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{(-a-j\omega)t}\, dt \\
&= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\
&= \frac{2a}{a^2+\omega^2}
\end{aligned}$$

$X(j\omega)$ *is real and even since* $x(t)$ *is real and even*



X(jω)
2/a
1/a
−a
a
ω

Fourier Transform : Example 3

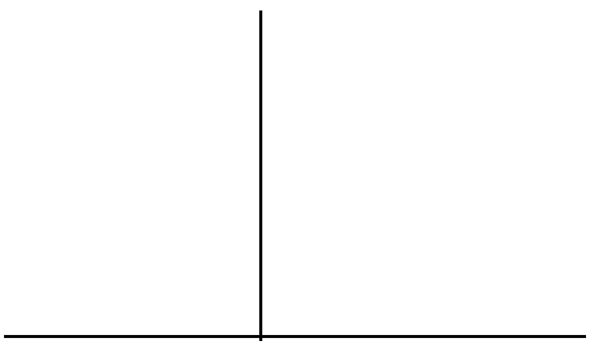
$$x(t) = \delta(t)$$

Fourier Transform : Example 3

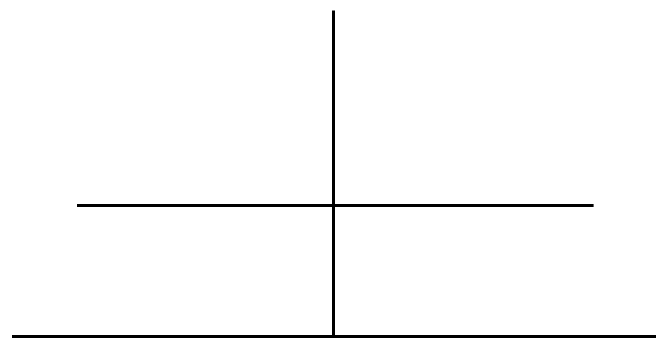
$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \delta(t)$$



$$X(j\omega) = 1$$



Fourier Transform : Example 3

$$x(t) = \delta(t)$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt \end{aligned}$$

Fourier Transform : Example 3

$$x(t) = \delta(t)$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt \\ &= 1 \end{aligned}$$

Fourier Transform : Example 3

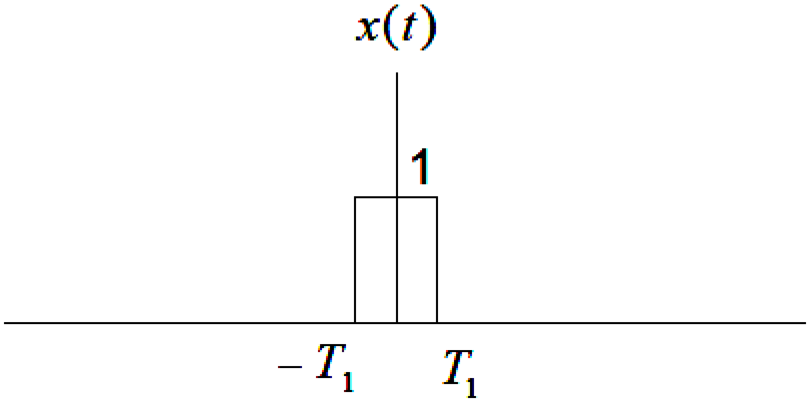
$$x(t) = \delta(t)$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt \\ &= 1 \end{aligned}$$

FT of unit impulse contains equal contributions at all frequencies

Fourier Transform : Example 4

$$x(t) = \begin{cases} 1 & , \quad |t| < T_1 \\ 0 & , \quad |t| > T_1 \end{cases}$$



Fourier Transform : Example 4

$$x(t) = \begin{cases} 1 & , \quad |t| < T_1 \\ 0 & , \quad |t| > T_1 \end{cases}$$

$$X(j\omega) \;=\; \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Transform : Example 4

$$x(t) = \begin{cases} 1 & , \quad |t| < T_1 \\ 0 & , \quad |t| > T_1 \end{cases}$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-T_1}^{T_1} e^{-j\omega t} dt \end{aligned}$$

Fourier Transform : Example 4

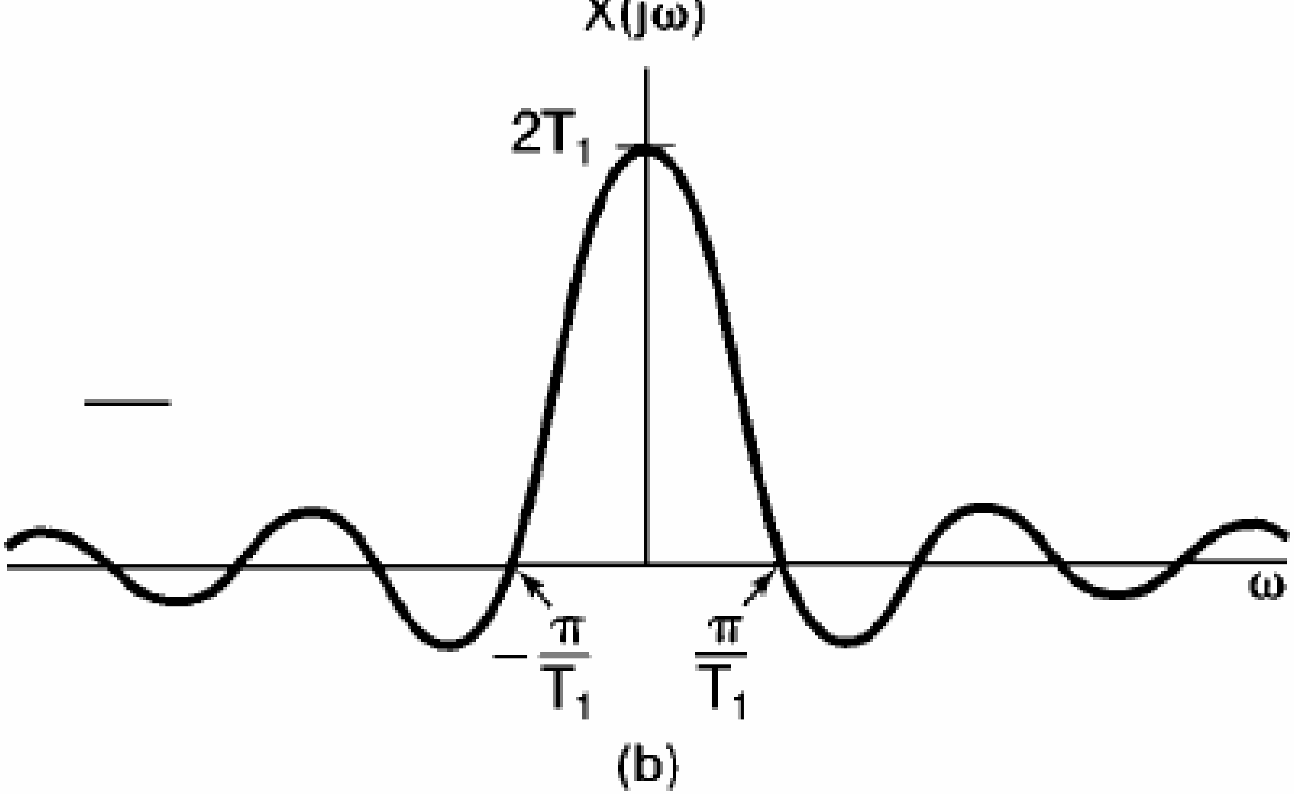
$$x(t) = \begin{cases} 1 & , \quad |t| < T_1 \\ 0 & , \quad |t| > T_1 \end{cases}$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-T_1}^{T_1} e^{-j\omega t} dt \\ &= \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega} \end{aligned}$$

Fourier Transform : Example 4

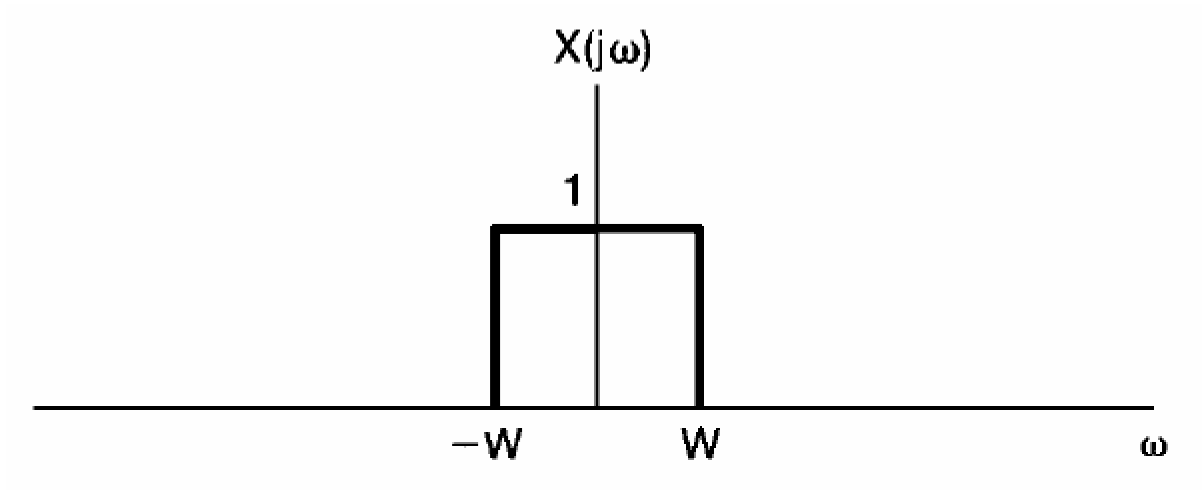
$$x(t) = \begin{cases} 1 \quad , & |t| < T_1 \\ 0 \quad , & |t| > T_1 \end{cases}$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-T_1}^{T_1} e^{-j\omega t} dt \\ &= \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega} \\ &= \frac{2\sin(\omega T_1)}{\omega} \end{aligned}$$


X(jω)
2T1
ω
−π/T1
π/T1
(b)

Fourier Transform : Example 5

$$X(j\omega) = \begin{cases} 1 \quad , & |\omega| < W \\ 0 \quad , & |\omega| < W \end{cases}$$



Fourier Transform : Example 5

$$X(j\omega) = \begin{cases} 1 & , \qquad |\omega| < W \\ 0 & , \qquad |\omega| < W \end{cases}$$

$$x(t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Fourier Transform : Example 5

$$X(j\omega) = \begin{cases} 1 & , \quad |\omega| < W \\ 0 & , \quad |\omega| < W \end{cases}$$

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega \end{aligned}$$

Fourier Transform : Example 5

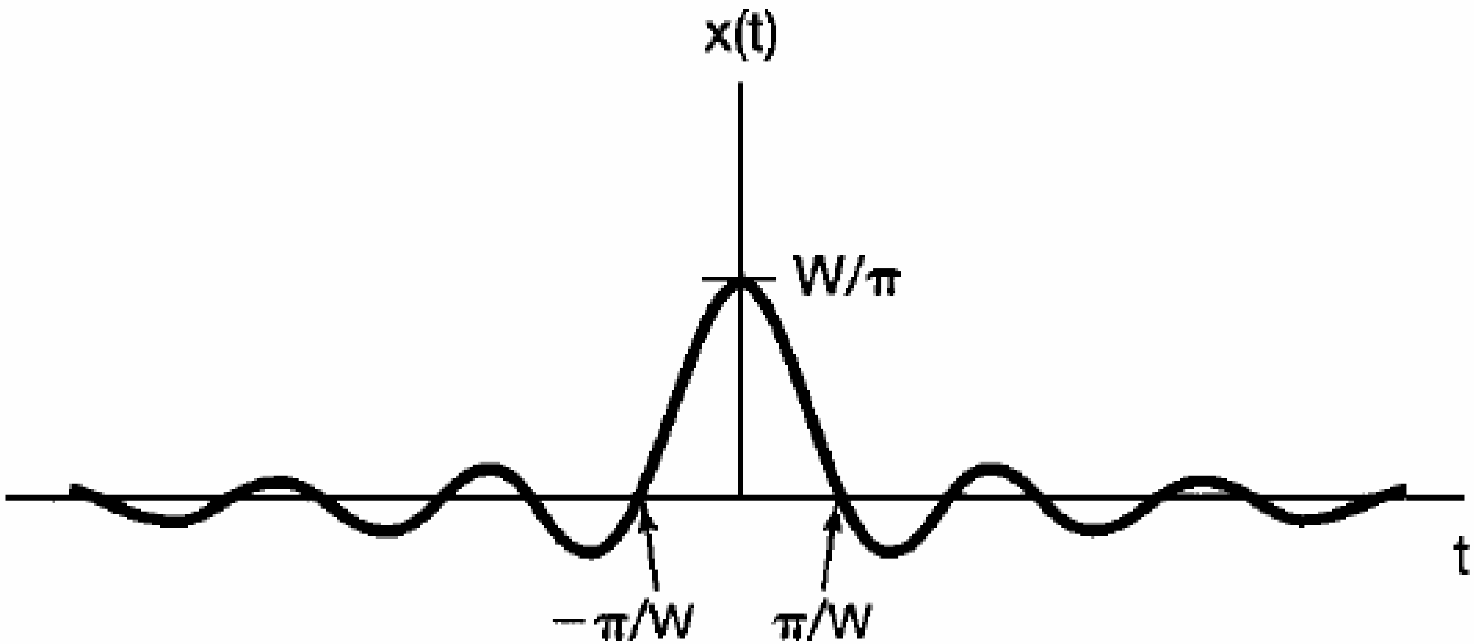
$$X(j\omega) = \begin{cases} 1 & , \quad |\omega| < W \\ 0 & , \quad |\omega| < W \end{cases}$$

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \\ &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega \\ &= \frac{e^{jWt} - e^{-jWt}}{2\pi jt} \end{aligned}$$

Fourier Transform : Example 5

$$X(j\omega) = \begin{cases} 1 & , \quad |\omega| < W \\ 0 & , \quad |\omega| < W \end{cases}$$

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega \\ &= \frac{e^{jWt} - e^{-jWt}}{2\pi j t} \\ &= \frac{\sin(Wt)}{\pi t} \end{aligned}$$



(b)

Rects and Sincs

$$x(t) = \begin{cases} 1 & , \quad |t| < T_1 \\ 0 & , \quad |t| > T_1 \end{cases}$$

$$X(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$$

$$X(j\omega) = \begin{cases} 1 & , \quad |\omega| < W \\ 0 & , \quad |\omega| < W \end{cases}$$

$$x(t) = \frac{\sin(Wt)}{\pi t}$$

$$\textit{Square wave} \underset{FT^{-1}}{\overset{FT}{\rightleftarrows}} \textit{Sinc function}$$

This means a square wave in the time domain,

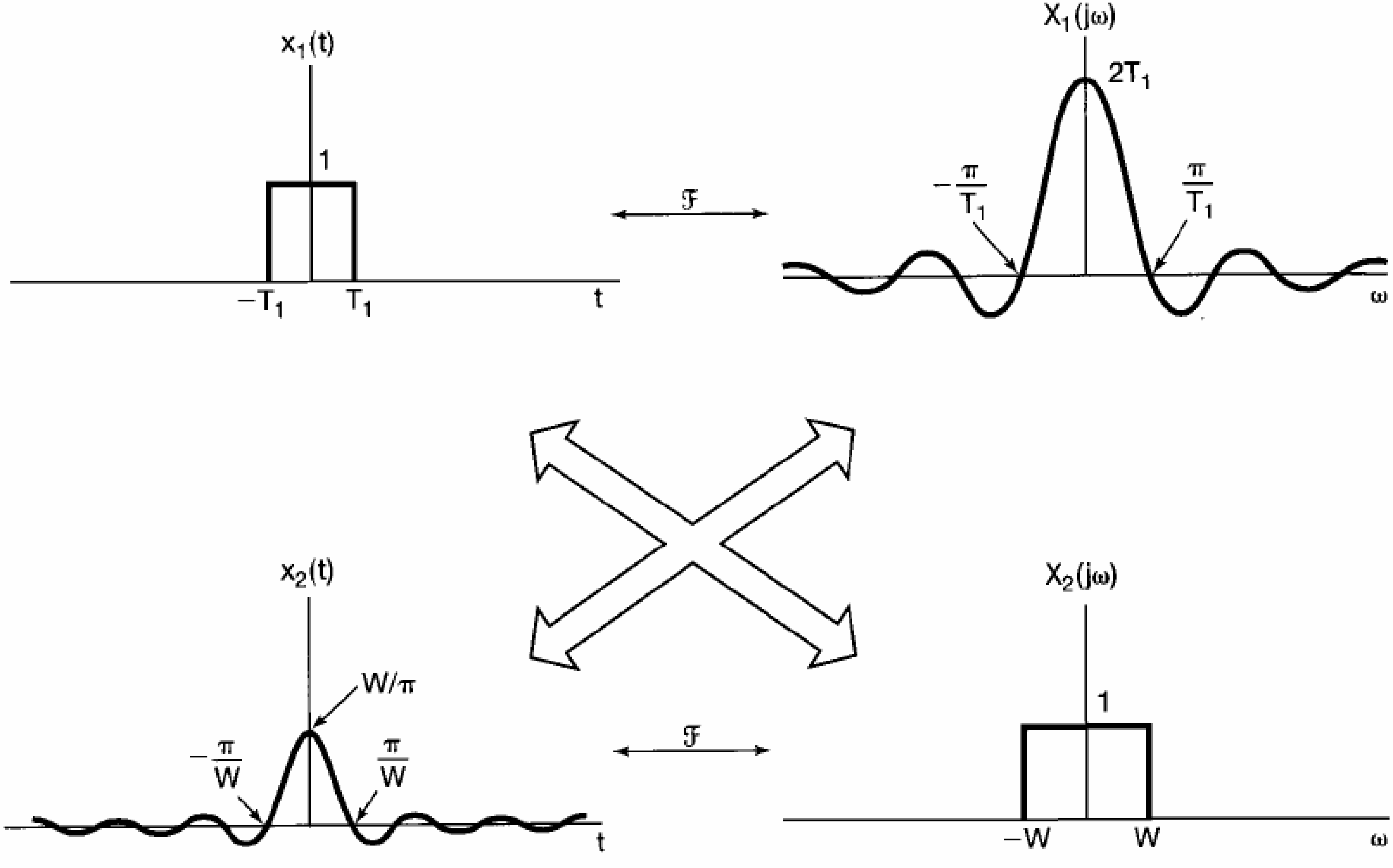
its Fourier transform is a *sinc* function. However,

if the signal in the time domain is a *sinc* function,

then its Fourier transform is a square wave.

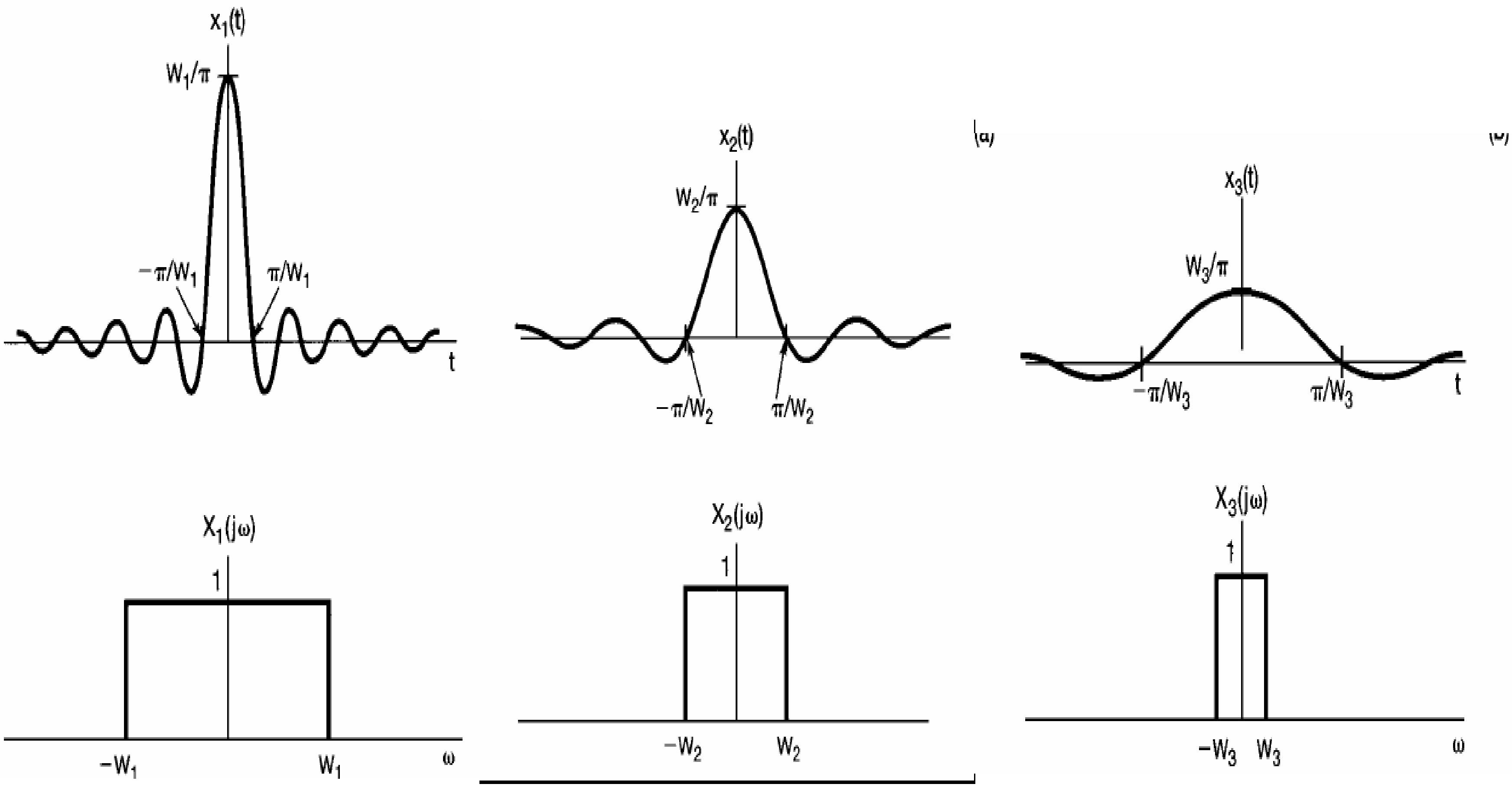
This property is referred to as ***Duality Property***.

Duality



$x_1(t)$
1
$-T_1$
T_1
t
$\mathfrak{F}$
$X_1(j\omega)$
$2T_1$
$-\frac{\pi}{T_1}$
$\frac{\pi}{T_1}$
ω
$x_2(t)$
W/π
$-\frac{\pi}{W}$
$\frac{\pi}{W}$
t
$\mathfrak{F}$
$X_2(j\omega)$
1
$-W$
W
ω

We also note that when the width of $X(j\omega)$ increases, its inverse Fourier transform $x(t)$ will be compressed. When $W \to \infty$, $X(j\omega)$ converges to an impulse.



Fourier transform of periodic signal

$$x(t) = x(t+T) \qquad T = 2\pi \big/ \omega_0$$

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$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) e^{-j\omega t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \left(\int_{-\infty}^{\infty} e^{j(k\omega_0 - \omega)t} dt \right) \\
&= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)
\end{aligned}$$

Fourier transform of periodic signal : Example 1

$$x(t) = \sin(\omega_0 t)$$

Fourier transform of periodic signal : Example 1

$$
\begin{aligned}
x(t) &= \sin(\omega_0 t) \\
&= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}
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$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

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$$\begin{aligned} X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) \\ &= 2\pi \left(a_1 \delta(\omega - \omega_0) + a_{-1} \delta(\omega + \omega_0) \right) \end{aligned}$$

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&= 2\pi \left(\frac{1}{2j} \delta(\omega - \omega_0) - \frac{1}{2j} \delta(\omega + \omega_0) \right)
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Fourier transform of periodic signal : Example 1

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\begin{aligned}
x(t) &= \sin(\omega_0 t) \\
&= \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \\
X(j\omega) &= \frac{1}{2j}\left(2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)\right)
\end{aligned}
$$

Fourier transform of periodic signal : Example 2

$$\begin{aligned} x(t) &= \sin(\omega_0 t) \\ &= \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \\ X(j\omega) &= \frac{1}{2j}\left(2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right) \end{aligned}$$

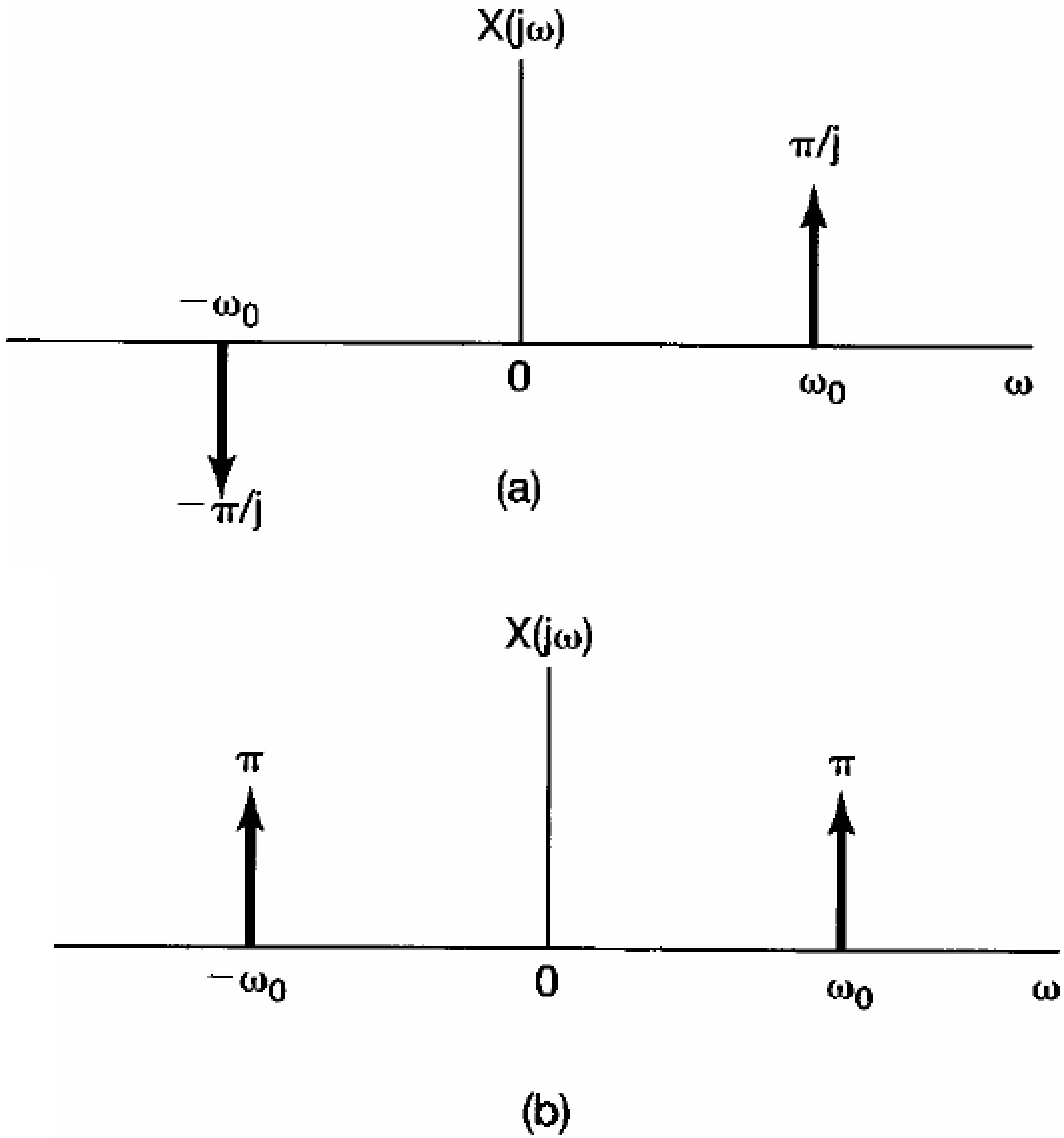
$$\begin{aligned} x(t) &= \cos(\omega_0 t) \\ &= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \\ X(j\omega) &= \frac{1}{2}\left(2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right) \end{aligned}$$

Fourier transform of periodic signal : Example 3

$$
\begin{aligned}
x(t) &= \sin(\omega_0 t) \\
&= \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \\
X(j\omega) &= \frac{1}{2j}\left(2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right)
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \cos(\omega_0 t) \\
&= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \\
X(j\omega) &= \frac{1}{2}\left(2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right)
\end{aligned}
$$

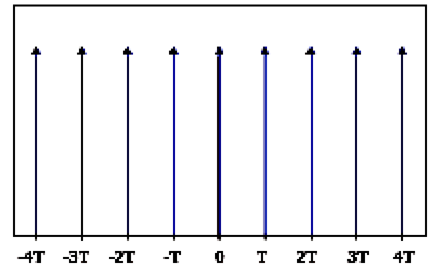
$$
\begin{aligned}
x(t) &= e^{j\omega_0 t} \\
X(j\omega) &= 2\pi\delta(\omega-\omega_0)
\end{aligned}
$$



Fourier transforms of (a) $x(t) = \sin \omega_0 t$; (b) $x(t) = \cos \omega_0 t$.

Fourier transform of periodic signal : Example 4

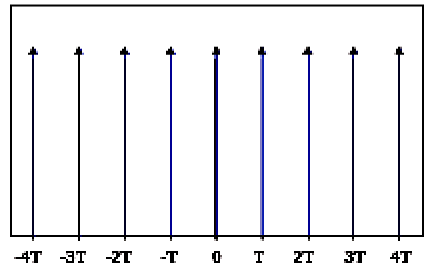
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



Fourier transform of periodic signal : Example 4

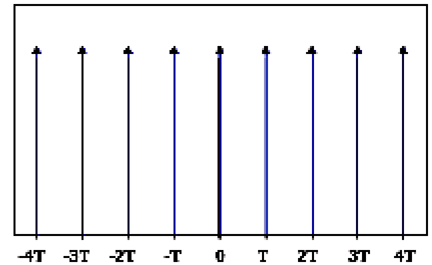
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$



$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt$$

Fourier transform of periodic signal : Example 4

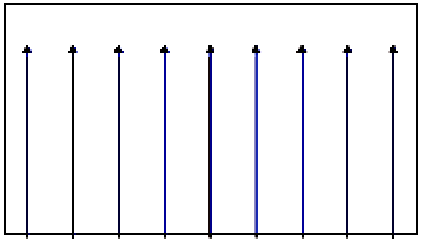


$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

$$\begin{aligned} a_k &= \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \end{aligned}$$

Fourier transform of periodic signal : Example 4



$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{T} \end{aligned}$$

$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

The Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\pi / T$

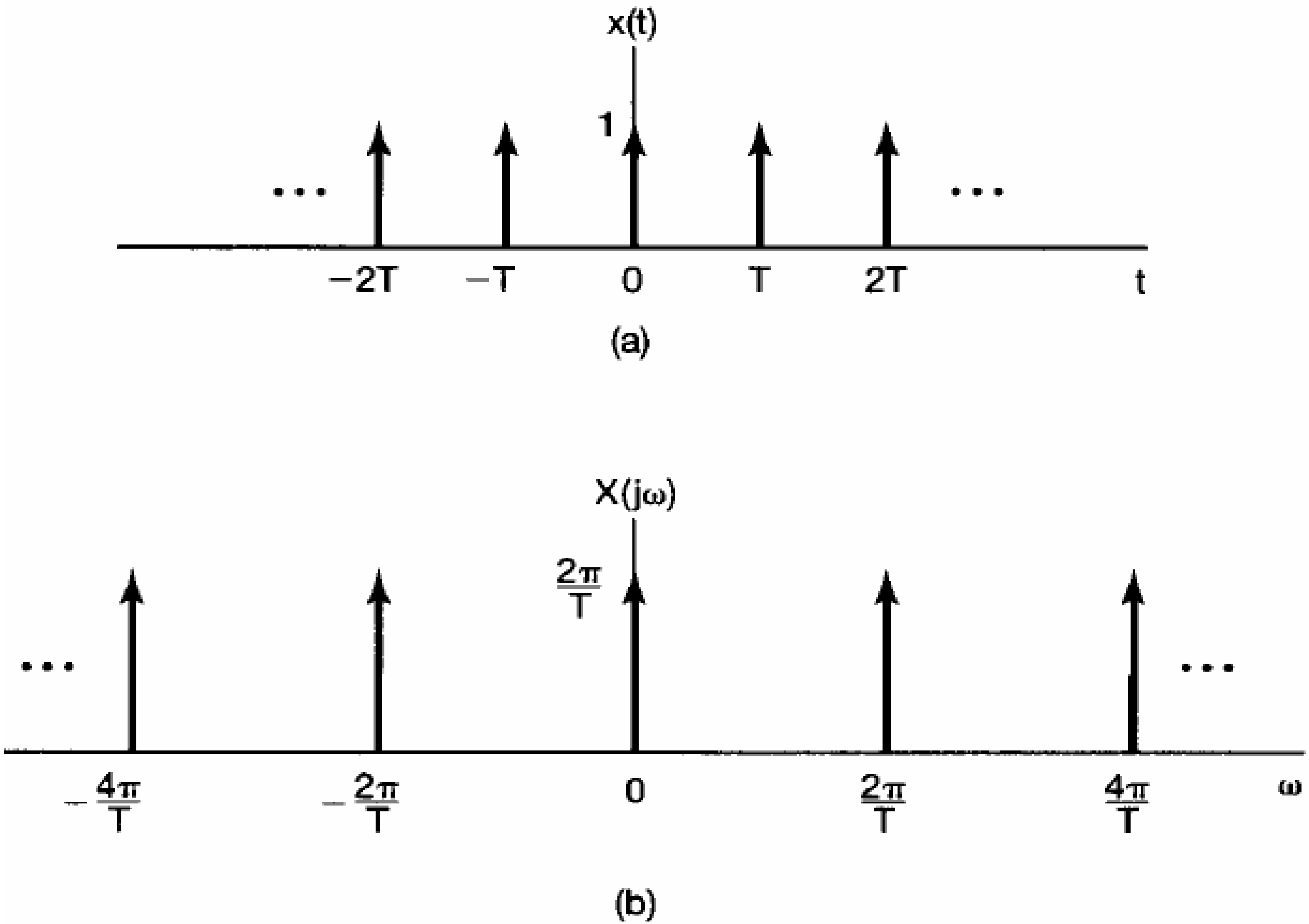


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.


THANK
YOU!