Signals and Systems UNIT-2

Reference Book – Simon Haykin

- The i/p o/p behaviour of a linear time invariant system is related by differential equation for CT signal and by difference equation for DT signal.
- A LTI system (Linear time invariant system) is characterized by it's impulse response.
- An impulse response of the system is the o/p of the system when the i/p is impulse function.

- **Convolution Sum** : an arbitrary signal is expressed as a weighted superposition of shifted impulses.
- $x[n]\delta[n - k] = x[k]\,\delta[n - k]$,
- $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n - k]$
- Let H denote the system to which i/p x[n] is applied. Then, the input x[n] to the system results in the output

  $y[n] = H\{x[n]\}$

  $y[n] = H\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n - k]\}$

  Using the property of linearity

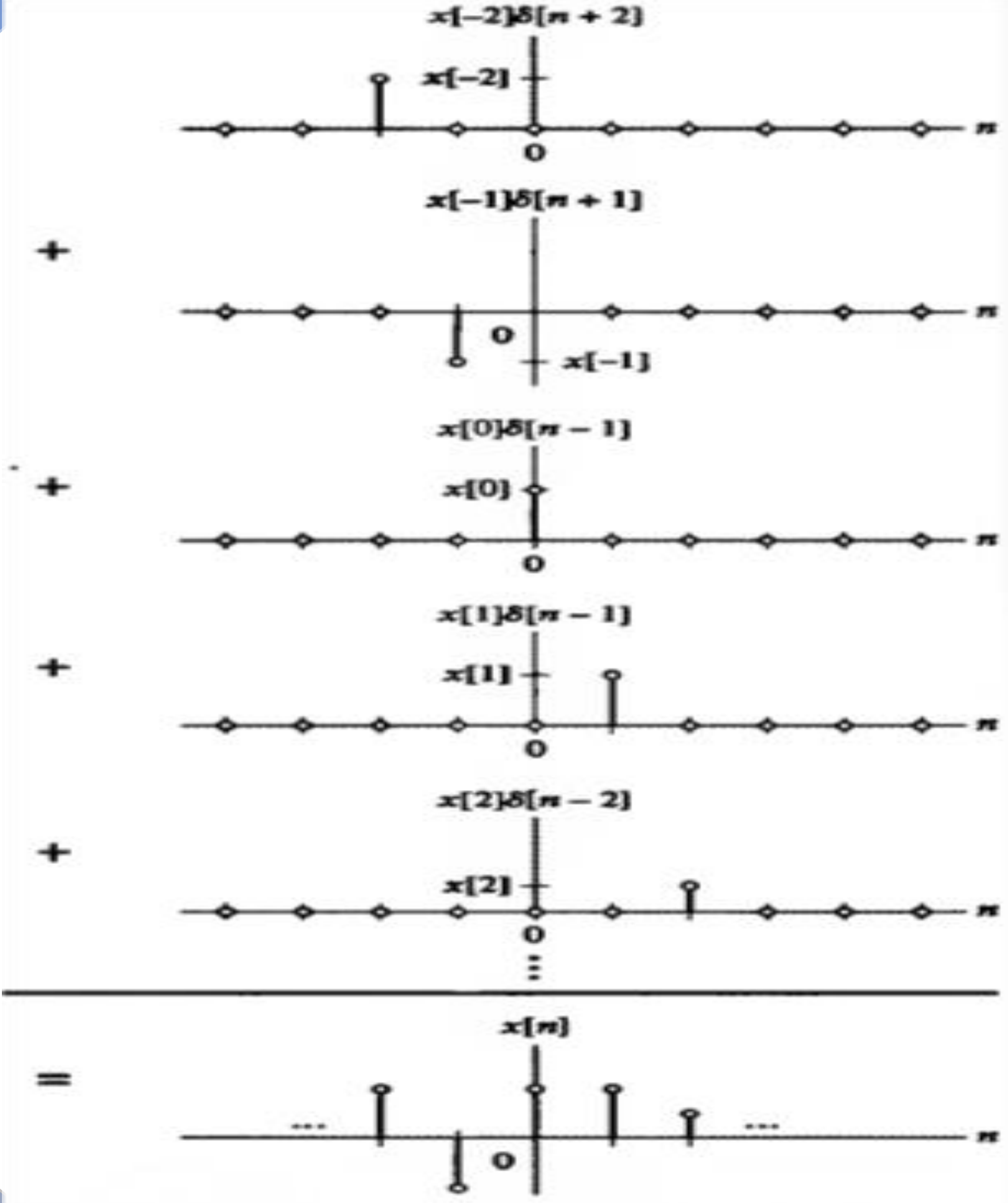
  $y[n] = \sum_{k=-\infty}^{\infty} H\{x[k]\,\delta[n - k]\}$
- Since n is the time index, the quantity x[k] is a constant with respect to the system operator H. Using linearity again, we interchange H with x[k] to obtain
- $Y[n]=\sum_{k=-\infty}^{\infty} x[k]\,H\{\delta[n - k]\}$

- assume that the system is time invariant, then a time shift in the input results in a time shift in the output
- H{δ[n — k]} = h[n - k].
- where h[n]= H{δ[n]} is the impulse response of the LTI system H.

The response of the system to each basis function is determined by the system impulse response (o/p of the system when the i/p is a unit impulse)

x[n] = $\sum_{k=-\infty}^{\infty}$ x[k] δ[n — k]

- the output of an LTI system is given by a weighted sum of time-shifted impulse responses.
- x[n] * h[n] = $\sum_{k=-\infty}^{\infty}$ x[k] h[n — k]



x[−2]δ[n + 2]
x[−2]
0
n
+
x[−1]δ[n + 1]
0
x[−1]
n
+
x[0]δ[n − 1]
x[0]
0
n
+
x[1]δ[n − 1]
x[1]
0
n
+
x[2]δ[n − 2]
x[2]
0
n
=
x[n]
0
n

- **<u>Evaluation procedure for convolution sum</u>**

Convolution sum is expressed as $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$

Suppose we define the intermediate signal $w_n[k] = x[k]h[n-k]$

- In this definition, k is the independent variable and we explicitly indicate that n is treated as a constant by writing n as a subscript on w. Now,

- h[n — k] = h[—(k — n)] is a reflected (because of —k) and time-shifted (by —n) version of h[k]. Hence, if n is negative, then h[n — k] is obtained by time shifting h[—k] to the left, while if n is positive, we time shift h[—k] to the right.

- O/p of the system is defined by

$$y[n] = \sum_{k=-\infty}^{\infty} w_n[k].$$

- **Convolution Sum evaluation by using an intermediate signal**

Consider a system with impulse response

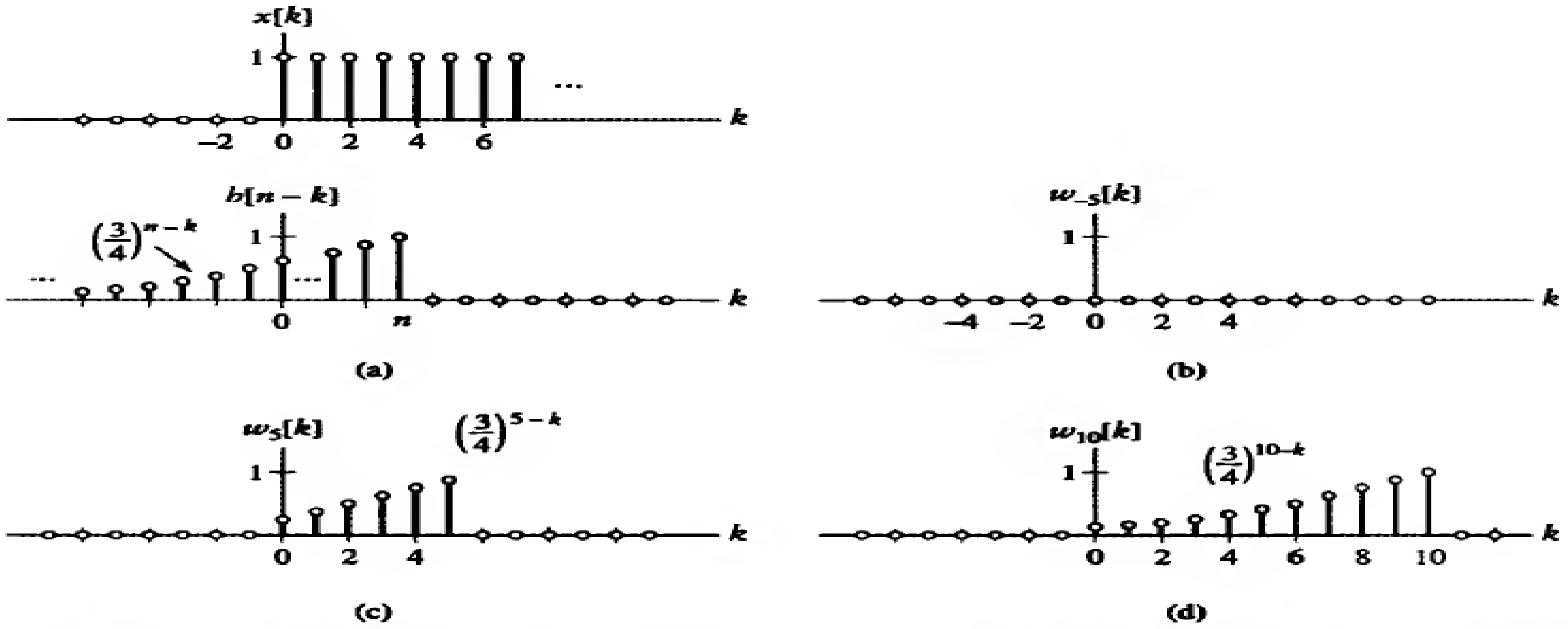
determine the output of the system at times n = —5, n = 5, and n = 10

when the input is x[n] = u[n].

$$h[n] = \left(\frac{3}{4}\right)^n u[n].$$

- **Ans**
- the impulse response and input are of infinite duration Figure 2.3(a) depicts x[k] superimposed on the reflected and time shifted impulse response h[n - k].

$$h[n-k] = \begin{cases} \left(\frac{3}{4}\right)^{n-k}, & k \leq n \\ 0, & \text{otherwise} \end{cases}.$$



- The figure represents:

a) The i/p signal x[k] above the reflected and time-shifted impulse response h[n-k], depicted as a function of k.

b) The product signal $w_{-5}[k]$ used to evaluate y[-5]

c) The product signal $w_5[k]$ used to evaluate y[5]

d) The product signal $w_{10}[k]$ used to evaluate y[10]

- Figures 2.3(b), (c), and (d) depict $w_n[k]$ for n = -5, n = 5, and n = 10, respectively. We have $w_{-5}[k] = 0,$
- $y[n] = \sum_{k=-\infty}^{\infty} w_n[k].$ yields y[-5] = 0
- For n=5 we have , $w_5[k] = \begin{cases} \left(\frac{3}{4}\right)^{5-k}, & 0 \le k \le 5 \\ 0, & \text{otherwise} \end{cases},$
- $y[n] = \sum_{k=-\infty}^{\infty} w_n[k].$ yields $y[5] = \sum_{k=0}^{5} \left(\frac{3}{4}\right)^{5-k},$

- which represents the sum of the nonzero values of the intermediate signal
- $w_5[k]$ shown in fig (c) . We then factor $\left(\frac{3}{4}\right)^5$ from the sum and apply the formula for the sum of a finite geometric series to obtain

$$y[5] = \left(\frac{3}{4}\right)^5 \sum_{k=0}^{5} \left(\frac{4}{3}\right)^k$$

$$= \left(\frac{3}{4}\right)^5 \frac{1 - \left(\frac{4}{3}\right)^6}{1 - \left(\frac{4}{3}\right)} = 3.288.$$

Last, for $n = 10$, we see that

$$w_{10}[k] = \begin{cases} \left(\frac{3}{4}\right)^{10-k}, & 0 \le k \le 10 \\ 0, & \text{otherwise} \end{cases},$$

and Eq. (2.6) gives

$$y[10] = \sum_{k=0}^{10} \left(\frac{3}{4}\right)^{10-k}$$

$$= \left(\frac{3}{4}\right)^{10} \sum_{k=0}^{10} \left(\frac{4}{3}\right)^k$$

$$= \left(\frac{3}{4}\right)^{10} \frac{1 - \left(\frac{4}{3}\right)^{11}}{1 - \left(\frac{4}{3}\right)} = 3.831.$$

Note that in this example $w_n[k]$ has only two different mathematical representations. For $n < 0$, we have $w_n[k] = 0$, since there is no overlap between the nonzero portions of $x[k]$ and $h[n - k]$. When $n \ge 0$, the nonzero portions of $x[k]$ and $h[n - k]$ overlap on the interval $0 \le k \le n$, and we may write

$$w_n[k] = \begin{cases} \left(\frac{3}{4}\right)^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}.$$

Hence, we may determine the output for an arbitrary n by using the appropriate mathematical representation for $w_n[k]$ in Eq. (2.6). ■

Procedure 2.1: Reflect and Shift Convolution Sum Evaluation

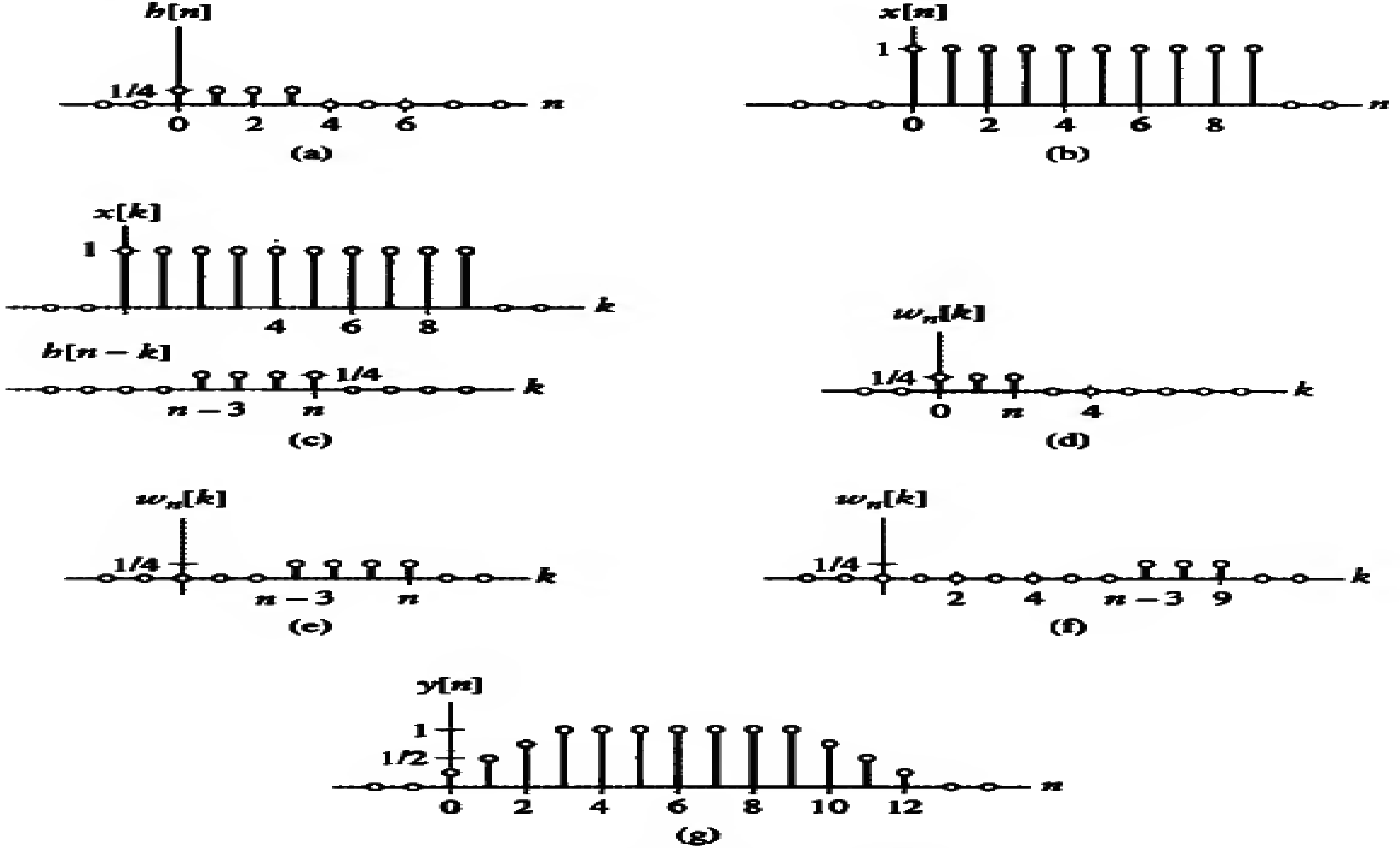
1. Graph both $x[k]$ and $h[n-k]$ as a function of the independent variable k. To determine $h[n-k]$, first reflect $h[k]$ about $k=0$ to obtain $h[-k]$. Then shift by $-n$.
2. Begin with n large and negative. That is, shift $h[-k]$ to the far left on the time axis.
3. Write the mathematical representation for the intermediate signal $w_n[k]$.
4. Increase the shift n (i.e., move $h[n-k]$ toward the right) until the mathematical representation for $w_n[k]$ changes. The value of n at which the change occurs defines the end of the current interval and the beginning of a new interval.
5. Let n be in the new interval. Repeat steps 3 and 4 until all intervals of time shifts and the corresponding mathematical representations for $w_n[k]$ are identified. This usually implies increasing n to a very large positive number.
6. For each interval of time shifts, sum all the values of the corresponding $w_n[k]$ to obtain $y[n]$ on that interval.

EXAMPLE 2.3 MOVING-AVERAGE SYSTEM: REFLECT-AND-SHIFT CONVOLUTION SUM EVALUATION The output $y[n]$ of the four-point moving-average system introduced in Section 1.10 is related to the input $x[n]$ according to the formula

$$y[n] = \frac{1}{4}\sum_{k=0}^{3} x[n-k].$$

The impulse response $h[n]$ of this system is obtained by letting $x[n] = \delta[n]$, which yields

$$h[n] = \frac{1}{4}(u[n] - u[n-4]),$$



h[n]
1/4
0 2 4 6
n
(a)
x[n]
1
0 2 4 6 8
n
(b)
x[k]
1
4 6 8
k
h[n − k]
1/4
n − 3
n
k
(c)
w_n[k]
1/4
0 n 4
k
(d)
w_n[k]
1/4
n − 3
n
k
(e)
w_n[k]
1/4
2 4 n − 3 9
k
(f)
y[n]
1
1/2
0 2 4 6 8 10 12
n
(g)

- **FIGURE 2.4**

Evaluation of the convolution sum for Example 2.3

a) The system impulse response h[n]

b) The input signal x[n]

c) The input above the reflected and time-shifted impulse response h[n-k], depicted as a function of k.

d) The product signal $w_n[k]$ for the interval of shifts $0 \leq n \leq 3$.

e)The product signal $w_n[k]$ for the interval of shifts $3 < n \leq 9$.

f) The product signal $w_n[k]$ for the interval of shifts $9 < n \leq 12$

g) The output y[n].

▶ **Problem 2.2** Evaluate the following discrete-time convolution sums:

(a) $y[n] = u[n] * u[n-3]$

(b) $y[n] = (1/2)^n u[n-2] * u[n]$

(c) $y[n] = \alpha^n\{u[n-2] - u[n-13]\} * 2\{u[n+2] - u[n-12]\}$

(d) $y[n] = (-u[n] + 2u[n-3] - u[n-6]) * (u[n+1] - u[n-10])$

(e) $y[n] = u[n-2] * h[n]$, where

$$h[n] = \begin{cases} \gamma^n, & n < 0, |\gamma| > 1 \\ \eta^n, & n \geq 0, |\eta| < 1 \end{cases}$$

(f) $y[n] = x[n] * h[n]$, where $x[n]$ and $h[n]$ are shown in Fig. 2.8.

Answers:

(a)

$$y[n] = \begin{cases} 0, & n < 3 \\ n-2, & n \geq 3 \end{cases}$$

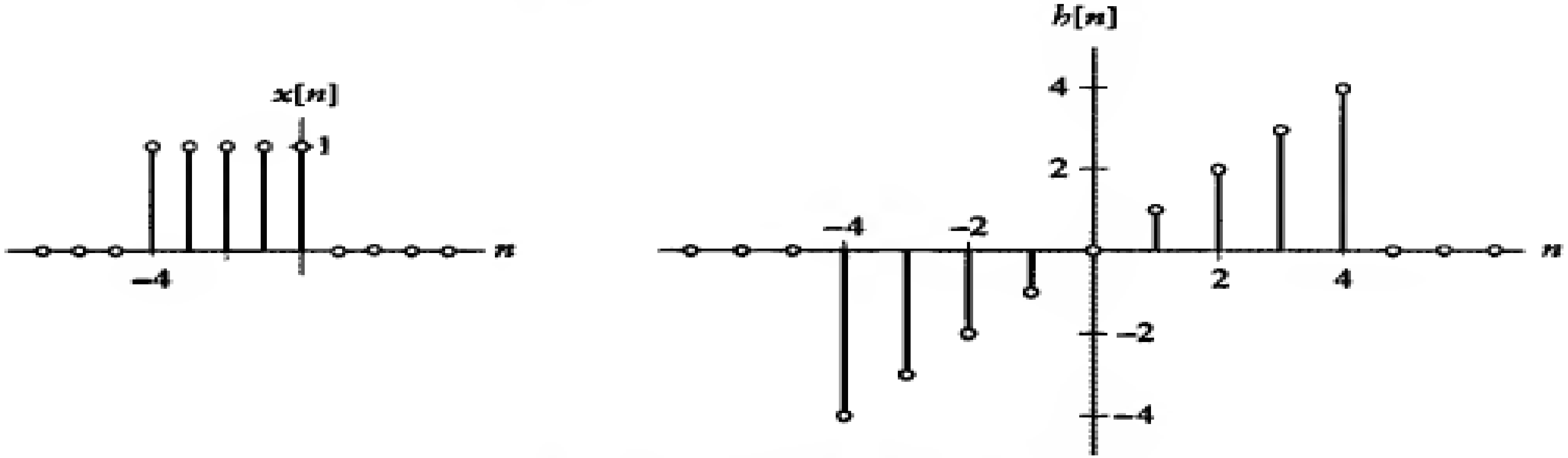


FIGURE 2.8 Signals for Problem 2.2(f).

(b)

$$y[n] = \begin{cases} 0, & n < 2 \\ 1/2 - (1/2)^n, & n \geq 2 \end{cases}$$

(c)

$$y[n] = \begin{cases} 0, & n < 0 \\ 2\alpha^{n+2}\dfrac{1 - (\alpha)^{-1-n}}{1 - \alpha^{-1}}, & 0 \leq n \leq 10 \\ 2\alpha^{12}\dfrac{1 - (\alpha)^{-11}}{1 - \alpha^{-1}}, & 11 \leq n \leq 13 \\ 2\alpha^{12}\dfrac{1 - (\alpha)^{n-24}}{1 - \alpha^{-1}}, & 14 \leq n \leq 23 \\ 0, & n \leq 24 \end{cases}$$

(d)

$$y[n] = \begin{cases} 0, & n < -1 \\ -(n+2), & -1 \le n \le 1 \\ n-4, & 2 \le n \le 4 \\ 0, & 5 \le n \le 9 \\ n-9, & 10 \le n \le 11 \\ 15-n, & 12 \le n \le 14 \\ 0, & n > 14 \end{cases}$$

(e)

$$y[n] = \begin{cases} \dfrac{\gamma^{n-1}}{\gamma - 1}, & n < 2 \\[2ex] \dfrac{1}{\gamma - 1} + \dfrac{1 - \eta^{n-1}}{1 - \eta}, & n \ge 2 \end{cases}$$

(f)

$$y[n] = \begin{cases} 0, & n < -8, n > 4 \\ -10 + (n+5)(n+4)/2, & -8 \le n \le -5 \\ 5(n+2), & -4 \le n \le 0 \\ 10 - n(n-1)/2, & 1 \le n \le 4 \end{cases} .$$

- **Convolution Integral**
- a continuous-time signal as the weighted superposition of time-shifted impulses

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau.$$

- The weights x(τ) d τ are derived from the value of the signal x(τ) at the time t at which each impulse occurs.
- Let the operator H denote the system to which the input x(t) is applied. We consider the system output in response to a general input expressed as the weighted superposition

$$\begin{aligned} y(t) &= H\{x(t)\} \\ &= H\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right\}. \end{aligned}$$

- Using the linearty property of the system, we may interchange the order of the operator H and integration to obtain $y(t) = \int_{-\infty}^{\infty} x(\tau) H\{\delta(t-\tau)\}\, d\tau.$

we define the impulse response h(t) = H{∂(t)} as the output of the system in response to a unit impulse input. If the system is also time invariant, then

$$H\{\delta(t-\tau)\} = h(t-\tau).$$

- time invariance implies that a time-shifted impulse input generates a time-shifted impulse response output
- output of an LTI system in response to an input signal can be expressed as

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau.}$$

The above equation is termed as convolution integral.

- The output y(t) is given as a weighted superposition of impulse response time shifted by τ.

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.$$

- **<u>Convolution Integral Evaluation procedure</u>**

The procedure for evaluating the convolution integral is based on defining an intermediate signal that simplifies the evaluation of the integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.$$

We redefine the integrand as the intermediate signal $w_t(\tau) = x(\tau)h(t-\tau).$

In this definition, τ is the independent variable and time t is treated as a constant.

- This is explicitly indicated by writing t as a subscript and τ within the parentheses of $w_t(\tau)$

- Hence, h(t- τ) = h(-(t- τ)) is reflected and shifted (by -t) version of h(τ).
- If t < 0, then
- h(—t) is time shifted to the left, while if t > 0, then h(-t) is shifted to the right.

The time shift t determines the time at which we evaluate the o/p of the system,

Equation given below

the system output at any time t is the area under the signal wt(τ)

$$y(t) = \int_{-\infty}^{\infty} w_t(\tau)\, d\tau.$$

Procedure 2.2: Reflect-and-Shift Convolution Integral Evaluation

1. Graph $x(\tau)$ and $h(t - \tau)$ as a function of the independent variable τ. To obtain $h(t - \tau)$, reflect $h(\tau)$ about $\tau = 0$ to obtain $h(-\tau)$, and then shift $h(-\tau)$, by $-t$.
2. Begin with the shift t large and negative, that is, shift $h(-\tau)$ to the far left on the time axis.
3. Write the mathematical representation of $w_t(\tau)$.
4. Increase the shift t by moving $h(t - \tau)$ towards the right until the mathematical representation of $w_t(\tau)$ changes. The value t at which the change occurs defines the end of the current set of shifts and the beginning of a new set.
5. Let t be in the new set. Repeat steps 3 and 4 until all sets of shifts t and the corresponding representations of $w_t(\tau)$ are identified. This usually implies increasing t to a large positive value.
6. For each set of shifts t, integrate $w_t(\tau)$ from $\tau = -\infty$ to $\tau = \infty$ to obtain $y(t)$.

- The effect of increasing t from a large negative value to a large positive value is to slide $h(-\tau)$ past $x(\tau)$ from left to right.

EXAMPLE 2.6 REFLECT-AND-SHIFT CONVOLUTION EVALUATION Evaluate the convolution integral for a system with input $x(t)$ and impulse response $h(t)$, respectively, given by

$$x(t) = u(t-1) - u(t-3)$$

and

$$h(t) = u(t) - u(t-2),$$

as depicted in Fig. 2.10.

Solution: To evaluate the convolution integral, we first graph $h(t-\tau)$ beneath the graph of $x(\tau)$, as shown in Fig. 2.11(a). Next, we identify the intervals of time shifts for which the mathematical representation of $w_t(\tau)$ does not change, beginning with t large and negative. Provided that $t < 1$, we have $w_t(\tau) = 0$, since there are no values τ for which both $x(\tau)$ and $h(t-\tau)$ are nonzero. Hence, the first interval of time shifts is $t < 1$.

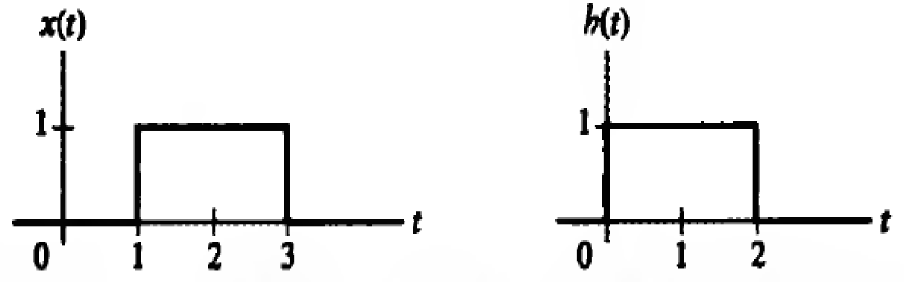


FIGURE 2.10 Input signal and LTI system impulse response for Example 2.6.

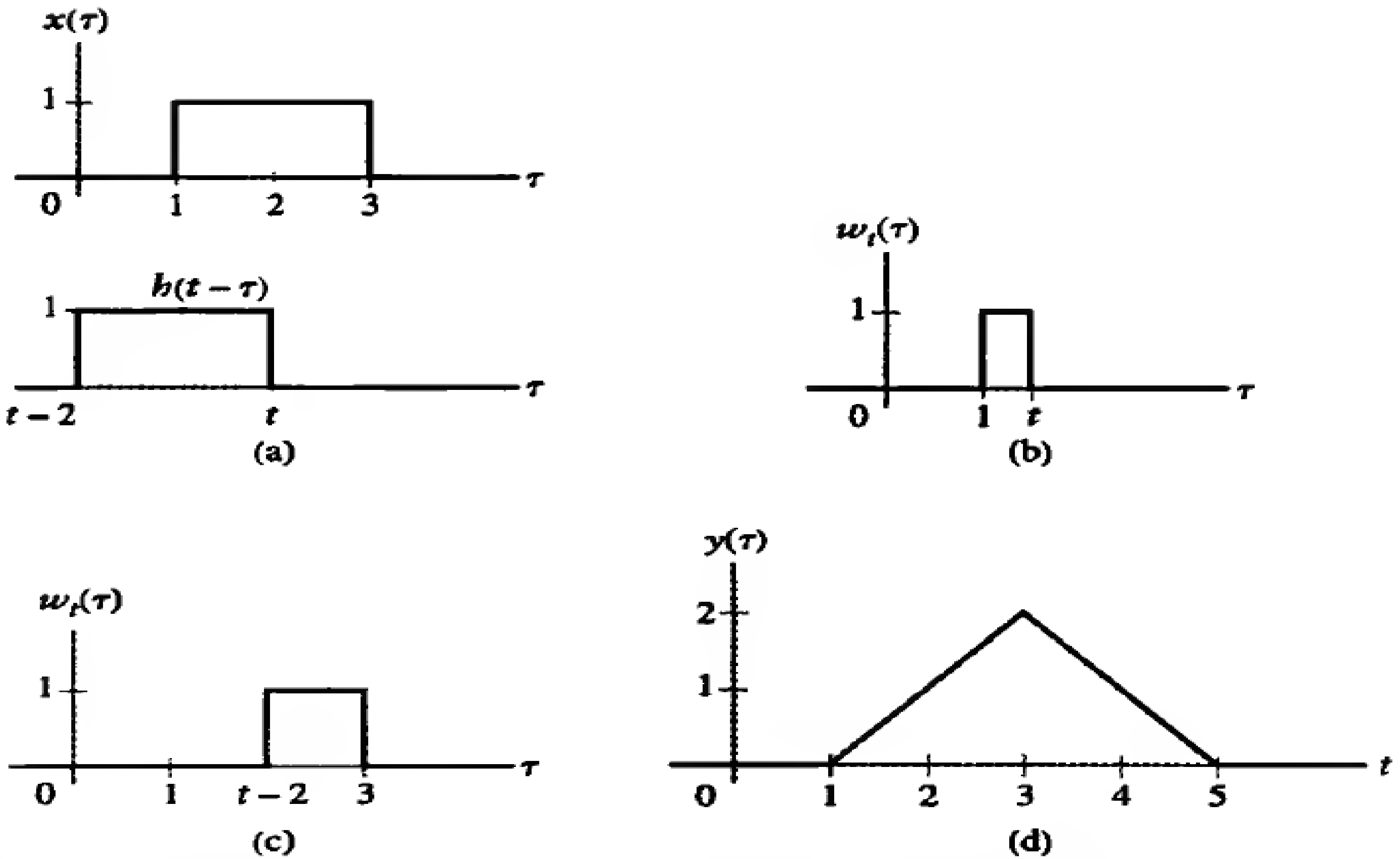


FIGURE 2.11 Evaluation of the convolution integral for Example 2.6. (a) The input $x(\tau)$ depicted above the reflected and time-shifted impulse response $h(t - \tau)$, depicted as a function of τ. (b) The product signal $w_t(\tau)$ for $1 \leq t < 3$. (c) The product signal $w_t(\tau)$ for $3 \leq t < 5$. (d) The system output $y(t)$.

Note that at $t = 1$ the right edge of $h(t - \tau)$ coincides with the left edge of $x(\tau)$. Therefore, as we increase the time shift t beyond 1, we have

$$w_t(\tau) = \begin{cases} 1, & 1 < \tau < t \\ 0, & \text{otherwise} \end{cases}.$$

This representation for $w_t(\tau)$ is depicted in Fig. 2.11(b). It does not change until $t > 3$, at which point both edges of $h(t - \tau)$ pass through the edges of $x(\tau)$. The second interval of time shifts t is thus $1 \leq t < 3$.

As we increase the time shift t beyond 3, we have

$$w_t(\tau) = \begin{cases} 1, & t - 2 < \tau < 3 \\ 0, & \text{otherwise} \end{cases},$$

as depicted in Fig. 2.11(c). This mathematical representation for $w_t(\tau)$ does not change until $t = 5$; thus, the third interval of time shifts is $3 \leq t < 5$.

At $t = 5$, the left edge of $h(t - \tau)$ passes through the right edge of $x(\tau)$, and $w_t(\tau)$ becomes zero. As we continue to increase t beyond 5, $w_t(\tau)$ remains zero, since there are no values τ for which both $x(\tau)$ and $h(t - \tau)$ are nonzero. Hence, the final interval of shifts is $t \geq 5$.

We now determine the output $y(t)$ for each of these four intervals of time shifts by integrating $w_t(\tau)$ over τ (i.e., finding the area under $w_t(\tau)$):

- For $t < 1$ and $t > 5$, we have $y(t) = 0$, since $w_t(\tau)$ is zero.
- For the second interval, $1 \leq t < 3$, the area under $w_t(\tau)$ shown in Fig. 2.11(b) is $y(t) = t - 1$.
- For $3 \leq t < 5$, the area under $w_t(\tau)$ shown in Fig. 2.11(c) is $y(t) = 3 - (t - 2)$.

Combining the solutions for each interval of time shifts gives the output

$$y(t) = \begin{cases} 0, & t < 1 \\ t - 1, & 1 \le t < 3 \\ 5 - t, & 3 \le t < 5 \\ 0, & t \ge 5 \end{cases}$$

EXAMPLE 2.8 ANOTHER REFLECT-AND-SHIFT CONVOLUTION EVALUATION Suppose the input $x(t)$ and impulse response $h(t)$ of an LTI system are, respectively, given by

$$x(t) = (t - 1)[u(t - 1) - u(t - 3)]$$

and

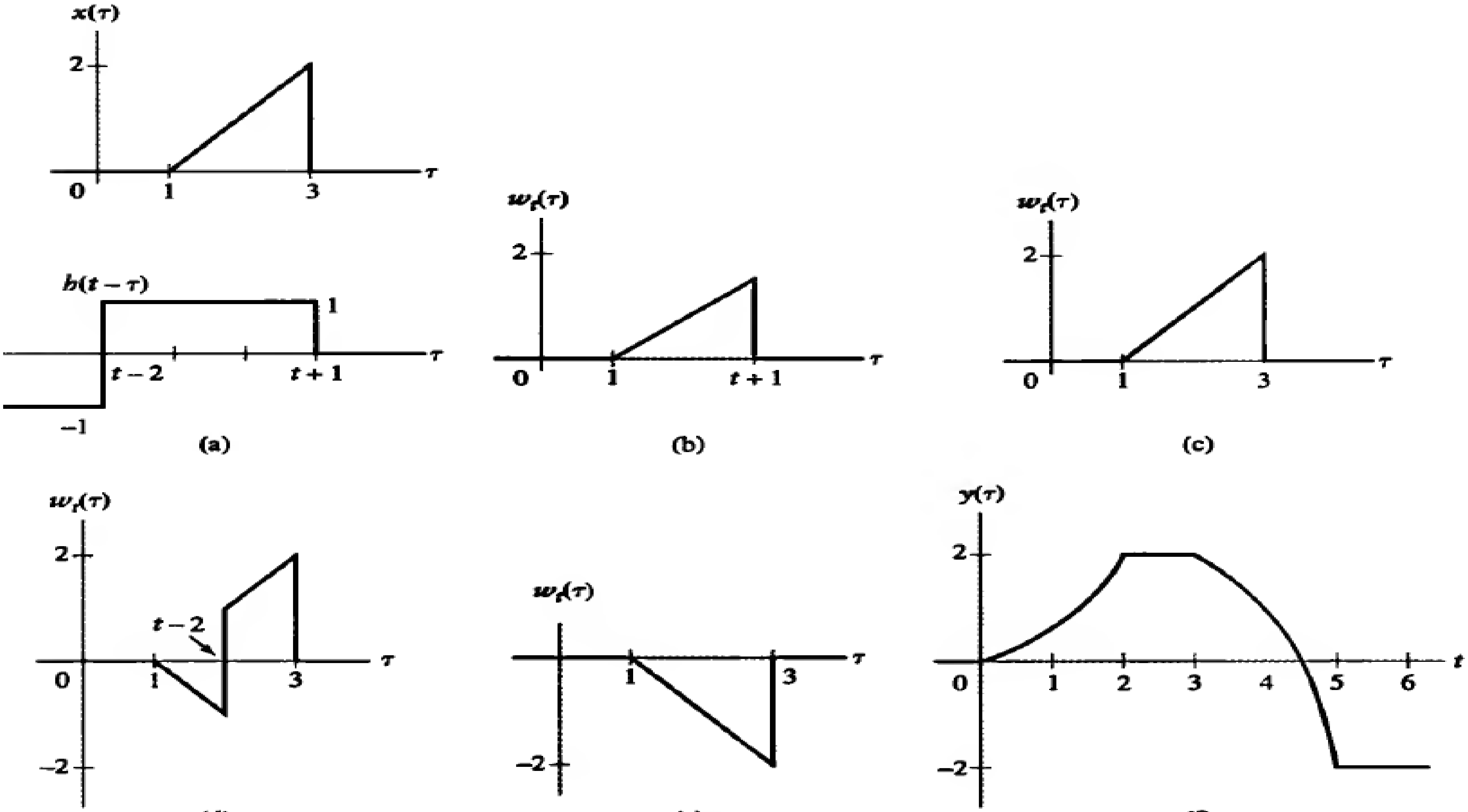
$$h(t) = u(t + 1) - 2u(t - 2).$$

Find the output of this system.

Solution: Graph $x(\tau)$ and $h(t - \tau)$ as shown in Fig. 2.14(a). From these graphical representations, we determine the intervals of time shifts, t, on which the mathematical representation of $w_t(\tau)$ does not change. We begin with t large and negative. For $t + 1 < 1$ or $t < 0$, the right edge of $h(t - \tau)$ is to the left of the nonzero portion of $x(\tau)$, and consequently, $w_t(\tau) = 0$.

For $t > 0$, the right edge of $h(t - \tau)$ overlaps with the nonzero portion of $x(\tau)$, and we have

$$w_t(\tau) = \begin{cases} \tau - 1, & 1 < \tau < t + 1 \\ 0, & \text{otherwise} \end{cases}.$$


$x(\tau)$
2
0
1
3
τ
$h(t-\tau)$
1
$t-2$
$t+1$
-1
τ
$w_t(\tau)$
2
0
1
$t+1$
τ
$w_t(\tau)$
2
0
1
3
τ
$w_t(\tau)$
2
$t-2$
0
1
3
-2
τ
$w_t(\tau)$
1
3
-2
τ
$y(\tau)$
2
0
1
2
3
4
5
6
-2
t

(a) (b) (c) (d) (e) (f)

- **FIGURE 2.14**

Evaluation of convolution integral for example 2.8

a) The input x(τ) superimposed on the reflected and time-shifted impulse response h(t- τ), depicted as a function of τ.
b) The product signal $w_t(\tau)$ for $0 \leq t < 2$
c) The product signal $w_t(\tau)$ for $2 \leq t < 3$
d) The product signal $w_t(\tau)$ for $3 \leq t < 5$
e) The product signal $w_t(\tau)$ for $t \geq 5$
f) The system output y(t)

This representation for $w_t(\tau)$ holds provided that $t + 1 < 3$, or $t < 2$, and is depicted in Fig. 2.14(b).

For $t > 2$, the right edge of $h(t - \tau)$ is to the right of the nonzero portion of $x(\tau)$. In this case, we have

$$w_t(\tau) = \begin{cases} \tau - 1, & 1 < \tau < 3 \\ 0, & \text{otherwise} \end{cases}.$$

This representation for $w_t(\tau)$ holds provided that $t - 2 < 1$, or $t < 3$, and is depicted in Fig. 2.14(c).

For $t \geq 3$, the edge of $h(t - \tau)$ at $\tau = t - 2$ is within the nonzero portion of $x(\tau)$, and we have

$$w_t(\tau) = \begin{cases} -(\tau - 1), & 1 < \tau < t - 2 \\ \tau - 1, & t - 2 < \tau < 3. \\ 0, & \text{otherwise} \end{cases}$$

This representation for $w_t(\tau)$ is depicted in Fig. 2.14(d) and holds provided that $t - 2 < 3$, or $t < 5$.

For $t \geq 5$, we have

$$w_t(\tau) = \begin{cases} -(\tau - 1), & 1 < \tau < 3 \\ 0, & \text{otherwise} \end{cases},$$

as depicted in Fig. 2.14(e).

The system output $y(t)$ is obtained by integrating $w_t(\tau)$ from $\tau = -\infty$ to $\tau = \infty$ for each interval of time shifts just identified. Beginning with $t < 0$, we have $y(t) = 0$, since $w_t(\tau) = 0$. For $0 \leq t < 2$,

$$
\begin{aligned}
y(t) &= \int_1^{t+1} (\tau - 1)\, d\tau \\
&= \left(\frac{\tau^2}{2} - \tau \Big|_1^{t+1} \right) \\
&= \frac{t^2}{2}.
\end{aligned}
$$

For $2 \leq t < 3$, the area under $w_t(\tau)$ is $y(t) = 2$. On the next interval, $3 \leq t < 5$, we have

$$
\begin{aligned}
y(t) &= -\int_1^{t-2} (\tau - 1)\, d\tau + \int_{t-2}^{3} (\tau - 1)\, d\tau \\
&= -t^2 + 6t - 7.
\end{aligned}
$$

Finally, for $t \geq 5$, the area under $w_t(\tau)$ is $y(t) = -2$. Combining the outputs for the different intervals of time shifts gives the result

$$
y(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t^2}{2}, & 0 \leq t < 2 \\ 2, & 2 \leq t < 3, \\ -t^2 + 6t - 7, & 3 \leq t < 5 \\ -2, & t \geq 5 \end{cases}
$$

as depicted in Fig. 2.14(f). ■

Eg :1

Let the impulse response of an LTT system be $h(t) = e^{-t}u(t)$. Find the output y(t) if the input is x(t) = u(t).

- Answer : $y(t) = (1 - e^{-t})u(t)$.

Eg : 2

- Let the impulse response of an LTI system be $h(t) = e^{-2(t-1)}u(t+1)$. Find the output y(t) if the input is $x(t) = e^{-|t|}$.

Answer: For $t < -1$,

$$w_t(\tau) = \begin{cases} e^{-2(t+1)}e^{3\tau}, & -\infty < \tau < t+1 \\ 0, & \text{otherwise} \end{cases},$$

so

$$y(t) = \frac{1}{3}e^{t+1}.$$

For $t > -1$,

$$w_t(\tau) = \begin{cases} e^{-2(t+1)}e^{3\tau}, & -\infty < \tau < 0 \\ e^{-2(t+1)}e^{\tau}, & 0 < \tau < t+1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$y(t) = e^{-(t+1)} - \frac{2}{3}e^{-2(t+1)}.$$

- **Eg**: Let x(t) be the input to an LTI system with impulse response h(t) be given in fig 2.15 . Find output y(t).

Answer:

$$y(t) = \begin{cases} 0, & t < -4, t > 2 \\ (1/2)t^2 + 4t + 8, & -4 \le t < -3 \\ t + 7/2, & -3 \le t < -2 \\ (-1/2)t^2 - t + 3/2, & -2 \le t < -1 \\ (-1/2)t^2 - t + 3/2, & -1 \le t < 0 \\ 3/2 - t, & 0 \le t < 1 \\ (1/2)t^2 - 2t + 2, & 1 \le t < 2 \end{cases}.$$

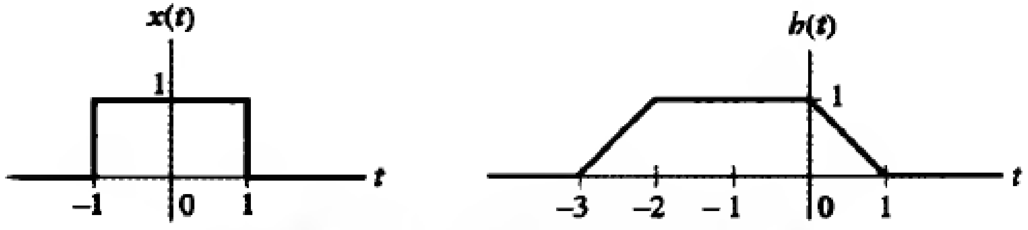


Figure 2.15 Signals for Problem 2.5.

▶ **Problem 2.6** Let the impulse response of an LTI system be given by $h(t) = u(t-1) - u(t-4)$. Find the output of this system in response to the input $x(t) = u(t) + u(t-1) - 2u(t-2)$.

Answer:

$$y(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 \le t < 2 \\ 2t-3, & 2 \le t < 3 \\ 3, & 3 \le t < 4 \\ 7-t, & 4 \le t < 5 \\ 12-2t, & 5 \le t < 6 \\ 0, & t \ge 6 \end{cases}.$$

◀

- **Properties**

Distributive property - CT $$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * \{h_1(t) + h_2(t)\}.$$

- DT $$x[n] * h_1[n] + x[n] * h_2[n] = x[n] * \{h_1[n] + h_2[n]\}.$$

Associative property – $$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}.$$

Commutative property – $$h_1(t) * h_2(t) = h_2(t) * h_1(t).$$

DT $$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\},$$

$$h_1[n] * h_2[n] = h_2[n] * h_1[n].$$

Property	Continuous-time system	Discrete-time system
Distributive	$x(t) * h_1(t) + x(t) * h_2(t) =$ $x(t) * \{h_1(t) + h_2(t)\}$	$x[n] * h_1[n] + x[n] * h_2[n] =$ $x[n] * \{h_1[n] + h_2[n]\}$
Associative	$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$	$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$
Commutative	$h_1(t) * h_2(t) = h_2(t) * h_1(t)$	$h_1[n] * h_2[n] = h_2[n] * h_1[n]$

- The impulse response completely characterizes the input-output behaviour of an LTI system. Hence, properties of the system, such as memory, causality, and stability, are related to the system's impulse response.

If the system is causal then $\boxed{h(\tau) = 0 \quad \text{for} \quad \tau < 0.}$

The o/p of an continuous-time causal LTI system is thus expressed as

$$y(t) = \int_0^{\infty} h(\tau)x(t-\tau)\,d\tau.$$

- **Stability :** The magnitude of the output is given by

$$|y[n]| = |h[n] * x[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right|.$$

- We conclude that the impulse response of a stable discrete-time LTI system satisfies the bound

- For DT and CT :

$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty.}$$

$$\boxed{\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty.}$$

▶ **Problem 2.10** For each of the following impulse responses, determine whether the corresponding system is (i) memoryless, (ii) causal, and (iii) stable. Justify your answers.

(a) $h(t) = u(t+1) - u(t-1)$

(b) $h(t) = u(t) - 2u(t-1)$

(c) $h(t) = e^{-2|t|}$

(d) $h(t) = e^{at}u(t)$

(e) $h[n] = 2^n u[-n]$

(f) $h[n] = e^{2n}u[n-1]$

(g) $h[n] = (1/2)^n u[n]$

Answers:

(a) not memoryless, not causal, stable.

(b) not memoryless, causal, not stable.

(c) not memoryless, not causal, stable.

(d) not memoryless, causal, stable provided that $a < 0$.

(e) not memoryless, not causal, stable.

(f) not memoryless, causal, not stable.

(g) not memoryless, causal, stable. ◀

- **Invertibility**
- The relationship between the impulse response of an LTI system, h(t), and that of the corresponding inverse system, $h^{inv}(t)$, is easily derived.
- The impulse response of the cascade connection is the convolution of h(t) and $h^{inv}(t)$,
- We require the o/p of the system to equal the input $x(t) * (h(t) * h^{inv}(t)) = x(t)$.
- The requirement implies that $\boxed{h(t) * h^{inv}(t) = \delta(t).}$

- Similarly, the impulse response of a discrete-time LTI inverse system,

$h^{inv}[n]$, must satisfy $\boxed{h[n] * h^{inv}[n] = \delta[n].}$

FIGURE 2.24 Cascade of LTI system with impulse response $h(t)$ and inverse system with impulse response $h^{inv}(t)$.

Inverse system

TABLE 2.2 ***Properties of the Impulse Response Representation for LTI Systems.***

Property	*Continuous-time system*	*Discrete-time system*
Memoryless	$h(t) = c\delta(t)$	$h[n] = c\delta[n]$
Causal	$h(t) = 0 \quad \text{for} \quad t < 0$	$h[n] = 0 \quad \text{for} \quad n < 0$
Stability	$\int_{-\infty}^{\infty} \lvert h(t) \rvert \, dt < \infty$	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$
Invertibility	$h(t) * h^{\text{inv}}(t) = \delta(t)$	$h[n] * h^{\text{inv}}[n] = \delta[n]$

STEP RESPONSE

- Let h[n] be the impulse response of a discrete-time LTI system, and denote the step response as s[n]. We thus write

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k].$$

$$\boxed{s[n] = \sum_{k=-\infty}^{n} h[k].}$$ - DT

$$\boxed{s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau.}$$ - CT

Note that we may invert these relationships to express the impulse response in terms of the step response as

$$h[n] = s[n] - s[n-1]$$

$$h(t) = \frac{d}{dt}s(t).$$

- Evaluate the step responses for the LTI systems represented by the following impulse responses:

(a) $h[n] = (1/2)^n u[n]$

(b) $h(t) = e^{-|t|}$

(c) $h(t) = \delta(t) - \delta(t-1)$

Answers:

(a) $s[n] = (2 - (1/2)^n)u[n]$

(b) $s(t) = e^t u(-t) + (2 - e^{-t})u(t)$

(c) $s(t) = u(t) - u(t-1)$

- **Equations Representations of LTI systems**
- Linear constant-coefficient difference and differential equations provide another representation for the input-output characteristics of LTI systems.
- Difference equations are used to represent discrete-time systems, while differential equations represent continuous-time systems.

$$y[n] = x[n] + 2x[n-1] - y[n-1] - \frac{1}{4}y[n-2]. \tag{2.38}$$

Beginning with $n = 0$, we may determine the output by evaluating the sequence of equations

$$y[0] = x[0] + 2x[-1] - y[-1] - \frac{1}{4}y[-2], \tag{2.39}$$

$$y[1] = x[1] + 2x[0] - y[0] - \frac{1}{4}y[-1], \tag{2.40}$$

$$y[2] = x[2] + 2x[1] - y[1] - \frac{1}{4}y[0],$$

$$y[3] = x[3] + 2x[2] - y[2] - \frac{1}{4}y[1],$$

EXAMPLE 2.15 RECURSIVE EVALUATION OF A DIFFERENCE EQUATION Find the first two output values $y[0]$ and $y[1]$ for the system described by Eq. (2.38), assuming that the input is $x[n] = (1/2)^n u[n]$ and the initial conditions are $y[-1] = 1$ and $y[-2] = -2$.

Solution: Substitute the appropriate values into Eq. (2.39) to obtain

$$y[0] = 1 + 2 \times 0 - 1 - \frac{1}{4} \times (-2) = \frac{1}{2}.$$

Now substitute for $y[0]$ in Eq. (2.40) to find

$$y[1] = \frac{1}{2} + 2 \times 1 - \frac{1}{2} - \frac{1}{4} \times (1) = 1\frac{3}{4}. \qquad \blacksquare$$

- The homogeneous form of a differential or difference equation is obtained by setting all terms involving the input to zero.

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(h)}(t) = 0.$$

- For DT $\boxed{\sum_{k=0}^{N} a_k r^{N-k} = 0.}$

Example 2.18 First-Order Recursive System: Homogeneous Solution Find the homogeneous solution for the first-order recursive system described by the difference equation

$$y[n] - \rho y[n-1] = x[n].$$

Solution: The homogeneous equation is

$$y[n] - \rho y[n-1] = 0,$$

and its solution is given by Eq. (2.43) for $N = 1$:

$$y^{(h)}[n] = c_1 r_1^n.$$

The parameter r_1 is obtained from the root of the characteristic equation given by Eq. (2.44) with $N = 1$:

$$r_1 - \rho = 0.$$

Hence, $r_1 = \rho$, and the homogeneous solution is

$$y^{(h)}[n] = c_1\rho^n. \qquad \blacksquare$$

► **Problem 2.16** Determine the homogeneous solution for the systems described by the following differential or difference equations:

(a)

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 2x(t) + \frac{d}{dt}x(t)$$

(b)

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t) + \frac{d}{dt}x(t)$$

(c)

$$y[n] - (9/16)y[n-2] = x[n-1]$$

(d)

$$y[n] + (1/4)y[n-2] = x[n] + 2x[n-2]$$

Answers:

(a)

$$y^{(h)}(t) = c_1e^{-3t} + c_2e^{-2t}$$

(b)

$$y^{(h)}(t) = c_1e^{-t} + c_2e^{-2t}$$

(c)

$$y^{(h)}[n] = c_1(3/4)^n + c_2(-3/4)^n$$

(d)

$$y^{(h)}[n] = c_1(1/2e^{j\pi/2})^n + c_2(1/2e^{-j\pi/2})^n$$ ◄

Continuous Time		Discrete Time	
Input	Particular Solution	Input	Particular Solution
1	c	1	c
t	$c_1 t + c_2$	n	$c_1 n + c_2$
e^{-at}	ce^{-at}	α^n	$c\alpha^n$
$\cos(\omega t + \phi)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$	$\cos(\Omega n + \phi)$	$c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$

- The particular solution y(p) represents any solution of the differential or difference equation for the given input.
- A particular solution is usually obtained by assuming an output of the same general form as the input. For example, if the input to a discrete time system is $x[n] = \alpha^n$, then we assume that the output is of the form $y^{(p)}[n] = c\alpha^n$ and find the constant c so that $y^{(p)}[n]$ is a solution of the system's difference equation.

EXAMPLE 2.19 FIRST-ORDER RECURSIVE SYSTEM (CONTINUED): PARTICULAR SOLUTION
Find a particular solution for the first-order recursive system described by the difference equation

$$y[n] - \rho y[n-1] = x[n]$$

if the input is $x[n] = (1/2)^n$.

Solution: We assume a particular solution of the form $y^{(p)}[n] = c_p\left(\frac{1}{2}\right)^n$. Substituting $y^{(p)}[n]$ and $x[n]$ into the given difference equation yields

$$c_p\left(\frac{1}{2}\right)^n - \rho c_p\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n.$$

We multiply both sides of the equation by $(1/2)^{-n}$ to obtain

$$c_p(1 - 2\rho) = 1. \tag{2.45}$$

Solving this equation for c_p gives the particular solution

$$y^{(p)}[n] = \frac{1}{1 - 2\rho}\left(\frac{1}{2}\right)^n.$$

If $\rho = \left(\frac{1}{2}\right)$, then the particular solution has the same form as the homogeneous solution found in Example 2.18. Note that in this case no coefficient c_p satisfies Eq. (2.45), and we must assume a particular solution of the form $y^{(p)}[n] = c_p n(1/2)^n$. Substituting this particular solution into the difference equation gives $c_p n(1 - 2\rho) + 2\rho c_p = 1$. Using $\rho = (1/2)$ we find that $c_p = 1$. ∎

► **Problem 2.18** Determine the particular solution associated with the specified input for the systems described by the following differential or difference equations:

(a) $x(t) = e^{-t}$:

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 2x(t) + \frac{d}{dt}x(t)$$

(b) $x(t) = \cos(2t)$:

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t) + \frac{d}{dt}x(t)$$

(c) $x[n] = 2$:

$$y[n] - (9/16)y[n-2] = x[n-1]$$

(d) $x[n] = (1/2)^n$:

$$y[n] + (1/4)y[n-2] = x[n] + 2x[n-2]$$

Answers:

(a) $y^{(p)}(t) = (1/2)e^{-t}$

(b) $y^{(p)}(t) = (1/4)\cos(2t) + (1/4)\sin(2t)$

(c) $y^{(p)}[n] = 32/7$

(d) $y^{(p)}[n] = (9/2)(1/2)^n$

EXAMPLE 2.21 FIRST-ORDER RECURSIVE SYSTEM (CONTINUED): COMPLETE SOLUTION Find the solution for the first-order recursive system described by the difference equation

$$y[n] - \frac{1}{4}y[n-1] = x[n] \tag{2.46}$$

if the input is $x[n] = (1/2)^n u[n]$ and the initial condition is $y[-1] = 8$.

Solution: The form of the solution is obtained by summing the homogeneous solution determined in Example 2.18 with the particular solution determined in Example 2.19 after setting $\rho = 1/4$:

$$y[n] = 2\left(\frac{1}{2}\right)^n + c_1\left(\frac{1}{4}\right)^n, \quad \text{for } n \geq 0. \tag{2.47}$$

The coefficient c_1 is obtained from the initial condition. First, we translate the initial condition to time $n = 0$ by rewriting Eq. (2.46) in recursive form and substituting $n = 0$ to obtain

$$y[0] = x[0] + (1/4)y[-1],$$

which implies that $y[0] = 1 + (1/4) \times 8 = 3$. Then we substitute $y[0] = 3$ into Eq. (2.47), yielding

$$3 = 2\left(\frac{1}{2}\right)^0 + c_1\left(\frac{1}{4}\right)^0,$$

from which we find that $c_1 = 1$. Thus, we may write the complete solution as

$$y[n] = 2\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n, \quad \text{for } n \geq 0.$$

■

- Find the output, given the input and initial conditions, for the systems described by the following differential or difference equations:

(a) $x(t) = e^{-t}u(t), y(0) = -\frac{1}{2}, \frac{d}{dt}y(t)\big|_{t=0} = \frac{1}{2}$:

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = x(t)$$

(b) $x(t) = \cos(t)u(t), y(0) = -\frac{4}{5}, \frac{d}{dt}y(t)\big|_{t=0} = \frac{3}{5}$:

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 2x(t)$$

(c) $x[n] = u[n], y[-2] = 8, y[-1] = 0$:

$$y[n] - \frac{1}{4}y[n-2] = 2x[n] + x[n-1]$$

(d) $x[n] = 2^n u[n], y[-2] = 26, y[-1] = -1$:

$$y[n] - \left(\frac{1}{4}\right)y[n-1] - \left(\frac{1}{8}\right)y[n-2] = x[n] + \left(\frac{11}{8}\right)x[n-1]$$

(a)

$$y(t) = \left(\left(\frac{1}{2}\right)e^{-t} + e^{-3t} - 2e^{-2t}\right)u(t)$$

(b)

$$y(t) = \left(\left(\frac{1}{5}\right)\cos(t) + \left(\frac{3}{5}\right)\sin(t) - 2e^{-t} + e^{-2t}\right)u(t)$$

(c)

$$y[n] = \left(-\left(\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^n + 4\right)u[n]$$

(d)

$$y[n] = \left(2(2)^n + \left(-\frac{1}{4}\right)^n + \left(\frac{1}{2}\right)^n\right)u[n]$$

- **<u>The impulse response</u>**:
- For a continuous-time system, the impulse response h(t) is related to the step response s(t) via the formula $h(t) = \frac{d}{dt}s(t).$
- For a discrete-time system, $h[n] = s[n] - s[n-1].$