Signals And Systems (UE17EC204)

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Unit V
Z transforms

(Chapter 10 of prescribed Textbook – Sections 10.1 – 10.3, 10.5 – 10.7, 10.9)

The Z transform

- For a DT LTI system with impulse response $h[n]$, the response $y[n]$ of the system to a complex exponential input of the form z^n is $y[n] = H(z)z^n$, where

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

- For $z = e^{j\omega}$, where ω is real ($|z| = 1$), the above summation corresponds to the DTFT of $h[n]$.
- When $|z|$ is not restricted to unity, the summation is called the **Z-Transform of** $h[n]$
- The Z-Transform of a general DT signal $x[n]$ is given by the following, where z is a complex variable:

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n},$$

The Z transform

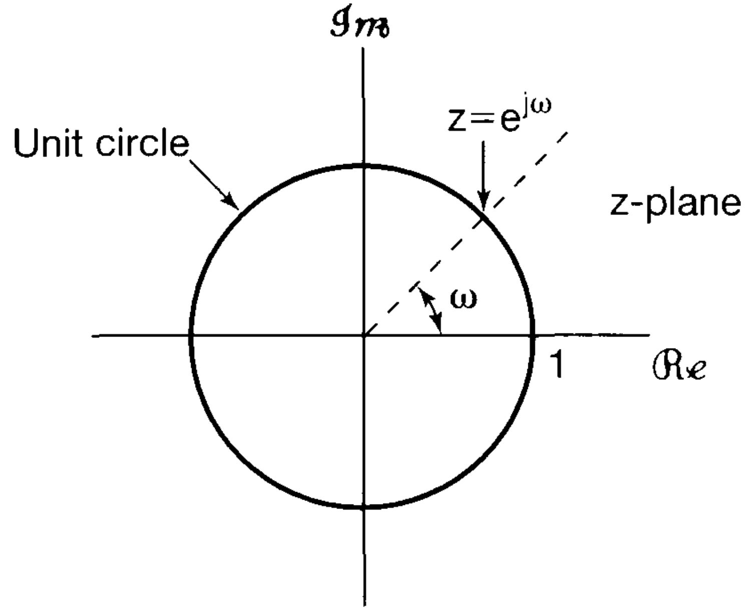
- For convenience, the ZT of $x[n]$ will be represented as $Z(x[n])$ and the relationship between $x[n]$ and its Z-Transform will be indicated as $x[n] \overset{Z}{\leftrightarrow} X(z)$
- There are important relations between ZT and the DTFT. Let $z = re^{j\omega}$ where r is the magnitude and ω is the angle of z .
- The ZT equation becomes

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-j\omega n}$$

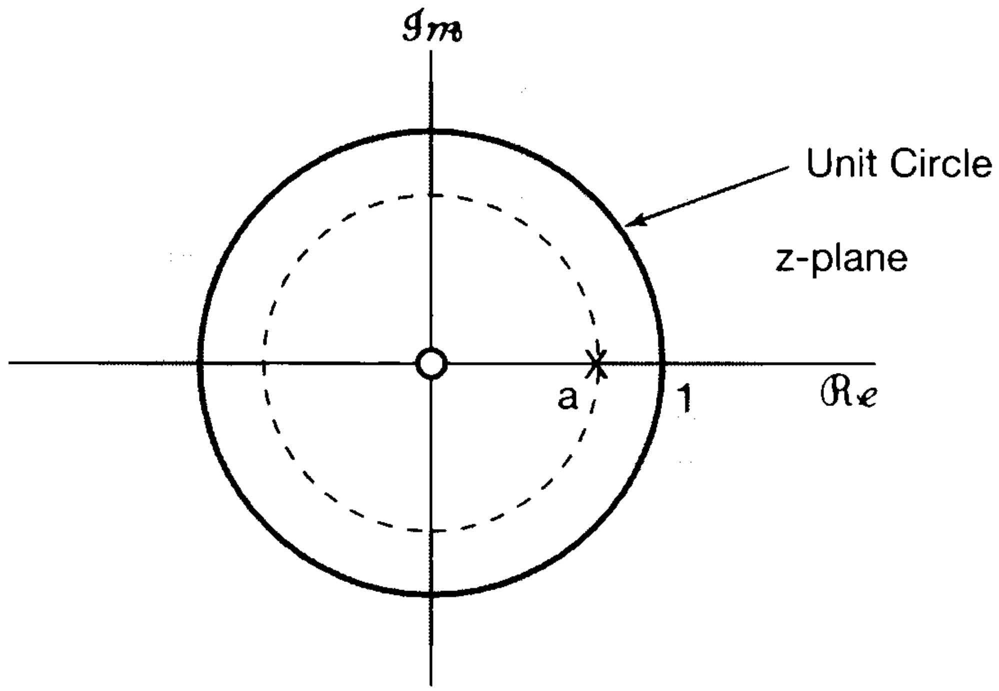
- $X\left(re^{j\omega}\right)$ is the FT of $x[n]$ multiplied by a real exponential r^{-n}, i.e., $X(re^{j\omega}) = F(x[n]r^{-n})$
- For $r = 1$ or $|z| = 1$, $X(e^{j\omega}) = F(x[n])$

The Z transform

- The ZT reduces to the FT on the contour in the complex z-plane corresponding to a circle with a circle of unit radius (called the **Unit Circle**).
- For any specific sequence, we would expect convergence for **some** r. The ZT is associated with a range of values for which it converges (**Region of Convergence – ROC**).
- If ROC contains the unit circle, the FT converges too.



Example 1



One zero at $z = 0$, and one pole at $z = a$

For $|a| > 1$, the ROC does not include the unit circle, no convergence

$$x[n] = a^n u[n].$$

The Z-Transform is

$$X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

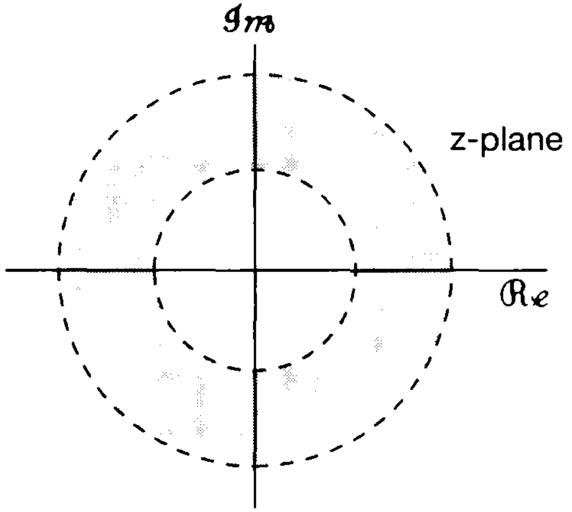
$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|.$$

When $a = 1$,

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1.$$

The Region of Convergence for the Z-Transform

- **Property 1:** The ROC of $X(z)$ consists of a ring in the z plane centered about origin.
- **Property 2:** The ROC contains no poles.
- **Property 3:** If $x[n]$ is of finite duration, then ROC is the entire z plane, except $z = 0$ and/or $z = \infty$.



The Region of Convergence for the Z-Transform

- **Property 4:** If $x[n]$ is a right-sided sequence and the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.
- **Property 5:** If $x[n]$ is a left-sided sequence and the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.
- **Property 6:** If $x[n]$ is two-sided sequence and the circle $|z| = r_0$ is in the ROC, then ROC consists of a ring in the z plane that includes the circle $|z| = r_0$.

The Region of Convergence for the Z-Transform

- **Property 7:** If the ZT $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.
- **Property 8:** If the ZT $X(z)$ of $x[n]$ is rational, and if $x[n]$ is a right-sided sequence, then the ROC is the region in the z plane outside the outermost pole (outside the circle of radius equal to the largest magnitude of the poles of $X(z)$). Furthermore, if $x[n]$ is causal, then ROC also includes $z = \infty$.
- **Property 9:** If the ZT $X(z)$ of $x[n]$ is rational, and if $x[n]$ is a left-sided sequence, then the ROC is the region in the z plane inside the innermost pole (inside the circle of radius equal to the smallest magnitude of the poles of $X(z)$). Furthermore, if $x[n]$ is anti-causal, then ROC also includes $z = 0$.

Example 2

$$\delta[n] \stackrel{z}{\longleftrightarrow} \sum_{n=-\infty}^{+\infty} \delta[n] z^{-n} = 1$$

ROC consisting of entire z-plane, including $z = \infty$ and $z = 0$

$$\delta[n-1] \stackrel{z}{\longleftrightarrow} \sum_{n=-\infty}^{+\infty} \delta[n-1] z^{-n} = z^{-1}.$$

Z-Transform well defined except at $z = 0$, where there's a pole.

ROC – Entire z-plane including $z = \infty$ but excluding $z = 0$

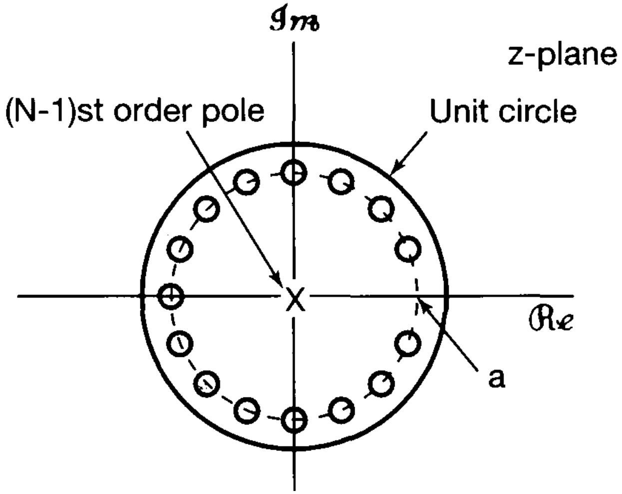
Example 3

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \ a > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$$

$$= \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$



Explanation:

$z = 0$ of finite length and ROC includes entire z plane except maybe 0 or ∞

$x[n]$ is zero for $n < 0$, the ROC extends to ∞

$x[n]$ is nonzero for some $n > 0$, the ROC excludes origin (pole of order $N - 1$at $z = 0$)

N roots of numerator at:

$$z_k = ae^{j(2\pi k/N)}, \quad k = 0, 1, \ldots, N-1.$$

The root for $k = 0$ cancels pole at $z = a$

No poles other than at origin

Remaining zeros at

$$z_k = ae^{j(2\pi k/N)}, \quad k = 1, \ldots, N-1$$

The Inverse Z-Transform

- Expression of a sequence in terms of its Z-Transform. We know that $X(re^{j\omega}) = F(x[n]r^{-n})$ for any value of r such that $z = re^{j\omega}$ is inside the ROC. Application of IFT to both sides yields $x[n]r^{-n} = F^{-1}\left(X(re^{j\omega})\right)$ or $x[n] = r^n F^{-1}\left(X(re^{j\omega})\right)$
- Using IFT expression,

$$x[n] = r^n \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re^{j\omega})^n d\omega$$

The Inverse Z-Transform

- Recovery of $x[n]$ possible from its ZT evaluated along a contour $z = re^{j\omega}$ in the ROC with r fixed and ω varying over 2π interval.
- Consider $z = re^{j\omega}$ and fixed r and integrating in the z plane, we get

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

This is the formal expression for the **inverse Z-Transform**

Example 4

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}.$$

Two poles, one at $(z = \frac{1}{3})$ and $(z = \frac{1}{4})$ ROC outside outermost pole (it consists of all points with magnitude greater than the pole with largest magnitude $(z = \frac{1}{3})$. The Inverse transform is a right-sided sequence. Expanding by method of partial fractions,

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

$$x[n] = x_1[n] + x_2[n]$$

$$x_1[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - \frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$$

$$x_2[n] \overset{z}{\longleftrightarrow} \frac{2}{1 - \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$x_2[n] = 2\left(\frac{1}{3}\right)^n u[n],$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

Properties of the Z-Transform

- **Linearity**

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z), \quad \text{with ROC} = R_1$$

$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z), \quad \text{with ROC} = R_2$$

$$a x_1[n] + b x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} a X_1(z) + b X_2(z), \quad \text{with ROC containing } R_1 \cap R_2$$

- **Time Shifting**

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \quad \text{with ROC} = R$$

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z), \quad \text{with ROC} = R\text{, except for the possible addition or deletion of the origin or infinity.}$$

Properties of the Z-Transform

- **Scaling in the z-Domain**

$$x[n] \xleftrightarrow{Z} X(z), \quad \text{with ROC} = R,$$

$$z_0^n x[n] \xleftrightarrow{Z} X\left(\frac{z}{z_0}\right), \quad \text{with ROC} = |z_0|R$$

- **Time Reversal**

$$x[n] \xleftrightarrow{Z} X(z), \quad \text{with ROC} = R.$$

$$x[-n] \xleftrightarrow{Z} X\left(\frac{1}{z}\right), \quad \text{with ROC} = \frac{1}{R}.$$

Properties of the Z-Transform

- **Time Expansion**

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R$$

$$x_{(k)}[n] \stackrel{z}{\longleftrightarrow} X(z^k), \quad \text{with ROC} = R^{1/k}$$

- **Conjugation**

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R$$

$$x^*[n] \stackrel{z}{\longleftrightarrow} X^*(z^*), \quad \text{with ROC} = R.$$

Properties of the Z-Transform

- Convolution

$$x_1[n] \xleftrightarrow{z} X_1(z), \quad \text{with ROC} = R_1$$

$$x_2[n] \xleftrightarrow{z} X_2(z), \quad \text{with ROC} = R_2$$

$$x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z)X_2(z), \quad \text{with ROC containing } R_1 \cap R_2.$$

- Differentiation in the z-domain

$$x[n] \xleftrightarrow{z} X(z), \quad \text{with ROC} = R$$

$$nx[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz}, \quad \text{with ROC} = R$$

Properties of the Z-Transform

- **Initial Value Theorem**

if $x[n] = 0, n < 0$ then

$$x[0] = \lim_{z \to \infty} X(z).$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \to \infty} X(z)$			

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m>0$) or ∞ (if $m<0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Analysis and Characterization of LTI Systems Using Z-Transforms

- From the convolution property,

$$Y(z) = H(z)X(z)$$

$H(z)$ is the transfer function of the system.

For z on the unit circle, $H(z)$ reduces to the frequency response of the system, provided that the unit circle is in the ROC for $H(z)$.

- If the input to an LTI system is the complex exponential $x[n] = z^n$ then the output will be $H(z)z^n$ (z^n is an eigenfunction of the system with eigenvalue given by $H(z)$.

Analysis and Characterization of LTI Systems Using Z-Transforms

- **Causality**

A causal LTI system has impulse response $h[n]$, zero for $n < 0$ and is therefore right-sided.

For a causal system, the series

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Has no positive powers of z. Consequently, ROC includes infinity.

A DT LTI system is causal iff the ROC of its system function is the exterior of the circle, including infinity.

A DT LTI system with rational system function $H(z)$ is causal iff:

(a) the ROC is the exterior of a circle outside the outermost pole; and

(b) With $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator.

Example 5

Consider a system with system function whose algebraic expression is

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}.$$

Without even knowing the ROC for this system, we can conclude that the system is not causal, because the numerator of $H(z)$ is of higher order than the denominator.

Analysis and Characterization of LTI Systems Using Z-Transforms

- **Stability**

The stability of a DT LTI system is equivalent to its impulse response being absolutely summable.

Here, the Fourier transform of $h[n]$ converges and so, the ROC of $h[n]$ must include the unit circle.

An LTI is stable iff the ROC of its system function $H(z)$ includes the unit circle $|z| = 1$.

A causal LTI system with rational system function $H(z)$ is stable iff all of the poles of $H(z)$ lie inside the unit circle - i.e., they must all have magnitude smaller than 1.

Analysis and Characterization of LTI Systems Using Z-Transforms

- **LTI Systems Characterized by Linear Constant-Coefficient Difference Equations (LCCDEs)**

For an Nth order difference equation, apply ZT to both sides, and use linearity and time-shift properties.

Considering LTI system for which input and output satisfy an LCCDE of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Taking ZT and using linearity and time-shift,

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

Analysis and Characterization of LTI Systems Using Z-Transforms

Or

$$(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{k=0}^{M} b_k z^{-k}$$

So that

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

The system function satisfying an LCCDE is always rational.

Constraints like causality and stability of the system are needed along with the difference equation to provide ROC specifications associated with $H(z)$.

The Unilateral Z-Transform

- The form of the Z-Transform used so far is referred to as the **Bilateral** Z-Transform.
- The **Unilateral** Z-Transform helps analyse causal systems specified by LCCDEs with non-zero initial conditions.

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x[n] \overset{UZ}{\leftrightarrow} \mathcal{X}(z) = UZ\{x[n]\}$$

- The summation is carrier out over nonnegative values of n, irrespective of whether $x[n]$ is zero for $n < 0$ or not.
- Think of UZT as a BZT of $x[n]u[n]$, then both will be identical.

Example 6

$$x[n] = a^n u[n].$$

Since $x[n] = 0, n < 0$ the UZT and BZT are equal.

$$\mathcal{X}(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

Example 7

$$x[n] = a^{n+1} u[n+1]$$

Since $x[-1] = 1$, not 0 the UZT and BZT are **not** equal.

$$X(z) = \frac{z}{1 - az^{-1}}, \qquad |z| > |a|$$

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{n+1} z^{-n}$$

$$\mathcal{X}(z) = \frac{a}{1 - az^{-1}}, \qquad |z| > |a|$$

TABLE 10.3 PROPERTIES OF THE UNILATERAL z-TRANSFORM

Property	Signal	Unilateral z-Transform
—	$x[n]$	$\mathcal{X}(z)$
—	$x_1[n]$	$\mathcal{X}_1(z)$
—	$x_2[n]$	$\mathcal{X}_2(z)$
Linearity	$ax_1[n] + bx_2[n]$	$a\mathcal{X}_1(z) + b\mathcal{X}_2(z)$
Time delay	$x[n-1]$	$z^{-1}\mathcal{X}(z) + x[-1]$
Time advance	$x[n+1]$	$z\mathcal{X}(z) - zx[0]$
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$\mathcal{X}(e^{-j\omega_0}z)$
	$z_0^n x[n]$	$\mathcal{X}(z/z_0)$
	$a^n x[n]$	$\mathcal{X}(a^{-1}z)$
Time expansion	$x_k[n] = \begin{cases} x[m], & n = mk \\ 0, & n \neq mk \quad \text{for any } m \end{cases}$	$\mathcal{X}(z^k)$
Conjugation	$x^*[n]$	$\mathcal{X}^*(z^*)$
Convolution (assuming that $x_1[n]$ and $x_2[n]$ are identically zero for $n < 0$)	$x_1[n] * x_2[n]$	$\mathcal{X}_1(z)\mathcal{X}_2(z)$
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})\mathcal{X}(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{1}{1 - z^{-1}}\mathcal{X}(z)$
Differentiation in the z-domain	$nx[n]$	$-z\frac{d\mathcal{X}(z)}{dz}$

Initial Value Theorem

$$x[0] = \lim_{z\to\infty} \mathcal{X}(z)$$

Example 8: Solving D. E. using the UZT

Considering the difference equation $y[n] + 3y[n-1] = x[n]$, $x[n] = \alpha\, u[n]$ and $y[-1] = \beta$

Application of UZT yields

$$\mathcal{Y}(z) + 3\beta + 3z^{-1}\mathcal{Y}(z) = \frac{\alpha}{1 - z^{-1}}$$

Solving, we get,

$$\mathcal{Y}(z) = -\frac{3\beta}{1 + 3z^{-1}} + \frac{\alpha}{(1 + 3z^{-1})(1 - z^{-1})}.$$

The second term is the UZT of the response of the system when $\beta = 0$ (**Zero State Response)**

The first term is the UZT of the **Zero Input Response** $(\alpha = 0)$

If $\alpha = 8$ and $\beta = 0$

$$\mathcal{Y}(z) = \frac{3}{1 + 3z^{-1}} + \frac{2}{1 - z^{-1}}$$

Applying UZT pairs

$$y[n] = [3(-3)^n + 2]u[n], \quad \text{for } n \geq 0$$