UNIT IV

Continuous Time Fourier Transform (CTFT)…. continued

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Properties of CTFT

1. Linearity

$$\text{If } x(t) \xleftrightarrow{F} X(j\omega) \text{ and } y(t) \xleftrightarrow{F} Y(j\omega)$$

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$$

Proof: $$Z(j\omega) = \int_{-\infty}^{+\infty} [ax(t) + by(t)]\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} ax(t)\, e^{-j\omega t} dt + \int_{-\infty}^{+\infty} by(t)\, e^{-j\omega t} dt$$

$$= a \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt + b \int_{-\infty}^{+\infty} y(t)\, e^{-j\omega t} dt$$

$$Z(j\omega) = aX(j\omega) + bY(j\omega)$$ ....Proved

2. Time shifting

If $x(t) \xleftrightarrow{F} X(j\omega)$

Then

$$x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega).$$

Or

$$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega) = |X(j\omega)| e^{j[\angle X(j\omega) - \omega t_0]}.$$

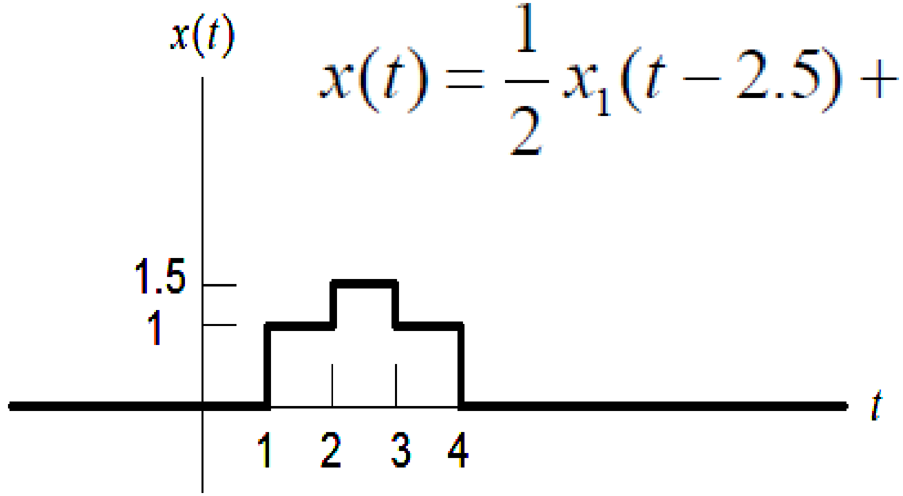
$No\ Change\ in\ Magnitude\ observed,$

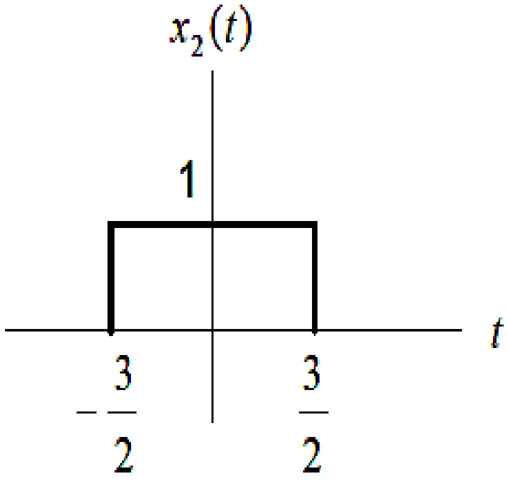
$but\ produces\ \ phase\ shift\ of + \angle\ e^{-j\omega To}$

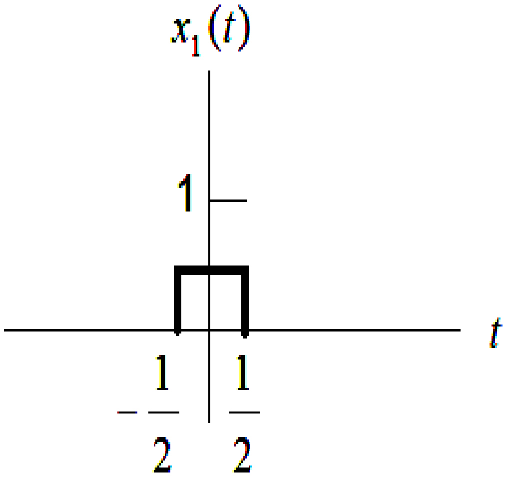
$i.e\ Phase\ Shift = \ \angle X(j\omega)\ + \angle\ e^{-j\omega To} \quad = \ \angle X(j\omega)\ + \omega To$

Example: To evaluate the Fourier transform of the signal $x(t)$ shown in the figure below.

$$x(t)=\frac{1}{2}x_1(t-2.5)+x_2(t-2.5)\,.$$







$x_1(t)$ and $x_2(t)$ are rectangular pulse signals and their Fourier transforms are

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-T_1}^{T_1} 1 e^{-j\omega t} dt = 2\frac{\sin \omega T_1}{\omega}.$$

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega} \text{ and } X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

Using the linearity and time-shifting properties of the Fourier transform yields

$$X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}$$

3. Conjugation and conjugation symmetry

If $x(t) \xleftrightarrow{F} X(j\omega)$

Then

$$x^*(t) \xleftrightarrow{F} X^*(-j\omega).$$

3. Conjugation and conjugation symmetry

We can also prove that

- if *x(t) is both real and even, then X(jω) will also be real and even.*
- if *x(t) is both real and odd, then X(jω) will also be purely imaginary and odd.*

A real function $x(t)$ can be expressed in terms of the sum of an even function $x_e(t) = Ev\{x(t)\}$ and an odd function $x_o(t) = Od\{x(t)\}$

$$x(t) = x_e(t) + x_o(t)$$

Form the Linearity property,

$$F\{x(t)\} = F\{x_e(t)\} + F\{x_o(t)\},$$

$F\{x_e(t)\}$ is real function

$F\{x_o(t)\}$ is purely imaginary.

we conclude with $x(t)$ real,

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$Ev\{x(t)\} \xleftrightarrow{F} \mathrm{Re}\{X(j\omega)\}$$

$$Od\{x(t)\} \xleftrightarrow{F} j\,\mathrm{Im}\{X(j\omega)\}$$

Example: Using the symmetry properties of the Fourier transform and the result $e^{-at}u(t) \xleftrightarrow{F} \dfrac{1}{a+j\omega}$ evaluate the Fourier transform of the signal $x(t) = e^{-a|t|}$ where $a > 0$.

Since $x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) =$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] = 2Ev\left\{e^{-at}u(t)\right\},$$

So $X(j\omega) = 2\,\mathrm{Re}\left(\dfrac{1}{a + j\omega}\right) = \dfrac{2a}{a^2 + \omega^2}$

4. Differentiation and Integration

If $x(t) \xleftrightarrow{F} X(j\omega)$

Then

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

Time Differentiation Property

If

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

Then,

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$$

Proof :

we know,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$

Differentiating both sides

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)(j\omega\, e^{j\omega t})d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,(j\omega)e^{j\omega t}d\omega \quad = j\omega X(j\omega)$$

Frequency Differentiation Property

$$If \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$then \quad -jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(j\omega)$$

Example: Consider the Fourier transform of the unit step $x(t) = u(t)$.

$$g(t) = \delta(t) \xleftrightarrow{F} 1$$

Also note that

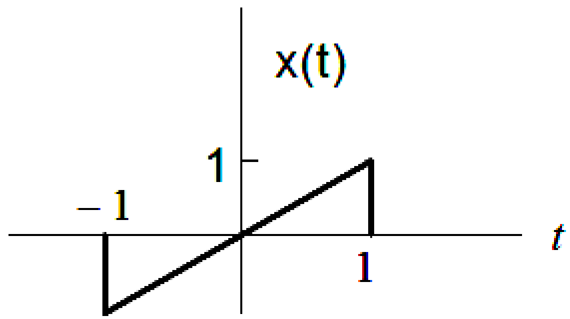
$$x(t) = \int_{-\infty}^{t} g(\tau)\,d\tau$$

The Fourier transform of this function is

$$X(j\omega) = \frac{1}{j\omega} + \pi G(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

where $G(0) = 1$.

Example: Consider the Fourier transform of the function $x(t)$



$$g(t) = \frac{dx(t)}{dt}$$

$g(t)$ is the sum of a rectangular pulse and two impulses.

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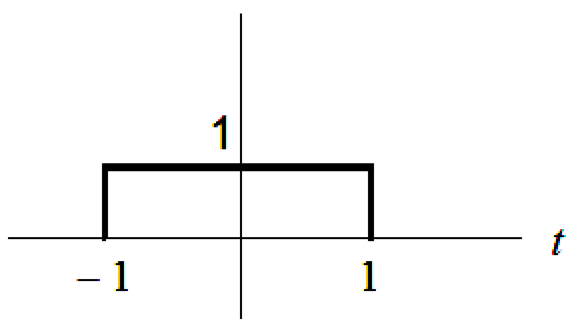

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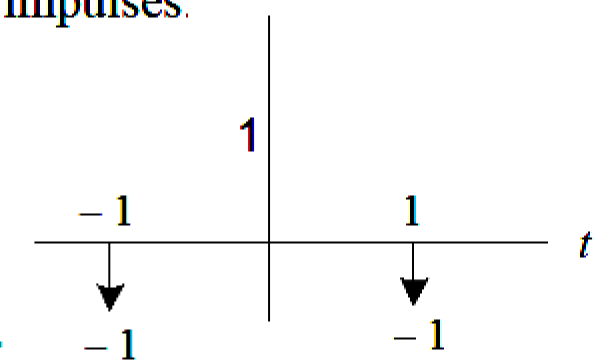


$$G(j\omega)=\left(\frac{2\sin\omega}{\omega}\right)-e^{j\omega}-e^{-j\omega}$$

Note that $G(0)=0$

using the integration property, we obtain

$$X(j\omega)=\frac{G(j\omega)}{j\omega}+\pi G(0)\delta(\omega)=\frac{2\sin\omega}{j\omega^2}-\frac{2\cos\omega}{j\omega}.$$

It can be found $X(j\omega)$ is purely imaginary and odd,

with the fact that $x(t)$ is real and odd.

5. Time and frequency scaling

$$x(t) \xleftrightarrow{F} X(j\omega),$$

Then

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X(\frac{j\omega}{a}).$$

From the equation we see that the signal is compressed in the time domain, the spectrum will be extended in the frequency domain.

Conversely, if the signal is extended, the corresponding spectrum will be compressed.

Proof: $$Y(j\omega) = \int_{-\infty}^{+\infty} y(t)\, e^{-j\omega t} dt$$

$$i.e\ \ Y(j\omega) = \int_{-\infty}^{+\infty} x(at)\, e^{-j\omega t} dt$$

$$Let\ a > 0;\ \ put\ at = \lambda$$

$$Then, \qquad dt = \frac{d\lambda}{a}$$

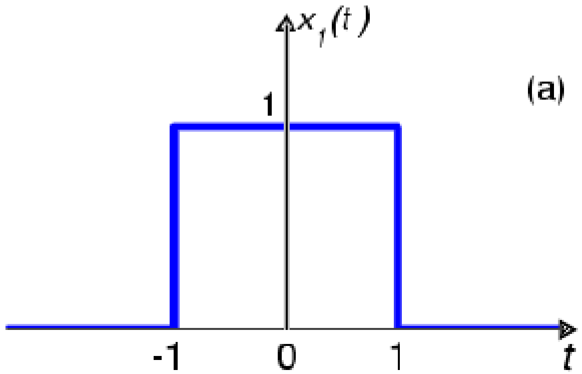
$$accordingly, \qquad Y(j\omega) = \int_{-\infty}^{+\infty} x(\lambda)\, e^{-j(\lambda/a)\omega} \frac{d\lambda}{a}$$

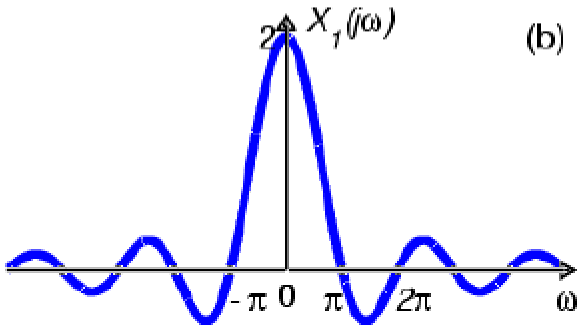
$$= \frac{1}{a} \int_{-\infty}^{+\infty} x(\lambda)\, e^{-j\omega(\lambda/a)} d\lambda \qquad = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

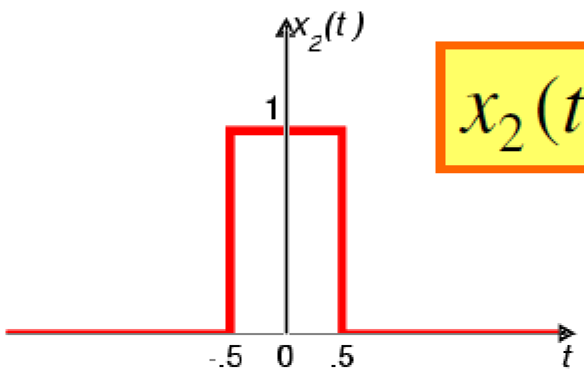
…… proved

Scaling Property

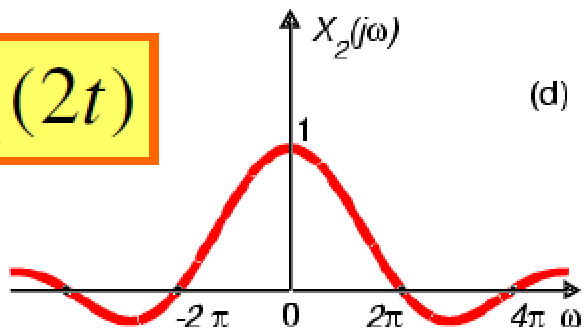
$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$







$$x_2(t) = x_1(2t)$$



If $a = -1$, we get from the above equation,

$$x(-t) \stackrel{F}{\longleftrightarrow} X(-j\omega).$$

That is, reversing a signal in time also reverses its Fourier transform.

6. Duality Property

If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then

$$X(t) \xleftrightarrow{\text{CTFT}} 2\pi x(-\omega)$$

There is a high degree of symmetry in Forward And Inverse Fourier transform equation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Forward and inverse FT equations are identical except the factor 2π and different sign in ω of exponential function *we know,* $$x(t) = \frac{1}{2\pi} \int_{\omega-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$

Interchanging t and ω, we get

$$x(j\omega) = \frac{1}{2\pi} \int_{t=-\infty}^{+\infty} X(t)\, e^{jt\omega} d\omega$$

we know, $$x(t) = \frac{1}{2\pi}\int_{\omega-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$

Interchanging t and ω, we get

we know, $$x(t) = \frac{1}{2\pi}\int_{\omega-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$

Interchanging t and ω, we get

$$x(j\omega) = \frac{1}{2\pi}\int_{t=-\infty}^{+\infty} X(t)\, e^{jt\omega} d\omega$$

Replacing ω with − ω, we get

we know, $$x(t) = \frac{1}{2\pi} \int_{\omega-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$

Interchanging t and ω, we get

$$x(j\omega) = \frac{1}{2\pi} \int_{t=-\infty}^{+\infty} X(t)\, e^{jt\omega}\, dt$$

Replacing ω with − ω, we get

$$x(-j\omega) = \frac{1}{2\pi} \int_{t=-\infty}^{+\infty} X(t)\, e^{-jt\omega} dt$$

$$\text{we know,} \qquad x(t) = \frac{1}{2\pi}\int_{\omega-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$

Interchanging t and ω, we get

$$x(j\omega) = \frac{1}{2\pi}\int_{t=-\infty}^{+\infty} X(t)\, e^{jt\omega}\, dt$$

Replacing ω with − ω, we get

$$x(-j\omega) = \frac{1}{2\pi}\int_{t=-\infty}^{+\infty} X(t)\, e^{-jt\omega} dt$$

$$2\pi x(-j\omega) = \int_{t=-\infty}^{+\infty} X(t)\, e^{-jt\omega} dt$$

we know, $$x(t) = \frac{1}{2\pi} \int_{\omega-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$

Interchanging t and ω, we get

$$x(j\omega) = \frac{1}{2\pi} \int_{t=-\infty}^{+\infty} X(t)\, e^{jt\omega}\, dt$$

Replacing ω with − ω, we get

$$x(-j\omega) = \frac{1}{2\pi} \int_{t=-\infty}^{+\infty} X(t)\, e^{-jt\omega} dt$$

$$2\pi x(-j\omega) = \int_{t=-\infty}^{+\infty} X(t)\, e^{-jt\omega} dt$$

Hence, then $X(t) \xleftrightarrow{FT} 2\pi x(-j\omega)$ ...proved

7. Frequency shift Property

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

Then, $$y(t) = e^{j\beta t}x(t) \xleftrightarrow{FT} Y(j\omega) = X(j(\omega - \beta))$$

Proof : We know $$Y(j\omega) = \int_{-\infty}^{+\infty} y(t)\, e^{-j\omega t}dt$$

$$= \int_{-\infty}^{+\infty} e^{j\beta t}x(t)\, e^{-j\omega t}dt \qquad = \int_{-\infty}^{+\infty} x(t)\, e^{-j(\omega-\beta)t}dt$$

$$= Xj(\omega - \beta)$$ ....proved

8. Convolution property

$$If \quad x(t) \xleftrightarrow{FT} X(j\omega) \quad and \quad y(t) \xleftrightarrow{FT} Y(j\omega)$$

$$then \; z(t) = x(t) * y(t) \xleftrightarrow{FT} Z(j\omega) = X(j\omega)Y(j\omega)$$

Proof :

$$we\ know, \quad Z(j\omega) = \int_{t=-\infty}^{+\infty} z(t)\, e^{-j\omega t} dt$$

$$= \int_{t=-\infty}^{+\infty} x(t) * y(t)\, e^{-j\omega t} dt$$

$$= \int_{t=-\infty}^{+\infty} \left[\int_{T=-\infty}^{+\infty} x(T) y(t-T) dT \right] e^{-j\omega t} dt$$

rearranging the above, we get

$$Z(j\omega) = \int_{T=-\infty}^{+\infty} x(T)\left[\int_{t=-\infty}^{+\infty} y(t-T)e^{-j\omega t}dt\right]dT$$

---

put $t - T = \lambda$, *then* $dt = d\lambda$,

Accordingly we get,

$$= \int_{T=-\infty}^{+\infty} x(T)e^{-j\omega(T)}dT\left[\int_{\lambda=-\infty}^{+\infty} y(\lambda)e^{-j\omega\lambda)}d\lambda\right]$$

$Z(j\omega) = X(j\omega)Y(j\omega) \ldots\ldots$ *Proved*

9. Integration or accumulation Property

$$If \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$then, \quad \int_{-\infty}^{t} x(T) \xleftrightarrow{FT} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

Proof : $we\ can\ write, \quad \int_{-\infty}^{t} x(T) = x(t) * u(t)$

$$Hence, \quad \int_{-\infty}^{t} x(T)dT \xleftrightarrow{FT} X(j\omega)U(j\omega) \ldots . (1)$$

It has been proved that

$$u(t) \xleftrightarrow{FT} U(j\omega) = \pi\delta(\omega) + \frac{1}{j\omega} \ldots\ldots (2)$$

Substituting (2) in (1)we get

$$\int_{-\infty}^{t} x(T)dT \xleftrightarrow{FT} X(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

please note that

$$X(j\omega)\delta(\omega) = X(0)\delta(\omega)$$

$$\int_{-\infty}^{t} x(T)dT \xleftrightarrow{FT} X(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

$$\xleftrightarrow{FT} \pi X(j\omega)\delta(\omega) + X(j\omega)\frac{1}{j\omega}$$

$$\xleftrightarrow{FT} \pi X(0)\delta(\omega) + X(j\omega)\frac{1}{j\omega} \ldots\ldots proved$$

9. Modulation or Multiplication

$$If \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$and \quad y(t) \xleftrightarrow{FT} Y(j\omega)$$

$$then \quad z(t) = x(t)y(t) \xleftrightarrow{FT} \frac{1}{2\pi}[X(j\omega) * Y(j\omega)]$$

Proof :

we know, $$Z(j\omega) = \int_{t=-\infty}^{+\infty} z(t)\, e^{-j\omega t} dt$$

$$= \int_{t=-\infty}^{+\infty} x(t) y(t)\, e^{-j\omega t} dt \;\ldots\ldots (3)$$

we also know, $$x(t) = \frac{1}{2\pi} \int_{\lambda-\infty}^{+\infty} X(\lambda)\, e^{j\lambda t} d\lambda \;\ldots\ldots (4)$$

Substituting Eqn (4) in (3)we get,

$$Z(j\omega) = \int_{t=-\infty}^{+\infty} \left[\frac{1}{2\pi} \int_{\lambda=-\infty}^{+\infty} X(\lambda)\, e^{j\omega t} d\lambda \right] y(t)\, e^{-j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{\lambda=-\infty}^{+\infty} X(\lambda) \int_{t=-\infty}^{+\infty} y(t)\, e^{-(j\omega-\lambda)t}\, dt\, d\lambda$$

$$= \frac{1}{2\pi} \int_{\lambda-\infty}^{+\infty} X(\lambda) Y(\omega - \lambda) d\lambda \quad = \frac{1}{2\pi} [X(j\omega) * Y(j\omega)] \ldots .proved$$

Hence, multiplication in Time Domain is

equivalent *to convolution in frequency Domain.*

They are said to be Duals of each other

10) Parsevals theorem or Rayleigh's theorem

$$If \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$then, \; E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

where E is the total energy content

of the signal x(t). Also $|x(\omega)|^2$ *is defined*

as the energy density spectrum of the signal x(t)

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t)\,dt \\
&= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(j\omega)\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega
\end{aligned}$$


THANK
YOU!