Chapter

10

NETWORK SYNTHESIS

10.1 INTRODUCTION

The previous chapters were devoted to network analysis. In the analysis work, the network and the input are given and the response is computed. The network synthesis is a design problem. The input and response are specified and the network has to be synthesized, i.e., network element types, values and their topography is required to be found. It is known that the input and response are related by the network function $H(s)$. Thus, the network is designed for a given $H(s)$. The analysis problem has a unique solution. However, the synthesis problem does not have a unique solution. For a given network function, one can synthesize many different networks.

10.2 CAUSALITY AND STABILITY

10.2.1 Causality

The word causality means that a current cannot appear in a network before voltage is impressed or vice versa. In other words, the impulse response of the network must be zero for $t < 0$, *i.e.*,

$$h(t) = 0 \text{ for } t < 0 \qquad \text{...(10.1)}$$

The impulse response

$$h(t) = e^{-at}\, u(t) \qquad \text{...(10.2)}$$

is causal and

$$h(t) = e^{-a|t|} \qquad \text{...(10.3)}$$

is not causal. Sometimes the impulse response can be made causal (realizable) by delaying it suitably. The impulse response shown in Fig. 10.1(a) is not realizable but if we delay the response by a second, the delayed response $h(t - a)$ is realizable (Fig. 10.1 b).

In the frequency domain causality is implied when Paley-Wiener Criterion is satisfied for the amplitude function $|H(j\omega)|$.

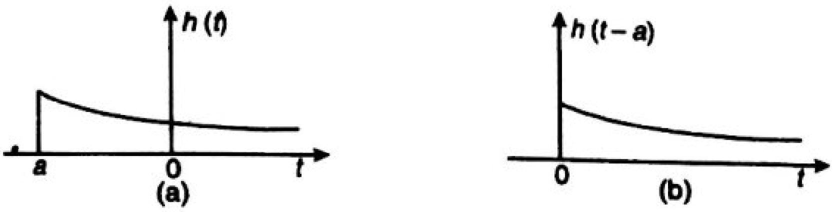


Fig. 10.1 (a) Non realizable impulse response (b) Realizable impulse response

This criterion states that a necessary and sufficient condition for an amplitude function $|H(j\omega)|$ to be causal (realizable) is that

$$\int_{-\infty}^{\infty} \frac{\left|\log |H(j\omega)|\right|}{1+\omega^2}\, d\omega < \infty \qquad ...(10.4)$$

For Paley-Wiener criterion to be satisfied, $h(t)$ must possess a Fourier transform $H(j\omega)$ and the factor $|H(j\omega)|^2$ must be integrable, *i.e.*,

$$\int_{-\infty}^{\infty} |H(j\omega)|^2\, d\omega < \infty \qquad ...(10.5)$$

The physical implication of Paley-Wiener criterion is that the amplitude $|H(j\omega)|$ of the realizable circuit must not be zero over a finite frequency band. The ideal low pass filter characteristic (Fig. 10.2) is not realizable because beyond ω_c the amplitude is zero. The Gaussian shaped curve (Fig. 10.3) has the characteristic.

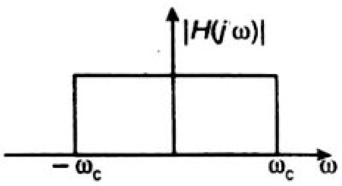


Fig. 10.2 Ideal filter characteristic

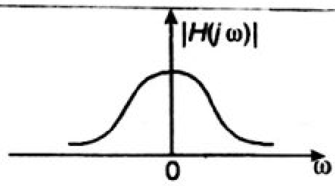


Fig. 10.3 Gaussian filter characteristic

$$|H(j\omega)|^2 = e^{-\omega^2} \qquad ...(10.6)$$

This characteristic is also not realizable because

$$\left|\log |H(j\omega)|\right| = \omega^2 \qquad ...(10.7)$$

so that the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\omega^2}{1+\omega^2}\, d\omega$ is not finite.

On the other hand the amplitude function

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}} \qquad ...(10.8)$$

represents a realizable circuit. Figure 10.4 shows an *RC* circuit whose voltage transfer ratio resembles Eq. (10.8).

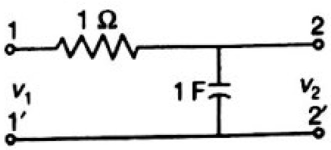


Fig. 10.4

10.2.2 Stability

A circuit is stable if a bounded input $e(t)$ causes a bounded response $r(t)$, *i.e.*,

If $\quad |e(t)| < K_1 \quad 0 \le t < \infty$

Then $\quad |r(t)| < K_2 \quad 0 \le t < \infty$

where K_1 and K_2 are real, positive and finite quantities.

If a linear system is stable, then from convolution theorem we get

$$|r(t)| < K_1 \int_0^\infty |h(\tau)|\, d\tau < K_2 \qquad ...(10.9)$$

Equation (10.9) requires that the impulse response be absolutely integrable or

$$\int_0^\infty |h(\tau)|\, d\tau < \infty \qquad ...(10.10)$$

If $h(t)$ is to be absolutely integrable, the impulse response should approach zero as t approaches infinity, *i.e.*,

$$\underset{t \to \infty}{Lt}\ h(t) = 0 \text{ for all values of } t \qquad ...(10.11)$$

Generally, it can be said that with the exception of isolated impulses, the impulse response must be bounded for all t, *i.e.*,

$$|h(t)| < K \text{ for all values of } t \qquad ...(10.12)$$

where K is a real, positive, finite number.

10.3 POSITIVE REAL FUNCTION

The class of driving point immittances (i.e., impedance or admittance) which can be synthesized by linear time invariant passive *RLC* elements is known as positive real functions. Such functions were first defined by *O* Brune in 1931 and are sometimes called Brune functions.

A driving point immittance can be expressed as a ratio of coefficients of polynomials, i.e.,

$$H(s) = \frac{P(s)}{Q(s)} = \frac{a_0}{b_0}\frac{(s-z_1)(s-z_2)......(s-z_n)}{(s-p_1)(s-p_2).......(s-p_m)} \qquad ...(10.13)$$

where z_1, z_2, etc., are the zeros, p_1, p_2, etc., are the poles and a_0/b_0 is the scale factor.

Theorem 1. $H(s)$ is a positive real function if and only if

(*i*) $H(s)$ is real when s is real

(*ii*) Re $[H(s)] \ge 0$ when Re $(s) \ge 0$.

Theorem 2. A practical and useful set of necessary and sufficient conditions for $H(s)$ to be positive real is

(*a*) $H(s)$ has no poles or zeros in the right half s plane, i.e., its poles and zeros can only be of the from $-\sigma$, $\pm j\omega$ and $-\sigma \pm j\omega$ where σ and ω real and positive.

(*b*) The poles of $H(s)$ on the imaginary axis are simple with real and positive residues.

(*c*) *Re* $[H(j\omega)] \ge 0$ for $0 \le \omega \le \infty$

The conditions of theorem 2 are much simpler to test than those of theorem 1. However, it is desirable to find a set of necessary conditions so that we can single out the functions which are not positive real.

Theorem 3. If $H(s)$ is a positive real function it must have the following properties:

1. All coefficients of numerator and denominator of $H(s)$ are real and positive.
2. All poles and zeros of $H(s)$ have either negative or zero parts.
3. Those poles of $H(s)$ *and* $1/H(s)$ which lie on the imaginary axis are simple and their residues are real and positive.
4. The poles and zeros are real or occur in conjugate pair.
5. The highest degrees of the numerator and denominator polynomials in $H(s)$ differ at the most by one.
6. The lowest degrees of the numerator and denominator polynomials in $H(s)$ differ at the most by one.
7. The numerator and denominator polynomials do not have missing terms between those of the highest and lowest degree unless all the odd or even terms are missing.
8. If $H(s)$ is positive real, $1/H(s)$ is also positive real.
9. The sum of positive real functions is positive real.

All the above properties have been discussed in Chapter 9. However, it is important to note that the above properties are only necessary conditions for a function to be positive real. If a function does not possess any of the above properties, it is not positive real. However, if it possesses the above properties, it is necessary to check if it satisfies condition (*c*) of theorem 2. It is positive real only if it possesses all the properties listed in theorem 3 and satisfies condition (*c*) of theorem 2.

Example 10.1. Test the following functions for positive realness.

(*a*) $\dfrac{12s^2+5}{2s^3+s}$ (*b*) $\dfrac{1}{s^2+1}$ (*c*) $\dfrac{s+4}{s^2+2s+5}$

(*d*) $\dfrac{s^2+7}{(s+1)^3}$ (*e*) $\dfrac{s+1}{s+5}$ (*f*) $\dfrac{s^3+5s}{s^4+2s^2+1}$

Solution. (*a*)
$$H(s) = \frac{12s^2+5}{2s^3+s}$$

This function possesses all the properties listed in Theorem 3. Therefore it is necessary to test if it satisfies condition (*c*) of Theorem 2.

$$Re\,[H(j\omega)] = Re\left[\frac{12(j\omega)^2+5}{2(j\omega)^3+j\omega}\right]$$
$$= Re\left[\frac{5-12\omega^2}{j\omega(1-2\omega^2)}\right] = 0 \text{ for all values of } \omega$$

Hence, this function is positive real.

(*b*)
$$H(s) = \frac{1}{s^2+1}$$

The degrees of numerator and denominator differ by 2. Hence, it is not positive real.

(c) $$H(s) = \frac{s+4}{s^2+2s+5}$$

This function possesses all the properties listed in Theorem 3. It is therefore necessary to test it for condition (c) of Theorem 2.

$$H(j\omega) = \frac{4+j\omega}{5-\omega^2+2j\omega}$$

$$Re\,[H(j\omega)] = Re\left[\frac{(4+j\omega)(5-\omega^2-2j\omega)}{(5-\omega^2+2j\omega)(5-\omega^2-2j\omega)}\right] = \frac{20-2\omega^2}{\omega^4-6\omega^2+25}$$

$Re\,[H(j\omega)]$ is negative for many values of ω (say for $\omega = 5$). Hence this function is not positive real.

(d) $$H(s) = \frac{s^2+7}{(s+1)^3}$$

It possesses all the properties listed in Theorem 3. It is therefore necessary to test it for condition (c) of Theorem 2.

$$Re\,[H(j\omega)] = Re\left[\frac{7-\omega^2}{(1+j\omega)^3}\right] = Re\left[\frac{7-\omega^2}{(1-3\omega^2)+j(3\omega-\omega^3)}\right]$$

$$= \frac{(7-\omega^2)(1-3\omega^2)}{(1-3\omega^2)^2+(3\omega-\omega^3)^2} = \frac{3\omega^4-22\omega^2+7}{(1-3\omega^2)^2+(3\omega-\omega^3)^2}$$

$Re\,[H(j\omega)]$ is negative for many values of ω (say for $\omega = 1$). Hence this function is not positive real.

(e) $$H(s) = \frac{s+1}{s+5}$$

It possesses all properties listed in Theorem 3. Therefore it is necessary to test it for condition (c) of Theorem 2.

$$H(j\omega) = \frac{1+j\omega}{5+j\omega} = \frac{(1+j\omega)(5-j\omega)}{(5+j\omega)(5-j\omega)}$$

$$Re\,[H(j\omega)] = Re\left[\frac{(1+j\omega)(5-j\omega)}{25+\omega^2}\right] = \frac{5+\omega^2}{25+\omega^2}$$

$Re\,[H(j\omega)]$ is positive for $0 < \omega < \infty$. Hence it is a positive real function.

(f) $$H(s) = \frac{s^3+5s}{s^4+2s^2+1} = \frac{2(s^2+5)}{(s^2+1)^2}$$

$H(s)$ has multiple poles on $j\omega$ axis. Hence it is not positive real.

10.4 HURWITZ POLYNOMIAL

A number of testing procedures are available to determine the sign of the real part of the roots of a polynomial without the necessity of finding the roots. One of these procedures is the concept of Hurwitz polynomial.

A polynomial $f(s)$ which may be the numerator or denominator of a network function, having all its zeros in the left half plane is known as a strictly Hurwitz polynomial. If its zeros are in the left half plane or simple on the imaginary axis, it is a Hurwitz polynomial.

A Hurwitz polynomial possesses the following properties :

1. All the coefficients are non-negative.
2. Both the even and odd parts of a Hurwitz polynomial have roots on the $j\omega$ axis.
3. The continued fraction expansion of the ratio of odd to even or even to odd parts of a Hurwitz polynomial yields all positive quotient terms.

The last of the above properties is a necessary and sufficient condition for a polynomial to be Hurwitz. The continued fraction expansion is obtained by synthetic division. For this purpose the polynomial $f(s)$ is written as

$$f(s) = m(s) + n(s)$$

where $m(s)$ and $n(s)$ are the even and odd parts of $f(s)$. Then

$$\frac{n(s)}{m(s)} = h_1 s + \cfrac{1}{h_2 s + \cfrac{1}{h_3 s + \cfrac{1}{h_4 s + \ldots}}} \qquad \ldots(10.14)$$

if h_1, h_2, h_3...... are all real and positive, $f(s)$ is a Hurwitz polynomial.

The process of synthetic division $n(s)/m(s)$ may terminate prematurely if $m(s)$ and $n(s)$ contain a common factor $W(s)$. In such a case $f(s)$ can be written as

$$f(s) = [W(s)]\,[f_1(s)] \qquad \ldots(10.15)$$

In such a case $f(s)$ is Hurwitz if both $W(s)$ and $f_1(s)$ are Hurwitz. The imaginary axis zeros of the function $f(s)$ are contributed by this common factor $W(s)$.

If $f(s)$ is an odd or even function, the continued fraction expansion is formed from the ratio of $f(s)$ to its derivative $f'(s)$.

Example 10.2. Test if the polynomial $s^3 + 6s^2 + 12s + 8$ is Hurwitz.

Solution.

$$n(s) = s^3 + 12s$$

$$m(s) = 6s^2 + 8$$

$$\frac{n(s)}{m(s)} = \frac{s^3 + 12s}{6s^2 + 8} = \frac{1}{6}s + \cfrac{1}{\cfrac{9}{16}s + \cfrac{1}{\cfrac{4}{3}s}}$$

$$6s^2 + 8 \,\overline{)\, s^3 + 12s \,}\, \left(\frac{1}{6}s \text{ first division}\right.$$
$$\underline{s^2 + 4s/3}$$

$$\frac{32}{3}s \,\overline{)\, 6s^2 + 8 \,}\, \left(\frac{9}{16}s \text{ second division}\right.$$
$$\underline{6s^2}$$

$$8 \,\overline{)\, \frac{32}{3}s \,}\, \left(\frac{4}{3}s \text{ third division}\right.$$
$$\frac{32}{3}s$$

Thus $h_1 = 1/6$, $h_2 = 9/16$ and $h_3 = 4/3$ which are all positive. Hence the polynomial is Hurwitz.

Example 10.3. Find the limits of K so that the polynomial $s^3 + 14s^2 + 56s + K$ may be Hurwitz.

Solution.

$$\frac{n(s)}{m(s)} = \frac{s^3+56s}{14s^2+K} = \frac{1}{14}s + \frac{\left(56-\frac{K}{14}\right)s}{14s^2+K}$$

$$= \frac{1}{14}s + \cfrac{1}{\left(\cfrac{14}{56-\frac{K}{14}}\right)s + \cfrac{1}{\cfrac{56-\frac{K}{14}}{K}s}}$$

If the function is Hurwitz

$$\frac{14}{56-\frac{K}{14}} > 0 \text{ and } \frac{56-\frac{K}{14}}{K} > 0$$

or $$56-\frac{K}{14} > 0 \quad \text{and} \quad K > 0$$

or $$\mathbf{0 < K < 784.}$$

Example 10.4. Test the polynomial $s^4 + 4s^3 + 8s^2 + 16s + 32$ and find out if all its roots have negative real parts.

Solution.

$$\frac{m(s)}{n(s)} = \frac{s^4+8s^2+32}{4s^3+16s}$$

$$= \frac{1}{4}s + \cfrac{1}{s + \cfrac{1}{-\frac{1}{4}s + \cfrac{1}{-\frac{1}{2}s}}}$$

Thus $h_1 = 1/4$, $h_2 = 1$, $h_3 = -1/4$ and $h_4 = 1/2$.

Since all these coefficients are not positive, all the roots do not have negative real parts.

Example 10.5. Find out if the polynomials $s^5 + 12s^4 + 45s^3 + 60s^2 + 44s + 48$ is Hurwitz.

Solution.

$$\frac{n(s)}{m(s)} = \frac{s^5+45s^3+44s}{12s^4+60s^2+48}$$

$$= \frac{1}{12}s + \cfrac{1}{\frac{3s}{10} + \cfrac{1}{[5s(s^2+1)]/[6(s^2+1)]}}$$

The expansion terminates prematurely due to cancellation of the common factor $(s^2 + 1)$. The common factor has a pair of roots on the imaginary axis, *i.e.*, at $\pm j\,1$. The coefficients in the continued expansion are positive. Hence the function is Hurwitz.

10.5 TESTING DRIVING POINT IMMITTANCES

We have discussed the necessary and sufficient conditions for a driving point immittance to be positive real in Section 10.3. It has been found that the use of Theorem 2 is the most convenient method to find out if a given function is positive real.

The first requirement is that $H(s)$ should not have any pole in the right half plane. If denominator of $H(s)$ is a Hurwitz polynomial, this requirement is satisfied. If $H(s)$ is positive real, $1/H(s)$ is also positive real. Therefore, it is preferable to check that the numerator of the function is also a Hurwitz polynomial.

The second condition requires the computation of residues at the poles on the $j\omega$ axis. These poles have already been identified in the form of the common factor $W(s)$ in the odd and even parts of $H(s)$ while testing for Hurwitz nature. The residues are found by the partial fraction expansion. At a pole $s = p_1$, the residue is given by

$$\text{Residue} = [H(s)][s+p_1]\big|_{s=p_1} \qquad ...(10.16)$$

The third condition is that $Re\,[H(j\omega)]$ is positive or zero for all values of ω. To find $Re\,[H(j\omega)]$ we separate the odd and even parts of $H(s)$.

$$H(s) = \frac{P(s)}{Q(s)} = \frac{M_1(s)+N_1(s)}{M_2(s)+N_2(s)} \qquad ...(10.17)$$

where $M_1(s)$ and $N_1(s)$ are the even and odd parts of $P(s)$ and $M_2(s)$ and $N_2(s)$ are the even and odd parts of $Q(s)$.

$$H(s) = \frac{[M_1(s)+N_1(s)][M_2(s)-N_2(s)]}{[M_2(s)+N_2(s)][M_2(s)-N_2(s)]}$$

$$= \frac{[M_1(s)\,M_2(s)-N_1(s)N_2(s)]+[M_2(s)\,N_1(s)-M_1(s)\,N_2(s)]}{[M_2(s)]^2-[N_2(s)]^2} \qquad ...(10.18)$$

The product of two even or two odd functions is an even function whereas the product of an even and odd function is an odd function. Therefore,

$$\text{Even part of } H(s) = \frac{M_1(s)\,M_2(s)-N_1(s)\,N_2(s)}{[M_2(s)]^2-[N_2(s)]^2} \qquad ...(10.19)$$

$$\text{Odd part of } H(s) = \frac{M_2(s)\,N_1(s)-M_1(s)\,N_2(s)}{[M_2(s)]^2-[N_2(s)]^2} \qquad ...(10.20)$$

when we substitute $s = j\omega$, the even part of $H(s)$ gives a real number while the odd part of $H(s)$ gives an imaginary number.

Therefore,

$$\text{Even}\,[H(s)]\big|_{s=j\omega} = Re(H(j\omega)] \qquad ...(10.21)$$

$$\text{Odd}\,[H(s)]\big|_{s=j\omega} = j\,I_m[H(j\omega)] \qquad ...(10.22)$$

when $s = j\omega$, the denominator of Eq. (10.19) is always positive. Therefore it is sufficient to test the numerator of Eq. (10.19). The numerator is

$$M_1(j\omega)\,M_2(j\omega) - N_1(j\omega)N_2(j\omega) = A(\omega) \qquad ...(10.23)$$

To check if $A(\omega)$ is positive, it is necessary to factorize it and finds its roots. The criterion for $A(\omega) \geq 0$ is that $A(\omega)$ has no positive real roots of odd multiplicity.

Example 10.6. Show that the function

$$H(s) = H\frac{s^2 + a_1 s + a_0}{s^2 + b_1 s + b_0} \text{ is positive real if}$$

$$a_1 b_1 \geq \left(\sqrt{a_0} - \sqrt{b_0}\right)^2$$

Solution. If b_1 and b_0 are positive, the denominator is Hurwitz and there are no poles on the $j\omega$ axis. To check if $Re[H(j\omega) \geq 0$ we find the even part of $H(s)$.

$$\text{Even } [H(s)] = \frac{(s^2 + a_0)(s^2 + b_0) - a_1 b_1 s^2}{(s^2 + b_0)^2 - b_1^2 s^2}$$

$$= \frac{s^4 + [(a_0 + b_0) - a_1 b_1]s^2 + a_0 b_0}{(s^2 + b_0)^2 - b_1^2 s^2}$$

$$Re\,[H(j\omega)] = \frac{\omega^4 - [(a_0 + b_0) - a_1 b_1]\omega^2 + a_0 b_0}{(b_0 - \omega^2)^2 + b_1^2 \omega^2}$$

The denominator of above expression is always positive. The roots of the numerator are

$$\omega_{1,2}^2 = \frac{(a_0 + b_0) - a_1 b_1}{2} \pm \frac{1}{2}\sqrt{(a_0 + b_0) - a_1 b_1]^2 - 4a_0 b_0}$$

There are two situations when $Re\,[H(j\omega)]$ does not have simple real roots.

(*a*) When the quantity under radical sign in the above equation is zero (giving double real roots) or negative (giving complex roots) i.e.,

$$[(a_0 + b_0) - a_1 b_1]^2 - 4a_0 b_0 \leq 0$$

or $$[(a_0 + b_0) - a_1 b_1]^2 \leq 4a_0 b_0$$

If $$(a_0 + b_0) - a_1 b_1 \geq 0$$

Then $$(a_0 + b_0) - a_1 b_1 \leq 2\sqrt{a_0 b_0}$$

or $$a_1 b_1 \geq \left(\sqrt{a_0} - \sqrt{b_0}\right)^2$$

If $$(a_0 + b_0) - a_1 b_1 \leq 0$$

Then $$a_1 b_1 - (a_0 + b_0) \leq 2\sqrt{a_0 b_0}$$

or $$(a_0 + b_0) - a_1 b_1 < 0 < a_1 b_1 - (a_0 + b_0)$$

or $$a_1 b_1 \geq \left(\sqrt{a_0} - \sqrt{b_0}\right)^2$$

(*b*) When $\omega_{1,2}^2$ is negative, the roots are imaginary and $Re[H(j\omega)]$ will not have simple real roots. This happens

when $$[(a_0 + b_0) - a_1 b_1]^2 - 4a_0 b_0 > 0$$

and $$(a_0 + b_0) - a_1 b_1 < 0.$$

The above equations give

$$a_1b_1 - (a_0 + b_0) > 2\sqrt{a_0b_0} > (a_0 + b_0) - a_1b_1$$

or
$$a_1b_1 \geq \left(\sqrt{a_0} - \sqrt{b_0}\right)^2$$

Hence the function $H(s)$ is positive real when

$$a_1b_1 \geq \left(\sqrt{a_0} - \sqrt{b_0}\right)^2$$

Example 10.7. Identify the functions which cannot be realized as driving point immittances of ordinary finite lumped linear bilateral *RLCM* circuits giving one reason for each of your identification.

(*i*) $s^2 + s + 1$ (*ii*) $1/(s+j)(s+2j)$ (*iii*) $\dfrac{s^3+2s^2+3}{s^3+2s^2+2s+1}$

(*iv*) $\dfrac{s^3+2s^2+2s+1}{s^3-3s^2+2s+2}$ (*v*) $\dfrac{s^2+s}{s^3+2s^2+2}$ (*vi*) $s - 1$

(*vii*) $s^{0.5}$

Solution. (*i*) $$H(s) = \frac{s^2+s+1}{s^0}$$

The difference between highest degrees of numerator and denominator is more than 1. Hence it cannot be realized as a passive network

(*ii*) $$H(s) = \frac{1}{(s+j)(s+2j)}$$

The poles of $H(s)$ are at $s = -j$ and $s = 2j$. Since the imaginary axis poles are not in conjugate pair, the function cannot realized.

(*iii*) $$H(s) = \frac{s^3+2s^2+3}{s^3+2s^2+2s+1}$$

In the numerator the term s is missing. Hence it cannot be realized.

(*iv*) $$H(s) = \frac{s^3+2s^2+2s+1}{s^3-3s^2+2s+2}$$

The coefficient of one term in the denominator is negative. Hence it cannot be realized.

(*v*) $$H(s) = \frac{s^2+s}{s^3+2s^2+2}$$

The term s is missing in the denominator. Hence it cannot be realized.

(*vi*) $$H(s) = s - 1$$

The coefficient of one term is negative. Hence it cannot be realized.

(*vii*) $$H(s) = s^{1/2}$$

The degree of s is fractional. Hence it cannot be realized.

Example 10.8. Test for positive reality the function.

$$H(s) = \frac{s^4 + 2s^3 + 3s^2 + s + 1}{s^4 + s^3 + 3s^2 + 2s + 1}$$

Solution. For the numerator

$$\text{Even part} = M_1(s) = s^4 + 3s^2 + 1$$

$$\text{Odd part} = N_1(s) = 2s^3 + s$$

The continued fraction expansion is

$$2s^3 + s \,\big)\, s^4 + 3s^2 + 1 \,\big(\, \frac{1}{2}s$$
$$s^4 + \frac{1}{2}s^2$$
$$\overline{\frac{5}{2}s^2 + 1 \,\big)\, 2s^3 + s \,\big(\, \frac{4}{5}s}$$
$$2s^3 + \frac{4}{5}s$$
$$\overline{\frac{s}{5} \,\big)\, \frac{5}{2}s^2 + 1 \,\big(\, \frac{25}{2}s}$$
$$\frac{5}{2}s^2$$
$$\overline{1 \,\big)\, \frac{s}{5} \,\big(\, \frac{s}{5}}$$
$$\frac{s}{5}$$
$$\overline{\times}$$

Since all coefficients are positive and real, the numerator is a Hurwitz polynomial.

For the denominator

$$\text{Even part} = M_2(s) = s^4 + 3s^2 + 1$$

$$\text{Odd part} = N_2(s) = s^3 + 2s$$

The continued fraction expansion is

$$s^3 + 2s \,\big)\, s^4 + 3s^2 + 1 \,\big(\, s$$
$$s^4 + 2s^2$$
$$\overline{s^2 + 1 \,\big)\, s^3 + 2s \,\big(\, s}$$
$$s^3 + s$$
$$\overline{s \,\big)\, s^2 + 1 \,\big(\, s}$$
$$s^2$$
$$\overline{1 \,\big)\, s \,\big(\, s}$$
$$s$$
$$\overline{\times}$$

All the coefficients are positive and real. Therefore, the denominator is also a Hurwitz polynomial. Moreover the continued fraction expansion of denominator has not terminated prematurely. Therefore the function $H(s)$ does not have poles on the imaginary axis.

The third condition for positive real quality is that $Re\ H(j\omega)] \geq 0$ for all value of ω.

$$\text{Even part of } H(s) = \frac{M_1(s)M_2(s) - N_1(s)N_2(s)}{[M_2(s)]^2 - [N_2(s)]^2}$$

When we substitute $s = j\omega$, the denominator of above expression is always positive. Then $M_1(s)\,M_2(s) - N_1(s)\,N_2(s) = (s^4 + 3s^2 + 1)\,(s^4 + 3s^2 + 1) - (2s^3 + s)\,(s^3 + 2s)$

Substituting $s = j\omega$, the expression becomes

$$(\omega^4 - 3\omega^2 + 1)^2 - (j\omega)\,(1 - 2\,\omega^2)\,(j\omega)\,(2 - \omega^2)$$
$$= \omega^8 - 4\omega^6 + 6\omega^4 - 4\omega^2 + 1$$

A little exercise reveals that $\omega^8 - 4\omega^6 + 6\omega^4 - 4\omega^2 + 1 = (\omega - 1)^4\,(\omega + 1)^4$. Evidently this expression is positive for $\omega \geq 0$.

Hence $H(s)$ is a positive real function.

10.6 ELEMENTARY SYNTHESIS PROCEDURES

The circuit synthesis problem is solved by breaking up the given driving point immittance into simpler positive real functions. Let a driving point impedance $Z(s)$ be broken up into two impedances $Z_1\,(s)$ and $Z_2\,(s)$, such that

$$Z(s) = Z_1\,(s) + Z_2\,(s) \qquad ...(10.24)$$

or

$$Z_2\,(s) = Z(s) - Z_1\,(s) \qquad ...(10.25)$$

Then $Z_1\,(s)$ is said to have been removed from $Z(s)$. In this process it is important that $Z_1(s)$ and $Z_2(s)$ should be both positive real. There are four important removal operations.

Removal of a pole at infinity. If the numerator of $Z(s)$ is one degree higher than the denominator, then $Z(s)$ tends to infinity as s tends to infinity. In this case, $Z(s)$ is said to have a pole at infinity. Let

$$Z(s) = \frac{a_{n+1}s^{n+1} + a_n s^n + ... + a_0}{b_n s^n + b_{n-1}s^{n-1} + ...b_0} \qquad ...(10.26)$$

If we divide the numerator by the denominator, we have

$$Z(s) = sL_1 + \frac{a_n' s^n + a_{n-1}' s^{n-1} + ... + a_0'}{b_n s^n + b_{n-1}s^{n-1} + ...b_0} = sL_1 + Z_1(s) \qquad ...(10.27)$$

where

$$L_1 = (a_n + 1)/b_n.$$

It is seen that sL_1 is the driving point impedance of an inductor with inductance L_1. Thus $Z(s)$ can be considered as the driving point impedance of an inductor L_1 in series with another impedance $Z_1(s)$ as shown in Fig. 10.5(a).

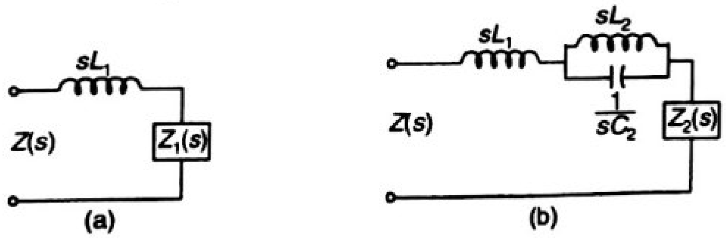


Fig. 10.5 (a) Removal of pole at infinity from $Z(s)$ leaving the remaining function positive real (b) Removal of a pair of imaginary axis poles from $Z(s)$ leaving the remaining function positive real

It is easy to see that $Z_1(s)$ is positive real function. Applying the conditions of Theorem 2 (Section 10.3) we observe that

1. The poles of $Z_1(s)$ are the same as the poles of $Z(s)$. Since $Z(s)$ does not have any pole in right half plane, $Z_1(s)$ does not have any pole in right half plane.
2. The removal of a pole at $s = \infty$, does note affect the poles on the $j\omega$ axis and the residus at these poles.
3. $$Re[Z(j\omega)] = Re[j\omega L_1] + Re[Z_1(j\omega)]$$

Since $$Re[j\omega L_1] = 0 \quad \text{we get}$$

$$Re[Z_1(j\omega)] = Re[Z(j\omega)] \geq 0 \text{ for } 0 \leq \omega \leq \infty$$

Thus $Z_1(s)$ is a positive real function. $Z_1(s)$ can be broke up further by removing the poles on the $j\omega$ axis.

Removal of poles on jω axis If $Z(s)$ contains a pair of poles on the $j\omega$ axis (say at $\omega = \pm j\omega_1$), then using partial fraction expansion $Z_1(s)$ can be written as

$$Z_1(s) = \frac{K_1}{s+j\omega_1} + \frac{K_2}{s-j\omega_1} + Z_2 s \qquad ...(10.28)$$

where K_1 and K_2 are the residues at the poles $s = -j\omega_1$ and $s = j\omega_1$ respectively. Since $Z_1(s)$ is positive real, the residues at both the imaginary axis poles are real and equal or $K_1 = K_2$ and Eq. (10.28) can be rewritten as

$$Z_1(s) = \frac{2K_1 s}{s^2+\omega_1^2} + Z_2(s) \qquad ...(10.29)$$

The first term is the driving point impedance of a parallel combination of an inductor L_2 and capacitor C_2 where

$$L_2 = \frac{2K_1}{\omega_1^2} \quad \text{and} \quad C_2 = \frac{1}{2K_1}$$

This is illustrated in Fig. 10.5(*b*). By the same method as used in the previous case it can be shown that $Z_2(s)$ is positive real.

Removal of pole at origin If a driving point immittance (say Z_s) has a pole at origin, it can be written as

$$Z(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0}{s(b_n s^n + b_{n-1}s^{n-1} + ... b_1 s + b_0)} \qquad ...(10.30)$$

Using partial fraction expansion, Eq. (10.30) can be written as

$$Z(s) = \frac{1}{sC_1} + Z_1(s) \qquad ...(10.31)$$

where $$C_1 = b_0/a_0$$

and $$Z_1(s) = \frac{a_n' s^n + a_{n-1}' s^{n-1} + ... + a_1' s + a_0'}{b_n s^n + b_{n-1}s^{n-1} + ... b_1 s + b_0} \qquad ...(10.32)$$

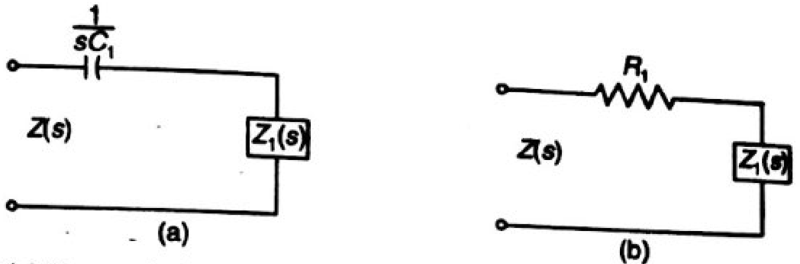


Fig. 10.6 (*a*) Removal of a pole at origin from $Z(s)$ (*b*) Removal of a constant from $Z(s)$

The function $Z_1(s)$ is also positive real. Fig. 10.6 (*a*) illustrates this process.

Removal of a constant. If a real number R_1 is subtracted from $Z(s)$ we have

$$Z_1(s) = Z(s) - R_1 \quad \text{...(10.33)}$$

or

$$Z(s) = R_1 + Z_1(s) \quad \text{...(10.34)}$$

where R_1 is a real number which can be realized as a resistor having resistance R_1 and $Z_1(s)$ is a positive real function. R_1 must be chosen so that $Z_1(s)$ is a positive real function. This requirements impose the following constrain on the value of resistance R_1.

$$R_1 \le [\min (\text{Re } Z(j\omega)] \quad \text{...(10.35)}$$

i.e., R_1 must be less than or equal to the minimum value of the real part of $Z(j\omega)$. This minimum value may occur at any value of ω in the range from zero to infinity. The process of removing a constant from function $Z(s)$ is illustrated in Fig. 10.6(*b*).

The operations discussed above can be applied repeatedly till the given function is broken up into simpler function which can be synthesized individually.

10.7 PROPERTIES OF LC IMMITTANCES

A network composed of inductances and capacitances only is known as an *LC* or reactance network. The driving point immittance of such a network is known as a reactance function. A rational function $H(s)$ is a reactance function if and only if

(*i*) All the poles of $H(s)$ are simple and lie on the $j\omega$ axis with real and positive residues.

(*ii*) *Re* $[H(j\omega)] = 0$ for $\omega \ge 0$.

From the above conditions it is easy to see that a reactance function can be written either as

$$H(s) = K\frac{(s^2+\omega_1^2)(s^2+\omega_3^2)......}{s(s^2+\omega_2^2)(s^2+\omega_4^2)......} \quad \text{...(10.36)}$$

or as

$$H(s) = K\frac{s(s^2+\omega_3^2)(s^2+\omega_5^2)......}{(s^2+\omega_2^2)(s^2+\omega_4^2)......} \quad \text{...(10.37)}$$

A reactance function has the following properties :

1. All the poles and zeros are simple and lie on the $j\omega$ axis only.
2. It has pole or zero at origin.
3. It has a pole or zero at infinity.
4. Its poles and zeros alternate on the $j\omega$ axis, i.e., in Eq. (10.36) we must have $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4$ ...... and in Eq. (10.37) we must have $0 < \omega_2 < \omega_3 < \omega_4 < \omega_5$ ...... This is known as interlacing property of a reactance function.

5. It is always a quotient of even to odd, or odd to even polynomials. If the odd polynomial is the numerator, then $H(s)$ has a pole at $s = 0$.
zero
6. $Re\,[H\,(j\omega)] = 0$ for $\omega \geq 0$.
7. Since all poles are on the $j\omega$ axis, the residues at all the poles are real and positive.
8. The slope of $H(j\omega)/d\omega$ is always positive.

Consider the following driving point impedance.

$$Z(s) = \frac{Ks(s^2+\omega_3^2)}{(s^2+\omega_2^2)(s^2+\omega_4^2)} \qquad \text{...(10.38)}$$

Substituting $s = j\omega$ we have

$$Z(j\omega) = jX(\omega) = +j\frac{K\omega(-\omega^2+\omega_3^2)}{(-\omega^2+\omega_2^2)(-\omega^2+\omega_4^2)} \qquad \text{...(10.39)}$$

Figure 10.7 shows the variation of $X(\omega)$ as ω varies from 0 to ∞. At $\omega = 0$, $X(\omega) = 0$. As ω increases $X(\omega)$ increases. Since $dX/d\omega$ is always positive, the next critical frequency is at ω_2 when $X(\omega)$ becomes infinitely large. As we pass through ω_2, $X(\omega)$ changes sign from positive to negative but the slope always remains positive. As ω increases, $X(\omega)$ continues to increase. At $\omega = \omega_3$, $X(\omega)$ changes sign from negative to positive. The next critical frequency is the pole ω_4 when the function changes sign from positive to negative. Finally, at $\omega = \infty$, the function again becomes zero.

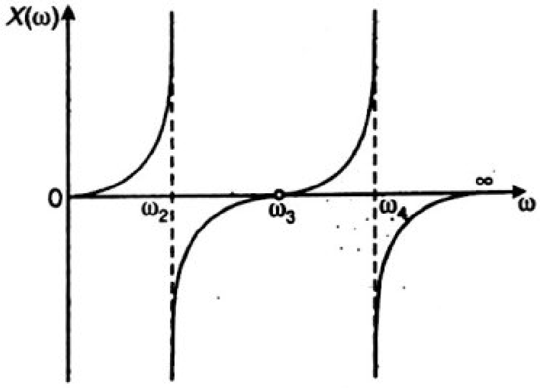


Fig. 10.7 Variation of $X(\omega)$ for a reactance function

Example 10.9. Which of the following are LC immittances? Discuss the reasons in each case.

$(a)\ \dfrac{s(s^2+1)(s^2+9)}{(s^2+4)(s^2+16)} \quad (b)\ \dfrac{(s^2+1)(s^2+9)}{s(s^2+4)} \quad (c)\ \dfrac{s(s^2+9)}{(s^2+1)(s^2+16)} \quad (d)\ \dfrac{s^4+3s^2+2}{s^3+3s}$

Solution. (*a*)
$$H(s) = \frac{s(s^2+1)(s^2+9)}{(s^2+4)(s^2+16)}$$

This function has poles at $s = \pm j2$ and $s = \pm 4$. The residue at $s = j2$ is

$$\text{Residue} = \frac{(-j2)(1-4)(9-4)}{(-j4)(16-4)} = \frac{-15}{24}$$

Since the residue at this pole is not positive real, $H(s)$ is not an LC immittance.

(b)
$$H(s) = \frac{(s^2+1)(s^2+9)}{s(s^2+4)}$$

This function has poles at $s = 0$ and $s = \pm j2$. Thus all the poles are simple and lie on $j\omega$ axis.

$$(\text{Residue at } s = 0) = \frac{1\times 9}{4} = \frac{9}{4}$$

$$(\text{Residue at } s = +j2) = \frac{(1-4)(9-4)}{(j2)(j4)} = \frac{15}{8}$$

$$(\text{Residue at } s = -j2) = \frac{(1-4)(9-4)}{(-j2)(-j4)} = \frac{15}{8}$$

Thus the residues at all the poles are positive and real.

$$H(j\omega) = \frac{(1-\omega^2)(9-\omega^2)}{j\omega(4-\omega^2)} = \frac{-j(1-\omega^2)(9-\omega^2)}{\omega(4-\omega^2)}$$

$$Re\,[H(j\omega)] = 0 \text{ for } \omega \geq 0$$

Hence it is an *LC* immittance

(c)
$$H(s) = \frac{s(s^2+9)}{(s^2+1)(s^2+16)}$$

This function has poles at $s = \pm j$ and $s = \pm j4$ which are simple and lie on the $j\omega$ axis.

$$(\text{Residue at } s = +j) = \frac{j(-1+9)}{2j(16-1)} = \frac{4}{15} = (\text{Residue at } s = -j)$$

$$(\text{Residue at } s = +j4) = \frac{j4(-16+9)}{(-16+1)(j8)} = \frac{7}{30} = (\text{Residue at } s = -4j)$$

Thus residues at all poles are real and positive.

$$H(j\omega) = \frac{j\omega(9-\omega^2)}{(1-\omega^2)(16-\omega^2)}$$

$$Re\,[H(j\omega)] = 0 \text{ for } \omega \geq 0$$

Hence it is an *LC* immittance.

(d)
$$H(s) = \frac{s^4+3s^2+2}{s^3+3s} = \frac{s^4+3s+2}{s(s^2+3)}$$

The poles are at $s = 0$ and $s = \pm j\sqrt{3}$

$$(\text{Residue at } s = \pm j\sqrt{3}) = \frac{9-9+2}{(j\sqrt{3})(2j\sqrt{3})} = -\frac{1}{3}$$

Since this residue is not positive real $H(s)$ is not an *LC* immittance.

10.7.1 Foster's Synthesis of LC Driving Point Immittances

Let $H(s)$ be the driving point impedance $Z(s)$. Using partial fraction expansion, Eq. (10.36) or (10.37) can be written as

$$Z(s) = K_\infty s + \frac{K_0}{s} + \frac{2K_2 s}{s^2+\omega_2^2} + \frac{2K_4 s}{s^2+\omega_4^2} + \ldots + \frac{2K_m s}{s^2+\omega_m^2} \quad \ldots(10.40)$$

where

$$K_\infty = \left.\frac{Z(s)}{s}\right|_{s\to\infty}$$

$$K_0 = sZ(s)|_{s=0}$$

$$K_2 = (s+j\omega_2)Z(s)|_{s=-j\omega_2}$$

$$= \left.\frac{(s^2+j\omega_2^2)\,Z(s)}{2s}\right|_{s^2=-\omega_2^2}$$

$$K_4 = (s+j\omega_4)\,Z(s)|_{s=-j\omega_4} = \left.\frac{(s^2+\omega_4^2)\,Z(s)}{2s}\right|_{s^2=-\omega_4^2}$$

and so on.

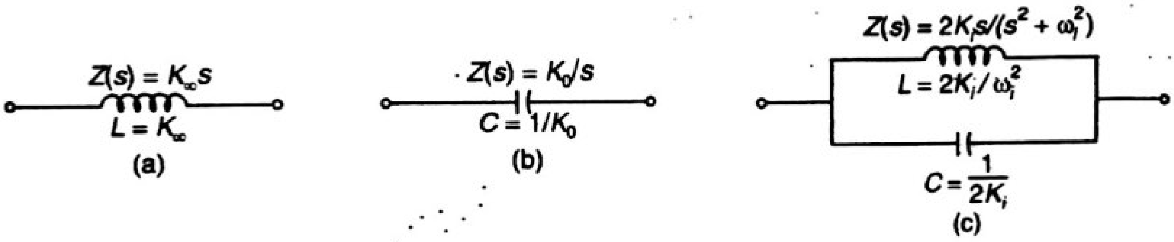


Fig. 10.8

All the constants in Eq. (10.40) are real and positive. The first term is the impedance of an inductor with inductance K_∞ Henrys (Fig. 10.8*a*). The second term is the impedance of a capacitor with capacitance $1/K_0$ Farads (Fig. 10.8*b*). Each of the subsequent terms is the impedance of a parallel tank circuit of a capacitor with capacitance $1/2K_i$ farads and an inductor with inductance $2K_i/\omega_i^2$ Henrys where i = 2, 4, 6 ...... m (Fig. 18.8*c*). The realization of the final network is obtained by a series connection of these impedances. The resulting network is known as first Foster form (or Foster series circuit) and is shown in Fig. 10.9.

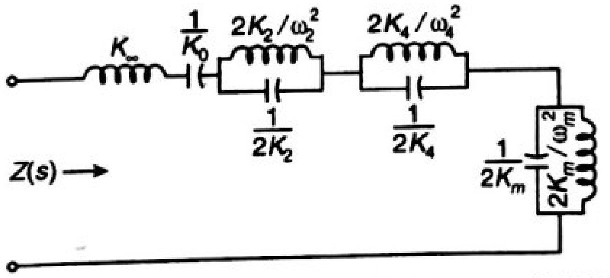


Fig. 10.9 First Foster realization of a reactance function. K_∞ is present if $Z(s)$ has a pole at infinity and $\frac{1}{K_0}$ is present if $Z(s)$ has a pole at origin.

To obtain second Foster form, we invert $Z(s)$ to give $Y(s)$. Evidently $Y(s)$ is also a reactance function and the expansion can be written as

$$Y(s) = K'_\infty s + \frac{K'_0}{s} + \frac{2K'_2 s}{s^2+\omega_2^2} + \frac{2K'_4 s}{s^2+\omega_4^2} + \ldots + \frac{2K'_m s}{s^2+\omega_m^2} \quad \ldots(10.41)$$

The constants in Eq. (10.41) are also real and positive. The first term is the admittance of a capacitor with capacitance K'_∞ Farads (Fig. 10.10*a*). The second term is the admittance of an inductor with inductance $1K'_0$ Henrys (Fig. 10.10*b*). Each subsequent term represents the series connection of an inductor with admittance $1/2K'_i$ Henrys and capacitor with capacitance $2K'_i/\omega_i^2$ Farads where $i = 2, 4, 6 \ldots\ldots m$ (Fig. 10.10*c*). The final network obtained by a parallel connection of these admittances is known as second Foster form (or Foster parallel circuit) and is shown in Fig. 10.11.

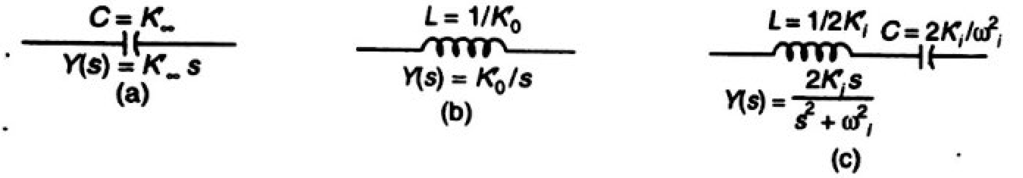


Fig. 10.10

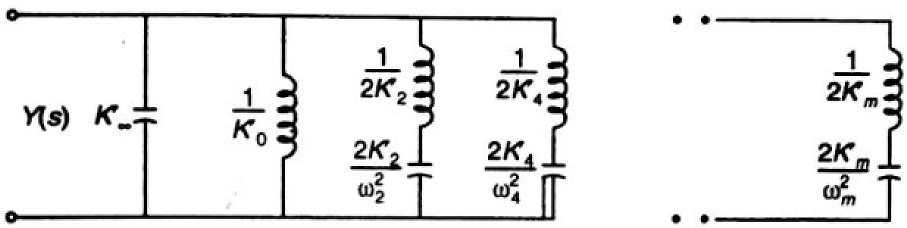


Fig. 10.11 Second Foster realization of a reactance function. K_∞ is present if $Z(s)$ has a zero at infinity and $1/k'_0$ is present if $Z(s)$ has a zero at origin

10.7.2 Cauer's Synthesis Method for LC Networks

Another method of realizing a reactance function is by removing poles at zero and at infinity, and realizing them as inductors or capacitors. Assume that the function of Eq. (10.37) describes an impedance $Z(s)$. In the equation, the numerator polynomial is one degree higher than the denominator polynomial, *i.e.*, $Z(s)$ has a pole at infinity. We can write $Z(s)$ as

$$Z(s) = s L_1 + Z_1(s) \quad \ldots(10.42)$$

$Z_1(s)$ is another reactance function whose denominator polynomial is one degree higher than its numerator polynomial. $Z_1(s)$ can be inverted and written as

$$\frac{1}{Z_1(s)} = C_2 S + Y_2(s) \quad \ldots(10.43)$$

where $Y_2(s)$ is a reactance function with a pole at ∞. This process is continued till the remainder is K_i/s or $k_i s$. In this way $Z(s)$ can be written as

$$Z(s) = L_1 s + \cfrac{1}{C_2 s + \cfrac{1}{L_3 s + \cfrac{1}{C_4 s + \ldots\ldots}}} \quad \ldots(10.44)$$

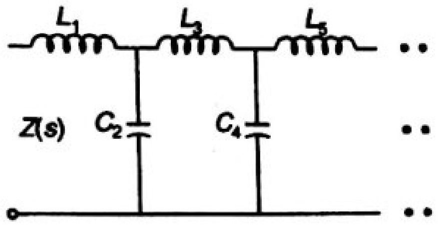


Fig. 10.12 First Cauer realization of a reactance function

It is seen that in Eq. (10.44) $L_1 s$ is the impedance of an inductor with inductance L_1 Henrys, $C_2\, s$ is the admittance of a capacitor with capacitance C_2 Farads and so on. The resulting network, known as first Cauer's form is shown in Fig. 10.12. It is ladder network with inductors in series arms and capacitors in shunt arms.

In the second Cauer's from successive poles are removed from origin. For this purpose the numerator and denominator are first arranged in ascending order in s, *i.e.*,

$$Z(s) = \frac{a_0 + a_1\, s + a_2\, s^2 + ...+ a_n\, s^n}{b_1\, s + b_2\, s^2 + ...+ b_{n+1} s^{n+1}} \qquad ...(10.45)$$

We divide the numerator by the denominator, invert the remainder and divide its numerator by the denominator again and continue. The resulting expansion takes the form

$$Z(s) = \frac{1}{C_1 s} + \cfrac{1}{1/L_2 s + \cfrac{1}{1/C_3 s + \cfrac{1}{1/L_4 s} + ...}} \qquad ...(10.46)$$

The network corresponding to Eq. (10.46) is shown in Fig. 10.13.

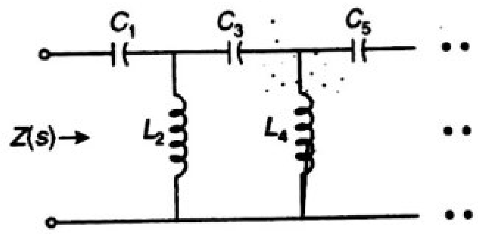


Fig. 10.13 Second Cauer realization of a reactance function

In all the four networks, the number of elements is the same and is equal to the highest power of s in $H(s)$. It is not possible to realize an LC circuit with lesser number of elements and as such these forms are known as cononic forms.

Example 10.10. (*a*) Test if $Z(s) = \dfrac{s^2+1}{s(s^2+2)}$ is a reactance function

(*b*) If yes, obtain both the Foster realizations of this function.

Solution. (*a*) The poles of $Z(s)$ are at $s = 0$ and $s = \pm j\sqrt{2}$

(Residue at $s = 0$) $= \dfrac{1}{2}$

(Residue at $s = +j\sqrt{2}$) $= \dfrac{1}{4}$ = Residue at $s = -j\sqrt{2}$

$$Re\,[Z\,(j\omega)] = \text{Re}\left[\frac{1-\omega^2}{j\omega(2-\omega^2)}\right] = \text{Re}\left[\frac{-j(1-\omega^2)}{\omega(2-\omega^2)}\right] = 0$$

All the poles are the $j\omega$ axis with real and positive residues and $Re\,[Z(j\omega)] = 0$ for $\omega \geq 0$. Therefore it is a reactance function.

(*b*) First Foster form:

Expanding $Z(s)$ by partial fraction expansion

$$Z(s) = K_\infty s + \frac{K_0}{s} + \frac{2K_2 s}{s^2+2}$$

$$K_\infty = \left.\frac{Z(s)}{s}\right|_{s\to\infty} = \left.\frac{s^2+1}{s^2(s^2+2)}\right|_{s\to\infty} = 0$$

Similarly, $K_0 = \frac{1}{2}$ and $K_2 = \frac{1}{4}$

Therefore, $Z(s) = \frac{1/2}{s} + \frac{(1/2)s}{s^2+2}$

The resulting first forster form is shown in Fig. 10.14(*a*).

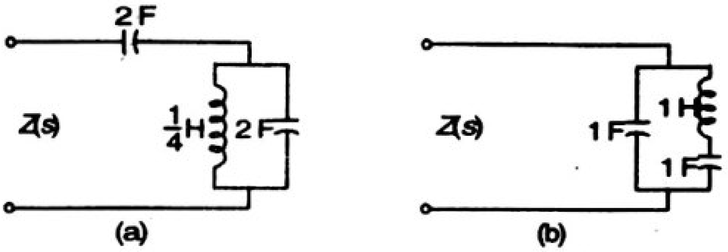


Fig. 10.14

Second Foster form:

To get second Foster form we inverts $Z(s)$ and then expand it. Thus,

$$Y(s) = \frac{1}{Z(s)} = \frac{s(s^2+2)}{s^2+1}$$

$$= K'_\infty\, s + \frac{K'_0}{s} + \frac{2K'_2 s}{s^2+1}$$

$$K'_\infty = \left.\frac{Y(s)}{s}\right|_{s\to\infty} = \left.\frac{s^2+2}{s^2+1}\right|_{s\to\infty}$$

$$= \left.\frac{1+2/s^2}{1+1/s^2}\right|_{s\to\infty} = 1$$

$$K'_0 = 0$$

$$K'_2 = \left.\frac{(s^2+1)Y(s)}{2s}\right|_{s^2=-1} = \left.\frac{s^2+2}{2}\right|_{s^2=-1} = \frac{1}{2}$$

Therefore, $$Y(s) = s + \frac{s}{s^2+1}$$

The resulting second Foster from is shown in Fig. 10.14(b).

Example 10.11. Obtain all the Foster and Cauer realizations of the driving point impedance.

[UPTU 2007]

$$Z(s) = \frac{s(s^2+4)}{(s^2+1)(s^2+9)}$$

Solution. It is seen that $Z(s)$ is a reactance function.

(a) First Foster realization

$$Z(s) = K_\infty s + \frac{K_0}{s} + \frac{2K_2 s}{s^2+1} + \frac{2K_4 s}{s^2+9}$$

$$K_\infty = \left.\frac{Z(s)}{s}\right|_{s\to\infty} = 0$$

$$K_0 = s\,Z(s)\big|_{s=0} = 0$$

$$K_2 = \left.\frac{(s^2+1)\,[Z(s)]}{2s}\right|_{s^2=-1} = \frac{3}{16}$$

$$K_4 = \left.\frac{(s^2+9)[Z(s)]}{2s}\right|_{s^2=-9} = \frac{5}{16}$$

The resulting network is shown in Fig. 10.15 (a).

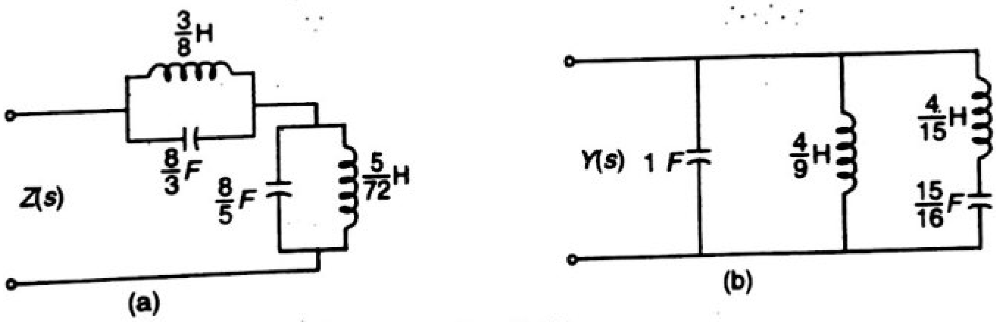


Fig. 10.15

(b) Second Foster realization

$$Y(s) = \frac{1}{Z(s)} = \frac{(s^2+1)(s^2+9)}{s(s^2+4)}$$

$$Y(s) = K'_\infty s + \frac{K'_0}{s} + \frac{2K'_2 s}{s^2+4}$$

or $$K'_\infty = \left.\frac{Y(s)}{s}\right|_{s\to\infty} = \left.\frac{(s^2+1)(s^2+9)}{s^2(s^2+4)}\right|_{s\to\infty}$$

$$K'_\infty = \left.\frac{s^4\left(1+\dfrac{1}{s^2}\right)\left(1+\dfrac{9}{s^2}\right)}{s^4\left(1+\dfrac{4}{s^2}\right)}\right|_{s\to\infty} = 1$$

$$K'_0 = sY(s)\big|_{s=0} = \frac{9}{4}$$

$$K'_2 = \left.\frac{(s^2+4)}{2s}Y(s)\right|_{s^2=-4} = \left.\frac{(s^2+1)(s^2+9)}{2s^2}\right|_{s^2=-4} = \frac{15}{8}$$

Therefore $$Y(s) = s + \frac{9}{4s} + \frac{(15/4)s}{s^2+4}$$

The resulting network is shown in Fig. 10.15(*b*).

(*c*) First Cauer form

$$Z(s) = \frac{s(s^2+4)}{(s^2+1)(s^2+9)} = \frac{1}{(s^4+10s^2+9)/(s^3+4s)}$$

The continued fraction expansion is

$$\begin{array}{l}
s^3+4s\,\big)\,s^4+10s^2+9\,\big(\,s \to Y_1 \\
\qquad\qquad s^4+4s^2 \\
\qquad\quad 6s^2+9\,\big)\,s^3+4s\,\big(\,s/6 \to Z_2 \\
\qquad\qquad\qquad s^3+\frac{3}{2}s \\
\qquad\qquad\quad \frac{5}{2}s\,\big)\,6s^2+9\,\big(\,\frac{12}{5}s \to Y_3 \\
\qquad\qquad\qquad\quad 6s^2 \\
\qquad\qquad\qquad\quad 9\,\big)\,\frac{5}{2}s\,\big(\,\frac{5}{18}s \to Z_4 \\
\qquad\qquad\qquad\qquad\quad \frac{5}{2}s \\
\qquad\qquad\qquad\qquad\quad \times
\end{array}$$

Hence $$Z(s) = \cfrac{1}{s + \cfrac{1}{\cfrac{s}{6} + \cfrac{1}{\cfrac{12}{5}s + \cfrac{1}{\cfrac{5}{18}s}}}}$$

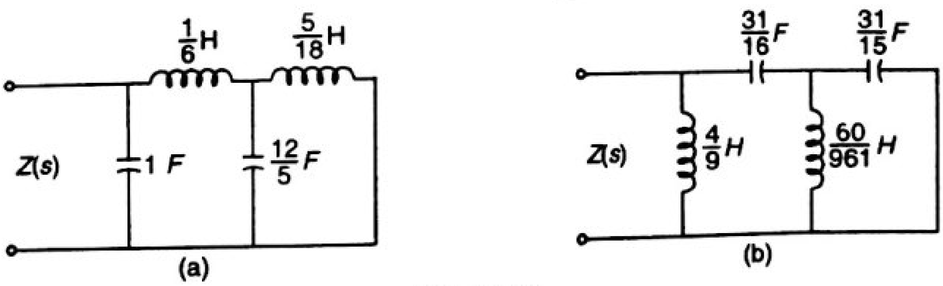


Fig. 10.16

The resulting network is shown in Fig. 10.16 (*a*).

(*d*) Second Cauer Form

$$Z(s) = \frac{s(s^2+4)}{(s^2+1)(s^2+9)} = \frac{s^3+4s}{s^4+10s^2+9} = \frac{1}{(9+10s^2+s^4)/(4s+s^3)}$$

The continued fraction expansion is

$$\begin{array}{l}
4s+s^3 \,\big)\, 9+10s^2+s^2 \,\big(\, \frac{9}{4s} \rightarrow Y_1 \\
\qquad\quad 9+\frac{9}{4}s^2 \\
\hline
\frac{31}{4}s^2+s^4 \,\big)\, 4s+s^3 \,\big(\, \frac{16}{31s} \rightarrow Z_2 \\
\qquad\quad 4s+\frac{16}{31}s^3 \\
\hline
\frac{15}{31}s \,\big)\, \frac{31}{4}s^2+s^4 \,\big(\, \frac{961}{60s}s \rightarrow Y_3 \\
\qquad\quad \frac{31}{4}s^2 \\
\hline
s^4 \,\big)\, \frac{15}{31}s^3 \,\big(\, \frac{15}{31}s \rightarrow Z_4 \\
\qquad\quad \frac{15}{31}s^3 \\
\hline
\qquad\quad \times
\end{array}$$

Hence

$$Z(s) = \cfrac{1}{\cfrac{9}{4s} + \cfrac{1}{\cfrac{16}{31s} + \cfrac{1}{\cfrac{961}{60s} + \cfrac{1}{\cfrac{15}{31s}}}}}$$

The resulting network is shown in Fig. 10.16(*b*).

Example 10.12. Obtain any three realizations of the driving point admittance.

$$Y(s) = \frac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)}$$

Solution. (*a*) First foster form

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)}$$

$Z(s)$ has poles at 0, $\pm j\sqrt{2}$ and $\pm j2$. Using partial fraction expansion,

$$Z(s) = K_\infty s + \frac{K_0}{s} + \frac{2K_2}{s^2+2} + \frac{2K_4}{s^2+4}$$

$$K_\infty = \left.\frac{Z(s)}{s}\right|_{s\to\infty} = 0$$

$$K_0 = sZ(s)\Big|_{s=0} = \frac{3}{8}$$

$$K_2 = \frac{(s^2+2)\,Z(s)}{2s}\Bigg|_{s^2=-2} = \frac{(s^2+1)(s^2+3)}{2s^2(s^2+4)}\Bigg|_{s^2=-2} = \frac{1}{8}$$

$$K_4 = \frac{(s^2+4)\,Z(s)}{2s}\Bigg|_{s^2=-4} = \frac{(s^2+1)(s^2+3)}{2s^2(s^2+4)}\Bigg|_{s^2=-4} = \frac{3}{16}$$

Therefore, $$Z(s) = \frac{3}{8s} + \frac{1/4}{s^2+2} + \frac{3/8}{s^2+4}$$

The resulting network is shown in Fig. 10.17 (*a*).

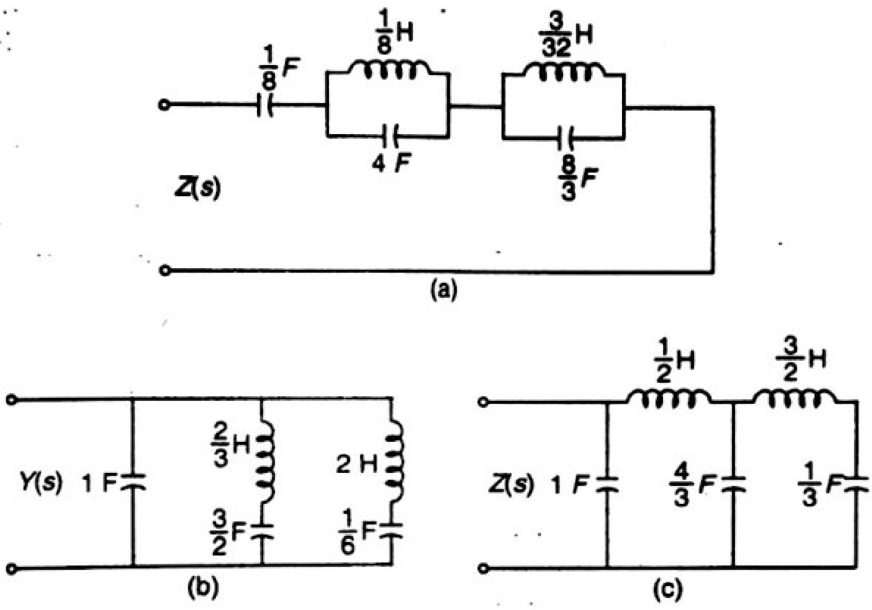


Fig. 10.17

(*b*) Second Foster form

$$Y(s) = \frac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)}$$

$Y(s)$ has poles at $\pm j1$ and $\pm j\sqrt{3}$. Using partial fraction expansion.

$$Y(s) = K'_\infty s + \frac{K'_0}{s} + \frac{2K'_2\, s}{s^2+1} + \frac{2K'_4 s}{s^2+3}$$

$$K'_\infty = \frac{Y(s)}{s}\Bigg|_{s\to\infty} = \frac{(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)}\Bigg|_{s\to\infty}$$

$$= \frac{\left(1+\frac{2}{s^2}\right)\left(1+\frac{4}{s^2}\right)}{\left(1+\frac{1}{s^2}\right)\left(1+\frac{3}{s^2}\right)}\Bigg|_{s\to\infty} = 1$$

$$K'_0 = 0$$

$$K'_2 = \left.\frac{(s^2+1)Y(s)}{2s}\right|_{s^2=-1} = \left.\frac{(s^2+2)(s^2+4)}{2(s^2+3)}\right|_{s^2=-1} = \frac{3}{4}$$

$$K'_4 = \left.\frac{(s^2+3)\,Y(s)}{2s}\right|_{s^2=-3} = \left.\frac{(s^2+2)(s^2+4)}{2(s^2+1)}\right|_{s^2=-3} = \frac{1}{4}$$

Therefore,
$$Y(s) = s + \frac{\frac{3}{2}s}{s^2+1} + \frac{\frac{1}{2}s}{s^2+3}$$

The resulting network is shown in Fig. 10.17(*b*).

(*c*) First Cauer form

$$Z(s) = \frac{1}{Y(s)} = \frac{1}{[s(s^2+2)(s^2+4)]/[(s^2+1)(s^2+3)]}$$

$$= \frac{1}{(s^5+6s^3+8s)/(s^4+4s^2+3)}$$

The continued fraction expansion gives

$$Z(s) = \cfrac{1}{s + \cfrac{1}{\cfrac{s}{2} + \cfrac{1}{\cfrac{4}{3}s + \cfrac{1}{\cfrac{3}{2}s + \cfrac{1}{\cfrac{s}{3}}}}}}$$

The resulting network is shown in Fig. 10.17 (*c*).

10.8 PROPERTIES OF RC IMPEDANCE AND ADMITTANCE FUNCTIONS

Consider the *RC* circuit shown in Fig. 10.18. The driving point impedance of this circuit is

$$Z_{RC}(s) = \frac{K_0}{s} + K_\infty + \frac{K_1}{s+\sigma_1} + \frac{K_3}{s+\sigma_3} + \ldots\ldots \qquad \ldots(10.47)$$

where $K_\infty = R_\infty, K_0 = 1/C_0, K_1 = 1/C_1, \sigma_1 = K_1/R_1,$ etc. All the constants in Eq. (10.47) must be real and positive. If $K_0 = 0$, Eq. (10.47) can be written as under

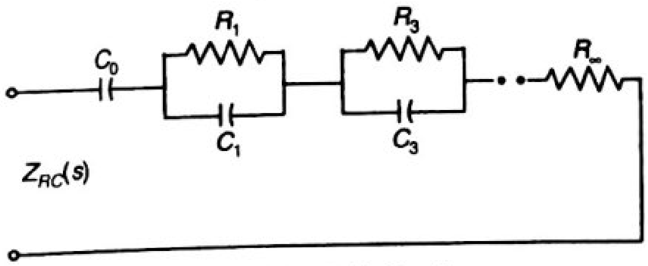


Fig. 10.18 R-C Circuit

$$Z_{RC}(s) = H\frac{(s+\sigma_2)(s+\sigma_4)......}{(s+\sigma_1)(s+\sigma_3)......} \quad ...(10.48)$$

RC impedance functions have the following properties:

1. All the poles are simple and lie on the negative real axis.
2. The poles and zeros are interlaced, *i.e.*, in Eq. (10.48), $\sigma_1 < \sigma_2 < \sigma_3 < \sigma_4$, etc.
3. The lowest critical frequency (*i.e.*, the one nearest the origin) is a pole which is at $s = 0$ if $K_0 \neq 0$.
4. The highest critical frequency is a zero which is at infinity if $K_\infty = 0$.
5. The residues at all the poles are real and positive,
6. The slope $dZ/d\sigma$ is negative.
7. $Z(\infty) \le Z(0)$. The property that $dZ/d\sigma$ is negative can be seen by substituting $s = \sigma$ in Eq. (10.47) and taking the differential.

$$Z(s) = \frac{K_0}{\sigma} + K_\infty + \frac{K_1}{\sigma+\sigma_1} + \frac{K_3}{\sigma+\sigma_3} + ... \quad ...(10.49)$$

$$\frac{dZ(\sigma)}{d\sigma} = \frac{-K_0}{\sigma^2} - \frac{K_1}{(\sigma+\sigma_1)^2} - \frac{K_3}{(\sigma+\sigma_3)^2} + ... \quad ....(10.50)$$

Since K_0, K_1, K_3 .... are all positive, $dZ(\sigma)/d\sigma$ must be negative.

At $\sigma = 0$ (*i.e.*, *dc*) all the capacitors (Fig. 10.18) are open circuits. If C_0 is in the circuit then $Z(0) = \infty$...... If C_0 is not in the circuit then $Z(0) = R_1 + R_3 +$ .... At $\sigma = \infty$, all the capacitors are short circuits. It R_∞ is in the circuit $Z(\infty) = R_\infty$. If R_∞ is not in the circuit then $Z(\infty) = 0$. Thus $Z(\infty) \le Z(0)$.

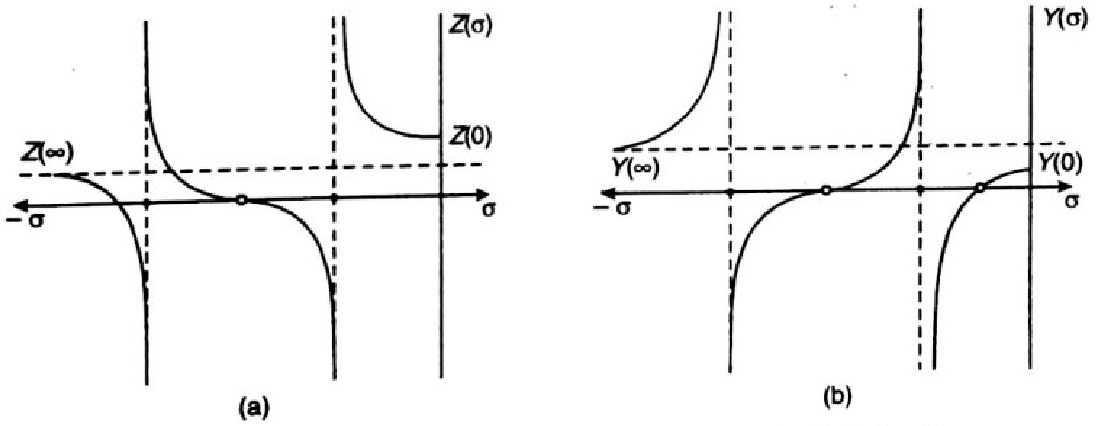


Fig. 10.19 (a) Variation of Z (σ) (b) Variation of Y(σ) for R-C circuit

Figure 10.19(*a*) shows a plot of $Z(\sigma)$. It is seen that $dZ/d\sigma$ is negative and $Z_\infty < Z(0)$.

The driving point admittance function of an *RC* network, *i.e.*, $Y_{RC}(s)$ has the following properties.

1. All the poles and zeros are simple and are located on the negative real axis.
2. The poles and zeros are interlaced.
3. The lowest critical frequency is a zero which may be at $s = 0$.

4. The highest critical frequency is a pole which may be at infinity.
5. The residues at the poles of $Y_{RC}(s)$ are real and negative but the residues of $[Y_{RC}(s)]/s$ are real and positive.
6. The slope $dY/d\sigma$ is positive.
7. $Y(0) < Y(\infty)$

Fig. 10.19 (*b*) shows a plot of $Y(\sigma)$.

10.8.1 Synthesis of RC Circuits

First Foster form. *RC* circuits can also be synthesized in four different ways. To get first Foster form we expand $Z_{RC}(s)$ by partial fraction expansion. This expansion will give an equation similar to Eq. (10.47) which can be re-written as

$$Z_{RC}(s) = \frac{K_0}{s} + K_\infty + \sum_{i=1}^{n} \frac{K_i}{s+\sigma_i} \qquad \text{...(10.51)}$$

where

$$K_0 = s\,Z_{RC}(s)\big|_{s=0}$$

$$K_\infty = \underset{s\to\infty}{Lt}\, Z_{RC}(s)$$

$$K_i = (s+\sigma_i)\, Z_{RC}(\ )\big|_{s=-\sigma_i}$$

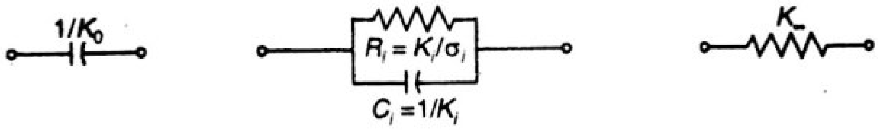


Fig. 10.20 Elemental impedances of $Z_{RC}(s)$

The firs term in Eq. (10.51) is the impedance of a capacitor having capacitance $1/K_0$, the second term is the impedance of a resistor having resistance K_∞. Each of the remaining terms represents the impedance of a parallel connection of a resistance R_i (Equal to K_i/σ_i) and a capacitance C_i (equal to $1/K_i$). These elemental impedances are shown in Fig. 10.20. The final network is obtained by a series connection of all these impedances (Fig. 10.18).

Second Foster Form As mentioned in the properties of *RC* admittance function, the residues at the poles of $Y_{RC}(s)$ are real and negative while the residues at the poles of $[Y_{RC}(s)]/s$ are real and positive. Therefore the proper function to expand, to get second Foster form, is $[Y_{RC}(s)]/s$. This expansion will give

$$\frac{Y_{RC}(s)}{s} = \frac{K'_0}{s} + K'_\infty + \sum_{i=1}^{n} \frac{K'_i}{s+\sigma'_i} \qquad \text{...(10.52)}$$

where

$$K'_0 = Y_{RC}(s)\big|_{s=0}$$

$$K'_\infty = \underset{s\to\infty}{Lt}\, Y_{RC}(s)/s$$

$$K'_i = \left[\frac{Y_{RC}(s)}{s}\right](s+\sigma'_i)\Big|_{s=-\sigma'_i}$$

From Eq. (10.52)

$$Y_{RC}(s) = K'_0 + K'_\infty s + \sum_{i=1}^{n} \frac{K'_i s}{s+\sigma'_i} \quad ...(10.53)$$

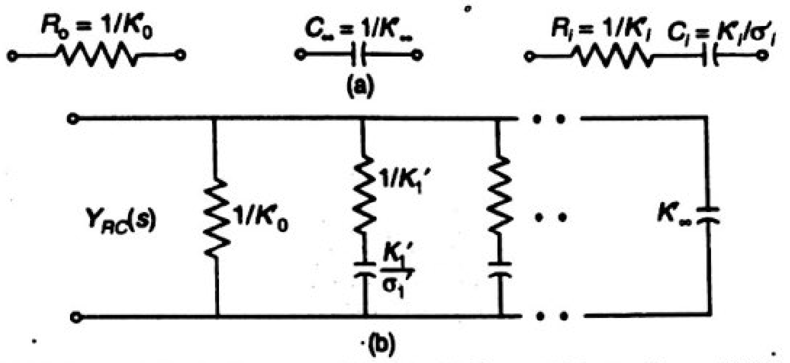


Fig. 10.21 (a) Elemental admittances of $Y_{RC}(s)$ (b) Second Foster form of *RC* admittance

The first term in Eq. (10.53) is the admittances of a resistor having a conductance K'_0 or resistance $1/K'_0$, the second term is the admittance of a capacitor having a capacitance K'_∞, and each subsequent term represents the admittance of a series connection of a resistance R_i (equal to $1/K'_i$) and a capacitance Ci (equal to K'_i/σ'_i). These elemental admittances are shown in Fig. 10.21(*a*). The final network is realized by a parallel connection of these elemental admittance (Fig. 10.21*b*).

The first and the last elements of the two Foster forms depend on the behaviour of $Z_{RC}(s)$ at $s = 0$ and $s = \infty$. These are indicated in Table 10.1.

Table 10.1 First and last elements in Foster realization of *RC* circuit

Form	Behaviour of $Z_{RC}(s)$ at $s = 0$	First element	Behaviour of $Z_{RC}(s)$ at $s = \infty$	Last element
First	Pole	*C*	constant	*R*
Foster	constant	None	Zero	None
Second	constant	*R*	Zero	*C*
Foster	pole	None	constant	None

First Cauer Form The first Cauer form of *RC* driving point impedance function is obtained by expanding it in continued fractions about infinity. If we remove Min [$Re\, Z(j\omega)$] = $Z_{(\infty)}$ from $Z(s)$ we create a zero at $s = \infty$ for the remainder $Z_1(s)$. If we invert $Z_1(s)$ we have a pole $s = \infty$ which we can remove to give $Z_2(s)$. Since min Re($Y_2(\omega)$] = $Y_2(\infty)$, if we remove $Y_2(\infty)$ we again have a zero at $s = \infty$ which we again invert and remove. Thus each time we remove a constant and then remove the pole at infinity of the remainder admittance. The quotients of the continued fraction expansion given the elements of ladder network.

In this form, resistors are in series branches and capacitors are in the shunt branches. The continued fraction expansion of an *RC* impedance takes the form

$$Z_{RC}(s) = \alpha_1 + \cfrac{1}{\alpha_2 s + \cfrac{1}{\alpha_3 + \cfrac{1}{\alpha_4 s \cfrac{1}{\dots + \cfrac{1}{\alpha_n s}}}}} \quad ...(10.54)$$

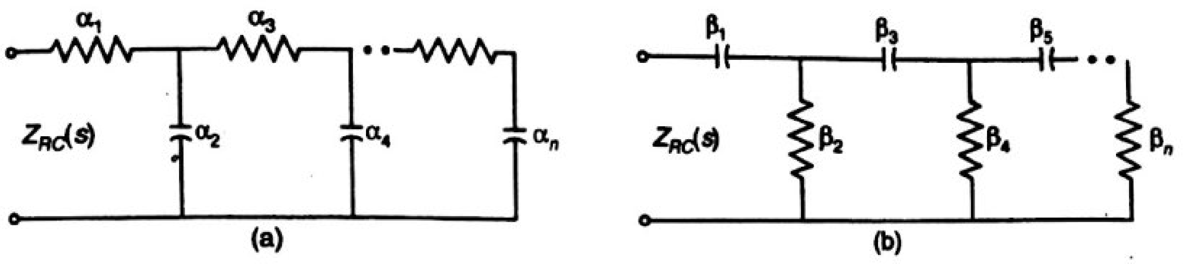


Fig.10.22. (a) First Cauer form of RC circuit (b) Second Cauer form of *RC* circuit

α_1, the first constant removed from $Z_{RC}(s)$, is the minimum value of $Z(j\omega)$ and is equal to $Z(\infty)$. If $Z(s)$ has a zero at $s = \infty$, then $\alpha_1 = 0$.

The resulting network is shown in Fig. 10.22(*a*).

Second Cauer form. The second Cauer form is obtained by expanding the function about the origin in continued fraction expansion. For this purpose, we arrange the numerator and denominator in ascending powers of *s* and divide the numerator by the denominator. We invert the remainder and carry out the continued fraction expansion till the remainder is zero. The quotients of the expansion give the elements of the ladder network. In this form, capacitors are in series branches and resistors are in shunt branches. The expansion takes the form

$$Z_{RC}(s) = \frac{1}{\beta_1 s} + \cfrac{1}{\cfrac{1}{\beta_2} + \cfrac{1}{\cfrac{1}{\beta_3 s} + \cfrac{1}{\cfrac{1}{\beta_4} + \dots}}} \qquad \dots(10.55)$$

The resulting network is shown in Fig. 10.22(*b*). If $Z_{RC}(s)$ is constant at $s = 0$, then β_1 is zero and the first element is the resistor β_2.

The first and last element of Cauer *RC* circuits take the following form.

First Cauer form:

If $Z_{RC}(s)$ has a zero at $s = \infty$, the first element is capacitor.

If $Z_{RC}(s)$ is a constant at $s = \infty$, the first element is resistance.

If $Z_{RC}(s)$ has a pole at $s = 0$, the last element is a capacitor.

If $Z_{RC}(s)$ is a constant at $s = 0$, the last element is a resistor.

Second Cauer form

If $Z_{RC}(s)$ has a pole at $s = 0$, the first element is a capacitor.

If $Z_{RC}(s)$ is constant at $s = 0$, the first element is a resistor.

If $Z_{RC}(s)$ has zero at $s = \infty$, the last element is a capacitor.

If $Z_{RC}(s)$ is constant at $s = \infty$, the last element is resistor.

All the four forms have the same number of elements and are canonic.

Example 10.13. Obtain any three realizations of the function

$$Z(s) = \frac{s^2 + 6s + 8}{s^2 + 4s + 3}$$

Solution. $$Z(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)}$$

If it easy to see that $Z(s)$ is an RC impedance.

(*a*) First Foster form

$$Z(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)} = \frac{K_0}{s} + K_\infty + \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

$$K_0 = s\,Z(s)\big|_{s=0} = 0$$

$$K_\infty = \underset{s\to\infty}{Lt}\; Z(s) = \underset{s\to\infty}{Lt}\; \frac{\left(1+\frac{2}{s}\right)\left(1+\frac{4}{s}\right)}{\left(1+\frac{1}{s}\right)\left(1+\frac{3}{s}\right)} = 1$$

$$K_1 = (s+1)\,Z(s)\big|_{s=-1} = \frac{3}{2}$$

$$K_2 = (s+3)\,Z(s)\big|_{s=-3} = \frac{1}{2}$$

The resulting circuit is shown in Fig. 10.23(*a*).

(*b*) Second Foster form

$$\frac{Y(s)}{s} = \frac{(s+1)(s+3)}{s(s+2)(s+4)}$$

$$= \frac{K'_0}{s} + K'_\infty + \frac{K'_1}{s+2} + \frac{K'_2}{s+4}$$

$$K'_0 = Y(s)\big|_{s=0} = \frac{3}{8}$$

$$K'_\infty = \underset{s\to\infty}{Lt}\; \frac{Y(s)}{s} = 0$$

$$K'_1 = (s+2)\frac{Y(s)}{s}\bigg|_{s=-2} = \frac{1}{4}$$

$$K'_2 = (s+4)\frac{Y(s)}{s}\bigg|_{s=-4} = \frac{3}{8}$$

Therefore $$Y(s) = \frac{3}{8} + \frac{s}{4(s+2)} + \frac{3s}{8(s+4)}$$

The resulting network is shown in Fig. 10.23(*b*).

(*c*) First Cauer form–expanding $Z(s)$ into continuous fraction about infinity, we have

$$
\begin{array}{l}
s^2+4s+3\,\big)\,s^2+6s+8\,\big(\,1 \\
\qquad\qquad\; s^2+4s+3 \\
\hline
\qquad\quad 2s+5\,\big)\,s^2+4s+3\,\big(\,\frac{s}{2} \\
\qquad\qquad\qquad s^2+\frac{5}{2}s \\
\hline
\qquad\qquad \frac{3}{2}s+3\,\big)\,2s+5\,\big(\,\frac{4}{3} \\
\qquad\qquad\qquad\quad 2s+4 \\
\hline
\qquad\qquad\qquad 1\,\big)\,\frac{3}{2}s+3\,\big(\,\frac{3}{2}s \\
\qquad\qquad\qquad\qquad \frac{3}{2}s \\
\hline
\qquad\qquad\qquad\qquad 3\,\big)\,1\,\big(\,\frac{1}{3} \\
\qquad\qquad\qquad\qquad\quad 1 \\
\hline
\qquad\qquad\qquad\qquad\quad \times
\end{array}
$$

Thus,
$$Z(s) = 1+\cfrac{1}{\cfrac{s}{2}+\cfrac{1}{\cfrac{4}{3}+\cfrac{1}{\cfrac{3}{2}s+\cfrac{1}{\cfrac{1}{3}}}}}$$

The resulting circuit is shown in Fig. 10.23(c).

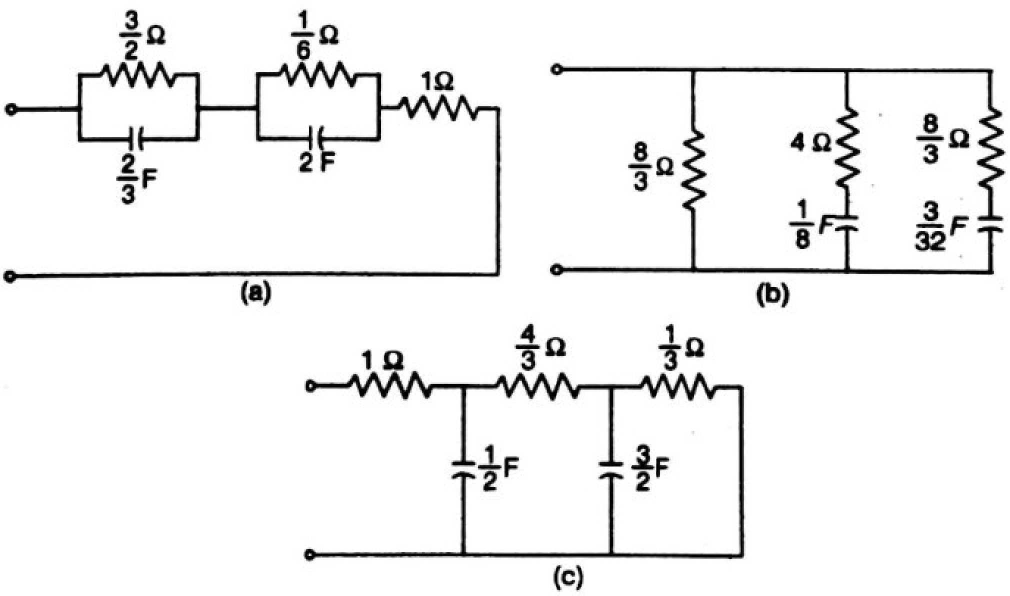


Fig. 10.23

Example 10.14. Synthesize in Foster form II

$$Z(s) = \frac{(s+5)(s+7)}{(s+1)(s+6)(s+8)}$$ [UPSC 2009]

Solution.

$$Y(s) = \frac{(s+1)(s+6)(s+8)}{(s+5)(s+7)}$$

$$\frac{Y(s)}{s} = \frac{(s+1)(s+6)(s+8)}{s(s+5)(s+7)}$$

$$= \frac{K'_0}{s} + K'_\infty + \frac{K'_1}{s+5} + \frac{K'_2}{s+7}$$

$$K'_0 = Y(s)\Big|_{s=0} = \frac{48}{35}$$

$$K'_\infty = \frac{Y(s)}{s}\Bigg|_{s\to\infty} = \frac{s^3\left(1+\frac{1}{s}\right)\left(1+\frac{6}{s}\right)\left(1+\frac{8}{s}\right)}{s^3\left(1+\frac{5}{s}\right)\left(1+\frac{7}{s}\right)}\Bigg|_{s\to\infty} = 1$$

$$K'_1 = \left[\frac{Y(s)}{s}\right](s+5)\Bigg|_{s=-5} = \frac{6}{5}$$

$$K'_2 = \left[\frac{Y(s)}{s}\right](s+7)\Bigg|_{s=-7} = \frac{3}{7}$$

Therefore,

$$\frac{Y(s)}{s} = \frac{48}{35s} + 1 + \frac{6/5}{s+5} + \frac{3/7}{s+7}$$

or

$$Y(s) = \frac{48}{35} + s + \frac{6s/5}{s+5} + \frac{3s/7}{s+7}$$

There resulting circuit is shown in Fig. 10.24.

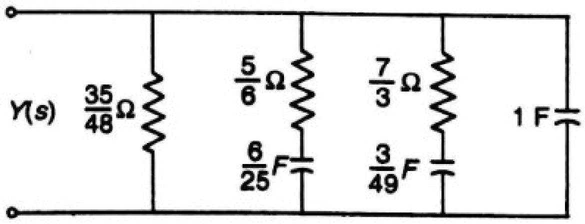


Fig. 10.24

Example 10.15. Synthesis $Y(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$ in Cauer Form II.

Solution.

$$Y(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)} = \frac{6+8s+2s^2}{12+8s+s^2}$$

The continued fraction expansion is

$$
\begin{array}{l}
12 + 8s + s^2 \,\big)\, 6 + 8s + 2s^2 \,\big(\, \frac{1}{2} \\
\qquad\qquad\qquad\;\; 6 + 4s + s^2/2 \\
\qquad\qquad\quad \overline{4s + \frac{3}{2}s^2 \,\big)\, 12 + 8s + s^2 \,\big(\, \frac{3}{s}} \\
\qquad\qquad\qquad\qquad\qquad 12 + 9s/2 \\
\qquad\qquad\qquad\quad \overline{\frac{7}{2}s + s^2 \,\big)\, 2s + \frac{3}{2}s^2 \,\big(\, \frac{8}{7}} \\
\qquad\qquad\qquad\qquad\qquad\qquad 4s + \frac{8}{7}s^2 \\
\qquad\qquad\qquad\qquad\quad \overline{\frac{5}{14}s^2 \,\big)\, \frac{7}{2}s + s^2 \,\big(\, \frac{49}{5s}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \frac{7}{2}s \\
\qquad\qquad\qquad\qquad\qquad\qquad \overline{s^2 \,\big)\, \frac{5}{14}s^2 \,\big(\, \frac{5}{14}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \frac{5}{14}s^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\quad\times\quad}
\end{array}
$$

Therefore,

$$Y(s) = \frac{1}{2} + \cfrac{1}{\cfrac{3}{s} + \cfrac{1}{\cfrac{8}{7} + \cfrac{1}{\cfrac{49}{5s} + \cfrac{1}{\cfrac{5}{14}}}}}$$

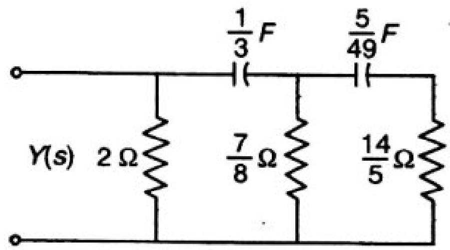


Fig. 10.25

The resulting circuit is shown in Fig. 10.25.

Example 10.16. If $Z(s) = \dfrac{s^2 + 4s + 3}{s^2 + 2s}$ obtain all Foster and cauer realizations.

Solution. (*a*) First Foster form

$$Z(s) = \frac{s^2 + 4s + 3}{s(s + 2)}$$

$$= \frac{K_0}{s} + K_\infty + \frac{K_1}{s + 2}$$

$$K_0 = sZ(s)\big|_{s=0} = \frac{3}{2}$$

$$K_\infty = \underset{s\to\infty}{Lt}\, Z(s) = \underset{s\to\infty}{Lt} \frac{1+\frac{4}{s}+\frac{3}{s^2}}{1+\frac{2}{s}} = 1$$

$$K_1 = [Z(s)][(s+2)]\Big|_{s=-2} = \frac{1}{2}$$

Therefore, $$Z(s) = \frac{3}{2s}+1+\frac{1/2}{s+2}$$

The resulting circuit is shown in Fig. 10.26(*a*).

(*b*) Second Foster form

$$Y(s) = \frac{s(s+2)}{s^2+4s+3} = \frac{s(s+2)}{(s+3)(s+1)}$$

$$\frac{Y(s)}{s} = \frac{s+2}{(s+3)(s+1)}$$

$$= \frac{K'_0}{s} + K'_\infty + \frac{K'_1}{s+3} + \frac{K'_2}{s+1}$$

$$K'_0 = Y(s)\big|_{s=0} = 0$$

$$K'_\infty = \underset{s\to\infty}{Lt} \frac{Y(s)}{s} = 0$$

$$K'_1 = \left[\frac{Y(s)}{s}\right](s+3)\Big|_{s=-3} = \frac{1}{2}$$

$$K'_2 = \left[\frac{Y(s)}{s}\right](s+1)\Big|_{s=-1} = \frac{1}{2}$$

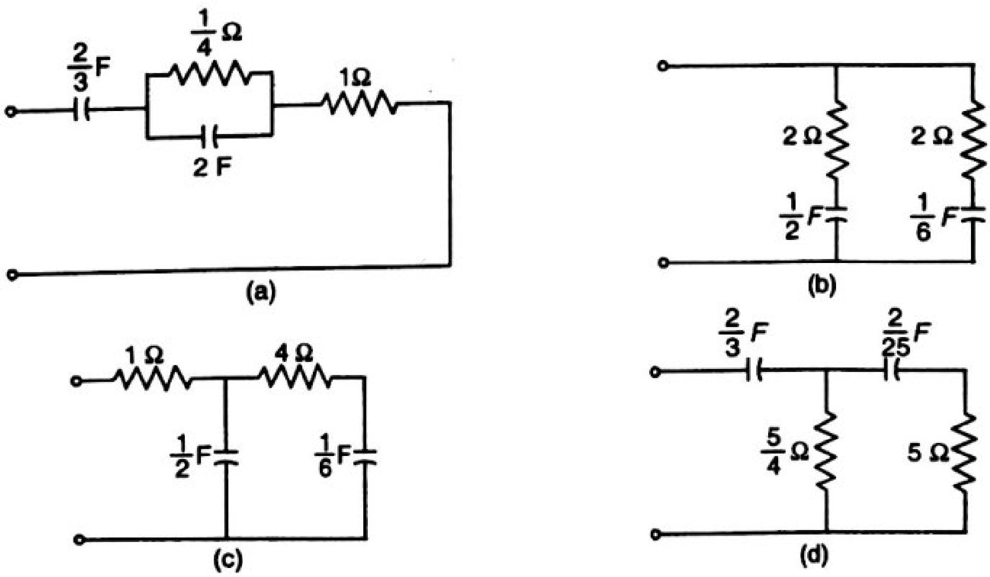


Fig. 10.26

Therefore,
$$\frac{Y(s)}{s} = \frac{1/2}{s+1}+\frac{1/2}{s+3}$$

or
$$Y(s) = \frac{s/2}{s+1}+\frac{s/2}{s+3}$$

The resulting circuit is shown in Fig. 10.26(*b*).

(*c*) First Cauer form – The continued fraction expansion of $Z(s)$ takes the form

$$\begin{array}{l}
s^2+2s\,\big)\,s^2+4s+3\,\big(\,1 \\
\qquad\quad s^2+2s \\
\hline
\qquad\quad 2s+3\,\big)\,s^2+2s\,\big(\,\frac{s}{2} \\
\qquad\qquad\qquad s^2+\frac{3}{2}s \\
\hline
\qquad\qquad\quad \frac{1}{2}s\,\big)\,2s+3\,\big(\,4 \\
\qquad\qquad\qquad\quad 2s \\
\hline
\qquad\qquad\qquad\quad 3\,\big)\,\frac{1}{2}s\,\big(\,\frac{s}{6} \\
\qquad\qquad\qquad\qquad \frac{1}{2}s \\
\hline
\qquad\qquad\qquad\qquad \times
\end{array}$$

Hence
$$Z(s) = 1+\cfrac{1}{\cfrac{s}{2}+\cfrac{1}{4+\cfrac{1}{s/6}}}$$

The resulting circuit is shown in Fig. 10.26(*c*).

(*d*) Second cauer form – Arranging the numerator and denominator in ascending power of s

$$Z(s) = \frac{3+4s+s^2}{2s+s^2}$$

The continued fraction expansion is

$$\begin{array}{l}
2s+s^2\,\big)\,3+4s+s^2\,\big(\,\frac{3}{2s} \\
\qquad\quad 3+\frac{3}{2}s \\
\hline
\qquad\quad \frac{5}{2}s+s^2\,\big)\,2s+s^2\,\big(\,\frac{4}{5} \\
\qquad\qquad\qquad 2s+\frac{4}{5}s^2 \\
\hline
\qquad\qquad\quad \frac{1}{5}s^2\,\big)\,\frac{5}{2}s+s^2\,\big(\,\frac{25}{2s} \\
\qquad\qquad\qquad\quad \frac{5}{2}s \\
\hline
\qquad\qquad\qquad\quad s^2\,\big)\,\frac{1}{5}s^2\,\big(\,\frac{1}{5} \\
\qquad\qquad\qquad\qquad \frac{1}{5}s^2 \\
\hline
\qquad\qquad\qquad\qquad \times
\end{array}$$

Hence
$$Z(s) = \frac{3}{2s}+\cfrac{1}{\frac{4}{5}+\cfrac{1}{\frac{25}{2s}+\cfrac{1}{1/5}}}$$

The resulting circuit in Fig. 10.26(*d*).

Example 10.17. A function $F(s)$ has poles and zeros as under:

Poles 0,— 4, —6

Zeros —2, —5,

Taking scale factor as *l*, synthesize $F(s)$ (*a*) as an impedance in Foster form (*b*) as an admittance in Cauer form.

Solution.
$$F(s) = \frac{(s+2)(s+5)}{s(s+4)(s+6)}$$

(*a*)
$$Z(s) = \frac{(s+2)(s+5)}{s(s+4)(s+6)}$$
$$= \frac{K_0}{s}+K_\infty+\frac{K_1}{s+4}+\frac{K_2}{s+6}$$
$$K_0 = s\,Z(s)\big|_{s=0} = \frac{5}{12}$$
$$K_\infty = \underset{s\to\infty}{Lt}\ Z(s) = 0$$
$$K_1 = (s+4)\,Z(s)\big|_{s=-4} = \left.\frac{(s+2)(s+5)}{s(s+6)}\right|_{s=-4} = \frac{1}{4}$$
$$K_2 = (s+6)\,Z(s)\big|_{s=-6} = \left.\frac{(s+2)(s+5)}{s(s+4)}\right|_{s=-6} = \frac{1}{3}$$

The resulting circuit is shown in Fig. 10.27(*a*).

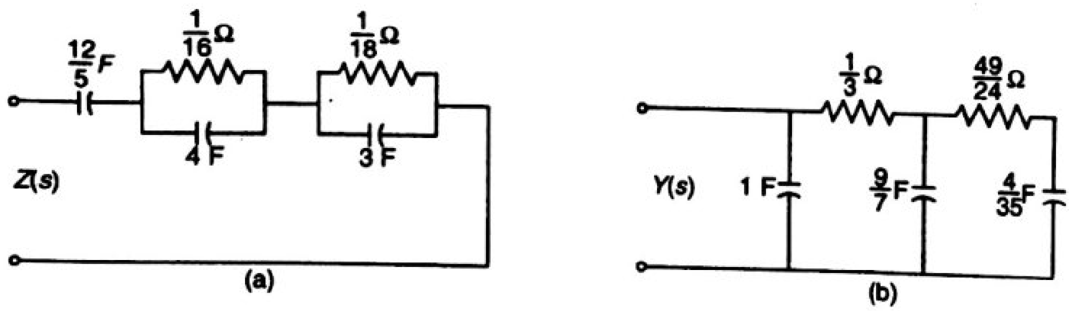


Fig. 10.27

(*b*)
$$Y(s) = \frac{s(s+4)(s+6)}{(s+2)(s+5)} = \frac{s^3+10s^2+24s}{s^2+7s+10}$$

$$
\begin{array}{l}
s^2+7s+10\overline{\big)\, s^3+10s^2+24s\,}\big(\, s \\
\qquad\qquad\quad\;\; s^3+7s^2+10s \\
\qquad\quad 3s^2+14s\,\overline{\big)\, s^2+7s+10\,}\big(\,\frac{1}{3} \\
\qquad\qquad\qquad\quad\;\; s^2+\frac{14}{3}s \\
\qquad\qquad\quad \frac{7}{3}s+10\,\overline{\big)\, 3s^2+14s\,}\big(\,\frac{9}{7}s \\
\qquad\qquad\qquad\qquad\quad 3s^2+\frac{90}{7}s \\
\qquad\qquad\qquad\quad \frac{8}{7}s\,\overline{\big)\,\frac{7}{3}s+10\,}\big(\,\frac{49}{24} \\
\qquad\qquad\qquad\qquad\quad\;\; \frac{7}{3}s \\
\qquad\qquad\qquad\qquad\quad 10\,\overline{\big)\,\frac{8}{7}s\,}\big(\,\frac{4}{35}s \\
\qquad\qquad\qquad\qquad\qquad\;\; \frac{8}{7}s \\
\qquad\qquad\qquad\qquad\qquad\;\; \times
\end{array}
$$

Therefore,

$$Y(s) = s+\cfrac{1}{\cfrac{1}{3}+\cfrac{1}{\cfrac{9}{7}s+\cfrac{1}{\cfrac{49}{24}+\cfrac{1}{\cfrac{4s}{35}}}}}$$

The resulting circuit is shown in Fig. 10.27(*b*).

10.9 PROPERTIES AND SYNTHESIS OF *RL* FUNCTIONS

Consider the *RL* circuit shown in Fig. 18.28(*a*).

The admittance of this circuit is

$$Y_{RL}(s) = \frac{K_0}{s}+K_\infty+\frac{K_1}{s+\sigma_1}+\ldots \qquad \ldots(10.56)$$

Equation (10.56) has the same form as Eq. (10.47). Thus the admittance of *R–L* circuit is of a form similar to the impedance of *RC* circuit.

A series *RL* circuit is shown in Fig. 10.28 (*b*).

The impedance of this circuit is

$$Z_{RL}(s) = K'_0+K'_\infty s+\frac{K'_1 s}{s+\sigma'_1}+\frac{K'_2 s}{s+\sigma'_2}+\ldots\ldots$$

$$= K'_0+K'_\infty s+\sum_{i=1}^{n}\frac{K'_i s}{s+\sigma'_i} \qquad \ldots(10.57)$$

Equation (10.57) is of the same form as Eq. (10.53). Thus the impedance of an *RL* circuit has the same Form as the admittance of an *RC* circuit. Therefore all the properties of *RL* admittances are the same as the properties of *RC* impedances and all the properties of *RC* admittances are the same as the properties of *RL* impedances. In view of this we observe that :

1. If a given driving point impedance function is positive real, has ll its poles and zeros on the negative real axis in the s plane with poles and zeros interlaced, and the smallest and highest critical frequencies are of opposite kinds (i.e., one a pole and the other a zero), then $Z(s)$ can be realized as an *RL* network or *RC* network but not both.
2. If the smallest critical frequency of $Z(s)$ is a pole, $Z(s)$ can be realizes as an *RC* network. If the smallest critical frequency is zero, it can be realized as an *RL* network.
3. An *RL* impedance function can be realized as an *RC* admittance function. An *RL* admittance function can be realized as an *RC* impedance function.

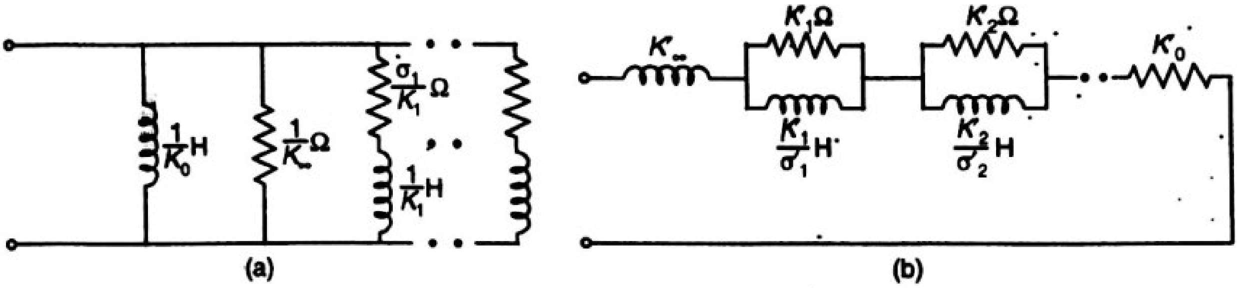


Fig. 10.28 *R–L* circuits

The synthesis procedures for *RL* networks are similar to those for *RC* networks. The first Foster form is obtained by partial fraction expansion of $Y_{RL}(s)$. For the second Foster form, the proper function to be expanded is $[Z_{RL}(s)]/s$. The Cauer forms are obtained through continued fraction expansion. It is necessary to specify whether the function is to be realized as an *RL* impedance or *RC* admittance.

Example 10.18. Obtain any two *RL* realizations of the function

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

Solution. (*i*)

$$\frac{Z(s)}{s} = \frac{2(s+1)(s+3)}{s(s+2)(s+6)}$$

$$= \frac{K'_0}{s} + K'_\infty + \frac{K'_1}{s+2} + \frac{K'_2}{s+6}$$

$$K'_0 = Z(s)\Big|_{s=0} = \frac{1}{2}$$

$$K'_\infty = \underset{s\to\infty}{Lt}\left(\frac{Z(s)}{s}\right) = 0$$

$$K'_1 = \left[\frac{Z(s)}{s}\right](s+2)\Big|_{s=-2} = \frac{1}{4}$$

$$K'_2 = \left[\frac{Z(s)}{s}\right](s+6)\Big|_{s=-6} = \frac{5}{4}$$

Hence $$\frac{Z(s)}{s} = \frac{1/2}{s} + \frac{1/4}{s+2} + \frac{5/4}{s+6}$$

or $$Z(s) = \frac{1}{2} + \frac{s/4}{s+2} + \frac{5s/4}{s+6}$$

The resulting circuit is shown in Fig. 10.29(*a*).

(*ii*) $$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)} = \frac{6+8s+2s^2}{12+8s+s^2}$$

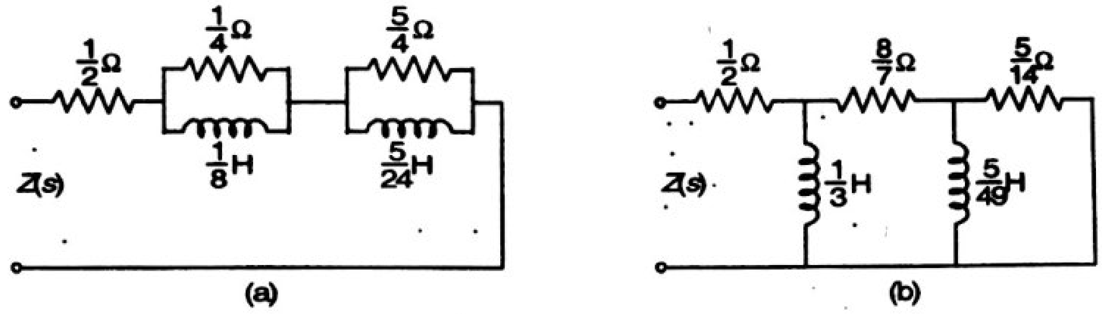


Fig. 10.29

The continued fraction expansion is

$$12 + 8s + s^2 \overline{\big)\, 6 + 8s + 2s^2 \,\big(}\ \frac{1}{2}$$
$$6 + 4s + \frac{1}{2}s^2$$
$$4s + \frac{3}{2}s^2 \overline{\big)\, 12 + 8s + s^2 \,\big(}\ \frac{3}{s}$$
$$12 + \frac{9}{2}s$$
$$\frac{7}{2}s + s^2 \overline{\big)\, 4s + \frac{3}{2}s^2 \,\big(}\ \frac{8}{7}$$
$$4s + \frac{8}{7}s^2$$
$$\frac{5}{14}s^2 \overline{\big)\, \frac{7}{2}s + s^2 \,\big(}\ \frac{49}{5s}$$
$$\frac{7}{2}s$$
$$s^2 \overline{\big)\, \frac{5}{14}s^2 \,\big(}\ \frac{5}{14}$$
$$\frac{5}{14}s^2$$
$$\times$$

Hence $$Z(s) = \frac{1}{2} + \cfrac{1}{\cfrac{3}{s} + \cfrac{1}{\cfrac{8}{7} + \cfrac{1}{\cfrac{49}{5s} + \cfrac{1}{\cfrac{5}{14}}}}}$$

The resulting network is shown in Fig. 10.29(*b*).

Example 10.19. A function $Z(s)$ has the following poles and zeros: Poles -2, -5, zeros, $0, -4, -6$. Taking scale factor as unity synthesize $Z(s)$ (*a*) as an impedance function in Foster form (*b*) as an admittance function in Cauer form.

Solution. $$Z(s) = \frac{s(s+4)(s+6)}{(s+2)(s+5)}$$

Since the lowest critical frequency is a zero, it is an *RL* function

$$\frac{Z(s)}{s} = \frac{(s+4)(s+6)}{(s+2)(s+5)} = \frac{K'_0}{s} + K'_\infty + \frac{K'_1}{s+2} + \frac{K'_2}{s+5}$$

$$K'_0 = Z(s)\big|_{s=0} = 0$$

$$K'_\infty = \underset{s\to\infty}{Lt}\left[\frac{Z(s)}{s}\right] = 1$$

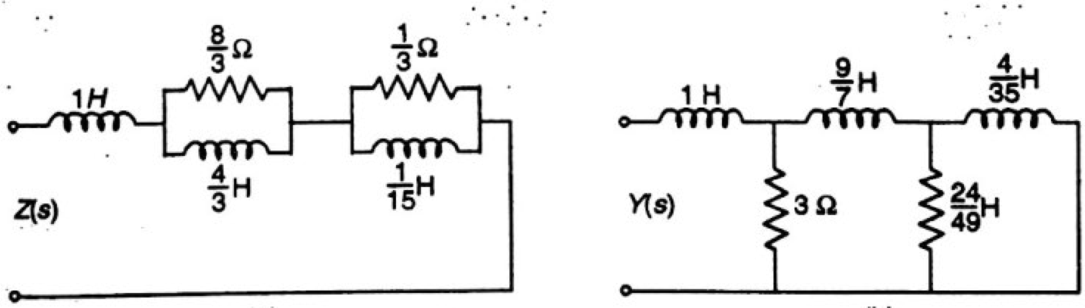
$$K'_1 = (s+2)\frac{Z(s)}{s}\bigg|_{s=-2} = \frac{8}{3}$$

$$K'_2 = (s+5)\frac{Z(s)}{s}\bigg|_{s=-5} = \frac{1}{3}$$

Therefore, $$\frac{Z(s)}{s} = 1 + \frac{8/3}{s+2} + \frac{s/3}{s+5}$$

or $$Z(s) = s + \frac{8s/3}{s+2} + \frac{s/3}{s+5}$$

The resulting circuit is shown in Fig. 10.30(*a*).



(a) (b)

Fig. 10.30

(*b*) $$Y(s) = \frac{(s+2)(s+5)}{s(s+4)(s+6)} = \frac{1}{\dfrac{s(s+4)(s+6)}{(s+2)(s+5)}}$$

$$= \frac{1}{(s^3+10s^2+24s)/(s^2+7s+10)}$$

The continued fraction expansion gives