UNIT-3

[(Fourier) $f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} (a_n \sin n\pi x).$]

I) 3D space.

Ex- $V = 4\hat{i} + 5\hat{j} + 6\hat{k}$

we can use here i, j, k → 1) Linearly independent.
(cannot express one vector in terms of the other)

2) span the entire 3 dimensional space.

[If set of vectors satisfy these 2 properties then that set is called BASIS]

BASIS is not unique.

BASIS formed here is orthonormal basis.

⇒ Dimension - is number of vectors in the basis.
(Unique)

⇒ Coordinates - The coefficients of basis vectors.

II) → The reason we choose sin and cosine in fourier ⇒ They form a basis (though it's not unique).

⇒ we choose sin and cosine as it behaves differently for LTI system.

⇒ The response of LTI system to complex exponentials.

study of

The LTI systems to represent signals as linear combinations of basis signals should satisfy the following 2 properties.

1) The set of basis signals can be used to construct a broad and useful class of signals.

2) The response of an LTI system to each signal should be simple enough in structure to provide us with convenient representation for the response of the system to any signal constructed as a linear combination of basis signals.

LTI

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

Basic Signal. $\qquad y(t) = x(t) * h(t).$

→ Consider a LTI system with the impulse response $h(t)$

Let I/P $x(t) = e^{st}$, $s \longrightarrow$ complex.

$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$$y(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$\left[H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau. \right]$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$= e^{st} H(s).$$

$e^{st} \longrightarrow \boxed{} \longrightarrow e^{st} H(s).$

[Eigen function] - (The function is preserved).

⇒ It is shown that complex exponentials are Eigen functions of LTI system.

The constant $H(s)$ for a specific value of s is then the eigenvalue associated with the Eigen function e^{st}.

$[CTS]\ T[e^{st}] = H(s)\, e^{st}$ ← Eigen function.

↑ Eigen value

→ A signal for which the system output is constant times the input is referred to as ~~the system's~~ an eigen ~~value~~ function. and the amplitude factor is referred to as the system's eigenvalue.

PTS → Consider an LTI System with its impulse response $h[n]$. Let z^n be the input to the system then

$$y(n) = h(n) * x(n)$$
$$= \sum_{k=-\infty}^{\infty} h(k) \times \cancel{x}(n-k)$$
$$= \sum_{k=-\infty}^{\infty} h(k) \cdot z^{n-k}$$
$$= z^n \sum_{k=-\infty}^{\infty} h(k) \cdot z^{-k}$$
$$y(n) = z^n H(z)$$
$$H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$$
↓
Z transform.

* z^n is the eigen function, $H(Z)$ is the eigenvalue

Q) $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$, find $y(t)$

$$e^{st} \longrightarrow e^{st} \cdot H(s)$$
$$ae^{st} \longrightarrow ae^{st} H(s)$$
$$a_1 e^{s_1 t} \longrightarrow a_1 e^{s_1 t} H(s_1)$$

$$x(t) = \left\{\begin{array}{lcl} a_1 e^{s_1 t} & \longrightarrow & a_1 H(s_1) e^{s_1 t} \\ + a_2 e^{s_2 t} & \longrightarrow & a_2 e^{s_2 t} H(s_2) \\ + a_3 e^{s_3 t} & \longrightarrow & a_3 e^{s_3 t} H(s_3) \end{array}\right\} = y(t)$$

NOTE-

- In general, if $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t}$ — If input (Linear combination of complex exponentials)

  then $y(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t} \cdot H(s_k)$

  where k is an Integer.

- If $x(n) = \sum_{k=-\infty}^{\infty} a_k z_k^n$

  then $y(n) = \sum_{k=-\infty}^{\infty} a_k z_k^n H(z_k)$

Q. Consider an LTI system for which input & output are related $\rightarrow y(t) = x(t-3)$. Find output $y(t)$ for $x(t) = e^{j2t}$

$$e^{j2t} \longrightarrow \boxed{\text{LTI}} \longrightarrow e^{j2t} \cdot e^{-6j}$$

↪ H S

$$y(t) = x(t-3)$$
$$= e^{j2(t-3)}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, d\tau\, e^{-s\tau}$$

$$h(t) = \delta(t-3)$$
$$h(\tau) = \delta(\tau - 3)$$

$$\left[H(s) = \int_{-\infty}^{\infty} \delta(\tau-3) e^{-st}\, d\tau \right.$$
$$\left. = \int_{-\infty}^{\infty} \phi(t) \cdot \delta(t) \right]$$

$$[\phi(t)]_{t=0}$$
$$= e^{-st} = e^{-3s}$$

$$H(s) = e^{-3s}$$

$$H(2j) = e^{-6j}$$

$$y(t) = e^{j2t} \cdot H(2j)$$
$$= e^{j2t} \cdot e^{-6j}$$

Q) Find $y(t)$ for $x(t) = \cos 4t + \cos 7t$

$$y(t) = x(t-3)$$

$$x(t-3) = \cos 4(t-3) + \cos 7(t-3)$$

$$x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-j4t} + \frac{1}{2} e^{j7t} + \frac{1}{2} e^{-j7t}$$

$$\frac{1}{2} e^{j4t} \longrightarrow \frac{1}{2} e^{j4t} \cdot e^{-12j}$$

$$\frac{1}{2} e^{-4jt} \longrightarrow \frac{1}{2} e^{-j4t} \cdot e^{+12j}$$

$$\frac{1}{2} e^{7jt} \longrightarrow \frac{1}{2} e^{7jt} \cdot e^{-21j}$$

$$\frac{1}{2} e^{-7jt} \longrightarrow \frac{1}{2} e^{-7jt} \cdot e^{21j}$$

$$y(t) = \frac{1}{2}\left[e^{j4(t-3)} + e^{-j4(t-3)}\right] + \frac{1}{2}\left[e^{j7(t-3)} + e^{-j7(t-3)}\right]$$

$$= \cos 4(t-3) + \cos 7(t-3)$$

NOTE –

if $x(t)$ is continuous periodic → Fourier Series
continuous aperiodic → Fourier Transform
discrete periodic → Discrete Time FS
discrete aperiodic → Discrete Time FT

Fourier series –

Representation of periodic signals of complex exponentials

All harmonically related signals are periodic with the periodicity T

$$x(t) = \{e^{j\omega t}, e^{2j\omega t}, e^{3j\omega_0 t}\}$$

harmonically related with the periodicity T_0

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \underbrace{a_{-2}}_{\text{weighting factor.}} e^{-2j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_1 e^{j\omega_0 t} + a_0 + a_2 e^{2j\omega_0 t}$$

$x(t) = e^{j\omega_0 t}$ is a periodic complex exponential with $T = 2\pi/\omega_0$, then the harmonically related complex exponentials is –

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}$$

Thus a linear combination of harmonically related complex exponentials of the form–

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ is also periodic with

(Fourier series representation of continuous Time)

Consider a periodic signal $x(t)$ with fundamental frequency 2π expressed in the form-

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk\omega_0 t}$$

$$a_0 = 1, \quad a_1 = a_{-1} = 1/4, \quad a_2 = a_{-2} = 1/2, \quad a_3 = a_{-3} = 1/3$$

$$= \frac{1}{3}\left[e^{-6jt\pi} + e^{6jt\pi}\right] + \frac{1}{2}\left[e^{-4jt\pi} + e^{4jt\pi}\right]$$

$$+ 1 + \frac{1}{4}\left[e^{-2jt\pi} + e^{2jt\pi}\right]$$

$$= \frac{2}{3}\cos 6\pi t + \cos 4\pi t + 1 + \frac{1}{2}\cos 2\pi t$$

* For real signals $\Rightarrow a_k = a_{-k}$

If $x(t)$ is real,

then $x(t) = x^*(t)$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = x(t) \quad \rightarrow i)$$

replace k with $-k$ —

$$x(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} \quad \rightarrow ii)$$

Comparing i) & ii)

$$a_k = a_{-k}$$

<u>Alternate forms of Fourier Series-</u>

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k e^{j\omega_0 kt} + a_{-k} e^{-j\omega_0 kt}\right]$$

$$= a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\right]$$

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left[a_k e^{jk\omega_0 t}\right]$$

Case I — If $a_k = A_k e^{j\theta_k}$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left[A_k e^{j(k\omega_0 t + \theta_k)}\right]$$

$$= a_0 + 2A_k \cos(k\omega t + \theta_k)$$

Case II — If $a_k = B_k + jC_k$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left[(B_k + jC_k)\, e^{jk\omega_0 t}\right]$$

$$= a_0 + 2\sum_{k=1}^{\infty}\left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right]$$

Fourier Series –

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_0 = \frac{1}{T}\int_{\langle T\rangle} x(t)\, dt$$

Ex - 3.5

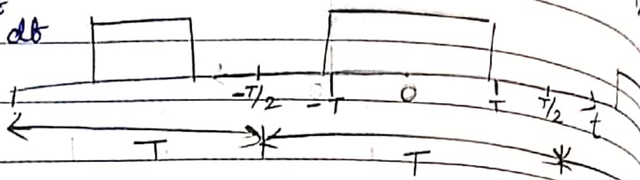
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

find the spectrum.

[$-T/2$ to $T/2$ → symmetric, ∴ we chose T b/w $-T/2$ to $T/2$]

Soln – $a_k = \dfrac{1}{T}\displaystyle\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt.$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$



$$= \frac{1}{T}\int_{-T_1}^{T_1} 1 \cdot e^{-jk\omega_0 t}\, dt.$$

$$= \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1}$$

$$\frac{e^{-x} - e^{x}}{2} = \sin x$$

$$= \frac{-1}{Tjk\omega_0}\left(e^{-jk\omega_0 T_1} - e^{+jk\omega_0 T_1}\right)$$

$$= \frac{2\sin k\omega_0 T_1}{k\omega_0 T}$$

$$T = \frac{2\pi}{\omega_0}$$

$$= \quad (\omega_0 T = 2\pi)$$

$$a_k = \frac{2\sin k\omega_0 T_1}{k\,2\pi}$$

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

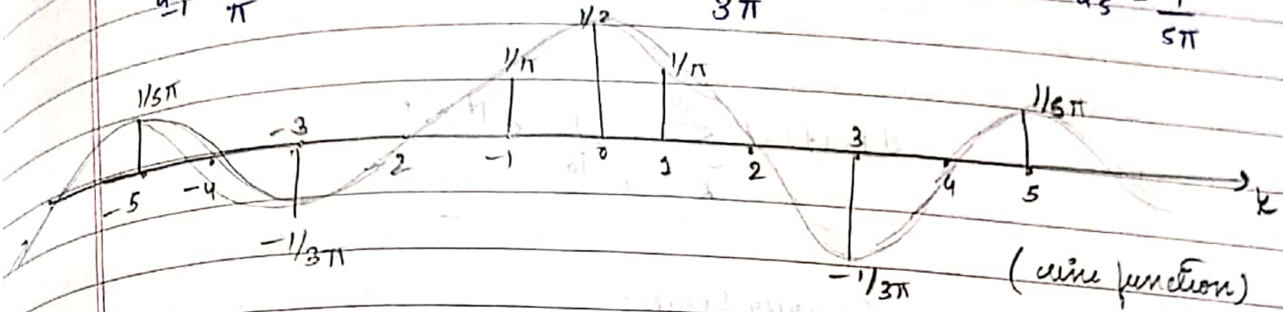
Let $T = 4T_1$

$a_k = \frac{\sin(k\pi/2)}{k\pi}$; $k \neq 0$ $\left\{ a_0 = \frac{2T_1}{4T_1} = \frac{1}{2} \right\}$

$a_0 = 1/2$

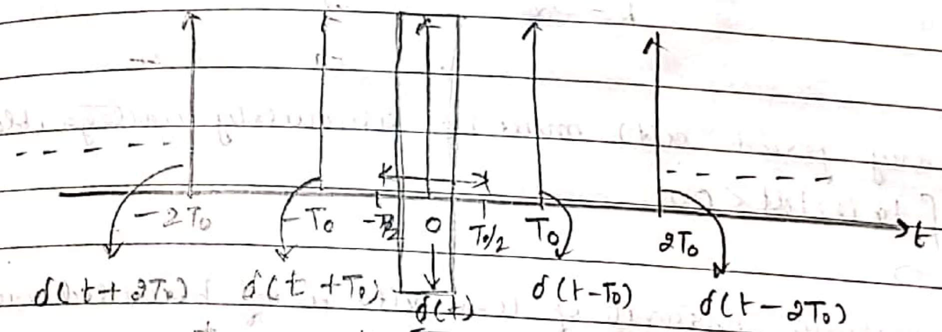
$a_1 = \frac{1}{\pi}$ $a_2 = 0$ $a_3 = \frac{-1}{3\pi}$ $a_4 = 0$ $a_5 = \frac{1}{5\pi}$

$a_{-1} = \frac{1}{\pi}$ $a_{-2} = 0$ $a_{-3} = \frac{-1}{3\pi}$ $a_{-4} = 0$ $a_5 = \frac{1}{5\pi}$



(sinc function)

Q) determine the fourier series representation of an impulse train represented by $\delta_{T_0}(t)$ (Dirac comb)



$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) \qquad k = 0, \pm1, \pm2 \dots$$

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt.$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jk\omega_0 t} dt.$$

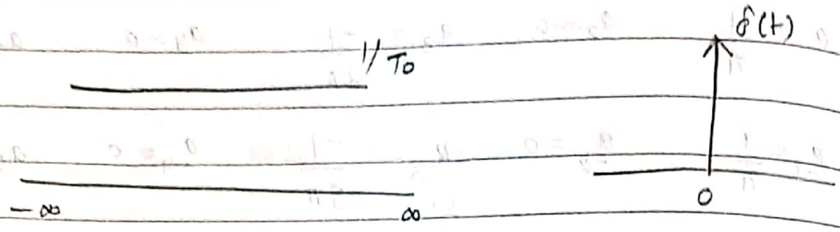
$$\left[\text{Sampling property} - \int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(t)\Big|_{t=0} \right]$$

$$a_k = \frac{1}{T_0} e^{-jk\omega_0 t}\Big|_{t=0} = \frac{1}{T_0}$$

Every circuit that you design, if its analog in nature noise comes into the picture

$$a_k = \frac{1}{T_0}$$

$$\underline{\underline{x(t) = \delta_{T_0}(t)}}$$



$$\boxed{\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t}}$$

Convergence of Fourier Series -

(infinite series should give finite values)

$$\underline{\underline{x(t)}} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

Dirichlet conditions → for convergence.

1) over any period $x(t)$ must be absolutely integrable i.e
$$\int_{\langle T \rangle} |x(t)|\,dt < \infty$$

2) In any finite interval of time $x(t)$ is of bounded variation, there are no more than finite number of maxima and minima during any ~~per~~ single period of signal.

3) In any finite interval of time there are only finite number of discontinuities and each of these discontinuities is finite

Properties of Fourier Series -

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt.$$

$$x(t) \xleftrightarrow{F.S} a_k$$

1) Linearity –

if $x(t) \xleftrightarrow{F.S} a_k$ & $y(t) \xleftrightarrow{F.S} b_k$

then $z(t) = \alpha x(t) + \beta y(t) \xleftrightarrow{F.S} c_k = \alpha a_k + \beta b_k$

Proof –

$$c_k = \frac{1}{T}\int_T z(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_T [\alpha x(t) + \beta y(t)] e^{-jk\omega_0 t} dt = \alpha \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt + \beta \frac{1}{T}\int_T y(t) e^{-jk\omega_0 t} dt.$$

$$= \alpha a_k + \beta b_k.$$

2) Time-shift –

If $x(t) \xleftrightarrow{F.S} a_k$

then $y(t) = x(t-t_0) \xleftrightarrow{F.S} b_k = e^{-jk\omega_0 t_0} a_k$

Proof – $b_k = \frac{1}{T}\int_T x(t-t_0) e^{-jk\omega_0 t} dt$

let $t - t_0 = \lambda \Rightarrow dt = d\lambda$

$t = t_0 + \lambda$.

$$b_k = \frac{1}{T}\int_T x(\lambda) e^{-jk\omega_0 (t_0+\lambda)} d\lambda$$

$$= e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\lambda) e^{-jk\omega_0 \lambda} d\lambda$$

$$b_k = e^{-jk\omega_0 t_0} a_k.$$

3) Time-Reversal –

if $x(t) \xleftrightarrow{F.S} a_k$

then $y(t) = x(-t) \xleftrightarrow{F.S} b_k = a_{-k}$

Proof –

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t} \quad \text{let } k = -m$$

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$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jw_0 m t}$$

$$\Rightarrow x(-t) \xleftrightarrow{F.S} a_{-k}$$

4. <u>Time differentiation</u> -

If $x(t) \xleftrightarrow{F.S} a_k$

then $y(t) = \frac{dx(t)}{dt} \xleftrightarrow{F.S} b_k = jkw_0 a_k$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} (jkw_0 a_k) \cdot e^{jkw_0 t}$$

$$\frac{dx(t)}{dt} \xleftrightarrow{F.S} jkw_0 a_k$$

5. <u>Multiplication</u> - / <u>Modulation</u> -

If $x(t) \xleftrightarrow{F.S} a_k$

& $y(t) \xleftrightarrow{F.S} b_k$

then $z(t) = x(t) y(t) \xleftrightarrow{F.S} c_k = a_k * b_k$

<u>Proof</u> -

$$c_k = \frac{1}{T} \int_T z(t) e^{-jkw_0 t} dt$$

$$= \frac{1}{T} \int_T x(t) y(t) e^{-jkw_0 t} dt$$

$$= \frac{1}{T} \int_T \left(\sum_{l=-\infty}^{\infty} a_l e^{jlw_0 t} \right) y(t) e^{-jkw_0 t} dt$$

(product in time domain becomes convolution in frequency domain and vice versa)

$$= \sum_{l=-\infty}^{\infty} a_l \left[\frac{1}{T} \int_T y(t) e^{-j(k-l)\omega_0 t} dt \right]$$

$$= \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$= a_k * b_k.$$

6) Conjugation and Conjugate symmetry -

If $x(t) \xleftrightarrow{F.S} a_k$

then $x^*(t) \xleftrightarrow{F.S}$

Proof - $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

Replacing k with -k

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

$$x^*(t) \xleftrightarrow{F.S} a_{-k}^*$$

1) if $x(t)$ is real

$x(t) = x^*(t) \Rightarrow a_k = a_{-k}^*$

(or)

$$\boxed{a_k^* = a_{-k}}$$

7) Parseval's Relation -

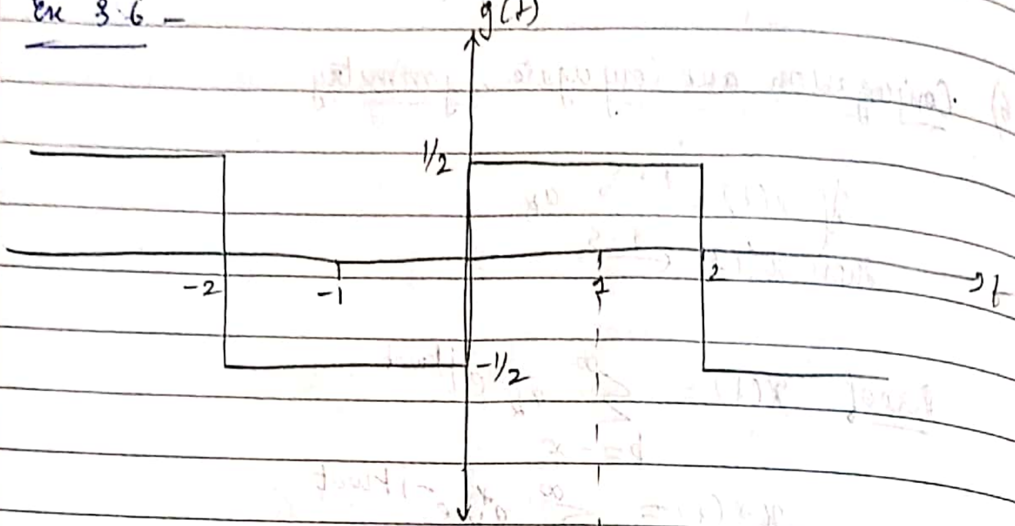
$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Proof-

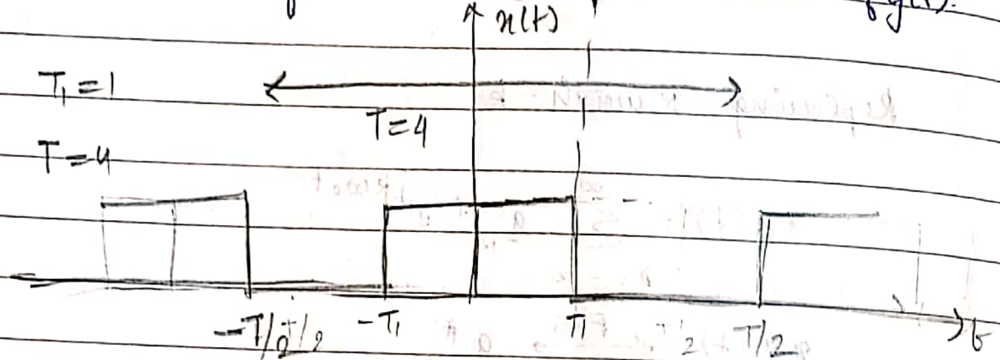
LHS. $\frac{1}{T}\int_T \left|\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right|^2 dt$

$= \sum_{k=-\infty}^{\infty} |a_k|^2 = RHS.$

Ex 3.6 -



Determine the fourier Series representation of y(t).



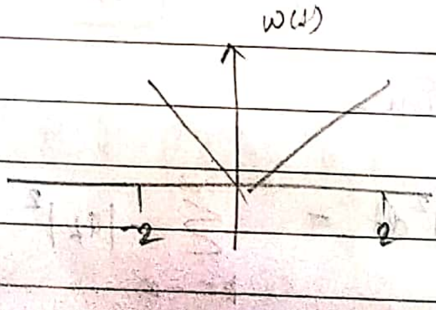
$T_1 = 1$

$T = 4$

$g(t) = x(t-1) - 1/2$ | $T_1 = 1$, $T = 4$

$x(t) \longleftrightarrow a_k$

$x(t-1) \longleftrightarrow e^{-jk\omega_0} a_k.$



F.S representation of g(t).

Suppose we are given the following facts of a signal $x(t)$.

1) $x(t)$ is a real signal

2) $x(t)$ is periodic, $T=4$ or $x(t) \xleftrightarrow{F.S} a_k$

3) $a_k = 0,\ |k|>1 \Rightarrow a_{-1}, a_0, a_1$

4) The signal with Fourier coefficients $b_k = e^{-j\pi k/2}\, a_{-k}$ is odd

5) $\frac{1}{4}\int_{\langle 4\rangle} |x(t)|^2 dt = 1/2.$

find $x(t)$.

soln - Using fact 3).

a_{-1}, a_0, a_1

$T=4 \Rightarrow \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \pi/2$

$x(t) = a_{-1} e^{-j\pi/2 t} + a_0 + a_1 e^{j\pi/2 t} = \sum_{k=-1}^{1} a_k e^{jk\pi/2 t}$

$x(t)$ is real

$\Rightarrow x(t) = x^*(t)$

$\downarrow \qquad \downarrow$

$a_k = a^*_{-k}$

or $\boxed{a^*_k = a_{-k}} \Rightarrow a_{-1} = a^*_1$

$x(t) = a_0 + a_1 e^{j\pi/2 t} + a^*_1 e^{-j\pi/2 t}$

$x(t) = a_0 + 2\,\mathrm{Re}\left[a_1 e^{j\pi/2 t}\right]$

Using 4)

$a_k \xleftrightarrow{F.S} x(t)$

$a_{-k} \xleftrightarrow{F.S} x(-t)$

$b_k = a_{-k}\, e^{-jk\pi/2} \xleftrightarrow{F.S} x[-(t-1)]$

$= x(-t+1)$

$$e^{-jk\omega t_0} a_k \longleftrightarrow x[-(t-t_0)]$$

$$x(-t+t_0)$$

$$b_k \xleftrightarrow{F.S} x(-t+1)$$

since b_k is odd

$\Rightarrow b_0 = 0$

b_k has value for $k = -1, 1$

$\Rightarrow$ $x(t)$	a_k
real or even	real or even
real or odd	purely imaginary or odd

Using ① & ⑤ $\Rightarrow$ since b_k is odd $\Rightarrow b_{-k} = -b_k$

$\Rightarrow b_k$ is purely imaginary

$k=1$ $\boxed{b_{-1} = -b_1}$

Using ③ $\frac{1}{4}\int_{\langle 4\rangle} |x(t)|^2 dt = 1/2$ $\qquad \frac{1}{T}\int_{\langle T\rangle} |x(t)|^2 dt = \sum |a_k|^2$

$$= \sum_{k=-\infty}^{\infty} |b_k|^2 = |b_{-1}|^2 + |b_1|^2$$

$$|b_{-1}|^2 + |b_1|^2 = 1/2$$

$$|b_{-1}| = |b_1|$$

$$|b_1|^2 + |b_1|^2 = 1/2$$

$$2|b_1|^2 = 1/2 \qquad a+jb$$

$$|b_1| = \frac{1}{2} \qquad b_1 = j/2$$

$(|b_1| = 1/2)$

Since b_1 is purely imaginary $\qquad b_1 = -j/2$

$b_1 = j/2$ or $b_1 = -j/2$ $\qquad (|b_1| = 1/2)$

$$x(t) = a_0 + 2\,\mathrm{Re}\left[a_1 e^{j\pi/2 t}\right]$$

$$b_k = e^{-jk\pi/2 t} a_{-k}$$

$k \rightarrow -k$

$a_k = e^{-jk\pi/2} b_k$

$a_1 = e^{-j\pi/2} b_{-1} = e^{-\pi/2}\left(\frac{-j}{2}\right) = \frac{1}{2}$

$b_1 = j/2 \Rightarrow a_1 = 1/2$

$x(t) = 2Re\left(\frac{1}{2} e^{j\pi/2 t}\right) = \cos\frac{\pi}{2}t$

$b_1 = -j/2 \Rightarrow a_1 = -1/2$

$x(t) = -\cos\frac{\pi}{2}t.$

Discrete-time Fourier series –

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

a_k is periodic.

$\Rightarrow x(n) = x(n+N), \forall n$

Let $\phi_k[n] = e^{jk\omega_0 n}$ → set of harmonically related signals.

$$x(n) = \sum_{k=\langle n\rangle} a_k e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N} \longrightarrow (1)$$

$k = 0, 1, \ldots N-1$

Eqn (1) is DTFS representation of $x(n)$.

$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x(n) e^{-jk\omega_0 n}$	$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t) e^{-jk\omega_0 t} dt$
$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$

$\Rightarrow$ PT the fourier series coefficients a_k for discrete time fourier signals are periodic.

1) $a_k = a_{k+N}$

$$a_{k+N} = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-j(k+N)w_0 n}$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-jk w_0 n}\, \underline{e^{-jNw_0 n}}$$

$$\Downarrow$$

$$\cancel{e^{-jN\frac{2\pi}{N}n}} \to 1$$

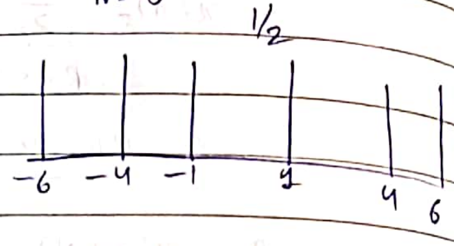
$$= a_k$$

$\Rightarrow$ Ex (3.10) $x(n) = \sin(w_0 n)$

Find a_k

$$= \underbrace{\frac{1}{2j}e^{jw_0 n}}_{k=1} - \underbrace{\frac{1}{2j}e^{-jw_0 n}}_{k=-1}$$

$N = 5.$



$$\Rightarrow a_1 = \frac{1}{2j} \qquad a_{-1} = \frac{-1}{2j}$$

$$|a_1| = \frac{1}{2} \qquad |a_{-1}| = \frac{1}{2}.$$

$$a_1 = a_{1+5} = a_6$$

$$a_{-1} = a_{-1+5} = a_4$$

$\Rightarrow$ Ex 3.11

$$x(n) = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right) + \cos\left(\frac{4\pi}{N} + \frac{\pi}{2}\right)$$

Find a_k.

$w_0 = \frac{2\pi}{N}$

$a_k = a$

$= 1 + \sin w_0 n + 2\cos w_0 n + \cos(2w_0 n + \pi/2)$

$= 1 + \frac{1}{2j} e^{jw_0 n} - \frac{1}{2j} e^{-jw_0 n} + \frac{2}{2} e^{jw_0 n} + \frac{2}{2} e^{-jw_0 n} + \frac{1}{2} e^{j\pi/2} e^{j2w_0 n} + \frac{1}{2} e^{-j\pi/2} e^{-j2w_0 n}$

$a_0 = 1$

$a_1 = \frac{1}{2j} + \frac{2}{2}$

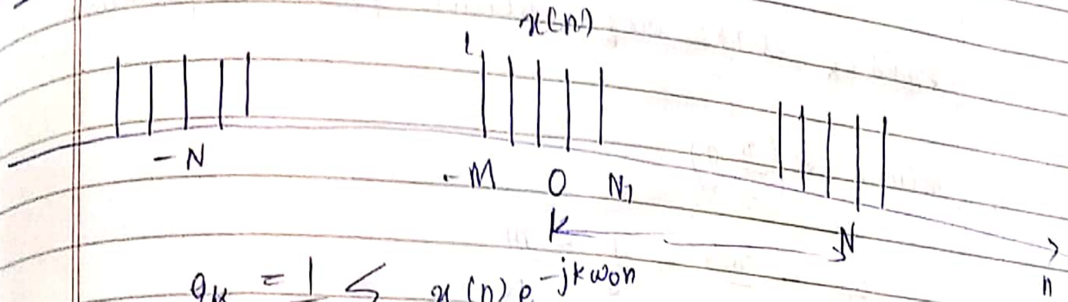
$a_{-1} = \frac{-1}{2j} + \frac{2}{2}$

$a_1^* = a_{-1} \Rightarrow x(n)$ is real

$a_2 = \frac{1}{2} e^{j\pi/2}$

$a_{-2} = \frac{1}{2} e^{-j\pi/2}$

Ex 3.12 Find a_k.



$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-jkw_0 n}$; $w_0 = \frac{2\pi}{N}$

$= \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 \cdot e^{-jkw_0 n}$

$\sum_{m=0}^{n} a^m = \frac{1-a^{n+1}}{1-a}$

let $n + N_1 = m$

$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jkw_0(m-N_1)} = \frac{1}{N} e^{jkw_0 N_1} \sum_{m=0}^{2N_1} e^{-jkw_0 m}$

$= \frac{1}{N} e^{jkw_0 N_1} \left[\frac{1-e^{-jkw_0(2N_1+1)}}{1-e^{-jkw_0}} \right]$

$$a_k = \frac{1}{N} \frac{\sin[(N_1+1/2)/N]}{\sin(\pi k/N)} \quad ; k \neq 0, \pm N, \pm 2N \ldots$$

$k=0 \qquad a_0 = \frac{1}{N}\sum_{m=0}^{2N_1} 1 = \frac{2N_1+1}{N}$

(or) $\quad a_k = \frac{2N_1+1}{N}, \quad k = 0, \pm N, \pm 2N \ldots$

⇒ Evaluate discrete fourier coefficient for the sequence

i) $x(n) = \cos\left(\frac{\pi}{4}n\right) \rightarrow \frac{\omega_0}{2\pi} = \frac{m}{N}$

ii) $x(n) = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}\right)n + 1$

Find a_k, DTFS representation, plot the spectrum.

1) $x(n) = \cos\left(\frac{\pi}{4}n\right)$

$$\frac{\omega_0}{2\pi} = \frac{\pi/4}{2\pi} = \frac{1}{8} = \frac{m}{N}$$

$$N = 8.$$

$$x(n) = \frac{1}{2}e^{j\pi/4 n} + \frac{1}{2}e^{-j\pi/4 n}$$

$$= \frac{1}{2}e^{j2\pi/8 n} + \frac{1}{2}e^{-j2\pi/8 n}$$

$$= \underbrace{\frac{1}{2}e^{j\omega_0 n}}_{k=1} + \underbrace{\frac{1}{2}e^{-j\omega_0 n}}_{k=-1}$$

$a_{-1} = \frac{1}{2}$ $\quad a_1 = \frac{1}{2}$

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

$a_k = a_{k+N}$

$a_{-1} = a_{-1+8}$

$= a_7$

$$x(n) = \sum_{k=-3}^{4} a_k e^{jk\omega_0 n}, \quad k = -3 \cdots 4$$

⇒ Determine the DTFS Representation for a sequence

1. $x(n) = \cos^2\left(\frac{\pi}{8} n\right)$

DTFS $x(n) = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$

2. $\cancel{x(n) = \sum_{m=}}$ $\quad \cancel{\frac{\omega_0}{2\pi}} = \cancel{\frac{\pi/8}{2\pi}} = \cancel{\frac{1}{16}} = \cancel{\frac{m}{N}}$

$\cancel{N = 16}$

$\cancel{x(n) =}$

$$= \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{8} n\right)$$

$$\frac{\omega_0}{2\pi} = \frac{\cancel{2}\pi}{8 \times \cancel{2\pi}} = \frac{1}{8} = \frac{m}{N}$$

$N = 8$

$\omega_1 = \frac{2\pi}{8}$

$$x(n) = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2} e^{j2\pi/8 n} + \frac{1}{2} e^{-j2\pi/8 n}\right]$$

$$= \frac{1}{2} + \frac{1}{4} e^{j2\pi/8 n} + \frac{1}{4} e^{-j2\pi/8 n}$$

$\Rightarrow a_0 = 1/2 \quad a_1 = 1/4 \quad a_{-1} = 1/4$

classmate

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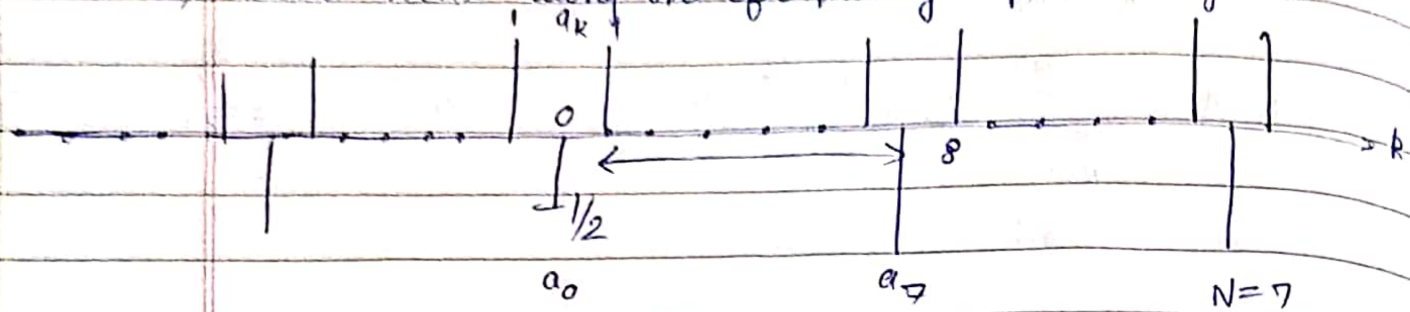
Page

$$x(n) = \sum_{k=-3}^{4} a_k e^{jk2\pi/e\, n}$$

Q) Find $x(n)$ using the frequency spectrum given



$N=7$

$$\left\{ \begin{aligned} a_0 &= a_{0+7} = a_7 \\ a_k &= a_{k+N} \\ N &= 7 \end{aligned} \right\}$$

$$\omega_0 = \frac{2\pi}{7}$$

$a_{-3} = 0$

$a_{-2} = 0$

$a_{-1} = 1$

$a_0 = -1/2$

$a_1 = 1$

$a_2 = 0$

$a_3 = 0.$

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk2\pi/N n}$$

$$= \sum_{k=-3}^{3} a_k e^{jk2\pi/N n}$$

$$= a_{-1} e^{-j\omega_0 n} + a_0 + a_1 e^{j\omega_0 n}$$

$$= 1 e^{-j\omega_0 n} + (-1/2) + 1 e^{j\omega_0 n}$$

$$= e^{-j\omega_0 n} + e^{j\omega_0 n} - \frac{1}{2}$$

Properties of DTFS

$x(n) \xleftrightarrow{DTFS} a_k$

1. Linearity

$Ax(n) + By(n) \xleftrightarrow{DTFS} Aa_k + Bb_k.$

2. Time shift

$x(n-n_0) \longleftrightarrow a_k e^{-jk\omega_0 n_0}$

3. Frequency shift

$e^{jM\omega_0 n} \cdot x(n) \longleftrightarrow a_{k-M}$

4. $x^*(n) \longleftrightarrow a^*_{-k}$

5. $x(-n) \longleftrightarrow a_{-k}$

6. $x_m(n) = \begin{cases} x(n/m), & \text{if } n \text{ is a multiple of } m \\ 0, & \text{otherwise} \end{cases} \longleftrightarrow \frac{1}{m} a_k$

7. (periodic convolution) $x(n) * y(n) \longleftrightarrow N a_k b_k$ [x(n) & y(n) should have same periodicity N)]

8. $x(n)\,y(n) \longleftrightarrow \sum_{l=\langle N \rangle} a_l b_{k-l}$

9. $x(n) - x(n-1) \longleftrightarrow (1-e^{-jk\omega_0}) a_k$

First difference

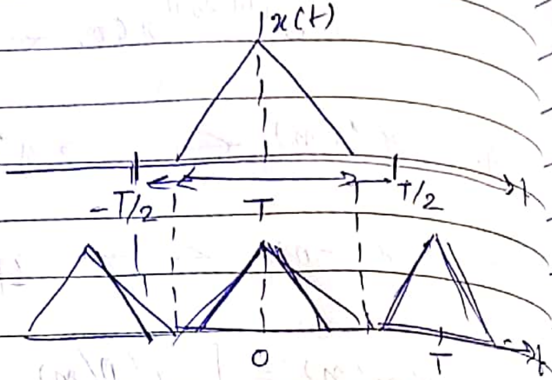
$\longrightarrow n$

Syllabus-
Ch- 3
3.1 - 3.7

H.W- Exercise-
{3.1 — 3.12}
{3.21 — 3.31}

Ch-4: Fourier Transform -

Fourier representation of Aperiodic Signal



The fourier representation of $x_T(T)$ will also represent $x(t)$ in the limit $T \to \infty$

$$\Rightarrow \lim_{T\to\infty} x_T(t) = x(t) \longrightarrow (1)$$

$$F.S \text{ of } x_T(t) \quad x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow (2)$$

$$\text{when } a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)\, e^{-jk\omega_0 t}\, dt \quad ; \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_{T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt.$$

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Let we define $x(j\omega_0) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega_0 t}\, dt. \longrightarrow 3)$

$\Rightarrow a_k = \frac{1}{T} X(jk\omega_0) \longrightarrow 4)$

Sub (4) in (2)

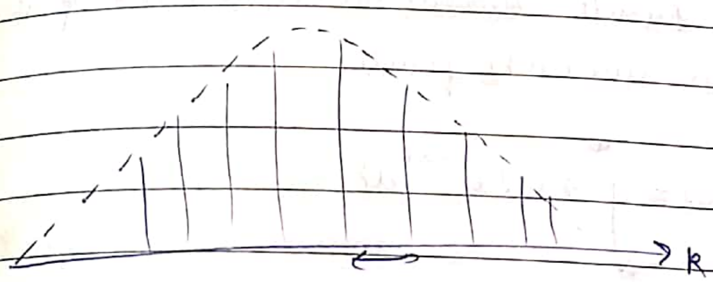
$$x_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)\, e^{jk\omega_0 t}$$

$$T = \frac{2\pi}{\omega_0}$$

$$x_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} (jk\omega_0) e^{jk\omega_0 t} \times \omega_0$$

$$x(t) = \lim_{T \to \infty} x_T(t)$$

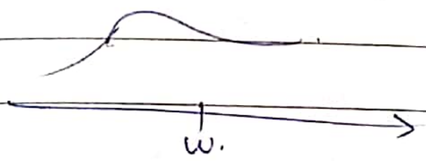
$$x(t) = \lim_{T \to \infty} \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} (jk\omega_0) e^{jk\omega_0 t}\, \omega_0$$



$T \to \infty$

$\omega_0 \to 0 \equiv \Delta\omega \to 0$

$k\omega_0 \to \omega.$



since $\Delta\omega \to 0$

(The samples of become very small hence the signal becomes continuous)

$$x(t) = \lim_{\Delta\omega \to 0} \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} X(j\omega) e^{j\omega t}\, \Delta\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega.$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

&

$$x(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

$$x(t) \xleftrightarrow{FT} X(j\omega).$$

$$F[x(t)] = X(j\omega)$$

$$\boxed{x(t) = F^{-1}\, X(j\omega).}$$

<u>Existence of FT</u> -

1) $\int |x(t)|\, dt < \infty$

2) Finite Maxima & Minima

3) Finite discontinuities & each of those discontinuities should be finite.

$\Rightarrow$ $L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

where $\Rightarrow$ $s = \sigma + j\omega$

Obtain the fourier transform of the signal $x(t)$

$$x(t) = e^{-at} u(t) \; ; \; a>0$$

$$x(t) \xleftrightarrow{F.T} X(j\omega).$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

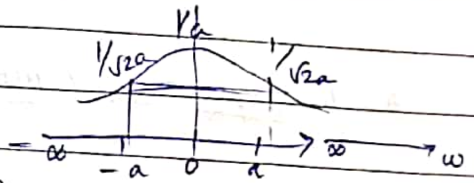
$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

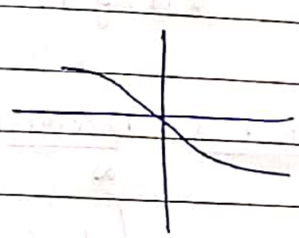
$$X(j\omega) = \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \left. \frac{e^{-(a+j\omega)t}}{a+j\omega} \right]_0^{\infty}$$

$$X(j\omega) = \frac{1}{a+j\omega} \; ; \; a>0$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$



$$\angle X(j\omega) = -\tan^{-1}(\omega/a)$$



Properties of Fourier Transform -

1. Linearity -

$$a x(t) + b y(t) \xleftrightarrow{FT} a X(j\omega) + b Y(j\omega)$$

2. Time shift -

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$$

3. Frequency shift -

$$e^{j\beta t} x(t) \xleftrightarrow{FT} X(j(\omega-\beta))$$

4. Scaling -

$$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

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5. Time differentiation

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$$

6. Frequency differentiation

$$-jt\,x(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(j\omega).$$

7. Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$$

8. Convolution:-

$$x(t) * y(t) \xleftrightarrow{FT} X(j\omega)Y(j\omega)$$

9. Multiplication / Modulation -

$$x(t)y(t) \xleftrightarrow{FT} \frac{1}{2\pi}\left[X(j\omega) * Y(j\omega)\right]$$

10. Parseval's theorem -

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |x(j\omega)|^2 d\omega.$$

11) Duality -

If $x(t) \xleftrightarrow{FT} X(j\omega)$

then $X(jt) \xleftrightarrow{FT} 2\pi x(-\omega)$

12) Symmetry -

$x(t)$: real $\xleftrightarrow{FT} X^*(j\omega) = X(-j\omega)$

$x(t)$: real or even $\xleftrightarrow{FT} \text{Img}[X(j\omega)] = 0$

$x(t)$: real or odd $\xleftrightarrow{FT} \text{Re}[X(j\omega)] = 0$

Q) Obtain the Fourier transform of the following signals.

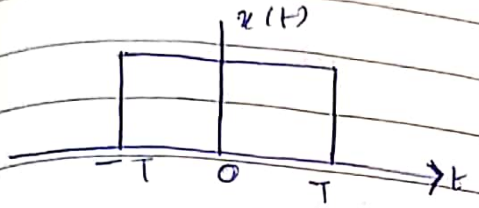
1) $x(t) = e^{-a|t|}$; $a > 0$

2) $x(t) = \delta(t)$

3) $x(t) = 1$

4) $x(t) = s$

5) $x(t) = u(t)$

6) 

1) $x(t) \xleftrightarrow{FT} X(j\omega)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

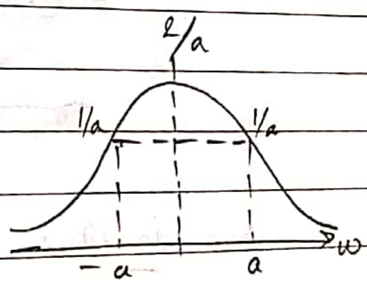
$$= \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-\infty}^{0} e^{t(a-j\omega)}\, dt + \int_{0}^{\infty} e^{t(-a-j\omega)}\, dt$$

$$= \left. \frac{e^{t(a-j\omega)}}{a-j\omega} \right]_{-\infty}^{0} + \left. \frac{e^{t(-a-j\omega)}}{-a-j\omega} \right]_{0}^{\infty}$$

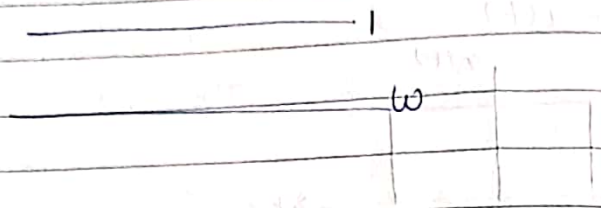
$$= \frac{2a}{a^2+\omega^2}$$



2) $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=0} = 1$$



3) Duality.

$$x(t) \longleftrightarrow X(j\omega)$$
$$x(jt) \longleftrightarrow 2\pi x(-\omega)$$

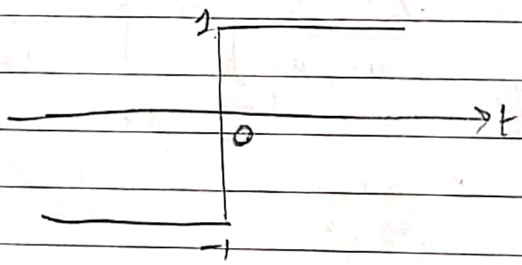
$$\delta(t) \longleftrightarrow 1$$
$$1 \longleftrightarrow 2\pi\,\delta(-\omega) = 2\pi\,\delta\omega.$$

$$\therefore 1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)$$

4) Signum function -

$$x(t) = \mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

$x(t) = \mathrm{sgn}(t)$



$$\frac{dx(t)}{dt} = 2\delta(t)$$

$\Downarrow FT \qquad \Downarrow FT$

$j\omega X(j\omega) \qquad 2$

$$X(j\omega) = \frac{2}{j\omega}$$

5) $x(t) = u(t)$

$sgn(t) = 2u(t) - 1$

$$x(t) = u(t) = \frac{sgn(t)}{2} + \frac{1}{2} \cdot 1$$

$\Downarrow$

$$X(j\omega) = \frac{\cancel{2}}{j\omega\cancel{2}} + \frac{1}{\cancel{2}} \cancel{2}\pi\delta(\omega)$$

$$= \frac{1}{j\omega} + \pi\delta(\omega)$$

6) $$x(j\omega) = \int_{-T}^{T} 1 \cdot e^{-j\omega t}\, dt \qquad \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right]$$

Q) compute the fourier transform of the following signals—

1. $x(t) = e^{-t-1}\, u(t)$
2. $x(t) = e^{-t - j\pi t}\, u(t)$.

1. $x(t) = e^{-t-1}\, u(t)$.

$$X(j\omega) = e^{-1} \int_{-\infty}^{\infty} e^{-t} u(t) e^{-j\omega t}\, dt$$

$$= e^{-1} \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt$$

$$= e^{-1} \int_{0}^{\infty} e^{(-1-j\omega)t}\, dt$$

$$X(j\omega) = \frac{e^{-1}}{1 + j\omega}$$

$$X^*(j\omega) = \frac{e^{-1}}{1-j\omega} = X(-j\omega)$$

$$X^*(j\omega) = X(-j\omega)$$

$\hookrightarrow$ Conjugate symmetry.

2) $x(t) = e^{-t} e^{-j\pi t} u(t)$

$= e^{-j\pi t} y(t)$

$y(t) = e^{-t} u(t) \longleftrightarrow Y(j\omega) = \frac{1}{1+j\omega}$.

$x(t) = e^{-j\pi t} y(t) \longleftrightarrow X(j\omega) = Y(j(\omega+\pi))$

(from frequency shift) $= \frac{1}{1+j(\omega+\pi)}$.

$$\left\{ \begin{array}{l} e^{-at} u(t) \longleftrightarrow \frac{1}{a+j\omega} \\ e^{-j\beta t} x(t) \longleftrightarrow X(j(\omega-\beta)) \end{array} \right\}$$

3) $x(t) = \frac{1}{a^2+t^2}$, $a>0$

$$e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{a^2+\omega^2}.$$

$$X(jt) = \frac{2a}{a^2+t^2} \xleftrightarrow{FT} \left[2\pi x(-\omega)\right]$$

$$2\pi e^{-a|-\omega|}$$

$$= 2\pi e^{-a|-\omega|}$$

$$\frac{\cancel{2a}\,1}{a^2+t^2} \xleftrightarrow{FT} \frac{\cancel{2}\pi e^{-a|-\omega|}}{\cancel{2}a} = \frac{\pi}{a} e^{-|\omega|}$$

(Using linearity, frequency shift)

1) $x(t) = \cos\omega_0 t$

$= \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \dfrac{e^{j\omega_0 t}}{2} + \dfrac{e^{-j\omega_0 t}}{2}$

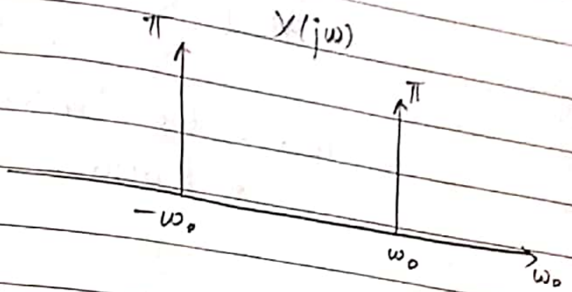
$= \frac{1}{2} e^{j\omega_0 t}$

$1/2 \longrightarrow \pi\delta(\omega)$

$1/2\, e^{j\omega_0 t} \longleftrightarrow \pi\delta(\omega-\omega_0)$

$1/2\, e^{-j\omega_0 t} \longleftrightarrow \pi\delta(\omega+\omega_0)$

$X(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$



Q) Using the properties of fourier transform find the FT.

1) $x(t) = \sin(\pi t)\, e^{-2t} u(t)$.

$= y(t)\, e^{-2t} u(t)$.

$y(t) = \sin(\pi t)$

$= \dfrac{e^{j\pi t} - e^{-j\pi t}}{2j}$

$e^{-at} u(t) \overset{FT}{\longleftrightarrow} \dfrac{1}{a+j\omega}$

$e^{-2t} u(t) \overset{FT}{\longleftrightarrow} \dfrac{1}{2+j\omega}$

$\dfrac{1}{2j} e^{j\pi t} e^{-2t} u(t) \longleftrightarrow \dfrac{1}{2j}\left[\dfrac{1}{2+j(\omega-\pi)}\right]$

$-\dfrac{1}{2j} e^{-j\pi t} e^{-2t} u(t) \longleftrightarrow \dfrac{1}{-2j}\left[\dfrac{1}{2+j(\omega+\pi)}\right]$

$X(j\omega) = \dfrac{1}{2j}\left[\dfrac{1}{2+j(\omega-\pi)} - \dfrac{1}{2+j(\omega+\pi)}\right]$ //.

2) $x(t) = e^{-3|t-2|}$

$= e^{-3} \cdot e^{|t-2|}$

$x(t) = e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{a^2+\omega^2}$

$x(t) = e^{-3|t|} \xleftrightarrow{FT} X(j\omega) = \frac{2 \times 3}{9+\omega^2} = \frac{6}{9+\omega^2}$

$x(t-t_0) = e^{-3|t-2|} \xleftrightarrow{FT} X(j\omega) = e^{-j\omega 2} \frac{6}{9+\omega^2}$

3) $x(t) = \frac{d}{dt}\left[t e^{-2t} \sin t u(t)\right]$

$= \frac{d}{dt}\left[\underbrace{t e^{-2t} \sin t u(t)}_{y(t)}\right]$

$y(t) = t e^{-2t} \sin t u(t)$

$= t \sin t \, e^{-2t} u(t)$

$\left\{ e^{-2t} u(t) \xleftrightarrow{FT} \frac{1}{2+j\omega} \right\}$

$\sin t \, e^{-2t} u(t) \xleftrightarrow{FT}$

$\frac{e^{jt} - e^{-jt}}{2j}$

$\frac{e^{jt}}{2j} e^{-2t} u(t) \longleftrightarrow \frac{1}{2j}\left[\frac{1}{2+j(\omega-1)}\right]$

$\frac{e^{-jt}}{-2j} e^{-2t} u(t) \longleftrightarrow \frac{1}{-2j}\left[\frac{1}{2+j(\omega+1)}\right]$

$= \frac{1}{2j}\left[\frac{1}{2+j(\omega-1)} - \frac{1}{2+j(\omega+1)}\right]$

Scaling =

$t \sin t e^{-2t} u(t) = \frac{1}{t}$

$$t e^{-at} u(t) \xleftrightarrow{FT} \frac{1}{(a+j\omega)^2}$$

$$t e^{-2t} u(t) \xleftrightarrow{FT} \frac{1}{(2+j\omega)^2}$$

$$\frac{1}{2j} e^{jt}\, t e^{-2t} u(t) \xleftrightarrow{FT} \frac{1}{2j} \frac{1}{[2+j(\omega-1)]^2}$$

$$\frac{-1}{2j} e^{jt}\, t e^{-2t} u(t) \xleftrightarrow{FT} \frac{-1}{2j} \cdot \frac{1}{[2+j(\omega+1)]^2} \quad + Y(j\omega)$$

$$Y(j\omega) = \frac{1}{2j}\left[\frac{1}{[2+j(\omega-1)]^2} - \frac{1}{2+j(\omega+1)^2}\right]$$

$$\frac{d}{dt} y(t) \longleftrightarrow j\omega Y(j\omega).$$

$$= j\omega\left[\frac{1}{2j}\left[\frac{1}{2+j(\omega-1)^2}\right] - \frac{1}{2j}\frac{1}{2+j(\omega+1)^2}\right]$$

Fourier Transform of Periodic Signals-

$$\left[x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right]$$

Let $x(t)$ be a period signal.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

$$a_k \xleftrightarrow{FT} 2\pi a_k \delta(\omega).$$

$$e^{jk\omega_0 t} a_k \xleftrightarrow{FT} 2\pi a_k \delta(\omega - k\omega_0)$$

$$\left[\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} a_k \xleftrightarrow{FT} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)\right]$$

$\Rightarrow$ Compute the fourier transform of an impulse Train -

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(\omega - kT)$$

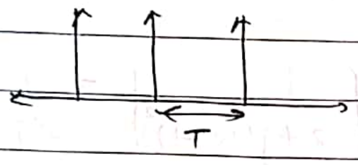
$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt.$$

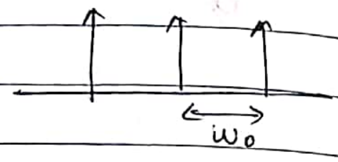
$$\boxed{a_k = \frac{1}{T}}$$

FT of periodic signal

$$x(t) \longleftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(\omega - k\omega_0)$$

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \xleftrightarrow{\text{FT.}} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$





Q) Find the FT of the periodic signal $x(t)$ and sketch the spectrum

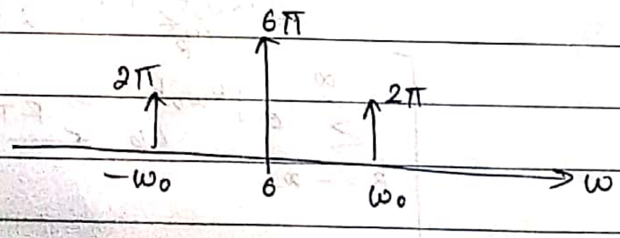
$$x(t) = 3 + 2\cos(10\pi t).$$

$$x(t) = 3 + \cancel{2}\left[\frac{e^{j10\pi t} + e^{-j10\pi t}}{\cancel{2}}\right] \qquad \omega_0 = 10\pi.$$

$$a_0 = 3 \qquad a_1 = 1 \qquad a_{-1} = 1$$

$$x(t) \xleftrightarrow{\text{FT}} \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(\omega - k\omega_0) \qquad \delta(n+1) + 3\delta(n) + \delta(n-1)$$

$$x(t) \longleftrightarrow 2\pi\, \delta(\omega + \omega_0) + 6\pi\delta(\omega) + 2\pi\delta(\omega - \omega_0)$$

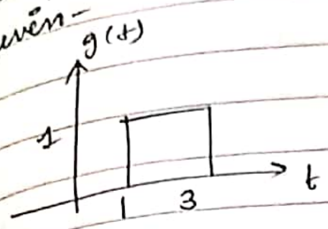


8) Given the FT pair

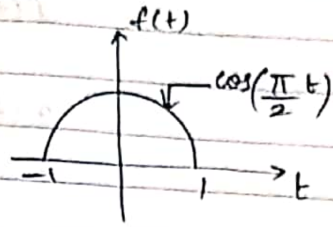
$$x(t) = \begin{cases} 1, & |t|<1 \\ 0, & |t|>1 \end{cases} \xleftrightarrow{FT} X(j\omega) = \frac{2\sin\omega}{\omega}$$

using the properties of FT, find the FT of the functions given–

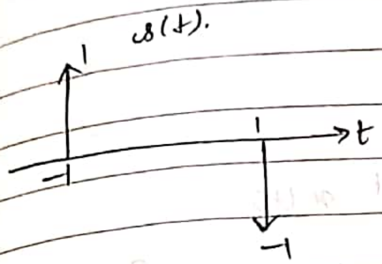
1)



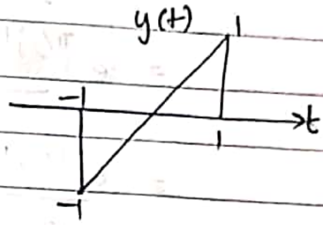
2)



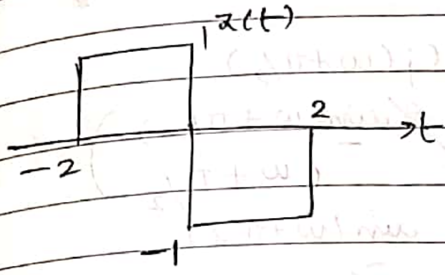
3)



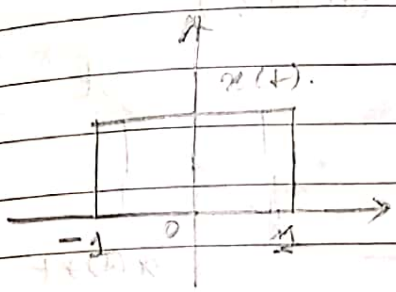
4)



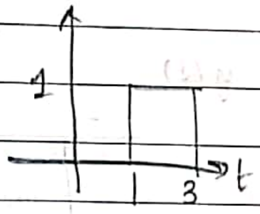
5)



1) $x(t)$.



$g(t) =$



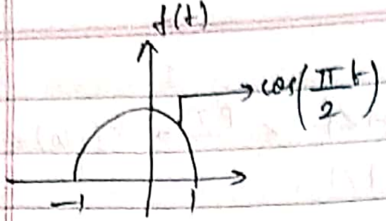
$$g(t) = x(t-3)$$

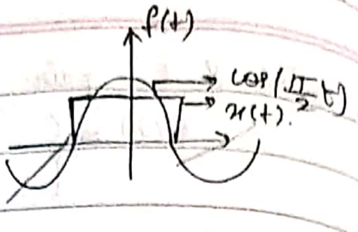
$$x(t) \longleftrightarrow X(j\omega)$$

$$\longleftrightarrow \frac{2\sin\omega}{\omega}$$

$$x(t-2) \longleftrightarrow e^{-j\omega 2}\,\frac{2\sin\omega}{\omega} = g(j\omega)$$

2)





$$f(t) = \cos\left(\frac{\pi t}{2}\right) x(t).$$

$$x(t) \longleftrightarrow X(j\omega)$$

$$\frac{2\sin\omega}{\omega}.$$

$$\cos\left(\frac{\pi t}{2}\right) x(t)$$

$$= \frac{e^{\pi/2 jt} + e^{-\pi/2 jt}}{2} \cdot x(t)$$

$$= \frac{e^{\pi/2 jt}}{2} x(t) + \frac{e^{-\pi/2 jt}}{2} x(t)$$

$$\beta = \frac{\pi}{2}$$

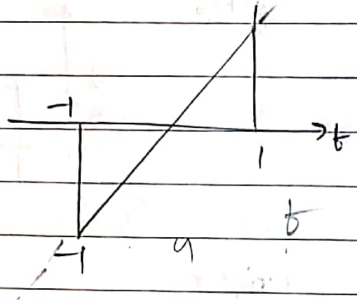
$$\Downarrow$$

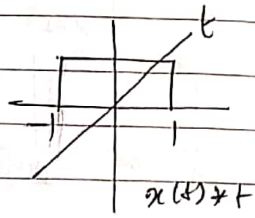
$$= X(j(\omega - \pi/2)) + X(j(\omega + \pi/2))$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}\sin(\omega - \pi/2)}{(\omega - \pi/2)} + \frac{1}{\cancel{2}} \frac{\cancel{2}\sin(\omega + \pi/2)}{(\omega + \pi/2)}$$

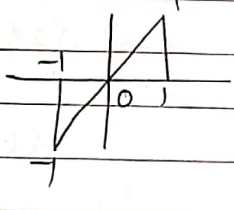
$$= \frac{\sin(\omega - \pi/2)}{(\omega - \pi/2)} + \frac{\sin(\omega + \pi/2)}{(\omega + \pi/2)}$$

3) $y(t)$





$\Downarrow$



$y(t) = t\,x(t).$

$$x(t) \longleftrightarrow X(j\omega)$$

$$(-jt)\,x(t) \longleftrightarrow \frac{d}{d\omega} X(j\omega)$$

$$t\,x(t) \xleftrightarrow{FT} j\frac{d}{d\omega} X(j\omega)$$

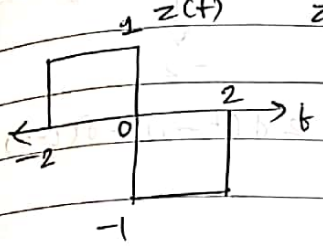
$$\boxed{t^n x(t) \longleftrightarrow j^n \frac{d^n}{d\omega^n} X(j\omega)}$$

$$y(t) = t\,x(t)$$

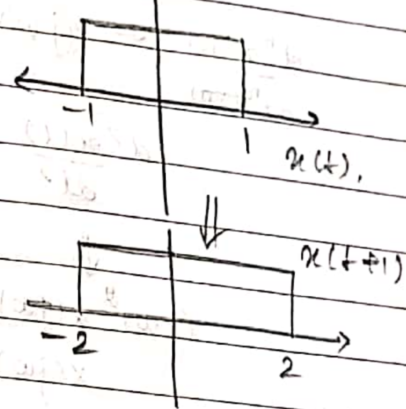
$$t\,x(t) \longleftrightarrow j\frac{d}{d\omega}\left(\frac{2\sin\omega}{\omega}\right)$$

$$j\left[\frac{\omega\, 2\cos\omega - 2\sin\omega}{\omega^2}\right]$$

ii) $z(t)$



$z(t) \Rightarrow$



$$z(t) = x(t+1) - x(t-1)$$

$$Z(j\omega) = e^{j\omega}X(j\omega) - e^{-j\omega}X(j\omega)$$

$$= \frac{2e^{j\omega}\sin\omega}{\omega} - \frac{2e^{-j\omega}\sin\omega}{\omega}$$

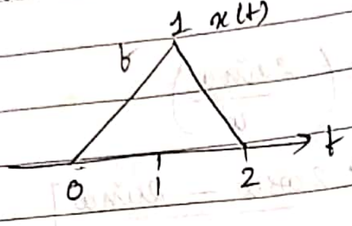
$$= \frac{2\sin\omega}{\omega}\left[2j\sin\omega\right]$$

$$= \frac{4j\sin^2\omega}{\omega}.$$

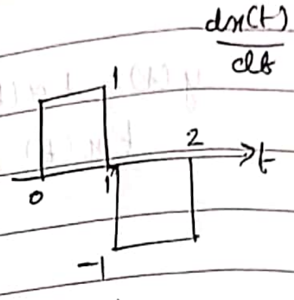
5) $s(t) = \frac{d}{dt} x(t)$

$$S(j\omega) = j\omega\, X(j\omega)$$
$$= j\omega \frac{2\sin\omega}{\omega}$$
$$= 2j\sin\omega$$

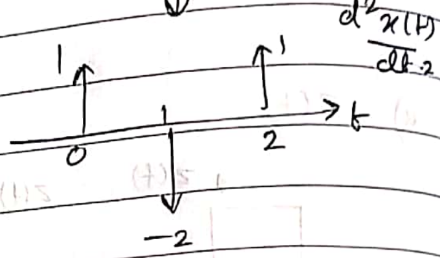
Q) Find the FT of.



$\longleftrightarrow$



$\downarrow$



$$\frac{dx(t)}{dt} \longleftrightarrow j\omega\, X(j\omega)$$

$$\frac{d^n x(n)}{dt^n} \longleftrightarrow (j\omega)^n X(j\omega)$$

$$\frac{d^2 x(t)}{dt^2} = \delta(t) - 2\delta(t-1) + \delta(t-2)$$

$\Downarrow$

$$(j\omega)^2 X(j\omega) = 1 - 2e^{-j\omega} + e^{-j2\omega}$$

$$X(j\omega) = \frac{1 - 2e^{-j\omega} + e^{-j2\omega}}{(j\omega)^2}$$

Discrete - Time Fourier Transform -

$x_T(t) \xleftrightarrow{FS} a_k$ | $x(n) \xleftrightarrow{FS} a_k$

$x(t) \xleftrightarrow{FT} X(j\omega)$ | $x(n) \xleftrightarrow{FT} X(e^{j\omega})$

Syllabus - 3.1 — 3.7
4.1 — 4.5
5.1 — 5.3

Aperiodic discrete signal -

$x_T(t) \longrightarrow T \longrightarrow \infty \Rightarrow$ [aperiodic signal]

$N \longrightarrow \infty$

$$F[x(n)] = X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega.$$

Condition for DTFT to exist, is $x(n)$ must be absolutely summable

i.e $\sum_{k=-\infty}^{\infty} |x(k)| < \infty.$

Q⇒ Find the DTFT of the following series-

1. $x(n) = \delta(n)$
2. $x(n) = a^n u(n).$, $|a| < 1$

1. If $x(n) \xrightarrow{DTFT} X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n}$$

$$= e^{-j\omega n}\Big|_{n=0} = 1$$

2. $x(n) = a^n u(n).$ $|a|<1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}, \quad |a|<1$$

$$\left[\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad a<1 \right]$$

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3. $x(n) = (1, 2, \underset{\uparrow}{3}, 2, 1)$

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-2}^{2} x(n) e^{-j\omega n}$$

$$= 1e^{+j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + 1e^{-2j\omega}$$

$$X(e^{j\omega}) = 3 + 2\cos 2\omega + 4\cos\omega.$$

4. $x(n) = (0.5)^{n+2} u(n)$

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (0.5)^{n+2} u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^{n+2} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^n (0.5)^2 e^{-j\omega n}$$

$$= (0.5)^2 \sum_{n=0}^{\infty} (0.5 e^{-j\omega})^n$$

$$= (0.5)^2 \frac{1}{1 - 0.5e^{-j\omega}}$$

$$= \frac{0.25}{1 - 0.5e^{-j\omega}}$$

5. $x(n) = n(0.5)^{2n} u(n)$

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} n(0.5)^{2n} u(n) e^{-j\omega n}$$

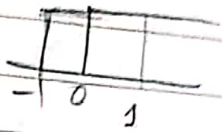
$$= \sum_{n=0}^{\infty} n\,(0.5)^{2n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} n\,(0.25)^{n} e^{-j\omega n} \qquad \left[\sum_{n=0}^{n} n a^{n} = \frac{a}{(1-a)^2} \right]$$

$$= \sum_{n=0}^{\infty} n\,(0.25\, e^{-j\omega})^{n}$$

$$= \frac{0.25\, e^{-j\omega}}{(1-0.25\, e^{-j\omega})^2}$$

$x(n) = -a^{n} u(-n-1)$, a is real.



6. $x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-1}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-1}^{\infty} -a^{n} \cancel{u(-n+} \; e^{-j\omega n}$$

$$= \sum_{n=-1}^{\infty} -a^{n} e^{-j\omega n} \qquad m \quad n=-m \quad n=-m$$

$$= -\sum_{m=1}^{\infty} a^{(-m)} e^{j\omega m} \qquad -1 \quad -1$$

$$= \sum_{k=0}^{\infty} a^{(k-1)} e^{-j\omega(k-1)} \qquad k = 1-m \quad k-1 \quad k=0$$

$$= -\sum_{k=0}^{\infty} (a e^{-j\omega})^{k-1}$$

$$= -\sum_{k=0}^{\infty} (a e^{-j\omega})^{k} (a e^{-j\omega})^{-1}$$

$$= \frac{-1}{a e^{-j\omega}} \sum_{k=0}^{\infty} (a e^{-j\omega})^{k} \qquad \left[\sum_{k=0}^{n} a^{k} = \frac{1-a^{n+1}}{1-a} \right]$$

$$= -\frac{1}{a e^{-j\omega}} \cdot \frac{1}{1-a e^{-j\omega}}$$

6) $X(e^{j\omega}) = \sum_{n=-\infty}^{-1} (-a^n) e^{-j\omega n}$

$n = -m$

$= \sum_{m=1}^{\infty} -(a^{-1}e^{j\omega})^m + 1 - 1$

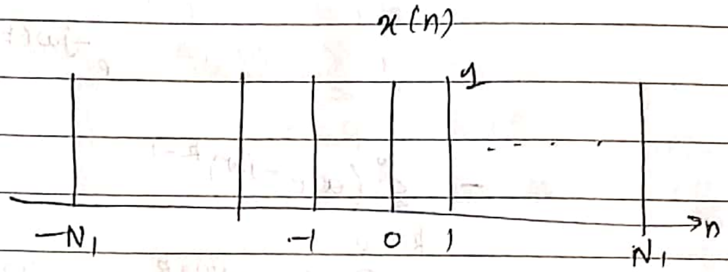
$= 1 - \sum_{m=0}^{\infty} (a^{-1}e^{j\omega})^m$

$= 1 - \frac{1}{1-a^{-1}e^{j\omega}}$

$= \frac{\cancel{1} - a^{-1}e^{j\omega} - \cancel{1}}{1-a^{-1}e^{j\omega}}$

$= \frac{1}{\frac{(1-a^{-1}e^{j\omega})}{-a^{-1}e^{j\omega}}}$

$= \frac{1}{-\frac{1}{a^{-1}e^{j\omega}} + 1} = \frac{1}{1-ae^{-j\omega}}$

7) 

$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$

$m = n + N_1$

$m - n = N_1$

$n + N_1 + N_1 = N_1$

$m = n + N_1$

$= \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)}$

$$= e^{j\omega N_1}\left[\sum_{m=0}^{2N_1}(e^{-j\omega})^m\right]$$

$$= e^{j\omega N_1}\left[\frac{1-e^{-j\omega(2N_1+1)}}{1-e^{-j\omega}}\right]$$

Properties of DTFT-

1. Linearity-

$$ax(n)+by(n) \xleftrightarrow{DTFT} aX(e^{j\omega})+bY(e^{j\omega})$$

2. Time-shift-

$$x(n-n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0}X(e^{j\omega})$$

3. Frequency-shift-

$$e^{-j\beta n}x(n) \xleftrightarrow{DTFT} X(e^{j(\omega-\beta)})$$

4. Scaling-

$$x(pn) \xleftrightarrow{DTFT} X(e^{j(\omega/p)})$$

5. Frequency Differentiation-

$$-j\,nx(n) \xleftrightarrow{DTFT} \frac{d}{d\omega}X(e^{j\omega})$$

$$nx(n) \longleftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

6. Summation-

$$\sum_{k=-\infty}^{n}x(k) \xleftrightarrow{DTFT} \frac{X(e^{j\omega})}{1-e^{-j\omega}}+\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

7. Convolution-

$$x(n)*y(n) \xleftrightarrow{DTFT} X(e^{j\omega})Y(e^{j\omega})$$

8. Modulation -

$$x(n)\,y(n) \xleftrightarrow{DTFT} \frac{1}{2\pi}\left[X(e^{j\omega}) \circledast Y(e^{j\omega})\right]$$

Periodic Convolution

9. Parseval's theorem

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 \, d\omega.$$

10. Symmetry -

If $x(n) \longleftrightarrow X(e^{j\omega})$

$x(n)$ - real $\longleftrightarrow X^*(e^{j\omega}) = X(e^{-j\omega})$

$x(n)$ - real or even $\longleftrightarrow X(e^{j\omega})$ is purely real

$x(n)$ - real or odd $\longleftrightarrow X(e^{j\omega})$ is purely imaginary.

Q) 1. Show that DTFT of $x(n)=1$ is $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$

2. $x(n) = (-1)^n u(n)$.
3. $x(n) = u(n) - u(n-6)$
4. $x(n) = \left(\frac{1}{2}\right)^n u(n-4)$
5. $x(n) = 2^n u(-n)$
6. $x(n) = a^{|n|}, |a|<1$.

2. $x(n) = (-1)^n u(n)$

$\left[= a^n (u(n))\right]$

$$= \frac{1}{1+1e^{-j\omega}}$$

$x(n) = u(n) - u(n-6)$. $\qquad (1/2)^n u(n-4)$.

$x(n) \xrightarrow{DTFT} X(e^{-j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} e^{-j\omega n} - \sum_{n=0}$$

$$=$$

$x(n) = u(n) - u(n-6)$

$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{5} e^{-j\omega n}$$

$$\left[\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \right]$$

$$= \frac{1 - e^{-j\omega 6}}{1 - e^{-j\omega}}$$

4. $x(n) = (1/2)^n u(n-4)$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (1/2)^n u(n-4)\, e^{-j\omega n}$$

$$= \sum_{n=4}^{\infty} (1/2)^n e^{-j\omega n}$$

$m = n-4$

$$= \sum_{m=0}^{\infty} \left(1/2\right)^{m+4} e^{-j\omega(n+4)}$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m \left(1/2\right)^4 e^{-j\omega m}\, e^{-j\omega 4}$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^m \left(\frac{1}{2} e^{-j\omega}\right)^4$$

$$= \left(\frac{1}{2} e^{-j\omega}\right)^4 \frac{1}{1 - 1/2\, e^{-j\omega}}$$

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5) $x(n) = 2^n u(-n)$

$$= \sum_{n=-\infty}^{\infty} 2^n u(-n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{0} 2^n e^{-j\omega n}$$

$$= \sum_{l=0}^{\infty} 2^{-l} e^{j\omega l} \qquad (n = -l)$$

$$= \sum_{l=0}^{\infty} \left(2e^{-j\omega}\right)^{l}$$

$$= \frac{1}{1 - 2e^{-j\omega}}$$

6. $x(n) = a^{|n|}, \quad |a|<1$

$$a^{n} = \begin{cases} a^{-n}, & n<0 \longrightarrow u(-n-1) \\ a^{n}, & n \geq 0 \longrightarrow u(n) \end{cases}$$

$$a^{|n|} = a^{-n} u(-n-1) + a^{n} u(n)$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$$= \frac{ae^{j\omega}}{1 - ae^{j\omega}} + \frac{1}{1 - ae^{-j\omega}}$$

$$a^n u(n) = \frac{1}{1 - ae^{-j\omega}}$$

Using properties, find DTFT-

$$\frac{1}{2}u(\cdot \qquad \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

1. $x(n) = \left(\frac{1}{2}\right)^{n} u(n-2)$

$a^n u(n)$

$$= \left(\frac{1}{2}\right)^{2} \left(\frac{1}{2}\right)^{n-2} u(n-2)$$

$$\underline{a^n u(n)} \rightarrow \left(\frac{1}{2}\right)^{n-2} u(n-2)$$

$$= \frac{1}{4} e^{-j\omega 2}\, X(e^{j\omega}) = \frac{1}{4} e^{-j\omega 2} \left[\frac{1}{1 - \frac{1}{2}e^{-j\omega}} \right.$$

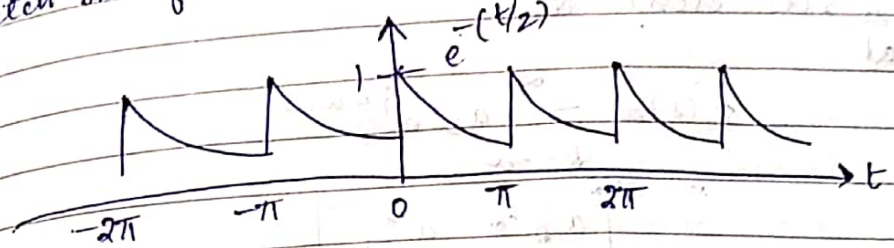
$$x(n) = \sin\left(\frac{\pi}{4}n\right)\left(\frac{1}{4}\right)^n u(n-1)$$

$$= \frac{1}{2j} e^{j\pi/4n} \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} u(n-1) - \frac{1}{2j} e^{-j\pi/4n} \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} u(n-1)$$

$$x(n) = a^n u(n) \longleftrightarrow X(j\omega)$$

$$x(n-1) \longleftrightarrow X(j\omega)e^{-j\omega}$$

→ sketch the fourier spectrum for the given signal



$$T_0 = \pi \Rightarrow \omega_0 = \frac{2\pi}{T_0} = 2.$$

$x(t) \xrightarrow{P}$ CTFS ✓
$\searrow$ FT

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt.$$

$x(n) \longrightarrow$ DTFS
$\searrow$ FT

$\cancel{\sin}$ $\sin(2\pi jt) + \cos(2\pi jt)$.

$\sin(2\pi j$

0

$$= \frac{1}{\pi}\int_0^{\pi} e^{-(t/2)} e^{-jk2t}\, dt.$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(e^{-t/2} e^{-jk2t}\right)dt$$

$$-\frac{t}{2} - jk2t$$

$$-t\left(\frac{1}{2} - jk2\right)$$

$$= \frac{1}{\pi}\int_0^{\pi} e^{-t/2 - jk2t}\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} e^{\cancel{2jk} - t(1/2 + 2kj)}\, dt$$

$$\boxed{a_k = \frac{0.504}{1 + j4k}}$$

$$= \frac{1}{\pi}\left[\frac{e^{-t(1/2 + 2kj)}}{\frac{1}{2} + 2kj}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-\pi(1/2 + 2kj)}}{1/2 + 2kj} - \frac{1}{\frac{1}{2} + 2kj}\right]$$

$$= \frac{1}{\pi}\left[e \cancel{\frac{-\pi(1/2 + 2kj)}{1/2 + 2kj}}\right. \quad = \cancel{\sin}$$

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$= \frac{1}{\pi}\Big[e$

Q) Let $x(t)$ be a periodic signal whose Fourier Series coefficients are $a_k = \begin{cases} 2 & ; k=0 \\ j\left(\frac{1}{2}\right)^{|k|} & ; \text{otherwise} \end{cases}$

use Fourier Series properties to answer the foll-

a) is $x(t)$ real? No

b) is $x(t)$ even? Yes $a_{-k} = a_k$

c) is $\frac{d}{dt}x(t)$ even? No

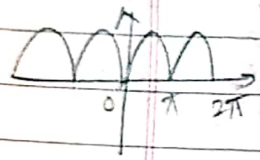
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\left[a_k = a^*_{-k}\right]$$

$$\frac{d}{dt}x(t) = jk\omega_0 a_k$$

Q) Sketch the phase spectrum for the full wave rectified sine wave.

$$x(t) = |A\sin(t)|. \qquad T = \pi$$



$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt \qquad \omega_0 = \frac{2\pi}{\pi} = 2$$

$$= \frac{1}{\pi}\int_0^{\pi} |a\sin(t)|\, e^{-jk2t}\, dt.$$

$$= \frac{a}{\pi}\int_0^{\pi}\left(\frac{e^{t} - e^{-t}}{2}\right) e^{-jk2t}\, dt.$$

$$e^{t(-2jk)} \qquad e^{-t(2kj)}$$

$$= \frac{a}{\pi}\int_0^{\pi}\left[\frac{e^{t(-jk2t)} - e^{-t-jk2t}}{2}\right] dt$$

$$= \frac{a}{2\pi}\left[\frac{e^{t(-2jk)}}{2jk} - \frac{e^{-t(2jk)}}{-2kj}\right]_0^{\pi} d$$

$$= \frac{a}{2\pi}\left[\frac{e^{-2jk\pi}}{2jk} + \cancel{e}\,\frac{1}{+2jk}\right]$$

$$a_k = \frac{2}{2\pi}\left[\frac{e^{-j\frac{k\pi}{2}}+1}{2jk}\right] \qquad a_k = \frac{2j}{\pi(1-k^2)}$$

no phase
no complex part

Suppose we are given the following info

1. $x(t)$ is real & odd
2. $x(t)$, $T=2$ and has a_k
3. $a_k = 0$ for $|k|>1$
4. $\frac{1}{2}\int_0^2 |x(t)|^2\,dt = 1$

Specify 2 different signals that satisfy these conditions

$\Rightarrow$ i) $x(t)$ is real

$$x(t) = x^*(t)$$

$$a_k = a^*_{-k}$$

$a_0 = 0$

$$T = 2 \Rightarrow \omega_0 = \frac{2\pi}{2} = \pi \qquad \Rightarrow \boxed{a_0 = 0} \qquad a_1 = a_{-1}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi t} \qquad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi t}$$

$$a_{-1} = a^*_{1}$$

ii) $$\frac{1}{2}\int_0^2 |x(t)|^2\,dt = 1$$

$$= \sum_{k=-\infty}^{\infty} |b_k|^2 = |a_{-1}|^2 + |a_1|^2$$

$a_1 = a_{-1}$

since a_k is purely imaginary

$|a_1| = 1$

$$|a_{-1}|^2 + |a_1|^2 = 1$$

$$a_{-1} = a_1$$

$$|a_{-1}|^2 + |a_1|^2 = 1$$

$$2|a_1|^2 = 1$$

$$\boxed{a_1 = \frac{1}{\sqrt{2}}}$$

$x(t) = -\sqrt{2}\sin \pi t.$

$a_1 = \frac{+j}{\sqrt{2}}$ $=$

$a_1 = \frac{-j}{\sqrt{2}}$

Q) Given the $x(t)$ has a fourier transform $X(j\omega)$ Express the fourier transforms of the signal listed below in terms of $X(j\omega)$.

1) $x_1(t) = x(1-t) + x(-1-t)$

2) $x_2(t) = x(3t-6)$

3) $x_3(t) = \frac{d^2}{dt^2} x(t-1)$

1)

$x(t) \longleftrightarrow X(j\omega)$.

$x(t+1) \longleftrightarrow e^{j\omega_0 t_0} X(j\omega)$.

$= e^{j\omega 1} X(j\omega)$

$x(-t+1) \longleftrightarrow \frac{1}{|-1|} \frac{e^{-j\omega} X(j\omega)}{(-1)}$

$= \cancel{2\cos\omega X(-j\omega)}$

$x(-1-t)$

$x(-t-1)$

$x(t-1) \longleftrightarrow e^{-j\omega_0} X(j\omega)$

$x(-t-1) \longleftrightarrow e^{\frac{-j\omega}{-1}} X(-j\omega)$

$= e^{j\omega} X(-j\omega)$

$= e^{-j\omega} X(-j\omega) + e^{j\omega} X(-j\omega)$

$= 2\cos\omega \, X(-j\omega)$

2) $x_2(t) = x(3t-6)$

$x(t-6) \longleftrightarrow e^{-j\omega 6} X(j\omega)$

$x(3t-6) \longleftrightarrow \frac{1}{|3|} e^{-j6\omega/3} X\left(\frac{j\omega}{3}\right)$

$= \frac{1}{3} e^{-2j\omega} X\left(\frac{j\omega}{3}\right)$

3) $x_3(t) = \frac{d^2}{dt^2} x(t-1).$

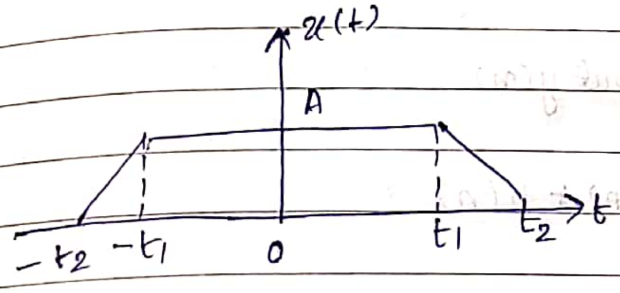
$=$

$x(t-1) \longrightarrow e^{-j\omega} X(j\omega).$

$\frac{d}{dt} x(t-1) \longrightarrow j\omega \left[e^{-j\omega} X(j\omega)\right]$

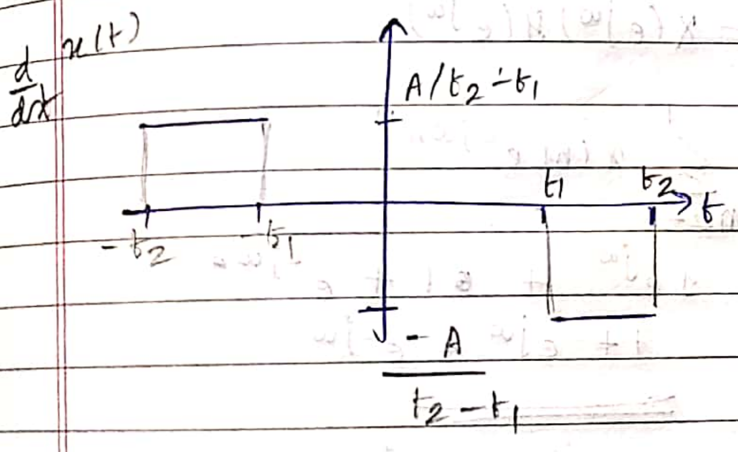
$\frac{d^2}{dt^2} x(t-1) \longleftrightarrow j^2\omega^2 \left[e^{-j\omega} X(j\omega)\right]$

Q) Find the fourier transform of the signal given-

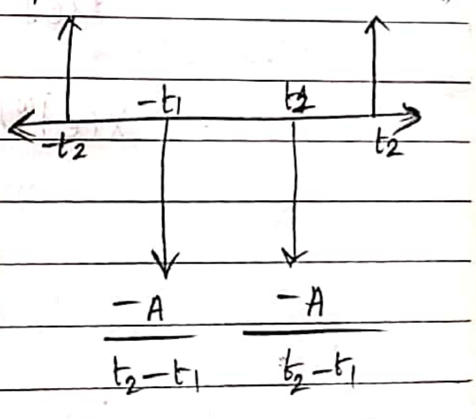


$x(t) = \{$

$\Rightarrow \frac{d}{dx} x(t)$



$\Rightarrow \frac{d^2 x(t)}{dt^2}$



$$\frac{d^2}{dt^2} = \frac{A}{t_2-t_1}\delta(t+t_2) - \frac{A}{t_2-t_1}\delta(t+t_1) - \frac{A}{t_2-t_1}\delta(t-t_1)$$

$$+ \frac{A}{t_2-t_1}\delta(t-t_2).$$

$$x(t) = \delta(t) \xleftrightarrow{FT} 1$$

$$\delta(t-t_0) \longleftrightarrow e^{-j\omega_0 t_0}$$

$$= \frac{A}{t_2-t_1}\left[\frac{e^{j\omega t_2} - e^{j\omega t_1} - e^{-j\omega t_1} + e^{-j\omega t_2}}{(j\omega)^2}\right]$$

$$= \Rightarrow \frac{\cancel{2A}\,A}{t_2-t_1}\left[\frac{2\cos(\omega t_2) - 2\cos(\omega t_1)}{(j\omega)^2}\right]$$

$$= \frac{2A}{(\omega^2)t_2-t_1}\left[\frac{\cos(\omega t_1) - \cos(\omega t_2)}{\cancel{2}}\right] //$$

Q) Given the input $x(n) = \{1, \underset{\uparrow}{1}, 1\}$ and impulse response $h(n) = \{1, \underset{\uparrow}{1}, 1\}$.

Evaluate the output $y(n)$

$$y(n) = x(n) * h(n)$$

$$\Downarrow$$

$$Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-1}^{1} x(n)\,e^{-j\omega n}$$

$$= 1e^{j\omega} + 1 + e^{-j\omega}$$

$$= 1 + e^{j\omega} + e^{-j\omega}$$

$$H(e^{j\omega}) = 1 + e^{j\omega} + e^{-j\omega}$$

$$\delta(n) \longleftrightarrow$$

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$$Y(e^{j\omega}) = 1 + e^{j\omega} + e^{-j\omega} + e^{j\omega} + e^{2j\omega} + 1 + e^{-j\omega} + 1 + e^{-2j\omega}$$

$$= 3 + 2e^{j\omega} + 2e^{-j\omega} + e^{2j\omega} + e^{-2j\omega}$$

$$Y(e^{j\omega}) = 3 + 2e^{j\omega} + 2e^{-j\omega} + e^{2j\omega} + e^{-2j\omega}$$

$$y(n) = 3\delta(n) + 2\delta(n+1) + 2\delta(n-1) + \delta(n+2) + \delta(n-2)$$

$$\delta(n-2) \longleftrightarrow e^{-j2\omega}.$$