Network Analysis & Systems

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Network Analysis and Synthesis

Part V: Network Synthesis

Causality (1)

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where $H(s)$ could be associated with a one-port or a two-port.
- Since $\mathcal{L}\{\delta(t)\} = 1$, if $u(t) = \delta(t)$,

$$Y(s) = H(s)$$

- Accordingly, the network function $H(s)$ is also referred to as the Laplace transform of the impulse response.

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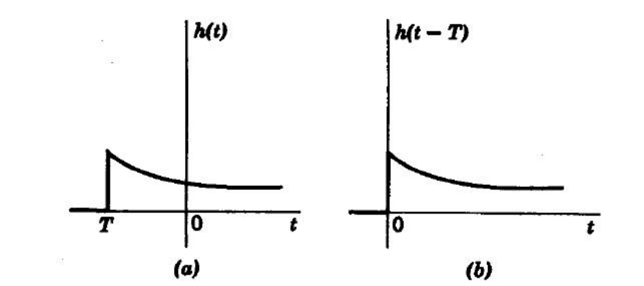
By definition of causality, for all $t < 0$,

$$h(t) = 0$$

Causality (3)



- If the impulse response is causal or non-anticipative, then it is realisable using passive components.
- If the impulse response is non-causal or anticipative, then it is not realisable using passive components.

Causality (4)

Theorem (Payley-Wiener)

A necessary and sufficient condition for a function to be realisable (i.e., causal) is that its magnitude function is such that

$$\int_{-\infty}^{\infty} \frac{|\ln |H(j\omega)||}{1+\omega^2} d\omega < \infty$$

- This implies that the amplitude cannot be zero over a finite range of frequencies. For example, the ideal low-pass filter.
- Equivalently, the amplitude function cannot fall off to zero faster than exponential order. For example, $|H(j\omega)| = e^{-\omega^2}$, resulting in $\int_{-\infty}^{\infty} \frac{\omega^2}{1+\omega^2} d\omega = \infty$.

Stability (1)

Definition

A network function is said to bounded-input bounded-output (BIBO) stable if the response of the network is bounded for every bounded applied voltage or current.

That is for any $u(t)$ s.t.

$$|u(t)| < m_1 \implies |y(t)| < m_2 \quad \forall\, t \in [0, \infty)$$

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Since for a realisable system $h(t) = 0$ for $t < 0$,

$$|y(t)| \leq m_1 \int_0^{\infty} |h(\tau)| d\tau < \infty$$

- That is the impulse response must be absolutely integrable.
- An important requirement for this is

$$\lim_{t \longrightarrow \infty} h(t) = 0$$

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- Otherwise, they are said to be unstable.
- For network functions of passive linear time-invariant networks with lumped parameters, stability implies causality.

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- This definition shall be used here.
- The class of polynomials following Kuo's definition can be called as modified Hurwitz polynomials.
- The class of polynomials following the latter definition can be called strictly Hurwitz polynomials.

Stability (4)

- A necessary condition for a polynomial to be Hurwitz is that none of the coefficients should be zero.
- A necessary condition for a polynomial to be modified Hurwitz is that none of the coefficients should be zero. If any coefficient is zero, then the coefficients of all the even powers of the odd powers are zeros.

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- A necessary condition for a polynomial to be modified Hurwitz is that none of the coefficients should be zero. If any coefficient is zero, then the coefficients of all the even powers of the odd powers are zeros.
- Specifically, the odd and even parts of a modified Hurwitz polynomial have roots on the imaginary axis.

Stability (5)

- Let $P(s)$ be a polynomial. Then

$$P(s) = p_o(s) + p_e(s)$$

where $p_o(s)$ contains all the odd powers of $P(s)$ and $p_e(s)$ contains all the even powers of $P(s)$.

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- Consider the continued fraction expansion of either $\frac{p_o(s)}{p_e(s)}$ or $\frac{p_e(s)}{p_o(s)}$.
- The polynomial is modified Hurwitz if all the quotient polynomials are positive.
- The CFE is finite in length.

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Examples:

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$$P(s) = s + \cfrac{1}{-s + \cfrac{1}{-\frac{s}{5} + \frac{1}{5s/2}}}$$

Stability (7)

- More generally, if the CFE of $P(s)$ yields positive quotient terms, then $P(s)$ is modified Hurwitz to within a multiplicative factor $W(s)$:

$$P(s) = W(s)P_1(s)$$

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Example: $P(s) = s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$:

$$P(s) = \frac{s}{2} + \cfrac{1}{\frac{4s}{3} + \cfrac{1}{\frac{\frac{3s}{2}(s^4+4)}{s^4+4}}}$$

Note that $W(s) = s^4 + 4 = (s^2 + 2s + 2)(s^2 - 2s + 2)$.

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- What if $P(s)$ is an odd or even function?
- Determine the CFE of $P(s)/P'(s)$, and draw appropriate conclusions.
- Example: $P(s) = s^7 + 3s^5 + 2s^3 + s$.

Positive Real Functions (1)

Definition

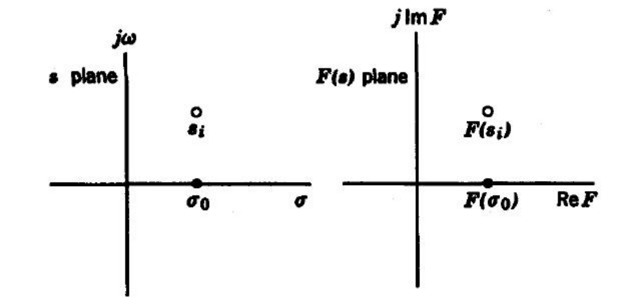
A function $H(s)$ is positive real (PR) if

1. $H(s)$ is real for real s.
2. For all s with $\operatorname{Re} s \geq 0$,

$$\operatorname{Re} H(s) \geq 0$$

- These represent physically realisable passive driving-point immittances.
- The first condition is obvious if $H(s)$ has real coefficients.

Positive Real Functions (2)



- The first condition indicates that the real axis of the s-plane maps onto the real axis of the $F(s)$-plane.
- The second condition indicates that the right half of the s-plane maps onto the right half of the $F(s)$-plane.

Positive Real Functions (3)

Examples:

- Ls.

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- If $H_1(s)$ and $H_2(s)$ are PR, so is $H_1(s) + H_2(s)$.
- Examples: Two impedances in series or two admittances in parallel.

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- The poles and zeros cannot have positive real parts.
- Only simple poles are permitted on the imaginary axis, and with real and positive residues.
- The relative degree can be -1, 0, or $+1$.
- The lowest powers of the denominator and numerator polynomials may differ at most by unity. This prevents multiple poles or zeros at the origin.

Positive Real Functions (5)

Fact

The necessary and sufficient conditions for a real-rational function $H(s)$ to be PR are

1. *$H(s)$ must have no poles in the right-half of s-plane.*
2. *$H(s)$ may have simples poles on the imaginary axis with real and positive residues.*
3. *Re $H(j\omega) \geq 0$ for all ω.*

- The denominator polynomial of $H(s)$ must be modified Hurwitz.

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Elementary Synthesis Procedures (1)

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$$H(s) = H_1(s) + H_2(s) + \cdots + H_n(s)$$

- Each H_i must be PR.
- Given the RHS, the synthesis procedure is easier.
- Given the LHS, the following can be attempted:
  - PFE of $H(s)$ or $1/H(s)$ (cleverly done!).
  - Removing $\min \mathrm{Re}\, H(j\omega)$ so that the remainder is PR.

Elementary Synthesis Procedures (2)

Examples:

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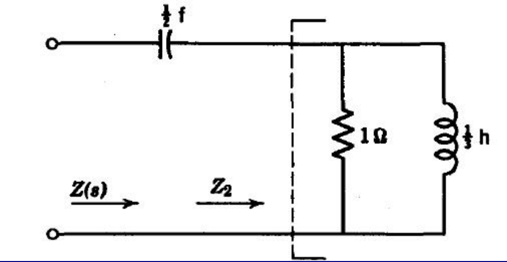
assuming $H_1(s)$ is an impedance.



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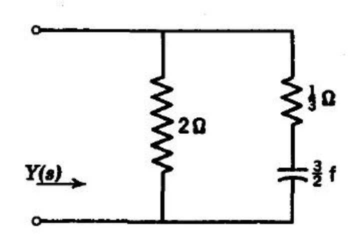

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