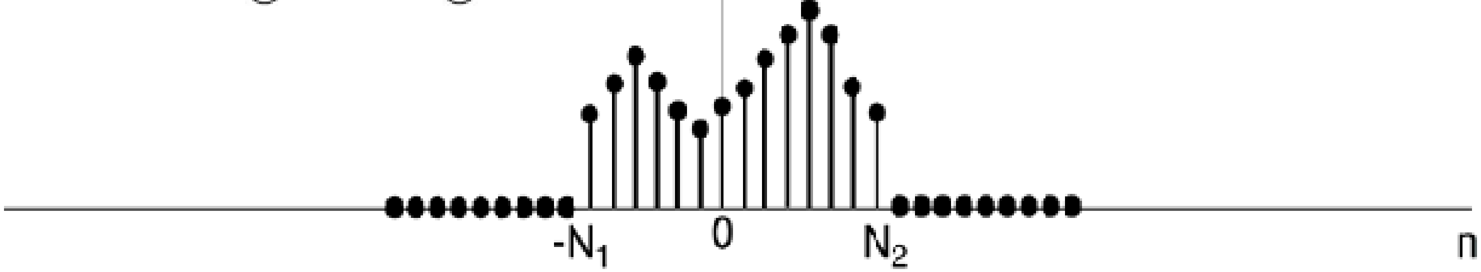
Discrete-time Fourier Transform

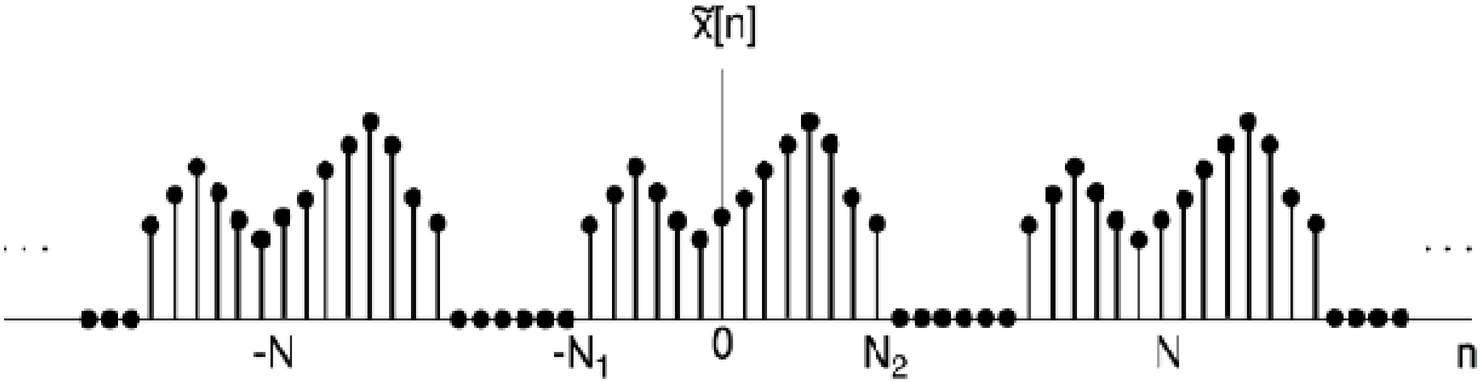
Derivation of the Discrete-time Fourier Transform

$x[n]$ - aperiodic and of finite duration

$x[n] = 0$ if $|n| \geq N/2$

N is large enough





$\tilde{x}[n] = x[n]$ for $|n| \leq N/2$ and periodic with period N

$\tilde{x}[n] = x[n]$ for *any* n as $N \rightarrow \infty$

Recall DTFS pair

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \ \omega_0 = \frac{2\pi}{N}$$ DTFS synthesis eq.

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\omega_0 n}$$ DTFS analysis eq.

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{N} X(e^{jk\omega_0})$$

where $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\tilde{x}[n] = \sum_{k=<N>} \underbrace{\frac{1}{N} X(e^{jk\omega_0})}_{a_k} e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

As $N \to \infty$: $\qquad \tilde{x}[n] \to x[n]$ for every n

$$\omega_0 \to 0, \ \sum \omega_0 \to \int d\omega$$

Thus, $x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

The limit of integration is over any interval of 2π in ω

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$ Periodic in ω with period 2π

DTFT Pair

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

– Analysis Equation
– FT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

– Synthesis Equation
– Inverse FT

Conditions for Convergence

Need conditions analogous to CTFT, e.g.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \quad \text{— Finite energy}$$

or $$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{— Absolutely summable}$$

Examples

1) $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

2) $x[n] = \delta[n - n_0]$ - shifted unit sample

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

Same amplitude (=1) as above,
but with a *linear* phase $-\omega n_0$

3) $x[n] = a^n u[n], |a| < 1$ - Exponentially decaying function

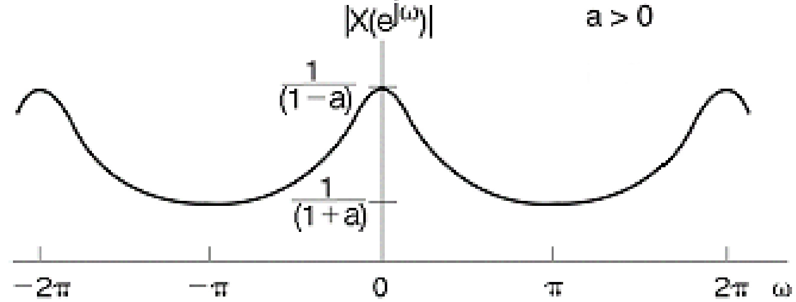
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \underbrace{\left(ae^{-j\omega}\right)^n}_{|ae^{-j\omega}|<1}$$

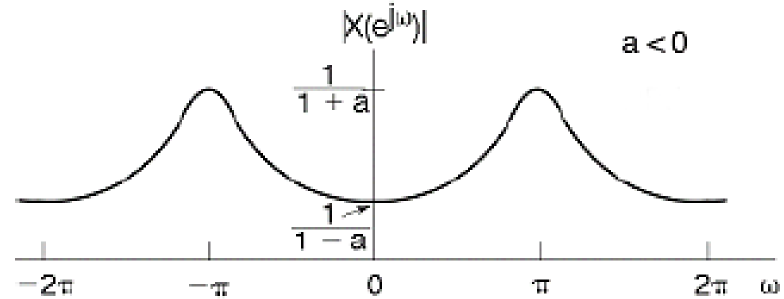
Infinite sum formula

$$= \frac{1}{1 - ae^{-j\omega}} = \frac{1}{(1 - a\cos\omega) + ja\sin\omega}$$

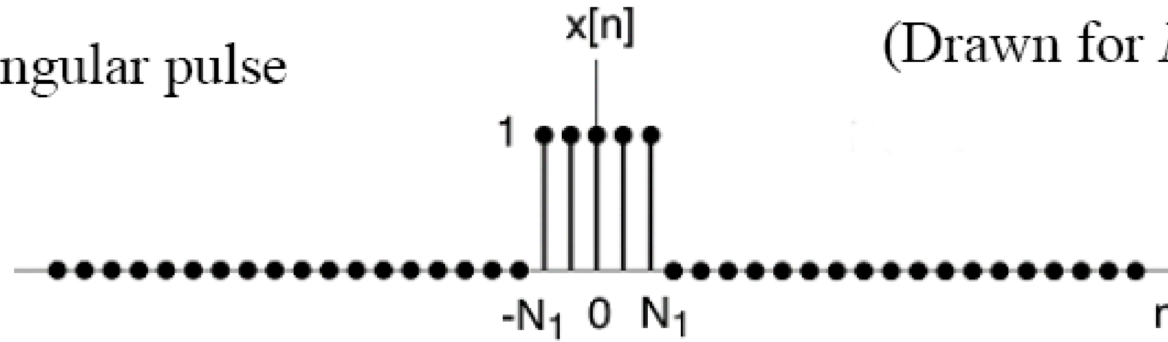
$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$= \begin{cases} \dfrac{1}{1-a}, & \omega = 0 \\ \dfrac{1}{1+a}, & \omega = \pi \end{cases}$$

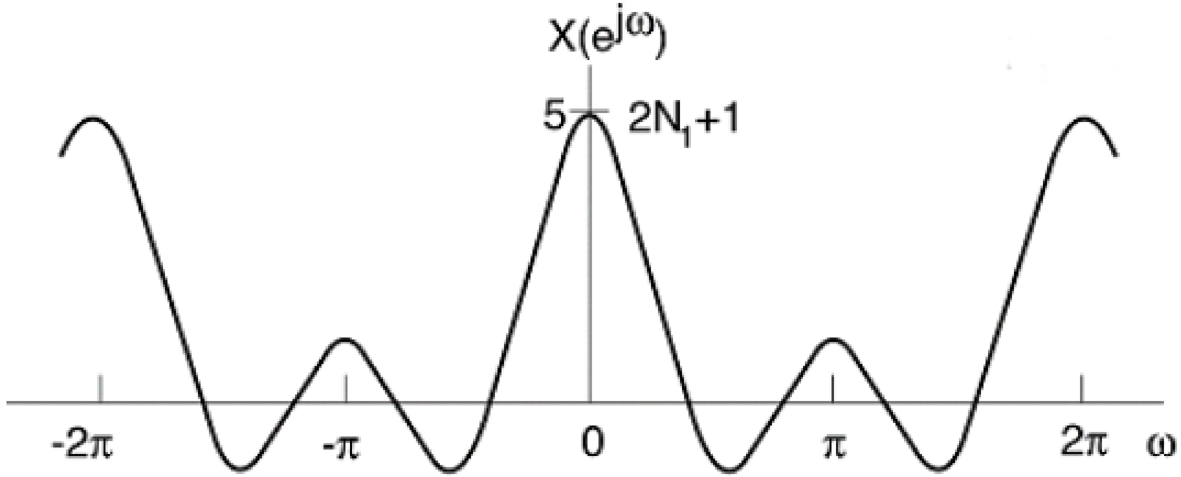




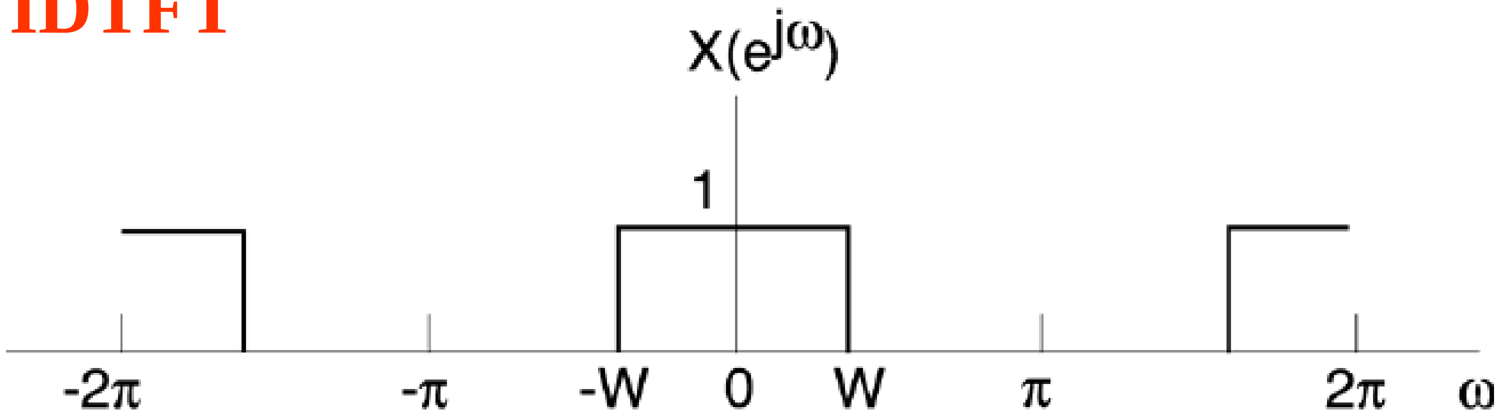
4) DT Rectangular pulse (Drawn for $N_1 = 2$)



$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{n=-N_1}^{N_1} (e^{-j\omega})^n = \frac{\sin\omega\left(N_1+\frac{1}{2}\right)}{\sin(\omega/2)} = X(e^{j(\omega-2\pi)})$$

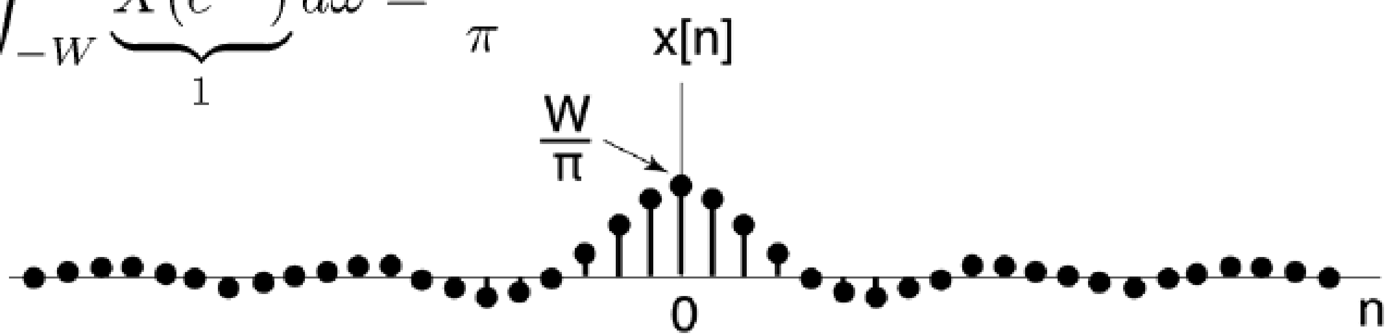


5) IDTFT



$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\,d\omega = \frac{\sin Wn}{\pi n}$$

$$x[0] = \frac{1}{2\pi}\int_{-W}^{W} \underbrace{X(e^{j\omega})}_{1}\,d\omega = \frac{W}{\pi}$$



6) Complex Exponentials

Recall CT result: $x(t) = e^{j\omega_0 t} \longleftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$

What about DT: $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = ?$

a) We expect an impulse (of area 2π) at $\omega = \omega_o$

b) But $X(e^{j\omega})$ must be periodic with period 2π

In fact

$$X(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)$$

Note: The integration in the synthesis equation is over 2π period, only need $X(e^{j\omega})$ in *one* 2π period. Thus,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)}_{X(e^{j\omega})} e^{j\omega n} d\omega = e^{j\omega_0 n}$$

DTFT of Periodic Signals

Recall the following DTFT pair:

$$e^{jk\omega_0 n} \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m)$$

Represent periodic signal $x[n]$ in terms of DTFS:

$$x[n] = x[n+N] = \sum_{k=<N>} a_k e^{jk\omega_0 n}\ ,\ \omega_0 = \frac{2\pi}{N}$$

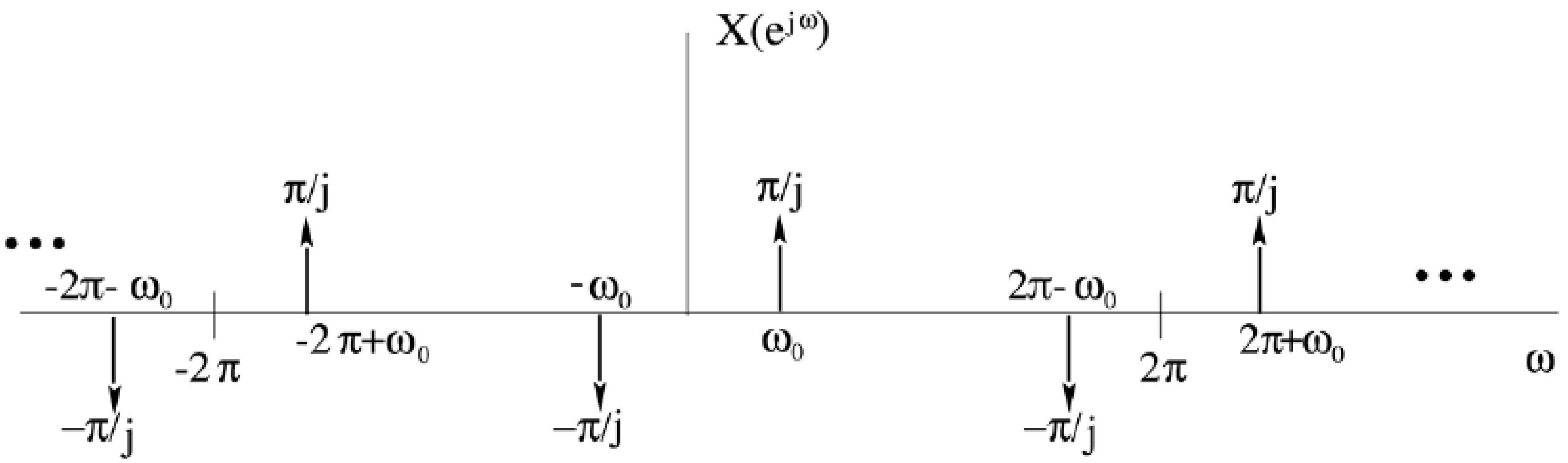
$$\begin{aligned} X(e^{j\omega}) &= \sum_{k=<N>} a_k \left[2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m) \right] \\ &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right) \end{aligned}$$

Linearity of DTFT

Example: A discrete-time Sine Function

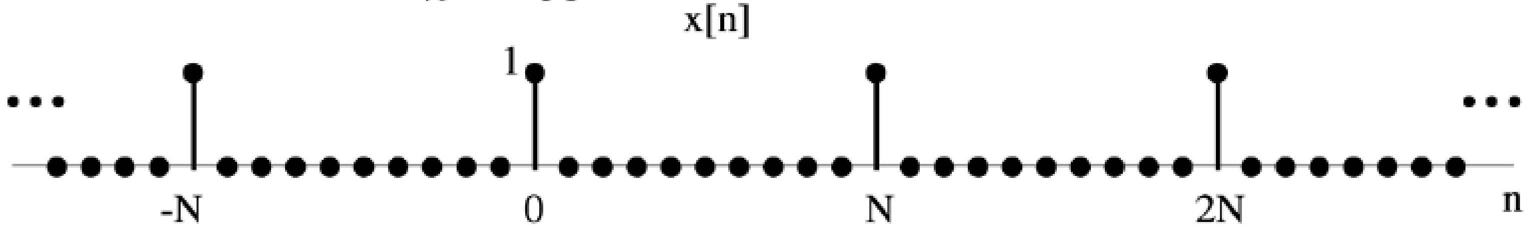
$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m) - \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi m)$$



Example: A discrete-time Periodic Impulse Train

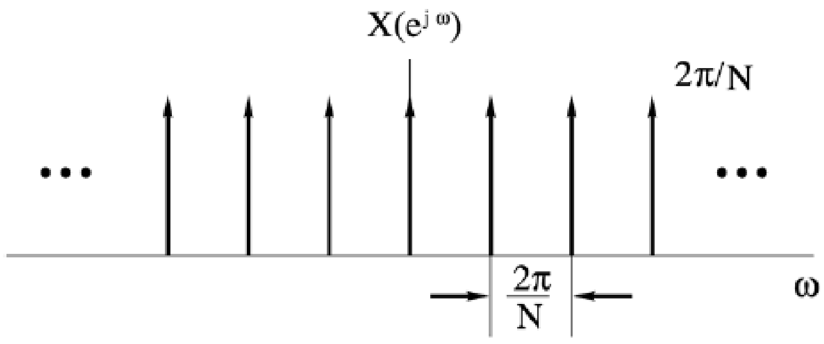
$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \qquad\qquad \omega_0 = 2\pi/N$$



The DTFS coefficients for this signal are:

$$c_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{x[n]}_{=\delta[n]} e^{-jk\omega_0 n} = \frac{1}{N}$$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



Properties of DTFT

Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Linearity: $ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

Time Shifting: $x[n-n_0] \longleftrightarrow e^{-j\omega n_0}X(e^{j\omega})$

Frequency Shifting: $e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$

Time Reversal: $x[-n] \longleftrightarrow X(e^{-j\omega})$

Properties of DTFT

Conjugate Symmetry: $x[n]$ real $\Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$

$|X(e^{j\omega})|$ and $\Re e\left\{X(e^{j\omega})\right\}$ are even functions

$\angle X(e^{j\omega})$ and $\Im m\left\{X(e^{j\omega})\right\}$ are odd functions

Parseval's Relation

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\text{Total energy in time domain}} = \underbrace{\frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega}_{\text{Total energy in frequency domain}}$$

Convolution Property

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = h[n] * x[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$H(e^{j\omega}) = \text{DTFT of } h[n]$$

$$\text{Frequency response} = \text{DTFT of the unit sample response}$$

Multiplication Property

$$y[n] = x_1[n] \cdot x_2[n]$$

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega}) \\ &\hookrightarrow \text{ Periodic Convolution} \end{aligned}$$