A little rearranging results in

$$\frac{dv}{dt} + 3.092 \times 10^3 v = 72.67 \times 10^3\, e^{-2000t}$$

which, upon comparison with Eqs. [28] and [30], allows us to write the complete response as

$$v(t) = e^{-Pt} \int Qe^{Pt} dt + Ae^{-Pt}$$

where in our case $P = 1/\tau = 3.092 \times 10^3$ and $Q(t) = 72.67 \times 10^3 e^{-2000t}$. We therefore find that

$$v(t) = e^{-3092t} \int 72.67 \times 10^3 e^{-2000t} e^{3092t} dt + Ae^{-3092t} \qquad \text{V}$$

Performing the indicated integration,

$$v(t) = 66.55e^{-2000t} + Ae^{-3092t} \qquad \text{V} \qquad [38]$$

Our only source is controlled by a step function with zero value for $t < 0$, so we know that $v(0^-) = 0$. Since v is a capacitor voltage, $v(0^+) = v(0^-)$, and we therefore find our initial condition $v(0) = 0$ easily enough. Substituting this into Eq. [38], we find $A = -66.55$ V and so

$$v(t) = 66.55(e^{-2000t} - e^{-3092t}) \text{ V} \qquad t > 0$$

PRACTICE

8.13 Determine the capacitor voltage v in the circuit of Fig. 8.46 for $t > 0$.

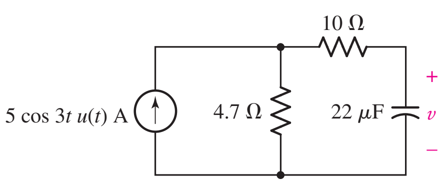


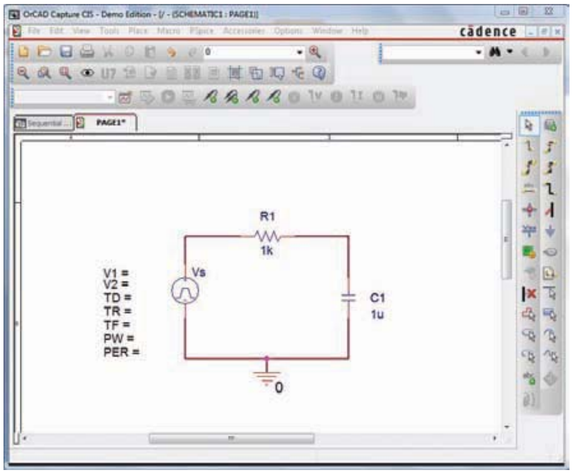
FIGURE 8.46 A simple *RC* circuit driven by a sinusoidal forcing function.

Ans: $23.5 \cos 3t + 22.8 \times 10^{-3} \sin 3t - 23.5e^{-3092t}$ V.

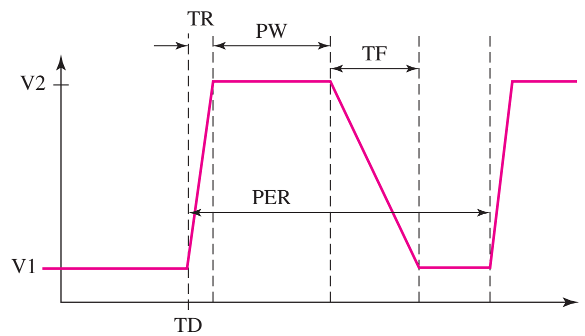
8.9 PREDICTING THE RESPONSE OF SEQUENTIALLY SWITCHED CIRCUITS

In Example 8.9 we briefly considered the response of an *RL* circuit to a pulse waveform, in which a source was effectively switched into and subsequently switched out of the circuit. This type of situation is common in practice, as few circuits are designed to be energized only once (passenger vehicle airbag triggering circuits, for example). In predicting the response of simple *RL* and *RC* circuits subjected to pulses and series of pulses—sometimes referred to as ***sequentially switched circuits***—the key is the relative size of the circuit time constant to the various times that define the pulse sequence. The underlying principle behind the analysis will be whether the energy storage element has time to fully charge before the pulse ends, and whether it has time to fully discharge before the next pulse begins.

Consider the circuit shown in Fig. 8.47*a*, which is connected to a pulsed voltage source described by seven separate parameters defined in Fig. 8.47*b*. The waveform is bounded by two values, **V1** and **V2.** The time t_r required to change from **V1** to **V2** is called the ***rise time (TR),*** and the time t_f required to change from **V2** to **V1** is called the ***fall time (TF).*** The duration W_p of the pulse is referred to as the ***pulse width (PW),*** and the ***period*** T of the waveform **(PER)** is the time it takes for the pulse to repeat. Note also that SPICE allows a time delay **(TD)** before the pulse train



(*a*)



(*b*)

■ FIGURE 8.47 (*a*) Schematic of a simple *RC* circuit connected to a pulsed voltage waveform. (*b*) Diagram of the SPICE **VPULSE** parameter definitions.

begins, which can be useful in allowing initial transient responses to decay for some circuit configurations.

For the purposes of this discussion, we set a zero time delay, V1 = 0, and V2 = 9 V. The circuit time constant is $\tau = RC = 1$ ms, so we set the rise and fall times to be 1 ns. Although SPICE will not allow a voltage to change in zero time since it solves the differential equations using discrete time intervals, compared to our circuit time constant 1 ns is a reasonable approximation to "instantaneous."

We will consider four basic cases, summarized in Table 8.1. In the first two cases, the pulse width W_p is much longer than the circuit time constant τ, so we expect the transients resulting from the beginning of the pulse to die out before the pulse is over. In the latter two cases, the opposite is true: the pulse width is so short that the capacitor does not have time to fully charge before the pulse ends. A similar issue arises when we consider the response of the circuit when the time between pulses $(T - W_p)$ is either short (Case II) or long (Case III) compared to the circuit time constant.

TABLE 8.1 Four Separate Cases of Pulse Width and Period Relative to the Circuit Time Constant of 1 ms

Case	Pulse Width W_p	Period T
I	10 ms ($\tau \ll W_p$)	20 ms ($\tau \ll T - W_p$)
II	10 ms ($\tau \ll W_p$)	10.1 ms ($\tau \gg T - W_p$)
III	0.1 ms ($\tau \gg W_p$)	10.1 ms ($\tau \ll T - W_p$)
IV	0.1 ms ($\tau \gg W_p$)	0.2 ms ($\tau \gg T - W_p$)

We qualitatively sketch the circuit response for each of the four cases in Fig. 8.48, arbitrarily selecting the capacitor voltage as the quantity of interest as any voltage or current is expected to have the same time dependence.

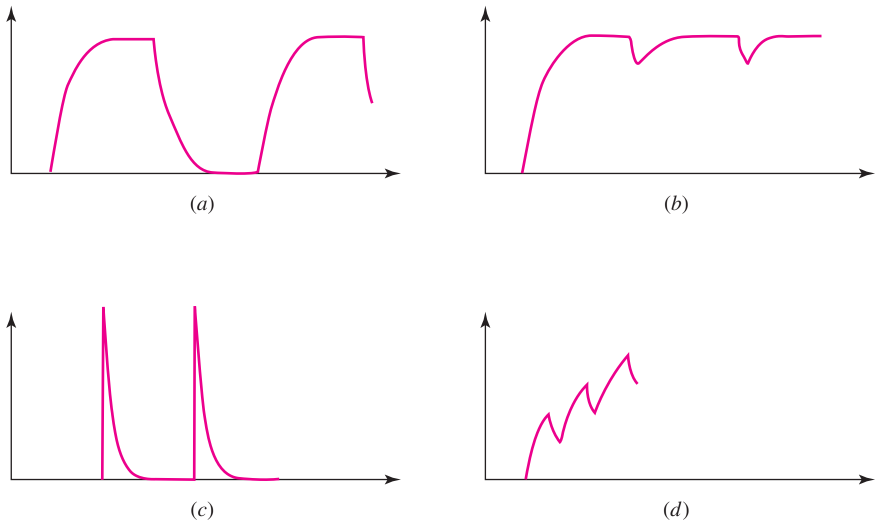


FIGURE 8.48 Capacitor voltage for the *RC* circuit, with pulse width and period as in (*a*) Case I; (*b*) Case II; (*c*) Case III; and (*d*) Case IV.

In Case I, the capacitor has time to both fully charge and fully discharge (Fig. 8.48*a*), whereas in Case II (Fig. 8.48*b*), when the time between pulses is reduced, it no longer has time to fully discharge. In contrast, the capacitor does not have time to fully charge in either Case III (Fig. 8.48*c*) or Case IV (Fig. 8.48*d*).

Case I: Time Enough to Fully Charge and Fully Discharge

We can obtain exact values for the response in each case, of course, by performing a series of analyses. We consider Case I first. Since the capacitor has time to fully charge, the forced response will correspond to the 9 V dc driving voltage. The complete response to the first pulse is therefore

$$v_C(t) = 9 + Ae^{-1000t} \qquad \text{V}$$

With $v_C(0) = 0$, $A = -9$ V and so

$$v_C(t) = 9(1 - e^{-1000t}) \qquad \text{V} \qquad [39]$$

in the interval of $0 < t < 10$ ms. At $t = 10$ ms, the source drops suddenly to 0 V, and the capacitor begins to discharge through the resistor. In this time interval we are faced with a simple "source-free" *RC* circuit, and we can write the response as

$$v_C(t) = Be^{-1000(t-0.01)} \qquad 10 < t < 20 \text{ ms} \qquad [40]$$

where $B = 8.99959$ V is found by substituting $t = 10$ ms in Eq. [39]; we will be pragmatic here and round this to 9 V, noting that the value calculated is consistent with our assumption that the initial transient dissipates before the pulse ends.

At $t = 20$ ms, the voltage source jumps immediately back to 9 V. The capacitor voltage just prior to this event is given by substituting $t = 20$ ms in Eq. [40], leading to $v_C(20\text{ ms}) = 408.6\ \mu\text{V}$, essentially zero compared to the peak value of 9 V.

If we keep to our convention of rounding to four significant digits, the capacitor voltage at the beginning of the second pulse is zero, which is the same as our starting point. Thus, Eqs. [39] and [40] form the basis of the response for all subsequent pulses, and we may write

$$v_C(t) = \begin{cases} 9(1 - e^{-1000t}) \text{ V} & 0 \le t \le 10 \text{ ms} \\ 9e^{-1000(t-0.01)} \text{ V} & 10 < t \le 20 \text{ ms} \\ 9(1 - e^{-1000(t-0.02)}) \text{ V} & 20 < t \le 30 \text{ ms} \\ 9e^{-1000(t-0.03)} \text{ V} & 30 < t \le 40 \text{ ms} \end{cases}$$

and so on.

Case II: Time Enough to Fully Charge But Not Fully Discharge

Next we consider what happens if the capacitor is not allowed to completely discharge (Case II). Equation [39] still describes the situation in the interval of $0 < t < 10$ ms, and Eq. [40] describes the capacitor voltage in the interval between pulses, which has been reduced to $10 < t < 10.1$ ms.

Just prior to the onset of the second pulse at $t = 10.1$ ms, v_C is now 8.144 V; the capacitor has only had 0.1 ms to discharge, and therefore still retains 82 percent of its maximum energy when the next pulse begins. Thus, in the next interval,

$$v_C(t) = 9 + Ce^{-1000(t-10.1\times10^{-3})} \quad \text{V} \qquad 10.1 < t < 20.1 \text{ ms}$$

where $v_C(10.1 \text{ ms}) = 9 + C = 8.144$ V, so $C = -0.856$ V and

$$v_C(t) = 9 - 0.856e^{-1000(t-10.1\times10^{-3})} \quad \text{V} \qquad 10.1 < t < 20.1 \text{ ms}$$

which reaches the peak value of 9 V much more quickly than for the previous pulse.

Case III: No Time to Fully Charge But Time to Fully Discharge

What if it isn't clear that the transient will dissipate before the end of the voltage pulse? In fact, this situation arises in Case III. Just as we wrote for Case I,

$$v_C(t) = 9 + Ae^{-1000t} \quad \text{V} \qquad [41]$$

still applies to this situation, but now only in the interval $0 < t < 0.1$ ms. Our initial condition has not changed, so $A = -9$ V as before. Now, however, just before this first pulse ends at $t = 0.1$ ms, we find that $v_C = 0.8565$ V. This is a far cry from the maximum of 9 V possible if we allow the capacitor time to fully charge, and is a direct result of the pulse lasting only one-tenth of the circuit time constant.

The capacitor now begins to discharge, so that

$$v_C(t) = Be^{-1000(t-1\times10^{-4})} \quad \text{V} \qquad 0.1 < t < 10.1 \text{ ms} \qquad [42]$$

We have already determined that $v_C(0.1^- \text{ ms}) = 0.8565$ V, so $v_C(0.1^+ \text{ ms}) = 0.8565$ V and substitution into Eq. [42] yields $B = 0.8565$ V. Just prior to the onset of the second pulse at $t = 10.1$ ms, the capacitor voltage has decayed to essentially 0 V; this is the initial condition at the start of the second pulse and so Eq. [41] can be rewritten as

$$v_C(t) = 9 - 9e^{-1000(t-10.1\times10^{-3})} \quad \text{V} \qquad 10.1 < t < 10.2 \text{ ms} \qquad [43]$$

to describe the corresponding response.

Case IV: No Time to Fully Charge or Even Fully Discharge

In the last case, we consider the situation where the pulse width and period are so short that the capacitor can neither fully charge nor fully discharge in any one period. Based on experience, we can write

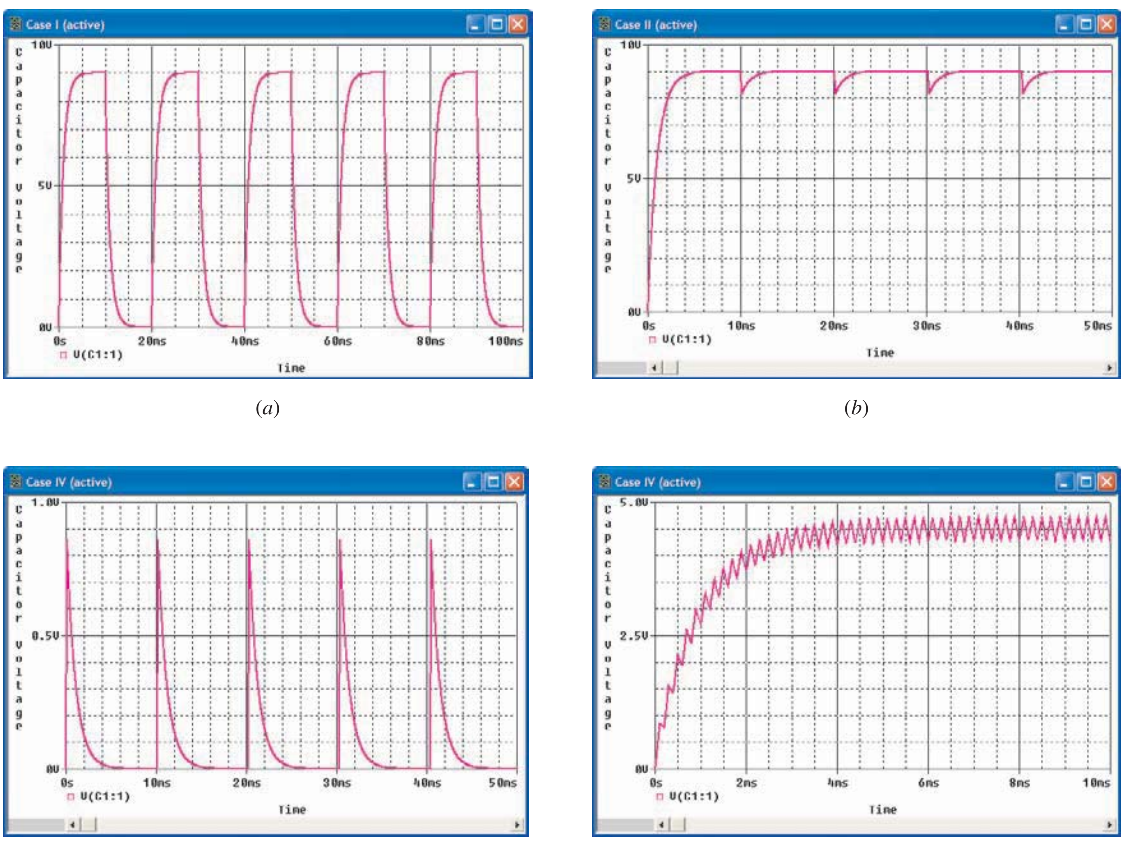
$$v_C(t) = 9 - 9e^{-1000t} \quad \text{V} \qquad 0 < t < 0.1 \text{ ms} \qquad [44]$$

$$v_C(t) = 0.8565e^{-1000(t-1\times10^{-4})} \quad \text{V} \qquad 0.1 < t < 0.2 \text{ ms} \qquad [45]$$

$$v_C(t) = 9 + Ce^{-1000(t-2\times10^{-4})} \quad \text{V} \qquad 0.2 < t < 0.3 \text{ ms} \qquad [46]$$

$$v_C(t) = De^{-1000(t-3\times10^{-4})} \quad \text{V} \qquad 0.3 < t < 0.4 \text{ ms} \qquad [47]$$

Just prior to the onset of the second pulse at $t = 0.2$ ms, the capacitor voltage has decayed to $v_C = 0.7750$ V; with insufficient time to fully discharge, it retains a large fraction of the little energy it had time to store initially. For the interval of $0.2 < t < 0.3$ ms, substitution of $v_C(0.2^+) = v_C(0.2^-) = 0.7750$ V into Eq. [46] yields $C = -8.225$ V. Continuing, we evaluate Eq. [46] at $t = 0.3$ ms and calculate $v_C = 1.558$ V just prior to the end of the second pulse. Thus, $D = 1.558$ V and our



(*a*) (*b*)

(*c*) (*d*)

■ FIGURE 8.49 PSpice simulation results corresponding to (*a*) Case I; (*b*) Case II; (*c*) Case III; (*d*) Case IV.

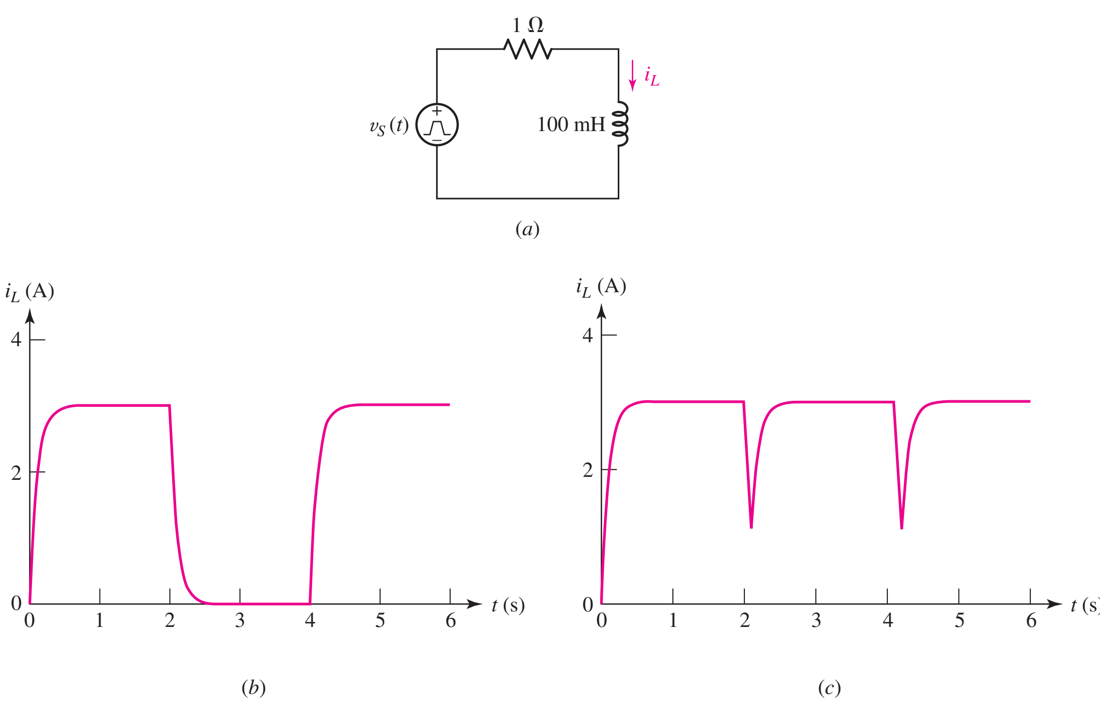
capacitor is slowly charging to ever increase voltage levels over several pulses. At this stage it might be useful if we plot the detailed responses, so we show the PSpice simulation results of Cases I through IV in Fig. 8.49. Note in particular that in Fig. 8.49*d*, the small charge/discharge transient response similar in shape to that shown in Fig. 8.49*a*–*c* is superimposed on a charging-type response of the form $(1 - e^{-t/\tau})$. Thus, it takes about 3 to 5 circuit time constants for the capacitor to charge to its maximum value in situations where a single period does not allow it to fully charge or discharge!

What we have not yet done is predict the behavior of the response for $t \gg 5\tau$, although we would be interested in doing so, especially if it was not necessary to consider a very long sequence of pulses one at a time. We note that the response of Fig. 8.49*d* has an *average* value of 4.50 V from about 4 ms onward. This is exactly half the value we would expect if the voltage source pulse width allowed the capacitor to fully charge. In fact, this long-term average value can be computed by multiplying the dc capacitor voltage by the ratio of the pulse width to the period.

PRACTICE

8.14 With regard to Fig. 8.50a, sketch $i_L(t)$ in the range of $0 < t < 6$ s for (*a*) $v_S(t) = 3u(t) - 3u(t-2) + 3u(t-4) - 3u(t-6) + \cdots$; (*b*) $v_S(t) = 3u(t) - 3u(t-2) + 3u(t-2.1) - 3u(t-4.1) + \cdots$.

Ans: See Fig. 8.50*b*; see Fig. 8.50*c*.



■ FIGURE 8.50 (*a*) Circuit for Practice Problem 8.14; (*b*) solution to part (*a*); (*c*) solution to part (*b*).