Network Analysis & Systems

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UE18EC201: Network Analysis & Systems

Network Analysis and Synthesis

Unit II: Transient Behaviour — First Order Circuits

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Syllabus of Unit II

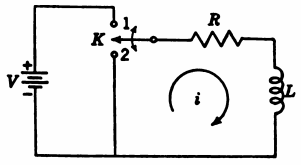
Unit II (11+4 hours) Transient Characteristics:

- Damping and time constants.
- First and Second Order Circuits:
  - Time-domain analysis.
  - Frequency-domain analysis.

Ref. A: Chapters 4–6; parts of 7-10.

Ref. B: Chapters 8–11, 13–15.

First-Order Circuits (1)



- The circuit or system is altered when the switch K transitions from 1 to 2.
- There is an equilibrium before switching.
- The transition is "make-before-break" so that there is no interruption of the current $i(t)$.
- The reference instant of time (when transition of switch takes place) can be designated $t = 0$.

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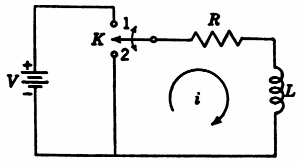

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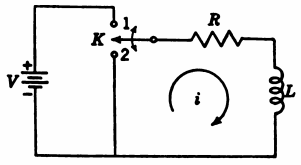
- There is an equilibrium before switching.
- The transition is "make-before-break" so that there is no interruption of the current $i(t)$.
- The reference instant of time (when transition of switch takes place) can be designated $t = 0$.
- Objective of Analysis: Obtain the equations that govern the behaviour of the current from the moment that equilibrium is altered.

First-Order Circuits (2)



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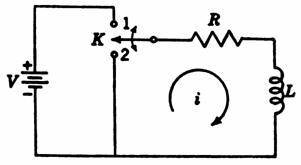

- The time immediately before transition of switch is designated $t = 0-$ and the time immediately after transition of switch is designated $t = 0+$.
- Since there is equilibrium at $t = 0-$, the current in this circuit is constant in the interval $[-\epsilon, 0-]$ for sufficiently large $\epsilon > 0$.
- Before the transition, for all time $t \in [-\epsilon, 0-]$,

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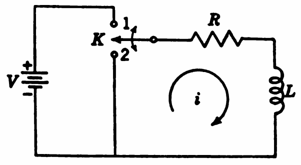

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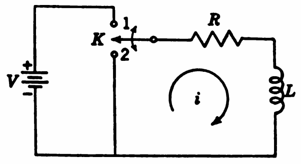
- Since there is equilibrium at $t = 0-$, the current in this circuit is constant in the interval $[-\epsilon, 0-]$ for sufficiently large $\epsilon > 0$.
- Before the transition, for all time $t \in [-\epsilon, 0-]$,

$$i(t) = \frac{V}{R} \triangleq I_0$$

Therefore, $i(0-) = I_0$.

First-Order Circuits (3)



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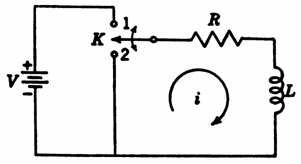

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$$L\frac{di}{dt} + Ri = 0$$

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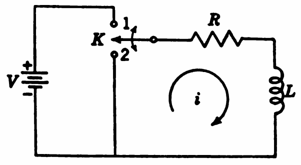

- After the transition of the switch from position 1 to 2, using KVL,

$$L\frac{di}{dt} + Ri = 0$$

- This is a homogeneous first-order ordinary differential equation (ODE) with constant coefficients representing an LTI system.

$$I_0 \longrightarrow \boxed{f(\cdot)} \longrightarrow i(t) = f(I_0)$$

This ODE governs the behaviour of the current in the circuit.

First-Order Circuits (4)

$$L\frac{di}{dt} + Ri = 0$$

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$$\frac{di}{i} = -\frac{R}{L}dt$$

- Integrating from $t = 0+$ to some arbitrary time t,

$$\int_{0+}^{t} \frac{di}{i} = -\int_{0+}^{t} \frac{R}{L}d\tau$$

we get

First-Order Circuits (4)

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- The solution to this ODE may be obtained by separation of variables:
$$\frac{di}{i} = -\frac{R}{L}dt$$
- Integrating from $t = 0+$ to some arbitrary time t,
$$\int_{0+}^{t} \frac{di}{i} = -\int_{0+}^{t} \frac{R}{L}d\tau$$
we get
$$\ln i(t) - \ln i(0+) = -\frac{R}{L}t$$
- Equivalently,
$$i(t) = i(0+)e^{-\frac{R}{L}t}, \quad \forall t \geq 0+$$

First-Order Circuits (5)

Thus, the general solution of

$$L\frac{di}{dt} + Ri = 0$$

is

$$i(t) = ce^{-\frac{R}{L}t}$$

where $c = i(0+)$ is some constant that is to be determined.

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- In particular for the circuit, we know that the current in the inductor cannot change instantaneously.
- Therefore,

$$c = i(0+) = i(0-) = I_0 = \frac{V}{R}$$

- Accordingly,

$$i(t) = \frac{V}{R}e^{-\frac{R}{L}t}, \quad \forall t \geq 0-$$

First-Order Circuits (6)

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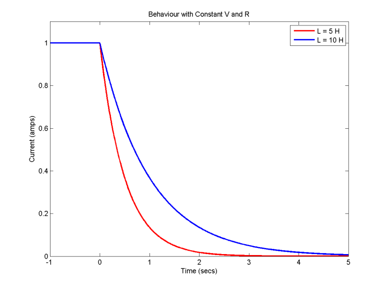

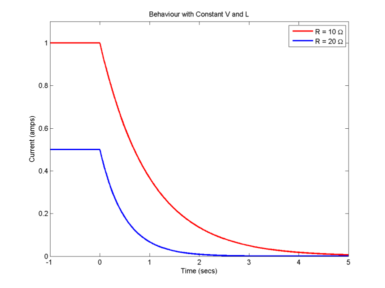

- The rate of decrease is controlled by the ratio $\frac{L}{R}$.

First-Order Circuits (7)

$$i(t) = \frac{V}{R} e^{-\frac{R}{L}t}, \quad \forall t \geq 0-$$

Or,

$$i(t) = I_0 e^{-\frac{1}{T}t}, \quad \forall t \geq 0-$$

where

$$T \triangleq \frac{L}{R}$$

is called the time constant.

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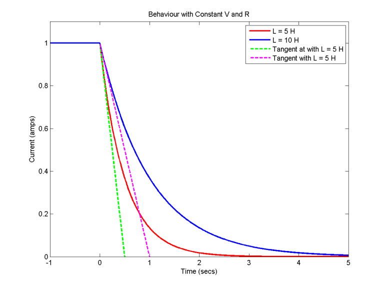
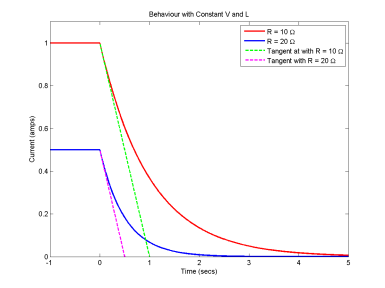
- The solution to any first order differential equation has this form.
- Only I_0 and T varies from problem to problem.

First-Order Circuits (8)

$$\left.\frac{di(t)}{dt}\right|_{t=0} = -\frac{V}{L}e^{-\frac{R}{L}t}\Big|_{t=0} = -\frac{V}{L}$$





$$\tan\theta = \frac{V}{L} = \frac{V/R}{T} \Longrightarrow \frac{1}{T} = \frac{R}{L}$$

First-Order Circuits (9)

For a general first-order differential equation,

$$\frac{dy}{dt} + \frac{1}{T} y(t) = 0$$

the solution is therefore,

$$y(t) = y_0 e^{-\frac{t}{T}}, \quad \forall t \geq 0+$$

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- Since

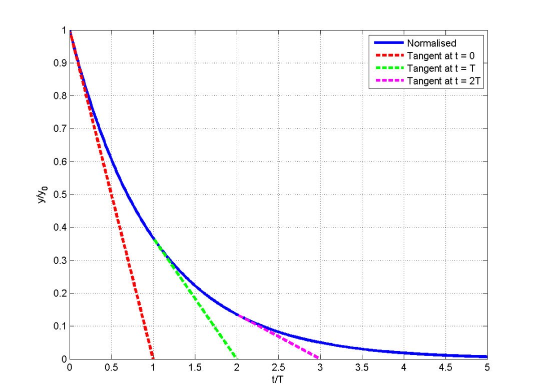
$$y(T) = \frac{y_0}{e} \approx 0.37 y_0$$

the value of the dependent variable reduces to 37% of its initial value in one time constant.

First-Order Circuits (10)

$$\frac{dy}{dt} + \frac{1}{T}y(t) = 0$$

t/T	$y(t)/y_0$
0	1.0
1	0.37
2	0.14
3	0.05
4	0.018
5	0.0067



First-Order Circuits (11)

Consider an R-L circuit with a constant applied voltage:

$$L\frac{di}{dt} + Ri = V$$

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Multiplying throughout by the integrating factor $e^{t/T}$,

$$e^{t/T}\frac{dy}{dt} + e^{t/T}\frac{1}{T}y = Ee^{t/T}$$

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Clearly,

$$\frac{d}{dt}\left(e^{t/T}y(t)\right) = Ee^{t/T}$$

First-Order Circuits (12)

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Integrating from $t = 0$ to t,

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Integrating from $t = 0$ to t,

$$e^{t/T}y(t) - y(0) = TE\left(e^{t/T} - 1\right)$$

Or,

$$y(t) = y(0)e^{-t/T} + TE\left(1 - e^{-t/T}\right), \quad t \geq 0$$

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Thus, for

$$L\frac{di}{dt} + Ri = V$$

the solution is

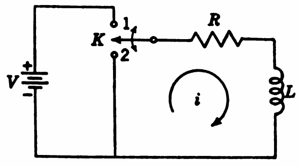
$$i(t) = i(0)e^{-t/T} + \frac{V}{R}\left(1 - e^{-t/T}\right), \quad t \geq 0$$

First-Order Circuits (13)

$$
\begin{aligned}
y(t) &= \underbrace{y(0)e^{-t/T}}_{\text{A}} + \underbrace{TE\left(1 - e^{-t/T}\right)}_{\text{B}} \\
&= \underbrace{\left(y(0) - TE\right)e^{-t/T}}_{\text{C}} + \underbrace{TE}_{\text{D}}
\end{aligned}
$$

- A: response due to initial condition — complementary solution or zero-input response
- B: response due to forcing function — particular solution or zero-state response.
- C: transient or natural response.
- D: steady-state response: $\lim_{t\longrightarrow\infty} y(t)$.

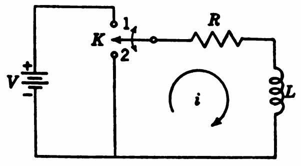
First-Order Circuits (14)



- Afer a long time, the switch K transitions from 2 to 1.

Therefore,

First-Order Circuits (14)



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Therefore, $i(0-) = 0$,

$$L\frac{di}{dt} + Ri = V$$

and

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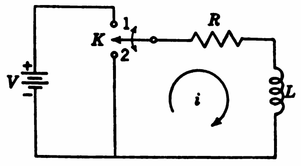

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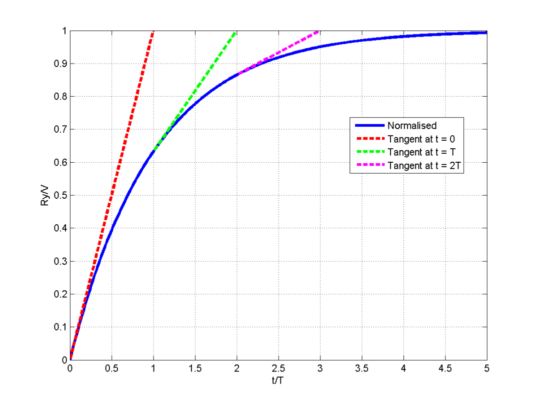
First-Order Circuits (15)

For an R-L circuit, if the initial current is zero,

$$i(t) = \frac{V}{R}\left(1 - e^{-t/T}\right), \quad t \geq 0$$



First-Order Circuits (16)

For an R-L circuit, if the initial current is zero,

$$i(t) = \frac{V}{R}\left(1 - e^{-t/T}\right), \quad t \geq 0$$

- The steady-state part is $i(\infty) = \frac{V}{R}$.
- This is established at $t = 0$ itself.
- However, since $i(0-) = i(0+) = 0$, the transient part must adjust itself so that this is maintained.

First-Order Circuits (17)

For an R-L circuit, $L\frac{di}{dt} + Ri = V$, and

$$i(t) = i(0)e^{-t/T} + \frac{V}{R}\left(1 - e^{-t/T}\right), \quad t \geq 0$$

Therefore, the steady-state value

$$i(\infty) = \frac{V}{R}$$

Now,

$$\begin{aligned} i(t) &= i(0)e^{-t/T} + \frac{V}{R}\left(1 - e^{-t/T}\right) \\ &= \frac{V}{R} - \left(\frac{V}{R} - i(0)\right)e^{-t/T} \\ &= i(\infty) - \left(i(\infty) - i(0)\right)e^{-t/T} \end{aligned}$$

Example (1)

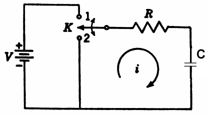
$$i(t) = i(\infty) - \big(i(\infty) - i(0)\big)\, e^{-t/T}, \quad t \geq 0$$

Example 4.2, Valkenburg:

Case 1: After a long time, the switch is closed.

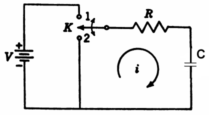
Case 2: Again, after a long time, the switch is opened. The new reference time is $t = 0$.

First-Order Circuits (18)



- Switch is at position 1: The zero-state of the capacitor $v_c(0-) = V$. Moreover, $i(0-) = 0$.
- Switch transitions from 1 to 2. Therefore,

First-Order Circuits (18)



- Switch is at position 1: The zero-state of the capacitor $v_c(0-) = V$. Moreover, $i(0-) = 0$.
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$$\frac{di}{dt} + \frac{1}{RC} i = 0$$

- Clearly,

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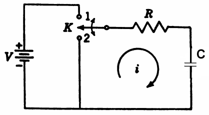

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- Switch transitions from 1 to 2. Therefore,

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- Clearly,

$$i(t) = -\frac{V}{R} e^{-t/RC}, \quad t \geq 0$$

- The time constant $T = RC$.

First-Order Circuits (19)

- First-order differential equations have only one time-constant.

First-Order Circuits (19)

- First-order differential equations have only one time-constant.
- These equations result from networks with a single inductor or capacitor in combination with any number of resistors, or
- if more complex networks can be reduced to a single equivalent resistor and a single equivalent inductor or capacitor.

Laplace Transform (1)

Definition

Given a time signal $f(t)$, its Laplace transform is a function of the complex number $s = \sigma + j\omega$ defined as

$$F(s) = \mathcal{L}[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}dt$$

provided the integral exists.

- This is called the one-sided Laplace transform.
- Note that the lower limit is $0-$.

Laplace Transform (2)

$$F(s) = \mathcal{L}[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Sufficient conditions for the existence of the Laplace transform are

1. $f(t)$ should be piecewise continuous.
2. $f(t)$ should be of exponential order: There exists $\sigma > 0$ s.t.

$$\int_{0-}^{\infty} |f(t)|e^{-\sigma t}dt < \infty$$

Laplace Transform (3)

- Unit step:

$$f(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

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- The exponential:

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- The derivative: $\frac{df(t)}{dt}$

Dirac Delta Impulse (1)

Definition

The Dirac delta impulse[1] by definition is a distribution[2] or generalised function s.t.

$$\int_{-\infty}^{\infty} \delta(t)dt = 1, \quad \delta(t) = 0 \quad \forall\, t \neq 0$$

s.t. for any continuous function $f(t)$,

$$\int_{-\infty}^{\infty} \delta(t)f(t)dt = f(0)$$

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th edition, Oxford University Press, 1958.

[2] These are not regular functions. Essentially, a distribution maps a function to a number.

Dirac Delta Impulse (2)

- Unlike a regular function $f(\cdot)$ which associates a unique value $f(t)$ for every t, it is usually not possible to associate a "value" to a distribution for each t.
- Many sequences of functions approximate δ. For example, a sequence of pulses of vanishing width and increasing amplitude.
- One should never "define" the value of an impulse at $t = 0$.
- Any operations with δ must be interpreted with reference to an integral.
- $\delta(t) = \frac{d}{dt}1(t)$, where $1(t)$ is the unit-step function.
- $\mathcal{L}[\delta(t)] = 1$.

First-Order Circuits (20)

$$\frac{dy}{dt} + \frac{1}{T}y = E$$

Assuming all signals are Laplace transformable,

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Assuming all signals are Laplace transformable,

$$sY(s) - y(0-) + \frac{1}{T}Y(s) = E(s)$$

That is,

$$Y(s) = \frac{y(0-)}{s + 1/T} + \frac{E(s)}{s + 1/T}$$

If E is a constant, then

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If E is a constant, then

$$Y(s) = \frac{y(0-)}{s+1/T} + \frac{E}{s(s+1/T)} = \frac{y(0-)}{s+1/T} + TE\left(\frac{1}{s} - \frac{1}{s+1/T}\right)$$

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$$\Longrightarrow y(t) = y(0-)e^{-t/T} + TE\left(1 - e^{-t/T}\right), \quad t \geq 0$$

First-Order Circuits (21)

- R-L circuit excited by a DC source and $i(0-) = 0$:

$$\begin{aligned} L\frac{di}{dt} + Ri &= V \\ \Longrightarrow sLI(s) + RI(s) &= \frac{V}{s} \end{aligned}$$

Therefore,

$$\begin{aligned} I(s) &= \frac{V/L}{s + L/R} \\ \Longrightarrow i(t) &= \frac{V}{R}\left(1 - e^{-tR/L}\right), \quad t \geq 0 \end{aligned}$$

First-Order Circuits (22)

- R-C circuit excited by a DC source and $v_C(0-) = 0$:

$$\begin{aligned} \frac{di}{dt} + \frac{1}{RC} i &= 0 \\ \implies sI(s) - i(0) + \frac{1}{RC} I(s) &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} I(s) &= \frac{i(0)}{s + 1/RC} \\ \implies i(t) &= \frac{V}{R} e^{-t/RC}, \quad t \geq 0 \end{aligned}$$

Examples (1)

4-20. In the network shown, the switch K is closed at $t = 0$. The current waveform is observed with a cathode ray oscilloscope. The initial value of the current is measured to be 0.01 amp. The transient appears to disappear in 0.1 sec. Find (a) the value of R, (b) the value of C, and (c) the equation of $i(t)$.

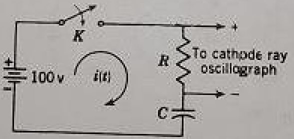

Fig. P4-20.

4-21. The circuit shown in the accompanying figure consists of a resistor and a relay with inductance L. The relay is adjusted so that it is actuated when the current through the coil is 0.008 amp. The switch K is closed at $t = 0$, and it is observed that the relay is actuated when $t = 0.1$ sec. Find: (a) the inductance L of the coil, (b) the equation of $i(t)$ with all terms evaluated.

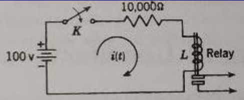

Fig. P4-21.

Examples (2)

4-22. A switch is closed at $t = 0$, connecting a battery of voltage V with a series RC circuit. (a) Determine the ratio of energy delivered to the capacitor to the total energy supplied by the source as a function of time. (b) Show that this ratio approaches 0.50 as $t \longrightarrow \infty$.

4-23. Consider the exponentially decreasing function $i = Ke^{-t/T}$ where T is the time constant. Let the tangent drawn from the curve at $t = t_1$ intersect the line $i = 0$ at t_2. Show that for any such point, $i(t_1)$, $t_2 - t_1 = T$.