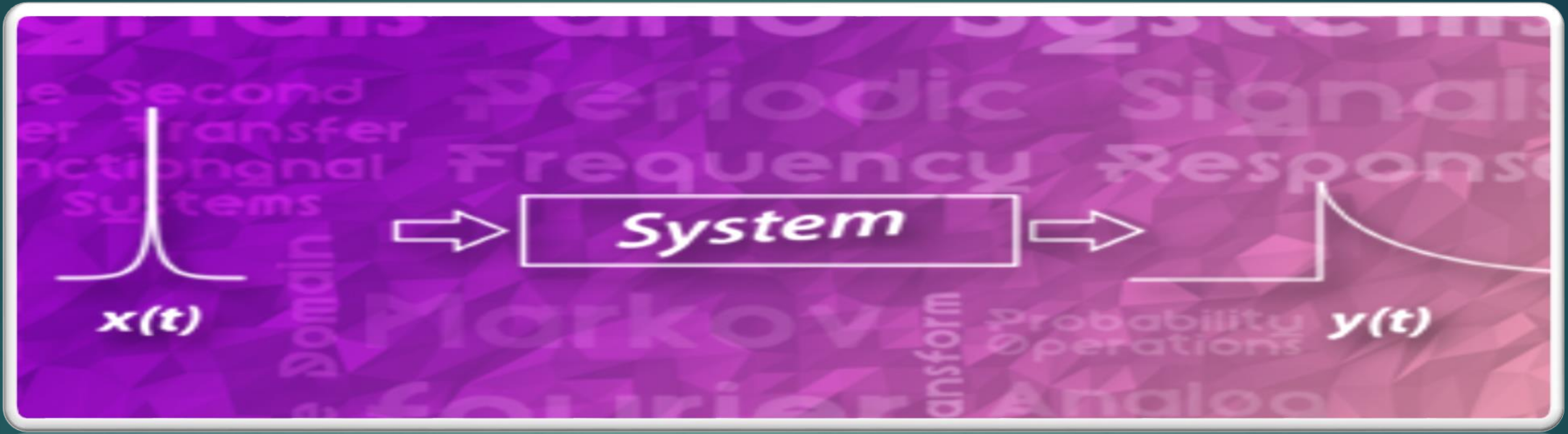

Signals and Systems UE18EC204

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UNIT 2

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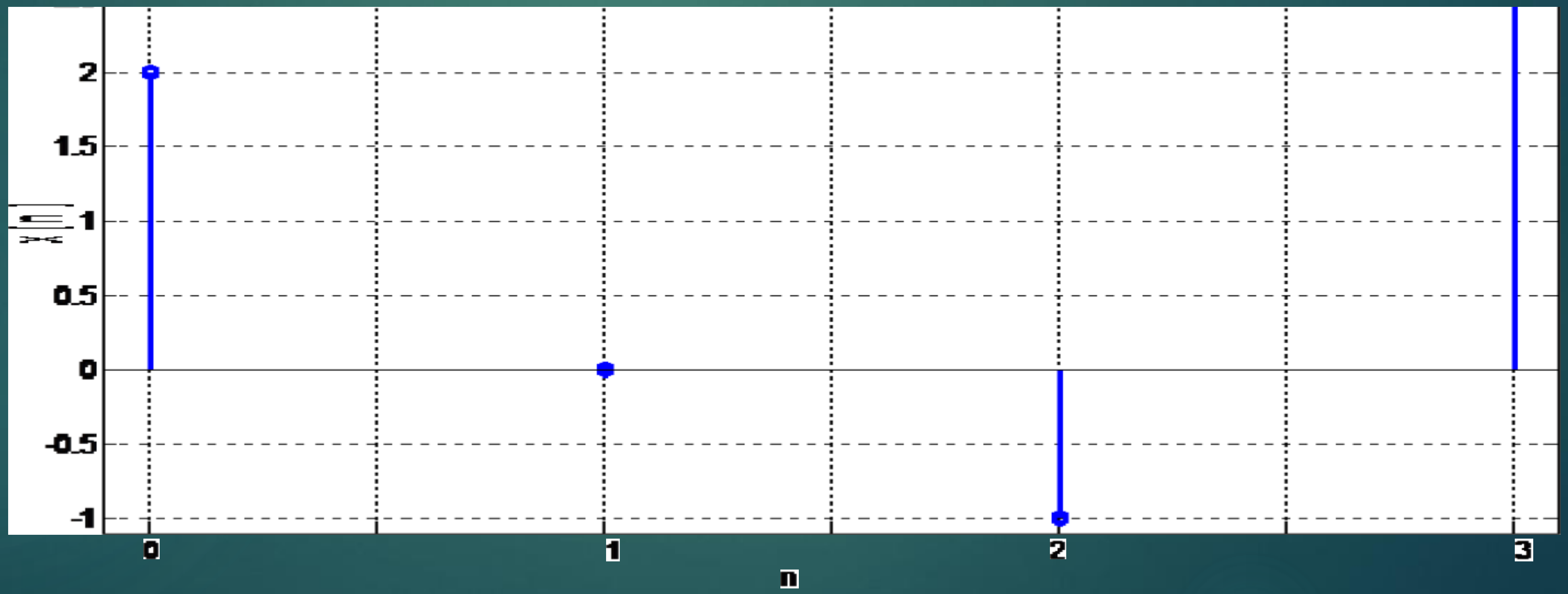
Why Linear Time-Invariant (LTI) Systems?

- In engineering, linear-time invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI.
- In practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.

Discrete-time signals

A discrete-time signal is a set of numbers

$$x = [2\ \ 0 - 1\ 3]$$



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Resolution of a DT Signal into pulses

x = [2 0 -1 3]

Impulses at n = 0, 1, 2, and 3 with amplitudes

x[0] = 2, x[1] = 0, x[2] = -1, x[3] = 3

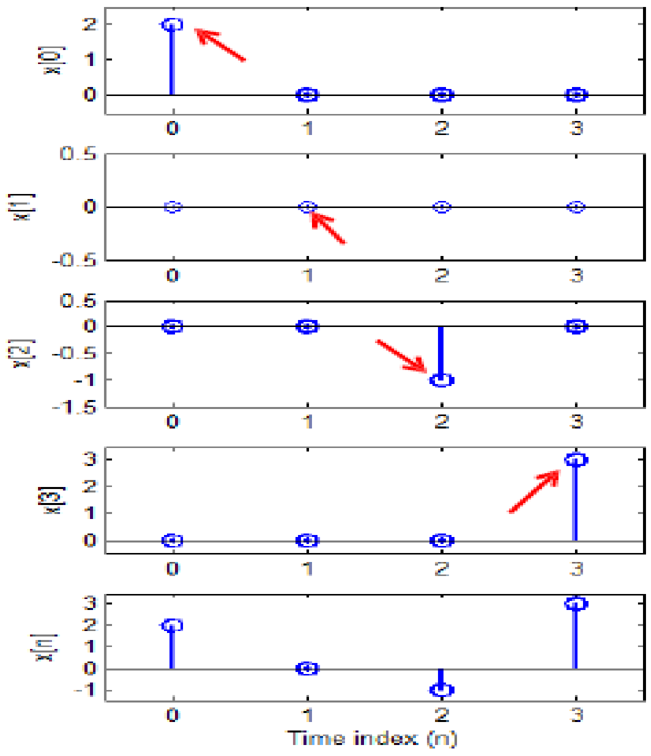
This can be written as,

$$x[n] = 2\delta[n] - \delta[n-2] + 3\delta[n-3]$$

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3]$$

$$x[n] = \sum_{k=0}^{K-1} x[k]\delta[n-k]$$ *K is the length of x*

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$ *For infinite pulses*



- This corresponds to the representation of an arbitrary sequence as a linear combination of shifted unit impulses $\boldsymbol{\delta}[n-k]$, where the weights are $x[n]$ in the linear combination.
- The above equation is called Sifting property of the discrete-time unit impulse.
- As an example, when $x[n] = u[n]$, the unit step, then we have

$$x\,[n] = \sum_{k=0}^{\infty} \boldsymbol{\delta}[n-k] = u[n]$$

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Example 1: Resolve the following discrete-time signals into impulses

$$x[n] = 2\ \underset{\uparrow}{4}\ 0\ 3 \qquad\qquad r[n] = 2\ 4\ \underset{\uparrow}{0}\ 3$$

Impulses occur at n = -1, 0, 1, 2 with amplitudes x[-1] = 2, x[0] = 4, x[1] = 0, x[2] = 3

$$x[n] = \sum_{k=-1}^{2} x[m]\delta[n-k]$$

$$= x[-1]\delta[n-(-1)] + x[0]\delta[n-0] + x[1]\delta[n-1] + x[2]\delta[n-2]$$

$$x[n] = 2\delta[n+1)] + 4\delta[n] + 3\delta[n-2]$$

Follow the same procedure for r[n]

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- **Output $y[n]$ for input $x[n]$** $$y[n] = T\{x[n]\}$$
- **Any signal can be decomposed into sum of discrete impulses** $$y[n] = T\left\{\sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]\right\}$$
- **Apply linearity properties of homogeneity then additivity** $$y[n] = \sum_{m=-\infty}^{\infty} x[m]\,T\{\delta[n-m]\}$$
- **Apply shift-invariance** $$y[n] = \sum_{m=-\infty}^{\infty} x[m]\,h[n-m]$$
- **Apply change of variables** $$y[n] = \sum_{m=-\infty}^{\infty} h[m]\,x[n-m]$$

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Convolution Sum

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Convolution Sum

- The (DT) convolution of the sequences $x[n]$ and $h[n]$, denoted $x[n] * h[n]$, is defined as the sequence

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- The convolution $x[n] * h[n]$ evaluated at the point n is simply a weighted sum of elements of $x[n]$, where the weighting is given by $h[n]$ time reversed and shifted by n.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- Convolution is used extensively in systems theory and in particular, convolution has a special significance in the context of LTI systems.

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Step 1	List the index 'k' covering a sufficient range
Step 2	List the input x[k]
Step 3	Obtain the reversed sequence h[-k], and align the rightmost element of h[n-k] to the leftmost element of x[k]
Step 4	Cross-multiply and sum the nonzero overlap terms to produce y[n]
Step 5	Slide h[n-k] to the right by one position
Step 6	Repeat step 4; stop if all the output values are zero or if required.

$x[k] = [\underset{\uparrow}{3}\ 1\ 2] \qquad h[k] = [\underset{\uparrow}{3}\ 2\ 1]$

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

$y[0] = 3 \times 3 = 9$

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

y[0] = 3 x 3 = 9

y[1] = 3 x 2 + 3 x 1 = 9

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

$y[0] = 3 \times 3 = 9$

$y[1] = 3 \times 2 + 3 \times 1 = 9$

$y[2] = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

$y[0] = 3 \times 3 = 9$

$y[1] = 3 \times 2 + 3 \times 1 = 9$

$y[2] = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$

$y[3] = 1 \times 1 + 2 \times 2 = 5$

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

$y[0] = 3 \times 3 = 9$

$y[1] = 3 \times 2 + 3 \times 1 = 9$

$y[2] = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$

$y[3] = 1 \times 1 + 2 \times 2 = 5$

$y[4] = 2 \times 1 = 2$

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

y[0] = 3 x 3 = 9

y[1] = 3 x 2 + 3 x 1 = 9

y[2] = 3 x 1 + 1 x 2 + 2 x 3= 11

y[3] = 1 x 1 + 2 x 2 = 5

y[4] = 2 x 1 = 2

y[5] = 0 (no overlap)

$$\boxed{y[n] = \{9 \quad 9 \quad 11 \quad 5 \quad 2 \quad 0\}}$$

Representation of Signals Using Impulses

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- For any function x,

$$x\,[n] = \sum_{k=-\infty}^{\infty} x[k]\boldsymbol{\delta}[n-k] = x * \boldsymbol{\delta}[n]$$

- Thus, any function x can be written in terms of an expression involving $\boldsymbol{\delta}$.
- Moreover, $\boldsymbol{\delta}$ is the convolutional identity. That is, for any function x,

$$x * \boldsymbol{\delta} = x$$

Convolution Integral

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Convolution Integral

- The Continuous-time convolution of the functions x and h, denoted $x * h$, is defined as the function

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})h(t - \boldsymbol{\tau})d\,\boldsymbol{\tau}$$

- The convolution result $x * h$ evaluated at the point t is simply a weighted average of the function x, where the weighting is given by h time reversed and shifted by t.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- Convolution is used extensively in systems theory and in particular, convolution has a special significance in the context of LTI systems.

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Convolution Integral Computation

- To compute the convolution

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})h(t-\boldsymbol{\tau})d\,\boldsymbol{\tau}$$

we proceed as follows:

- Plot $x(\boldsymbol{\tau})$ and $h(t-\boldsymbol{\tau})$ as a function of $\boldsymbol{\tau}$.
- Initially, consider an arbitrarily large negative value for t. This will result in $h(t-\boldsymbol{\tau})$ being shifted very far to the left on the time axis.
- Write the mathematical expression for $x * h(t)$.
- Increase t gradually until the expression for $x * h(t)$ changes form. Record the interval over which the expression for $x * h(t)$ was valid.
- Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t-\boldsymbol{\tau})$ being shifted very far to the right on the time axis.
- The results for the various intervals can be combined in order to obtain an expression for $x * h(t)$ for all t.

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Representation of Signals Using Impulses

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- For any function x,

$$x(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\boldsymbol{\delta}(t-\boldsymbol{\tau})d\,\boldsymbol{\tau} = x * \boldsymbol{\delta}(t)$$

- Thus, any function x can be written in terms of an expression involving $\boldsymbol{\delta}$.
- Moreover, $\boldsymbol{\delta}$ is the convolutional identity. That is, for any function x,

$$x * \boldsymbol{\delta} = x$$

Properties of Convolution

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Properties of Convolution

- The convolution operation is commutative. That is, for any two functions x and h,

$$x * h = h * x.$$

- The convolution operation is associative. That is, for any signals $x, h1,$ and $h2$,

$$(x * h1) * h2 = x * (h1 * h2).$$

- The convolution operation is distributive with respect to addition. That is, for any signals $x, h1,$ and $h2$,

$$x * (h1 + h2) = x * h1 + x * h2.$$

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Properties of Convolution

- Shift: If $x_1(t) * x_2(t) = c(t)$, then $x_1(t) * x_2(t - T) = x_1(t - T) * x_2(t) = c(t - T)$ and $x_1(t - T_1) * x_2(t - T_2) = c(t - T_1 - T_2)$ (applies to convolution sum)
- Convolution with impulse, $x(t) * \boldsymbol{\delta}(t) = x(t)$ and $x(\mathrm{n}) * \boldsymbol{\delta}(\mathrm{n}) = x(n)$
- Convolution with shifted impulse, $x(t) * \boldsymbol{\delta}(t-T) = x(t-T)$
- Differentiation: If $x(t) * h(t) = y(t)$ then $\frac{dx(t)}{dt} * h(t) = x(t) * \frac{dh(t)}{dt} = \frac{dy(t)}{dt}$
- Time-scaling: If $x(t) * h(t) = y(t)$ then $x(at) * h(at) = \frac{1}{|a|} y(at)$
  - The Convolution of an odd and an even function is an odd function
  - The Convolution of two odd functions is an even function
  - The Convolution of two even functions is an even function

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Impulse Response

- The response h of a system H to the input $\boldsymbol{\delta}$ is called the impulse response of the system (i.e., $h = H\{\boldsymbol{\delta}\}$).
- For any LTI system with input x, output y, and impulse response h, the following relationship holds:
$$y = x * h.$$
- In other words, a LTI system simply computes a convolution.
- Furthermore, a LTI system is completely characterized by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.
- Since the impulse response of a LTI system is an extremely useful quantity, we often want to determine this quantity in a practical setting.
- Unfortunately, in practice, the impulse response of a system cannot be determined directly from the definition of the impulse response.

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Step Response

- The response s of a system H to the input u (unit step) is called the step response of the system (i.e., $s = H\{u\}$).
- The impulse response h and step response s of a system are related as

$$h(t) = \frac{ds(t)}{dt}$$

$$h[n] = s[n] - s[n-1].$$

- Therefore, the impulse response of a system can be determined from its step response by differentiation and (first-order) differencing for continuous time and discrete-time systems respectively.
- The step response provides a practical means for determining the impulse response of a system.

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Block Diagram Representation of LTI Systems

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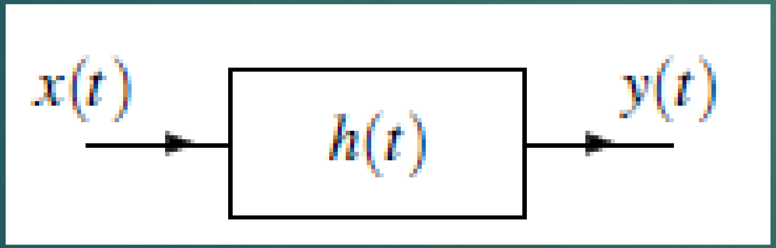
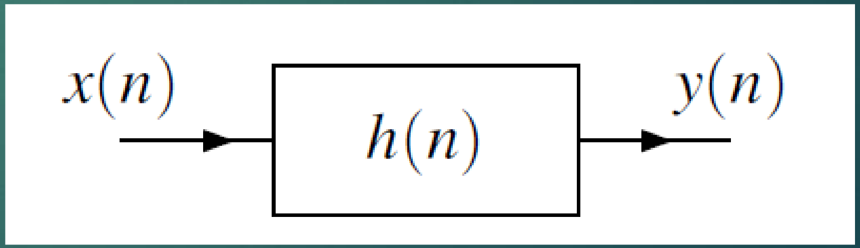
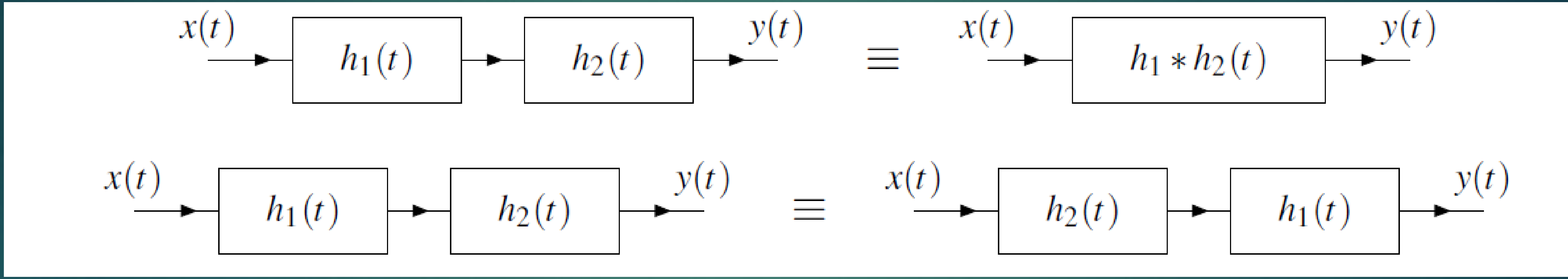
- Often, it is convenient to represent a LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input x, output y, and impulse response h, as shown below.

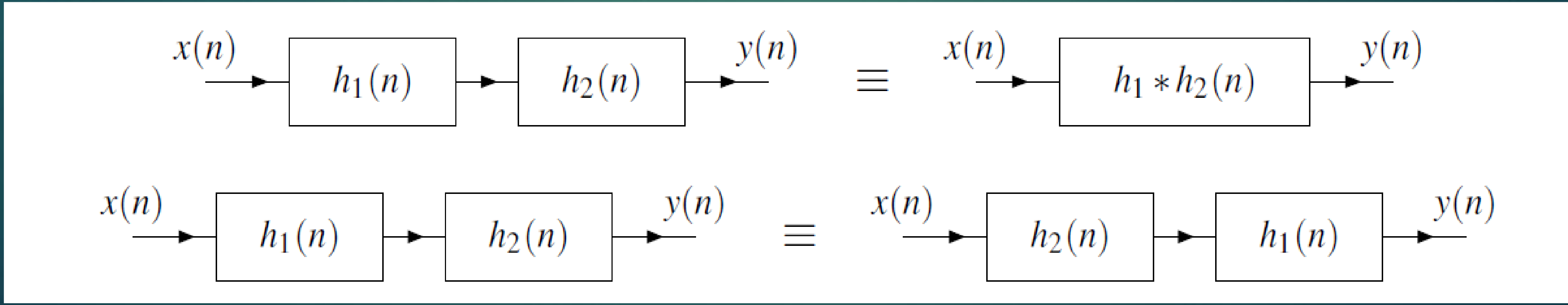




Interconnection of LTI Systems

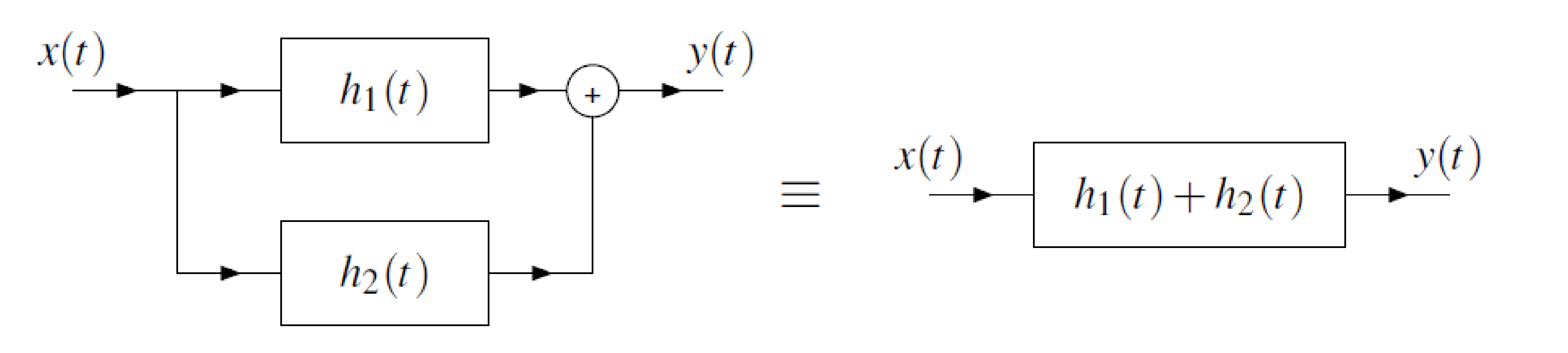
- The series interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h = h_1 * h_2$ That is, we have the equivalences shown below.

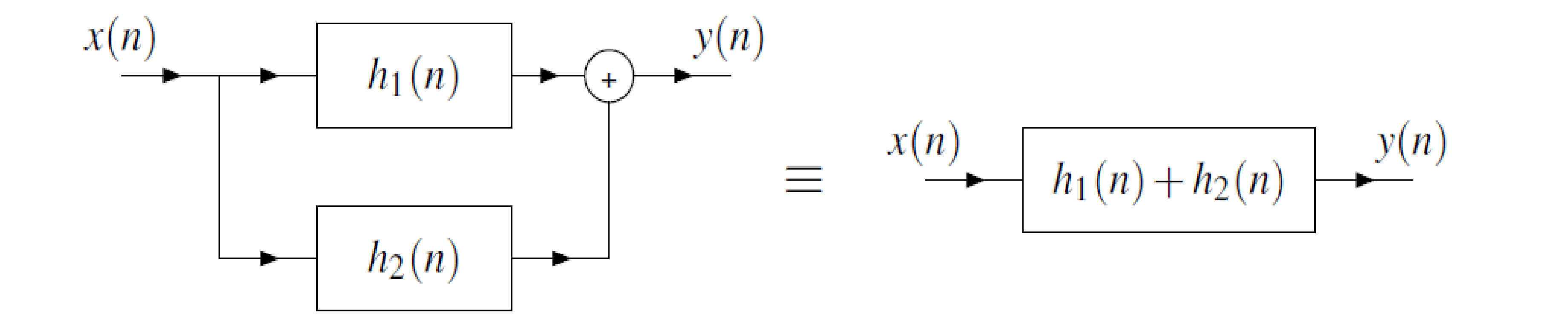




Interconnection of LTI Systems

- The parallel interconnection of the LTI systems with impulse responses h_1 and h_2 is a LTI system with the impulse response $h = h_1 + h_2$. That is, we have the equivalence shown below.





Properties of LTI Systems

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Memory

- A LTI system with impulse response h is memoryless if and only if

$$h(t) = 0 \; for \; all \; t \neq 0$$
$$h[n] = 0 \; for \; all \; n \neq 0$$

- That is, a LTI system is memoryless if and only if its impulse response h is of the form

$$h(t) = K\boldsymbol{\delta}(t),$$
$$h[n] = K\boldsymbol{\delta}[n],$$

- where K is a constant.
- Consequently, every memoryless LTI system with input x and output y is characterized by an equation of the form

$$y = x * (K\boldsymbol{\delta}) = Kx$$

- (i.e., the system is an ideal amplifier).
- For a LTI system, the memoryless constraint is extremely restrictive (as every memoryless LTI system is an ideal amplifier).

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Causality

- A LTI system with impulse response h is causal if and only if

$$h(t) = 0 \; for \; all \; t < 0$$

$$h[n] = 0 \; for \; all \; n < 0$$

(i.e., h is a causal signal).

- It is due to the above relationship that we call a signal x, satisfying

$$x(t) = 0 \; for \; all \; t < 0$$

$$\mathrm{x}[n] = 0 \; for \; all \; n < 0$$

a causal signal.

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Invertibility

- The inverse of a LTI system, if such a system exists, is a LTI system.
- Let h and h_{inv} denote the impulse responses of a LTI system and its (LTI) inverse, respectively. Then,

$$h * h_{inv} = \boldsymbol{\delta}$$

- Consequently, a LTI system with impulse response h is invertible if and only if there exists a function h_{inv} such that

$$h * h_{inv} = \boldsymbol{\delta}$$

- Except in simple cases, the above condition is often quite difficult to test.

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BIBO Stability

- A LTI system with impulse response $h(t)$ is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

(i.e., $h(t)$ is absolutely integrable).

- A LTI system with impulse response $h[n]$ is BIBO stable if and only if

$$\sum_{-\infty}^{\infty} |h[n]| < \infty$$

(i.e., $h[n]$ is absolutely summable)

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Convolution Sum Computation

- To compute the convolution

$$x * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

we proceed as follows:

- Plot $x[k]$ and $h[n-k]$ as a function of k.
- Initially, consider an arbitrarily large negative value for n. This will result in $h[n-k]$ being shifted very far to the left on the time axis.
- Write the mathematical expression for $x * h[n]$.
- Increase n gradually until the expression for $x * h[n]$) changes form. Record the interval over which the expression for $x * h[n]$ was valid.
- Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h[n-k]$ being shifted very far to the right on the time axis.
- The results for the various intervals can be combined in order to obtain an expression for $x * h[n]$ for all n.

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