Network Analysis & Systems

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UE18EC201: Network Analysis & Systems

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Network Analysis and Synthesis

Unit III: Network Theorems

Overview of Syllabus & Lesson Plan

Unit III (8+4 hours) Network Theorems:

- Superposition theorem.
- Thevenin's and Norton's theorems.
- Maximum power transfer and reciprocity theorems.
- Millmann's and Tellegen's theorems.

Ref. A: Chapters 9 & 14.

Ref. B: Chapter 5.

Complex Frequency (1)

Recall: The solution to differential equations had terms of the form

$$y(t) = k_n e^{s_n t}$$

- Complex frequency: $s_n = \sigma_n + j\omega_n$
- Radian frequency or angular frequency: ω_n rad/sec. Further,

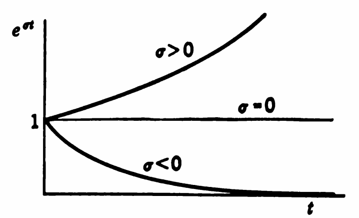
$$\omega_n = 2\pi f_n = \frac{2\pi}{T_n}$$

  where f_n is the frequency in cycles per second, or Hertz, and T_n is the corresponding period.
- Neper[1] frequency: σ_n neper per second.
- Note: the unit for the natural (or Naperian) logarithm is the *neper*.

[1] The word originates from Neperus, the Latin form of Napier, the 17th century mathematician.

Complex Frequency (2)

Case 1: $s_n = \sigma_n + j0$ (Neper frequency only)



- $\sigma > 0$: increases exponentially.
- $\sigma < 0$: decreases or decays exponentially.
- $\sigma = 0$: direct current or voltage.

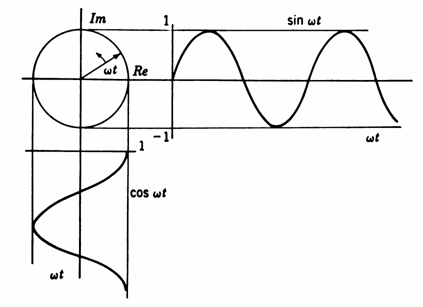
Complex Frequency (3)

Case 2: $s_n = 0 + j\omega_n$ (Radian frequency only)

- Thus,
$$k_n e^{\pm j\omega_n t} = k_n(\cos\omega_n t \pm j\sin\omega_n t)$$
- $e^{\pm j\omega_n t}$ is interpreted as a unit rotating *phasor*.
- A phasor has magnitude and phase w.r.t. to a reference.
- The unit phasor $e^{j\omega_n t}$ rotates counterclockwise or anti-clockwise and is considered positive.
- The unit phasor $e^{-j\omega_n t}$ rotates clockwise and is considered negative.

Complex Frequency (4)

Case 2: $s_n = 0 + j\omega_n$ (Radian frequency only)



- For positive rotation, the projection on the real axis is $\cos \omega_n t$.
- For positive rotation, the projection on the imaginary axis is $\sin \omega_n t$.

Complex Frequency (5)

Case 3: $s_n = \sigma_n + j\omega_n$.

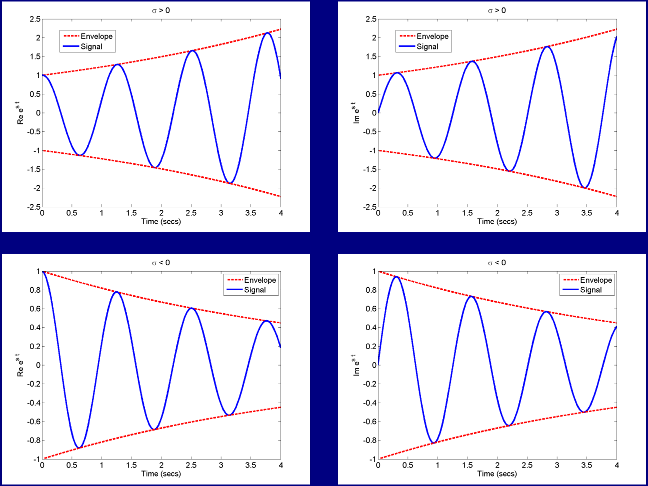
- Thus,

$$k_n e^{s_n t} = k_n e^{\sigma_n t} e^{\pm j\omega_n t} = k_n e^{\sigma_n t}(\cos\omega_n t \pm j\sin\omega_n t)$$

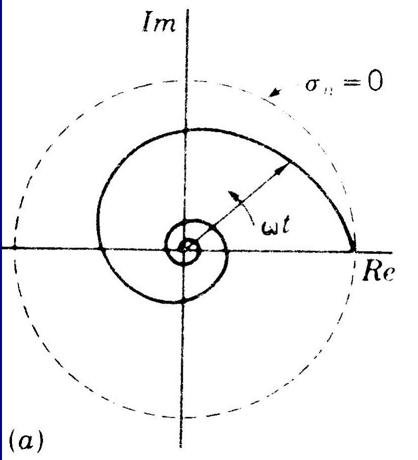
- Clearly, there is the rotating phasor, whose amplitude is varying with time.
- For positive rotation, for $k_n = 1$,

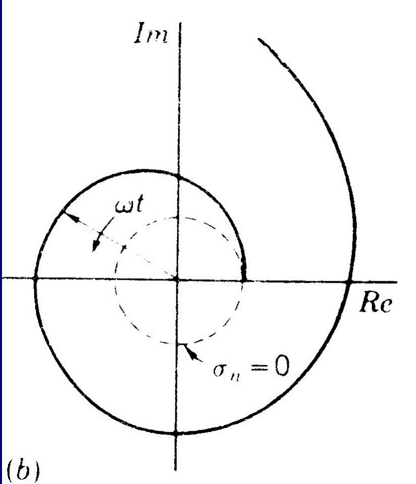
$$\text{Re } e^{s_n t} = e^{\sigma_n t}\cos\omega_n t, \quad \text{Im } e^{s_n t} = e^{\sigma_n t}\sin\omega_n t$$

Complex Frequency (5)

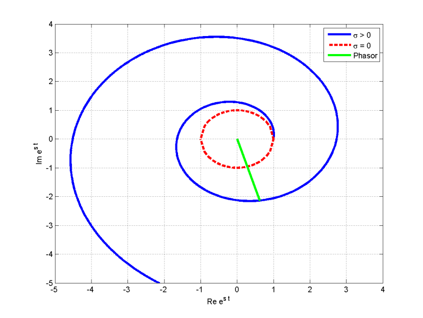


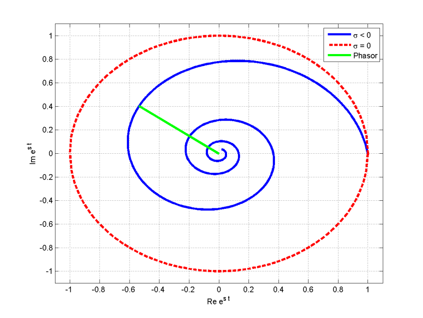
Complex Frequency (6)





Complex Frequency (7)





Transform Impedance and Admittance (1)

Resistance:

$$v_R(t) = Ri_R(t), \quad i_R(t) = \frac{1}{R}v_R(t) = Gv_R(t)$$

Or,

$$V_R(s) = RI_R(s), \quad I_R(s) = GV_R(s)$$

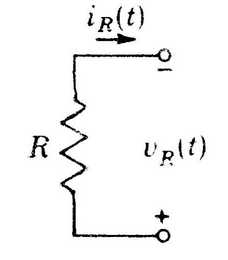
That is,

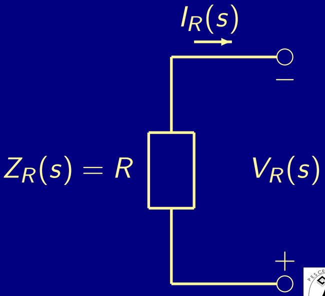
$$Z_R(s) \triangleq \frac{V_R(s)}{I_R(s)} = R, \quad Y_R(s) \triangleq \frac{I_R(s)}{V_R(s)} = G$$

- $Z_R(s)$ is called the transform *impedance*, and $Y_R(s)$ is called the transform *admittance*.

Transform Impedance and Admittance (2)

$$Z_R(s) \triangleq \frac{V_R(s)}{I_R(s)} = R, \quad Y_R(s) \triangleq \frac{I_R(s)}{V_R(s)} = G$$





Transform Impedance and Admittance (3)

Inductance:

$$v_L(t) = L\frac{di_L(t)}{dt}, \quad i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)d\tau$$

Or,

$$V_L(s) = L\left(sI_L(s) - i_L(0-)\right), \quad I_L(s) = \frac{1}{s}\left(\frac{1}{L}V_L(s) + i_L(0-)\right)$$

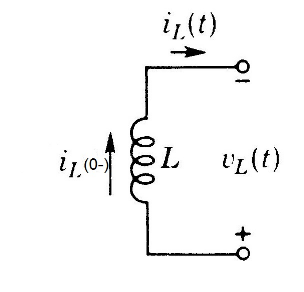
That is,

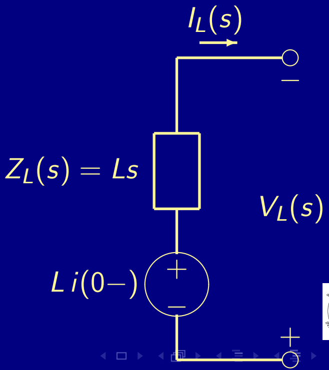
$$Z_L(s) \triangleq \frac{V_1(s)}{I_L(s)} = Ls, \quad Y_L(s) \triangleq \frac{I_1(s)}{V_L(s)} = \frac{1}{Ls}$$

where $V_1(s) \triangleq V_L(s) + Li_L(0-)$ and $I_1(s) = I_L(s) - \frac{i_L(0-)}{s}$.

Transform Impedance and Admittance (4)

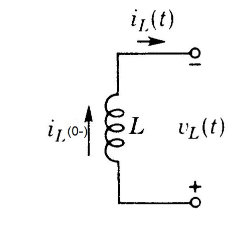
$$Z_L(s) = \frac{V_L(s) + L i_L(0-)}{I_L(s)} = Ls, \quad Y_L(s) = \frac{I_L(s) - \frac{i_L(0-)}{s}}{V_L(s)} = \frac{1}{Ls}$$

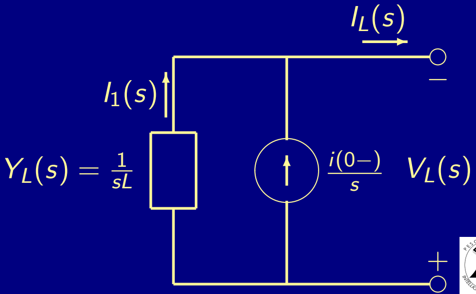




Transform Impedance and Admittance (5)

$$Z_L(s) = \frac{V_L(s) + Li_L(0-)}{I_L(s)} = Ls, \quad Y_L(s) = \frac{I_L(s) - \frac{i_L(0-)}{s}}{V_L(s)} = \frac{1}{Ls}$$





Transform Impedance and Admittance (6)

Capacitance:

$$i_C(t) = C\frac{dv_C(t)}{dt}, \quad v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau$$

Or,

$$I_C(s) = C\left(sV_C(s) - v_C(0-)\right), \quad V_C(s) = \frac{1}{s}\left(\frac{1}{C}I_C(s) + v_C(0-)\right)$$

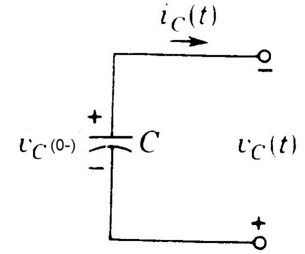
That is,

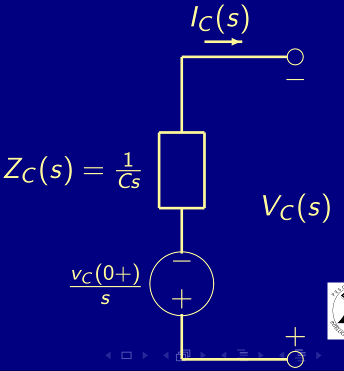
$$Y_C(s) \triangleq \frac{I_1(s)}{V_C(s)} = Cs, \quad Z_C(s) \triangleq \frac{V_1(s)}{I_C(s)} = \frac{1}{Cs}$$

where $I_1(s) \triangleq I_C(s) + Cv_C(0-)$ and $V_1(s) = V_C(s) - \frac{v_C(0-)}{s}$.

Transform Impedance and Admittance (7)

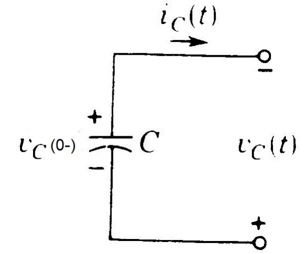
$$Y_C(s) = \frac{I_C(s) + Cv_C(0-)}{V_C(s)} = Cs, \quad Z_C(s) = \frac{V_C(s) - \frac{v_C(0-)}{s}}{I_C(s)} = \frac{1}{Cs}$$

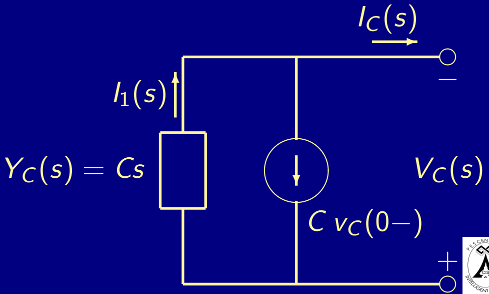




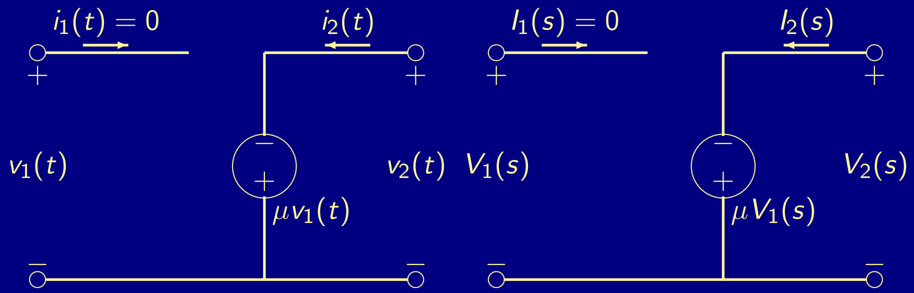
Transform Impedance and Admittance (8)

$$Y_C(s) = \frac{I_C(s) + C v_C(0-)}{V_C(s)} = Cs, \quad Z_C(s) = \frac{V_C(s) - \frac{v_C(0-)}{s}}{I_C(s)} = \frac{1}{Cs}$$

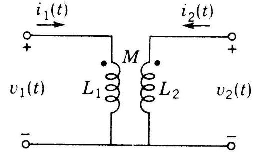


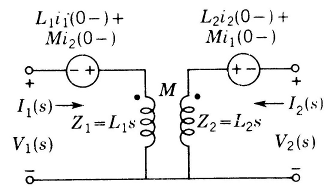


Transform Impedance and Admittance (9)



Transform Impedance and Admittance (10)





$$
\begin{aligned}
v_1(t) &= L_1 \frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt} \\
v_2(t) &= M \frac{di_1(t)}{dt} + L_2\frac{di_2(t)}{dt}
\end{aligned}
$$

$$
\begin{aligned}
V_1(s) &= L_1 sI_1(s) - L_1 i_1(0-) + MsI_2(s) - Mi_2(0-) \\
V_1(s) &= MsI_1(s) - Mi_1(0-) + L_2 sI_2(s) - L_2 i_2(0-)
\end{aligned}
$$

Transform Impedance and Admittance (11)

Remarks:

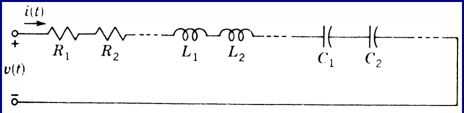
- For single elements, under zero initial conditions, the transform impedance is the ratio of the transform of the element voltage to the transform of the element current:

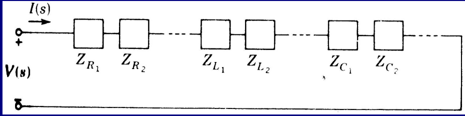
$$Z(s) = \frac{\mathcal{L}[v(t)]}{\mathcal{L}[i(t)]} = \frac{V(s)}{I(s)}$$

- Under zero initial conditions, the reciprocal ratio is the transform admittance.
- Initial conditions are represented by transform voltage sources or current sources.

Networks of Elements (1)

Series Combination (with zero initial conditions):





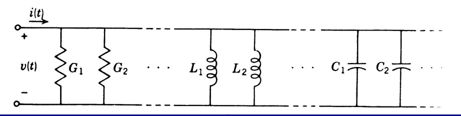
Applying KVL,

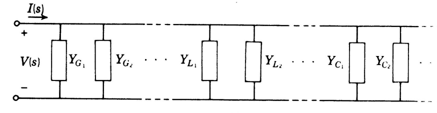
$$\begin{aligned} V(s) &= V_{R_1}(s) + \cdots + V_{L_1}(s) + \cdots + V_{C_1}(s) + \cdots + \\ \Longrightarrow Z(s) &= Z_{R_1}(s) + \cdots + Z_{L_1}(s) + \cdots + Z_{C_1}(s) + \cdots + \\ &= \sum_{k=1}^{n} Z_k(s) \end{aligned}$$

- Add all the impedances!

Networks of Elements (2)

Parallel Combination (with zero initial conditions):





Applying KCL,

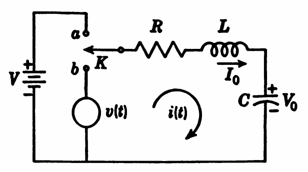
$$\begin{aligned} I(s) &= I_{G_1}(s) + \cdots + I_{L_1}(s) + \cdots + I_{C_1}(s) + \cdots + \\ \implies Y(s) &= Y_{G_1}(s) + \cdots + Y_{L_1}(s) + \cdots + Y_{C_1}(s) + \cdots + \\ &= \sum_{k=1}^{n} Y_k(s) \end{aligned}$$

- Add all the admittances!

Networks of Elements (3)

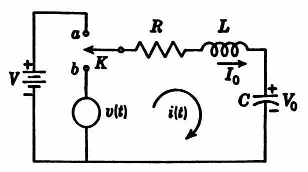
- An immittance is an impedance or an admittance.
- A network function relates currents or voltages at different parts of the network.
- Network functions are therefore an immittance or voltage gain or current gain.
- A transfer function is a network function under zero initial conditions.

Examples (1)



Source: Van Valkenburg, 1975.

Examples (1)



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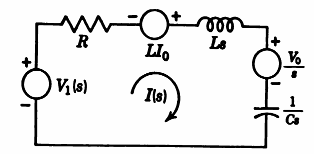
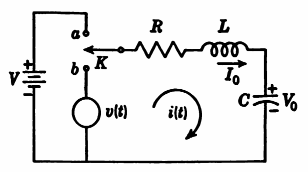


Examples (1)



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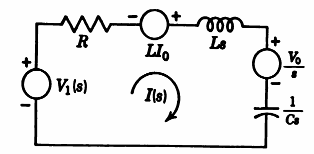
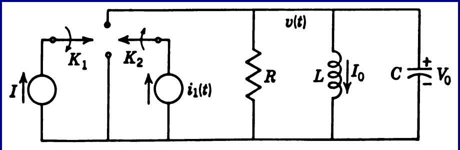


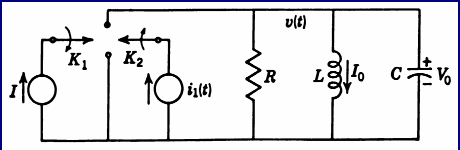
$$I(s) = \frac{V(s)}{Z(s)} = \frac{sV_1(s) + LI_0(s) - V_0}{Ls^2 + Rs + 1/C}$$

Examples (2)



Source: Van Valkenburg, 1975.

Examples (2)



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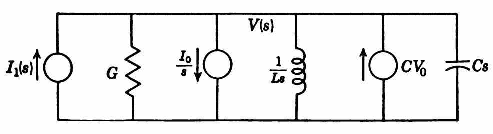
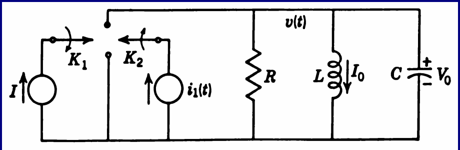


Examples (2)



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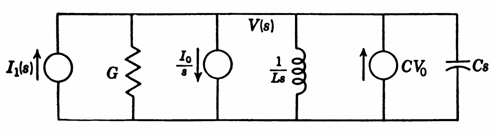
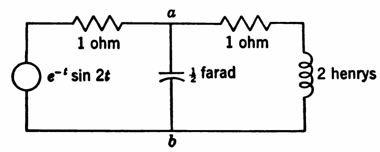
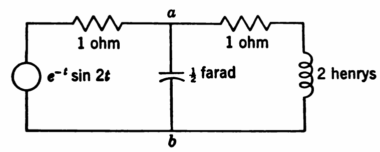


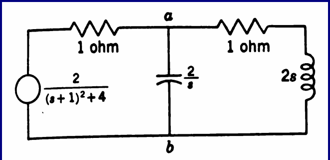
$$V(s) = \frac{I(s)}{Y(s)} = \frac{sI_1(s) + CV_0(s) - I_0}{Cs^2 + Gs + 1/L}$$

Examples (3)



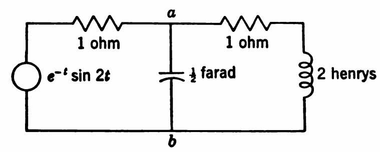
Examples (3)

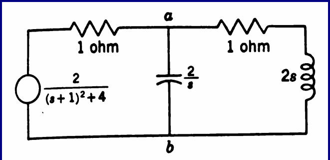




Source: Van Valkenburg, 1975.

Examples (3)





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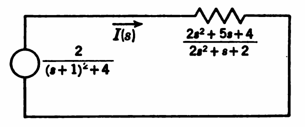
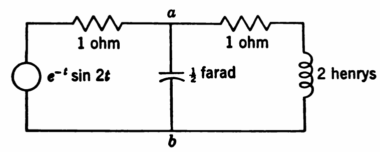
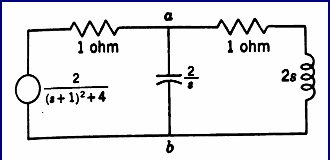


Examples (3)





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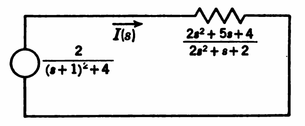
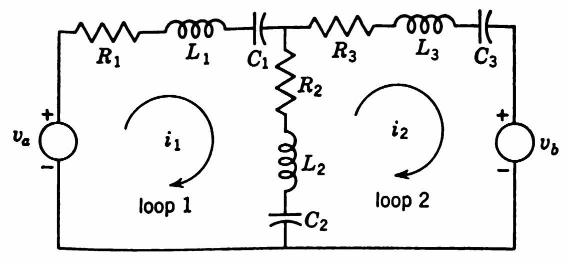


$$I(s) = \frac{V(s)}{Z(s)} = \frac{2(2s^2+s+2)}{((s+1)^2+4)(2s^2+5s+4)}$$

Examples (4)



Source: Van Valkenburg, 1975.

Examples (5)

$$
\begin{aligned}
v_a &= \left(\underbrace{(R_1 + R_2) + s(L_1 + L_2) + \frac{1}{s}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)}_{a_{11}}\right) I_1(s) \\
&\quad \left(\underbrace{-R_2 - L_2 s - \frac{1}{C_2 s}}_{a_{12}}\right) I_2(s)
\end{aligned}
$$

Examples (6)

$$-v_b = \left(\underbrace{-R_2 - L_2 s - \frac{1}{C_2 s}}_{a_{21}}\right) I_1(s)$$

$$\left(\underbrace{(R_2 + R_3) + s(L_2 + L_3) + \frac{1}{s}\left(\frac{1}{C_2} + \frac{1}{C_3}\right)}_{a_{22}}\right) I_2(s)$$

Thus,

$$\begin{aligned} a_{11}(s)I_1(s) + a_{12}(s)I_2(s) &= v_1(s) \\ a_{21}(s)I_1(s) + a_{22}(s)I_2(s) &= v_2(s) \end{aligned}$$

Network Theorems (1)

More generally,

$$\sum_{j=1}^{L} a_{kj}(s) I_j(s) = V_k(s), \quad k = 1, 2, \ldots, L$$

$$\sum_{j=1}^{N} b_{kj}(s) V_j(s) = I_k(s), \quad k = 1, 2, \ldots, N$$

- L is the number of loops and N is the number of nodes.
- The coefficients a_{kj} are network impedances plus initial-condition sources.
- The coefficients b_{kj} are network admittances plus initial-condition sources.

Network Theorems (2)

More compactly,

$$[Z][I] = [V] + [V_0] = [V']$$

where

$$[Z] = \begin{pmatrix} Z_{11}(s) & Z_{12}(s) & \cdots & Z_{1L}(s) \\ Z_{21}(s) & Z_{22}(s) & \cdots & Z_{2L}(s) \\ \vdots & \vdots & \ddots & \vdots \\ Z_{L1}(s) & Z_{L2}(s) & \cdots & Z_{LL}(s) \end{pmatrix}, \quad [V] = \begin{pmatrix} V_1(s) \\ V_2(s) \\ \vdots \\ V_L(s) \end{pmatrix},$$

$$[I] = \begin{pmatrix} I_1(s) & I_2(s) & \cdots & I_L(s) \end{pmatrix}^T$$

Network Theorems (3)

Also,

$$[Y][V] = [I] + [I_0] = [I']$$

where

$$[Y] = \begin{pmatrix} Y_{11}(s) & Y_{12}(s) & \cdots & Y_{1N}(s) \\ Y_{21}(s) & Y_{22}(s) & \cdots & Y_{2N}(s) \\ \vdots & \vdots & \ddots & \vdots \\ Y_{N1}(s) & Y_{N2}(s) & \cdots & Y_{NN}(s) \end{pmatrix}, \quad [V] = \begin{pmatrix} V_1(s) \\ V_2(s) \\ \vdots \\ V_N(s) \end{pmatrix},$$

$$[I] = \begin{pmatrix} I_1(s) & I_2(s) & \cdots & I_N(s) \end{pmatrix}^T$$

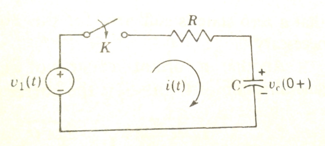
Network Theorems — Superposition (4)

$$[Z][I] = [V] + [V_0] = [V'], \quad [Y][V] = [I] + [I_0] = [I']$$

- Each current (or voltage) is obtained by determining the individual responses to a voltage (or current) and then summing up the responses.
- This is the superposition principle or principle of linearity.
- Recall: Linearity is
  1. additivity plus
  2. homogeneity.

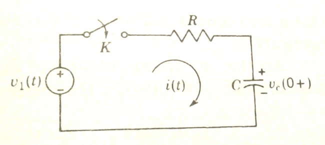
Examples (1)

Example 4, Van Valkenburg, 1975

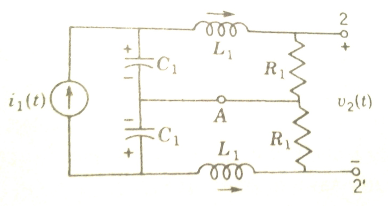


Examples (1)

Example 4, Van Valkenburg, 1975



Example 5, Van Valkenburg, 1975



Network Theorems (5)

Suppose that $[Z]$ is symmetric. For example,

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12} & Z_{22} & Z_{23} \\ Z_{13} & Z_{23} & Z_{33} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

Therefore, for example,

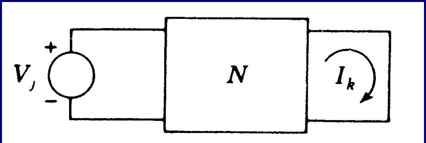
$$\begin{aligned} I_1 &= \frac{\Delta_{11}}{\Delta} V_1 + \frac{\Delta_{12}}{\Delta} V_2 + \frac{\Delta_{13}}{\Delta} V_3 \\ I_3 &= \frac{\Delta_{13}}{\Delta} V_1 + \frac{\Delta_{23}}{\Delta} V_2 + \frac{\Delta_{33}}{\Delta} V_3 \end{aligned}$$

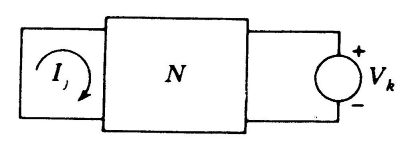
- If $V_1 = V_2 = 0$ for the first loop, $I_1 = \frac{\Delta_{13}}{\Delta} V_3$.
- If $V_2 = V_3 = 0$ for the third loop, $I_3 = \frac{\Delta_{13}}{\Delta} V_1$.
- Therefore, $\frac{I_1}{V_3} = \frac{I_3}{V_1}$.

Network Theorems — Reciprocity (6)

More generally, for a network with L loops,

$$\frac{I_k}{V_j} = \frac{I_j}{V_k}$$



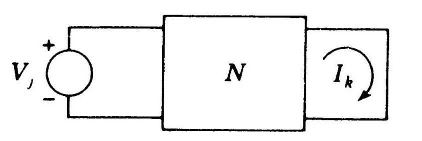


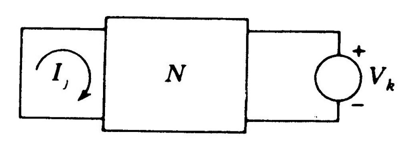
- Suppose that V_j is the only source in the network. This results in the current I_k in the kth loop.
- If this source is now moved to the loop k, then the current in the jth loop is $I_j = I_k$.

Network Theorems — Reciprocity (7)

More generally, for a network with L loops,

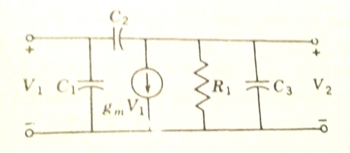
$$\frac{I_k}{V_j} = \frac{I_j}{V_k}$$



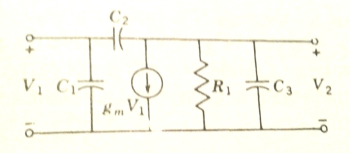


- This is the principle of reciprocity.
- The ratio of the response transform to the excitation transform is invariant to an interchange of the position in the network of the excitation and the response.
- When this property holds, we say that the network is reciprocal; otherwise the network is non-reciprocal.

Network Theorems — Reciprocity (8)



Network Theorems — Reciprocity (8)



- For a network to be reciprocal, the impedance matrix $[Z]$ must be symmetric.
- Therefore, the network must be in zero-state (initially relaxed).
- There should not be any dependent sources.
- There should be only R, L, C and transformers as elements.