Fourier Series and Transform

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- **<u>Introduction</u>**:
- If we represent a signal as a weighted superposition of complex sinusoids.
- Then when such a signal is applied to an LTI system, then the system output is a weighted superposition of the system response to each complex sinusoid.
- The study of signals and systems using sinusoidal representations is termed **Fourier analysis.**
- **<u>Complex sinusoids and frequency response of LTI systems</u>**
- The response of an LTI system to a sinusoidal input leads to a characterization of system behavior that is termed the frequency response of the system.
- Consider the output of a discrete-time LTI system with impulse response h[n] and unit amplitude complex sinusoidal input $x[n] = e^{j\Omega n}$. This output is given by

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{j\Omega(n-k)}.
\end{aligned}
$$

- We factor $e^{j\Omega n}$ from the sum to obtain
$$y[n] = e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k} = H(e^{j\Omega})e^{j\Omega n},$$

- where we have defined
$$\boxed{H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}.}$$

- Hence, the output of the system is a complex sinusoid of the same frequency as the input , multiplied by the complex number $H(e^{j\Omega})$
- The complex scaling factor $H(e^{j\Omega})$ is not a function of time n, but is only a function of frequency Ω and is termed the frequency response of the discrete-time system.

- **<u>Continuous time LTI systems</u>**
- Let the impulse response of such a system be h(t) and the input be x(t) = $e^{j\omega t}$.
- Then the convolution integral gives the output as

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)}\, d\tau \\
&= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau \\
&= H(j\omega) e^{j\omega t},
\end{aligned}
$$

- where we define $$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau.}$$

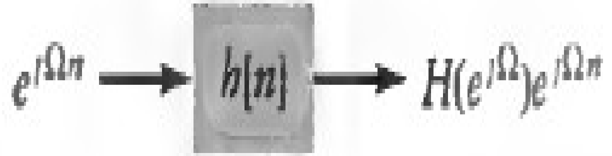


FIGURE 3.1 The output of a complex sinusoidal input to an LTI system is a complex sinusoid of the same frequency as the input, multiplied by the frequency response of the system.

- We say that the complex sinusoid $\psi(t) = e^{j\omega t}$ is an eigenfunction of the LTI system H associated with the eigenvalue $\lambda = H(j\omega)$, because satisfies an eigenvalue problem described by $H\{\psi(t)\} = \lambda\psi(t)$.

- The effect of the system on an eigenfunction input signal is scalar multiplication, the output is given by the product of the input and a complex number.

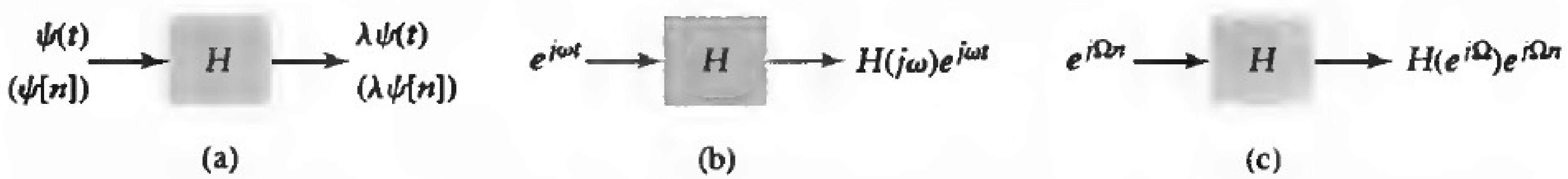


FIGURE 3.4 Illustration of the eigenfunction property of linear systems. The action of the system on an eigenfunction input is multiplication by the corresponding eigenvalue. (a) General eigenfunction $\psi(t)$ or $\psi[n]$ and eigenvalue λ. (b) Complex sinusoidal eigenfunction $e^{j\omega t}$ and eigenvalue $H(j\omega)$. (c) Complex sinusoidal eigenfunction $e^{j\Omega n}$ and eigenvalue $H(e^{j\Omega})$.

- consider expressing the input to an LTI system as the weighted sum of M complex sinusoids

$$x(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t}.$$

- The output is a weighted sum of M complex sinusoids, with the input weights a_k modified by the system frequency response $H(j\omega_k)$
- The operation of convolution $h(t) * x(t)$, becomes multiplication, $a_k H(j\omega_k)$ because x(t) is expressed as a sum of eigenfunctions.

If $e^{j\omega_k t}$ is an eigenfunction of the system with eigenvalue $H(j\omega_k)$, then each term in the input, $a_k e^{j\omega_k t}$, produces an output term $a_k H(j\omega_k)e^{j\omega_k t}$. Hence, we express the output of the system as

$$y(t) = \sum_{k=1}^{M} a_k H(j\omega_k) e^{j\omega_k t}.$$

TABLE 3.1 ***Relationship between Time Properties of a Signal and the Appropriate Fourier Representation.***

Time Property	*Periodic*	*Nonperiodic*
Continuous (t)	Fourier Series (FS)	Fourier Transform (FT)
Discrete $[n]$	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)

- If x[n] is a discrete-time signal with fundamental period N, then we seek to represent x[n] by the DTFS $\hat{x}[n] = \sum_k A[k] e^{jk\Omega_o n}$,

- where $\Omega_o = 2\pi/N$ is the fundamental frequency of x[n]. The frequency of the kth sinusoid in the superposition is $k\Omega_o$. Each of these sinusoids has a common period N. Similarly, if x(t) is a continuous-time signal of fundamental period T, we represent x(t) by the FS

$$\hat{x}(t) = \sum_k A[k] e^{jk\omega_o t},$$

- Where $\omega_0 = 2\pi/T$ is the fundamental frequency of x(t). Here, the frequency of the kth sinusoid is $k\omega_o$, and each sinusoid has a common period T.

- A sinusoid whose frequency is an integer multiple of a fundamental frequency is said to be a harmonic of the sinusoid at the fundamental frequency. Thus, $e^{jk\omega_o t}$ is the kth harmonic of $e^{j\omega_o t}$.
- The variable k indexes the frequency of the sinusoids, so we say that A[k] is a function of frequency.
- the complex sinusoids $e^{jk\Omega_o n}$ n are N-periodic in the frequency index k, as shown by the relationship

$$\begin{aligned} e^{j(N+k)\Omega_o n} &= e^{jN\Omega_o n} e^{jk\Omega_o n} \\ &= e^{j2\pi n} e^{jk\Omega_o n} \\ &= e^{jk\Omega_o n}. \end{aligned}$$

Hence, there are only N distinct complex sinusoids of the form $e^{jk\Omega_o n}$. A unique set of N distinct complex sinusoids is obtained by letting the frequency index k vary from $k = 0$ to $k = N - 1$. Accordingly, we may rewrite Eq. (3.4) as

$$\hat{x}[n] = \sum_{k=0}^{N-1} A[k] e^{jk\Omega_o n}. \tag{3.6}$$

- In contrast to the discrete-time case, continuous-time complex sinusoids $e^{jk\omega_0 t}$, with distinct frequencies $k\omega_0$ are always distinct. Hence, there are potentially an infinite number of distinct terms in the series $\hat{x}(t) = \sum_{k=-\infty}^{\infty} A[k]e^{jk\omega_0 t}$.

- Nonperiodic Signals: Fourier-Transform Representations
- In contrast to the case of the periodic signal, there are no restrictions on the period of the sinusoids used to represent nonperiodic signals.
- CT – $\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega.$

- Discrete-time sinusoids are unique only over a 2π interval of frequency, since discrete-time sinusoids with frequencies separated by an integer multiple of 2π are identical.

- DTFT involves sinusoidal frequencies within a 2π interval, as shown by the relationship $$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\, d\omega.$$

▶ **Problem 3.1** Identify the appropriate Fourier representation for each of the following signals:

(a) $x[n] = (1/2)^n u[n]$

(b) $x(t) = 1 - \cos(2\pi t) + \sin(3\pi t)$

(c) $x(t) = e^{-t}\cos(2\pi t)u(t)$

(d) $x[n] = \sum_{m=-\infty}^{\infty} \delta[n - 20m] - 2\delta[n - 2 - 20m]$

Answers:

(a) DTFT

(b) FS

(c) FT

(d) DTFS

- **<u>Discrete-Time Periodic Signals:The Discrete-Time Fourier Series</u>**
- The DTFS representation of a periodic signal x[n] with fundamental period N and fundamental frequency $\Omega_o = 2\pi/N$ is given by

$$\boxed{x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n},}$$

where $$\boxed{X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}}$$ are the DTFS coefficients of the signal x[n]

- We say that x[n] and X[k] are a DTFS pair and denote this relationship as

$$x[n] \xleftrightarrow{DTFS;\Omega_o} X[k].$$

- The DTFS coefficients X[k] are termed a frequency-domain representation for x[n], because each coefficient is associated with a complex sinusoid of a different frequency.
- The variable k determines the frequency of the sinusoid associated with X[k], so we say that X[k] is a function of frequency.

EXAMPLE 3.2 DETERMINING DTFS COEFFICIENTS Find the frequency-domain representation of the signal depicted in Fig. 3.5

Solution: The signal has period $N = 5$, so $\Omega_o = 2\pi/5$. Also, the signal has odd symmetry, so we sum over $n = -2$ to $n = 2$ in Eq. (3.11) to obtain

$$\begin{aligned} X[k] &= \frac{1}{5}\sum_{n=-2}^{2} x[n]e^{-jk2\pi n/5} \\ &= \frac{1}{5}\{x[-2]e^{jk4\pi/5} + x[-1]e^{jk2\pi/5} + x[0]e^{j0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5}\}. \end{aligned}$$

Using the values of $x[n]$, we get

$$\begin{aligned} X[k] &= \frac{1}{5}\left\{1 + \frac{1}{2}e^{jk2\pi/5} - \frac{1}{2}e^{-jk2\pi/5}\right\} \\ &= \frac{1}{5}\{1 + j\sin(k2\pi/5)\}. \end{aligned} \tag{3.12}$$

From this equation, we identify one period of the DTFS coefficients $X[k]$, $k = -2$ to $k = 2$, in rectangular and polar coordinates as

$$X[-2] = \frac{1}{5} - j\frac{\sin(4\pi/5)}{5} = 0.232e^{-j0.531}$$

$$X[-1] = \frac{1}{5} - j\frac{\sin(2\pi/5)}{5} = 0.276e^{-j0.760}$$

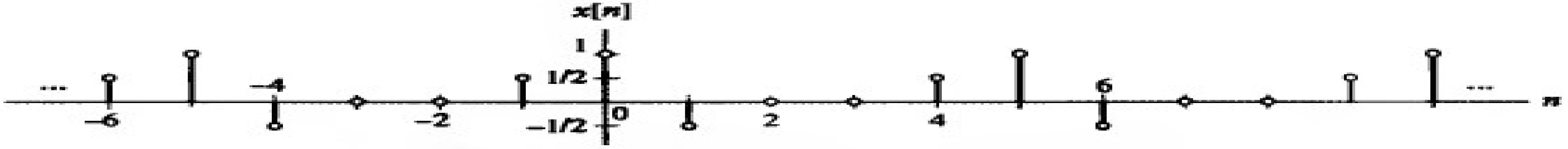


FIGURE 3.5 Time-domain signal for Example 3.2.

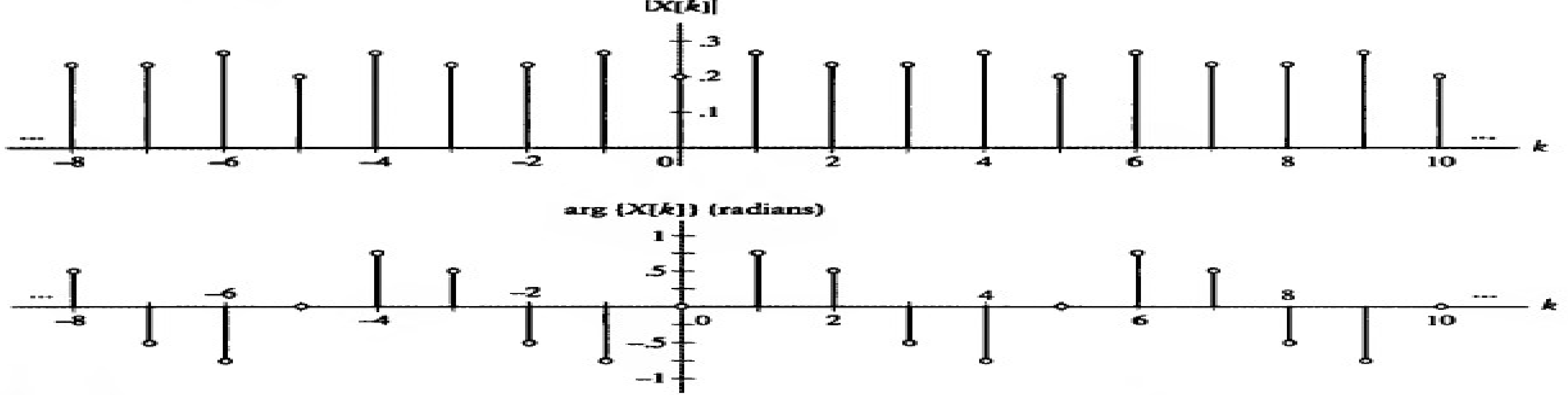


FIGURE 3.6 Magnitude and phase of the DTFS coefficients for the signal in Fig. 3.5.

$$X[0] = \frac{1}{5} = 0.2e^{j0}$$
$$X[1] = \frac{1}{5} + j\frac{\sin(2\pi/5)}{5} = 0.276e^{j0.760}$$
$$X[2] = \frac{1}{5} + j\frac{\sin(4\pi/5)}{5} = 0.232e^{j0.531}.$$

Figure 3.6 depicts the magnitude and phase of $X[k]$ as functions of the frequency index k.

Now suppose we calculate $X[k]$ using $n = 0$ to $n = 4$ for the limits on the sum in Eq. (3.11), to obtain

$$\begin{aligned} X[k] &= \frac{1}{5}\{x[0]e^{j0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5} + x[3]e^{-jk6\pi/5} + x[4]e^{-jk8\pi/5}\} \\ &= \frac{1}{5}\left\{1 - \frac{1}{2}e^{-jk2\pi/5} + \frac{1}{2}e^{-jk8\pi/5}\right\}. \end{aligned}$$

This expression appears to differ from Eq. (3.12), which was obtained using $n = -2$ to $n = 2$. However, noting that

$$\begin{aligned} e^{-jk8\pi/5} &= e^{-jk2\pi}e^{jk2\pi/5} \\ &= e^{jk2\pi/5}, \end{aligned}$$

we see that both intervals, $n = -2$ to $n = 2$ and $n = 0$ to $n = 4$, yield equivalent expressions for the DTFS coefficients. ■

- The magnitude of $X[k]$, denoted $|X[k]|$ and plotted against the frequency index k, is known as the **magnitude spectrum** of $x[n]$. Similarly, the phase of $X[k]$, termed $\arg\{X[k]\}$, is known as the **phase spectrum** of $x[n]$. Note that in the previous example $|X[k]|$ is even while $\arg\{X[k]\}$ is odd.

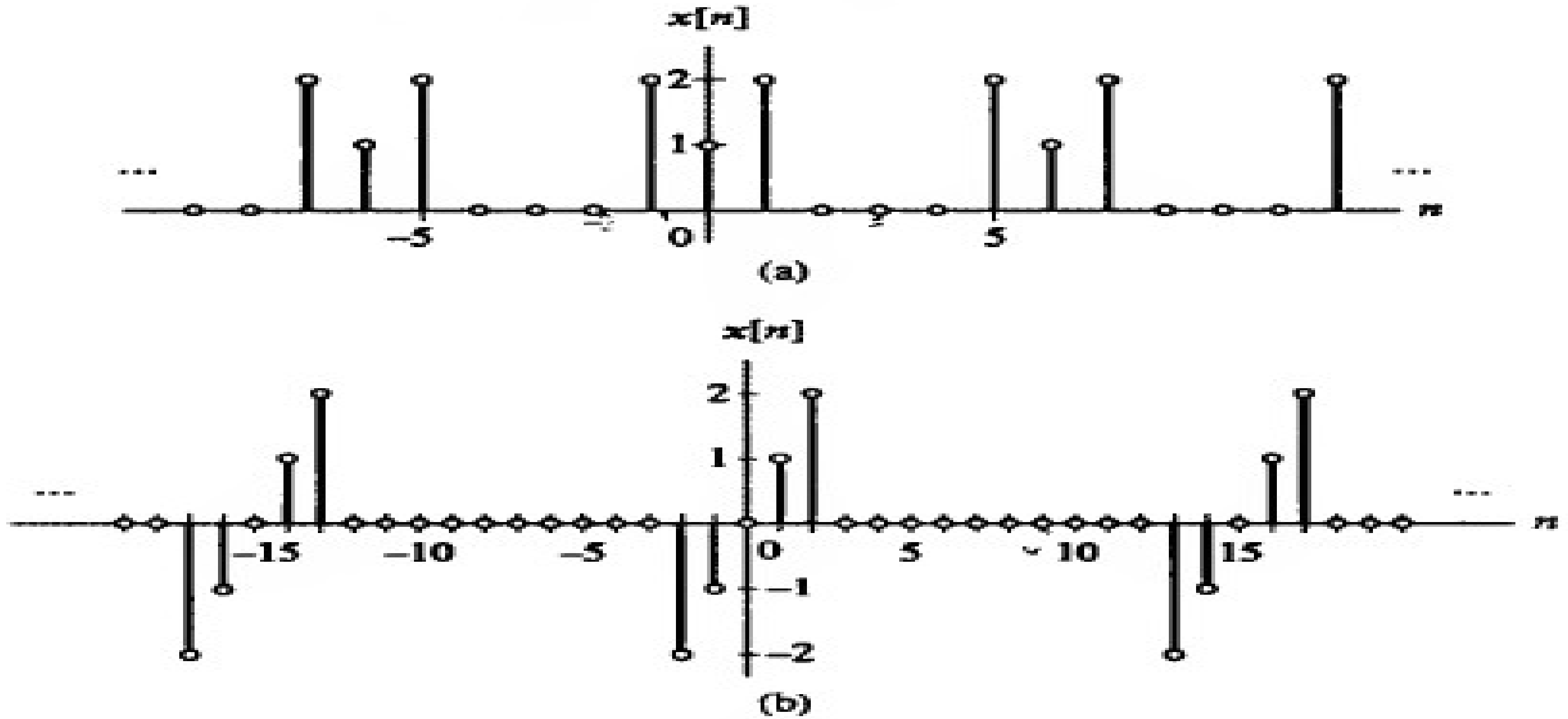


FIGURE 3.7 Signals $x[n]$ for Problem 3.2.

► **Problem 3.2** Determine the DTFS coefficients of the periodic signals depicted in Figs. 3.7(a) and (b).

Answers:

Fig. 3.7(a):

$$x[n] \xleftrightarrow{DTFS;\,\pi/3} X[k] = \frac{1}{6} + \frac{2}{3}\cos(k\pi/3)$$

Fig. 3.7(b):

$$x[n] \xleftrightarrow{DTFS;\,2\pi/15} X[k] = \frac{-2j}{15}(\sin(k2\pi/15) + 2\sin(k4\pi/15))$$ ◄

EXAMPLE 3.3 COMPUTATION OF DTFS COEFFICIENTS BY INSPECTION Determine the DTFS coefficients of $x[n] = \cos(\pi n/3 + \phi)$, using the method of inspection.

Solution: The period of $x[n]$ is $N = 6$. We expand the cosine by using Euler's formula and move any phase shifts in front of the complex sinusoids. The result is

$$\begin{aligned} x[n] &= \frac{e^{j\left(\frac{\pi}{3}n+\phi\right)} + e^{-j\left(\frac{\pi}{3}n+\phi\right)}}{2} \\ &= \frac{1}{2}e^{-j\phi}e^{-j\frac{\pi}{3}n} + \frac{1}{2}e^{j\phi}e^{j\frac{\pi}{3}n}. \end{aligned} \tag{3.13}$$

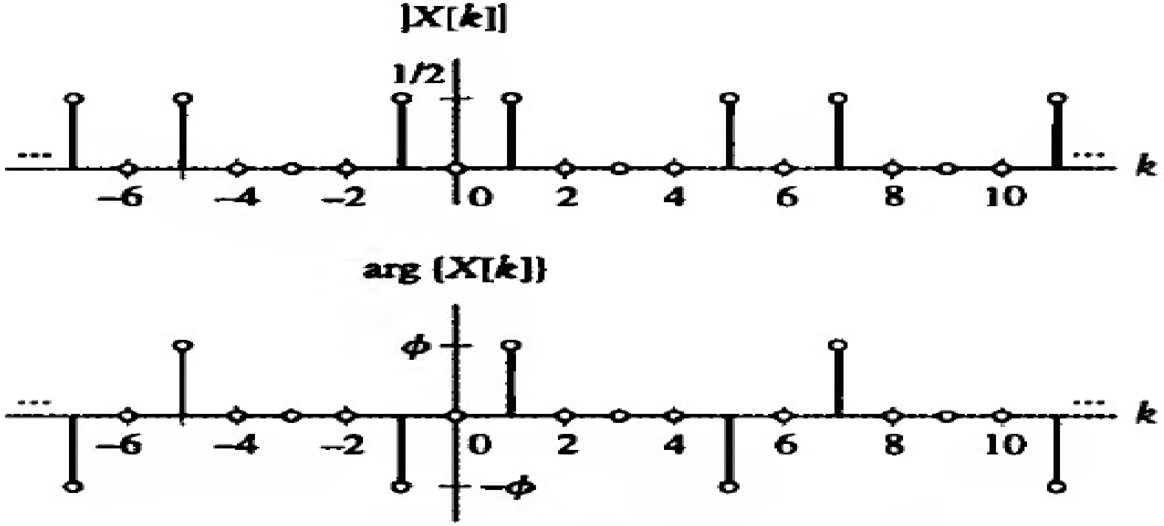


FIGURE 3.8 Magnitude and phase of DTFS coefficients for Example 3.3.

Now we compare Eq. (3.13) with the DTFS of Eq. (3.10) with $\Omega_o = 2\pi/6 = \pi/3$, written by summing from $k = -2$ to $k = 3$:

$$\begin{aligned} x[n] &= \sum_{k=-2}^{3} X[k]e^{jk\pi n/3} \\ &= X[-2]e^{-j2\pi n/3} + X[-1]e^{-j\pi n/3} + X[0] + X[1]e^{j\pi n/3} + X[2]e^{j2\pi n/3} + X[3]e^{j\pi n}. \end{aligned} \tag{3.14}$$

Equating terms in Eq. (3.13) with those in Eq. (3.14) having equal frequencies, $k\pi/3$, gives

$$x[n] \xleftrightarrow{DTFS;\frac{\pi}{3}} X[k] = \begin{cases} e^{-j\phi}/2, & k = -1 \\ e^{j\phi}/2, & k = 1 \\ 0, & \text{otherwise on } -2 \le k \le 3 \end{cases}.$$

The magnitude spectrum, $|X[k]|$, and phase spectrum, $\arg\{X[k]\}$, are depicted in Fig. 3.8. ■

▶ **Problem 3.3** Use the method of inspection to determine the DTFS coefficients for the following signals:

(a) $x[n] = 1 + \sin(n\pi/12 + 3\pi/8)$

(b) $x[n] = \cos(n\pi/30) + 2\sin(n\pi/90)$

Answers:

(a) $$x[n] \xleftrightarrow{DTFS;2\pi/24} X[k] = \begin{cases} -e^{-j3\pi/8}/(2j), & k = -1 \\ 1, & k = 0 \\ e^{j3\pi/8}/(2j), & k = 1 \\ 0, & \text{otherwise on } -11 \le k \le 12 \end{cases}$$

(b) $$x[n] \xleftrightarrow{DTFS;2\pi/180} X[k] = \begin{cases} -1/j, & k = -1 \\ 1/j, & k = 1 \\ 1/2, & k = \pm 3 \\ 0, & \text{otherwise on } -89 \le k \le 90 \end{cases}$$

◀

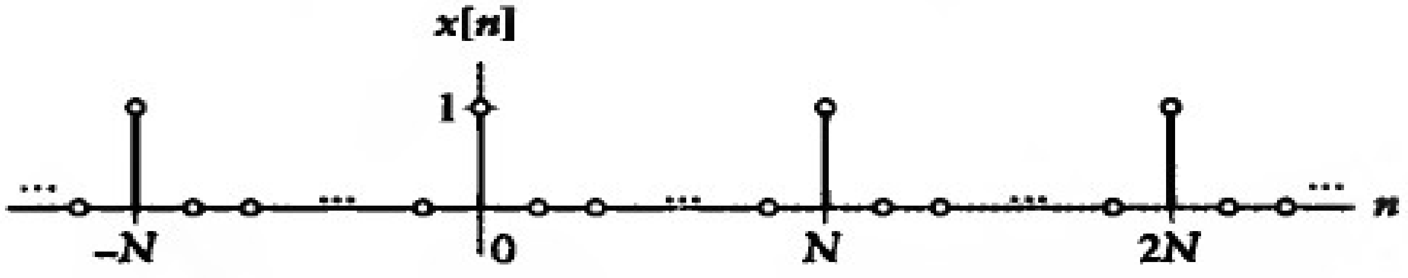


FIGURE 3.9 A discrete-time impulse train with period N.

EXAMPLE 3.4 DTFS REPRESENTATION OF AN IMPULSE TRAIN Find the DTFS coefficients of the N-periodic impulse train

$$x[n] = \sum_{l=-\infty}^{\infty} \delta[n - lN],$$

as shown in Fig. 3.9.

Solution: Since there is only one nonzero value in $x[n]$ per period, it is convenient to evaluate Eq. (3.11) over the interval $n = 0$ to $n = N - 1$ to obtain

$$\begin{aligned} X[k] &= \frac{1}{N}\sum_{n=0}^{N-1} \delta[n]e^{-jkn2\pi/N} \\ &= \frac{1}{N}. \end{aligned}$$

■

EXAMPLE 3.5 THE INVERSE DTFS Use Eq. (3.10) to determine the time-domain signal $x[n]$ from the DTFS coefficients depicted in Fig. 3.10.

Solution: The DTFS coefficients have period 9, so $\Omega_o = 2\pi/9$. It is convenient to evaluate Eq. (3.10) over the interval $k = -4$ to $k = 4$ to obtain

$$\begin{aligned} x[n] &= \sum_{k=-4}^{4} X[k]e^{jk2\pi n/9} \\ &= e^{j2\pi/3}e^{-j6\pi n/9} + 2e^{j\pi/3}e^{-j4\pi n/9} - 1 + 2e^{-j\pi/3}e^{j4\pi n/9} + e^{-j2\pi/3}e^{j6\pi n/9} \\ &= 2\cos(6\pi n/9 - 2\pi/3) + 4\cos(4\pi n/9 - \pi/3) - 1. \end{aligned}$$ ■

$$\boxed{x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n},} \qquad (3.10)$$

► **Problem 3.4** One period of the DTFS coefficients of a signal is given by

$$X[k] = (1/2)^k, \quad \text{on } 0 \leq k \leq 9.$$

Find the time-domain signal $x[n]$ assuming $N = 10$.

Answer:

$$x[n] = \frac{1 - (1/2)^{10}}{1 - (1/2)e^{j(\pi/5)n}}$$

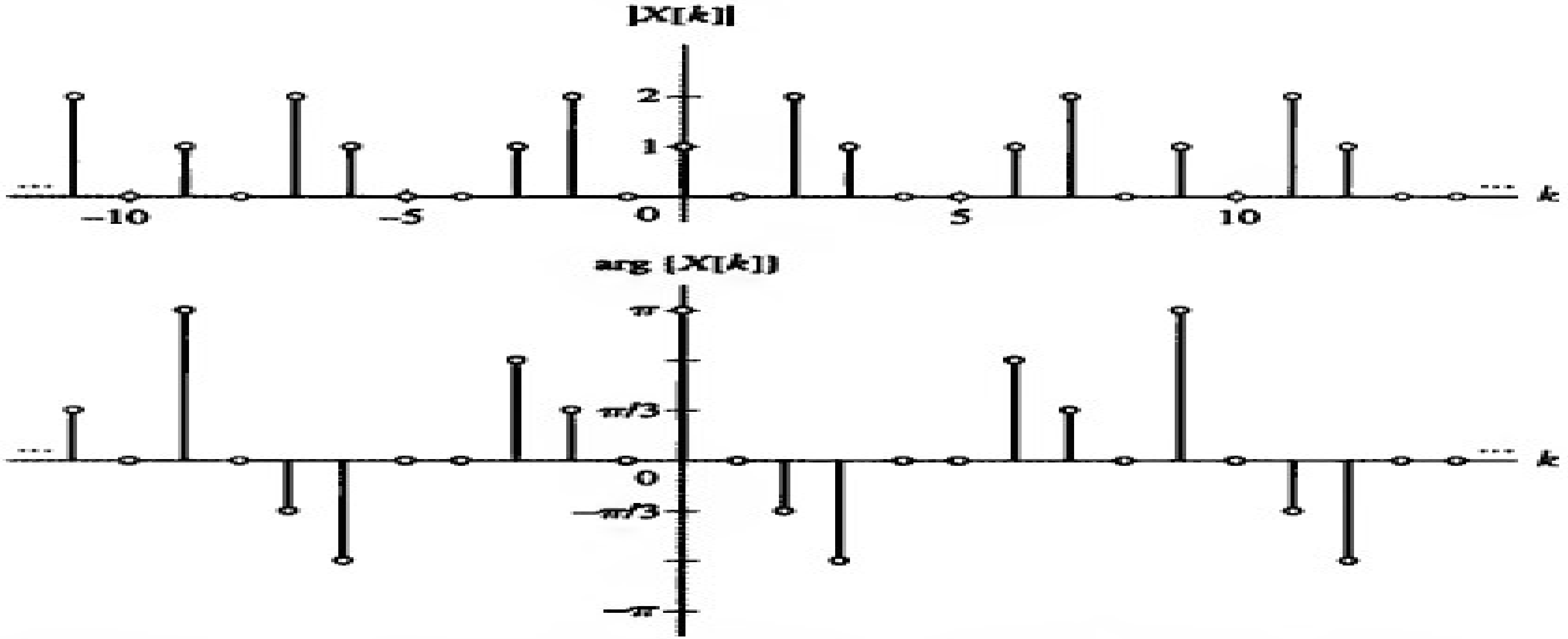


FIGURE 3.10 Magnitude and phase of DTFS coefficients for Example 3.5.

▶ **Problem 3.5** Use the method of inspection to find the time-domain signal corresponding to the DTFS coefficients

$$X[k] = \cos(k4\pi/11) + 2j\sin(k6\pi/11).$$

Answer:

$$x[n] = \begin{cases} 1/2, & n = \pm 2 \\ 1, & n = 3 \\ -1, & n = -3 \\ 0, & \text{otherwise on } -5 \le n \le 5 \end{cases}$$ ◀

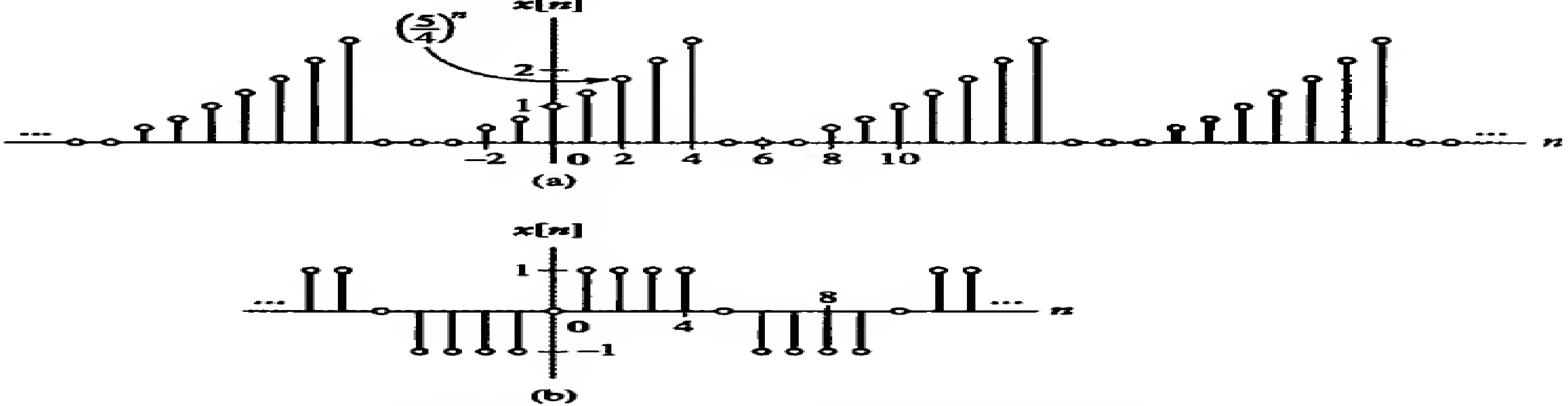


FIGURE 3.13 Signals $x[n]$ for Problem 3.6.

► **Problem 3.6** Find the DTFS coefficients of the signals depicted in Figs. 3.13(a) and (b).

Answers:

(a)

$$X[k] = \frac{8}{125} e^{jk2\pi/5} \frac{1 - \left(\frac{5}{4} e^{-jk\pi/5}\right)^7}{1 - \frac{5}{4} e^{-jk\pi/5}}$$

(b)

$$X[k] = -\frac{j}{5} \sin(k\pi/2) \frac{\sin(k2\pi/5)}{\sin(k\pi/10)}$$ ◄

- **Continuous-time periodic fourier series**
- Continuous-time periodic signals are represented by the Fourier series (FS). We may write the FS of a signal x(t) with fundamental period T and fundamental frequency $\omega_o = 2\pi/T$ as,

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t},}$$

Where $$\boxed{X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt}$$

- are the FS coefficients of the signal x(t). We say that x(t) and X[k] are an FS pair and denote this relationship as
- $$x(t) \xleftrightarrow{FS;\omega_o} X[k].$$

- Convergence of CTFS is guaranteed at all values of t except those coressponding to discontinuities if the Dirichlet conditions are satisfied.
- x(t) is bounded.
- x(t) has a finite number of maxima and minima in one period.
- x(t) has a finite number of discontinuities in one period.

EXAMPLE 3.9 DIRECT CALCULATION OF FS COEFFICIENTS Determine the FS coefficients for the signal $x(t)$ depicted in Fig. 3.16.

Solution: The period of $x(t)$ is $T = 2$, so $\omega_o = 2\pi/2 = \pi$. On the interval $0 \le t \le 2$, one period of $x(t)$ is expressed as $x(t) = e^{-2t}$, so Eq. (3.20) yields

$$X[k] = \frac{1}{2}\int_0^2 e^{-2t}e^{-jk\pi t}\,dt$$
$$= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\,dt.$$

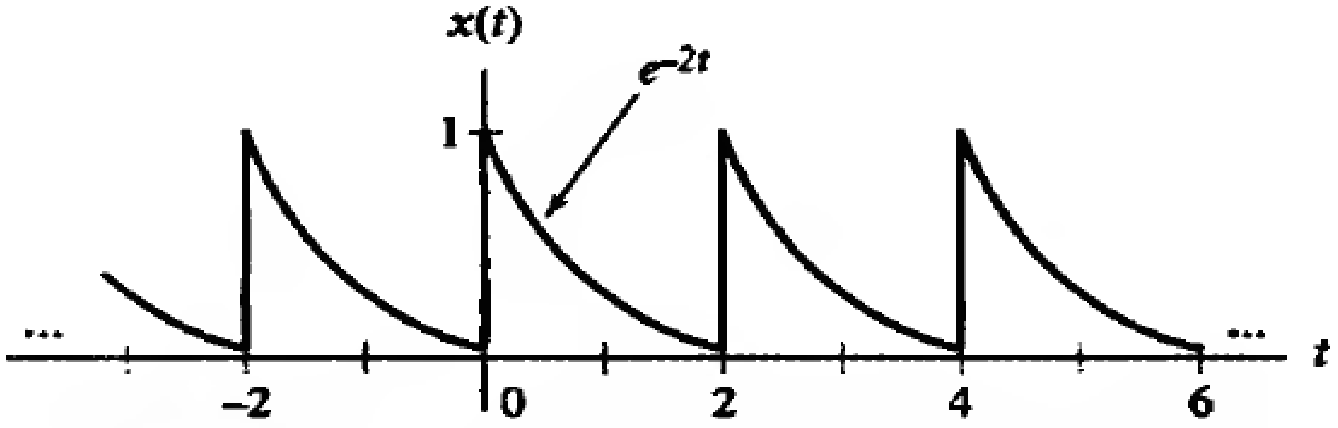


FIGURE 3.16 Time-domain signal for Example 3.9.

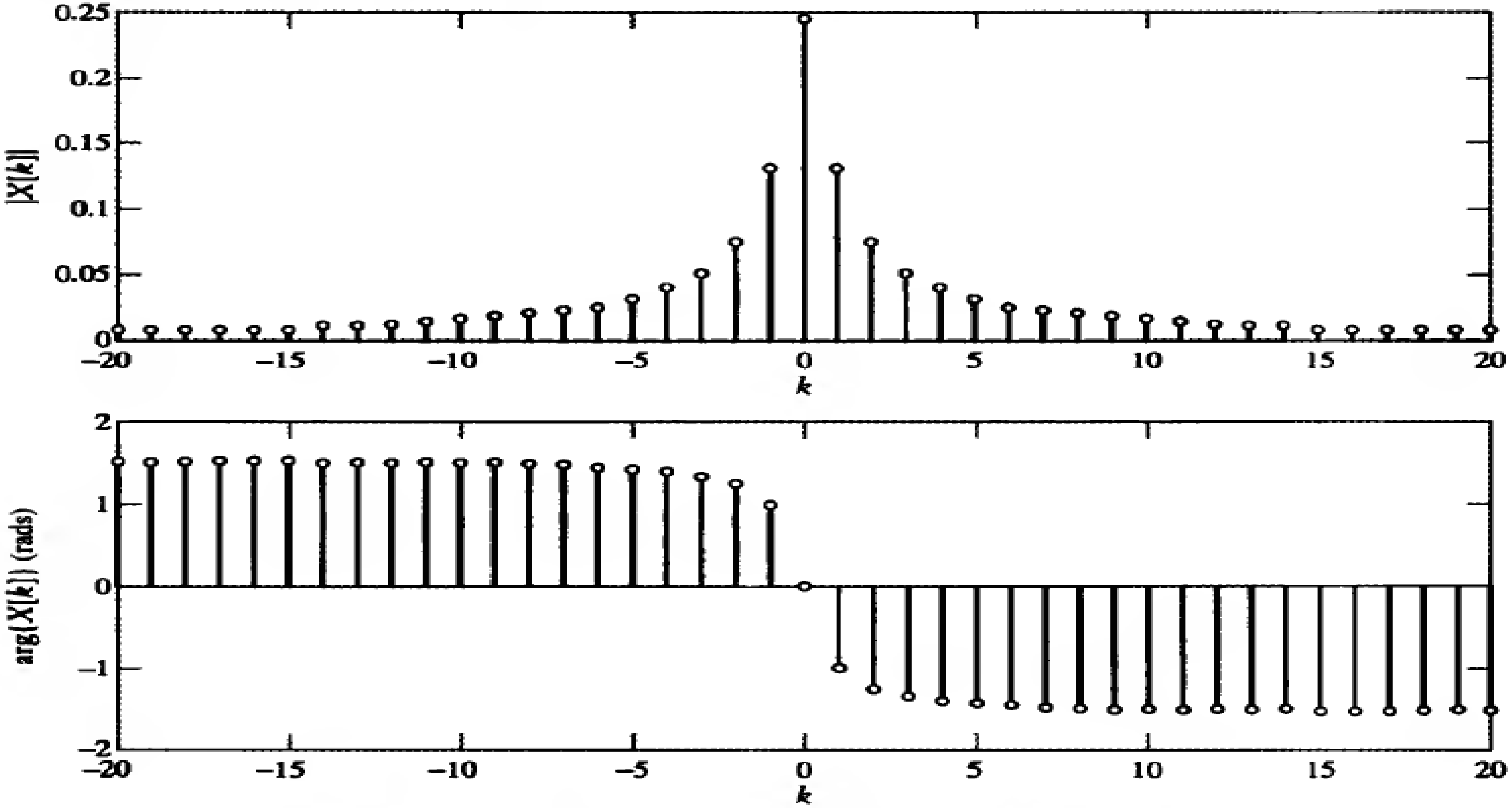


FIGURE 3.17 Magnitude and phase spectra for Example 3.9.

- As with the DTFS, the magnitude of X[k] is known as the magnitude spectrum of x(t), while the phase of X[k] is known as the phase spectrum of x(t).

We evaluate the integral to obtain

$$X[k] = \frac{-1}{2(2 + jk\pi)} e^{-(2+jk\pi)t} \Big|_0^2$$

$$= \frac{1}{4 + jk2\pi}(1 - e^{-4}e^{-jk2\pi})$$

$$= \frac{1 - e^{-4}}{4 + jk2\pi},$$

since $e^{-jk2\pi} = 1$. Figure 3.17 depicts the magnitude spectrum $|X[k]|$ and the phase spectrum $\arg\{X[k]\}$. ■

- Also, since x(t) is periodic, the interval of integration may be chosen as any interval one period in length.

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t},} \tag{3.19}$$

EXAMPLE 3.10 FS COEFFICIENTS FOR AN IMPULSE TRAIN Determine the FS coefficients for the signal defined by

$$x(t) = \sum_{l=-\infty}^{\infty} \delta(t - 4l).$$

Solution: The fundamental period is $T = 4$, and each period of this signal contains an impulse. The signal $x(t)$ has even symmetry, so it is easier to evaluate Eq. (3.20) by integrating over a period that is symmetric about the origin, $-2 < t \leq 2$, to obtain

$$X[k] = \frac{1}{4}\int_{-2}^{2} \delta(t)e^{-jk(\pi/2)t}\,dt$$
$$= \frac{1}{4}.$$

In this case, the magnitude spectrum is constant and the phase spectrum is zero. Note that we cannot evaluate the infinite sum in Eq. (3.19) in this case and that $x(t)$ does not satisfy the Dirichlet conditions. However, the FS expansion of an impulse train is useful in spite of convergence difficulties. ■

Example 3.11 Calculation of FS Coefficients by Inspection Determine the FS representation of the signal

$$x(t) = 3\cos(\pi t/2 + \pi/4),$$

using the method of inspection.

Solution: The fundamental period of $x(t)$ is $T = 4$. Hence, $\omega_o = 2\pi/4 = \pi/2$, and Eq. (3.19) is written as

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\pi t/2}. \tag{3.21}$$

Using Euler's formula to expand the cosine yields

$$\begin{aligned} x(t) &= 3\frac{e^{j(\pi t/2+\pi/4)} + e^{-j(\pi t/2+\pi/4)}}{2} \\ &= \frac{3}{2}e^{j\pi/4}e^{j\pi t/2} + \frac{3}{2}e^{-j\pi/4}e^{-j\pi t/2}. \end{aligned}$$

Equating each term in this expression to the terms in Eq. (3.21) gives the FS coefficients:

$$X[k] = \begin{cases} \frac{3}{2}e^{-j\pi/4}, & k = -1 \\ \frac{3}{2}e^{j\pi/4}, & k = 1 \\ 0, & \text{otherwise} \end{cases}. \tag{3.22}$$

The magnitude and phase spectra are depicted in Fig. 3.18. Note that all of the power in this signal is concentrated at two frequencies: $\omega = \pi/2$ and $\omega = -\pi/2$. ■

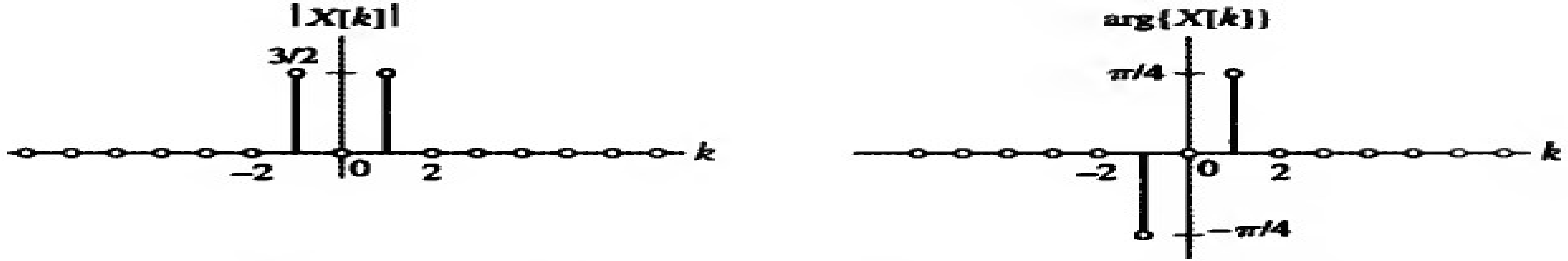


FIGURE 3.18 Magnitude and phase spectra for Example 3.11.

► **Problem 3.7** Determine the FS representation of

$$x(t) = 2\sin(2\pi t - 3) + \sin(6\pi t).$$

Answer:

$$x(t) \xleftrightarrow{FS;2\pi} X[k] = \begin{cases} j/2, & k = -3 \\ je^{j3}, & k = -1 \\ -je^{-j3}, & k = 1 \\ -j/2, & k = 3 \\ 0, & \text{otherwise} \end{cases}$$

► **Problem 3.8** Find the FS coefficients of the full-wave rectified cosine depicted in Fig. 3.19.

Answer:

$$X[k] = \frac{\sin(\pi(1-2k)/2)}{\pi(1-2k)} + \frac{\sin(\pi(1+2k)/2)}{\pi(1+2k)}$$ ◄

The time-domain signal represented by a set of FS coefficients is obtained by evaluating Eq. (3.19), as illustrated in the next example.

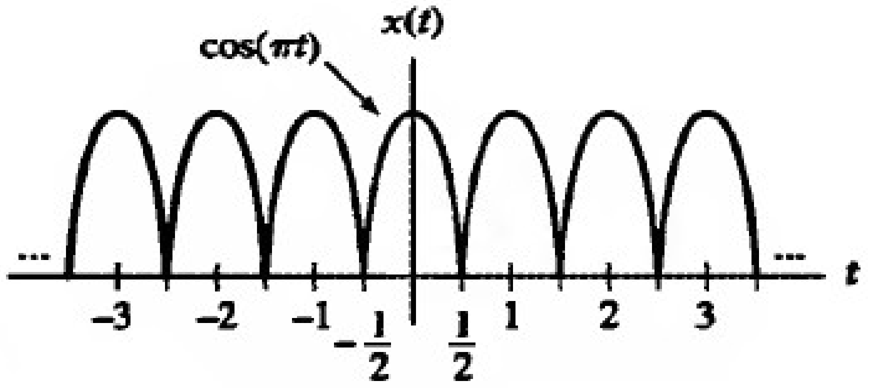


FIGURE 3.19 Full-wave rectified cosine for Problem 3.8.

Example 3.12 Inverse FS Find the time-domain signal $x(t)$ corresponding to the FS coefficients

$$X[k] = (1/2)^{|k|} e^{jk\pi/20}.$$

Assume that the fundamental period is $T = 2$.

Solution: Substituting the values given for $X[k]$ and $\omega_o = 2\pi/T = \pi$ into Eq. (3.19) yields

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} (1/2)^{k} e^{jk\pi/20} e^{jk\pi t} + \sum_{k=-1}^{-\infty} (1/2)^{-k} e^{jk\pi/20} e^{jk\pi t} \\ &= \sum_{k=0}^{\infty} (1/2)^{k} e^{jk\pi/20} e^{jk\pi t} + \sum_{l=1}^{\infty} (1/2)^{l} e^{-jl\pi/20} e^{-jl\pi t}. \end{aligned}$$

The second geometric series is evaluated by summing from $l = 0$ to $l = \infty$ and subtracting the $l = 0$ term. The result of summing both infinite geometric series is

$$x(t) = \frac{1}{1 - (1/2)e^{j(\pi t + \pi/20)}} + \frac{1}{1 - (1/2)e^{-j(\pi t + \pi/20)}} - 1.$$

Putting the fractions over a common denominator results in

$$x(t) = \frac{3}{5 - 4\cos(\pi t + \pi/20)}$$

■

► **Problem 3.9** Determine the time-domain signal represented by the following FS coefficients:

(a)

$$X[k] = -j\delta[k-2] + j\delta[k+2] + 2\delta[k-3] + 2\delta[k+3], \quad \omega_o = \pi$$

(b) $X[k]$ given in Fig. 3.20 with $\omega_o = \pi/2$

Answers:

(a)

$$x(t) = 2\sin(2\pi t) + 4\cos(3\pi t)$$

(b)

$$x(t) = \frac{\sin(9\pi(t-1)/4)}{\sin(\pi(t-1)/4)}$$ ◄

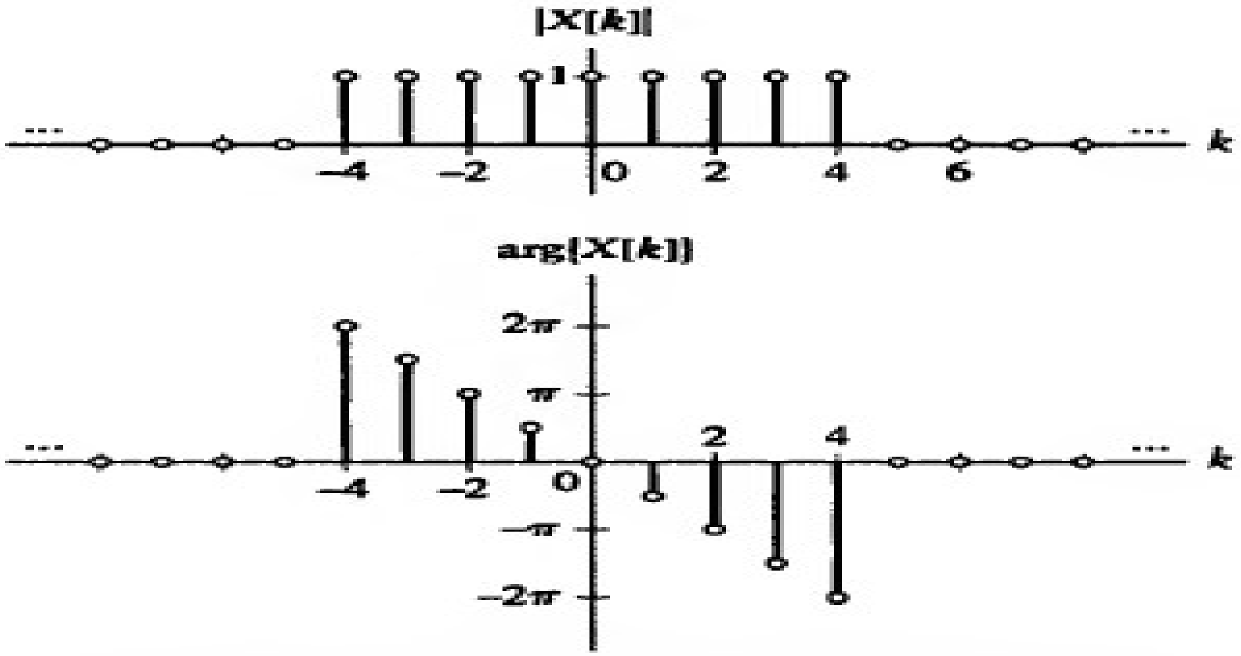


FIGURE 3.20 FS coefficients for Problem 3.9(b).

- ***Discrete-Time Nonperiodic Signals: The Discrete-Time Fourier Transform***
- The DTFT is used to represent a discrete-time nonperiodic signal as a superposition of complex sinusoids.

$$\boxed{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega,} \qquad (3.31)$$

Where

$$\boxed{X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}} \qquad (3.32)$$

- is the DTFT of the signal x[n]. We say that $X(e^{j\Omega})$ and x[n] are a DTFT pair

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}).$$

- The transform $X(e^{j\Omega})$ describes the signal x[n] as a function of a sinusoidal frequency Ω and is termed the frequency-domain representation of x[n].

- Equation (3.31) is usually termed the inverse DTFT, since it maps the frequency-domain representation back into the time domain.
- The infinite sum in Eq. (3.32) converges if x[n] has finite duration and is finite valued.
- If x[n] is of infinite duration, then the sum converges only for certain classes of signals.

EXAMPLE 3.17 DTFT OF AN EXPONENTIAL SEQUENCE Find the DTFT of the sequence $x[n] = \alpha^n u[n]$.

Solution: Using Eq. (3.32), we have

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n}. \end{aligned}$$

This sum diverges for $|\alpha| \geq 1$. For $|\alpha| < 1$, we have the convergent geometric series

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{n=0}^{\infty} (\alpha e^{-j\Omega})^n \\ &= \frac{1}{1 - \alpha e^{-j\Omega}}, \qquad |\alpha| < 1. \end{aligned} \tag{3.33}$$

If α is real valued, we may expand the denominator of Eq. (3.33) using Euler's formula to obtain

$$X(e^{j\Omega}) = \frac{1}{1 - \alpha \cos \Omega + j\alpha \sin \Omega}.$$

From this form, we see that the magnitude and phase spectra are given by

$$\begin{aligned} |X(e^{j\Omega})| &= \frac{1}{((1 - \alpha \cos \Omega)^2 + \alpha^2 \sin^2 \Omega)^{1/2}} \\ &= \frac{1}{(\alpha^2 + 1 - 2\alpha \cos \Omega)^{1/2}} \end{aligned}$$

and

$$\arg\{X(e^{j\Omega})\} = -\arctan\left(\frac{\alpha \sin \Omega}{1 - \alpha \cos \Omega}\right),$$

respectively. The magnitude and phase are depicted graphically in Fig. 3.29 for $\alpha = 0.5$ and $\alpha = 0.9$. The magnitude is even and the phase is odd. Note that both are 2π periodic. ■

▶ **Problem 3.11** Find the DTFT of $x[n] = 2(3)^n u[-n]$.

Answer:

$$X(e^{j\Omega}) = \frac{2}{1 - e^{j\Omega}/3}$$

- As with the other Fourier representations, the magnitude spectrum of a signal is the magnitude of $|X(e^{j\Omega})|$ plotted as a function of Ω. The phase spectrum is the phase of $X(e^{j\Omega})$ plotted as a function of Ω.

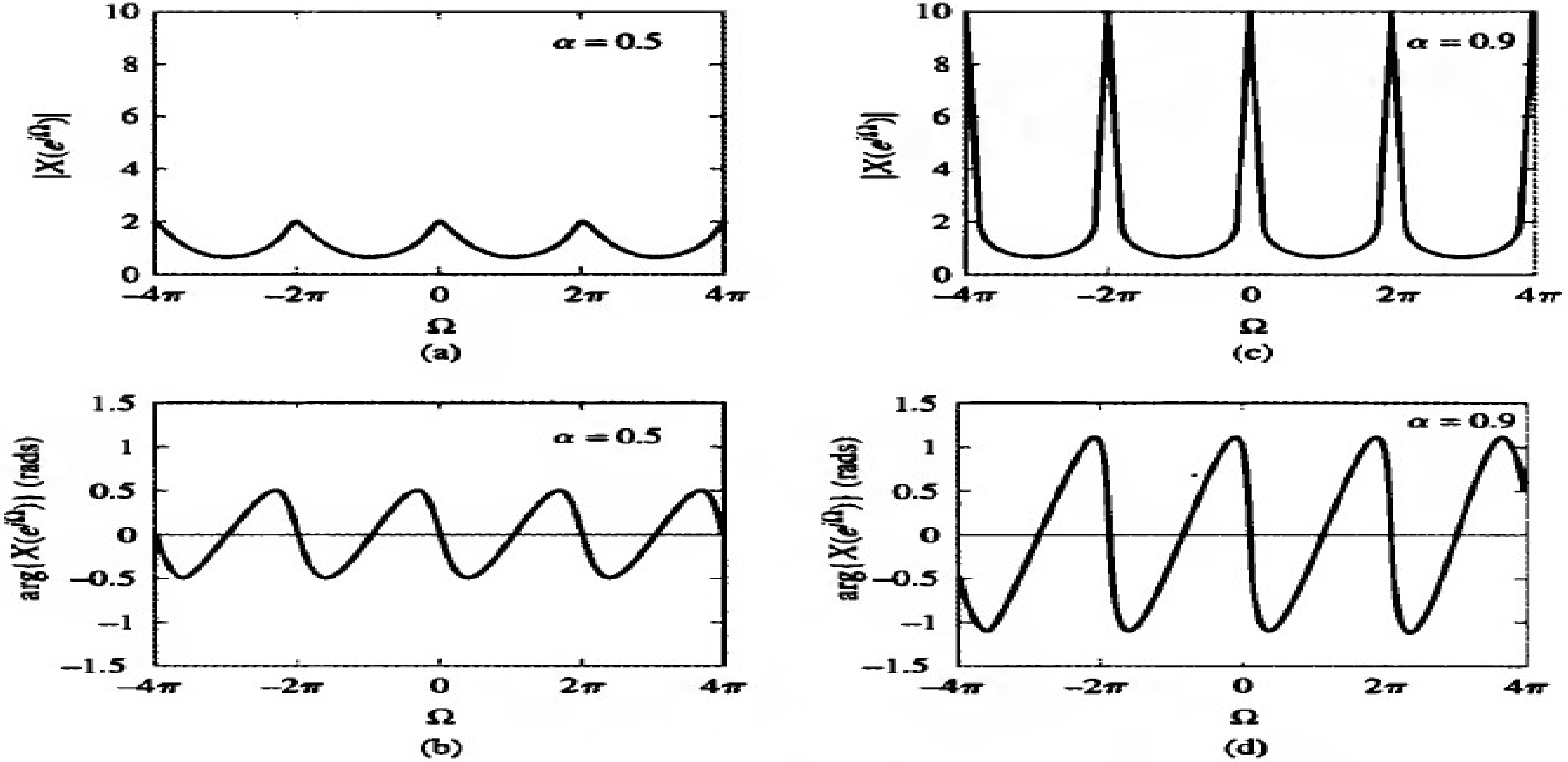


FIGURE 3.29 The DTFT of an exponential signal $x[n] = (\alpha)^n u[n]$. (a) Magnitude spectrum for $\alpha = 0.5$. (b) Phase spectrum for $\alpha = 0.5$. (c) Magnitude spectrum for $\alpha = 0.9$. (d) Phase spectrum for $\alpha = 0.9$.

EXAMPLE 3.18 DTFT OF A RECTANGULAR PULSE Let

$$x[n] = \begin{cases} 1, & |n| \le M \\ 0, & |n| > M \end{cases},$$

as depicted in Fig. 3.30(a). Find the DTFT of $x[n]$.

Solution: We substitute for $x[n]$ in Eq. (3.32) to obtain

$$X(e^{j\Omega}) = \sum_{n=-M}^{M} 1e^{-j\Omega n}.$$

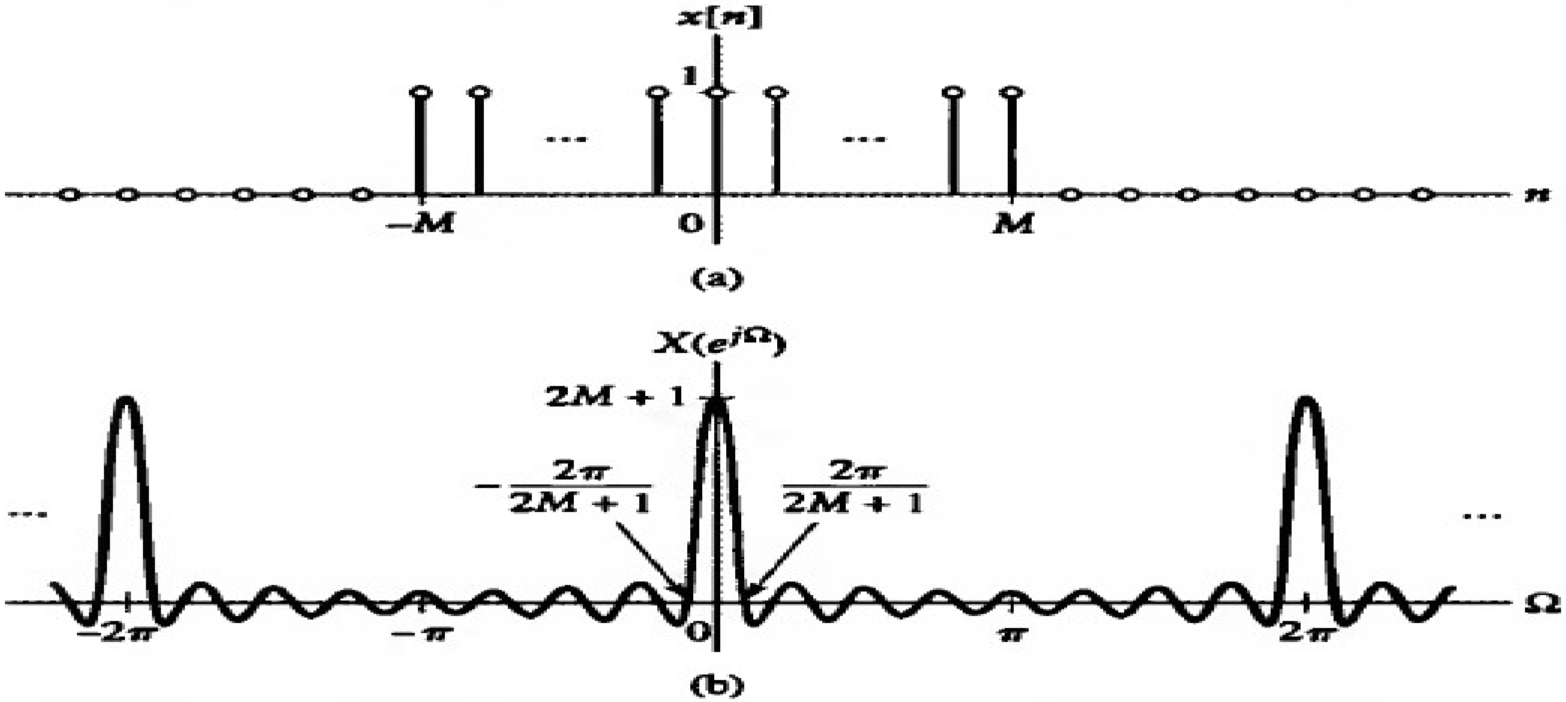


FIGURE 3.30 Example 3.18. (a) Rectangular pulse in the time domain. (b) DTFT in the frequency domain.

Now we perform the change of variable $m = n + M$, obtaining

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{m=0}^{2M} e^{-j\Omega(m-M)} \\ &= e^{j\Omega M} \sum_{m=0}^{2M} e^{-j\Omega m} \\ &= \begin{cases} e^{j\Omega M} \dfrac{1 - e^{-j\Omega(2M+1)}}{1 - e^{-j\Omega}}, & \Omega \neq 0, \pm 2\pi, \pm 4\pi, \ldots \\ 2M + 1, & \Omega = 0, \pm 2\pi, \pm 4\pi, \ldots \end{cases}. \end{aligned}$$

The expression for $X(e^{j\Omega})$ when $\Omega \neq 0, \pm 2\pi, \pm 4\pi, \ldots$, may be simplified by symmetrizing the powers of the exponential in the numerator and denominator as follows:

$$\begin{aligned} X(e^{j\Omega}) &= e^{j\Omega M} \frac{e^{-j\Omega(2M+1)/2}\left(e^{j\Omega(2M+1)/2} - e^{-j\Omega(2M+1)/2}\right)}{e^{-j\Omega/2}\left(e^{j\Omega/2} - e^{-j\Omega/2}\right)} \\ &= \frac{e^{j\Omega(2M+1)/2} - e^{-j\Omega(2M+1)/2}}{e^{j\Omega/2} - e^{-j\Omega/2}}. \end{aligned}$$

We may now write $X(e^{j\Omega})$ as a ratio of sine functions by dividing the numerator and denominator by $2j$ to obtain

$$X(e^{j\Omega}) = \frac{\sin(\Omega(2M + 1)/2)}{\sin(\Omega/2)}.$$

L'Hôpital's rule gives

$$\lim_{\Omega \to 0, \pm 2\pi, \pm 4\pi, \ldots,} \frac{\sin(\Omega(2M + 1)/2)}{\sin(\Omega/2)} = 2M + 1;$$

hence, rather than write $X(e^{j\Omega})$ as two forms dependent on the value of Ω, we simply write

$$X(e^{j\Omega}) = \frac{\sin(\Omega(2M+1)/2)}{\sin(\Omega/2)},$$

with the understanding that $X(e^{j\Omega})$ for $\Omega = 0, \pm 2\pi, \pm 4\pi, \ldots$, is obtained as a limit. In this example, $X(e^{j\Omega})$ is purely real. A graph of $X(e^{j\Omega})$ as a function of Ω is given in Fig. 3.30(b). We see that as M increases, the time extent of $x[n]$ increases, while the energy in $X(e^{j\Omega})$ becomes more concentrated near $\Omega = 0$. ■

Example 3.19 Inverse DTFT of a Rectangular Spectrum Find the inverse DTFT of

$$X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < W \\ 0, & W < |\Omega| < \pi \end{cases},$$

which is depicted in Fig. 3.31(a).

Solution: First, note that $X(e^{j\Omega})$ is specified only for $-\pi < \Omega \le \pi$. This is all that is needed, however, since $X(e^{j\Omega})$ is always 2π-periodic and the inverse DTFT depends solely on the values in the interval $-\pi < \Omega \le \pi$. Substituting for $X(e^{j\Omega})$ in Eq. (3.31) gives

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi n j} e^{j\Omega n}\Big|_{-W}^{W}, \quad n \neq 0 \\ &= \frac{1}{\pi n}\sin(Wn), \quad n \neq 0. \end{aligned}$$

For $n = 0$, the integrand is unity and we have $x[0] = W/\pi$. Using L'Hôpital's rule, we easily show that

$$\lim_{n\to 0} \frac{1}{\pi n}\sin(Wn) = \frac{W}{\pi},$$

and thus we usually write

$$x[n] = \frac{1}{\pi n}\sin(Wn)$$

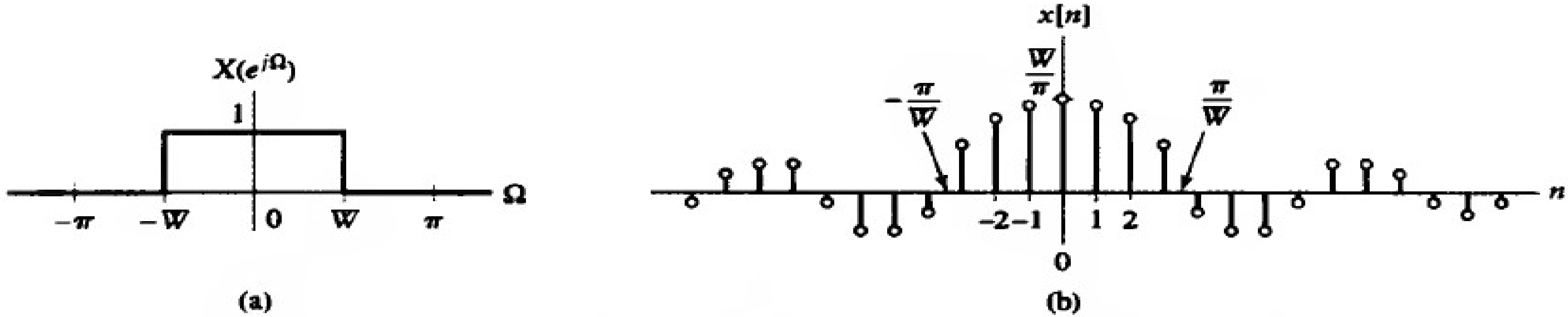


FIGURE 3.31 Example 3.19. (a) One period of rectangular pulse in the frequency domain. (b) Inverse DTFT in the time domain.

as the inverse DTFT of $X(e^{j\Omega})$, with the understanding that the value at $n = 0$ is obtained as the limit. We may also write

$$x[n] = \frac{W}{\pi}\operatorname{sinc}(Wn/\pi),$$

using the sinc function notation defined in Eq. (3.24). A graph depicting $x[n]$ versus time n is given in Fig. 3.31(b). ∎

EXAMPLE 3.20 DTFT OF THE UNIT IMPULSE Find the DTFT of $x[n] = \delta[n]$.

Solution: For $x[n] = \delta[n]$, we have

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} \\ &= 1. \end{aligned}$$

Hence

$$\delta[n] \xleftrightarrow{DTFT} 1.$$

This DTFT pair is depicted in Fig. 3.32.

EXAMPLE 3.21 INVERSE DTFT OF A UNIT IMPULSE SPECTRUM Find the inverse DTFT of $X(e^{j\Omega}) = \delta(\Omega), -\pi < \Omega \leq \pi$.

Solution: By definition, from Eq. (3.31),

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega)e^{j\Omega n}\, d\Omega.$$

We use the sifting property of the impulse function to obtain $x[n] = 1/(2\pi)$ and thus write

$$\frac{1}{2\pi} \xleftrightarrow{DTFT} \delta(\Omega), \quad -\pi < \Omega \leq \pi.$$

In this example, we have again defined only one period of $X(e^{j\Omega})$. Alternatively, we can define $X(e^{j\Omega})$ over all Ω by writing it as an infinite sum of delta functions shifted by integer multiples of 2π:

$$X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi).$$

Both definitions are common. This DTFT pair is depicted in Fig. 3.33. ■

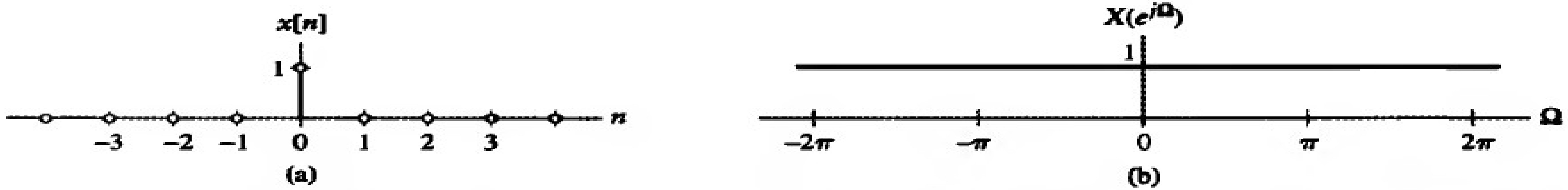


FIGURE 3.32 Example 3.20. (a) Unit impulse in the time domain. (b) DTFT of unit impulse in the frequency domain.

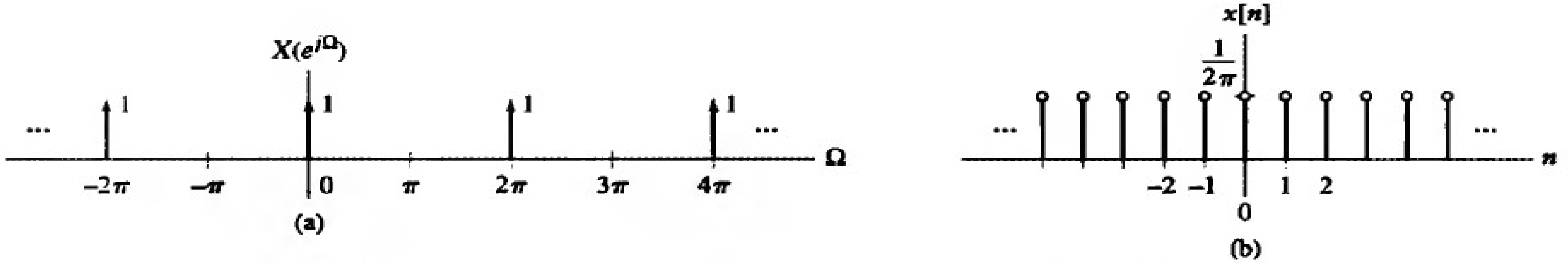


FIGURE 3.33 Example 3.21. (a) Unit impulse in the frequency domain. (b) Inverse DTFT in the time domain.

► **Problem 3.12** Find the DTFT of the following time-domain signals:

(a)

$$x[n] = \begin{cases} 2^n, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$x[n] = a^{|n|}, \qquad |a| < 1$$

(c)

$$x[n] = \delta[6 - 2n] + \delta[6 + 2n]$$

(d) $x[n]$ as depicted in Fig. 3.34.

(a)

$$X(e^{j\Omega}) = \frac{1 - 2^{10}e^{-j10\Omega}}{1 - 2e^{-j\Omega}}$$

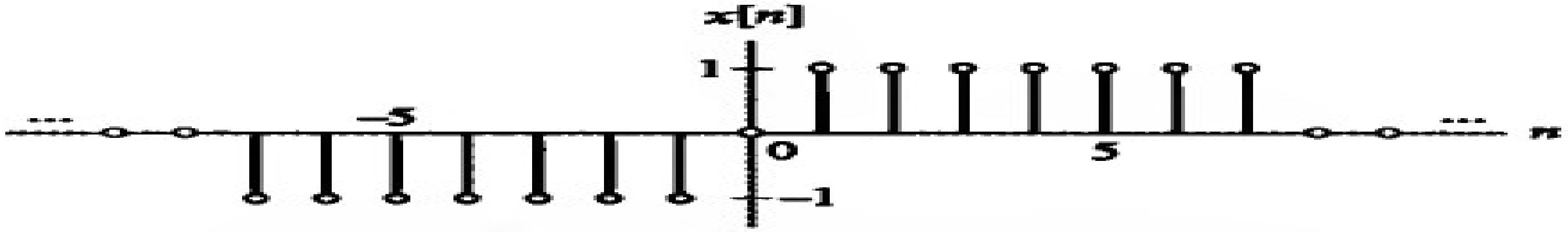


FIGURE 3.34 Signal $x[n]$ for Problem 3.12.

(b)

$$X(e^{j\Omega}) = \frac{1 - a^2}{1 - 2a\cos\Omega + a^2}$$

(c)

$$X(e^{j\Omega}) = 2\cos(3\Omega)$$

(d)

$$X(e^{j\Omega}) = -2j\sin(7\Omega/2)\frac{\sin(4\Omega)}{\sin(\Omega/2)}$$

▶ **Problem 3.13** Find the inverse DTFT of the following frequency-domain signals:

(a)

$$X(e^{j\Omega}) = 2\cos(2\Omega)$$

(b)

$$X(e^{j\Omega}) = \begin{cases} e^{-j4\Omega}, & \pi/2 < |\Omega| \leq \pi \\ 0, & \text{otherwise} \end{cases}, \quad \text{on } -\pi < \Omega \leq \pi$$

(c) $X(e^{j\Omega})$ as depicted in Fig. 3.35.

(a)

$$x[n] = \begin{cases} 1, & n = \pm 2 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$x[n] = \delta[n-4] - \frac{\sin(\pi(n-4)/2)}{\pi(n-4)}$$

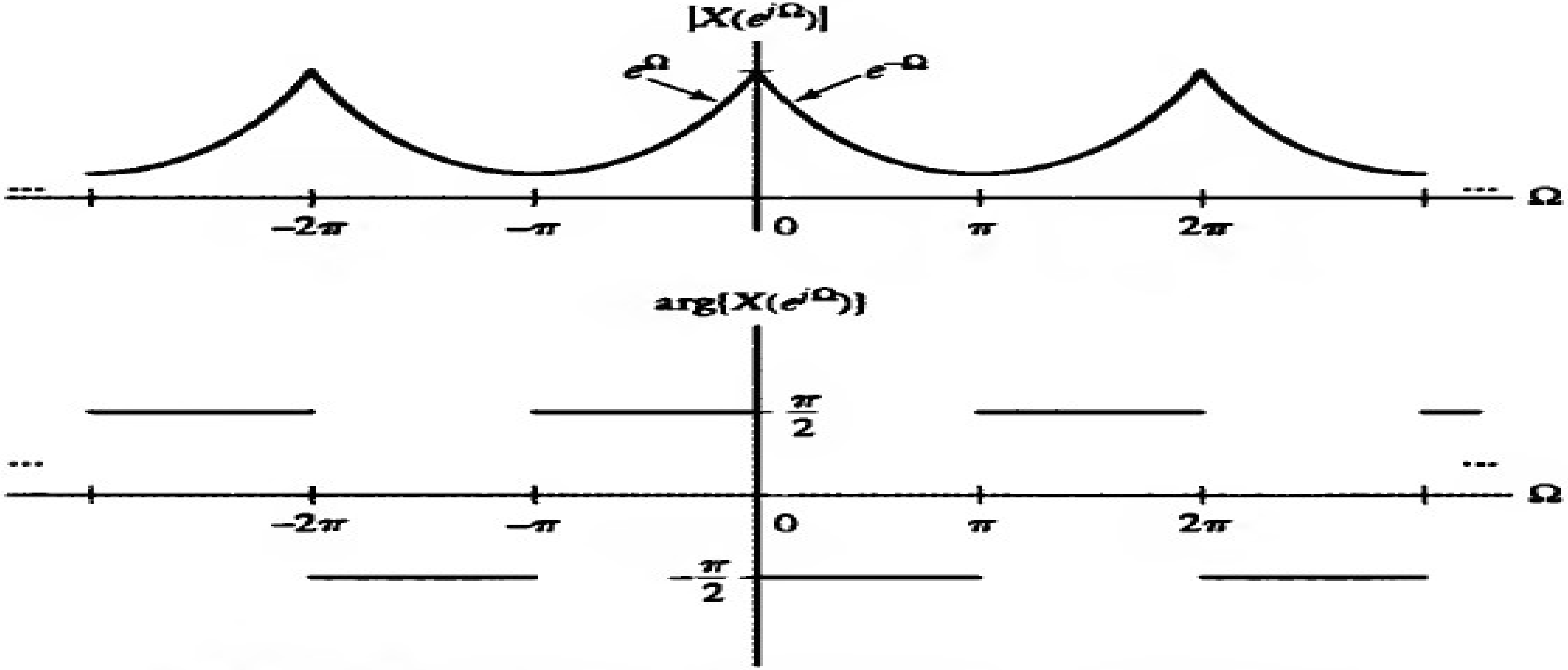


FIGURE 3.35 Frequency-domain signal for Problem 3.13(c).

- **Continuous-Time Nonperiodic Signals: The Fourier Transform**
- The Fourier transform (FT) is used to represent a continuous-time nonperiodic signal as a superposition of complex sinusoids
- the FT representation of a continuous-time signal involves an integral over the entire frequency interval

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega,} \tag{3.35}$$

- where $$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt} \tag{3.36}$$

We say that x(t) and $X(j\omega)$ are an FT pair and write $x(t) \xleftrightarrow{FT} X(j\omega).$

- The transform $X(j\omega)$ describes the signal x(t) as a function of frequency ω and is termed the frequency-domain representation of x(t).
- Equation (3.35) is termed the inverse FT, since it maps the frequency-domain representation $X(j\omega)$ back into the time domain.
- **convergence** is guaranteed at all values of t except those corresponding to discontinuities if x(t) satisfies the Dirichlet conditions for nonperiodic signals:
- x(t) is absolutely integrable: $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty.$
- x(t) has a finite number of maxima, minima, and discontinuities in any finite interval.
- The size of each discontinuity is finite.

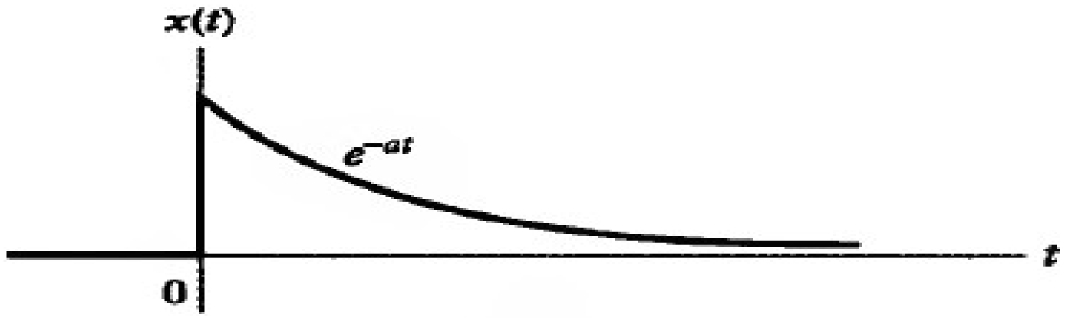
EXAMPLE 3.24 FT OF A REAL DECAYING EXPONENTIAL Find the FT of $x(t) = e^{-at}u(t)$, shown in Fig. 3.39(a).

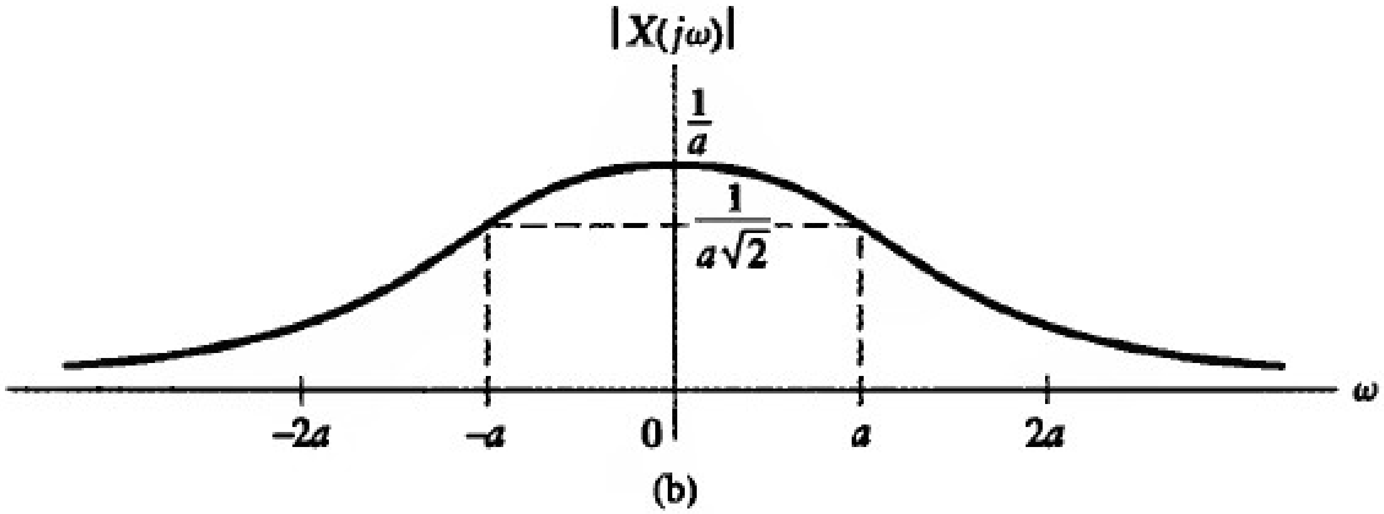
Solution: The FT does not converge for $a \le 0$, since $x(t)$ is not absolutely integrable; that is,

$$\int_0^{\infty} e^{-at}\,dt = \infty, \qquad a \le 0.$$

For $a > 0$, we have

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt \\ &= \int_0^{\infty} e^{-(a+j\omega)t}\,dt \\ &= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} \\ &= \frac{1}{a+j\omega}. \end{aligned}$$





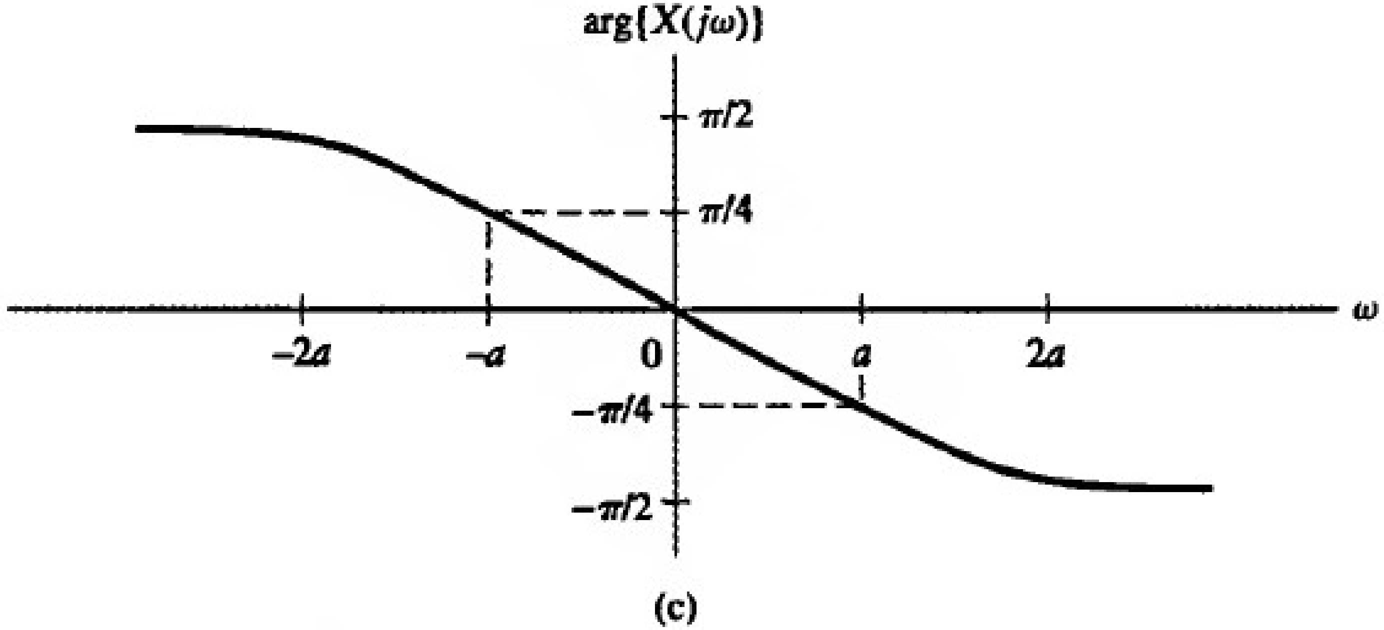


FIGURE 3.39 Example 3.24. (a) Real time-domain exponential signal. (b) Magnitude spectrum. (c) Phase spectrum.

Converting to polar form, we find that the magnitude and phase of $X(j\omega)$ are respectively given by

$$|X(j\omega)| = \frac{1}{(a^2 + \omega^2)^{\frac{1}{2}}}$$

and

$$\arg\{X(j\omega)\} = -\arctan(\omega/a),$$

as depicted in Figs. 3.39(b) and (c), respectively. ■

- As before, the magnitude of X(jω) plotted against ω is termed the magnitude spectrum of the signal x(t), and the phase of X(jω) plotted as a function of ω is termed the phase spectrum of x(t).

EXAMPLE 3.25 FT OF A RECTANGULAR PULSE Consider the rectangular pulse depicted in Fig. 3.40(a) and defined as

$$x(t) = \begin{cases} 1, & -T_o < t < T_o \\ 0, & |t| > T_o \end{cases}.$$

Find the FT of $x(t)$.

Solution: The rectangular pulse $x(t)$ is absolutely integrable, provided that $T_o < \infty$. We thus have

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \\ &= \int_{-T_o}^{T_o} e^{-j\omega t}\, dt \\ &= -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-T_o}^{T_o} \\ &= \frac{2}{\omega}\sin(\omega T_o), \qquad \omega \neq 0. \end{aligned}$$

For $\omega = 0$, the integral simplifies to $2T_o$. L'Hôpital's rule straightforwardly shows that

$$\lim_{\omega \to 0} \frac{2}{\omega}\sin(\omega T_o) = 2T_o.$$

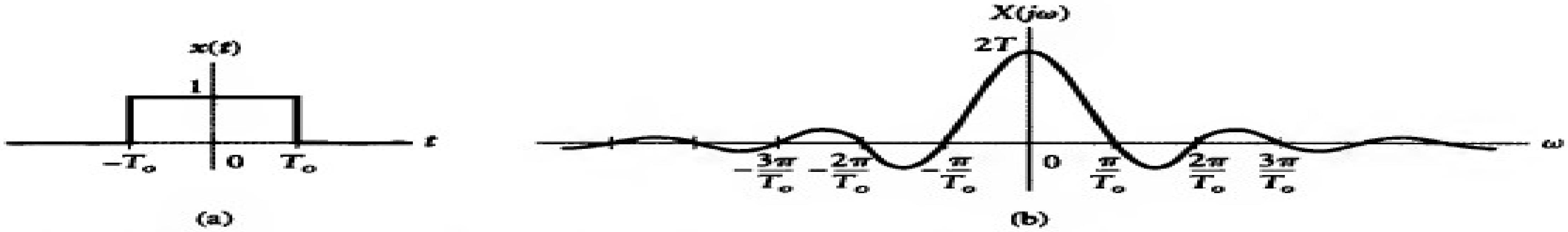


FIGURE 3.40 Example 3.25. (a) Rectangular pulse in the time domain. (b) FT in the frequency domain.

Thus, we usually write

$$X(j\omega) = \frac{2}{\omega}\sin(\omega T_o),$$

with the understanding that the value at $\omega = 0$ is obtained by evaluating a limit. In this case, $X(j\omega)$ is real. $X(j\omega)$ is depicted in Fig. 3.40(b). The magnitude spectrum is

$$|X(j\omega)| = 2\left|\frac{\sin(\omega T_o)}{\omega}\right|,$$

and the phase spectrum is

$$\arg\{X(j\omega)\} = \begin{cases} 0, & \sin(\omega T_o)/\omega > 0 \\ \pi, & \sin(\omega T_o)/\omega < 0 \end{cases}.$$

Using sinc function notation, we may write $X(j\omega)$ as

$$X(j\omega) = 2T_o \operatorname{sinc}(\omega T_o/\pi). \qquad \blacksquare$$

▶ **Problem 3.14** Find the FT of the following signals:

(a) $x(t) = e^{2t}u(-t)$

(b) $x(t) = e^{-|t|}$

(c) $x(t) = e^{-2t}u(t - 1)$

(d) $x(t)$ as shown in Fig. 3.41(a). (*Hint:* Use integration by parts.)

(e) $x(t)$ as shown in Fig. 3.41(b).

Answers:

(a) $X(j\omega) = -1/(j\omega - 2)$

(b) $X(j\omega) = 2/(1 + \omega^2)$

(c) $e^{-(j\omega+2)}/(j\omega + 2)$

(d) $X(j\omega) = j(2/\omega)\cos\omega - j(2/\omega^2)\sin\omega$

(e) $X(j\omega) = 2j(1 - \cos(2\omega))/\omega$

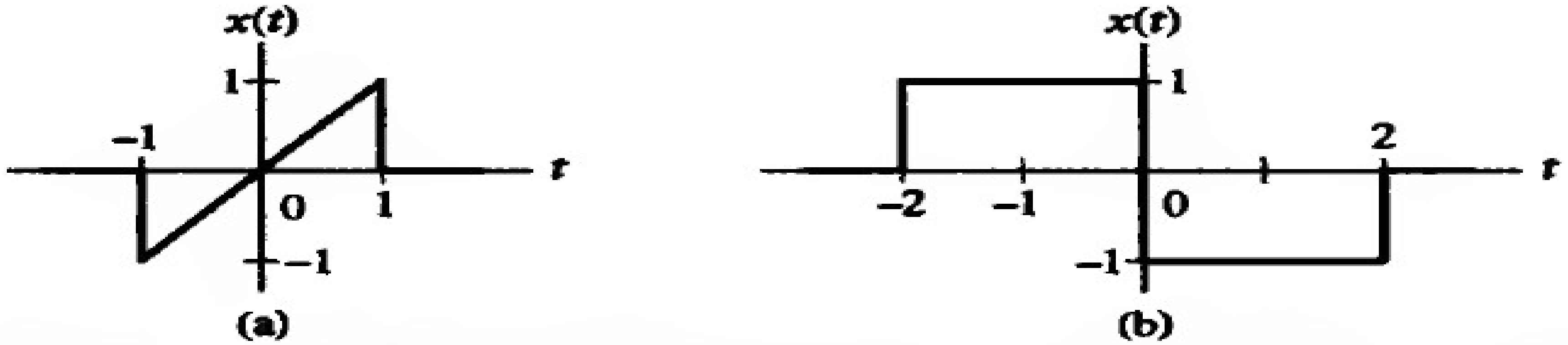


FIGURE 3.41 Time-domain signals for Problem 3.14. (a) Part (d). (b) Part (e).

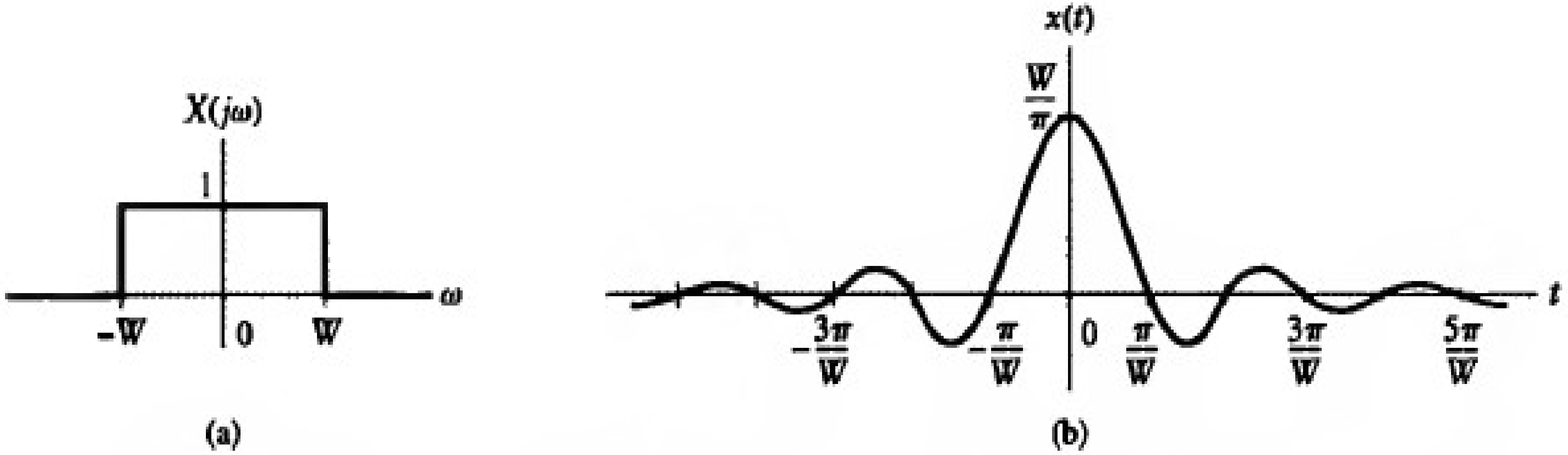


FIGURE 3.42 Example 3.26. (a) Rectangular spectrum in the frequency domain. (b) Inverse FT in the time domain.

Example 3.26 Inverse FT of a Rectangular Spectrum Find the inverse FT of the rectangular spectrum depicted in Fig. 3.42(a) and given by

$$X(j\omega) = \begin{cases} 1, & -W < \omega < W \\ 0, & |\omega| > W \end{cases}.$$

Solution: Using Eq. (3.35) for the inverse FT gives

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega \\ &= \frac{1}{2j\pi t}e^{j\omega t}\Big|_{-W}^{W} \\ &= \frac{1}{\pi t}\sin(Wt), \qquad t \neq 0. \end{aligned}$$

When $t = 0$, the integral simplifies to W/π. Since

$$\lim_{t\to 0}\frac{1}{\pi t}\sin(Wt) = W/\pi,$$

we usually write

$$x(t) = \frac{1}{\pi t}\sin(Wt),$$

or

$$x(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right),$$

with the understanding that the value at $t = 0$ is obtained as a limit. Figure 3.42(b) depicts $x(t)$. ■

- **Duality:** a rectangular time-domain pulse is transformed to a sinc function in frequency, a sinc function in time is transformed to a rectangular pulse in frequency. This is called "duality".

EXAMPLE 3.27 FT OF THE UNIT IMPULSE Find the FT of $x(t) = \delta(t)$.

Solution: This $x(t)$ does not satisfy the Dirichlet conditions, since the discontinuity at the origin is infinite. We attempt to proceed in spite of this potential problem, using Eq. (3.36) to write

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt$$
$$= 1.$$

The evaluation to unity follows from the sifting property of the impulse function. Hence,

$$\delta(t) \xleftrightarrow{FT} 1,$$

and the impulse contains unity contributions from complex sinusoids of all frequencies, from $\omega = -\infty$ to $\omega = \infty$. ■

EXAMPLE 3.28 INVERSE FT OF AN IMPULSE SPECTRUM Find the inverse FT of $X(j\omega) = 2\pi\delta(\omega)$.

Solution: Here again, we may expect convergence irregularities, since $X(j\omega)$ has an infinite discontinuity at the origin. Nevertheless, we may proceed by using Eq. (3.35) to write

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}\,d\omega \\ &= 1. \end{aligned}$$

Hence, we identify

$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

as an FT pair. This implies that the frequency content of a dc signal is concentrated entirely at $\omega = 0$, which is intuitively satisfying. ■

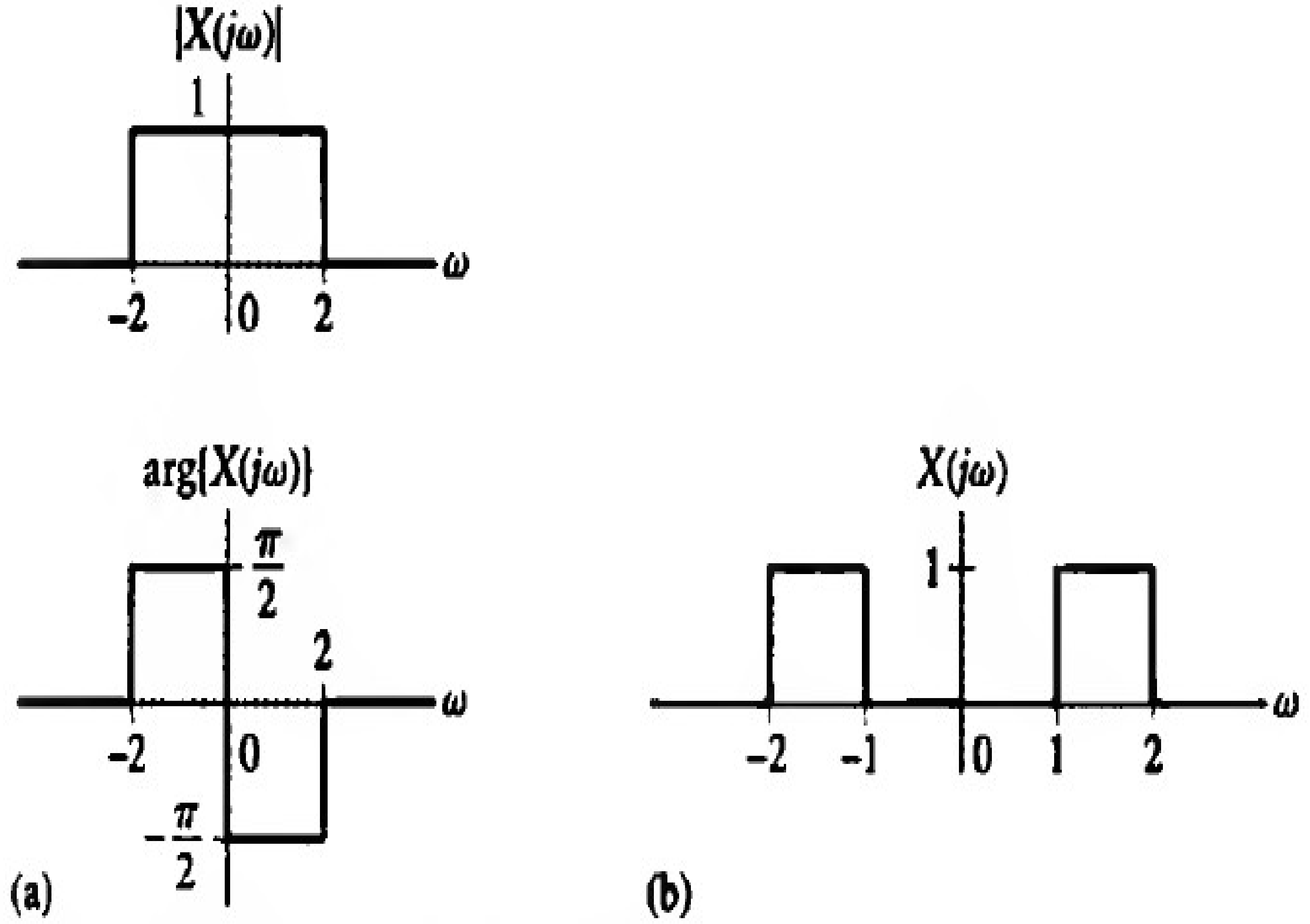


FIGURE 3.43 Frequency-domain signals for Problem 3.15. (a) Part (d). (b) Part (e).

► **Problem 3.15** Find the inverse FT of the following spectra:

(a)

$$X(j\omega) = \begin{cases} 2\cos\omega, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases}$$

(b) $X(j\omega) = 3\delta(\omega - 4)$

(c) $X(j\omega) = \pi e^{-|\omega|}$

(d) $X(j\omega)$ as depicted in Fig. 3.43(a).

(e) $X(j\omega)$ as depicted in Fig. 3.43(b).

Answers:

(a)

$$x(t) = \frac{\sin(\pi(t+1))}{\pi(t+1)} + \frac{\sin(\pi(t-1))}{\pi(t-1)}$$

(b) $x(t) = (3/2\pi)e^{j4t}$

(c) $x(t) = 1/(1 + t^2)$

(d) $x(t) = (1 - \cos(2t))/(\pi t)$

(e) $x(t) = (\sin(2t) - \sin t)/(\pi t)$

- **Properties of Fourier Representations**

Time Domain	Periodic (t, n)	Non periodic (t, n)	
Continuous (t)	**Fourier Series** $x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$ $X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt$ $x(t)$ has period T $\omega_o = \frac{2\pi}{T}$	**Fourier Transform** $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	Nonperiodic (k, ω)
Discrete (n)	**Discrete-Time Fourier Series** $x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n}$ $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_o n}$ $x[n]$ and $X[k]$ have period N $\Omega_o = \frac{2\pi}{N}$	**Discrete-Time Fourier Transform** $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ $X(e^{j\Omega})$ has period 2π	Periodic (k, Ω)
	Discrete (k)	Continuous (ω, Ω)	Frequency Domain

TABLE 3.3 ***Periodicity Properties of Fourier Representations.***

Time-Domain Property	*Frequency-Domain Property*
continuous	nonperiodic
discrete	periodic
periodic	discrete
nonperiodic	continuous

- ***Linearity and Symmetry Properties***

$$z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(j\omega) = aX(j\omega) + bY(j\omega)$$

$$z(t) = ax(t) + by(t) \xleftrightarrow{FS;\omega_o} Z[k] = aX[k] + bY[k]$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFT} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFS;\Omega_o} Z[k] = aX[k] + bY[k]$$

- assume that the uppercase symbols denote the Fourier representation of the corresponding lowercase symbols

EXAMPLE 3.30 LINEARITY IN THE FS Suppose $z(t)$ is the periodic signal depicted in Fig. 3.49(a). Use the linearity property and the results of Example 3.13 to determine the FS coefficients $Z[k]$.

Solution: Write $z(t)$ as a sum of signals; that is,

$$z(t) = \frac{3}{2}x(t) + \frac{1}{2}y(t),$$

where $x(t)$ and $y(t)$ are depicted in Figs. 3.49(b) and (c), respectively. From Example 3.13, we have

$$x(t) \xleftrightarrow{FS;2\pi} X[k] = (1/(k\pi))\sin(k\pi/4)$$
$$y(t) \xleftrightarrow{FS;2\pi} Y[k] = (1/(k\pi))\sin(k\pi/2)$$

The linearity property implies that

$$z(t) \xleftrightarrow{FS;2\pi} Z[k] = \frac{3}{2k\pi}\sin(k\pi/4) + \frac{1}{2k\pi}\sin(k\pi/2)$$

■

▶ **Problem 3.16** Use the linearity property and Tables C.1–4 in Appendix C to determine the Fourier representations of the following signals:

(a) $x(t) = 2e^{-t}u(t) - 3e^{-2t}u(t)$

(b) $x[n] = 4(1/2)^n u[n] - \frac{1}{\pi n}\sin(\pi n/4)$

(c) $x(t) = 2\cos(\pi t) + 3\sin(3\pi t)$

Answers:

(a) $X(j\omega) = 2/(j\omega + 1) - 3/(j\omega + 2)$

(b)

$$X(e^{j\Omega}) = \begin{cases} \dfrac{3 + (1/2)e^{-j\Omega}}{1 - (1/2)e^{-j\Omega}} & |\Omega| \le \pi/4 \\[2ex] \dfrac{4}{1 - (1/2)e^{-j\Omega}} & \pi/4 < |\Omega| \le \pi \end{cases}$$

(c) $\omega_o = \pi, X[k] = \delta[k-1] + \delta[k+1] + 3/(2j)\delta[k-3] - 3/(2j)\delta[k+3]$ ◀

Symmetry Properties: Real and Imaginary Signals

- $$X^*(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt\right]^*$$
- $$= \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}\,dt.$$
-

If x(t) is real $\boxed{X^*(j\omega) = X(-j\omega).}$

X(jω) is the complex-conjugate symmetric, or $X^*(j\omega) = X(-j\omega)$

- Taking the real and imaginary parts of this expression gives Re{X(jω)}=Re{X(-jω)} and Im{X(jω)} =-Im{X(-jω)}.

If x(t) is real valued, then the real part of the transform is an even function of frequency, while the imaginary part is an odd function of frequency.

The magnitude spectrum is an even function while the phase spectrum is an odd function.

The real part of the Fourier representation has even symmetry and the imaginary part has odd symmetry.

TABLE 3.4 ***Symmetry Properties for Fourier Representation of Real- and Imaginary-Valued Time Signals.***

Representation	*Real-Valued Time Signals*	*Imaginary-Valued Time Signals*
FT	$X^*(j\omega) = X(-j\omega)$	$X^*(j\omega) = -X(-j\omega)$
FS	$X^*[k] = X[-k]$	$X^*[k] = -X[-k]$
DTFT	$X^*(e^{j\Omega}) = X(e^{-j\Omega})$	$X^*(e^{j\Omega}) = -X(e^{-j\Omega})$
DTFS	$X^*[k] = X[-k]$	$X^*[k] = -X[-k]$

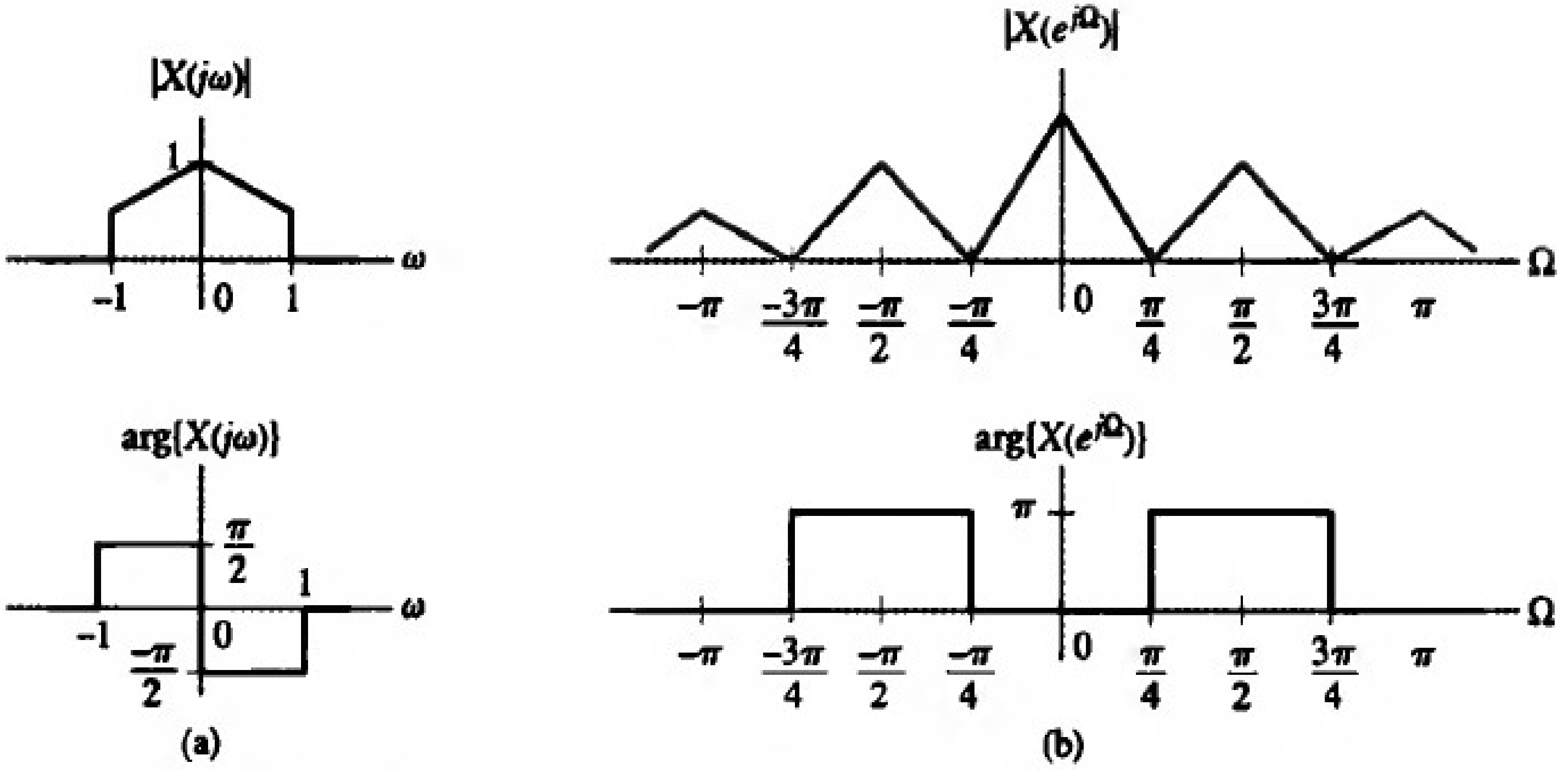


FIGURE 3.51 Frequency-domain representations for Problem 3.17.

▶ **Problem 3.17** Determine whether the time-domain signals corresponding to the following frequency-domain representations are real or complex valued and even or odd:

(a) $X(j\omega)$ as depicted in Fig. 3.51(a)

(b) $X(e^{j\Omega})$ as depicted in Fig. 3.51(b)

(c) FS: $X[k] = (1/2)^k u[k] + j2^k u[-k]$

(d) $X(j\omega) = \omega^{-2} + j\omega^{-3}$

(e) $X(e^{j\Omega}) = j\Omega^2 \cos(2\Omega)$

Answers:

(a) $x(t)$ is real and odd

(b) $x[n]$ is real and even

(c) $x(t)$ is complex valued

(d) $x(t)$ is real valued

(e) $x[n]$ is pure imaginary and even ◀

- **Convolution**:

$$y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega).$$

▶ **Problem 3.18** Use the convolution property to find the FT of the system output, either $Y(j\omega)$ or $Y(e^{j\Omega})$, for the following inputs and system impulse responses:

(a) $x(t) = 3e^{-t}u(t)$ and $h(t) = 2e^{-2t}u(t)$.

(b) $x[n] = (1/2)^n u[n]$ and $h[n] = (1/(\pi n))\sin(\pi n/2)$

Answers:

(a)

$$Y(j\omega) = \left(\frac{2}{j\omega + 2}\right)\left(\frac{3}{j\omega + 1}\right)$$

(b)

$$Y(e^{j\Omega}) = \begin{cases} 1/(1 - (1/2)e^{-j\Omega}), & |\Omega| \le \pi/2 \\ 0, & \pi/2 < |\Omega| \le \pi \end{cases}$$

TABLE 3.5 Convolution Properties.

$x(t) * z(t) \xleftrightarrow{FT} X(j\omega)Z(j\omega)$
$x(t) \circledast z(t) \xleftrightarrow{FS;\omega_o} TX[k]Z[k]$
$x[n] * z[n] \xleftrightarrow{DTFT} X(e^{j\Omega})Z(e^{j\Omega})$
$x[n] \circledast z[n] \xleftrightarrow{DTFS;\Omega_o} NX[k]Z[k]$

- **Differentiation**:

$$\boxed{\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega).}$$

EXAMPLE 3.37 VERIFYING THE DIFFERENTIATION PROPERTY The differentiation property implies that

$$\frac{d}{dt}(e^{-at}u(t)) \xleftrightarrow{FT} \frac{j\omega}{a + j\omega}.$$

Verify this result by differentiating and taking the FT of the result.

Solution: Using the product rule for differentiation, we have

$$\begin{aligned}\frac{d}{dt}(e^{-at}u(t)) &= -ae^{-at}u(t) + e^{-at}\delta(t)\\ &= -ae^{-at}u(t) + \delta(t).\end{aligned}$$

Taking the FT of each term and using linearity, we may write

$$\begin{aligned}\frac{d}{dt}(e^{-at}u(t)) &\xleftrightarrow{FT} \frac{-a}{a + j\omega} + 1\\ &\xleftrightarrow{FT} \frac{j\omega}{a + j\omega}.\end{aligned}$$

■

▶ **Problem 3.22** Use the differentiation property to find the FT of the following signals:

(a) $x(t) = \frac{d}{dt}e^{-2|t|}$

(b) $x(t) = \frac{d}{dt}(2te^{-2t}u(t))$

Answers:

(a) $X(j\omega) = (4j\omega)/(4 + \omega^2)$

(b) $X(j\omega) = (2j\omega)/(2 + j\omega)^2$ ◀

DIFFERENTIATION IN FREQUENCY

$$\boxed{-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega).}$$

▶ **Problem 3.25** Use the frequency-differentiation property to find the FT of

$$x(t) = te^{-at}u(t),$$

given that $e^{-at}u(t) \xleftrightarrow{FT} 1/(j\omega + a)$.

Answer:

$$X(j\omega) = \frac{1}{(a + j\omega)^2}$$

▶ **Problem 3.26** Use the time-differentiation and convolution properties to find the FT of

$$y(t) = \frac{d}{dt}\{te^{-3t}u(t) * e^{-2t}u(t)\}.$$

Answer:

$$Y(j\omega) = \frac{j\omega}{(3 + j\omega)^2(j\omega + 2)}$$

◀

- **Integration**:

$$Y(j\omega) = \frac{1}{j\omega}X(j\omega).$$

$$\boxed{\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega).}$$

► **Problem 3.25** Use the frequency-differentiation property to find the FT of

$$x(t) = te^{-at}u(t),$$

given that $e^{-at}u(t) \xleftrightarrow{FT} 1/(j\omega + a)$.

Answer:

$$X(j\omega) = \frac{1}{(a + j\omega)^2}$$

▶ **Problem 3.29** Use the integration property to find the inverse FT of

$$X(j\omega) = \frac{1}{j\omega(j\omega + 1)} + \pi\delta(\omega).$$

Answer:

$$x(t) = (1 - e^{-t})u(t)$$

TABLE 3.6 Commonly Used Differentiation and Integration Properties.

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FS;\,\omega_o} jk\omega_o X[k]$$

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$$

$$-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\Omega}X(e^{j\Omega})$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$$

- **Time and frequency shifting properties**

Table 3.7 *Time-Shift Properties of Fourier Representations.*

$$x(t - t_o) \xleftrightarrow{FT} e^{-j\omega t_o} X(j\omega)$$

$$x(t - t_o) \xleftrightarrow{FS;\omega_o} e^{-jk\omega_o t_o} X[k]$$

$$x[n - n_o] \xleftrightarrow{DTFT} e^{-j\Omega n_o} X(e^{j\Omega})$$

$$x[n - n_o] \xleftrightarrow{DTFS;\Omega_o} e^{-jk\Omega_o n_o} X[k]$$

▶ **Problem 3.31** Find the Fourier representation of the following time-domain signals:

(a) $x(t) = e^{-2t}u(t-3)$

(b) $y[n] = \sin(\pi(n+2)/3)/(\pi(n+2))$

Answers:

(a)

$$X(j\omega) = e^{-6}e^{-j3\omega}/(j\omega + 2)$$

(b)

$$Y(e^{j\Omega}) = \begin{cases} e^{j2\Omega}, & |\Omega| \le \pi/3 \\ 0, & \pi/3 < |\Omega| \le \pi \end{cases}$$

◀

TABLE 3.8 *Frequency-Shift Properties of Fourier Representations.*

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega - \gamma))$$

$$e^{jk_o\omega_o t}x(t) \xleftrightarrow{FS;\,\omega_o} X[k - k_o]$$

$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\Omega-\Gamma)})$$

$$e^{jk_o\Omega_o n}x[n] \xleftrightarrow{DTFS;\,\Omega_o} X[k - k_o]$$

▶ **Problem 3.34** Use the frequency-shift property to find the time-domain signals corresponding to the following Fourier representations:

(a)

$$Z(e^{j\Omega}) = \frac{1}{1 - \alpha e^{-j(\Omega+\pi/4)}}, \qquad |\alpha| < 1$$

(b)

$$X(j\omega) = \frac{1}{2 + j(\omega - 3)} + \frac{1}{2 + j(\omega + 3)}$$

Answers:

(a)

$$z[n] = e^{-j\pi/4n}\alpha^n u[n]$$

(b)

$$x(t) = 2\cos(3t)e^{-2t}u(t)$$

◀

▶ **Problem 3.36** Use partial-fraction expansions to determine the time-domain signals corresponding to the following FT's:

(a)

$$X(j\omega) = \frac{-j\omega}{(j\omega)^2 + 3j\omega + 2}$$

(b)

$$X(j\omega) = \frac{5j\omega + 12}{(j\omega)^2 + 5j\omega + 6}$$

(c)

$$X(j\omega) = \frac{2(j\omega)^2 + 5j\omega - 9}{(j\omega + 4)(-\omega^2 + 4j\omega + 3)}$$

Answers:

(a) $x(t) = e^{-t}u(t) - 2e^{-2t}u(t)$
(b) $x(t) = 3e^{-3t}u(t) + 2e^{-2t}u(t)$
(c) $x(t) = e^{-4t}u(t) - 2e^{-t}u(t) + 3e^{-3t}u(t)$ ◀

- **Multiplication property**

$$\boxed{y(t) = x(t)z(t) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{2\pi} X(j\omega) * Z(j\omega),}$$

TABLE 3.9 *Multiplication Properties of Fourier Representations.*

$$x(t)z(t) \xleftrightarrow{FT} \frac{1}{2\pi} X(j\omega) * Z(j\omega)$$

$$x(t)z(t) \xleftrightarrow{FS;\,\omega_o} X[k] * Z[k]$$

$$x[n]z[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} X(e^{j\Omega}) \circledast Z(e^{j\Omega})$$

$$x[n]z[n] \xleftrightarrow{DTFS;\,\Omega_o} X[k] \circledast Z[k]$$

- **Scaling property**

$$\boxed{z(t) = x(at) \xleftrightarrow{FT} (1/|a|)X(j\omega/a).}$$

- If x(t) is a periodic then x(at) is also periodic. If x(t) has a fundamental period T and the fundamental frequency $\omega 0$ then x(at) will have a fundamental period T/a and the fundamental frequency $k\omega 0$.
- The scaling property is slightly different character in DT. x[pn] is defined for integer values of p only.
- If p>1 then the scaling operation discards the information, since it retains only pth value of x[n].
- This loss of information prevents us from expressing the DTFT or DTFS of x[pn] interms of DTFT or DTFS of x[n].

- **Parseval Relationships**

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega.$$

- Energy in time-domain representations is equal to the energy in the frequency-domain representations, normalised by 2π. The quantity $|X(j\omega)|^2$ plotted against ω is the energy spectrum of the signal.

TABLE 3.10 *Parseval Relationships for the Four Fourier Representations.*

Representation	*Parseval Relation*
FT	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 \, d\omega$
FS	$\frac{1}{T} \int_0^T \lvert x(t)\rvert^2 \, dt = \sum_{k=-\infty}^{\infty} \lvert X[k]\rvert^2$
DTFT	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lvert X(e^{j\Omega})\rvert^2 \, d\Omega$
DTFS	$\frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \sum_{k=0}^{N-1} \lvert X[k]\rvert^2$

▶ **Problem 3.43** Use Parseval's theorem to evaluate the following quantities:

(a)

$$\chi_1 = \int_{-\infty}^{\infty} \frac{2}{|j\omega + 2|^2} d\omega$$

(b)

$$\chi_2 = \sum_{k=0}^{29} \frac{\sin^2(11\pi k/30)}{\sin^2(\pi k/30)}$$

Answers:

(a) $\chi_1 = \pi$

(b) $\chi_2 = 330$

- **Duality**
- A rectangular pulse in either time or frequency corresponds to a sinc function in either time or frequency.

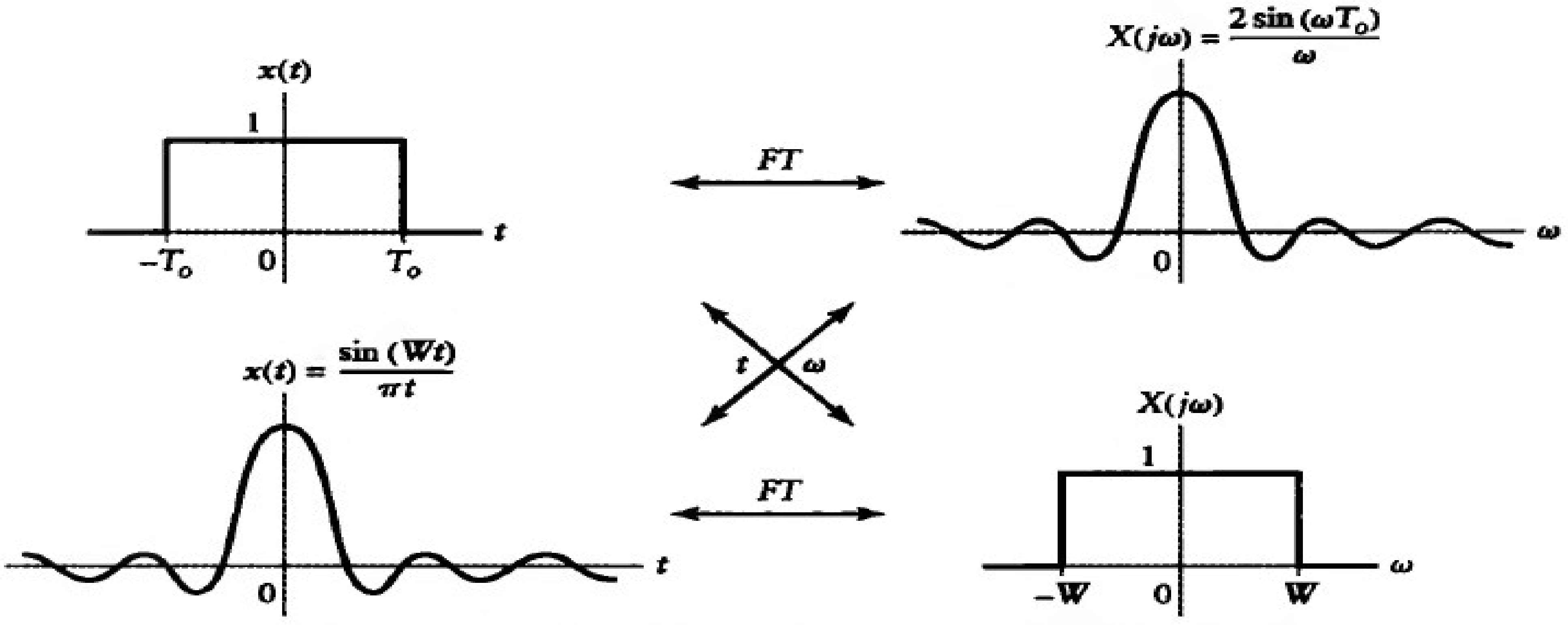


FIGURE 3.73 Duality of rectangular pulses and sinc functions.

- An impulse in time transforms to a constant in frequency, while a constant in time tranforms to impulse in frequency.

  This interchangeablity property is called duality.

EXAMPLE 3.52 APPLYING DUALITY Find the FT of

$$x(t) = \frac{1}{1 + jt}.$$

Solution: First, recognize that

$$f(t) = e^{-t}u(t) \xleftrightarrow{FT} F(j\omega) = \frac{1}{1 + j\omega}.$$

Replacing ω by t, we obtain

$$F(jt) = \frac{1}{1 + jt}.$$

Hence, we have expressed $x(t)$ as $F(jt)$. The duality property given by Eqs. (3.69) and (3.70) states that

$$F(jt) \xleftrightarrow{FT} 2\pi f(-\omega),$$

which implies that

$$\begin{aligned} X(j\omega) &= 2\pi f(-\omega) \\ &= 2\pi e^{\omega} u(-\omega). \end{aligned}$$ ■

TABLE 3.11 ***Duality Properties of Fourier Representations.***

FT	$f(t) \xleftrightarrow{FT} F(j\omega)$	$F(jt) \xleftrightarrow{FT} 2\pi f(-\omega)$
DTFS	$x[n] \xleftrightarrow{DTFS;\,2\pi/N} X[k]$	$X[n] \xleftrightarrow{DTFS;\,2\pi/N} (1/N)x[-k]$
FS–DTFT	$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$	$X(e^{jt}) \xleftrightarrow{FS;\,1} x[-k]$