Tracing of Curves

Polar form

Tracing of a Polar Curve

List of points to be examined

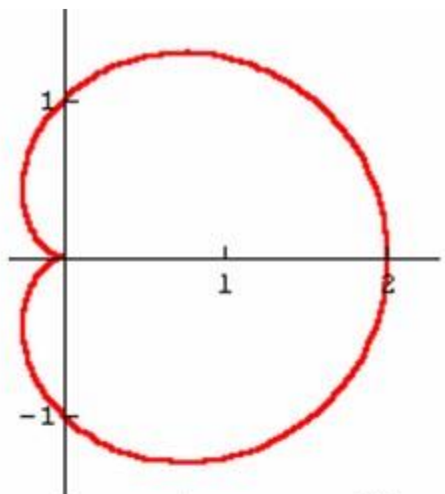
- Symmetry
- Tangents at the pole
- Asymptotes of the curve
- Points of intersection
- Direction of the tangent
- Points of intersection
- Sign of derivatives
- Loops
- Region of Existence

SYMMETRY:

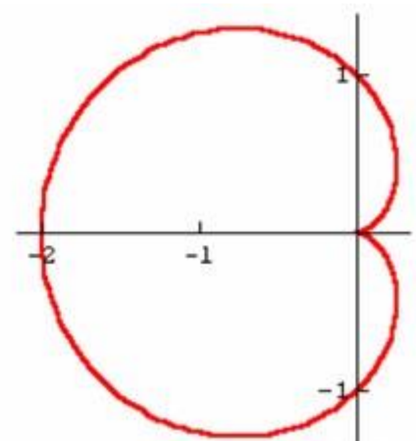
Symmetrical about the line $\theta = 0$: If the equation remains unaltered when θ is replaced by $-\theta$.

In other words if , $f(r, -\theta) = f(r, \theta)$

Example : $r = a(1 \pm cos\theta)$



$r = 1 + \cos(\theta)$



$r = 1 - \cos(\theta)$

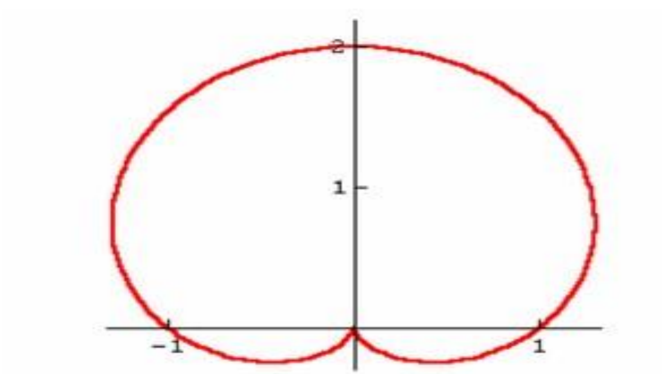
SYMMETRY:

- Symmetrical about the line $\theta = \frac{\pi}{2}$:

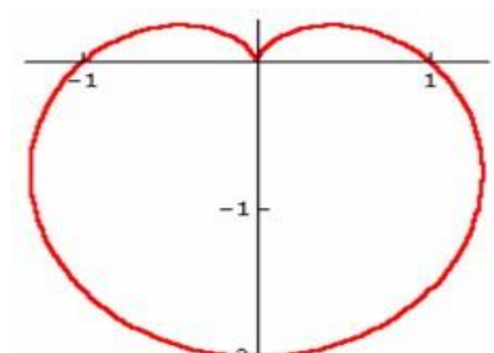
If the equation remains unaltered when θ is replaced by $\pi - \theta$.

In other words if , $f(r, \pi - \theta) = f(r, \theta)$.

Example : $r = a(1 \pm sin\theta)$



$r = 1 + \sin(\theta)$



$r = 1 - \sin(\theta)$

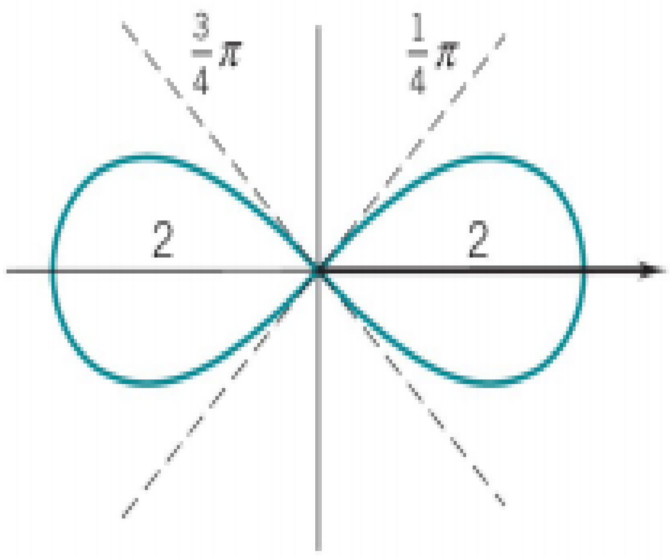
SYMMETRY:

- Symmetrical about the pole:

  If the equation remains unaltered when r is replaced by $-$ r. If the equation contains even powers of r.

  In other words if , $f(-r,\theta) = f(r,\theta)$.

Example : $r^2 = acos2\theta$



$r^2 = 4\cos 2\theta$

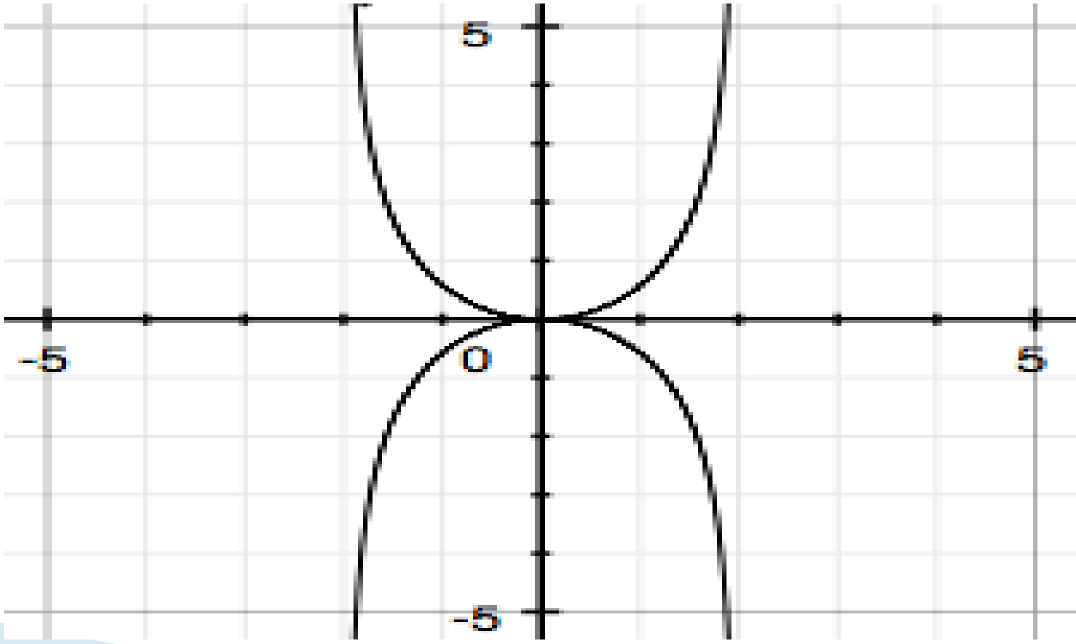
SYMMETRY:

- Symmetrical about the pole:

  If the equation remains unaltered when θ is replaced by $\pi + \theta$.

  In other words if , $f(r,\theta) = f(r,\pi+\theta)$.

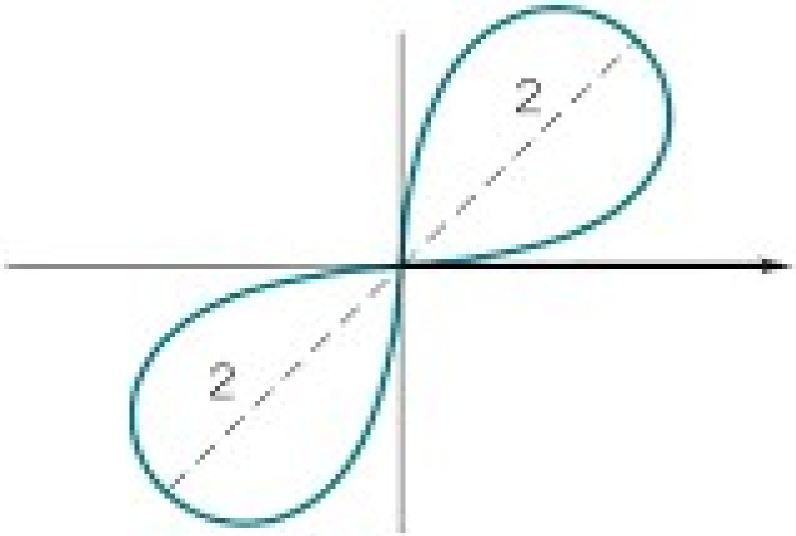
Example : $r = 4tan\theta$



SYMMETRY:

- Symmetrical about the line $\theta = \frac{\pi}{4}$:

  If the equation remains unaltered when θ is replaced by $\frac{\pi}{2} - \theta$. In other words if , $f(r, \theta) = f\left(r, \frac{\pi}{2} - \theta\right)$.



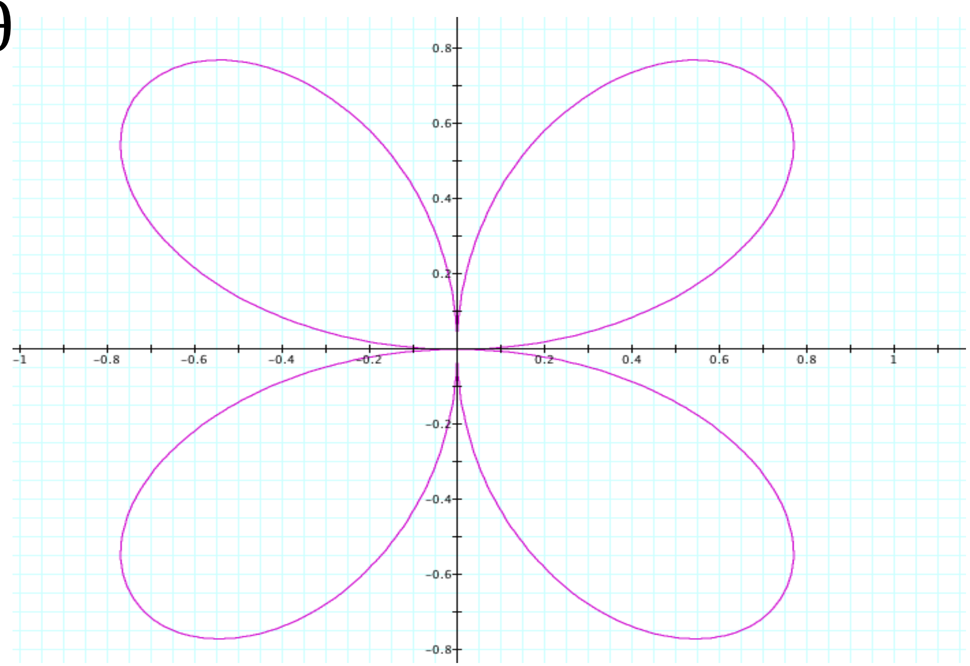
$r^2 = 4 \sin 2\theta$

SYMMETRY:

- Symmetrical about the line $\theta = \dfrac{3\pi}{4}$:

If the equation remains unaltered when θ is replaced by $\frac{3\pi}{2} - \theta$. In other words if , $f(r,\theta) = f\left(r, \frac{3\pi}{2} - \theta\right)$.

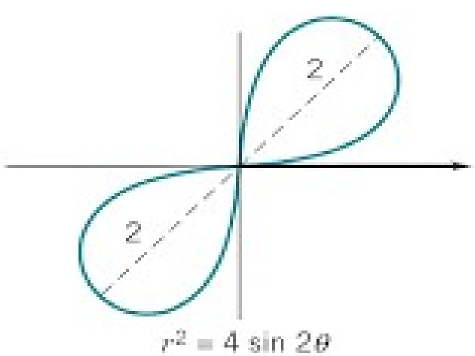
Example:r=asin2θ

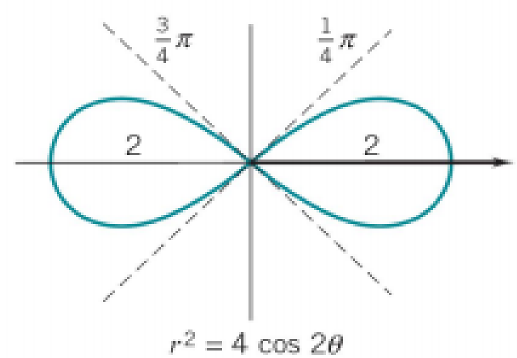


Pole :

If $r = f(\theta_1) = 0$ for some $\theta = \theta_1$= constant then the curve passes through the pole and the tangent at the pole is $\theta = \theta_1$.

- At $\theta = \pi, r^2 = 4sin2\theta = 0$. Therefore the curve through the pole and $\theta = \pi$ is the tangent at the pole.
- At $\theta = {}^{\pi}/_{4}, r^2 = 4cos2\theta = 0$. Therefore the curve through the pole and $\theta = {}^{\pi}/_{4}$ is the tangent at the pole.





Asymptote :

- An asymptote to the curve exists if $\lim_{\theta \to \theta_1} r = \infty$ and is given by the equation $rsin(\theta - \theta_1) = f'(\theta_1)$ where θ_1 is the solution of $\frac{1}{f(\theta)}$=0.
- Example : For the curve, $r^2 cos2\theta = a^2$, the asymptotes are $\theta = \pm\frac{\pi}{4}$

Points of Intersection :

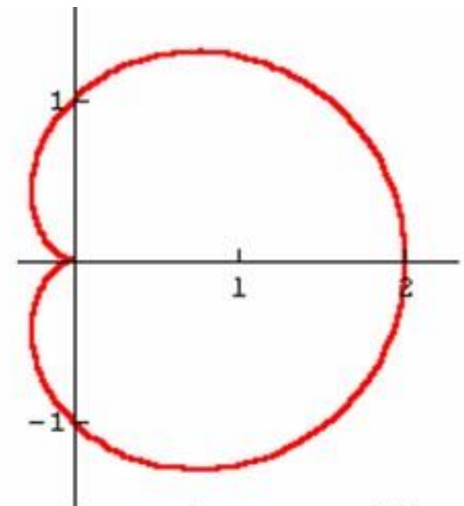
The points of intersection of the curve with the initial line ,the line $\theta = \pi/2$, the line $\theta = \frac{\pi}{4}$ and the line $\theta = \frac{3\pi}{4}$ can be obtained by putting $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ respectively in the given equation.

Points of Intersection :

Examples :

1. The curve $r = a(1 + cos\theta)$ intersects

- the line $\theta = 0$ at $(2a, 0)$
- the line $\theta = \pi$ at $(0, \pi)$
- the line $\theta = \frac{\pi}{2}$ at $\left(a, \frac{\pi}{2}\right)$
- the line $\theta = \frac{3\pi}{2}$ at $\left(a, \frac{3\pi}{2}\right)$



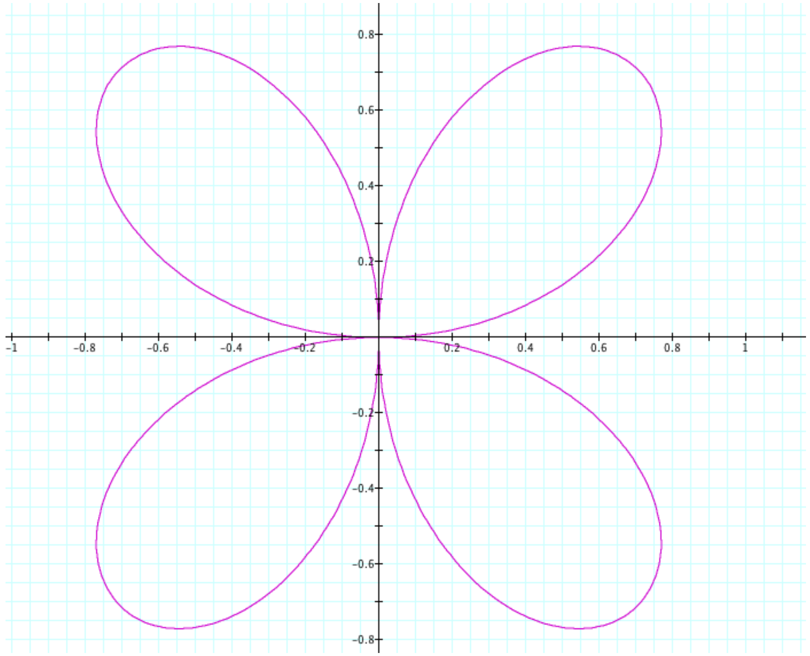
$r = 1 + \cos(\theta)$

Points of Intersection :

Examples :

2. The curve $r = a sin 2\theta$ intersects

- the line $\theta = \frac{\pi}{4}$ at $\left(a, \frac{\pi}{4}\right)$
- the line $\theta = \frac{3\pi}{4}$ at $\left(a, \frac{3\pi}{4}\right)$



Region or Extent:

- If a and b are the least and greatest values of r such that

$a < r < b$ then the curve lies in the annulus region between the two circles of radii a and b.

Example : For the curve $r = asin2\theta$, since maximum value of $sin2\theta$ is 1, the curve lies within the circle $r = a$.

- The curve does not exist for those values of θ at which r is imaginary.

Example : For the curve $r^2 = a^2cos2\theta$, in the interval $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, $cos2\theta$ is negative and hence r is imaginary.
Therefore the curve does not exist in this region.

- For equations involving periodic functions generally θ varies from 0 to 2π.

Direction of the tangent :

Determine ∅ where ∅ is the angle between the radius vector and the tangent.

Using $tan\emptyset = r\frac{d\theta}{dr}$

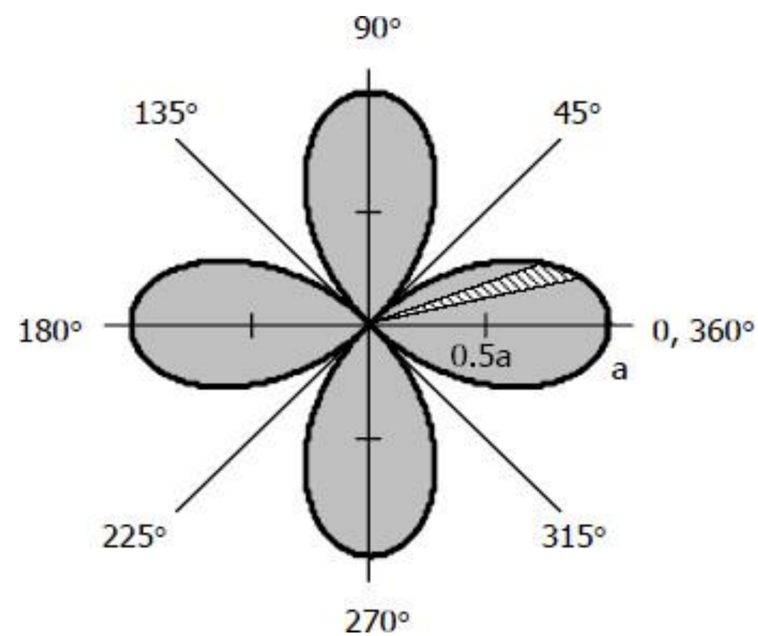
The angle ∅ gives the direction of the tangent at the point of intersection.

Derivative :

- If $\frac{dr}{d\theta} > 0$, then the curve increases.
- If $\frac{dr}{d\theta} < 0$, then the curve decreases.

Loop :

Example : The curve intersects the initial line at the points A(0,0) and $B(a, 0)$. Also the curve is symmetric about the initial line. Hence a loop of the curve exists between the points A and B.



Note :

If the curve,

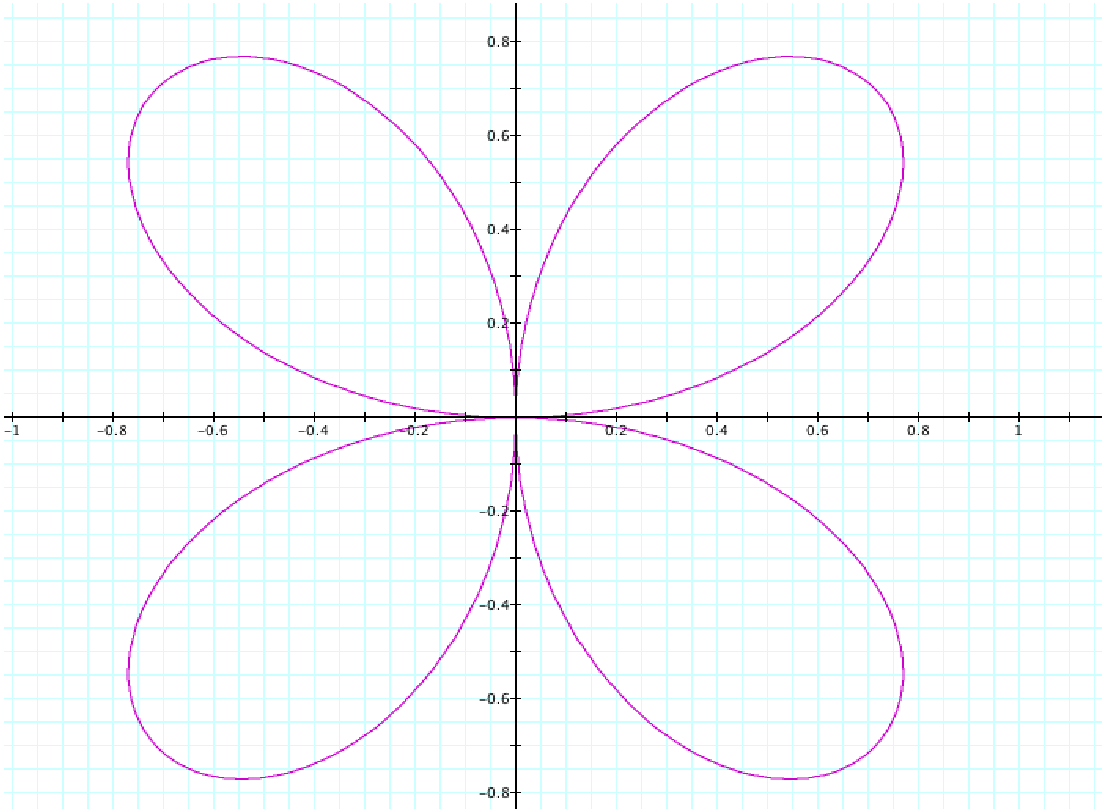
- intersects any line at the points A and B,
- is symmetric about that line

Then a loop of the curve exists between the points A and B.

Curves of the type $r = a\sin(n\theta)$ and $r = a\cos(n\theta)$ (are called roses) consist of either n and $2n$ similar loops (also called as leaves) according as n is odd or even.

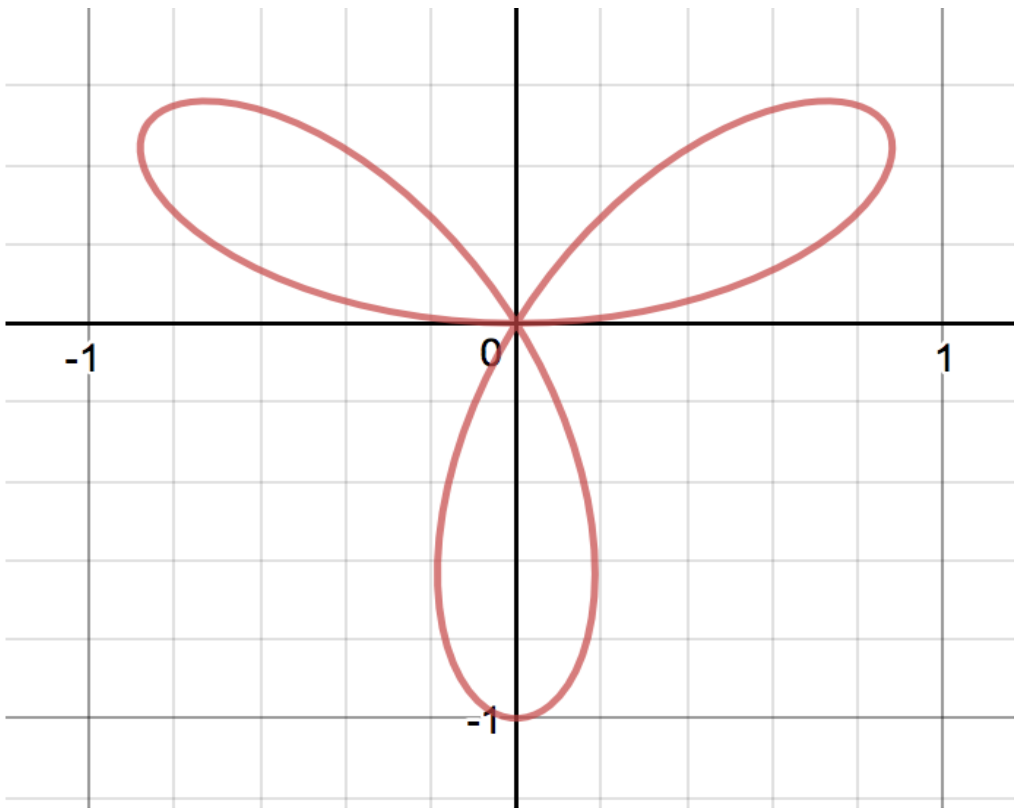
Examples :

- 1. $r = a\sin 2\theta$



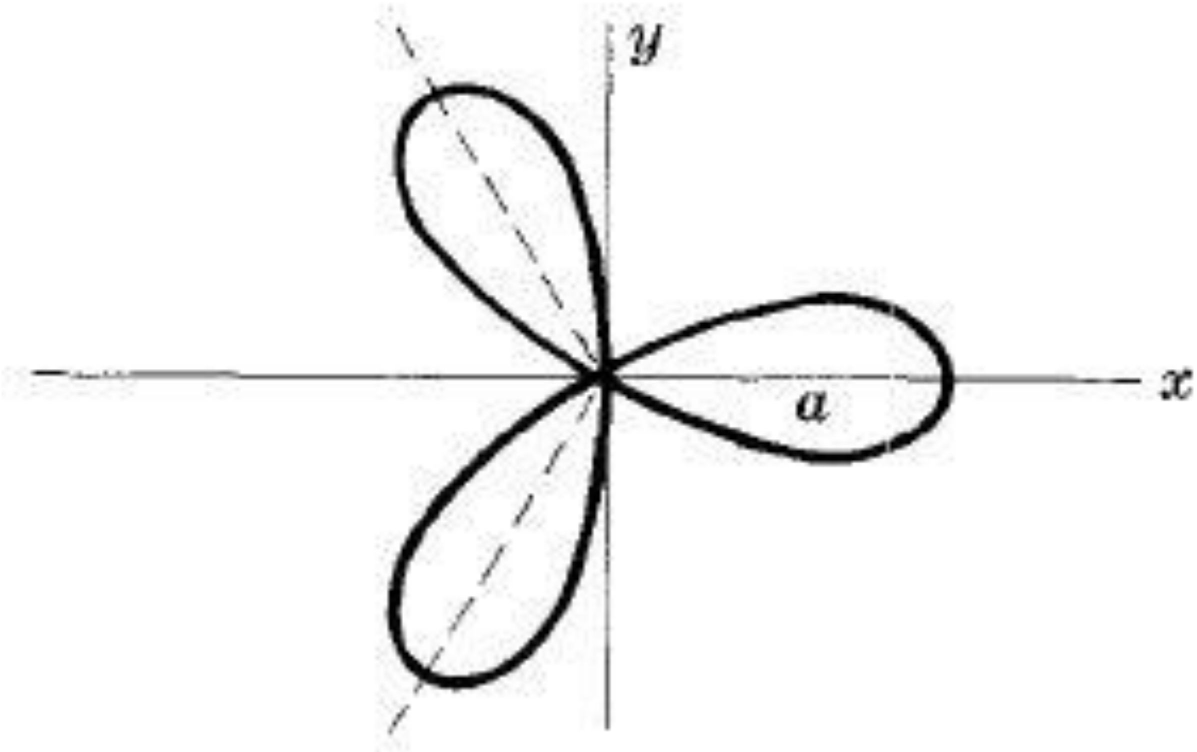
Examples :

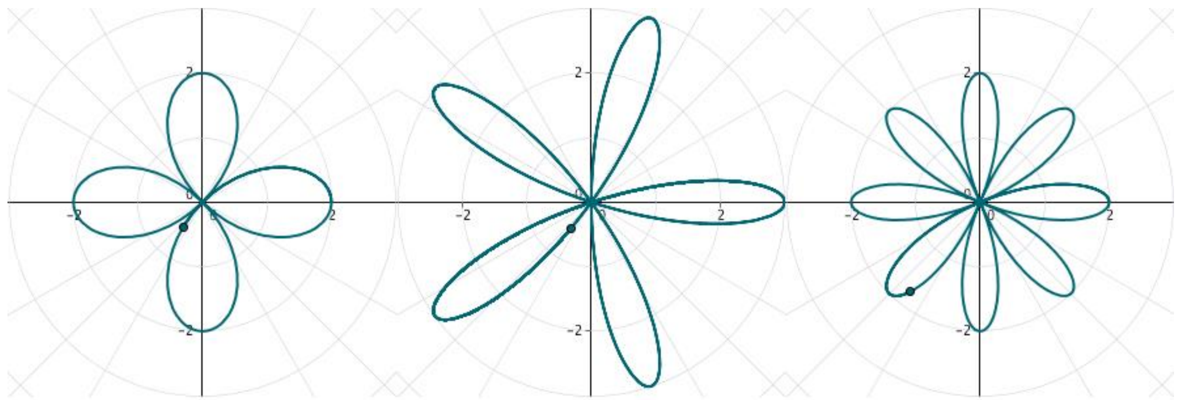
- 2. $r = a\sin 3\theta$



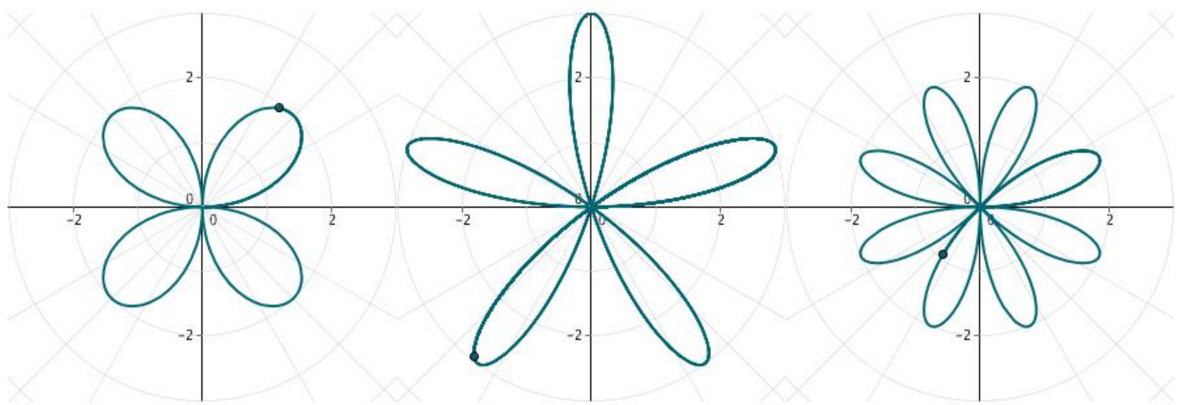
Examples :

- $3. r = a cos 3\theta$





$r = 2\cos 2\theta$ $r = 3\cos 5\theta$ $r = 2\cos 4\theta$



$r = 2\sin 2\theta$ $r = 3\sin 5\theta$ $r = 2\sin 4\theta$

Three leaved Rose: $r = a \sin 3\theta$, $a > 0$

1.Symmetry:

Curve is symmetric about the line $\theta = \frac{\pi}{2}$ passing through the pole.

2. Asymptote:

No asymptote since r is always finite for θ.

Three leaved Rose: r=a sin3θ

3. Pole & tangents at the pole:

The curve passes through the pole when $r = asin3\theta = 0$ i.e. when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$. The tangents to the curve at the pole are given by $\theta = 0, \theta = \frac{\pi}{3}, \theta = \frac{2\pi}{3}, \theta = \pi, \theta = \frac{4\pi}{3}, \theta = \frac{5\pi}{3}$.

4. Intersection:

Curve meets the line $\theta = \frac{\pi}{2}$ at $r = -a$.

Three leaved Rose: r=a sin3θ

5.Direction of the tangent:

$$tan\emptyset = r\frac{d\theta}{dr}.$$

$$\tan\emptyset = \frac{1}{3}\tan 3\theta.$$

At point(-a,π/2), $tan\emptyset$= $\frac{1}{3}\tan\frac{3\pi}{2} \to \infty, \emptyset = \frac{\pi}{2}$.Thus , the tangent is perpendicular to the line $\theta \ = \frac{\pi}{2}$.

6. Region:

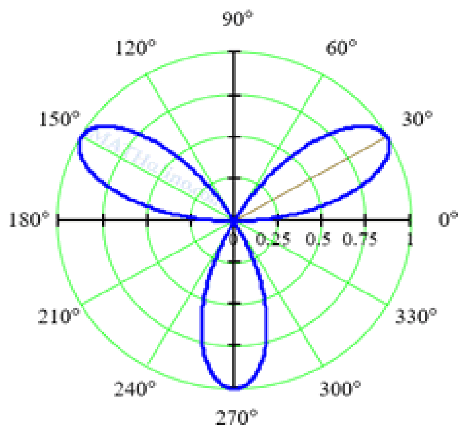
Maximum value of r is a. So the curve

Lies within the circle $r = a$. For n=3 the curve consists of **3 loops.**

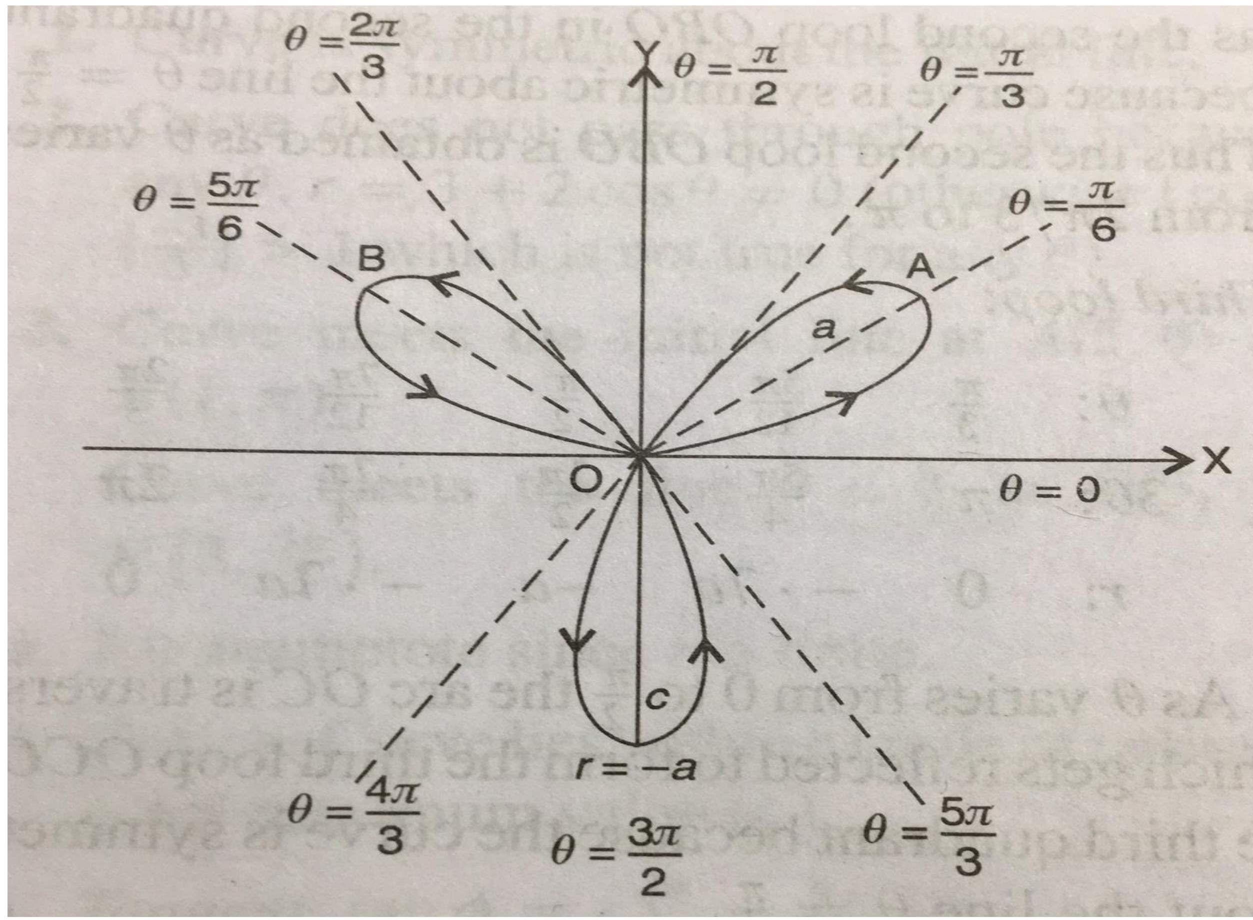
Three leaved Rose: r=a sin3θ

- **7. Variation of $\boldsymbol{r}$&$\boldsymbol{\theta}$**(period of the function is $T = \frac{2\pi}{3}$)

$\boldsymbol{\theta}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$
3θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0	.7a	a	.7a	0	−.7a	−a	−.7a	0



90°
120°
60°
150°
30°
180°
0°
0.25
0.5
0.75
1
210°
330°
240°
300°
270°



$\theta = \frac{2\pi}{3}$
Y
$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{3}$
$\theta = \frac{5\pi}{6}$
$\theta = \frac{\pi}{6}$
B
A
a
O
X
$\theta = 0$
c
$r = -a$
$\theta = \frac{4\pi}{3}$
$\theta = \frac{3\pi}{2}$
$\theta = \frac{5\pi}{3}$

Four leaved Rose : $r = acos2\theta, a > 0$

1.Symmetry:

a. Curve is symmetric about the initial line since $r(-\theta) = r(\theta)$

b. Curve is symmetric about the line $\theta = \frac{\pi}{2}$ since $r(\pi - \theta) = \text{a}\, cos\, 2(\pi - \theta) = acos2\theta = r(\theta)$

2. Asymptote:

No asymptote since r is always finite for θ

Four leaved Rose :$r = acos2\theta$

3. Pole & tangents at the pole:

The curve passes through the pole when $r = acos2\theta = 0$ i.e. when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. The tangents to the curve at the pole are given by $\theta = \frac{\pi}{4}, \theta = \frac{3\pi}{4}, \theta = \frac{5\pi}{4}, \theta = \frac{7\pi}{4}$.

4. Intersection:

Points of intersection of the curve with the initial line are (a,0) & (a, π) . The intersection with the line $\theta = \frac{\pi}{2}$ are (-a, $\frac{\pi}{2}$) &(-a, $\frac{3\pi}{2}$).

Four leaved Rose : $r = a\cos 2\theta$

5.Direction of the tangent:

$tan\emptyset = r\frac{d\theta}{dr}$.

$tan\emptyset = -\frac{1}{2}cot2\theta$.

At point (a,0) , $tan\emptyset = -\frac{1}{2}\cot(0) \rightarrow\rightarrow \infty, \emptyset = \frac{\pi}{2}$.

Thus, the tangent is perpendicular to the initial line.

6.Region:

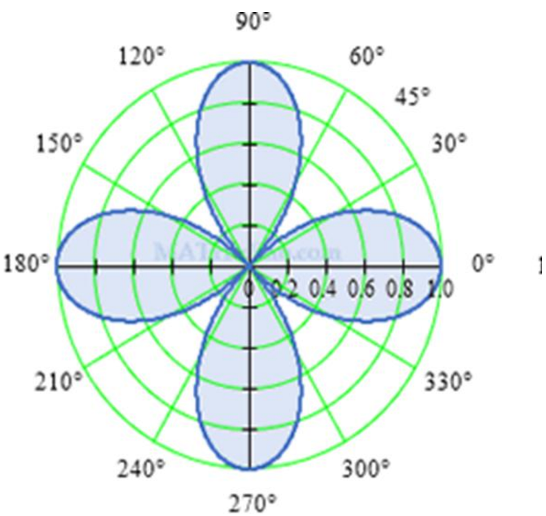
Maximum value of r is a. So the curve lies within the circle $r = a$. For n=2 the curve consists of **4 loops.**

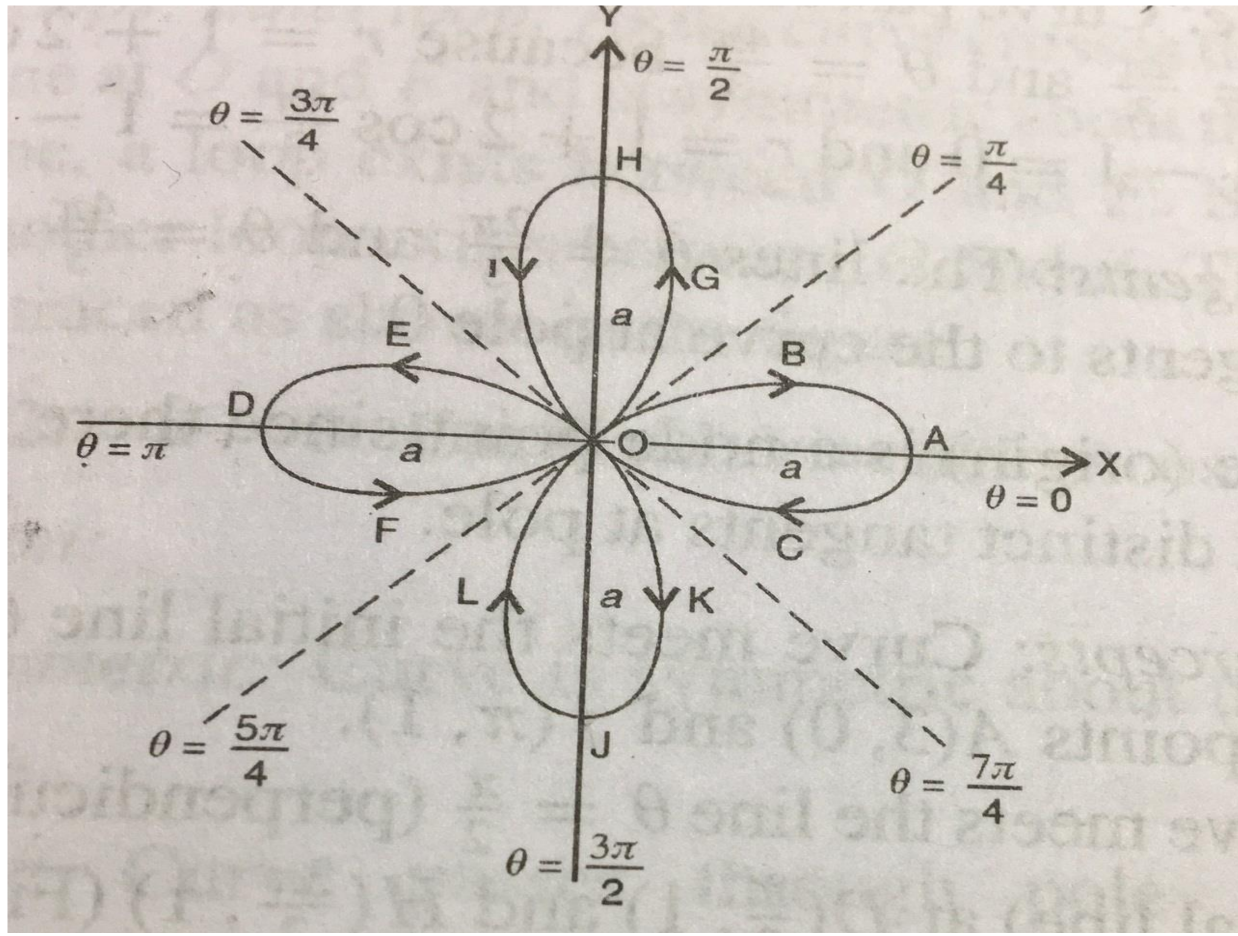
Four leaved Rose :$r = acos2\theta$

- **7. Variation of** $\boldsymbol{r}$&$\boldsymbol{\theta}$(period of the function is $T = \pi$)

$\boldsymbol{\theta}$	0	$\boldsymbol{\frac{\pi}{8}}$	$\boldsymbol{\frac{\pi}{4}}$	$\boldsymbol{\frac{3\pi}{8}}$	$\boldsymbol{\frac{\pi}{2}}$	$\boldsymbol{\frac{5\pi}{8}}$	$\boldsymbol{\frac{3\pi}{4}}$	$\boldsymbol{\frac{7\pi}{8}}$	$\boldsymbol{\pi}$
2θ	0	$\boldsymbol{\frac{\pi}{4}}$	$\boldsymbol{\frac{\pi}{2}}$	$\boldsymbol{\frac{3\pi}{4}}$	π	$\boldsymbol{\frac{5\pi}{4}}$	$\boldsymbol{\frac{3\pi}{2}}$	$\boldsymbol{\frac{7\pi}{4}}$	2π
r	a	.7a	0	−.7a	−a	−.7a	0	.7a	a

Four leaved Rose : $r = a\cos 2\theta$





Limacon

- Limacons are polar curves whose equations are of the form
- r = a+bsin $\boldsymbol{\theta}$, r= a−bsin $\boldsymbol{\theta}$,
- $r = a + bcos\boldsymbol{\theta}$, r= a−bcos $\boldsymbol{\theta}$ with a>0 ,b>0.

We get a Limacon with inner loop if a<b;

Cardiod if a=b;

Dimpled Limacon if a>b.

For the curve $r = a + bcos\boldsymbol{\theta}$ (Limacon of pascal) consider three cases a<b, a=b, a>b

Limacon r=a+bcosθ, $a > 0$ b> 0
Case 1: a<b

1.Symmetry: Curve is symmetric about the initial line $\theta = 0$.

2. Asymptote: No asymptote since r is always finite for θ.

3. Pole & tangents at the pole: It lies on the curve.

If r = 0 ,cos θ = $\frac{-a}{b}$ > -1. Therefore $\theta = cos^{-1}\left(\frac{-a}{b}\right)$ is the tangent at origin.

4.Intersection: Curve meets the initial line $\theta = 0$,

$\theta = \frac{\pi}{2}$, $\theta = \pi$ at (a+b,0), (a,$\frac{\pi}{2}$) & ($a -$ b, π) respectively .

Limacon r=a+bcosθ, $a > 0$ b> 0

5. Direction of the tangent: $tan\emptyset = r\frac{d\theta}{dr}$

$$tan\emptyset = \frac{a+bcos\theta}{-bsin\theta}$$

At point (a+b,0); $tan\emptyset \to \infty, \emptyset = \frac{\pi}{2}$.

Thus, the tangent is perpendicular to the initial line .

At point $(a, \frac{\pi}{2})$; $\emptyset = tan^{-1}\left(\frac{-a}{b}\right) = \pi - tan^{-1}\left(\frac{a}{b}\right)$

Thus the tangent makes an angle $\pi - tan^{-1}\left(\frac{a}{b}\right)$ with the line $\theta = \frac{\pi}{2}$.

At point (a – b, π) (2a, π); $tan\emptyset \to \infty, \emptyset = \frac{\pi}{2}$.

Thus the tangent is perpendicular to the line $\theta = \pi$.

Limacon r=a+bcosθ, $a > 0$ b> 0

6. Region: Since maximum value of cos θ= 1 , r ≤ a+b. Thus the entire curve lies inside the circle r = a+b.

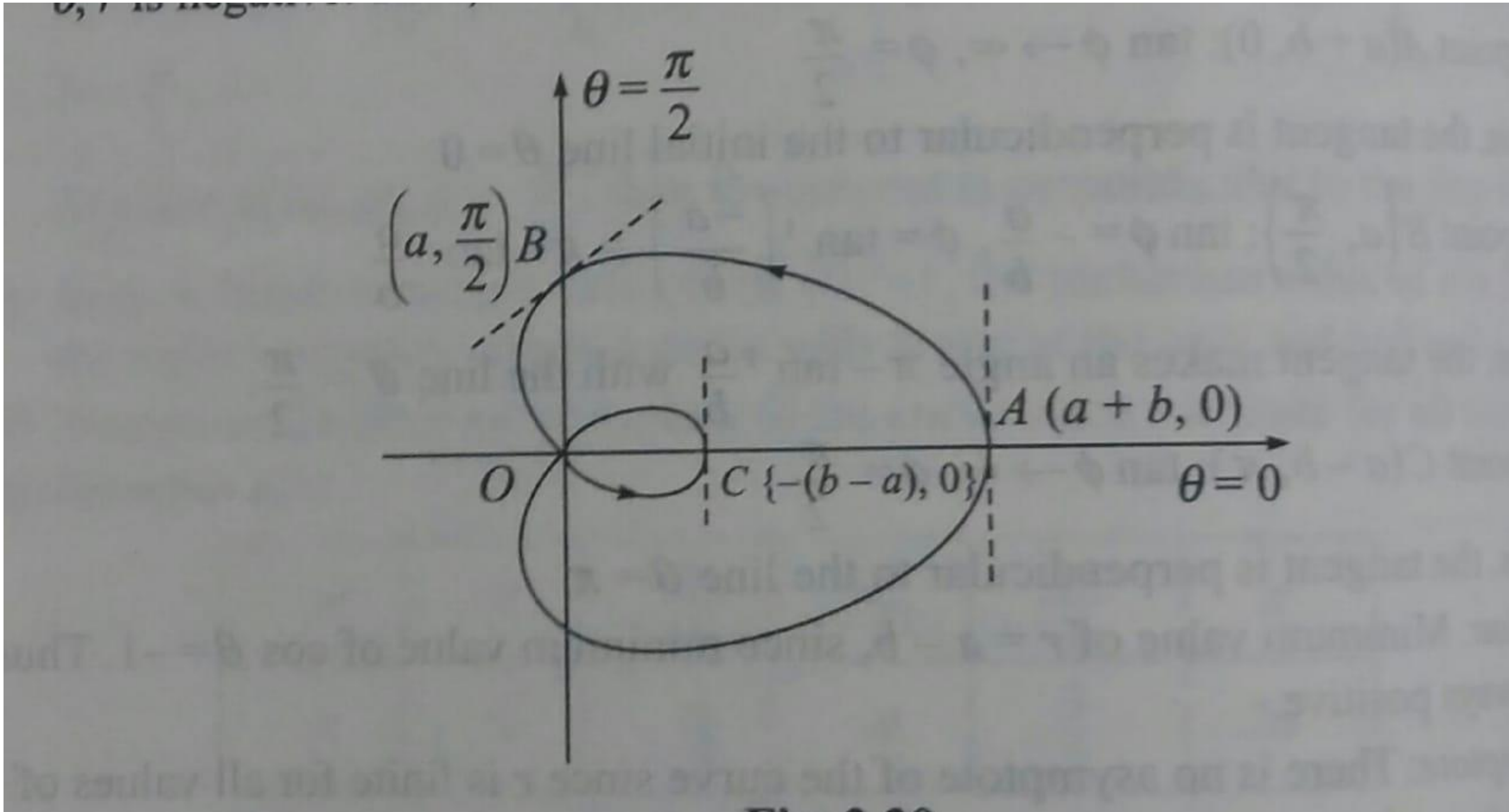
7.Variations of r and $\boldsymbol{\theta}$:

ϴ	0	π/3	π/2	2π/3	π
r	a+b	a+b/2	a	a–b/2	a–b

Limacon r=a+bcosθ, $a > 0$ b> 0

- Minimum value of r= a-b<0, thus r is negative for some values of θ.
- Here a-b<0.Therefore for some values of $\boldsymbol{\theta}$, r is negative. Thus a smaller loop exists between 0 and C.

Limacon r=a+bcosθ,a<b



Cardioid: $r = a(1 + cos\theta)$
Case2: a=b

1.Symmetry: Curve is symmetric about the initial line

2. Asymptote: No asymptote since r is always finite for θ.

3. Pole & tangents at the pole: The curve passes through the pole since $r = a(1 + cos\pi) = 0$ **.**The tangent to the curve at pole is $\theta = \pi$

4. Intersection: Curve meets the initial line at (2a,0) & (0, π) $and\ meets\ \theta = \frac{\pi}{2}$ at $(a, \frac{\pi}{2})$ & $(a, \frac{3\pi}{2})$

Cardioid: $r = a(1 + cos\theta)$

5. Direction of the tangent: $tan\emptyset = r\frac{d\theta}{dr}$

$tan\emptyset = r\frac{d\theta}{dr} = tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$. So $\emptyset = \frac{\pi}{2} + \frac{\theta}{2}$.

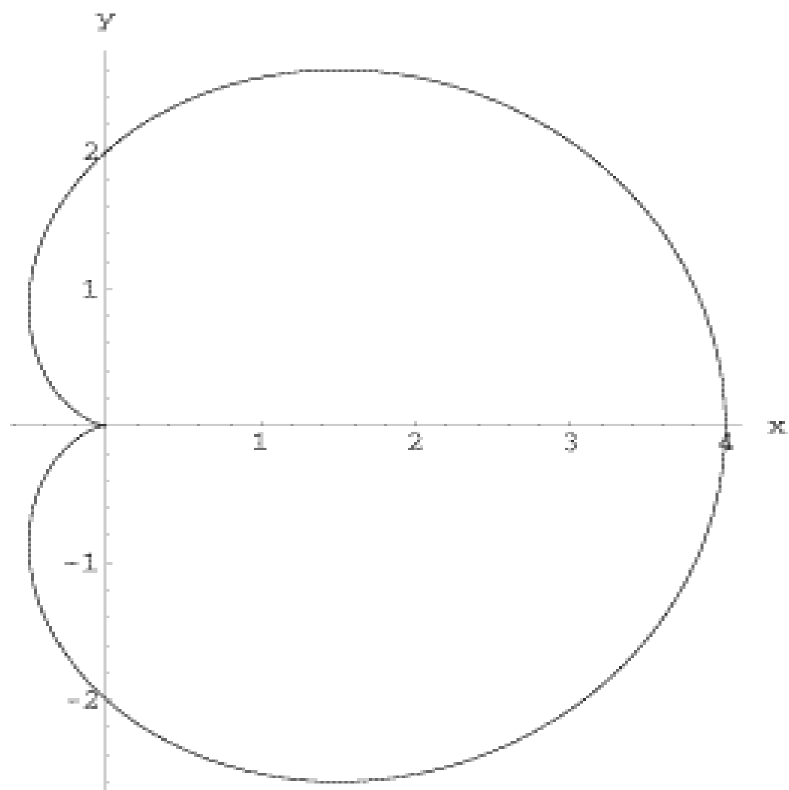
At point (2a, 0); Ø=π/2, thus the tangent is perpendicular to the line $\theta = 0$.

6. Region: Since maximum value of cos $\theta = +1$, the maximum value of r is 2a. Thus, the whole curve lies within a circle of radius 2a.

7. Variation of r&θ(period of the function is $T = 2\pi$)

ϴ	0	π/3	π/2	2π/3	π	4π/3	3π/2	11π/6	2π
r	2a	3a/2	a	a/2	0	a/2	a	3a/2	2a

Cardioid: $r = a(1 + cos\theta)$



Limacon r=a+bcos$\theta, a > 0$ b> 0
Case 3: a>b

1.Symmetry: Curve is symmetric about the initial line $\theta = 0$.

2. Asymptote: No asymptote since r is always finite for θ.

3. Pole : It does not lie on the curve. If r = 0 ,cos $\theta = \frac{-a}{b}$ < -1 which is not possible. Thus r $\neq$ 0 for any values of θ.

4. Intersection: Curve meets the initial line $\theta = 0$,
$\theta = \frac{\pi}{2}$, $\theta = \pi$ at (a+b,0), $(a, \frac{\pi}{2})$ & $(a -$ b, π) respectively.

Limacon r=a+bcosθ, $a > 0$ b> 0

5. Direction of the tangent: $tan\emptyset = r\frac{d\theta}{dr}$

$$tan\emptyset = \frac{a+bcos\theta}{-bsin\theta}$$

At point (a+b,0); $tan\emptyset \to \infty, \emptyset = \frac{\pi}{2}$.

Thus, the tangent is perpendicular to the initial line .

At point $(a, \frac{\pi}{2})$; $\emptyset = tan^{-1}\left(\frac{-a}{b}\right) = \pi - tan^{-1}\left(\frac{a}{b}\right)$

Thus the tangent makes an angle $\pi - tan^{-1}\left(\frac{a}{b}\right)$ with the line $\theta = \frac{\pi}{2}$.

At point (a – b, π) (2a, π); $tan\emptyset \to \infty, \emptyset = \frac{\pi}{2}$.

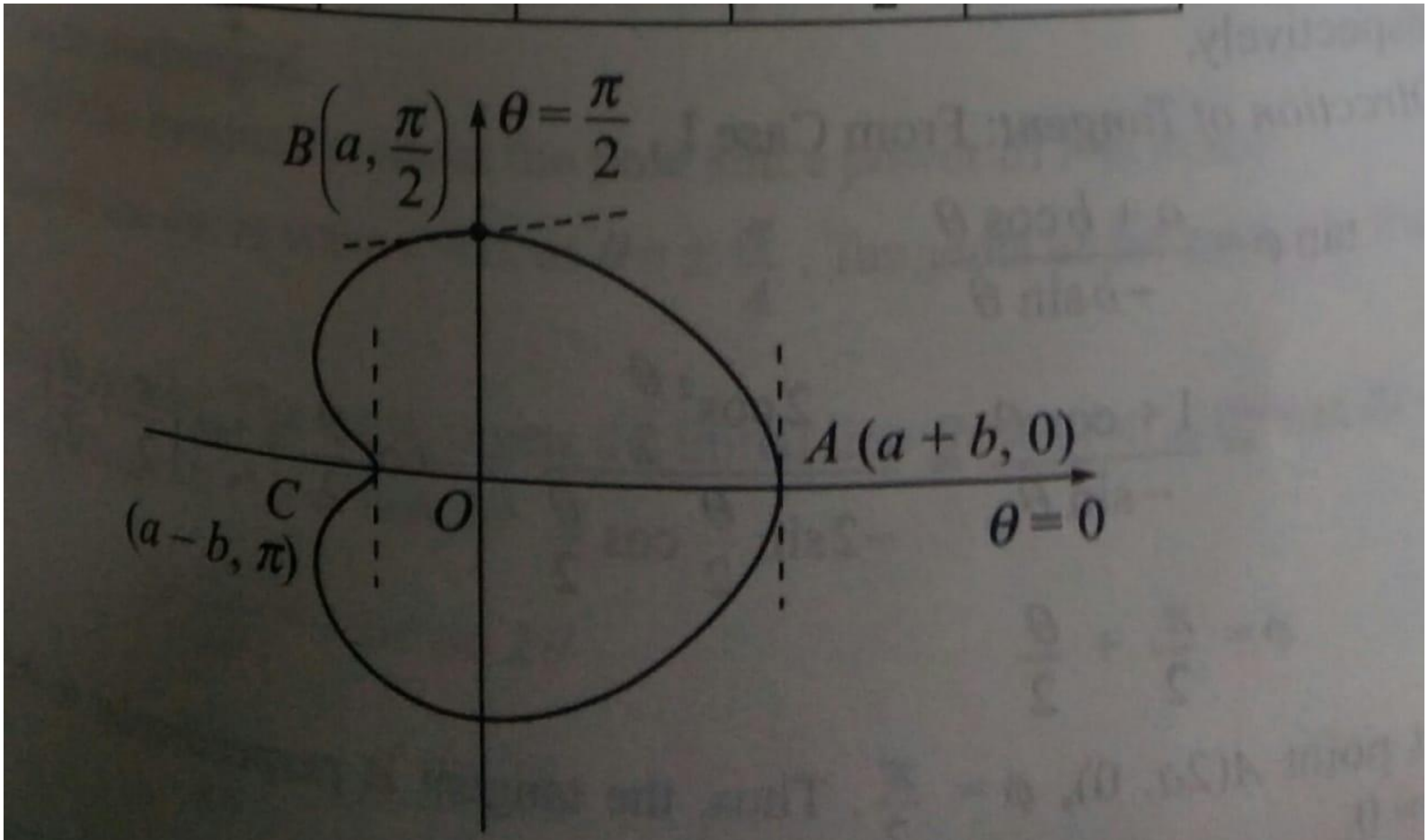
Thus the tangent is perpendicular to the line $\theta = \pi$.

Limacon $r=a+b\cos\theta, a > 0$ $b > 0$

- **6. Region:** Since minimum value of cos θ = -1 , the minimum value of r is a-b. Curve lies with in a circle of radius a+b since cos θ maximum value is 1.
- 7. Variation of $\boldsymbol{r}$&$\boldsymbol{\theta}$:
-

θ	0	π/3	π/2	2π/3	π
r	a+b	a+b/2	a	a−b/2	a−b

Limacon r=a+bcosθ, a>b



Cardioid: $r = a(1 - cos\theta)$
Case2: a=b

1.Symmetry: Curve is symmetric about the initial line

2. Asymptote: No asymptote since r is always finite for θ.

3. Pole & tangents at the pole: The curve passes through the pole since $r = a(1 - cos\theta) = 0$ **.**The tangent to the curve at pole is $\theta = 0$

4. Intersection: Curve meets the initial line at (0,0) & (2a, π) $and\ meets\ \theta = \frac{\pi}{2}$ at $(a, \frac{\pi}{2})$ & $(a, \frac{3\pi}{2})$

Cardioid: $r = a(1 - cos\theta)$

5. Direction of the tangent: $tan\emptyset = r\frac{d\theta}{dr}$

$tan\emptyset = r\frac{d\theta}{dr} = tan\frac{\theta}{2}$. So $\emptyset = \frac{\theta}{2}$.

At point(a,π/2); ∅=π/4, thus the tangent makes an angle π/4with the line $\theta = \frac{\pi}{2}$.

At point (2a, π); ∅=π/2, thus the tangent is perpendicular to the line $\theta = \pi$.

6. Region: Since minimum value of cos θ = -1 , the maximum value of r is 2a. Thus, the whole curve lies within a circle with centre at the pole and radius 2a.

7. Variation of r&θ(period of the function is $T = 2\pi$)

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{11}{6}$	2π
r	0	$\frac{a}{2}$	a	$\frac{3a}{2}$	2 a	$\frac{3a}{2}$	a	$\frac{a}{2}$	0

Cardioid: $r = a(1 - cos\theta)$