Network Analysis & Systems

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UE18EC201: Network Analysis & Systems

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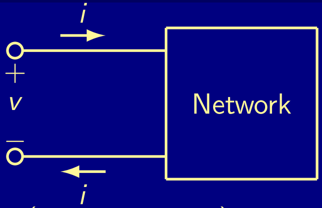
Network Analysis and Synthesis

Unit III: Network Theorems

Energy and Power (1)



- Energy absorbed by (or, delivered to) the network:

$$w = \int_{t_1}^{t_2} v(\tau)i(\tau)d\tau \text{ J}$$

- The rate at which energy is absorbed (i.e., instantaneous power) is

$$p(t) = \frac{dw}{dt} = v(t)i(t) \text{ W}$$

- Convention: For the voltage and current references shown, a positive p indicates a flow of energy into the network.

Energy and Power — Resistor (2)

- For a resistor, $v = Ri$, the energy absorbed is

$$w_R = R\int_{t_1}^{t_2} Ri^2(\tau) = \frac{1}{R}\int_{t_1}^{t_2} v^2(\tau)d\tau$$

and the instantaneous power

$$p_R(t) = Ri^2(t) = \frac{1}{R}v^2(t)$$

- For a sinusoidal current, $i(t) = I_m \sin\omega t$,

$$\begin{aligned} w_R &= RI_m^2\int_0^t sin^2\omega\tau d\tau = \frac{RI_m^2}{2}\left(t - \frac{\sin 2\omega t}{2\omega}\right) \\ p_R(t) &= RI_m^2 sin^2\omega t = \frac{RI_m^2}{2}(1-\cos 2\omega t) \end{aligned}$$

Energy and Power — Resistor (3)

- Both power and energy varies at twice the frequency of voltage or current.
- Power and energy are always positive.
- Energy increases with time.

Energy and Power — Inductor (4)

- For an inductor, $v = L\frac{di}{dt}$, the energy absorbed is

$$w_L = L\int_{t_1}^{t_2} i\, di = \frac{1}{2}L\left(i^2(t_2) - i^2(t_1)\right)$$

and the instantaneous power

$$p_L(t) = Li\frac{di}{dt}$$

- For a sinusoidal current, $i(t) = I_m \sin\omega t$, $t \geq 0$,

$$\begin{aligned} w_L(t) &= \frac{1}{2}LI_m^2\sin^2\omega t = \frac{1}{4}LI_m^2(1-\cos 2\omega t) \\ p_L(t) &= LI_m^2\,\omega\,\sin\omega t\,\cos\omega t = \frac{LI_m^2}{2}\,\omega\,\sin 2\omega t \end{aligned}$$

Energy and Power — Capacitor (5)

- For a capacitor, $i = C\frac{dv}{dt}$, the energy absorbed is

$$w_C = C\int_{t_1}^{t_2} v\,dv = \frac{1}{2}C\left(v^2(t_2) - v^2(t_1)\right)$$

and the instantaneous power

$$p_C(t) = Cv\frac{dv}{dt}$$

- For a sinusoidal voltage, $v(t) = V_m \sin\omega t,\ t \geq 0$,

$$\begin{aligned} w_C(t) &= \frac{1}{2}CV_m^2\sin^2\omega t = \frac{1}{4}CV_m^2(1-\cos 2\omega t) \\ p_C(t) &= CV_m^2\,\omega\,\sin\omega t\,\cos\omega t = \frac{CV_m^2}{2}\,\omega\,\sin 2\omega t \end{aligned}$$

Instantaneous Power (6)

Specifically, if $\mathbf{V} = V_m e^{j\theta_v} = V_m \angle \theta_v$ and $\mathbf{I} = I_m e^{j\theta_i} = I_m \angle \theta_i$, i.e.,

$$\begin{aligned} v(t) &= V_m \sin(\omega t + \theta_v) \\ i(t) &= I_m \sin(\omega t + \theta_i) \end{aligned}$$

then the instantaneous power is

$$\begin{aligned} p(t) &= v(t) i(t) \\ &= V_m I_m \sin(\omega t + \theta_v) \sin(\omega t + \theta_i) \\ &= \frac{V_m I_m}{2} \left(\cos(\theta_v - \theta_i) - \cos(2\omega t + \theta_v + \theta_i) \right) \end{aligned}$$

Average Power (7)

- For an inductor, if T is the period,

$$P_{av,L} = \frac{1}{nT} \int_0^{nT} \frac{1}{2} L\, I_m^2\, \omega\, \sin 2\omega t\, dt = 0$$

- Similarly, for a capacitor,

$$P_{av,C} = \frac{1}{nT} \int_0^{nT} \frac{1}{2} C\, V_m^2\, \omega\, \sin 2\omega t\, dt = 0$$

- For a resistor,

$$P_{av,R} = \frac{1}{nT} \int_0^{nT} \frac{1}{2} R\, I_m^2\, (1 - \cos 2\omega t)\, dt = \frac{1}{2} R\, I_m^2$$

Average Power (8)

Specifically, if $\mathbf{V} = V_m e^{j\theta_v} = V_m \angle \theta_v$ and $\mathbf{I} = I_m e^{j\theta_i} = I_m \angle \theta_i$, i.e.,

$$\begin{aligned} v(t) &= V_m \sin(\omega t + \theta_v) \\ i(t) &= I_m \sin(\omega t + \theta_i) \end{aligned}$$

then the instantaneous power is

$$\begin{aligned} p(t) &= v(t) i(t) \\ &= \frac{V_m I_m}{2} \left(\cos(\theta_v - \theta_i) - \cos(2\omega t + \theta_v + \theta_i) \right) \end{aligned}$$

Therefore, the average power

$$P_{av} = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$$

Average Energy (9)

- For each cycle, the energy supplied to a capacitor or an inductor is zero.
- For a resistor, the energy absorbed per cycle,

$$\Delta w_R = R \int_0^T i^2(t)dt = \frac{1}{2} R\, I_m^2\, T$$

- For a one-port network with n elements, if the energy stored or dissipated for element j is w_j, the total energy stored or dissipated is

$$w_t = w_1 + w_2 + \cdots + w_n = \int_{t_1}^{t_2} \sum_{k=1}^{n} v_k(t) i_k(t) dt = \int_{t_1}^{t_2} v^T(t) i(t) dt$$

- The total power is determined by differentiating this equation.

Effective or Root-Mean-Square Values (10)

- The effective value of a periodic current is defined as the constant value of current which will produce the same power in a resistor as is produced on the average by the periodic current.
- For a constant current I, $P = I^2 R$.
- In the sinusoidal steady-state, the average power is

$$P_{av,R} = \frac{1}{nT} \int_0^{nT} \frac{1}{2} R\, I_m^2 \,(1 - \cos 2\omega t)\, dt = \frac{1}{2} R\, I_m^2$$

- For $P = P_{av,R}$, $I = I_{\text{eff}} = \frac{I_m}{\sqrt{2}}$.
- Since $P = V^2/R$ and $V_m = I_m R$, $V_{\text{eff}} = \frac{V_m}{\sqrt{2}}$.
- Finally, $P_{av} = I_{\text{eff}}^2 R = I_{\text{eff}} V_{\text{eff}}$.

Effective or Root-Mean-Square Values (11)

For a non-sinusoidal but periodic current $i(t)$ of period T,

$$P_{av} = \frac{1}{T}\int_{t_0}^{t_0+T} Ri^2(t)dt$$

Therefore,

$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)dt} = I_{\text{rms}}$$

Similarly,

$$V_{\text{eff}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2(t)dt} = V_{\text{rms}}$$

Thus, $P_{av} = I_{\text{rms}}V_{\text{rms}}$.

- Thus, the more descriptive term is root-mean-square.

Examples (16)

- Example 1, p. 426, Van Valkenburg.

Examples (16)

- Example 1, p. 426, Van Valkenburg.
- Example 2, p. 426, Van Valkenburg.

Average Power and Complex Power (12)

- Recall: For a resistor,

$$P_{av} = \frac{1}{2} R\, I_m^2 = \frac{1}{2R}\, V_m^2 = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R} = I_{\text{rms}} V_{\text{rms}}$$

- Consider a one-port network with no independent sources.
- Let the driving-point impedance of the one-port network be

$$Z(j\omega) = R + j\, X(j\omega) = |Z| e^{j\theta}$$

- Then, $R = \text{Re } Z(j\omega) = |Z| \cos\theta$.
- Therefore,

$$P_{av} = I_{\text{rms}}^2 |Z| \cos\theta = V_{\text{rms}} I_{\text{rms}} \cos\theta$$

Power Factor (13)

$$P_{av} = I_{\text{rms}}^2 |Z| \cos\theta = V_{\text{rms}} I_{\text{rms}} \cos\theta$$

- The power factor (pf) is the cosine of the angle of the impedance.
- It is also the phase difference between voltage and current.
- The pf is said to be leading if current leads voltage; lagging otherwise.
- Let $V = V_m \sin\omega t$. If the pf is leading,

$$I = I_m \sin(\omega t + \theta)$$

and if the pf is lagging

$$I = I_m \sin(\omega t - \theta)$$

Average Power and Complex Power (14)

- Let the phasors representing the voltage and current be

$$\mathbf{V} = V_m e^{j\theta_v} = V_m \underline{/\theta_v}, \quad \mathbf{I} = I_m e^{j\theta_i} = I_m \underline{/\theta_i}$$

- The impedance is

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m}{I_m} \underline{/\theta_v - \theta_i} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \underline{/\theta_v - \theta_i} = |Z| \underline{/\theta}$$

- The complex power or phasor power or apparent power is

$$S = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i}$$

- Therefore,

$$P_{av} = \text{Re}\, \frac{1}{2}\mathbf{V}\mathbf{I}^*$$

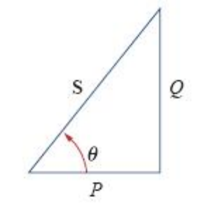
Average Power and Complex Power (15)

- The complex (or phasor) power or apparent power is

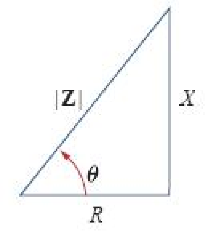
$$S = \frac{1}{2}\mathbf{VI}^* = P_{av} + jQ = V_{rms} I_{rms} \cos(\theta_v - \theta_i) + jV_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

  where Q is called the reactive power with unit volt-ampere reactive or var.
- If the reactance power is negative, power factor angle is negative, power factor is leading, and the reactance is capacitive in nature.
- If the reactance power is positive, power factor angle is positive, power factor is lagging, and the reactance is inductive in nature.

Summary (16)

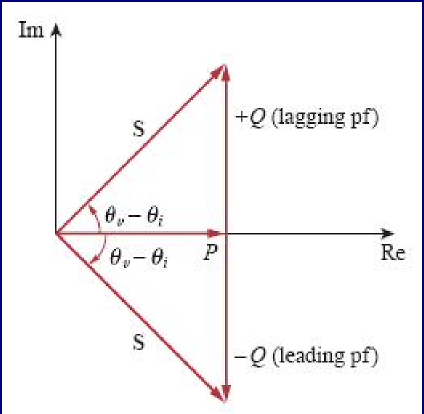


Power Triangle



Impedance Triangle

Summary (17)



Power Triangle

Examples (17)

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10^\circ)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20^\circ)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

- The pf angle is

Examples (17)

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10°)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20°)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

- The pf angle is $-20 - 10 = -30°$. Therefore pf is $\cos(-30) = \sqrt{3}/2$, leading.
- The apparent power is

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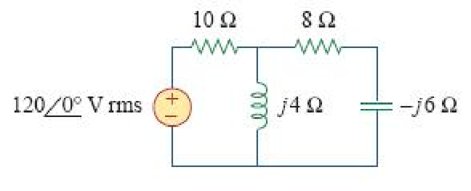
- The pf angle is $-20 - 10 = -30^\circ$. Therefore pf is $\cos(-30) = \sqrt{3}/2$, leading.
- The apparent power is $S = V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} = \frac{120}{\sqrt{2}} \frac{4}{\sqrt{2}} \underline{/-30^\circ}$ $= 240\underline{/-30^\circ}$ VA.
- The load impedance is

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\underline{/-20^\circ}}{4\underline{/10^\circ}} = 30\underline{/-30^\circ} = 25.98 - j15\Omega$$

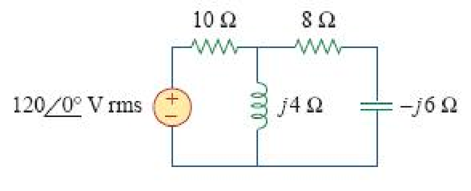
- Clearly $R = 25.98\ \Omega$ and $C = 212.2\mu$F as $X_C = -15$ $= -\frac{1}{\omega C} \implies C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi}$.

Examples (18)



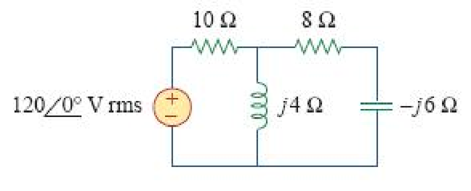
- The pf is

Examples (18)



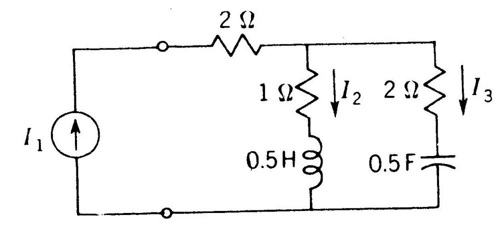
- The pf is 0.936, lagging.
- The average power supplied is

Examples (18)



- The pf is 0.936, lagging.
- The average power supplied is 1.062 kW.

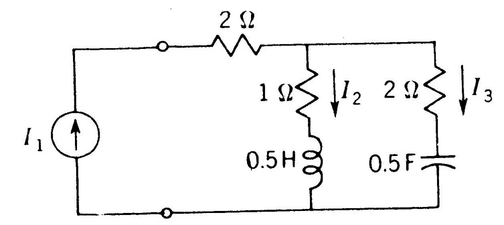
Examples (19)



$I_1 = 5\sqrt{2}\sin 2t.$

- $Z =$

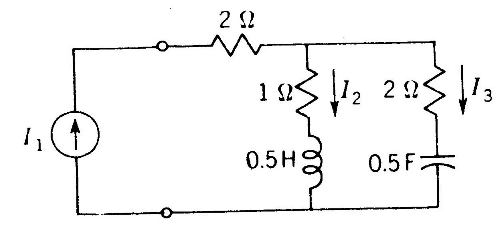
Examples (19)



$I_1 = 5\sqrt{2}\sin 2t$.

- $Z = 3 + j/3$.
- $P_{av} =$

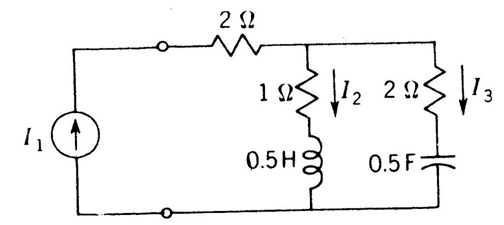
Examples (19)



$I_1 = 5\sqrt{2}\sin 2t.$

- $Z = 3 + j/3.$
- $P_{av} = 75$ W.
- $Q =$

Examples (19)



$I_1 = 5\sqrt{2}\sin 2t$.

- $Z = 3 + j/3$.
- $P_{av} = 75$ W.
- $Q = 25/3$ vars.

Optimising Power Transfer (19)

- Consider the Thévenin equivalent of a network, with V_1 V (rms) as the source and $Z_1 = R_1 + jX_1$ as the impedance.
- Let the impedance of Network B (i.e., load) be $Z_2 = R_2 + jX_2$.
- Thus, the phasor current is $\mathbf{I}_1 = \frac{V_1}{Z_1+Z_2}$.
- Thus, the power dissipated by the load is

$$P_2 = I_1^2 R_2 = \frac{V_1^2 R_2}{(R_1 + R_2)^2 + (X_1 + X_2)^2}$$

- What is the value of X_2 that will maximise P_2?

Optimising Power Transfer (20)

- Differentiating P_2 w.r.t. X_2,

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- Differentiating P_2 w.r.t. X_2,

$$\frac{dP_2}{dX_2} = \frac{-2(X_1 + X_2)V_1^2 R_2}{\left((R_1 + R_2)^2 + (X_1 + X_2)^2\right)^2}$$

- Thus, $X_2 = -X_1$, resulting in

$$P_2 = \frac{V_1^2 R_2}{(R_1 + R_2)^2}$$

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- Differentiating this w.r.t. R_2,

$$\frac{dP_2}{dR_2} = \frac{(R_1 + R_2)^2 V_1^2 - 2V_1^2 R_2(R_1 + R_2)}{\left((R_1 + R_2)^2 + (X_1 + X_2)^2\right)^2}$$

Optimising Power Transfer (21)

- Thus, $R_2 = R_1$, leading to

$$P_2 = \frac{V_1^2}{4R_1}$$

- In conclusion, maximum power is transferred when

$$Z_2 = Z_1^*$$

That is, the reactive components cancel and the real components are equal.

Maximum Power Transfer Theorem (22)

Theorem

The optimum load impedance is the complex conjugate of the Thévenin impedance.

Tellegen's Theorem (1)

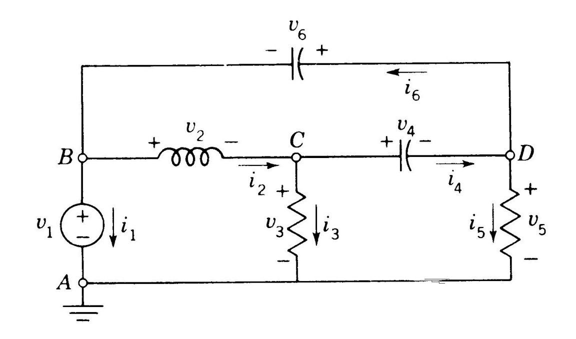
Theorem

For an arbitrary lumped network with b branches and n nodes, suppose that the kth branch is assigned a branch voltage v_k and a branch current i_k for $k = 1, 2, \ldots, b$ such that they satisfy arbitrarily picked associated reference directions. If the branch voltages $v_1, v_2, \ldots, v_b$ satisfy all the constraints imposed by KVL and if the branch currents $i_1, i_2, \ldots, i_b$ satisfy all the constraints imposed by KCL, then

$$\sum_{k=1}^{b} v_k i_k = 0$$

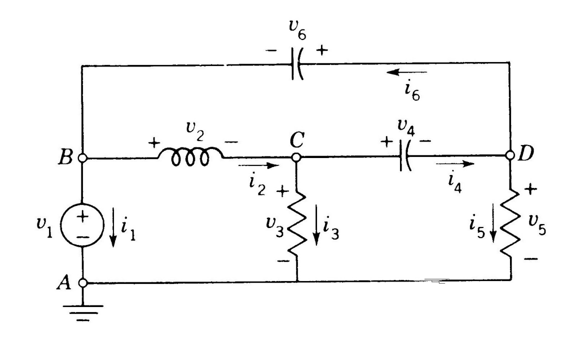
- Applicable to any lumped parameter network — linear or nonlinear, passive or active, time-varying or time-invariant.

Tellegen's Theorem (2)



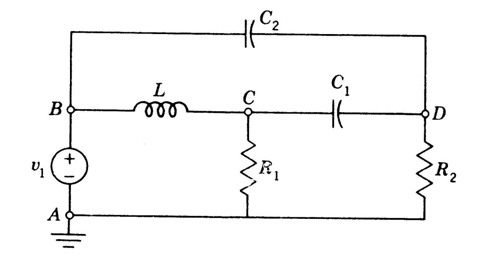
Item	1	2	3	4	5	6
v_k	4	2	2	3	−1	−5
i_k	2	2	4	−2	−6	4
$v_k i_k$	8	4	8	−6	6	−20

Tellegen's Theorem (3)



Item	1	2	3	4	5	6
v_k	4	2	2	3	-1	-5
i_k	-1	3	2	1	-1	2
$v_k i_k$	-4	6	4	3	1	-10

Tellegen's Theorem (4)



$$\begin{aligned}
\sum_{k=1}^{6} v_k i_k &= v_B i_{BA} + (v_B - v_C) i_{BC} + v_C i_{CA} + (v_C - v_D) i_{CD} \\
&\quad + v_D i_{DA} + (v_D - v_B) i_{DB} \\
&= v_B (i_{BA} + i_{BC} - i_{DB}) + v_C(-i_{BC} + i_{CA} + i_{CD}) \\
&\quad + v_D(-i_{CD} + i_{DA} + i_{DB}) \\
&= 0
\end{aligned}$$

Tellegen's Theorem — Implications (5)

- Given a network made of both active and passive components. Suppose that the network is anlysed using KCL and KVL to determine the actual voltages and currents. Then $v_k i_k$ is the instantaneous power in the kth branch. Tellegen's theorem states that the summation of all the instantananeous powers for the b branches must be equal to zero.

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- Equivalently, the energy must be supplied at a rate which is equal to the rate at which it is dissipated in resistors and stored in inductors and capacitors.
- Since, $\frac{1}{2}\sum_{k=1}^{b} \mathbf{V}_k \mathbf{I}^* = 0$, complex power is conserved.

Tellegen's Theorem — Implications (6)

- Let the network can be divided into two parts as shown. Then Tellegen's theorem states that the power delivered by the independent sources of the network is equal to the sum of the power absorbed in all of the other branches of the network.