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Monday

MODULE - 3

FOURIER SERIES REPRESENTATION OF PERIODIC SIGNA[LS]

$$x(t) \xrightarrow{\text{Transformation}} y(t)$$

time domain $\xrightarrow[\text{laplace}]{\text{fourier}}$ frequency domain

Signal :

continuous - periodic ⟶ fourier series

continuous - aperiodic ⟶ fourier transform

discrete - periodic ⟶ discrete time fourier series

discrete - aperiodic ⟶ discrete time fourier transform

3.2 : The response of LTI system to complex exponentials

The study of LTI systems to represent signals as linear combinations of basic signals should satisfy the following 2 properties.

1) The set of basic signal can be used to construct broad and useful set of signals

2) The response of an LTI system response should be simple in structure to provide us with a convinient representation for the system to any signal constructed as a linear combination of the basic signal

To show that complex exponentials are eigen functions of LTI systems, consider a system with impulse response $h(t)$

Let $x(t) = e^{st}$, s is a complex number

$$x(t) = e^{st} \longrightarrow \underset{\text{LTI}}{\boxed{h(t)}} \longrightarrow y(t) = ?$$

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$y(t) = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau$$

$$\boxed{y(t) = e^{st} H(s)}$$

$\Rightarrow$ e^{st} is the eigen function

$\Rightarrow$ $H(s)$ is the eigen value

* Hence we have shown that complex exponentials are eigens functions of LTI systems and $H(s)$ for eigen values of s is then the eigen value associated with the eigen function.

Consider a LTI system with input $x(n) = z^n$

Let $h(n)$ be the impulse response

Hence, $y(n) = h(n) * x(n)$

$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, z^{n-k}$$

$$y(n) = z^n \sum_{k=-\infty}^{\infty} h(k)\, z^{-k}$$

$$\boxed{y(n) = z^n H(z)}$$

where

$$H(z) = \sum_{k=-\infty}^{\infty} h(k)\, z^{-k}$$

$$x(n) = z^n,\ x(t) = e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(n) = z^n H(z),\ y(t) = e^{st} H(s)$$

$$\boxed{H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}}$$

$$\boxed{H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}$$

Example : Given $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$, find $y(t)$

Ans $x(t)$ is a linear combination of complex exponentials.

$$x(t) \left\{ \begin{array}{l} a_1 e^{s_1 t} \longrightarrow a_1 e^{s_1 t} H(s_1) \\ a_2 e^{s_2 t} \longrightarrow a_2 e^{s_2 t} H(s_2) \\ a_3 e^{s_3 t} \longrightarrow a_3 e^{s_3 t} H(s_3) \end{array} \right\} y(t)$$

$$y(t) = a_1 e^{s_1 t} H(s_1) + a_2 e^{s_2 t} H(s_2) + a_3 e^{s_3 t} H(s_3)$$

In general, if $x(t) = \sum_k a_k e^{s_k t}$

then $y(t) = \sum_k a_k H(s_k) e^{s_k t}$

similarly, if $x(n) = \sum_k a_k z_k^n$,

Then $y(n) = \sum_k a_k H(z_k) z_k^n$

for both continuous & discrete time signal if the input to an LTI system is represented as a linear combination function of complex exponential the output also can be represented as the same linear combination of complex exponential signals. Each coefficient in this representation of output is obtained as product of corresponding coefficients of input & system's eigen value.

Example 3.1 : $y(t) = x(t-3)$

if $x(t) = e^{j2t}$

Ans $x(t) = e^{j2t}$

$$x(t-3) = e^{j2(t-3)} = e^{j2t} e^{-j6}$$

(or)

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-s\tau} \delta(\tau-3) d\tau = e^{-s\tau}\Big|_{\tau=3} = e^{-3s}$$

$$H(s) = e^{-3s} \quad , \quad H(2j) = e^{-6j}$$

$$e^{st} \longrightarrow H(s) e^{st}$$

$$e^{j2t} \longrightarrow e^{j2t} H(2j)$$

where $\boxed{s = 2j}$

Example : $x(t) = \cos 4t + \cos 7t$

Ans $x(t) = \frac{1}{2} e^{4tj} + \frac{1}{2} e^{-j4t} + \frac{1}{2} e^{7jt} + \frac{1}{2} e^{-7jt}$

$$\frac{1}{2} e^{4tj} \longrightarrow \frac{1}{2} e^{4tj} H(4j)$$

$$\frac{1}{2} e^{-j4t} \longrightarrow \frac{1}{2} e^{-j4t} H(-4j)$$

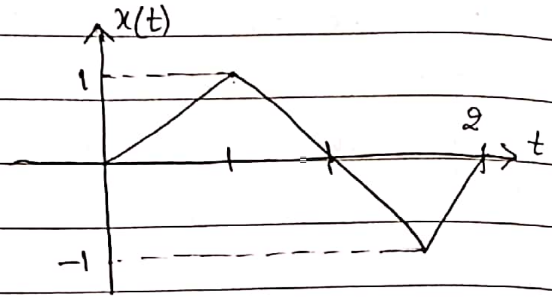
$$\frac{1}{2} e^{7jt} \longrightarrow \frac{1}{2} e^{7jt} H(7j)$$

$$\frac{1}{2} e^{-7jt} \longrightarrow \frac{1}{2} e^{7jt} H(-7j)$$

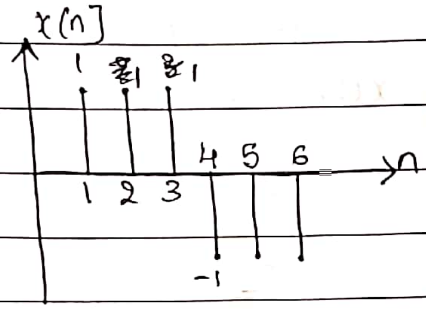
Deadline : 18/10/2019

Assignment - 1 Unit 1 & unit - 2

unit - 1 Q.NO : 3



Q.NO-4



unit - 2

QNO-5 :

$$h[n] = \{1, 1\}$$

$$h(t) = e^{-t}[u(t) - u(t-1)]$$

fourier Series representation of continuous time periodic signals

$$x(t) = x(t+T), \forall t$$

$$x(t) = \cos\omega_0 t, \quad x(t) = e^{j\omega_0 t}, \quad T_0 = \frac{2\pi}{\omega_0}$$

$$e^{jk\omega_0 t}, \text{ where } k = \text{integer}$$

set of harmonically related complex exponentials is given by

$$\phi_k(t) = e^{jk\omega_0 t} \quad \text{where } k = 0, \pm1, \pm2, \ldots$$

A linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \quad \text{—(1)}$$

It is also periodic with T

It is also periodic with T

$k=0 \Rightarrow$ constant

$k=1 \Rightarrow$ first harmonic

$k=2 \Rightarrow$ Second harmonic

The representation of a periodic signal in the form of equation (1) is refined to as fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$, where $a_k \rightarrow$ fourier coefficients.

1) consider a periodic signal $x(t)$ with fundamental frequency 2π and is expressed in the form

$$x(t) = \sum_{k=+3}^{-3} a_k e^{jk\omega_0 t}$$

Given $a_0 = 1$, $a_1 = a_{-1} = 1/4$

$a_2 = a_{-2} = 1/2$

$a_3 = a_{-3} = 1/3$

Ans

$$x(t) = a_{-3} e^{-j3(2\pi)t} + a_{-2} e^{-j2(2\pi)t} + a_{-1} e^{-j(2\pi)t} + a_0 + a_1 e^{j(2\pi)t} + a_2 e^{j2(2\pi)t} + a_3 e^{j3(2\pi)t}$$

$$x(t) = \frac{1}{3} e^{-j3(2\pi)t} + \frac{1}{2} e^{-j2(2\pi)t} + \frac{1}{4} e^{-j(2\pi)t} + 1 + \frac{1}{4} e^{j(2\pi)t} + \frac{1}{2} e^{j2(2\pi)t} + \frac{1}{3} e^{j3(2\pi)t}$$

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$$x(t) = \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left[e^{j4\pi t} + e^{-j4\pi t}\right] + \frac{1}{3}\left[e^{j6\pi t} + e^{-j6\pi t}\right] + 1$$

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$

$$\cancel{x^*(t) = x(t)}$$

* If $x(t)$ is real $\Rightarrow$ $a_K = a_{-K}$

$$x^*(t) = x(t)$$

$$x^*(t) = \sum_{K=-\infty}^{\infty} a_K^* e^{-jK\omega_0 t} = x(t) \quad (\because x(t) \text{ is real})$$

Replacing K with $-K$

$$x(t) = \sum_{K=-\infty}^{\infty} a_{-K}^* e^{jK\omega_0 t} \text{ ——— } (2)$$

comparing (1) & (2)

$$\boxed{a_K = a_{-K}^*}$$

If a_K is real

then $\boxed{a_K = a_{-K}}$.

Alternative forms of fourier series

$$x(t) = a_0 + \sum_{K=1}^{\infty}\left[a_K e^{jK\omega_0 t} + a_{-K} e^{-jK\omega_0 t}\right]$$

$$= a_0 + \sum_{K=1}^{\infty}\left[a_K e^{jK\omega_0 t} + a_{-K}^* e^{-jK\omega_0 t}\right]$$

$$= a_0 + 2\,\mathrm{Re}\left[\sum_{K=1}^{\infty} a_K e^{jK\omega_0 t}\right]$$

(1) if $a_K = A_K e^{j\theta_K}$

$$x(t) = a_0 + 2\sum_{K=1}^{\infty} A_K \cos(K\omega_0 t + \theta_K)$$

(ii) if $a_K = B_K + jC_K$

$$x(t) = a_0 + 2\sum_{K=1}^{\infty}\{B_K \cos K\omega_0 t - C_K \sin K\omega_0 t\}$$

Magnitude spectrum — A_K Vs ω

Phase spectrum — θ_K Vs ω

to find a_K :

Multiplying both sides of eq^n ① by $e^{-jn\omega_0 t}$, we obtain

$$x(t)e^{-jn\omega_0 t} = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t} e^{-jn\omega_0 t}$$

Integrating both sides from 0 to $T = 2\pi/\omega_0$, we get

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \int_0^T \sum_{K=-\infty}^{\infty} a_K e^{j(K-n)\omega_0 t}\,dt$$

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \sum_{K=-\infty}^{\infty} a_K \int_0^T e^{j(K-n)\omega_0 t}\,dt$$

Case-I

for $K \neq n$

$$\int_0^T e^{j(K-n)\omega_0 t}\,dt = \int_0^T \cos(K-n)\omega_0 t\,dt + j\int_0^T \sin(K-n)\omega_0 t\,dt = 0$$

Case-II

for $K = n$

$$\int_0^T dt = T$$

$$\int_0^T x(t)\,e^{-jn\omega_0 t} = \sum_{K=-\infty}^{\infty} a_K \begin{cases} 0 & K \neq n \\ T & K = n \end{cases}$$

$$\int_0^T x(t)\,e^{-jn\omega_0 t} = a_n T$$

(or) $a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$

$$\boxed{a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt}$$

$$\boxed{a_k = \frac{1}{T}\int_{<T>} x(t)\, e^{-jk\omega_0 t}\, dt} \longrightarrow \text{Analysis equation}$$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}} \longrightarrow \text{Synthesis equation}$$

The set of coefficients $\{a_k\}$ ⇒ fourier series coefficients (or) spectral coefficients of $x(t)$

$k = 0$

$$a_0 = \frac{1}{T}\int_{<T>} x(t)\, dt$$

Ex 3.3: $x(t) = \sin\omega_0 t$

Ans $x(t) = \frac{e^{j\omega_0 t}}{2j} - \frac{e^{-j\omega_0 t}}{2j} = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$

$$= \frac{1}{2j} e^{jk\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t k}$$

comparing with equation ①

$$a_1 = \frac{1}{2j} \qquad a_{-1} = \frac{-1}{2j}$$

magnitude spectrum is even and phase spectrum is odd

Ex : 3.4 $x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$

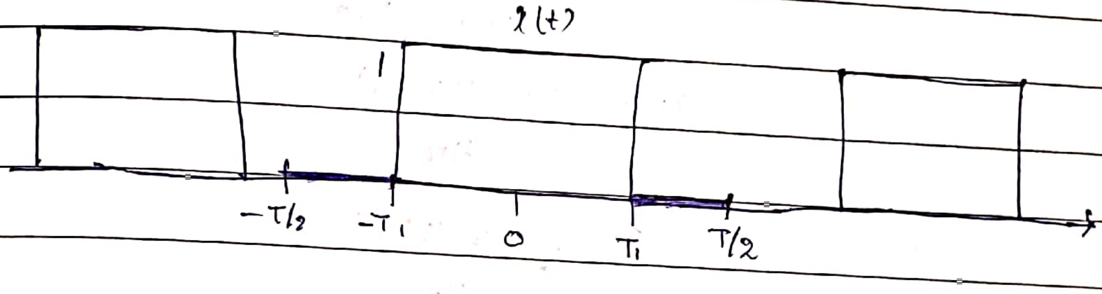
Ans $x(t) = 1 + \frac{e^{j\omega_0 t}}{2j} - \frac{e^{-j\omega_0 t}}{2j} + 2\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] +$

Ex : 3.5 :

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

★ It is periodic

Ans



$$a_K = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} 1 \cdot e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{T}\left[\frac{e^{-jK\omega_0 t}}{-jK\omega_0}\right]_{-T_1}^{T_1} = \frac{-1}{+jK\omega_0 T} e^{-jK\omega_0 T_1} + \frac{1}{jK\omega_0 T} e^{jK\omega_0 T_1}$$

$$a_K = -\frac{1}{jK\omega_0 T}\left[e^{-jK\omega_0 T_1} - e^{jK\omega_0 T_1}\right]$$

$$a_K = \frac{+2}{K\omega_0 T}\left[\frac{e^{jK\omega_0 T_1} - e^{-jK\omega_0 T_1}}{2j}\right]$$

$$a_K = \frac{2\sin(K\omega_0 T_1)}{K\omega_0 T}$$

The waveform for this a_K is similar to the sinc function.

Since $\omega_0 T = 2\pi$

$$a_K = \frac{\sin(K\omega_0 T_1)}{K\pi}, \text{ where } K \neq 0$$

★★ $\boxed{\text{Rect} \Leftrightarrow \text{Sinc}}$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = = \frac{2T_1}{T}$$

condition $T = 4T_1$

$\cancel{a_k} = \rightarrow \quad \omega_0 T_1 = \frac{2\pi T_1}{T}$

$$\omega_0 T_1 = \frac{2\pi T_1}{4T_1}$$

$$\omega_0 T_1 = \pi/2$$

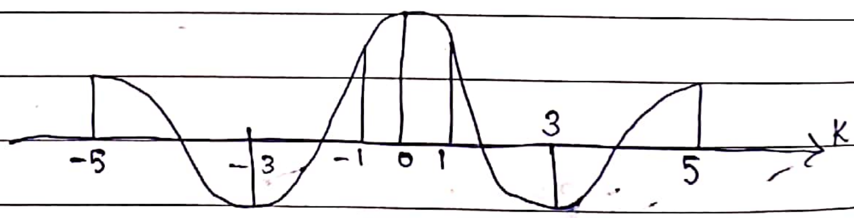
$$a_K = \frac{\sin(K\omega_0 T_1)}{K\pi}$$

$$a_K = \frac{\sin(K\pi/2)}{K\pi}$$

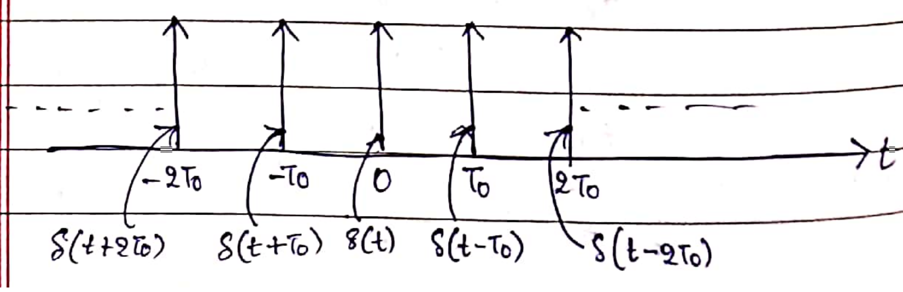
$$a_0 = \frac{1}{2}$$

$a_1 = \frac{1}{\pi} \quad a_2 = 0 \quad a_3 = \frac{1}{3\pi}, a_4 \ldots$

$a_{-1} = \frac{1}{\pi} \quad a_{-2} = 0 \quad a_{-3} = \frac{-1}{3\pi}, a_4 \ldots$



consider the periodic impulse ~~string~~ train $\delta_{T_0}(t)$ determine the complex fourier series representation



$$\boxed{\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)}$$

where $k = 0, \pm1, \pm2, \ldots$

$$\delta_{T_0}(t) = \sum a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt$$

$\delta(t)$ $\phi(t)$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) e^{-jk\omega_0 t} dt$$

Sampling property

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt = \phi(t)\Big|_{t=0}$$

$$\therefore\ a_k = \frac{1}{T_0}$$

3.4: Convergence of fourier series

Dirichlet condition

condition 1: over any period, $x(t)$ must be absolutely integrable that is,

$$\int_{\langle T\rangle} |x(t)|\,dt < \infty \rightarrow a_k$$

condition 2: In any finite interval of time $x(t)$ is of bounded variation that is there are no more than a finite number of maxima & minimum during any single period of a signal.

condition 3: In any finite interval of time there are only a finite number of discontinuities

Properties of fourier series

1) **linearity**: let $x(t) \xrightarrow{F.S} a_k$; $y(t) \xrightarrow{F.S} b_k$

$$z(t) = ax(t) + by(t) \xleftrightarrow{F.S} ?$$

Proof: we have $a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$

Let $z(t) \xleftrightarrow{F.S} z_k$

then $z_k = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{\langle T\rangle} [ax(t)+by(t)]\, e^{-jk\omega_0 t}\, dt$

$$= a\frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt + b \times \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$z_k = aa_k + bb_k$$

$$\Rightarrow ax(t) + by(t) \xleftrightarrow{F.S} aa_k + bb_k$$

2) **Time shifting**:

If $x(t) \xleftrightarrow{F.S} a_k$

then $y(t) = x(t-t_0) \xleftrightarrow{F.S} b_k$

Proof: let $y(t) \xleftrightarrow{F.S} b_k$

then $b_k = \frac{1}{T}\int_{\langle T\rangle} y(t)\, e^{-jk\omega_0 t}\, dt$

$$\Rightarrow \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

$$t - t_0 = \lambda \Rightarrow dt = d\lambda$$

$$t = t_0 + \lambda$$

$$= \frac{1}{T}\int_T x(\lambda)\, e^{-jk\omega_0 \lambda}\, e^{-jk\omega_0 t_0}\, d\lambda$$

$$= e^{-jk\omega_0 t_0}\, a_k$$

3) **Time revereal :**

If $x(t) \xrightarrow{F.S} a_K$

then $y(t) = x(-t) \xleftrightarrow{F.S} b_K = a_{-K}$

proof

$$x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jk\omega_0 t}$$

$$x(-t) = \sum_{K=-\infty}^{\infty} a_K e^{-jk\omega_0 t}$$

let $K = -m$

$$\Rightarrow x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t}$$

$$\boxed{x(-t) \xleftrightarrow{F.S} a_{-K}}$$

4) **Time scaling:**

If $x(t) \xleftrightarrow{F.S} a_K$

then $y(t) = x(\alpha t) \xleftrightarrow{F.S} b_K$

proof

$$a_K = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} \cdot dt$$

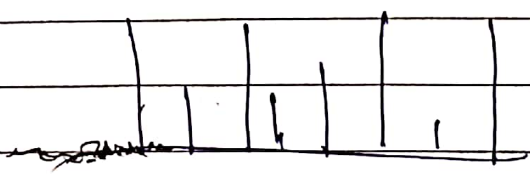
$$b_K = \frac{1}{(T/\alpha)} \int_{(T/\alpha)} x(\alpha t) e^{-jk\alpha\omega_0 t} dt$$

$$\alpha t = \lambda \Rightarrow dt = d\lambda/\alpha$$

$$b_K = \frac{\alpha}{T}\int_{\langle T/\alpha \rangle} x(\lambda) e^{-jk\omega_0\lambda} \frac{d\lambda}{\alpha}.$$

$$b_K = \frac{1}{T}\int_T x(\lambda) e^{-jk\omega_0\lambda} d\lambda$$

$$b_K = a_K$$



5) Time-differentiation

if $x(t) \xleftrightarrow{F.S} a_k$

then $y(t) = \frac{dx(t)}{dt} \xleftrightarrow{F.S} b_k = ?$

Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} a_k\, jk\omega_0\, e^{jk\omega_0 t}$$

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{F.S} jk\omega_0 a_k}$$

6) Multiplication:

If $x(t) \xrightarrow{F.S} a_k$

$y(t) \xrightarrow{F.S} b_k$

then $z(t) = x(t)\,y(t) \xleftrightarrow{F.S} C_k = ?$

Proof

$$C_k = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t)\, y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T \left[\sum_{l=-\infty}^{\infty} a_l\, e^{+jl\omega_0 t}\right] y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\sum_{l=-\infty}^{\infty} a_l \int_T y(t)\, e^{-j(k-l)\omega_0 t}\, dt$$

$$= \sum_{l}^{\infty} a_l\, b_{k-l}$$

$$\boxed{C_k = a_k * b_k}$$

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7) conjugation:

if $x(t) \xleftrightarrow{F.S} a_k$

then $x^*(t) \xleftrightarrow{F.S} a^*_{-k}$

Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a^*_k e^{-jk\omega_0 t}$$

Replacing ~~by~~ k with -k,

$$x^*(t) = \sum_{k=-\infty}^{\infty} a^*_{-k} e^{+jk\omega_0 t}$$

$$x^*(t) \xleftrightarrow{F.S} a^*_{-k}$$

further: if $x(t)$ is real,

then $x^*(t) = x(t)$

$\Rightarrow a_k = a^*_{-k}$

(or) $\boxed{a^*_k = a_{-k}}$

Parseval's Relation:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

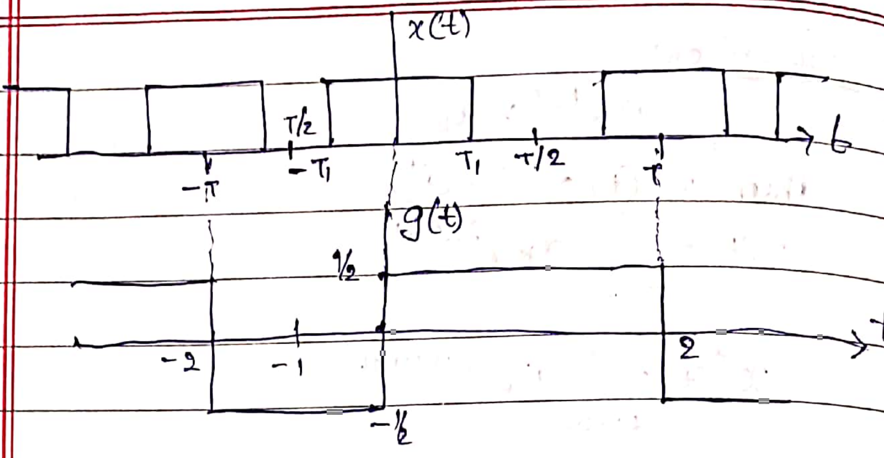
proof L.H.S

$$\frac{1}{T}\int_T x(t)\, x^*(t)\, dt = \frac{1}{T}\int_T x(t) \left[\sum_{k=-\infty}^{\infty} a^*_k e^{-jk\omega_0 t}\right] dt$$

$$= \sum_{k=-\infty}^{\infty} a^*_k \int_T x(t) e^{-jk\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a^*_k a_k = \sum_{k=-\infty}^{\infty} |a_k|^2$$

= RHS

8) Ex: 3.6 : find the FS representation of $g(t)$ knowing F.S representation of $x(t)$



referring to $x(t)$; $T = 4$ and $T_1 = 1$

$$x(t) \longrightarrow x(t-T_1) - 1/2 \longrightarrow g(t)$$

$$\Rightarrow g(t) = x(t-1) - 1/2$$

$$x(t) \xleftrightarrow{F.S} a_k$$

$$x(t-t_0) \xleftrightarrow{F.S} e^{-jk\omega_0 t_0} a_k$$

$$x(t-1) \xleftrightarrow{F.S} e^{-jk\omega_0} a_k = b_k$$

$$-1/2 \xleftrightarrow{F.S} C_k = -1/2,\ k=0$$

Let $g(t) \xleftrightarrow{F.S} d_k$

$$d_k = \begin{cases} e^{-jk\omega_0} a_k & , k \neq 0 \\ a_0 - 1/2 & ; k = 0 \end{cases}$$

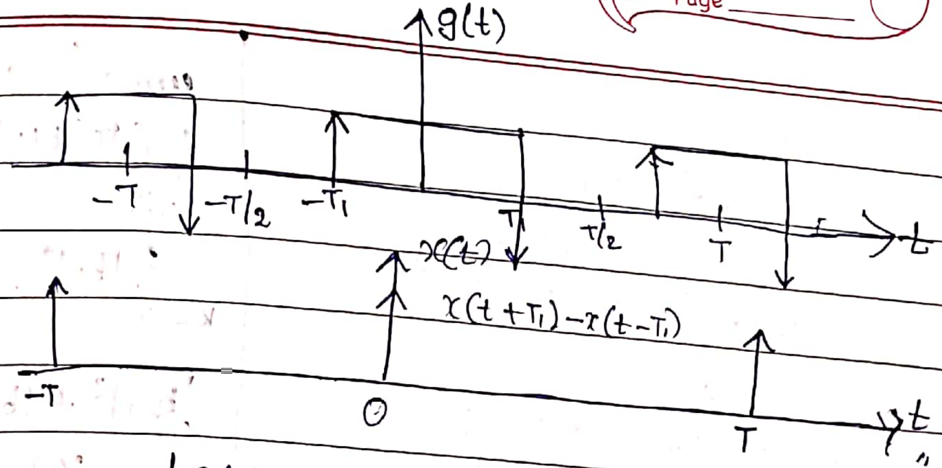
$$d_k = \begin{cases} e^{-jk\pi/2} \dfrac{\sin k\pi/2}{k\pi} & ; k \neq 0 \\ 0 & , k = 0. \end{cases}$$

2) Ex 3.7 : calculate/determine the fourier series response $x(t)$ with a time period $T = 4$ and $\omega_0 = \frac{\pi}{2}$

Ex 3.8 : $g(t) = \begin{cases} 1 & , |t| < T_1 \\ 0 & , T_1 < |t| < T/2 \end{cases}$

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Ans



$$\frac{dg(t)}{dt} = x(t+T_1) - x(t-T_1)$$

$$\frac{dg(t)}{dt} \xleftrightarrow{F.S} \frac{e^{jk\omega_0 T_1}}{T} - \frac{e^{-jk\omega_0 T_1}}{T} = \frac{2\sin(k\omega_0 T_1)}{2jT}$$

Ex 3.9 suppose we are given the following facts about signal $x(t)$

1) $x(t)$ is a real signal

2) $x(t)$ is periodic with $T=4$ and it has fourier coefficient a_k, 3) $a_k = 0$ for $|k|>1$

4) The signal with fourier coefficients $b_k = e^{-jk\pi/2} a_{-k}$ is odd

5) $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$

Ans $x(t) \xleftrightarrow{F.S} a_k \longrightarrow$ using (2)

$a_k = 0$; $|k|>1 \Rightarrow a_{-1}, a_0, a_1 \neq 0 \Rightarrow$ using (3)

$x(t) = \sum a_k e^{jk\omega_0 t}$

$x(t) = \sum_{k=-1}^{1} a_k e^{jk\omega_0 t} = a_0 + a_1 e^{-jk\frac{\pi}{2}t} + a_1 e^{jk\pi/2\, t}$ using (3)

$x(t) = x^*(t)$

$a_k = a^*_{-k} \Rightarrow a^*_k = a_{-k}$ — using (1)

$x(t) = a_0 + a_1 e^{j\phi\pi/2t} + a_1^* e^{-jk\pi/2t}$

$= a_0 + a_1 e^{jk\pi/2t} + (a_1 e^{jk\pi/2t})^*$

$x(t) = a_0 + 2Re(a_1 e^{jk\pi/2t})$

using (4)

$a_k \xleftrightarrow{F.S} x(t)$

$a_{-k} \xleftrightarrow{F.S} x(-t)$

$[x-(t-1)] \longrightarrow e^{-jk\pi/2} a_{-k} = b_k$

$\Downarrow$

$x(-t+1)$

$x(-t+1) \xleftrightarrow{F.S} b_k$

$\Downarrow$

$b_0 = 0$

$x(-t+1)$ is real & b_k is odd

$b_0 = 0 \Rightarrow b_{-1} = -b_1$

using (3) & Parseval's Relation

$\frac{1}{4}\int_{<4>} |x(-t+1)|^2 dt = 1/2$

$\sum_{k=-\infty}^{\infty} |b_k|^2 = |b_{-1}|^2 + |b_1|^2$

$|b_{-1}|^2 + |b_1|^2 = 1/2$

$i(t)$ – real

$\Rightarrow x(-t+1)$ is real

$x(-t+1)$ is R&O $\rightarrow b_k$ is O & imaginary

" R & E $\rightarrow$ "

$|-b_1|^2 + |b_1|^2 = 1/2$

$|b_1| = 1/2$

Discrete-Time fourier series

$x(n) = x(n+N), \forall n$

$\underbrace{\phi_k[n] = e^{jk\omega_0 n}}_{\text{Basis}} \longrightarrow$ set of harmonically related signals

$x(n) = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n}$

for DT periodic signals, the Summation varies as N

$\Rightarrow x(n) = \sum_{k=<N>} a_k e^{jk\omega_0 n}$

$x(n) = \sum_{<N>} a_k e^{jk\omega_0 n}$

$k = 0, 1, \ldots\ldots N-1$

$\longrightarrow$ Fourier series representation of DT periodic signals

a_k = fourier coefficients

$x(n) = \sum_{<N>} a_k e^{jk\omega_0 n}$	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
$a_k = \frac{1}{N} \sum_{n=<N>} a_k e^{-jk\omega_0 n}$	$a_k = \frac{1}{T}\int_{<T>} x(t) e^{-jk\omega_0 t} dt$

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check for the periodicity of fourier series a Coefficients

$$a_K = \frac{1}{N} \sum_{<N>} x(n) e^{-jK\omega_0 n}$$

$$a_{K+N} = \frac{1}{N} \sum_{<N>} x(n) e^{-j(K+N)\omega_0 n}$$

$$= \frac{1}{N} \sum_{<N>} x(n) e^{-jK\omega_0 n} \cancel{e^{-jKN\frac{2\pi}{N} n}}^{1}$$

$$= a_K$$

Q) determine the DT fourier series for the sequence

(i) $x(n) = \cos\left(\frac{\pi}{4} n\right)$

Ans (i) Periodic $\frac{\omega_0}{2\pi} = \frac{\pi/4}{2\pi} = \frac{1}{8} = \frac{m}{N}$

$$\boxed{N=8}$$

(ii) $x(n) = \frac{1}{2} e^{j\frac{\pi}{4} n} + \frac{1}{2} e^{-j\frac{\pi}{4} n}$

$$= \underbrace{a_1 e^{j\omega_0 n}}_{K=1} + \underbrace{a_{-1} e^{-j\omega_0 n}}_{K=-1}$$

$$x(n) = \sum_{N=0}^{7} a_K e^{jK\omega_0 n}$$

$$a_K = a_{K+N}$$

$$a_1 = a_{1+N} - a_1 + 8 = a_9$$

$$a_{-1} = a_{-1+8} = a_7$$

$$= a_1 e^{j\omega_0 n} + a_7 e^{j7\omega_0 n}$$

DTFS representation for $x(n)$

$$x(n) = \sum_{K=<N>} a_K e^{jK\omega_0 n}$$

$K = -3, -2, \ldots, 4$

$a_1 = 1/2 \quad a_1 = 1/2$

$$x(n) = \sum_{K=<N>} a_1 a^{jK\omega_0 n}$$

$K = 0, \ldots, 7$

$a_1 = 1/2 \quad a_7 = 1/2$

Q) Evaluate the DTFS coefficients for the signal

$$x(n) = \sin\left(\frac{4\pi}{21}\right)n + \cos\left(\frac{10\pi n}{21}\right) + 1$$

Ans

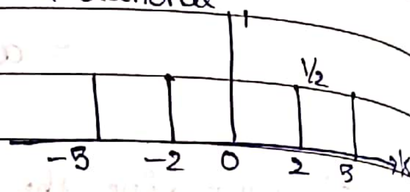
$$\omega_1 = \frac{4\pi}{21} \qquad \omega_2 = \frac{10\pi}{21}$$

$$\omega_1 = \frac{2\pi}{N} \qquad \omega_2 = \frac{2\pi}{N_2}$$

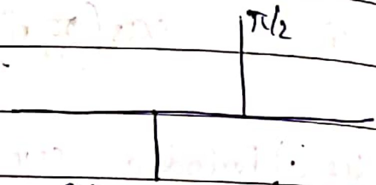
$N_1/N_2 \rightarrow$ rational $\Rightarrow \omega_1/\omega_2 \rightarrow$ rational

fundamental period of

$x(n)$ is $N = 21$



$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$



$$\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{21}$$

$$x(n) = \frac{1}{2j} e^{j\frac{4\pi}{21}n} - \frac{1}{2j} e^{-j\frac{4\pi}{21}n} + \frac{1}{2} e^{+j\frac{10\pi}{21}n} + \frac{1}{2} e^{-j\frac{10\pi}{21}n} + 1$$

$$= \frac{1}{2j} e^{j2\left(\frac{2\pi}{21}\right)n} - \frac{1}{2j} e^{-j2\omega_0 n} + \frac{1}{2} e^{j5\omega_0 n} + \frac{1}{2} e^{-j5\omega_0 n} + 1$$

$a=2 \qquad a=-2 \qquad a=5 \qquad a=5 \qquad a=0$

$a_2 = \frac{1}{2j} \qquad a_{-2} = -\frac{1}{2j} \qquad a_5 = \frac{1}{2} \qquad a_{-5} = \frac{1}{2} \qquad a_0 = 1$

$|a_2| = \frac{1}{2} \qquad |a_{-2}| = \frac{1}{2} \qquad |a_5| = \frac{1}{2} \qquad |a_{-5}| = \frac{1}{2} \qquad |a_0| = 1$

$\angle a_2 = \pi/2 \qquad \angle a_{-2} = -\pi/2 \qquad \angle a_5 = 0 \qquad \angle a_{-5} = 0 \qquad \angle a_0 = 0$

3) $x(n) = \cos^2\left(\frac{\pi n}{8}\right)$

Ans

$$\omega_1 = \frac{\pi}{8}$$

fundamental period is

$$\frac{\omega_0}{2\pi} = \frac{\pi}{8 \times 2\pi} = \frac{1}{16} = \frac{m}{N}$$

$$N = 16$$

$x(n) =$

$$N = 8$$

$$x(n) = \sum_{k=-3}^{4} a_k e^{jk\omega_0 n} \; ; \quad \omega_0 = \frac{2\pi}{8}$$

$$x(n) = \frac{1}{2} + \frac{1}{4} e^{j\frac{2\pi}{8}n} + \frac{1}{4} e^{-j\frac{2\pi}{8}n}$$

$$a_0 = 1/2 \quad a_1 = 1/4 \quad a_{-1} = 1/4$$

Q) determine the fourier series for $x(n) = \sin(\omega_0 n)$

Ans

$$x(n) = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

$$\omega_0 = \frac{2\pi}{N} \; ; \; N = \text{fundamental period}$$

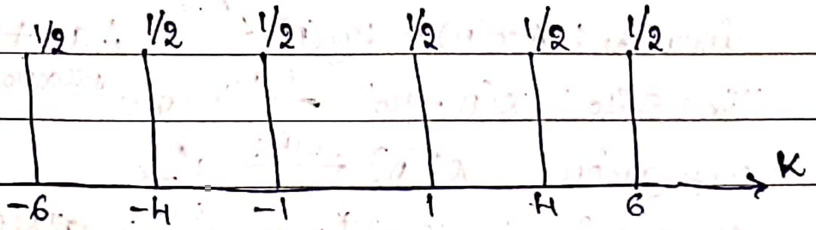
By comparing with definition of DTFS,

$$a_1 = \frac{1}{2j} \qquad a_{-1} = \frac{-1}{2j}$$

Magnitude spectrum

$$|a_1| = \frac{1}{2} \qquad |a_{-1}| = \frac{1}{2}$$

$N = 5$



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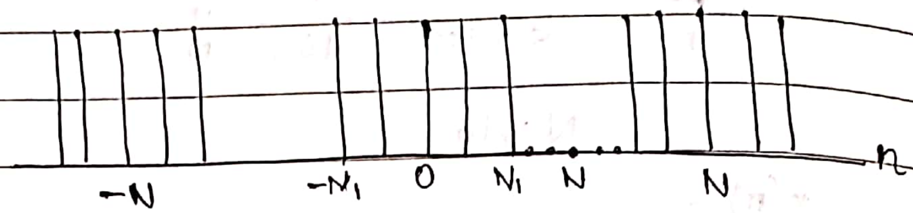
Ex: 3.12



$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-jk\omega_0 n}$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{M} 1 \cdot e^{-jk\left(\frac{2\pi}{N}\right)n}$$

let $m = n + N_1$

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)}$$

$$= \frac{1}{N} e^{jk\frac{2\pi}{N}N_1}\left[\sum_{m=0}^{2N_1} e^{-jk\frac{2\pi}{N}m}\right]$$

$$= \frac{1}{N} e^{jk\frac{2\pi}{N}N_1}\left[\frac{1 - e^{-jk\frac{2\pi}{N}(2N_1+1)}}{1 - e^{-jk\frac{2\pi}{N}}}\right]$$

$$a_k = \frac{1}{N}\left[\frac{\sin(2\pi k(N_1+1/2)/N}{\sin(\pi k/N)}\right], \quad k \neq 0, \pm N,$$

for $k=0$ $\quad a_0 = \frac{1}{N}\sum_{m=0}^{2N_1} 1 = \frac{2N_1+1}{N}, \quad k = 0, \pm N_1, \dots$

Properties of DTFs

$$x(n) \xleftrightarrow{DTF's} a_k$$

linearity: $A x(n) + B y(n) \xleftrightarrow{DTFS} A a_k + B b_k$

Time shift: $x(n-n_0) \xleftrightarrow{DTF's} a_k e^{-jk\omega_0 n_0}$

conjugation: $x^*[n] \xleftrightarrow{DTF's} a^*_{-k}$

convolution: $x(n) * y(n) \xleftrightarrow{DTF's} N a_k b_k$

Multiplication: $x(n)y(n) \longleftrightarrow \sum_{l=\langle N\rangle} a_l b_{k-l}$

First difference: $x(n) - x(n-1) \longleftrightarrow \left(1 - e^{jk\left(\frac{2\pi}{N}\right)}\right) a_k$

Parseval's Relation

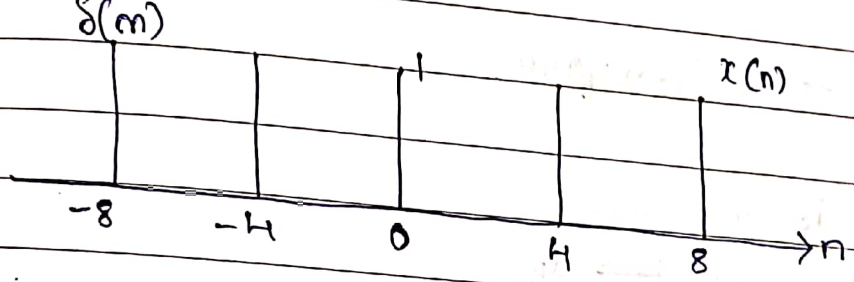
$$\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2 = \sum_{k=\langle N\rangle}|a_k|^2$$

Q) consider the sequence $x(n) = \sum_{m=-\infty}^{\infty} \delta(n-4m)$

(i) Sketch $x(n)$

(ii) find a_k

Ans



$N = 4$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-jk\omega_0 n}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = \frac{1}{4}\sum_{n=0}^{3} (1)\, e^{-jk\left(\frac{\pi}{2}\right)n}$$

$$a_k = \frac{1}{4}\sum_{n=0}^{3} e^{-jk\left(\frac{\pi}{2}\right)n} = \frac{1}{4}, \quad k = \{0,1,2,3\}$$

~~Q) Ex 3.9~~ ✗.

Ex: 3.9 (continuation)

$|b_1| = 1/2$

$b_1 = j/2 \qquad b_1 = -j/2$

(i) $b_1 = j/2$

(4) $\Rightarrow k = -k \Rightarrow a_k = b_{-k}\, e^{-jk\pi/2}$

$a_1 = b_{-1}\, e^{-j\pi/2}$

$a_1 = \left(\frac{-j}{2}\right) e^{-j\pi/2} = 1/2$

(ii) $b_1 = -j/2 \Rightarrow a_1 = -1/2$

$x(t) = 2\,\mathrm{Re}\left[\frac{1}{2} e^{j\pi/2t}\right] = \cos\frac{\pi}{2}t \quad (a_1 = 1/2)$

$x(t) = -\cos\pi/2t \quad (a_1 = -1/2)$

Fourier Transform

Periodic signal:

$CT \xrightarrow{FS} a_k$

$DT \xrightarrow{DTFS} a_k$

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