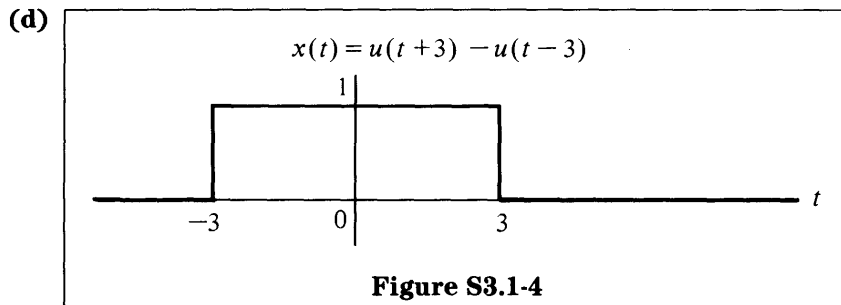
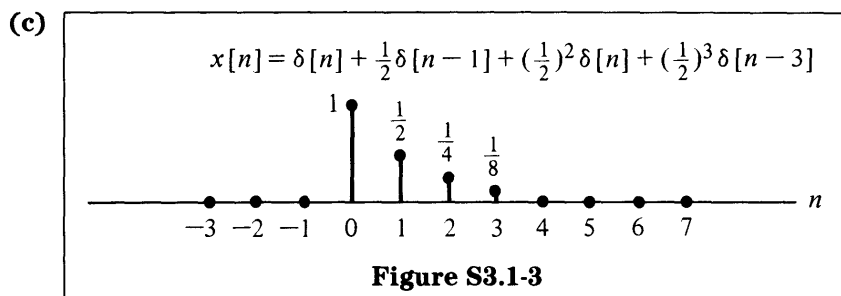
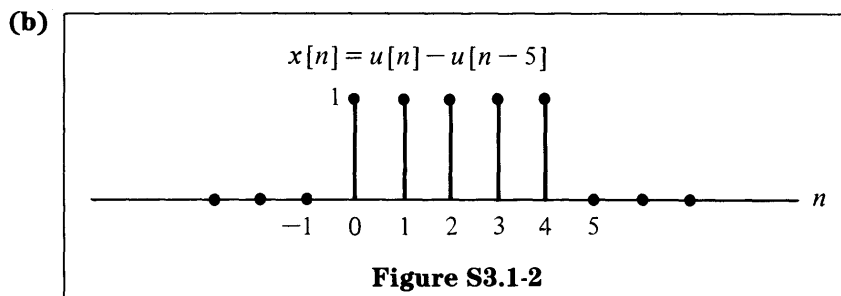
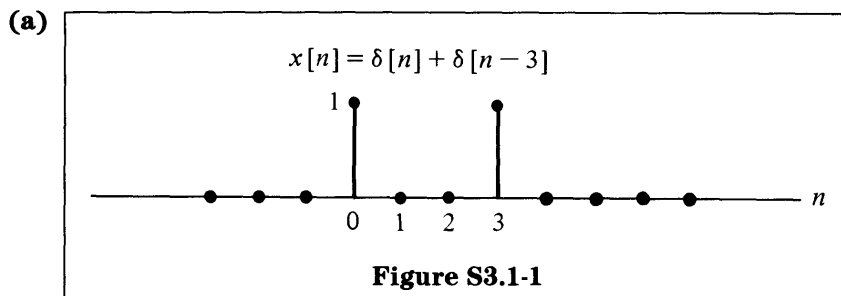
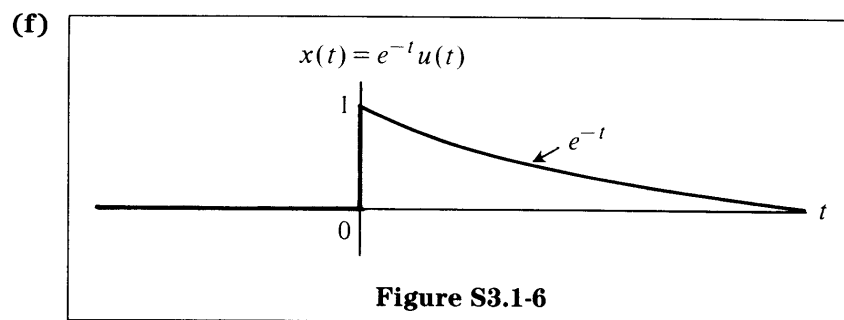
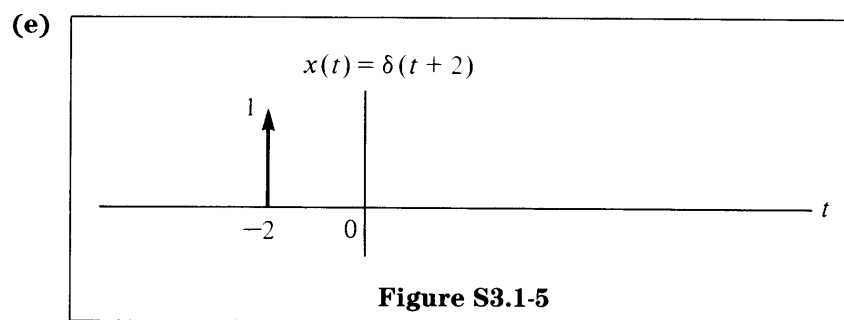


# 3 Signals and Systems: Part II

## Solutions to Recommended Problems

S3.1





**S3.2**

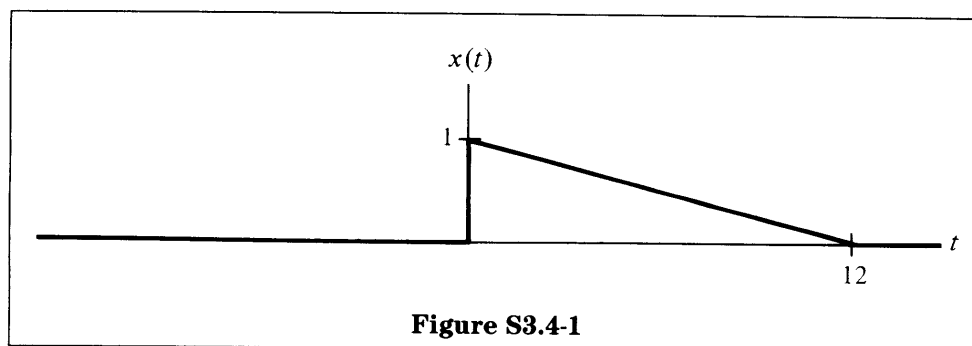
- (1) h
- (2) d
- (3) b
- (4) e
- (5) a, f
- (6) None

**S3.3**

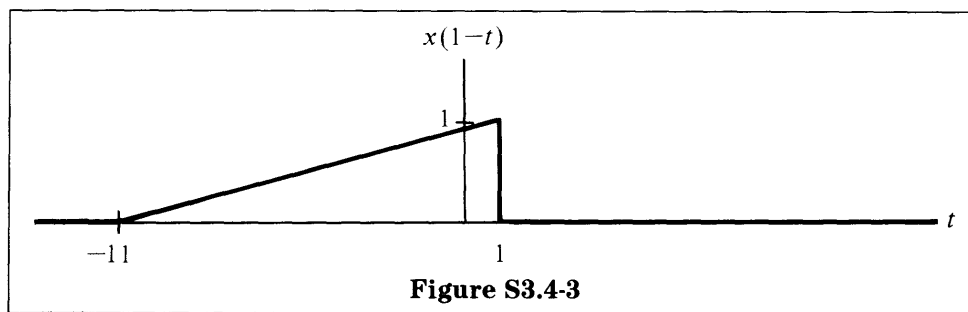
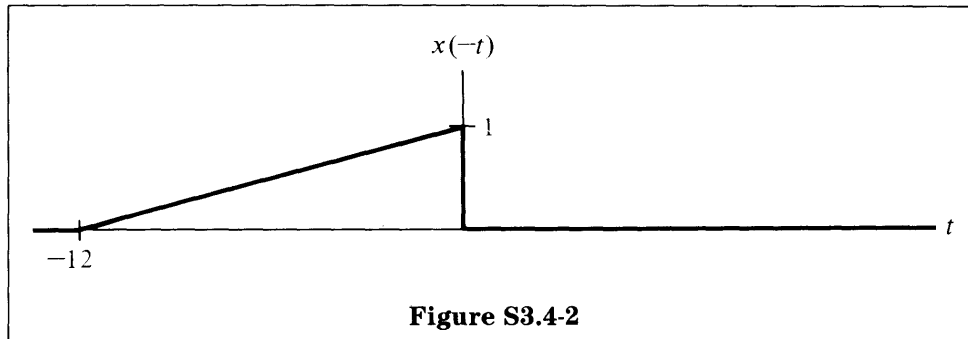
- (a)  $x[n] = \delta[n - 1] - 2\delta[n - 2] + 3\delta[n - 3] - 2\delta[n - 4] + \delta[n - 5]$
- (b)  $s[n] = -u[n + 3] + 4u[n + 1] - 4u[n - 2] + u[n - 4]$

**S3.4**

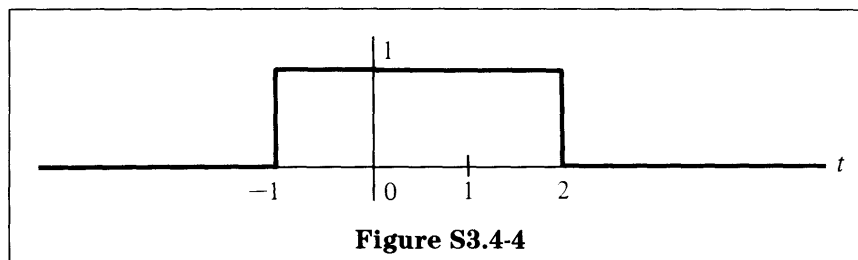
We are given Figure S3.4-1.



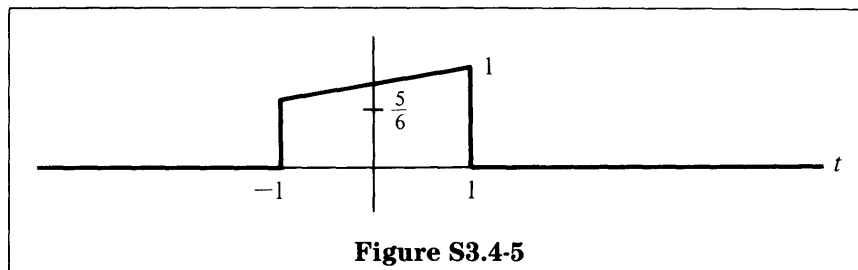
$x(-t)$  and  $x(1 - t)$  are as shown in Figures S3.4-2 and S3.4-3.



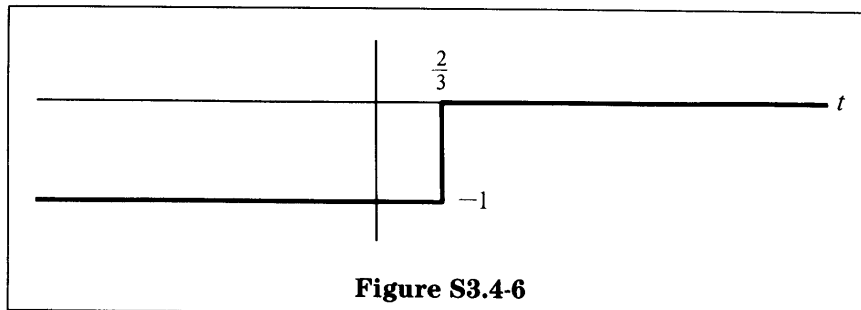
(a)  $u(t + 1) - u(t - 2)$  is as shown in Figure S3.4-4.



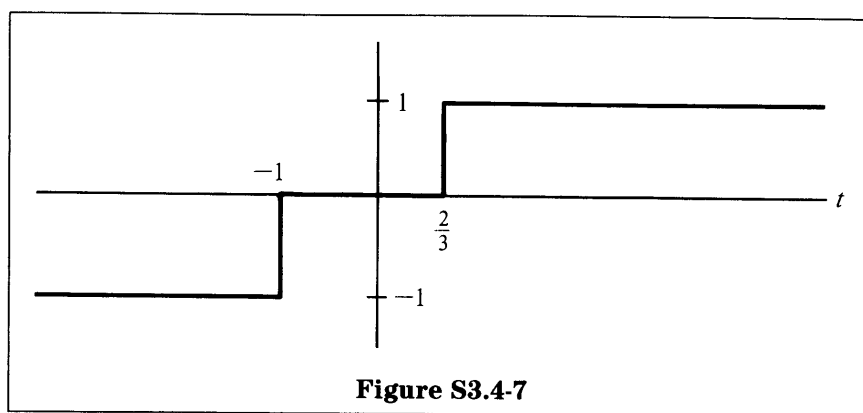
Hence,  $x(1 - t)[u(t + 1) - u(t - 2)]$  looks as in Figure S3.4-5.



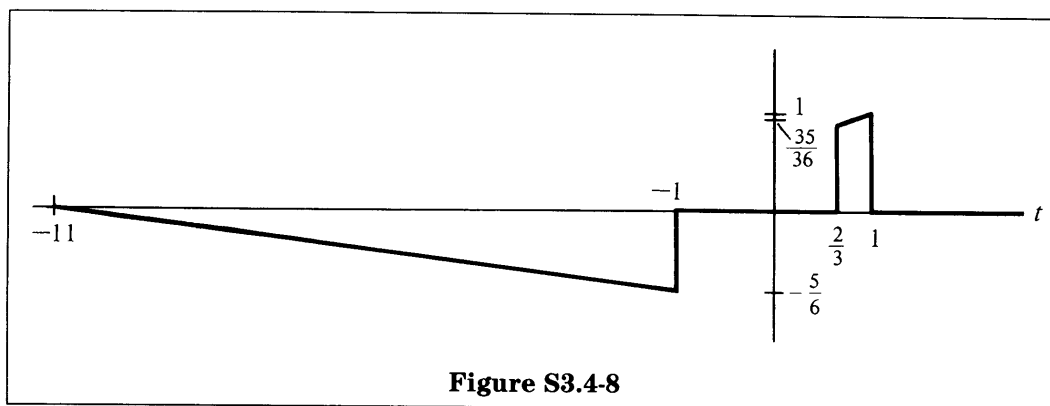
(b)  $-u(2 - 3t)$  looks as in Figure S3.4-6.



Hence,  $u(t + 1) - u(2 - 3t)$  is given as in Figure S3.4-7.



So  $x(1 - t)[u(t + 1) - u(2 - 3t)]$  is given as in Figure S3.4-8.



### S3.5

- (a)  $y[n] = x^2[n] + x[n] - x[n - 1]$
- (b)  $y[n] = x^2[n] + x[n] - x[n - 1]$
- (c)  $y[n] = H[x[n] - x[n - 1]]$   
 $= x^2[n] + x^2[n - 1] - 2x[n]x[n - 1]$
- (d)  $y[n] = G[x^2[n]]$   
 $= x^2[n] - x^2[n - 1]$

$$\begin{aligned}
 \text{(e)} \quad y[n] &= F[x[n] - x[n-1]] \\
 &= 2(x[n] - x[n-1]) + (x[n-1] - x[n-2]) \\
 y[n] &= 2x[n] - x[n-1] - x[n-2] \\
 \text{(f)} \quad y[n] &= G[2x[n] + x[n-1]] \\
 &= 2x[n] + x[n-1] - 2x[n-1] - x[n-2] \\
 &= 2x[n] - x[n-1] - x[n-2]
 \end{aligned}$$

(a) and (b) are equivalent. (e) and (f) are equivalent.

### S3.6

*Memoryless:*

- (a)  $y(t) = (2 + \sin t)x(t)$  is memoryless because  $y(t)$  depends only on  $x(t)$  and not on prior values of  $x(t)$ .
- (d)  $y[n] = \sum_{k=-\infty}^n x[k]$  is not memoryless because  $y[n]$  does depend on values of  $x[\cdot]$  before the time instant  $n$ .
- (f)  $y[n] = \max\{x[n], x[n-1], \dots, x[-\infty]\}$  is clearly not memoryless.

*Linear:*

$$\begin{aligned}
 \text{(a)} \quad y(t) &= (2 + \sin t)x(t) = T[x(t)], \\
 T[ax_1(t) + bx_2(t)] &= (2 + \sin t)[ax_1(t) + bx_2(t)] \\
 &= a(2 + \sin t)x_1(t) + b(2 + \sin t)x_2(t) \\
 &= aT[x_1(t)] + bT[x_2(t)]
 \end{aligned}$$

Therefore,  $y(t) = (2 + \sin t)x(t)$  is linear.

$$\begin{aligned}
 \text{(b)} \quad y(t) &= x(2t) = T[x(t)], \\
 T[ax_1(t) + bx_2(t)] &= ax_1(2t) + bx_2(2t) \\
 &= aT[x_1(t)] + bT[x_2(t)]
 \end{aligned}$$

Therefore,  $y(t) = x(2t)$  is linear.

$$\begin{aligned}
 \text{(c)} \quad y[n] &= \sum_{k=-\infty}^{\infty} x[k] = T[x[n]], \\
 T[ax_1[n] + bx_2[n]] &= a \sum_{k=-\infty}^{\infty} x_1[k] + b \sum_{k=-\infty}^{\infty} x_2[k] \\
 &= aT[x_1[n]] + bT[x_2[n]]
 \end{aligned}$$

Therefore,  $y[n] = \sum_{k=-\infty}^{\infty} x[k]$  is linear.

$$\text{(d)} \quad y[n] = \sum_{k=-\infty}^n x[k] \text{ is linear (see part c).}$$

$$\begin{aligned}
 \text{(e)} \quad y(t) &= \frac{dx(t)}{dt} = T[x(t)], \\
 T[ax_1(t) + bx_2(t)] &= \frac{d}{dt}[ax_1(t) + bx_2(t)] \\
 &= a \frac{dx_1(t)}{dt} + b \frac{dx_2(t)}{dt} = aT[x_1(t)] + bT[x_2(t)]
 \end{aligned}$$

Therefore,  $y(t) = dx(t)/dt$  is linear.

$$\begin{aligned}
 \text{(f)} \quad y[n] &= \max\{x[n], \dots, x[-\infty]\} = T[x[n]], \\
 T[ax_1[n] + bx_2[n]] &= \max\{ax_1[n] + bx_2[n], \dots, ax_1[-\infty] + bx_2[-\infty]\} \\
 &\neq a \max\{x_1[n], \dots, x_1[-\infty]\} + b \max\{x_2[n], \dots, x_2[-\infty]\}
 \end{aligned}$$

Therefore,  $y[n] = \max\{x[n], \dots, x[-\infty]\}$  is not linear.

*Time-invariant:*

$$\begin{aligned} \text{(a)} \quad y(t) &= (2 + \sin t)x(t) = T[x(t)], \\ T[x(t - T_0)] &= (2 + \sin t)x(t - T_0) \\ &\neq y(t - T_0) = (2 + \sin(t - T_0))x(t - T_0) \end{aligned}$$

Therefore,  $y(t) = (2 + \sin t)x(t)$  is not time-invariant.

$$\begin{aligned} \text{(b)} \quad y(t) &= x(2t) = T[x(t)], \\ T[x(t - T_0)] &= x(2t - 2T_0) \neq x(2t - T_0) = y(t - T_0) \end{aligned}$$

Therefore,  $y(t) = x(2t)$  is not time-invariant.

$$\begin{aligned} \text{(c)} \quad y[n] &= \sum_{k=-\infty}^{\infty} x[k] = T[x[n]], \\ T[x[n - N_0]] &= \sum_{k=-\infty}^{\infty} x[k - N_0] = y[n - N_0] \end{aligned}$$

Therefore,  $y[n] = \sum_{k=-\infty}^{\infty} x[k]$  is time-invariant.

$$\begin{aligned} \text{(d)} \quad y[n] &= \sum_{k=-\infty}^n x[k] = T[x[n]], \\ T[x[n - N_0]] &= \sum_{k=-\infty}^n x[k - N_0] = \sum_{l=-\infty}^{n-N_0} x[l] = y[n - N_0] \end{aligned}$$

Therefore,  $y[n] = \sum_{k=-\infty}^n x[k]$  is time-invariant.

$$\begin{aligned} \text{(e)} \quad y(t) &= \frac{dx(t)}{dt} = T[x(t)], \\ T[x(t - T_0)] &= \frac{d}{dt} x(t - T_0) = y(t - T_0) \end{aligned}$$

Therefore,  $y(t) = dx(t)/dt$  is time-invariant.

*Causal:*

$$\begin{aligned} \text{(b)} \quad y(t) &= x(2t), \\ y(1) &= x(2) \end{aligned}$$

The value of  $y(\cdot)$  at time = 1 depends on  $x(\cdot)$  at a future time = 2. Therefore,  $y(t) = x(2t)$  is not causal.

$$\text{(d)} \quad y[n] = \sum_{k=-\infty}^n x[k]$$

Yes,  $y[n] = \sum_{k=-\infty}^n x[k]$  is causal because the value of  $y[\cdot]$  at any instant  $n$  depends only on the previous (past) values of  $x[\cdot]$ .

*Invertible:*

$$\text{(b)} \quad y(t) = x(2t) \text{ is invertible; } x(t) = y(t/2).$$

$$\text{(c)} \quad y[n] = \sum_{k=-\infty}^{\infty} x[k] \text{ is not invertible. Summation is not generally an invertible operation.}$$

$$\text{(e)} \quad y(t) = dx(t)/dt \text{ is invertible to within a constant.}$$

*Stable:*

$$\text{(a)} \quad \text{If } |x(t)| < M, |y(t)| < (2 + \sin t)M. \text{ Therefore, } y(t) = (2 + \sin t)x(t) \text{ is stable.}$$

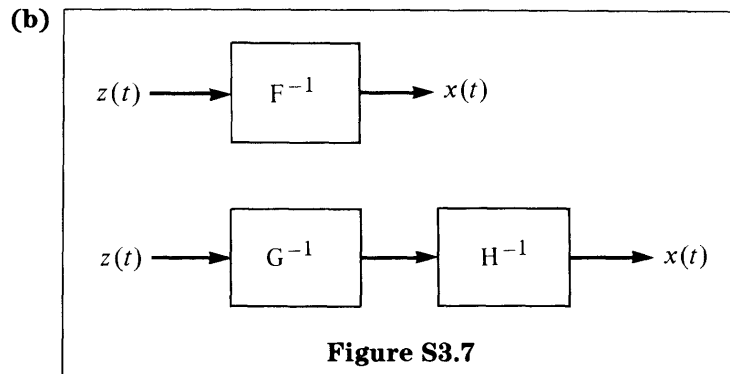
$$\text{(b)} \quad \text{If } |x(t)| < M, |x(2t)| < M \text{ and } |y(t)| < M. \text{ Therefore, } y(t) = x(2t) \text{ is stable.}$$

$$\text{(d)} \quad \text{If } |x[k]| \leq M, |y[n]| \leq M \cdot \sum_{k=-\infty}^n 1, \text{ which is unbounded. Therefore, } y[n] = \sum_{k=-\infty}^n x[k] \text{ is not stable.}$$

**S3.7**

(a) Since  $H$  is an integrator,  $H^{-1}$  must be a differentiator.

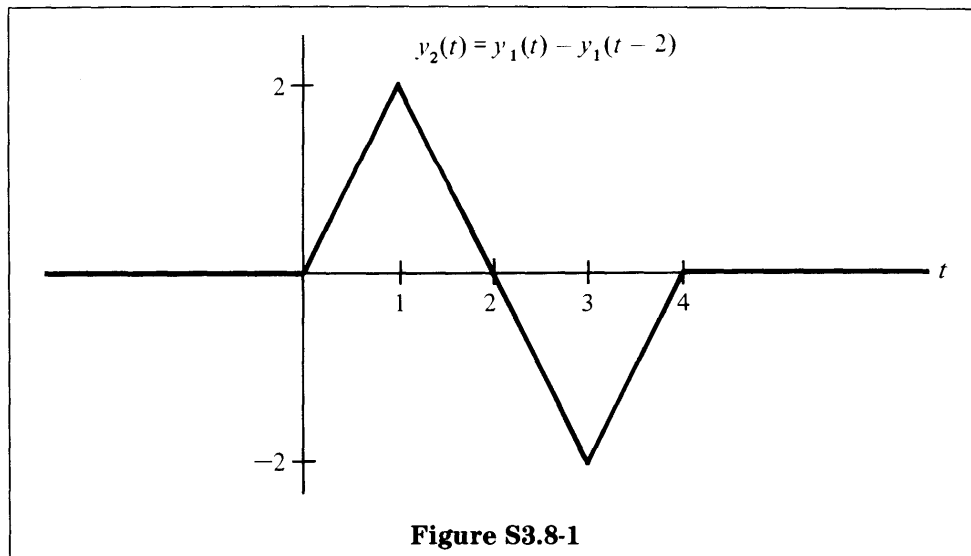
$$\begin{aligned} H^{-1}: \quad y(t) &= \frac{dx(t)}{dt} \\ G: \quad y(t) &= x(2t) \\ G^{-1}: \quad y(t) &= x(t/2) \end{aligned}$$



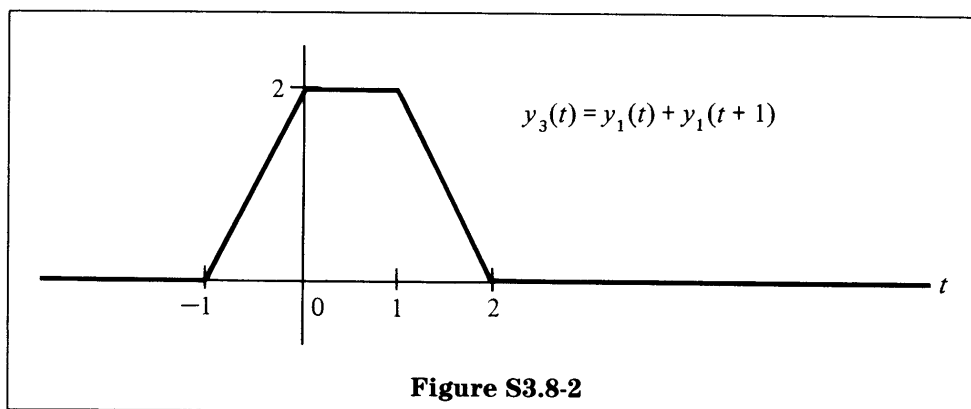
## Solutions to Optional Problems

**S3.8**

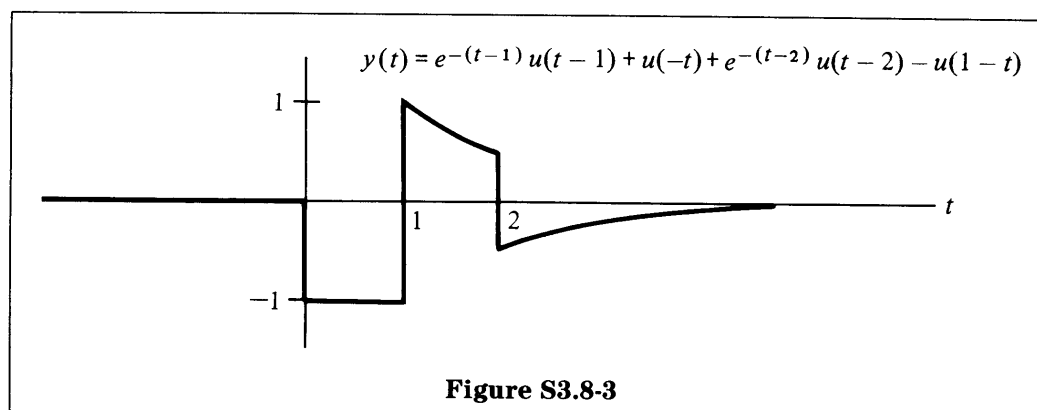
(a)  $x_2(t) = x_1(t) - x_1(t - 2)$



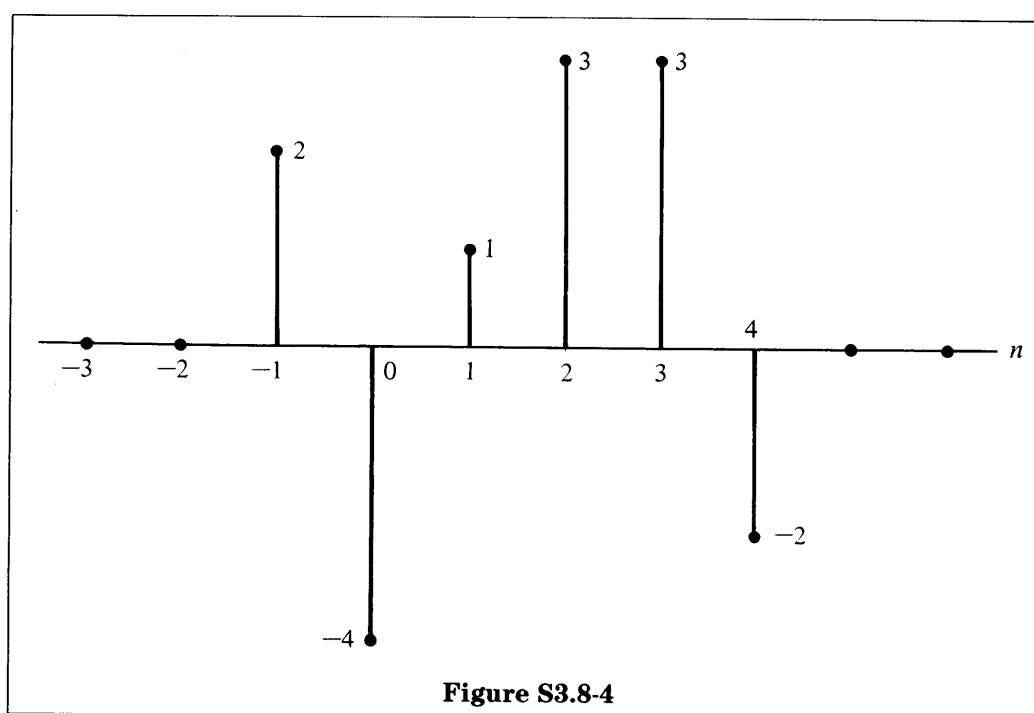
(b)  $x_3(t) = x_1(t) + x_1(t + 1)$



(c)  $x(t) = u(t - 1) - u(t - 2)$

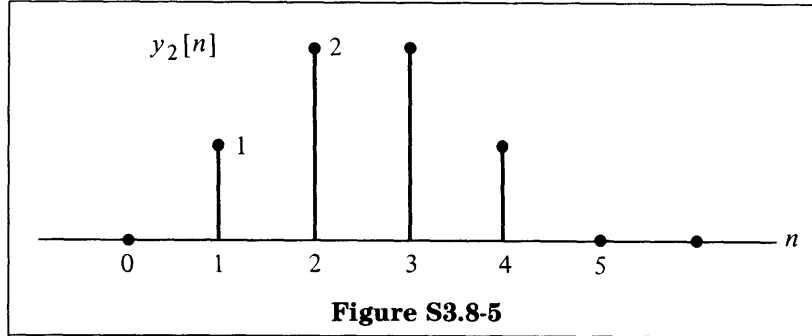


(d)  $y[n] = 3y_1[n] - 2y_2[n] + 2y_3[n]$

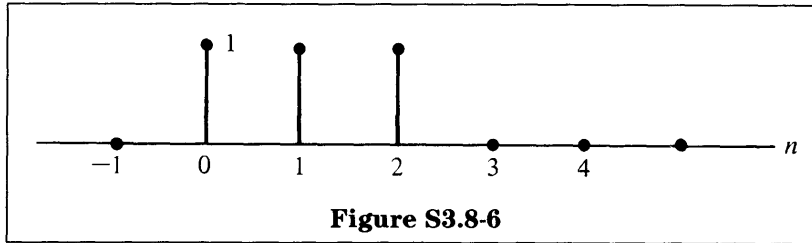




(e)  $y_2[n] = y_1[n] + y_1[n - 1]$



$y_3[n] = y_1[n + 1]$



(f) From linearity,

$$y_1(t) = \pi + 6 \cos(2t) - 47 \cos(5t) + \sqrt{e} \cos(6t),$$

$$x_2(t) = \frac{1 + t^{10}}{1 + t^2} = \sum_{n=0}^4 (-t^2)^n.$$

So  $y_2(t) = 1 - \cos(2t) + \cos(4t) - \cos(6t) + \cos(8t).$

### S3.9

(a) (i) The system is linear because

$$\begin{aligned} T[ax_1(t) + bx_2(t)] &= \sum_{n=-\infty}^{\infty} [ax_1(t) + bx_2(t)]\delta(t - nT) \\ &= a \sum_{n=-\infty}^{\infty} x_1(t)\delta(t - nT) + b \sum_{n=-\infty}^{\infty} x_2(t)\delta(t - nT) \\ &= aT[x_1(t)] + bT[x_2(t)] \end{aligned}$$

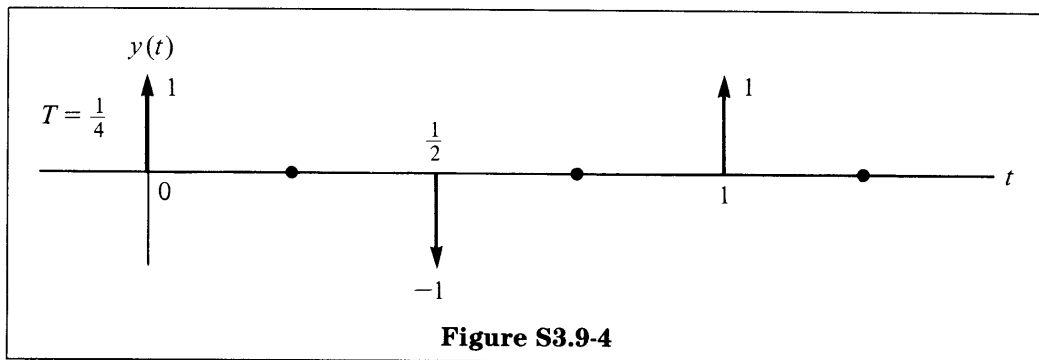
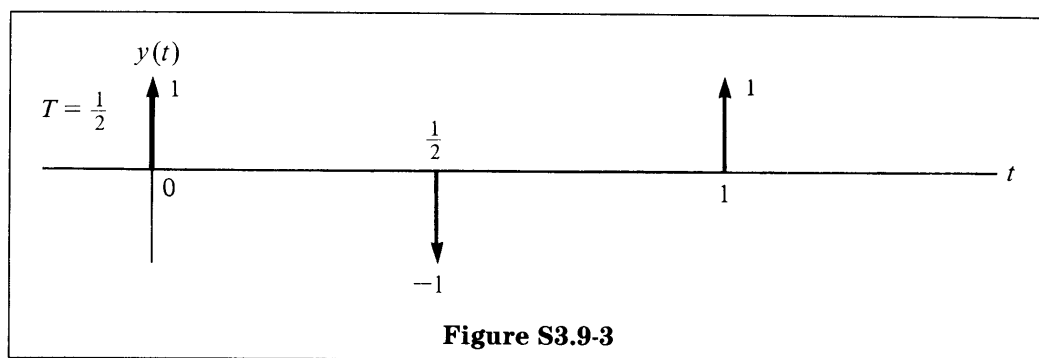
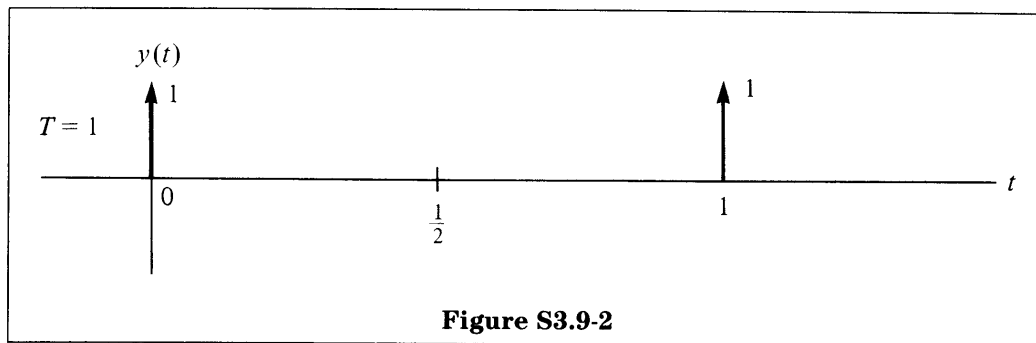
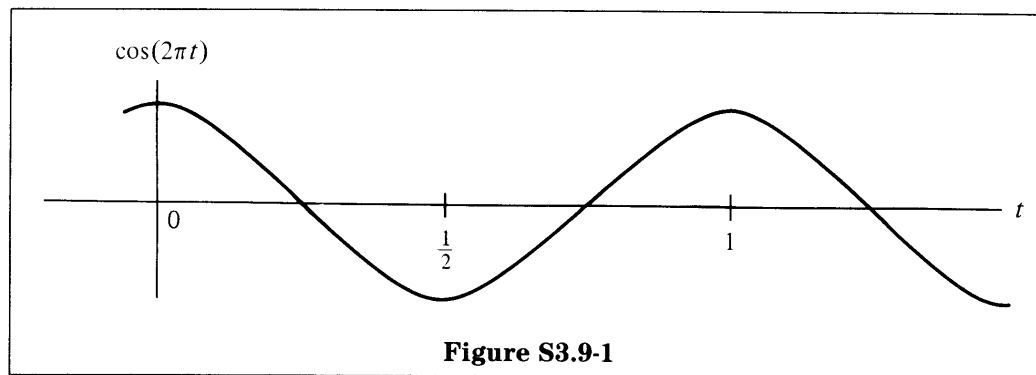
(ii) The system is not time-invariant. For example, let  $x_1(t) = \sin(2\pi t/T)$ . The corresponding output  $y_1(t) = 0$ . Now let us shift the input  $x_1(t)$  by  $\pi/2$  to get

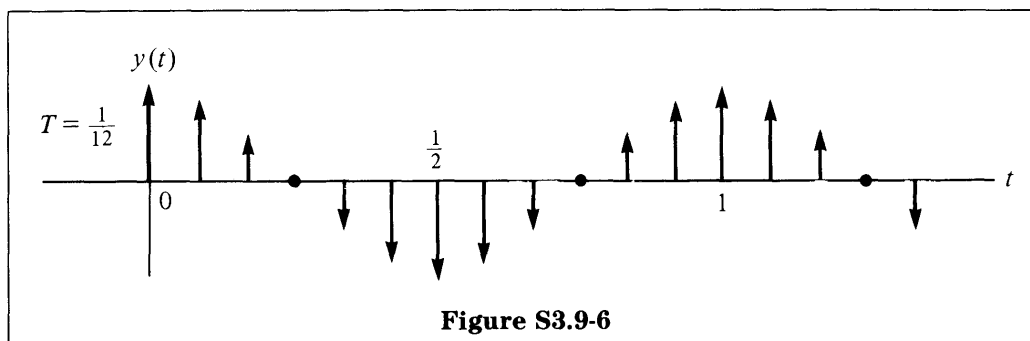
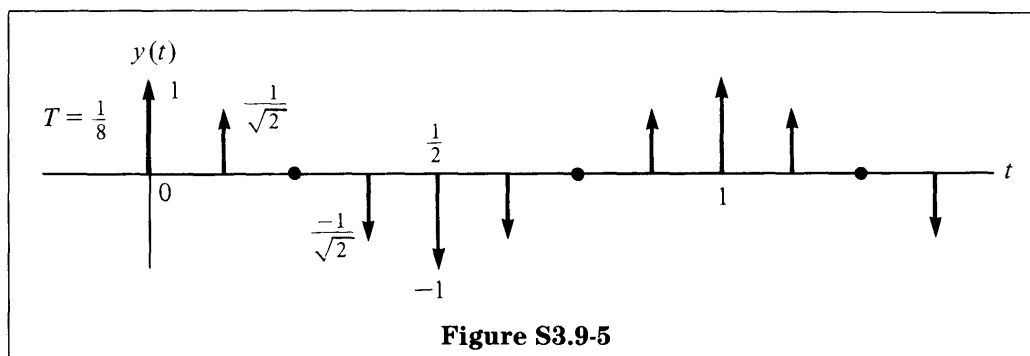
$$x_2(t) = \sin\left(\frac{2\pi t}{T} + \frac{\pi}{2}\right) = \cos\left(\frac{2\pi t}{T}\right)$$

Now the output

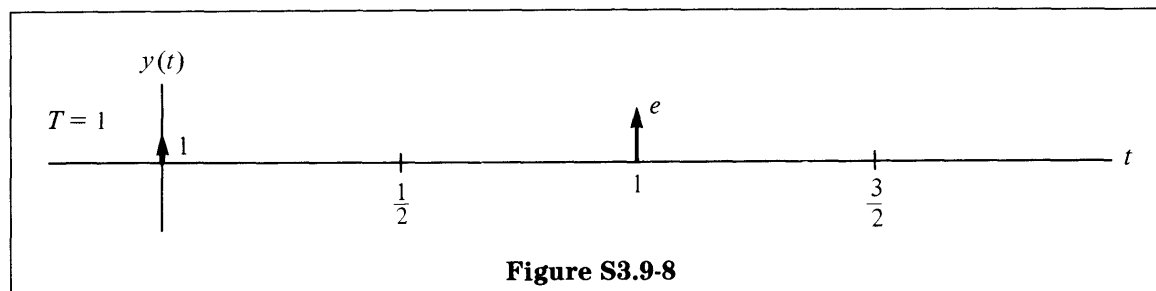
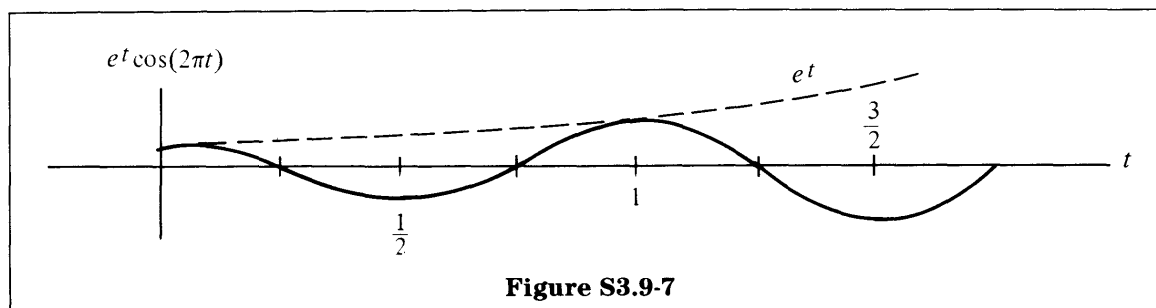
$$y_2(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \neq y_1\left(t + \frac{\pi}{2}\right) = 0$$

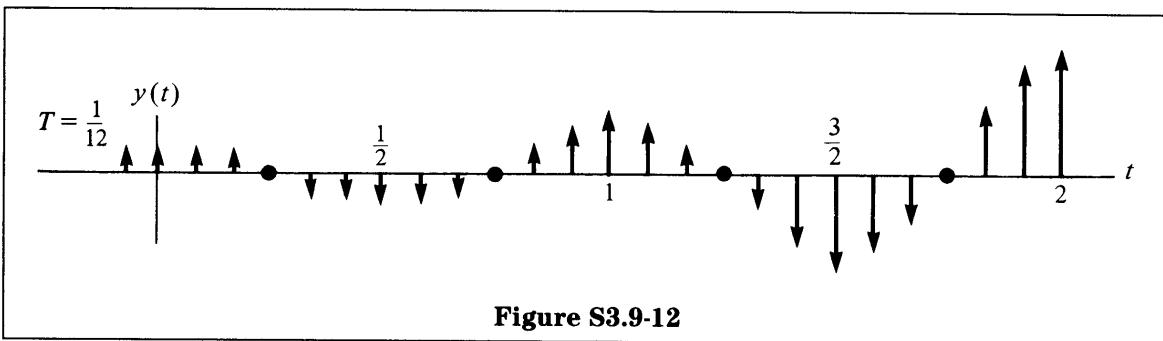
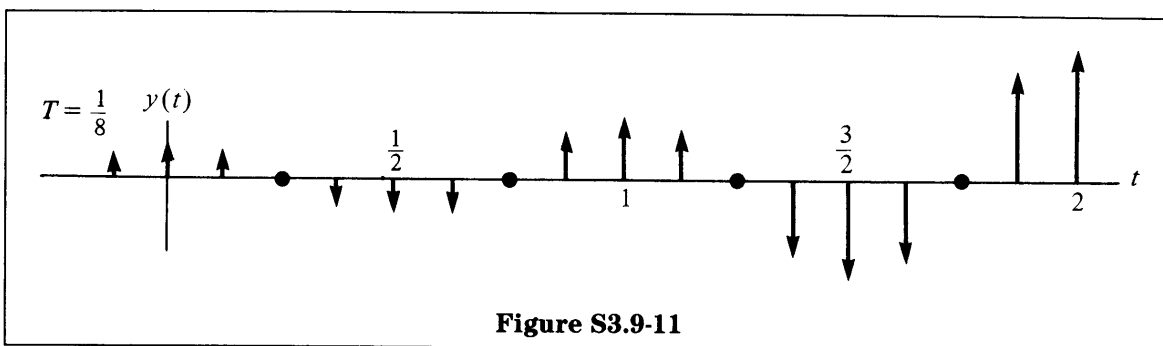
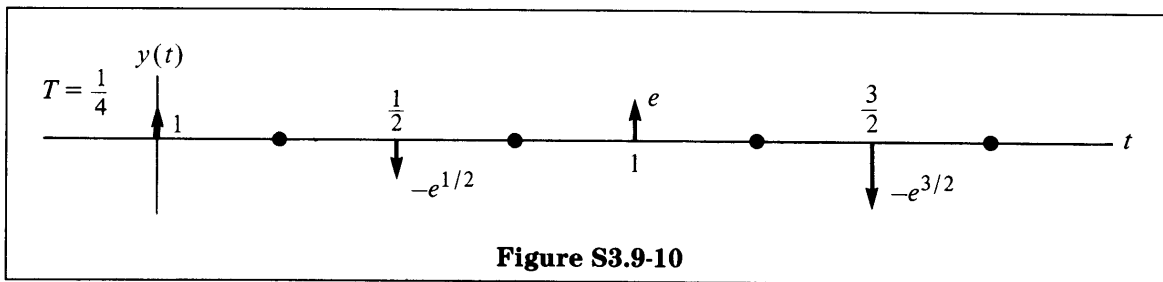
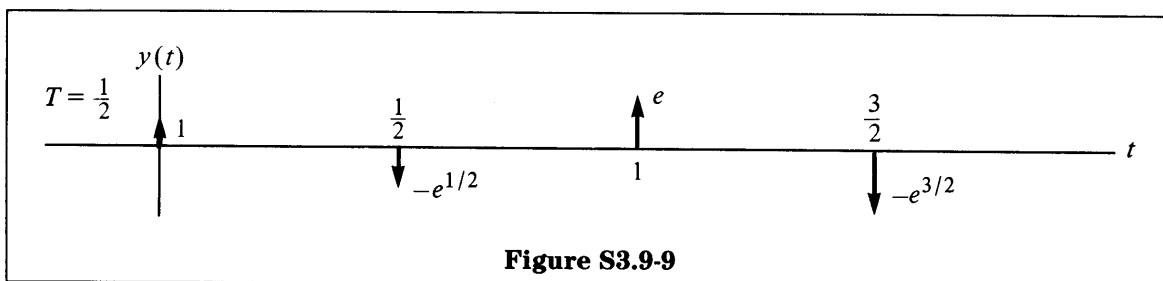
$$\begin{aligned} \text{(b) } y(t) &= \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} \cos(2\pi t)\delta(t - nT) \end{aligned}$$





(c) 
$$y(t) = \sum_{n=-\infty}^{\infty} e^t \cos(2\pi t) \delta(t - nT)$$





### S3.10

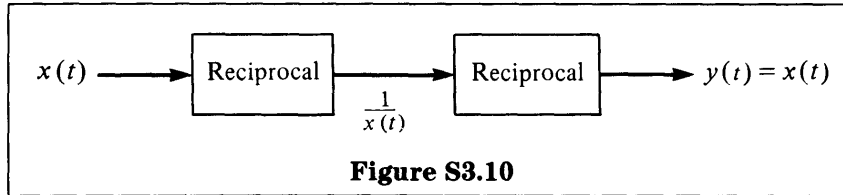
(a) True. To see that the system is linear, write

$$\begin{aligned}
 y_2(t) &= T_2[T_1[x(t)]] \triangleq T[x(t)], \\
 T_1[ax_1(t) + bx_2(t)] &= aT_1[x_1(t)] + bT_1[x_2(t)] \\
 &\Rightarrow T_2[T_1[ax_1(t) + bx_2(t)]] = T_2[aT_1[x_1(t)] + bT_1[x_2(t)]] \\
 &= aT_2[T_1[x_1(t)]] + bT_2[T_1[x_2(t)]] \\
 &= aT[x_1(t)] + bT[x_2(t)]
 \end{aligned}$$

We see that the system is time-invariant from

$$\begin{aligned} T_2[T_1[x(t - T)]] &= T_2[y_1(t - T)] \\ &= y_2(t - T), \\ T[x(t - T)] &= y_2(t - T) \end{aligned}$$

- (b) False. Two nonlinear systems in cascade can be linear, as shown in Figure S3.10. The overall system is identity, which is a linear system.



(c)  $y[n] = z[2n] = w[2n] + \frac{1}{2}w[2n - 1] + \frac{1}{4}w[2n - 2]$   
 $= x[n] + \frac{1}{4}x[n - 1]$

The system is linear and time-invariant.

(d)  $y[n] = z[-n] = aw[-n - 1] + bw[-n] + cw[-n + 1]$   
 $= ax[n + 1] + bx[n] + cx[n - 1]$

- (i) The overall system is linear and time-invariant for any choice of  $a$ ,  $b$ , and  $c$ .
- (ii)  $a = c$
- (iii)  $a = 0$

### S3.11

- (a)  $y[n] = x[n] + x[n - 1] = T[x[n]]$ . The system is linear because

$$\begin{aligned} T[ax_1[n] + bx_2[n]] &= ax_1[n] + ax_1[n - 1] + bx_2[n] + bx_2[n - 1] \\ &= aT[x_1[n]] + bT[x_2[n - 1]] \end{aligned}$$

The system is time-invariant because

$$\begin{aligned} y[n] &= x[n] + x[n - 1] = T[x[n]], \\ T[x[n - N]] &= x[n - N] + x[n - 1 - N] \\ &= y[n - N] \end{aligned}$$

- (b) The system is linear, shown by similar steps to those in part (a). It is not time-invariant because

$$\begin{aligned} T[x[n - N]] &= x[n - N] + x[n - N - 1] + x[0] \\ &\neq y[n - N] = x[n - N] + x[n - N - 1] + x[-N] \end{aligned}$$

### S3.12

- (a) To show that causality implies the statement, suppose

$$\begin{aligned} x_1(t) &\rightarrow y_1(t) \quad (\text{input } x_1(t) \text{ results in output } y_1(t)), \\ x_2(t) &\rightarrow y_2(t), \end{aligned}$$

where  $y_1(t)$  and  $y_2(t)$  depend on  $x_1(t)$  and  $x_2(t)$  for  $t < t_0$ . By linearity,

$$x_1(t) - x_2(t) \rightarrow y_1(t) - y_2(t)$$

If  $x_1(t) = x_2(t)$  for  $t < t_0$ , then  $y_1(t) = y_2(t)$  for  $t < t_0$ . Hence, if  $x(t) = 0$  for  $t < t_0$ ,  $y(t) = 0$  for  $t < t_0$ .

- (b)  $y(t) = x(t)x(t+1)$ ,  
 $x(t) = 0$  for  $t < t_0 \Rightarrow y(t) = 0$ , for  $t < t_0$

This is a nonlinear, noncausal system.

- (c)  $y(t) = x(t) + 1$  is a nonlinear, causal system.

- (d) We want to show the equivalence of the following two statements:

Statement 1 (S1): The system is invertible.

Statement 2 (S2): The only input that produces the output  $y[n] = 0$  for all  $n$  is  $x[n] = 0$  for all  $n$ .

To show the equivalence, we will show that

$$\begin{aligned} \text{S2 false} &\Rightarrow \text{S1 false} && \text{and} \\ \text{S1 false} &\Rightarrow \text{S2 false} \end{aligned}$$

S2 false  $\Rightarrow$  S1 false: Let  $x[n] \neq 0$  produce  $y[n] = 0$ . Then  $x[n] \Rightarrow y[n] = 0$ .

S1 false  $\Rightarrow$  S2 false: Let  $x_1 \Rightarrow y_1$  and  $x_2 \Rightarrow y_2$ . If  $x_1 \neq x_2$  but  $y_1 = y_2$ , then  $x_1 - x_2 \neq 0$  but  $y_1 - y_2 = 0$ .

- (e)  $y[n] = x^2[n]$  is nonlinear and not invertible.