The current i is related to the voltage v across the capacitor by Eq. [1]:

$$i = C\frac{dv}{dt}$$

For the voltage waveform depicted in Fig. 7.3*a*, $dv/dt = 0$, so $i = 0$; the result is plotted in Fig. 7.4*a*. For the case of the sinusoidal waveform of Fig. 7.3*b*, we expect a cosine current waveform to flow in response, having the same frequency and twice the magnitude (since $C = 2$ F). The result is plotted in Fig. 7.4*b*.

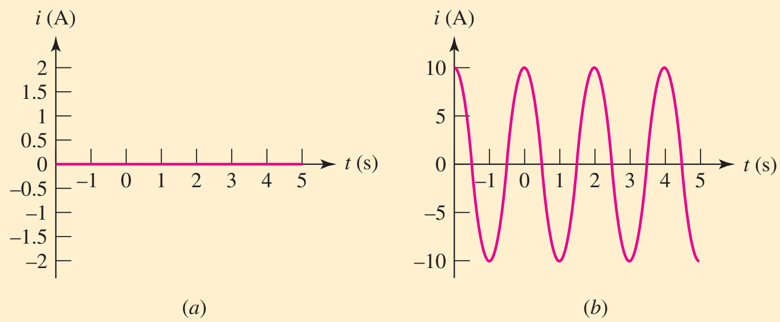


FIGURE 7.4 (*a*) $i = 0$ as the voltage applied is dc. (*b*) The current has a cosine form in response to a sine wave voltage.

PRACTICE

7.1 Determine the current flowing through a 5 mF capacitor in response to a voltage v equal to: (*a*) −20 V; (*b*) $2e^{-5t}$ V.

Ans: 0 A; $-50e^{-5t}$ mA.

Integral Voltage-Current Relationships

The capacitor voltage may be expressed in terms of the current by integrating Eq. [1]. We first obtain

$$dv = \frac{1}{C}i(t)\,dt$$

and then integrate[2] between the times t_0 and t and between the corresponding voltages $v(t_0)$ and $v(t)$:

$$\boxed{v(t) = \frac{1}{C}\int_{t_0}^{t} i(t')\,dt' + v(t_0)} \qquad [2]$$

Equation [2] may also be written as an indefinite integral plus a constant of integration:

$$v(t) = \frac{1}{C}\int i\,dt + k$$

(2) Note that we are employing the mathematically correct procedure of defining a *dummy variable* t' in situations where the integration variable t is also a limit.

If we now consider the time interval represented by the rectangular pulse, we obtain

$$v(t) = \frac{1}{5 \times 10^{-6}} \int_0^t 20 \times 10^{-3}\, dt' + v(0)$$

Since $v(0) = 0$,

$$v(t) = 4000t \qquad 0 \le t \le 2 \text{ ms}$$

For the semi-infinite interval following the pulse, the integral of $i(t)$ is once again zero, so that

$$v(t) = 8 \qquad t \ge 2 \text{ ms}$$

The results are expressed much more simply in a sketch than by these analytical expressions, as shown in Fig. 7.5*b*.

PRACTICE

7.2 Determine the current through a 100 pF capacitor if its voltage as a function of time is given by Fig. 7.6.

Ans: 0 A, $-\infty \le t \le 1$ ms; 200 nA, 1 ms $\le t \le 2$ ms; 0 A, $t \ge 2$ ms.

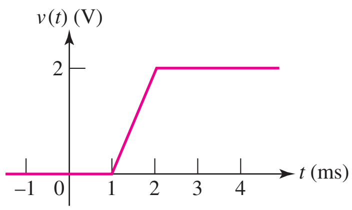


FIGURE 7.6

Energy Storage

To determine the energy stored in a capacitor, we begin with the power delivered to it:

$$p = vi = Cv\frac{dv}{dt}$$

The change in energy stored in its electric field is simply

$$\int_{t_0}^{t} p\, dt' = C\int_{t_0}^{t} v\frac{dv}{dt'}\, dt' = C\int_{v(t_0)}^{v(t)} v'\, dv' = \frac{1}{2}C\left\{[v(t)]^2 - [v(t_0)]^2\right\}$$

and thus

$$w_C(t) - w_C(t_0) = \tfrac{1}{2}C\left\{[v(t)]^2 - [v(t_0)]^2\right\} \qquad [3]$$

where the stored energy is $w_C(t_0)$ in joules (J) and the voltage at t_0 is $v(t_0)$. If we select a zero-energy reference at t_0, implying that the capacitor voltage is also zero at that instant, then

$$\boxed{w_C(t) = \tfrac{1}{2}Cv^2} \qquad [4]$$

Let us consider a simple numerical example. As sketched in Fig. 7.7, a sinusoidal voltage source is in parallel with a 1 MΩ resistor and a 20 μF capacitor. The parallel resistor may be assumed to represent the finite resistance of the dielectric between the plates of the physical capacitor (an *ideal* capacitor has infinite resistance).

Verify the solution. Is it reasonable or expected?

We do not expect to calculate a *negative* stored energy, which is borne out in our sketch. Further, since the maximum value of $\sin 2\pi t$ is 1, the maximum energy expected anywhere would be $(1/2)(20 \times 10^{-6})(100)^2 =$ 100 mJ.

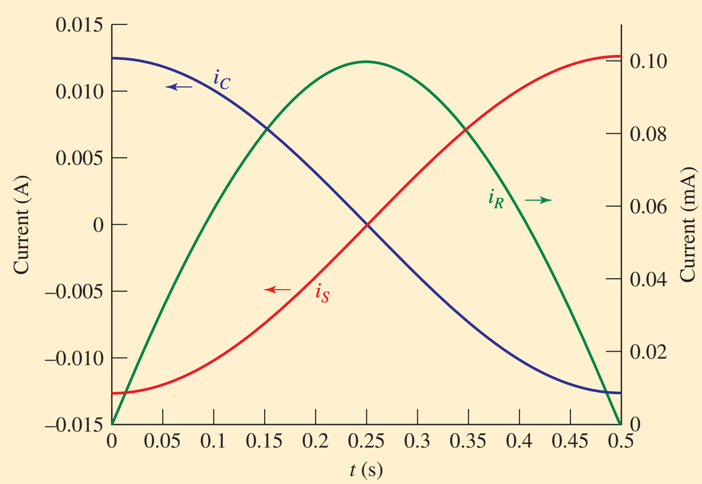
The resistor dissipated 2.5 mJ in the period of 0 to 500 ms, although the capacitor stored a maximum of 100 mJ at one point during that interval. What happened to the "other" 97.5 mJ? To answer this, we compute the capacitor current

$$i_C = 20 \times 10^{-6} \frac{dv}{dt} = 0.004\pi \cos 2\pi t$$

and the current i_s defined as flowing *into* the voltage source

$$i_s = -i_C - i_R$$

both of which are plotted in Fig. 7.9. We observe that the current flowing through the resistor is a small fraction of the source current, not entirely surprising as 1 MΩ is a relatively large resistance value. As current flows from the source, a small amount is diverted to the resistor, with the rest flowing into the capacitor as it charges. After $t = 250$ ms, the source current is seen to change sign; current is now flowing from the capacitor back into the source. Most of the energy stored in the capacitor is being returned to the ideal voltage source, except for the small fraction dissipated in the resistor.



■ FIGURE 7.9 Plot of the resistor, capacitor, and source currents during the interval of 0 to 500 ms.

PRACTICE

7.3 Calculate the energy stored in a 1000 μF capacitor at $t = 50\ \mu$s if the voltage across it is $1.5 \cos 10^5 t$ volts.

Ans: 90.52 μJ.

PRACTICE

7.4 The current through a 200 mH inductor is shown in Fig. 7.13. Assume the passive sign convention, and find v_L at t equal to (*a*) 0; (*b*) 2 ms; (*c*) 6 ms.

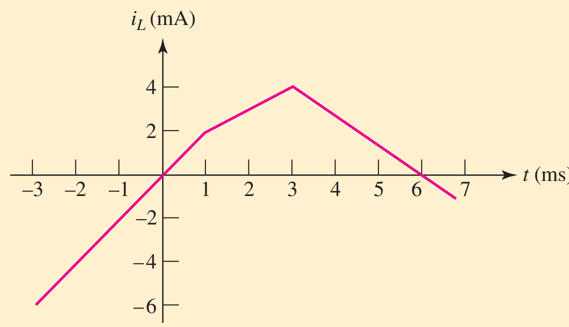


FIGURE 7.13

Ans: 0.4 V; 0.2 V; −0.267 V.

Let us now investigate the effect of a more rapid rise and decay of the current between the 0 and 1 A values.

EXAMPLE 7.5

Find the inductor voltage that results from applying the current waveform shown in Fig. 7.14*a* to the inductor of Example 7.4.

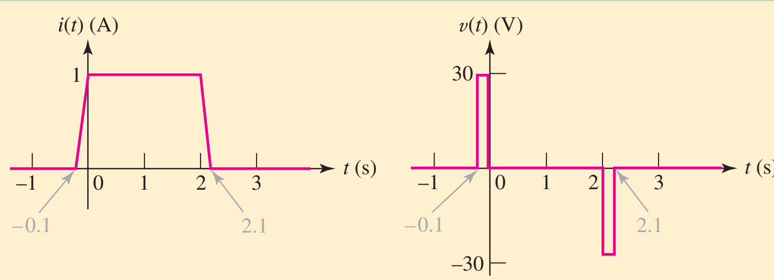
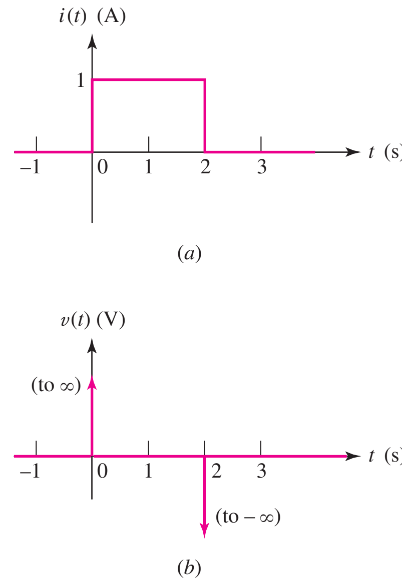


FIGURE 7.14 (*a*) The time required for the current of Fig. 7.12*a* to change from 0 to 1 and from 1 to 0 is decreased by a factor of 10. (*b*) The resultant voltage waveform. The pulse widths are exaggerated for clarity.

Note that the intervals for the rise and fall have decreased to 0.1 s. Thus, the magnitude of each derivative will be 10 times larger; this condition is shown in the current and voltage sketches of Fig. 7.14*a* and *b*. In the voltage waveforms of Fig. 7.13*b* and 7.14*b*, it is interesting to note that the area under each voltage pulse is 3 V · s.

Just for curiosity's sake, let's continue in the same vein for a moment. A further decrease in the rise and fall times of the current waveform will produce a proportionally larger voltage magnitude, but only within the interval in which the current is increasing or decreasing. An abrupt change in the current will cause the infinite voltage "spikes" (each having an area of 3 V · s) that are suggested by the waveforms of Fig. 7.15*a* and *b*; or, from the equally valid but opposite point of view, these infinite voltage spikes are required to produce the abrupt changes in the current.



■ FIGURE 7.15 (*a*) The time required for the current of Fig. 7.14*a* to change from 0 to 1 and from 1 to 0 is decreased to zero; the rise and fall are abrupt. (*b*) The resultant voltage across the 3 H inductor consists of a positive and a negative infinite spike.

PRACTICE

7.5 The current waveform of Fig. 7.14*a* has equal rise and fall times of duration 0.1 s (100 ms). Calculate the maximum positive and negative voltages across the same inductor if the rise and fall times, respectively, are changed to (*a*) 1 ms, 1 ms; (*b*) 12 μs, 64 μs; (*c*) 1 s, 1 ns.

Ans: 3 kV, −3 kV; 250 kV, −46.88 kV; 3 V, −3 GV.

Integral Voltage-Current Relationships

We have defined inductance by a simple differential equation,

$$v = L\frac{di}{dt}$$

and we have been able to draw several conclusions about the characteristics of an inductor from this relationship. For example, we have found that we may consider an inductor to be a short circuit to direct current, and we have agreed that we cannot permit an inductor current to change abruptly from one value to another, because this would require that an infinite voltage and power be associated with the inductor. The simple defining equation for inductance contains still more information, however. Rewritten in a slightly different form,

$$di = \frac{1}{L}v\,dt$$

it invites integration. Let us first consider the limits to be placed on the two integrals. We desire the current i at time t, and this pair of quantities therefore provides the upper limits on the integrals appearing on the left and right sides of the equation, respectively; the lower limits may also be kept general by merely assuming that the current is $i(t_0)$ at time t_0. Thus,

$$\int_{i(t_0)}^{i(t)} di' = \frac{1}{L}\int_{t_0}^{t} v(t')\,dt'$$

which leads to the equation

$$i(t) - i(t_0) = \frac{1}{L}\int_{t_0}^{t} v\,dt'$$

or

$$\boxed{i(t) = \frac{1}{L}\int_{t_0}^{t} v\,dt' + i(t_0)} \qquad [6]$$

Equation [5] expresses the inductor voltage in terms of the current, whereas Eq. [6] gives the current in terms of the voltage. Other forms are

Equation [8] is going to cause trouble with this particular voltage. We based the equation on the assumption that the current was zero when $t = -\infty$. To be sure, this must be true in the real, physical world, but we are working in the land of the mathematical model; our elements and forcing functions are all idealized. The difficulty arises after we integrate, obtaining

$$i(t) = 0.6 \sin 5t' \Big|_{-\infty}^{t}$$

and attempt to evaluate the integral at the lower limit:

$$i(t) = 0.6 \sin 5t - 0.6 \sin(-\infty)$$

The sine of $\pm\infty$ is indeterminate, and therefore we cannot evaluate our expression. Equation [8] is only useful if we are evaluating functions which approach zero as $t \to -\infty$.

PRACTICE

7.6 A 100 mH inductor has voltage $v_L = 2e^{-3t}$ V across its terminals. Determine the resulting inductor current if $i_L(-0.5) = 1$ A.

Ans: $-\frac{20}{3}e^{-3t} + 30.9$ A.

We should not make any snap judgments, however, as to which single form of Eqs. [6], [7], and [8] we are going to use forever after; each has its advantages, depending on the problem and the application. Equation [6] represents a long, general method, but it shows clearly that the constant of integration is a current. Equation [7] is a somewhat more concise expression of Eq. [6], but the nature of the integration constant is suppressed. Finally, Eq. [8] is an excellent expression, since no constant is necessary; however, it applies only when the current is zero at $t = -\infty$ and when the analytical expression for the current is not indeterminate there.

Energy Storage

Let us now turn our attention to power and energy. The absorbed power is given by the current-voltage product

$$p = vi = Li\,\frac{di}{dt}$$

The energy w_L accepted by the inductor is stored in the magnetic field around the coil. The change in this energy is expressed by the integral of the power over the desired time interval:

$$\int_{t_0}^{t} p\,dt' = L\int_{t_0}^{t} i\,\frac{di}{dt'}\,dt' = L\int_{i(t_0)}^{i(t)} i'\,di'$$

$$= \frac{1}{2}L\left\{[i(t)]^2 - [i(t_0)]^2\right\}$$

Thus,

$$w_L(t) - w_L(t_0) = \tfrac{1}{2}L\left\{[i(t)]^2 - [i(t_0)]^2\right\} \qquad [9]$$

or

$$w_R = \int_0^6 14.4\left(\frac{1}{2}\right)\left(1 - \cos\frac{\pi}{3}t\right) dt = 43.2 \text{ J}$$

Thus, we have expended 43.2 J in the process of storing and then recovering 216 J in a 6 s interval. This represents 20 percent of the maximum stored energy, but it is a reasonable value for many coils having this large an inductance. For coils having an inductance of about 100 μH, we might expect a figure closer to 2 or 3 percent.

PRACTICE

7.7 Let $L = 25$ mH for the inductor of Fig. 7.10. (*a*) Find v_L at $t = 12$ ms if $i_L = 10te^{-100t}$ A. (*b*) Find i_L at $t = 0.1$ s if $v_L = 6e^{-12t}$ V and $i_L(0) = 10$ A. If $i_L = 8(1 - e^{-40t})$ mA, find (*c*) the power being delivered to the inductor at $t = 50$ ms and (*d*) the energy stored in the inductor at $t = 40$ ms.

Ans: -15.06 mV; 24.0 A; 7.49 μW; 0.510 μJ.

We summarize by listing four key characteristics of an inductor which result from its defining equation $v = L\,di/dt$:

Important Characteristics of an Ideal Inductor

1. There is no voltage across an inductor if the current through it is not changing with time. An inductor is therefore a *short circuit to dc*.
2. A finite amount of energy can be stored in an inductor even if the voltage across the inductor is zero, such as when the current through it is constant.
3. It is impossible to change the current through an inductor by a finite amount in zero time, for this requires an infinite voltage across the inductor. (An inductor resists an abrupt change in the current through it in a manner analogous to the way a mass resists an abrupt change in its velocity.)
4. The inductor never dissipates energy, but only stores it. Although this is true for the *mathematical* model, it is not true for a *physical* inductor due to series resistances.

It is interesting to anticipate our discussion of ***duality*** in Sec. 7.6 by rereading the previous four statements with certain words replaced by their "duals." If *capacitor* and *inductor, capacitance* and *inductance, voltage* and *current, across* and *through, open circuit* and *short circuit, spring* and *mass,* and *displacement* and *velocity* are interchanged (in either direction), the four statements previously given for capacitors are obtained.