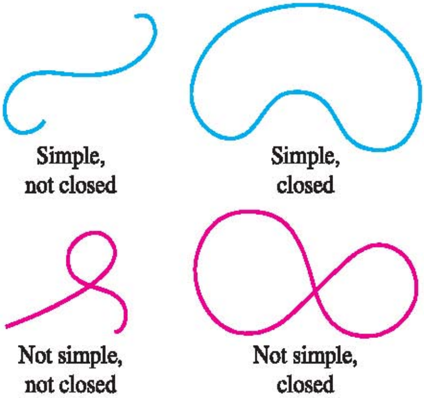
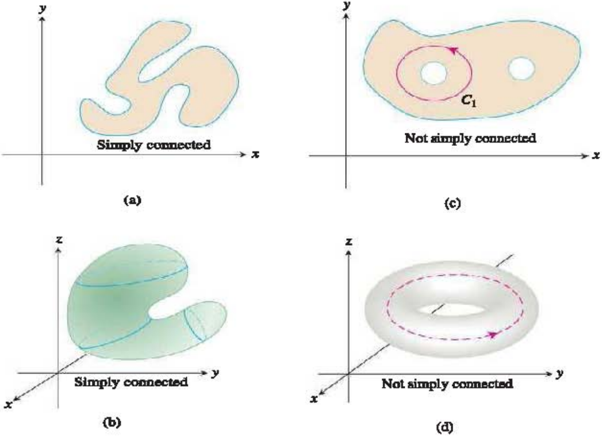
MA101

Green's Theorem & Surface Integral

Amit K. Verma

Department of Mathematics
IIT Patna


Simple, not closed
Simple, closed
Not simple, not closed
Not simple, closed


y
x
Simply connected
(a)
y
x
C_1
Not simply connected
(c)
z
y
x
Simply connected
(b)
z
y
x
Not simply connected
(d)

Green's Theorem says that under suitable conditions the **outward flux** of a vector field across a **simple closed curve** in the plane equals the **double integral of the divergence** of the field over the region enclosed by the curve.

Theorem 3 Green's Theorem (Flux-Divergence or Normal Form)

The outward flux of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ across a simple closed curve C equals the double integral of **div F** over the region R enclosed by C.

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy$$

Problem 15, 16.4

Find the outward flux of the field $F = xy\hat{i} + y^2\hat{j}$ around and over the boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.

Green's Theorem says that the **counterclockwise circulation** of a vector field around a **simple closed curve** is the **double integral** of the **k-component of the curl** of the field over the region enclosed by the curve.

Theorem 4 Green's Theorem (Circulation-Curl or Tangential Form)

The counterclockwise circulation of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ around a simple closed curve C in the plane equals the double integral of $Curl\ \mathbf{F} \cdot k$ over the region R enclosed by C.

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Remark

The two forms of Green's Theorem are equivalent. Applying Theorem 3 to the field $G_1 = N\mathbf{i} - M\mathbf{j}$ gives Theorem 4, and applying Theorem 4 to $G_2 = -N\mathbf{i} + M\mathbf{j}$ gives Theorem 3.

Problem 15, 16.4

Find the counterclockwise circulation of the field $F = xy\hat{i} + y^2\hat{j}$ around and over the boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.

Problems

Calculating Area with Green's Theorem

If a **simple closed curve** C in the plane and the region R it encloses satisfy the hypotheses of Green's Theorem, the area of R is given by

$$\textbf{Area of R} = \frac{1}{2}\oint_{C} \mathbf{xdy - ydx}.$$

Example: Compute area of Ellipse.

Surface Integral

Suppose, for example, that we have an **electrical charge** distributed over a surface $f(x,y,z)=c$ and that the function $g(x,y,z)$ gives the **charge per unit area (charge density)** at each point on S. Then we may calculate the total charge on S.

$$\text{Total Charge} \approx \sum g(x_k,y_k,z_k)\Delta P_k = \sum g(x_k,y_k,z_k)\frac{A_k}{|\cos\gamma_k|}$$

$$\text{which gives} \iint_R g(x,y,z)\frac{dA}{|\cos\gamma_k|} = \iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f\cdot p|}dA.$$

Surface Integral

If R is the shadow region of a surface S defined by the equation $f(x,y,z)=c$ and g is a **continuous function** defined at the points of S, then the integral of g over S is the integral

$$\iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f\cdot p|}dA$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla f\cdot\ \mathbf{p}\neq 0$. The integral itself is called a surface integral.

The integral in Equation

$$\iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot p|} dA$$

takes on different meanings in different applications.

- If g has the **constant value 1**, the integral gives the **area of surface** S.
- If g gives the **mass density** of a thin shell of material modeled by S, the integral gives the **mass of the shell**.

The Surface Area Differential and the Differential Form for Surface Integrals

Surface Area Differential:

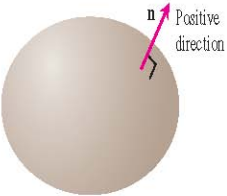
$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot p|} dA,$$

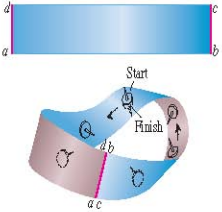
Differential formula for surface integrals:

$$\iint_S g(x, y, z) d\sigma.$$

Orientation

- For surfaces embedded in Euclidean space, an orientation is specified by the choice of a continuously varying surface normal $\mathbf{n}$ at every point.
- If such a normal exists at all, then there are always two ways to select it: $\mathbf{n}$ or $-\mathbf{n}$. More generally, an orientable surface admits exactly two orientations.
- Any patch or subportion of an orientable surface is orientable.
- Spheres and other smooth closed surfaces in space (smooth surfaces that enclose solids) are orientable.
- By convention, we choose $\mathbf{n}$ on a closed surface to point outward.
- **The Möbius band is not orientable.**





Surface Integral for Flux

Flux

The flux of a three-dimensional vector field **F** across an oriented surface S

$$\textbf{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

If S is part of a level surface $g(x, y, z) = c$ then **n** may be taken to be one of the two fields

$$\mathbf{n} = \pm \frac{\nabla g}{|\nabla g|}$$

depending on which one gives the preferred direction. The corresponding flux is

$$\textbf{Flux} = \iint_R \left(\mathbf{F} \cdot \pm \frac{\nabla g}{|\nabla g|} \right) \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} dA = \iint_R \left(\mathbf{F} \cdot \pm \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} \right) dA.$$

Special Formulas for Surface Area

Let $F = f(x, y) - z$ and R_{xy} is the projection of the surface area on xy-plane. The surface area of $z = f(x, y)$ lying above the region R_{xy} is

$$\iint\limits_{R_{xy}} \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA = \iint\limits_{\mathbf{R_{xy}}} \sqrt{\mathbf{f_x^2 + f_y^2 + 1dxdy}}.$$

Similarly, the area of a smooth surface $x = f(y, z)$ over a region in the $yz-$plane is

$$\iint\limits_{\mathbf{R_{yz}}} \sqrt{\mathbf{f_y^2 + f_z^2 + 1dxdy}},$$

and the area of a smooth surface $y = f(x, z)$ over a region in the $xz-$plane is

$$\iint\limits_{\mathbf{R_{xz}}} \sqrt{\mathbf{f_x^2 + f_z^2 + 1dxdz}}.$$

Green's Theorem

$\Downarrow$

$\swarrow \searrow$

Flux (Normal) Form (2D) $\qquad$ **Circulation (Tangential) Form (2D)**

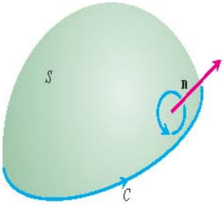
$\Downarrow$ $\qquad$ $\Downarrow$

Divergence Theorem in 3D $\qquad$ **Stokes' Theorem in 3D**

Stokes' Theorem

Stokes' Theorem says that, the **circulation** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ around the boundary C of an **oriented surface** S in space in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $(\nabla \times \mathbf{F} \cdot \mathbf{n})$ over the surface S.

$$\frac{\textbf{Counterclockwise}}{\textbf{circulation}} = \oint_C \mathbf{F} \cdot \, d\mathbf{r} = \iint_S (\nabla \times \, \mathbf{F}) \cdot \, \mathbf{n} d\sigma = \textbf{Curl integral}.$$



An Important Identity

$$\textit{Curl Grad } f \quad = \nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= (f_{zy} - f_{yz})\mathbf{i} + (f_{zx} - f_{xz})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k} = 0$$

If the second partial derivatives are continuous, the mixed second derivatives in parentheses are equal (Theorem 2, Section 14.3) and the vector is zero.

Curl **F**=0 Related to the Closed-Loop Property

If $\nabla \times \mathbf{F} = 0$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Zero curl yet Field is not conservative

Show that curl of

$$\mathbf{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{i} + z\hat{k}$$

is zero but $\oint\limits_C \mathbf{F} \cdot \, d\mathbf{r} \neq 0$ if C is the circle $x^2 + y^2 = 1$.

The Divergence Theorem and a Unified Theory

Divergence in Three Dimensions

The divergence of a vector field $F = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$ is the scalar function

$$div\ \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

The symbol "*div* **F**" is read as "divergence of **F**" or "div **F**." The notation is read "del dot F."

Divergence Theorem

The flux of a vector field $\mathbf{F}$ across a closed oriented surface S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the integral of $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\textbf{Outward Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V \nabla \cdot \mathbf{F} \, dV = \textbf{Divergence Integral}$$

Wishing all the best for End Semester Examination.