Convex Optimization - AI2101
Assignment - VII

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Question 4.1

Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & f_0(x_1, x_2) \\ \text{subject to} \quad & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

1. $f_0(x_1, x_2) = x_1 + x_2$
2. $f_0(x_1, x_2) = -x_1 - x_2$
3. $f_0(x_1, x_2) = x_1$
4. $f_0(x_1, x_2) = \max\{x_1, x_2\}$
5. $f_0(x_1, x_2) = x_1^2 + 9x_2^2$

Solution:

We have to minimize $f_0(x_1, x_2)$ subject to:

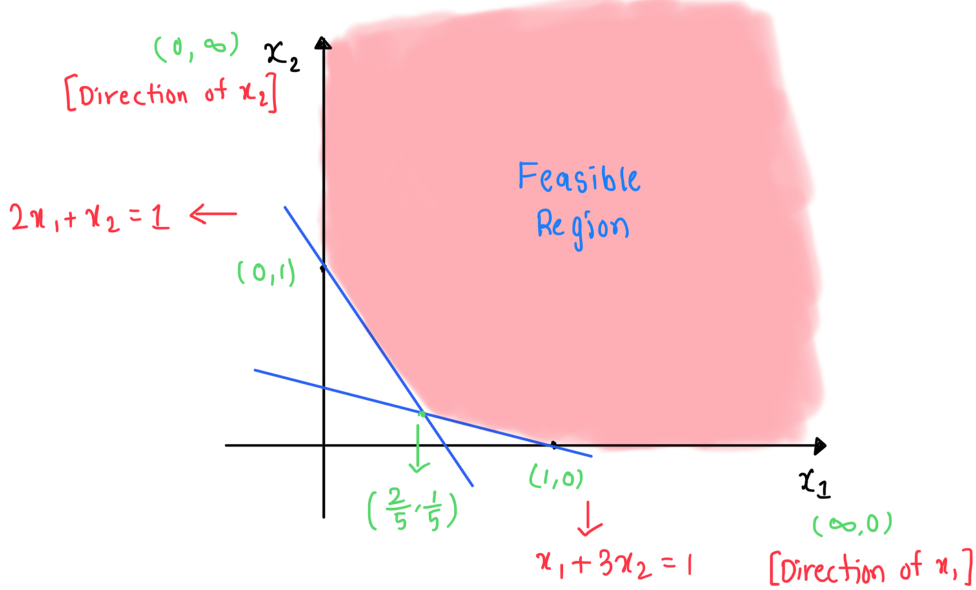
$$\begin{aligned} 2x_1 + x_2 &\geq 1 \\ x_1 + 3x_2 &\geq 1 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

The intersection of all these half planes (convex) will form the corresponding feasible region in the (x_1, x_2) plane. Feasible set follows all the constraints, and the boundary points of this region are:

$$\begin{aligned} x_1 = 0,\ 2x_1 + x_2 = 1 &\Rightarrow (0, 1) \\ x_2 = 0,\ x_1 + 3x_2 = 1 &\Rightarrow (1, 0) \\ 2x_1 + x_2 = 1,\ x_1 + 3x_2 = 1 &\Rightarrow \left(\frac{2}{5}, \frac{1}{5}\right) \end{aligned}$$

But we should note that the convex hull of only these three points will not result in the feasible set.
We also have to add the points $(0, \infty)$ and $(\infty, 0)$, which denote
the directions towards those points in the feasible set.
Feasible set is the convex hull of:

$$(0, \infty), (1, 0), (0, 1), \left(\frac{2}{5}, \frac{1}{5}\right), (\infty, 0)$$



(a) $f_0(x_1,x_2) = x_1 + x_2$

When we are trying to minimize $x_1 + x_2$, it should be close to the origin as much as possible, and from the feasible region, $\left(\frac{2}{5},\frac{1}{5}\right)$ is the closest.

$$x^* = \left(\frac{2}{5},\frac{1}{5}\right), \quad f_0(x^*) = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}$$

(b) $f_0(x_1,x_2) = -x_1 - x_2$

Minimizing $(-x_1 - x_2)$ is the same as maximizing $(x_1 + x_2)$.

So we can go to infinity (x_1 and x_2) to maximize $x_1 + x_2$. $-(x_1 + x_2)$ will go to negative infinity.
The function $f_0(x_1,x_2)$ is unbounded below.

(c) $f_0(x_1,x_2) = x_1$

Since we want the smallest x_1, we go to the very left of the feasible region (vertical edge or extreme left).

Minimum occurs on $\{(0,x_2) \mid x_2 \geq 1\}$, so $f_0 = x_1 = 0$.

So the point lies on $x_1 = 0$ above $x_2 \geq 1$.

(d) $f_0(x_1,x_2) = \max(x_1,x_2)$

$\max(x_1,x_2)$ is minimized when x_1 and x_2 are equal and as small as possible in the feasible region.

Considering the intersection of $x_1 = x_2 = k$ with both lines enclosing the feasible region:

$$x_1 = x_2 = k, \quad 2x_1 + x_2 \geq 1 \Rightarrow \left(\frac{1}{3},\frac{1}{3}\right), \quad (k \geq \frac{1}{3})$$
$$x_1 = x_2 = k, \quad x_1 + 3x_2 \geq 1 \Rightarrow \left(\frac{1}{4},\frac{1}{4}\right), \quad (k \geq \frac{1}{4})$$

$k \geq \frac{1}{3}$ would be the case (satisfying both).

Smallest such k is $\frac{1}{3}$, and $(x_1, x_2) = \left(\frac{1}{3}, \frac{1}{3}\right)$

$$x^* = \left(\frac{1}{3}, \frac{1}{3}\right), \quad f_0(x^*) = \max\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$$

(e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$

This quadratic function is shaped like an ellipse.

We want to minimize x_1, x_2, but smaller x_2 is better since it has a coefficient of 9 (i.e., $9x_2^2$ dominates).

The optimal case could be at a point whose tangent is one of the lines enclosing the feasible region.

$$\text{Let } x_1^2 + 9x_2^2 = t$$

(i) Line $x_1 + 3x_2 = 1$ is a tangent:

$$\begin{aligned} x_1 &= 1 - 3x_2 \\ (1 - 3x_2)^2 + 9x_2^2 &= t \\ 9x_2^2 + (1 - 3x_2)^2 &= t \\ 9x_2^2 + 1 - 6x_2 + 9x_2^2 &= t \\ 18x_2^2 - 6x_2 + 1 - t &= 0 \end{aligned}$$

Making $\Delta = 0$ (tangent condition):

$$\begin{aligned} (-6)^2 - 4(18)(1 - t) &= 0 \\ 36 = 72(1 - t) \Rightarrow 1 - t = \frac{1}{2} \Rightarrow t &= \frac{1}{2} \end{aligned}$$

So, $x_1^2 + 9x_2^2 = \frac{1}{2}$

(ii) Line $2x_1 + x_2 = 1$ is a tangent:

$$\begin{aligned} x_2 &= 1 - 2x_1 \\ x_1^2 + 9(1 - 2x_1)^2 &= t \\ x_1^2 + 9(1 - 4x_1 + 4x_1^2) &= t \\ \Rightarrow x_1^2 + 9 - 36x_1 + 36x_1^2 &= t \\ \Rightarrow 37x_1^2 - 36x_1 + 9 - t &= 0 \end{aligned}$$

Making $\Delta = 0$ (tangent condition):

$$\begin{aligned} (-36)^2 - 4(37)(9 - t) &= 0 \\ 1296 = 148(9 - t) \Rightarrow t &= \frac{9}{37} \end{aligned}$$

So, $x_1^2 + 9x_2^2 = \frac{9}{37}$

Even though $\frac{9}{37}$ is smaller, the corresponding point $(x_1, x_2) = (0.487, 0.027)$ is **not** in the feasible region.

When $x_1^2 + 9x_2^2 = \frac{1}{2}$:
The point lies in the feasible region and the Optimal Solution in this casse:

$$x_2 = \frac{1}{6}, \quad x_1 = \frac{1}{2}$$
$$x^* = \left(\frac{1}{2}, \frac{1}{6}\right), \quad f_0(x^*) = \frac{1}{2}$$

Question 4.3

Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}x^T P x + q^T x + r \\ \text{subject to} \quad & -1 \le x_i \le 1, \quad i = 1, 2, 3, \end{aligned}$$

where
$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \quad q = \begin{bmatrix} -22.0 \\ -14.5 \\ 13.0 \end{bmatrix}, \quad r = 1,$$

and the proposed solution is $x^* = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix}$.

Solution:

We have the optimization problem to minimize

$$f(x) = \frac{1}{2}x^T P x + q^T x + r$$

subject to $-1 \le x_i \le 1, \quad i = 1, 2, 3$
We first verify that the objective function is convex (since this is a convex optimization problem).
We need not check the convexity of $q^T x + r$ since it is a linear function.
So, this quadratic function is convex if and only if the matrix P is positive semidefinite (PSD).
Given:

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}$$

To check if P is PSD, we can check if all leading principal minors are positive (Sylvester's Criterion).

- First minor: $\det([13]) = 13 > 0$
- Second minor:
$$\det\left(\begin{bmatrix} 13 & 12 \\ 12 & 17 \end{bmatrix}\right) = 13 \cdot 17 - 12 \cdot 12 = 221 - 144 = 77 > 0$$
- Third minor:
$$\det(P) = \begin{vmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{vmatrix}$$

Expanding along the first row:

$$\det(\mathrm{P}) = 13 \cdot (17 \cdot 12 - 6 \cdot 6) - 12 \cdot (12 \cdot 12 - 6 \cdot (-2)) + (-2) \cdot (12 \cdot 6 - 17 \cdot (-2))$$

$$\det(\mathrm{P}) = 13(204 - 36) - 12(144 + 12) - 2(72 + 34)$$

$$\det(\mathrm{P}) = 13 \cdot 168 - 12 \cdot 156 - 2 \cdot 106 = 2184 - 1872 - 212 = 100 > 0$$

Since all leading principal minors are positive, P is positive semidefinite.
So, the function $\frac{1}{2}x^T Px + q^T x + r$ Objective Function is convex.
Now, to verify the optimal solution x, it is optimal if and only if

$$\nabla f_0(x)^T (y - x) \geq 0 \quad \forall y \in X$$

$$\nabla f_0(x) = Px + q$$

We put our optimal solution $x = x^* = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix}$

$$Px = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} 21 \\ \frac{29}{2} \\ -11 \end{bmatrix}$$

$$Px + q = \begin{bmatrix} 21 \\ \frac{29}{2} \\ -11 \end{bmatrix} + \begin{bmatrix} -22 \\ -\frac{29}{2} \\ 13 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \nabla f_0(x)^T = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}$$

Let

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Then,

$$\nabla f_0(x)^T (y - x) = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} \right)$$

$$\nabla f_0(x)^T (y - x) = (-1)(y_1 - 1) + 0(y_2 - \tfrac{1}{2}) + 2(y_3 + 1)$$

$$\nabla f_0(x)^T (y - x) = -y_1 + 1 + 2y_3 + 2 = 3 - y_1 + 2y_3$$

So,

$$\nabla f_0(x)^T (y - x) = 3 - y_1 + 2y_3$$

Since $\forall y \in X$, this condition should be satisfied:

$$-1 \leq x_i \leq 1 \quad \forall i = 1, 2, 3$$

We find the minimum of $3 - y_1 + 2y_3$ at $y_1 = 1$, $y_3 = -1$ (maximize y_1, minimize y_3).

$$\min(3 - y_1 + 2y_3) = 3 - 1 + 2(-1) = 0$$

So we have:

$$3 - y_1 + 2y_3 \geq 0 \quad \forall y \in X$$

So,

$$\nabla f(x^*)(y - x^*) \geq 0 \quad \forall y \in X$$

$$\text{when } x^* = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

$$\therefore \text{So } x^* = \left(1, \frac{1}{2}, -1\right) \text{ is an optimal solution.}$$

Question 4.12

Network flow problem. Consider a network of n nodes, with directed links connecting each pair of nodes. The variables in the problem are the flows on each link: x_{ij} will denote the flow from node i to node j. The cost of the flow along the link from node i to node j is given by $c_{ij}x_{ij}$, where c_{ij} are given constants. The total cost across the network is

$$C = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}.$$

Each link flow x_{ij} is also subject to a given lower bound l_{ij} (usually assumed to be nonnegative) and an upper bound u_{ij}.
The external supply at node i is given by b_i, where $b_i > 0$ means an external flow enters the network at node i, and $b_i < 0$ means that at node i, an amount $|b_i|$ flows out of the network. We assume that $\mathbf{1}^T b = 0$, i.e., the total external supply equals the total external demand.
At each node we have conservation of flow: the total flow into node i along links and the external supply, minus the total flow out along the links, equals zero.
The problem is to minimize the total cost of flow through the network, subject to the constraints described above. Formulate this problem as a linear program (LP).

Solution:

We are given that the total cost across the network

$$C = \sum_{i,j=1}^{n} c_{ij}x_{ij}$$

We intend to minimize C.
Here x_{ij} is the flow from node i to node j.
We have to reframe this into the form of Linear Programming
(Objective function and constraints should be linear)

$$\text{Minimize } f(x) = c^T x$$

$$\text{subject to constraints } a_i^T x \leq b_i$$

Here the objective function is:

$$C = \sum_{i,j=1}^{n} c_{ij}x_{ij} \quad \text{(Linear function)}$$

The constraints here are:

1. $l_{ij} \leq x_{ij} \leq u_{ij} \to$ linear constraint

2. Flow must be conserved.
   Inflow at node $i = b_i + \sum_{j=1}^{n} x_{ji}$
   Outflow from node $j = \sum_{j=1}^{n} x_{ij}$

$$b_i + \sum_{j=1}^{n} x_{ji} - \sum_{j=1}^{n} x_{ij} = 0, \quad i = 1, 2, \ldots, n$$

   THis is a linear constraint

Thus, reframed as LP,

$$\text{minimize } C = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to:

$$b_i + \sum_{j=1}^{n} x_{ji} - \sum_{j=1}^{n} x_{ij} = 0, \quad i = 1, 2, \ldots, n$$
$$l_{ij} \leq x_{ij} \leq u_{ij}$$

Question 4.15

Relaxation of Boolean LP.
In a Boolean linear program, the variable x is constrained to have components equal to zero or one:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0, 1\}, \quad i = 1, \ldots, n. \end{array} \tag{4.67}$$

In general, such problems are very difficult to solve, even though the feasible set is finite (containing at most 2^n points).
In a general method called *relaxation*, the constraint that x_i be zero or one is replaced with the linear inequalities $0 \leq x_i \leq 1$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \leq x_i \leq 1, \quad i = 1, \ldots, n. \end{array} \tag{4.68}$$

We refer to this problem as the LP relaxation of the Boolean LP (4.67). The LP relaxation is far easier to solve than the original Boolean LP.

(a) Show that the optimal value of the LP relaxation (4.68) is a lower bound on the optimal value of the Boolean LP (4.67). What can you say about the Boolean LP if the LP relaxation is infeasible?

(b) It sometimes happens that the LP relaxation has a solution with $x_i \in \{0, 1\}$. What can you say in this case?

Solution:

(a) Given, Boolean LP:

$$\text{minimize } c^T x$$

$$\text{subject to } Ax \preceq b$$

$$x_i \in \{0,1\}, \quad i = 1,2,\ldots,n$$

Relaxation:

$$\text{minimize } c^T x$$

$$\text{subject to } Ax \preceq b$$

$$0 \leq x_i \leq 1, \quad i = 1,2,\ldots,n$$

In the Boolean case, $x_i \in \{0,1\}$ $\forall i$ is a subset of the relaxation case $0 \leq x_i \leq 1$.
Now we can say that the feasible set of Boolean LP is (a subset) also in the feasible set of LP relaxation.
For the proof, we will consider the contradiction: Boolean produces a lesser optimum value than relaxation LP.
Let us take x_B^* as the solution such that

$$\text{opt}_B < \text{opt}_{LP}$$

But from our proposition, i.e., from LP relaxation:

$$\text{opt}_{LP} = \text{opt}_B$$

This is a contradiction.
We can say any solution reachable from Boolean is possible from LP relaxation too.
However, x_B^* may not be optimal for LP relaxation. Because of larger feasible set,

$$\text{opt}_{LP} \leq \text{opt}_B$$

$\therefore$ If LP relaxation is infeasible, from the proposition Boolean LP will also be infeasible.

(b) For an optimal solution of LP to be an optimal solution of Boolean LP,
it should be $x_i \in \{0,1\}$ (so that it resides in feasible set of Boolean LP too)