Convex Optimization - AI2101
Assignment - IX

Bhuvan Chandra K
AI23BTECH11013

May 7, 2025

Question 5.26

Consider the QCQP

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & (x_1-1)^2 + (x_2-1)^2 \leq 1 \\ & (x_1-1)^2 + (x_2+1)^2 \leq 1 \end{array}$$

with variable $x \in \mathbb{R}^2$.

(a) Sketch the feasible set and level sets of the objective. Find the optimal point $x^\star$ and optimal value $p^\star$.

(b) Give the KKT conditions. Do there exist Lagrange multipliers $\lambda_1^\star$ and $\lambda_2^\star$ that prove that $x^\star$ is optimal?

(c) Derive and solve the Lagrange dual problem. Does strong duality hold?

Solution:

(a)

We have two constraints,

$$g_1(x) = (x_1-1)^2 + (x_2-1)^2 - 1 \leq 0$$
$$g_2(x) = (x_1-1)^2 + (x_2+1)^2 - 1 \leq 0$$

These are closed unit discs with centres at $(1,1)$ and $(1,-1)$ respectively.
They only intersect at their point of contact. They touch each other (distance between centres $= \sqrt{1^2+2^2} = 2$)
Sum of radii $= 1 + 1 = 2$
Point of contact is the mid point of line joining the centres:

$$\left(\frac{1+1}{2}, \frac{1+(-1)}{2}\right) = (1,0)$$

The feasible set contains only one point $x^* = (1,0)$.
Since the feasible set has only one point, it is the optimum solution because

$$\min_{x \in \text{feasible set}} f(x) = f(x^*)$$

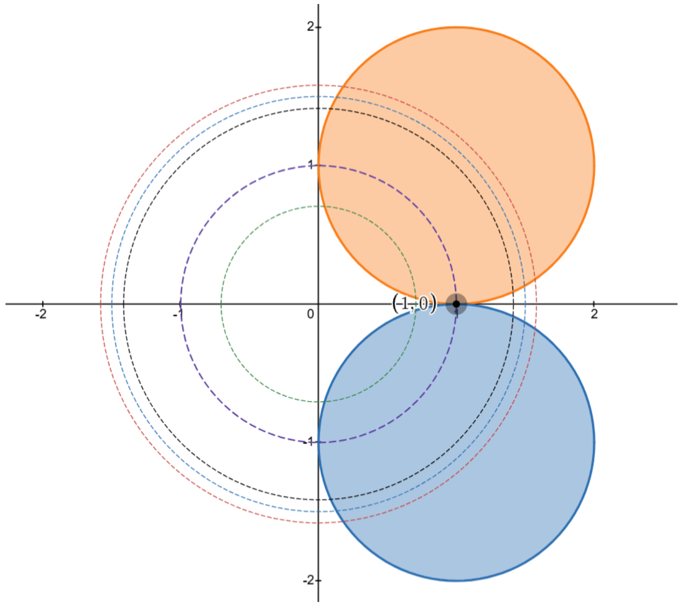
$$x^* = (1,0), \quad p^* = 1^2 + 0^2 = 1$$

Level sets are defined as:
For any constant $c \geq 0$,

$$x_1^2 + x_2^2 = c$$

The smallest circle that intersects the feasible set should pass through $(1,0)$.

$$x_1^2 + x_2^2 = 1^2 + 0^2 = 1$$



(b)

Let us consider the Lagrange multipliers $\lambda_1, \lambda_2 \geq 0$ for $g_1(x) \leq 0$ and $g_2(x) \leq 0$.

We take objective function as $f(x) = x_1^2 + x_2^2$

Here, KKT conditions at (x^*, λ^*) are:

$$\rightarrow g_1(x^*) \leq 0, \quad g_2(x^*) \leq 0$$
$$\rightarrow \lambda_1^* \geq 0, \quad \lambda_2^* \geq 0$$
$$\rightarrow \lambda_1^* g_1(x^*) = \lambda_2^* g_2(x^*) = 0$$

Define the Lagrangian $L(x, \lambda)$ as:

$$L(x,\lambda) = x_1^2 + x_2^2 + \lambda_1 \left[(x_1-1)^2 + (x_2-1)^2 - 1\right] + \lambda_2 \left[(x_1-1)^2 + (x_2+1)^2 - 1\right]$$

$$\rightarrow \nabla_x L(x^*, \lambda^*) = 0$$

We have $x^* = (1, 0)$

$$g_1(x^*) = (1-1)^2 + (0-1)^2 - 1 = 0 \quad [g_1(x^*) \leq 0]$$
$$g_2(x^*) = (1-1)^2 + (0+1)^2 - 1 = 0 \quad [g_2(x^*) \leq 0]$$

Since $g_1(x^*) = g_2(x^*) = 0$

$$\lambda_1^* g_1(x^*) = \lambda_2^* g_2(x^*) = 0 \quad \text{[irrespective of } \lambda_1^*, \lambda_2^*]$$

$L(x, \lambda) = f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x)$

$$\frac{\partial}{\partial x_1} L(x,\lambda) = \frac{\partial}{\partial x_1}\left(x_1^2 + x_2^2\right) + \lambda_1 \frac{\partial}{\partial x_1}\left((x_1-1)^2 + (x_2-1)^2 - 1\right) + \lambda_2 \frac{\partial}{\partial x_1}\left((x_1-1)^2 + (x_2+1)^2 - 1\right)$$

$$\frac{\partial}{\partial x_1} L(x,\lambda) = 2x_1 + \lambda_1 \cdot 2(x_1-1) + \lambda_2 \cdot 2(x_1-1)$$

$$\frac{\partial}{\partial x_2} L(x,\lambda) = \frac{\partial}{\partial x_2}(x_1^2 + x_2^2) + \lambda_1 \cdot \frac{\partial}{\partial x_2}\left((x_1-1)^2 + (x_2-1)^2 - 1\right) + \lambda_2 \cdot \frac{\partial}{\partial x_2}\left((x_1-1)^2 + (x_2+1)^2 - 1\right)$$

$$\frac{\partial}{\partial x_2} L(x, \lambda) = 2x_2 + \lambda_1 \cdot 2(x_2 - 1) + \lambda_2 \cdot 2(x_2 + 1)$$

We want:

$$\frac{\partial}{\partial x_1} L(x^*, \lambda^*) = 0, \quad \frac{\partial}{\partial x_2} L(x^*, \lambda^*) = 0$$

So, we put $x^* = (1, 0)$, $\lambda_1 = \lambda_1^*, \lambda_2 = \lambda_2^*$

$$2(1) + \lambda_1^* \cdot 2(1 - 1) + \lambda_2^* \cdot 2(1 - 1) = 0 \Rightarrow 2 = 0 \tag{1}$$

$$2(0) + \lambda_1^* \cdot 2(0 - 1) + \lambda_2^* \cdot 2(0 + 1) = 0 \Rightarrow -2\lambda_1^* + 2\lambda_2^* = 0 \tag{2}$$

Since (1) is not possible, this condition fails.
No choice of λ_1^*, λ_2^* can satisfy this condition to prove x^* is optimal.

(c)

We define the dual function as:

$$g(\lambda) = \inf_{x \in \mathbb{R}^2} L(x, \lambda)$$

$$L(x, \lambda) = f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x)$$

$$L(x, \lambda) = x_1^2 + x_2^2 + \lambda_1 \left((x_1 - 1)^2 + (x_2 - 1)^2 - 1 \right) + \lambda_2 \left((x_1 - 1)^2 + (x_2 + 1)^2 - 1 \right)$$

$$L(x, \lambda) = x_1^2 + x_2^2 + \lambda_1 (x_1^2 + x_2^2 - 2x_1 - 2x_2 + 1) + \lambda_2 (x_1^2 + x_2^2 - 2x_1 + 2x_2 + 1)$$

$$L(x, \lambda) = (1 + \lambda_1 + \lambda_2)(x_1^2 + x_2^2) - 2(\lambda_1 + \lambda_2)x_1 - 2(\lambda_1 - \lambda_2)x_2 + \lambda_1 + \lambda_2$$

Since $g(\lambda)$ is the infimum of $L(x, \lambda)$, we find $x = [x_1, x_2]$ to minimize $L(x, \lambda)$.
If $1 + \lambda_1 + \lambda_2 \geq 0$, then it is a convex (upward) quadratic with a finite minimum.
If $1 + \lambda_1 + \lambda_2 < 0$, then it is a concave (downward) quadratic with $\inf_x L(x, \lambda) = -\infty$
$L(x, \lambda)$ is minimum at $\nabla_x L(x, \lambda) = 0$,
That is,

$$\frac{\partial}{\partial x_1} L(x, \lambda) = 0 \quad \text{and} \quad \frac{\partial}{\partial x_2} L(x, \lambda) = 0$$

From (b), we have:

$$\frac{\partial}{\partial x_1} L(x, \lambda) = 2x_1 + 2\lambda_1 x_1 + 2\lambda_2 x_1 - 2(\lambda_1 + \lambda_2) = 0$$

$$x_1 = \frac{\lambda_1 + \lambda_2}{1 + \lambda_1 + \lambda_2}$$

$$\frac{\partial}{\partial x_2} L(x, \lambda) = 2x_2 + 2\lambda_1 x_2 + 2\lambda_2 x_2 - 2\lambda_1 + 2\lambda_2 = 0$$

$$x_2 = \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 + \lambda_2}$$

Thus, $L(x, \lambda)$ is minimized at:

$$x = (x_1, x_2) = \left(\frac{\lambda_1 + \lambda_2}{1 + \lambda_1 + \lambda_2}, \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 + \lambda_2} \right)$$

Second derivatives:

$$\frac{\partial^2}{\partial x_1^2} L(x, \lambda) = 2 + 2\lambda_1 + 2\lambda_2 = 2(1 + \lambda_1 + \lambda_2)$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} L(x, \lambda) = 0 \quad \frac{\partial^2}{\partial x_2 \partial x_1} L(x, \lambda) = 0$$

$$\frac{\partial^2}{\partial x_2^2} L(x,\lambda) = 2 + 2\lambda_1 + 2\lambda_2 = 2(1+\lambda_1+\lambda_2)$$

The Hessian matrix is:

$$\text{Hessian} = \begin{pmatrix} \frac{\partial^2 L(x,\lambda)}{\partial x_1^2} & \frac{\partial^2 L(x,\lambda)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L(x,\lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x,\lambda)}{\partial x_2^2} \end{pmatrix}$$

$$\text{Hessian} = \begin{pmatrix} 2(1+\lambda_1+\lambda_2) & 0 \\ 0 & 2(1+\lambda_1+\lambda_2) \end{pmatrix}$$

From our assumed constraint $1+\lambda_1+\lambda_2 \geq 0$, the Hessian is positive semi-definite.
We verified that the point is a minimum.
Substituting $x_{\min}$ in $L(x,\lambda)$:

$$g(\lambda) = (1+\lambda_1+\lambda_2)\left(\frac{\lambda_1+\lambda_2}{1+\lambda_1+\lambda_2}\right)^2 + \left(\frac{\lambda_1-\lambda_2}{1+\lambda_1+\lambda_2}\right)^2$$

$$-2(\lambda_1+\lambda_2)\left(\frac{\lambda_1+\lambda_2}{1+\lambda_1+\lambda_2}\right) - 2(\lambda_1-\lambda_2)\left(\frac{\lambda_1-\lambda_2}{1+\lambda_1+\lambda_2}\right)$$

$$g(\lambda) = \frac{-2(\lambda_1^2+\lambda_2^2)}{1+\lambda_1+\lambda_2} + (\lambda_1+\lambda_2) \qquad \text{where } \lambda_1, \lambda_2 \geq 0$$

So,

$$g(\lambda) = \begin{cases} \frac{-2(\lambda_1^2+\lambda_2^2)}{1+\lambda_1+\lambda_2} + (\lambda_1+\lambda_2) & \text{if } 1+\lambda_1+\lambda_2 \geq 0 \\ -\infty & \text{if } 1+\lambda_1+\lambda_2 < 0 \end{cases}$$

Now the Lagrange dual problem becomes:

$$\begin{aligned} &\text{maximize} && g(\lambda) \\ &\text{subject to} && \lambda_1 \geq 0, \quad \lambda_2 \geq 0 \end{aligned} \tag{1}$$

i.e.,

$$\begin{aligned} &\text{maximize} && \frac{-2(\lambda_1^2+\lambda_2^2)}{1+\lambda_1+\lambda_2} + \lambda_1 + \lambda_2 \\ &\text{subject to} && \lambda_1 \geq 0, \quad \lambda_2 \geq 0 \end{aligned} \tag{2}$$

Let

$$\lambda_1 + \lambda_2 = a, \quad \lambda_1^2 + \lambda_2^2 = b$$

Then,

$$g(a) = a - \frac{2b}{1+a}$$

$$\frac{\partial g}{\partial \lambda_i} = \frac{\partial a}{\partial \lambda_i} - \frac{2\left[\frac{\partial b}{\partial \lambda_i}(1+a) - b\frac{\partial(1+a)}{\partial \lambda_i}\right]}{(1+a)^2}$$

Here,

$$\frac{\partial a}{\partial \lambda_i} = 1, \quad \frac{\partial b}{\partial \lambda_i} = 2\lambda_i$$

So,

$$\frac{\partial g}{\partial \lambda_i} = 1 - 2\left(\frac{2\lambda_i(1) - b(1)}{(1+a)^2}\right)$$

Making $\frac{\partial g}{\partial \lambda_i} = 0$, we get:

$$4\lambda_i(1+a) - 2b = (1+a)^2 \qquad \text{for } i = 1, 2$$

i.e.,

$$4\lambda_1(1+a) - 2b = (1+a)^2$$
$$4\lambda_2(1+a) - 2b = (1+a)^2$$

Subtracting these equations gives: $\lambda_1 = \lambda_2 = \lambda$
Substituting $\lambda_1 = \lambda_2 = \lambda$ in a, b:

$$a = 2\lambda, \quad b = 2\lambda^2$$

$$4\lambda(1+2\lambda) - 2 \cdot 2\lambda^2 = (1+2\lambda)^2$$

$$4\lambda + 4 \cdot 2\lambda^2 - 4\lambda^2 = 1 + 4\lambda + 4\lambda^2$$

$$4\lambda + 4\lambda^2 = 1 + 4\lambda + 4\lambda^2 \Rightarrow 0 = 1$$

This is impossible for any finite λ.
We get:

$$g(\lambda) = 2\lambda - \frac{4\lambda^2}{1+2\lambda}$$
$$g(\lambda) = \frac{2\lambda}{1+2\lambda}$$

As $\lambda \to \infty$, $g(\lambda) \to 1$

$$\lambda = (\lambda_1, \lambda_2)$$

We have $p^* = 1$ and $d^* = 1$ but optimum is not attained in the case of dual problem.

$$\therefore \text{Strong Duality does not hold.}$$