Convex Optimization - AI2101
Assignment - VIII

Bhuvan Chandra K
AI23BTECH11013

May 6, 2025

Question 5.1

A simple example. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq 0, \end{array}$$

with variable $x \in \mathbb{R}$.

(a) **Analysis of primal problem.** Give the feasible set, the optimal value, and the optimal solution.

(b) **Lagrangian and dual function.** Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^\star \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

(c) **Lagrange dual problem.** State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution $\lambda^\star$. Does strong duality hold?

Solution:

Given the optimization problem, to minimize x^2+1, subject to

$$(x-2)(x-4) \leq 0, \quad x \in \mathbb{R}$$

(a)

The constraint is:

$$(x-2)(x-4) \leq 0$$

Roots of $(x-2)(x-4)$ are $2, 4$.
Feasible set (satisfying the constraint) is:

$$x \in [2, 4]$$

We want to minimize the objective function which is x^2+1.
Minimum of x^2+1 occurs at:

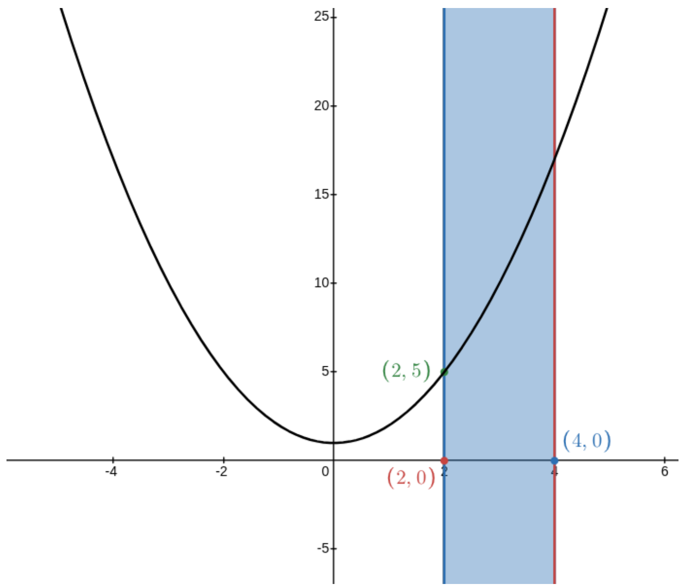
$$\frac{d}{dx}(x^2+1) = 2x = 0 \Rightarrow x = 0$$

But $x = 0$ is not in the feasible set.
The function is increasing ($2x > 0$) in the feasible set, so the minimum within the feasible set $[2, 4]$ happens at $x = 2$ (left endpoint).

$$x^* = 2 \Rightarrow f(x^*) = (2)^2 + 1 = 5$$

$$\therefore \text{Optimal solution is } x^* = 2 \quad \text{and} \quad \text{Optimal value is } f(x^*) = 5$$

This is the plot of the feasible set(blue region):



(b)

From the definition of Lagrangian:

Optimization problem in the form:

$$\text{Minimize } f_0(x)$$
$$\text{subject to } f_1(x) \leq 0$$

Let $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$L(x, \lambda) = f_0(x) + \lambda f_1(x)$$

Here,

$$f_0(x) = x^2 + 1, \quad f_1(x) = (x-2)(x-4)$$

So in this question, we define the Lagrangian as:

$$L(x, \lambda) = x^2 + 1 + \lambda(x-2)(x-4)$$

Expanding:

$$L(x, \lambda) = (1+\lambda)x^2 + (8\lambda)x + (1 - 6\lambda)$$

The plot of $L(x, \lambda)$ with $\lambda = 0, 0.5, 1, 2, 3, 6$ [from the left], also consisting of $f(x) = x^2 + 1$, the feasible region, and the optimal point, is shown below

In our feasible region, $f_1(x) \leq 0$, and with the condition $\lambda \geq 0$, we have:

$$\lambda f_1(x) \leq 0$$

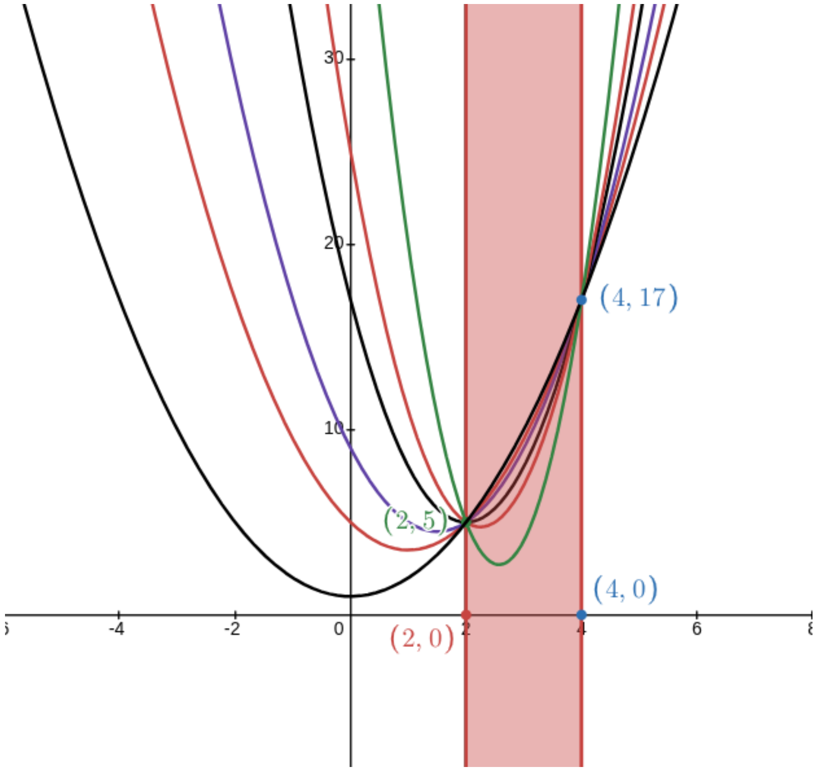
$$L(x, \lambda) = f_0(x) + \lambda f_1(x) \leq f_0(x)$$

This inequality holds even for infimums:

$$\inf_{x \in [2,4]} L(x, \lambda) \leq \inf_{x \in [2,4]} f_0(x)$$

From (a), we have:

$$\inf_{x \in [2,4]} f(x) = p^* \quad \text{(minimum and optimal)}$$



$$\therefore \text{ We have } \quad \inf_x L(x,\lambda) \le p^* \quad (\lambda \ge 0)$$

We define the dual function $g(\lambda)$ as:

$$g(\lambda) = \inf_x L(x,\lambda) = \inf_x \left[(1+\lambda)x^2 - 6\lambda x + (1+8\lambda)\right]$$

If $1+\lambda > 0$, then it is an upward parabola with a finite minimum.
If $1+\lambda \le 0$, then either it is flat or a downward parabola with $\inf_x L(x,\lambda) = -\infty$
$L(x,\lambda)$ is minimum at:

$$\frac{d}{dx}\left[(1+\lambda)x^2 - 6\lambda x + (1+8\lambda)\right] = 0$$

$$x(1+\lambda) = 3\lambda \quad \Rightarrow \quad x = \frac{3\lambda}{1+\lambda} \quad (\text{for } 1+\lambda > 0)$$

Substitute $x = \frac{3\lambda}{1+\lambda}$ into $L(x,\lambda)$ to evaluate $g(\lambda)$.
From previous substitution:

$$g(\lambda) = (1+\lambda)\left(\frac{3\lambda}{1+\lambda}\right)^2 - 6\lambda\left(\frac{3\lambda}{1+\lambda}\right) + (1+8\lambda)$$

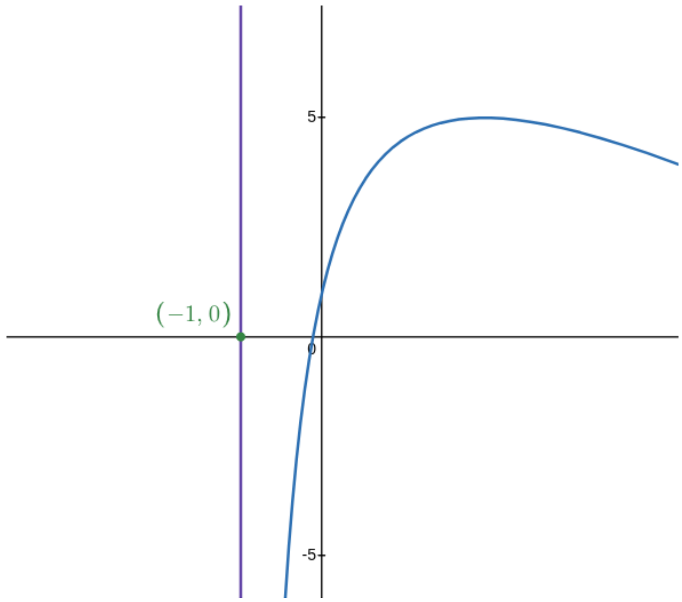
$$g(\lambda) = \frac{9\lambda^2}{1+\lambda} - \frac{18\lambda^2}{1+\lambda} + (1+8\lambda)$$

$$g(\lambda) = \frac{9\lambda^2 - 18\lambda^2 + (8\lambda+1)(1+\lambda)}{1+\lambda}$$

$$g(\lambda) = \frac{9\lambda - \lambda^2 + 1}{1+\lambda} \quad \text{for } 1+\lambda > 0$$

$$\therefore \quad g(\lambda) = \begin{cases} \frac{9\lambda - \lambda^2 + 1}{1+\lambda} & \text{if } 1 + \lambda > 0 \\ -\infty & \text{if } 1 + \lambda \leq 0 \end{cases}$$

The plot of the dual function is as follows:



(c)

We take $\lambda \geq 0$ for the problem (since our constraint is of the form $g(x) \leq 0$). The dual function is:

$$g(\lambda) = \frac{9\lambda - \lambda^2 + 1}{\lambda + 1}$$

The dual problem is:

$$\text{maximize } g(\lambda) = \frac{9\lambda - \lambda^2 + 1}{\lambda + 1}, \quad \text{subject to } \lambda \geq 0$$

We get:

$$g''(\lambda) = \frac{-18}{(1+\lambda)^3} < 0 \quad (\lambda \geq 0)$$

This displays concavity of the objective function.
For finding the optimum value, we solve:

$$g'(\lambda) = \frac{-\lambda^2 - 2\lambda + 8}{(\lambda + 1)^2} = 0$$

$$\Rightarrow \lambda^* = 2 \quad (\lambda = -4 \text{ is not an option})$$

Optimum value of $g(\lambda)$:

$$g(2) = \frac{18 - 4 + 1}{3} = 5$$

Since the primal was convex ($f(x) = x^2 + 1$ is convex also the constraint $(x-2)(x-4) \leq 0$ is a convex constraint set,) There exists a strictly feasible point (which satisfies $(x-2)(x-4) < 0$), so Slater's condition holds.

$$\text{So} \quad p^* = d^* = 5 \quad \text{(Optimal duality gap = 0)}$$

$$\therefore \text{Strong duality holds.}$$