Convex Optimization - AI2101
Assignment - X

Bhuvan Chandra K
AI23BTECH11013

May 7, 2025

Question 4.4

Symmetries and convex optimization. Suppose $G = \{Q_1, \dots, Q_k\} \subseteq \mathbb{R}^{n\times n}$ is a group, i.e., closed under products and inverse. We say that the function $f : \mathbb{R}^n \to \mathbb{R}$ is G-invariant, or symmetric with respect to G, if $f(Q_i x) = f(x)$ holds for all x and $i = 1, \dots, k$.

We define $\bar{x} = \frac{1}{k}\sum_{i=1}^{k} Q_i x$, which is the average of x over its G-orbit. We define the fixed subspace of G as

$$F = \{x \mid Q_i x = x,\ i = 1, \dots, k\}.$$

(a) Show that for any $x \in \mathbb{R}^n$, we have $\bar{x} \in F$.

(b) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and G-invariant, then $f(\bar{x}) \le f(x)$.

(c) We say the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \end{array} \tag{1}$$

is G-invariant if the objective f_0 is G-invariant, and the feasible set is G-invariant, which means

$$f_1(x) \le 0, \dots, f_m(x) \le 0 \Rightarrow f_1(Q_i x) \le 0, \dots, f_m(Q_i x) \le 0, \quad \text{for } i = 1, \dots, k.$$

Show that if the problem is convex and G-invariant, and there exists an optimal point, then there exists an optimal point in F. In other words, we can adjoin the equality constraints $x \in F$ to the problem, without loss of generality.

(d) As an example, suppose f is convex and symmetric, i.e., $f(Px) = f(x)$ for every permutation matrix P. Show that if f has a minimizer, then it has a minimizer of the form $\alpha\mathbf{1}$. (This means to minimize f over $x \in \mathbb{R}^n$, we can just as well minimize $f(t\mathbf{1})$ over $t \in \mathbb{R}$.)

Solution:

Let

$$G = \{Q_1, \dots, Q_k\} \subset \mathbb{R}^{n\times n}$$

be a finite group of invertible matrices.
Define for any $x \in \mathbb{R}^n$ its average over its G-orbit as

$$\bar{x} = \frac{1}{k}\sum_{i=1}^{k} Q_i x$$

The fixed subspace is denoted as

$$F = \{x : Q_i x = x\ \forall i = 1, \dots, k\}$$

(a)

We have to show:

$$Q_j\bar{x} = \bar{x} \quad \forall j$$

$$Q_j(\bar{x}) = Q_j\left(\frac{1}{k}\sum_{i=1}^{k} Q_i x\right) = \frac{1}{k}\sum_{i=1}^{k}(Q_jQ_ix)$$

For any element $Q_m \in G$, we want to find i such that

$$Q_jQ_i = Q_m$$

$$Q_i = Q_j^{-1}Q_m$$

Since $Q_j^{-1} \in G$ and $Q_m \in G$, their product $Q_j^{-1}Q_m$ must also be in G (by closure property of groups).
So $Q_j^{-1}Q_m = Q_n$ for some $n \in \{1, 2, \ldots, k\}$
So we choose the index n (i.e., $Q_i = Q_n$) as the choice for i. (Each Q_i is unique)
So $\{Q_jQ_i : i = 1, \ldots, k\}$ is just a permutation of $\{Q_1, Q_2, \ldots, Q_k\}$
From this result, we get:

$$\sum_i Q_jQ_ix = \sum_i Q_ix$$

$$Q_j\bar{x} = \frac{1}{k}\left(\sum_{i=1}^{k} Q_jQ_ix\right) = \frac{1}{k}\left(\sum_{i=1}^{k} Q_ix\right)$$

$$Q_j\bar{x} = \bar{x} \Rightarrow \bar{x} \in F$$

$$\therefore \quad \bar{x} \in F$$

(b)

Given $f : \mathbb{R}^n \to \mathbb{R}$ is convex and G-invariant

$$f(\bar{x}) = f\left(\frac{1}{k}\sum_i Q_ix\right)$$

We know that for any finite set of points $y_1, \ldots, y_k$ and weights $\alpha_i \geq 0$ summing to 1:

$$f\left(\sum_{i=1}^{k} \alpha_i y_i\right) \leq \sum_{i=1}^{k} \alpha_i f(y_i)$$

Here we take $\alpha_i = \frac{1}{k}$
So,

$$f\left(\frac{1}{k}\sum_i Q_ix\right) \leq \frac{1}{k}\sum_{i=1}^{k} f(Q_ix)$$

Since f is G-invariant, i.e., $f(Q_ix) = f(x)$ for all i, we get:

$$f(\bar{x}) \leq \frac{1}{k}\sum_{i=1}^{k} f(x) = f(x)$$

$$\therefore f(\bar{x}) \leq f(x)$$

(c)

We have the optimization problem:

$$\begin{aligned} &\text{minimize} && f_0(x) \\ &\text{subject to} && f_j(x) \le 0, \quad \text{for } j = 1, \dots, m \end{aligned} \tag{2}$$

and these assumptions:
(i) Each f_j is convex.
(ii) An optimal x^* exists.
(iii) The problem is G-invariant, i.e.,

$$f_0(Q_i x) = f_0(x), \quad f_j(Q_i x) \le 0 \Leftrightarrow f_j(x) \le 0 \quad \forall i.$$

We will define G-orbit average as:

$$\bar{x}^* = \frac{1}{k} \sum_i Q_i x^*$$

For each constraint $f_j(\bar{x}^*)$, by similar argument from (b):

$$f_j(\bar{x}^*) \le \frac{1}{k} \sum_{i=1}^{k} f_j(Q_i x^*)$$

From condition $f_j(Q_i x^*) = f_j(x^*)$

$$f_j(\bar{x}^*) \le f_j(x^*)$$

$$f_j(\bar{x}^*) \le 0$$

So $\bar{x}^*$ is feasible.

$$f_0(\bar{x}^*) \le \frac{1}{k} \sum_i f_0(Q_i x^*)$$

$$f_0(\bar{x}^*) \le f_0(x^*)$$

So $\bar{x}^*$ attains the same minimum value, also $\bar{x}^* \in F$ from (a).
Thus we may (without loss of generality) consider $x \in F$ when searching for an optimal solution.

(d)

Let x^* be a minimizer of $f : \mathbb{R}^n \to \mathbb{A}$

$$x' = \frac{1}{n!} \sum_{P \in S_n} P x^*$$

The sum is over all $n \times n$ permutation matrices P.
So for every permutation P,

$$P x' = \frac{1}{n!} \sum_{Q \in S_n} P(Q x^*) = \frac{1}{n!} \sum_{Q \in S_n} (PQ) x^* = x'$$

x' lies in the fixed subspace of all permutations:

$$x' = \alpha \mathbf{1} \quad \text{for some } \alpha \in \mathbb{R}$$

Since f is symmetric and using Jensen's inequality,

$$f(x') = f\left(\frac{1}{n!} \sum_p P x^* \right) \le \frac{1}{n!} \sum_p f(P x^*)$$

$$\frac{1}{n!}\sum_{p} f(Px^*) = \frac{1}{n!}\sum_{p} f(x^*) = f(x^*)$$

We have $f(x') \leq f(x^*)$

$$\therefore \quad \text{Since } x^* \text{ was a minimizer, } x' \text{ must also be a minimizer.}$$

Question 4.6

Handling convex equality constraints. A convex optimization problem can have only linear equality constraint functions. In some special cases, however, it is possible to handle convex equality constraint functions, i.e., constraints of the form $g(x) = 0$, where g is convex. We explore this idea in this problem.

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h(x) = 0 \end{array} \tag{3}$$

where f_i and h are convex functions with domain $\mathbb{R}^n$. Unless h is affine, this is not a convex optimization problem. Consider the related problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h(x) \leq 0 \end{array} \tag{4}$$

where the convex equality constraint has been relaxed to a convex inequality. This problem is, of course, convex.

Now suppose we can guarantee that at any optimal solution x^* of the convex problem, we have $h(x^*) = 0$, i.e., the inequality $h(x) \leq 0$ is always active at the solution. Then we can solve the (nonconvex) problem by solving the convex problem.

Show that this is the case if there is an index r such that

- f_0 is monotonically increasing in x_r,
- $f_1, \dots, f_m$ are nonincreasing in x_r,
- h is monotonically decreasing in x_r.

Solution:

Considering the given constraints:
Let x^* be an optimal solution to (4.66) and $h(x^*) < 0$.
We denote x^* as

$$x^* = (x_1, x_2, \dots, x_r, \dots, x_n)$$

We are given that h is monotonically decreasing in x_r. So if we decrease x_r, the value of h will increase (since it's monotone, and as we go right, h decreases).
Let's take a new point x'. This is a tweaked version of x^*. We decrease x_r by γ in x^* and take it as x'.

$$x' = (x_1, x_2, \dots, x_r - \gamma, \dots, x_n) \quad \text{for some } \gamma > 0$$

We know h is continuous (also a convex function), we will make sure to choose a small γ so that $h(x') \leq 0$ still holds. [This is from $h(x^*) < 0$]
From the condition that each f_i is non-increasing in x_r, we can write:

$$f_i(x') \leq f_i(x^*)$$

$$f_i(x') \leq 0$$

We can see that x' satisfies all inequalities.
Since f_0 is monotonically increasing in x_r and we have decreased x_r by γ,

$$f_0(x') < f_0(x^*)$$

So we minimized f_0 more with x' than x^*. But this contradicts that x^* is optimal.
So our assumption $h(x^*) < 0$ is false. We are left with $h(x^*) = 0$

$$\therefore \text{ At any optimal solution } x^* \text{ of (4.66), we must have } h(x^*) = 0.$$

Question 4.7 (a)

Convex-concave fractional problems. Consider a problem of the form

$$\text{minimize } \frac{f_0(x)}{c^T x + d} \quad \text{subject to } f_i(x) \leq 0, \; i = 1, \ldots, m \quad Ax = b$$

where $f_0, f_1, \ldots, f_m$ are convex, and the domain of the objective function is defined as

$$\{x \in \text{dom } f_0 \mid c^T x + d > 0\}.$$

(a) Show that this is a quasiconvex optimization problem.

Solution:

First we show that the objective function is quasiconvex.
A function φ is quasiconvex if and only if all its sublevel sets are convex.

$$S_\alpha = \{x \in D \mid \varphi(x) \leq \alpha\}$$

Here $D = \{x \mid x \in \text{dom } f_0, \; c^T x + d > 0\}$
Let

$$\varphi(x) = \frac{f_0(x)}{c^T x + d}$$

and we will show S_α is convex for all $\alpha \in \mathbb{R}$.

$$\text{dom } f_0 \text{ is convex since } f_0 \text{ is convex,}$$

$$\{x \mid c^T x + d > 0\} \text{ is convex [It is open half space]}$$

We know their intersection is also convex.
So we can write $\varphi(x) \leq \alpha$ as:

$$\varphi(x) \leq \alpha \Rightarrow \begin{cases} c^T x + d > 0 \\ f_0(x) \leq \alpha(c^T x + d) \end{cases}$$

Redefine S_α as:

$$S_\alpha = \{x \mid c^T x + d > 0,\ f_0(x) - \alpha(c^T x + d) \leq 0\}$$

As seen before, the set $\{x \mid c^T x + d > 0\}$ is convex.
$\alpha(c^T x + d)$ is affine, $f_0(x)$ is convex.
We know that convex minus affine is also convex:

$$f_0(x) - \alpha(c^T x + d) \text{ is convex.}$$

So the side $f_0(x) - \alpha(c^T x + d) \leq 0$ is also convex.
Intersection of the two convex sets giving S_α is also convex.

$\therefore$ We found S_α is convex, so φ is quasiconvex. This is a quasiconvex optimization problem.