Majorization Relation and Karamata's Inequality

By Bhuvanesh N

2. Majorization relation for finite sequences and Karamata's inequality

Let us introduce a majorization relation for finite sequences of real numbers.

Definition 1. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two (finite) sequences of real numbers. We say that the sequence a majorizes the sequence b and we write, if, after a possible renumeration, the terms of the sequences a and b satisfy the following three conditions:

1. $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$;
2. $a_1 + a_2 + \cdots + a_k \geq b_1 + b_2 + \cdots + b_k$, for each k, $1 \leq k \leq n-1$;
3. $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$.

The first condition is obviously no restriction, since we can always rearrange the sequence. The second condition is essential. Clearly, $a \succ a$ holds for an arbitrary sequence a.

Example 1.

(a) If $a = (a_i)_{i=1}^n$ is an arbitrary sequence of nonnegative numbers, having the sum equal to 1, then

$$(1, 0, \ldots, 0) \succ (a_1, a_2, \ldots, a_n) \succ \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right).$$

(b) The sequences $(4, 4, 1)$ and $(5, 2, 2)$ are incomparable in the sense of the relation $\succ$, i.e., none of the two majorizes the other one.

Abel's Transformation

Let $(a_k)_{k=1}^n$ and $(b_k)_{k=1}^n$ be sequences of real numbers. Define

$$A_k = \sum_{i=1}^{k} a_i, \quad k = 1, 2, \ldots, n, \quad and \quad A_0 = 0.$$

Abel's Transformation formula:

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} (A_k - A_{k-1}) b_k = \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}) + A_n b_n.$$

Explanation:

- Compute the partial sums A_k of the sequence (a_k).
- Rewrite the sum of products $\sum a_k b_k$ in terms of A_k and differences of b_k.
- This transformation is analogous to integration by parts in the discrete case.

Remark: Abel's transformation is especially useful when one sequence is monotone, allowing estimation of sums.

Karamata's Inequality

Definition 1 (Majorization). Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be sequences of real numbers. We say that $\mathbf{a}$ *majorizes* $\mathbf{b}$, denoted by $\mathbf{b} \prec \mathbf{a}$, if, after possible rearrangement, the sequences satisfy:

1. $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_n$;
2. $\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i$ for all $k = 1, 2, \ldots, n-1$;
3. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$.

Example 1.

(a) If $\mathbf{a} = (a_i)_{i=1}^{n}$ is nonnegative with $\sum a_i = 1$, then

$$(1, 0, \ldots, 0) \succ (a_1, \ldots, a_n) \succ \left(\frac{1}{n}, \ldots, \frac{1}{n}\right).$$

(b) The sequences $(4, 4, 1)$ and $(5, 2, 2)$ are incomparable: neither majorizes the other.

Theorem 1 (Karamata's Inequality). Let $\mathbf{a} = (a_i)_{i=1}^{n}$ and $\mathbf{b} = (b_i)_{i=1}^{n}$ be sequences in an interval (α, β) such that $\mathbf{b} \prec \mathbf{a}$, and let $f : (\alpha, \beta) \to R$ be convex. Then

$$\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i).$$

First Proof (using Abel's Transformation). Define the divided differences

$$c_i = \frac{f(b_i) - f(a_i)}{b_i - a_i}.$$

Since f is convex, (c_i) is decreasing. Let

$$A_k = \sum_{i=1}^{k} a_i, \quad B_k = \sum_{i=1}^{k} b_i, \quad A_0 = B_0 = 0.$$

Then

$$\sum_{i=1}^{n} f(a_i) - \sum_{i=1}^{n} f(b_i) = \sum_{i=1}^{n} c_i (A_i - B_i - (A_{i-1} - B_{i-1})) = \sum_{i=1}^{n-1} (c_i - c_{i+1})(A_i - B_i) \geq 0.$$

Second Proof (using Stieltjes Integral). Let $\psi_1, \psi_2 : [\alpha, \beta] \to R$ be integrable functions with $\psi_1 \prec \psi_2$, i.e.,

$$\int_\alpha^x \psi_1(t)dt \leq \int_\alpha^x \psi_2(t)dt \quad \forall x \in [\alpha, \beta), \quad \int_\alpha^\beta \psi_1 = \int_\alpha^\beta \psi_2.$$

Let ϕ be increasing. Then

$$\int_\alpha^\beta \phi\psi_1 dx \leq \int_\alpha^\beta \phi\psi_2 dx.$$

Applying this to the representation $f(x) = \int_\alpha^x \phi(t)dt$ for an increasing function ϕ yields Karamata's inequality.

Schur's and Muirhead's Inequalities

Definition 2 (Symmetric Sums). Let $F(x_1, \ldots, x_n)$ be a function in n nonnegative real variables. Define the *full symmetric sum*:

$$\mathcal{T}[F](x_1, \ldots, x_n) := \sum_{\sigma \in S_n} F(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

where S_n is the symmetric group on n elements.

Particularly, if $F(x_1, \ldots, x_n) = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ for a sequence of nonnegative exponents $\mathbf{a} = (a_1, \ldots, a_n)$, we write

$$T[a_1, \ldots, a_n](x_1, \ldots, x_n) := \mathcal{T}[x_1^{a_1} \ldots x_n^{a_n}].$$

Example 4.

$$T[1, 0, \ldots, 0] = (n-1)!\,(x_1 + \ldots + x_n), \quad T\left[\frac{1}{n}, \ldots, \frac{1}{n}\right] = n!\,\sqrt[n]{x_1 x_2 \ldots x_n}.$$

Theorem 3 (Schur). For positive a, b and any positive sequence (x, y, z):

$$T[a+2b, 0, 0] + T[a, b, b] \geq 2T[a+b, b, 0].$$

Proof.

$$\frac{1}{2}T[a+2b, 0, 0] + \frac{1}{2}T[a, b, b] - T[a+b, b, 0] = x^a(x^b - y^b)(x^b - z^b) + y^a(y^b - x^b)(y^b - z^b) + z^a(z^b - x^b)(z^b - y^b).$$

Assume $x \geq y \geq z$, then only the y-term may be negative. Factor:

$$(x^b - y^b)(x^a(x^b - z^b) - y^a(y^b - z^b)) \geq 0 x^{a+b} - y^{a+b} - z^b(x^a - y^a) \geq 0.$$

Corollary 4 (Schur for r). If $x, y, z \geq 0$ and $r \geq 0$:

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0.$$

Theorem 4 (Muirhead). Let $\mathbf{a}, \mathbf{b}$ be sequences of nonnegative integers. Then $T[\mathbf{a}] \geq T[\mathbf{b}]$ for all positive x_i iff $\mathbf{b} \prec \mathbf{a}$. Equality holds iff $\mathbf{a} = \mathbf{b}$ or $x_1 = \ldots = x_n$.

Proof Sketch. Consider $x_i = c$ constant:

$$c^{\sum a_i} \geq c^{\sum b_i} \sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i.$$

Sufficiency uses the linear operation L on exponents: for $b_k > b_l$, define

$$a_k = \rho + \sigma = \frac{\tau + \sigma}{2\tau} b_k + \frac{\tau - \sigma}{2\tau} b_l, \quad a_l = \rho - \sigma = \frac{\tau - \sigma}{2\tau} b_k + \frac{\tau + \sigma}{2\tau} b_l,$$

with $a_\nu = b_\nu$ for $\nu \neq k, l$, producing a sequence $\mathbf{a}$ majorized by $\mathbf{b}$.

Theorem 5 (Petrović). Let $f : [0, +\infty) \to R$ convex, and $x_1, \ldots, x_n > 0$. Let $s = \sum x_i$ and $\lambda_i = x_i / s$. Then

$$\sum_{i=1}^{n} f(x_i) \leq f(s) + (n-1) f(0),$$

because $x_i = (1 - \lambda_i) \cdot 0 + \lambda_i s$, convexity yields

$$f(x_i) \leq (1 - \lambda_i) f(0) + \lambda_i f(s).$$

Corollary (Szegö-type). For $f : [0, b_1] \to R$ convex, $b_1 \geq b_2 \geq \ldots \geq b_{2n+1} \geq 0$:

$$f(b_1 - b_2 + \ldots + b_{2n+1}) \leq f(b_1) - f(b_2) + \ldots + f(b_{2n+1}).$$

Example 5.

$$x^3 + y^3 + z^3 + 3xyz \geq x^2 y + x y^2 + y^2 z + y z^2 + z^2 x + z x^2.$$

The Ultimate Chaos: Karamata × Schur × Muirhead

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{i} x_j^{\alpha_j} \cdot \sum_{k=i}^{n} y_k^{\beta_k}}{\int_0^{x_i} \sqrt{t^2 + \gamma^2}\, dt + \sum_{\ell=1}^{i} z_\ell !} \quad \geq \quad \prod_{i=1}^{n} \left(\sum_{\sigma \in S_n} \cos\left(\pi x_{\sigma(i)} - \frac{y_i}{z_i + 1} \right) \right)^{\alpha_i}$$

$$T\Big[\underbrace{a_1, a_2, \ldots, a_n}_{majorized}, 0, \ldots, 0\Big](x_1, \ldots, x_n) + \sum_{i<j} (x_i - x_j)^r \frac{\sin(x_i x_j)}{1 + e^{x_i - x_j}} \;\geq\; \prod_{i=1}^{n} \left(\int_0^{x_i} f(t) dt \right)^{\lambda_i}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\Delta f(x_i, y_j, z_k)}{\sqrt[n]{x_i^2 + y_j^2 + z_k^2}} + \oint_{\partial\Omega} e^{i\theta} d\theta \;\leq\; \max_{1 \leq i \leq n} \left(\sum_{j=1}^{i} \frac{x_j^2}{y_j + z_j} \right)^{\frac{1}{2}}$$

$$\underbrace{\sum_{i_1<i_2<\ldots<i_m} \prod_{k=1}^{m} n i_k \frac{(-1)^k}{\log(x_{i_k}+1)} \cosh\left(\frac{\pi i_k}{n}\right)}_{Inspired by Max Verstappen} \geq \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{\sigma(i)}^{a_i} - \sum_{i=1}^{n} \frac{f^{(i)}(0)}{i!} \left(\sum_{j=1}^{i} y_j\right)^i$$

$$\int_0^1 \cdots \int_0^1 \frac{\prod_{i,j,k} \sin(\pi x_i y_j z_k)}{\prod_{\ell=1}^{n} \sqrt{\alpha_\ell^2+\beta_\ell^2+\gamma_\ell^2}} \, dx_1 \ldots dz_n \;\leq\; \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{T[a_i+b_j+c_k+d_l](x_i,y_j,z_k,w_l)}{(x_i+y_j+z_k+w_l)!}$$

$$\textstyle\sum_{i=1}^{n} \prod_{j=1}^{i} \left(\frac{a_j^{\alpha_j}+\sqrt[3]{b_j^{\beta_j}}}{c_j+\log(d_j)}\right)$$

$$+ \textstyle\int_0^{\pi} \int_0^{\pi/2} \left[\sum_{k=1}^{m} \frac{(-1)^k x_k^k}{\prod_{l\neq k}(x_k-x_l)}\right]^2 d\theta \, d\phi$$

$$+ \left(\textstyle\bigcup_{r=1}^{p} \bigcap_{s=1}^{q} F_{r,s}\right)^{\dagger}$$

$$+ \lim_{t\to\infty} \left(\textstyle\prod_{u=1}^{v} \sqrt[3]{\frac{\Gamma(\nu_u+t)}{\Gamma(\nu_u)}}\right)$$

$$+ \textstyle\sum_{i,j=1 i\neq j}^{n} \frac{2ii\,\zeta(2j)}{i^2+j^2} e^{i\pi\frac{i+j}{n}}$$

$$+ \textstyle\oint_{\mathcal{C}} \frac{\sin(z^2)\,dz}{\prod_{k=1}^{n}(z-z_k)^3}$$

$$+ \textstyle\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{(-1)^{k+\ell}}{k!\ell!} \left(\frac{\partial^{k+\ell} f(x,y)}{\partial x^k \partial y^\ell}\right)\Big|_{x=0,y=0}$$

$$+ \textstyle\bigotimes_{m=1}^{M} \bigoplus_{n=1}^{N} T[a_m,b_n,c_n]$$

$$+ \textstyle\sum_{i=1}^{n} \left(\frac{\int_0^{x_i} t^{i-1} e^{-t} dt}{\prod_{j\neq i}(x_i-x_j)}\right)^2$$

$$+ \sqrt[4]{\textstyle\sum_{k=1}^{K} \left|\sum_{n=1}^{N} \frac{(-1)^n}{n^k}\right|^2}$$

$$+ \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\cos(x_i-y_j)+i\sin(x_i+y_j)}{\sqrt{x_i^2+y_j^2+1}}$$

$$+ \textstyle\sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \left(\sum_{q=1}^{\infty} \frac{(-1)^q q^{2p}}{q!}\right)$$

$$+ \textstyle\prod_{r=1}^{R} \sum_{s=1}^{S} \int_0^{x_{r,s}} \frac{\partial^2}{\partial u_{r,s}^2} \left(e^{u_{r,s}^2} + \log(1+u_{r,s})\right) du_{r,s}$$

$+ \sum_{i,j,k=1 i\neq j\neq k}^{n} \frac{\Delta(x_i,x_j,x_k)\,T[i,j,k]}{(x_i+x_j+x_k)^2+\sqrt{x_i x_j x_k}}$

$+ \oint_{\mathcal{C}} \oint_{\mathcal{C}'} \frac{\exp(zw)}{(z^2+w^2+1)^2} dz dw$

$+ \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\prod_{k=1}^{n} \Gamma(k/2)}{n!} e^{in\pi/3}$

$+ \bigoplus_{i=1}^{I} \bigotimes_{j=1}^{J} \bigcup_{k=1}^{K} \bigcap_{l=1}^{L} H_{i,j,k,l}$

$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\sin(x_i y_j z_k)+\cos(x_i+y_j+z_k)}{1+x_i^2+y_j^2+z_k^2}$

$+ \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} \frac{(-1)^n n^m}{\prod_{k=1}^{m}(n+k)} \right)$

Explanation of the Equation

The above expression is a symbolic collage of various advanced mathematical operations, intended purely for illustrative purposes. Let us describe the components:

1. The first term,
$$\sum_{i=1}^{n} \prod_{j=1}^{i} \left(\frac{a_j^{\alpha_j} + \sqrt[3]{b_j^{\beta_j}}}{c_j + \log(d_j)} \right),$$
represents nested summation and product operations with powers and cube roots.

2. The second term,
$$\int_0^{\pi} \int_0^{\pi/2} \left[\sum_{k=1}^{m} \frac{(-1)^k\, x_k^k}{\prod_{l\neq k}(x_k - x_l)} \right]^2 d\theta\, d\phi,$$
involves a double integral of a squared Lagrange-like interpolation sum.

3. The third term,
$$\left(\bigcup_{r=1}^{p} \bigcap_{s=1}^{q} F_{r,s} \right)^{\dagger},$$
demonstrates set operations with a formal adjoint indicated by the dagger.

4. The fourth term,
$$\lim_{t\to\infty} \left(\prod_{u=1}^{v} \sqrt[3]{\frac{\Gamma(\nu_u + t)}{\Gamma(\nu_u)}} \right),$$
uses limits, factorial-like Gamma functions, and cube roots to create an asymptotic product.

5. The fifth term,

$$\sum_{i,j=1 i\neq j}^{n} \frac{2ii\,\zeta(2j)}{i^2+j^2} e^{i\pi\frac{i+j}{n}},$$

combines combinatorial coefficients, the Riemann zeta function, and complex exponentials.

6. Each successive line represents increasingly complex operations, including contour integrals, multiple sums, tensor products, derivative operations, and nested limits.

7. The final lines use hyperoperators such as $\bigoplus$, $\bigotimes$, $\bigcup$, $\bigcap$, and nested trigonometric and exponential functions to create an unending cascade of mathematical symbols.

Remark: This equation is not intended for computation or analysis; rather, it is a form of *mathematical visual art*, demonstrating the beauty and chaos possible in symbolic mathematics.

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n} \prod_{j=1}^{i} \left(\frac{a_j^{\alpha_j}}{b_j^{\beta_j}} \right) + \int_0^{\pi} \int_0^{\pi/2} \left[\sum_{k=1}^{m} \frac{(-1)^k\, x_k^k}{\prod_{l\neq k}(x_k - x_l)} \right]^2 d\theta\, d\phi \\
&+ \left(\textstyle\bigcup_{r=1}^{p} \bigcap_{s=1}^{q} F_{r,s} \right)^{\dagger} + \lim_{t\to\infty} \left(\textstyle\prod_{u=1}^{v} \sqrt[3]{\frac{\Gamma(\nu_u + t)}{\Gamma(\nu_u)}} \right) \\
&+ \textstyle\sum_{i,j=1 i\neq j}^{n} \frac{2ii\,\zeta(2j)}{i^2+j^2}\, e^{i\pi\frac{i+j}{n}} + \oint_{\mathcal{C}} \frac{\sin(z^2)\,dz}{\prod_{k=1}^{n}(z-z_k)^3} \\
&+ \textstyle\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{(-1)^{k+\ell}}{k!\ell!} \left(\frac{\partial^{k+\ell}}{\partial x^k \partial y^\ell} f(x,y) \right)\Big|_{x=0,y=0} + \bigotimes_{m=1}^{M} \bigoplus_{n=1}^{N} T[a_m, b_n, c_n] \\
&+ \textstyle\sum_{i=1}^{n} \left(\frac{\int_0^{x_i} t^{i-1} e^{-t} dt}{\prod_{j\neq i}(x_i - x_j)} \right)^2 + \sqrt[4]{\sum_{k=1}^{K} \left| \sum_{n=1}^{N} \frac{(-1)^n}{n^k} \right|^2} \\
&+ \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\cos(x_i - y_j) + i\sin(x_i + y_j)}{\sqrt{x_i^2 + y_j^2 + 1}} + \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \left(\sum_{q=1}^{\infty} \frac{(-1)^q q^{2p}}{q!} \right) \quad \sum_{i=1}^{n} \prod_{j=1}^{i} \left(\frac{a_j^{\alpha_j} + \sqrt[3]{b_j^{\beta_j}}}{c_j + \log(d_j)} \right) \\
&+ \textstyle\int_0^{\pi} \int_0^{\pi/2} \left[\sum_{k=1}^{m} \frac{(-1)^k\, x_k^k}{\prod_{l\neq k}(x_k - x_l)} \right]^2 d\theta\, d\phi \\
&+ \left(\textstyle\bigcup_{r=1}^{p} \bigcap_{s=1}^{q} F_{r,s} \right)^{\dagger} \\
&+ \lim_{t\to\infty} \left(\textstyle\prod_{u=1}^{v} \sqrt[3]{\frac{\Gamma(\nu_u + t)}{\Gamma(\nu_u)}} \right) \\
&+ \textstyle\sum_{i,j=1 i\neq j}^{n} \frac{2ii\,\zeta(2j)}{i^2+j^2} e^{i\pi\frac{i+j}{n}} \\
&+ \textstyle\oint_{\mathcal{C}} \frac{\sin(z^2)\,dz}{\prod_{k=1}^{n}(z-z_k)^3}
\end{aligned}$$

$+ \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{(-1)^{k+\ell}}{k!\ell!} \left(\frac{\partial^{k+\ell} f(x,y)}{\partial x^k \partial y^\ell} \right) \Big|_{x=0,y=0}$

$+ \bigotimes_{m=1}^{M} \bigoplus_{n=1}^{N} T[a_m, b_n, c_n]$

$+ \sum_{i=1}^{n} \left(\frac{\int_0^{x_i} t^{i-1} e^{-t} dt}{\prod_{j \neq i} (x_i - x_j)} \right)^2$

$+ \sqrt[4]{\sum_{k=1}^{K} \left| \sum_{n=1}^{N} \frac{(-1)^n}{n^k} \right|^2}$

$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\cos(x_i - y_j) + i \sin(x_i + y_j)}{\sqrt{x_i^2 + y_j^2 + 1}}$

$+ \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \left(\sum_{q=1}^{\infty} \frac{(-1)^q q^{2p}}{q!} \right)$

$+ \prod_{r=1}^{R} \sum_{s=1}^{S} \int_0^{x_{r,s}} \frac{\partial^2}{\partial u_{r,s}^2} \left(e^{u_{r,s}^2} + \log(1 + u_{r,s}) \right) du_{r,s}$

$+ \sum_{i,j,k=1 i \neq j \neq k}^{n} \frac{\Delta(x_i, x_j, x_k) \, T[i,j,k]}{(x_i + x_j + x_k)^2 + \sqrt{x_i x_j x_k}}$

$+ \oint_{\mathcal{C}} \oint_{\mathcal{C}'} \frac{\exp(zw)}{(z^2 + w^2 + 1)^2} dz dw$

$+ \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\prod_{k=1}^{n} \Gamma(k/2)}{n!} e^{in\pi/3}$

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$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\sin(x_i y_j z_k) + \cos(x_i + y_j + z_k)}{1 + x_i^2 + y_j^2 + z_k^2}$

$+ \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} \frac{(-1)^n n^m}{\prod_{k=1}^{m} (n+k)} \right)$

Explanation of the Equation

The above expression is a symbolic collage of various advanced mathematical operations, intended purely for mathematical reasons (well no shit):

1. The first term,
$$\sum_{i=1}^{n} \prod_{j=1}^{i} \left(\frac{a_j^{\alpha_j} + \sqrt[3]{b_j^{\beta_j}}}{c_j + \log(d_j)} \right),$$
represents nested summation and product operations with powers and cube roots.

2. The second term,

$$\int_0^{\pi} \int_0^{\pi/2} \left[\sum_{k=1}^{m} \frac{(-1)^k \, x_k^k}{\prod_{l \neq k} (x_k - x_l)} \right]^2 d\theta \, d\phi,$$

involves a double integral of a squared Lagrange-like interpolation sum.

3. The third term,

$$\left(\bigcup_{r=1}^{p} \bigcap_{s=1}^{q} F_{r,s} \right)^{\dagger},$$

demonstrates set operations with a formal adjoint indicated by the dagger.

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uses limits, factorial-like Gamma functions, and cube roots to create an asymptotic product.

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combines combinatorial coefficients, the Riemann zeta function, and complex exponentials.

6. Each successive line represents increasingly complex operations, including contour integrals, multiple sums, tensor products, derivative operations, and nested limits.

7. The final lines use hyperoperators such as $\bigoplus$, $\bigotimes$, $\bigcup$, $\bigcap$, and nested trigonometric and exponential functions to create an unending cascade of mathematical symbols.

Remark: This equation is not intended for computation or analysis; rather, it is a form of *Max Verstappen*, as it is just simply lovely.

4. Spectral density function

$$S(\omega) = 1.466 \, H_s^2 \, \frac{\omega_0^5}{\omega^6} \, e^{\left[-3^{\omega/(\omega_0)}\right]^2} \tag{1}$$

or better

$$S(\omega) = 1.466 \, H_s^2 \frac{\omega_0^5}{\omega^6} \exp\left[-3^{\frac{\omega}{\omega_0}}\right]^2 \tag{2}$$

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Spectral Density Function

The **spectral density function**, also called the **power spectral density (PSD)**, describes how the power of a signal or time series is distributed over frequency. It is widely used in physics, engineering, and signal processing.

1. Continuous-Time Case

For a **stationary random process** $X(t)$, the **autocorrelation function** is defined as:

$$R_X(\tau) = E[X(t)X(t+\tau)] \tag{3}$$

The **spectral density function** $S_X(\omega)$ is the **Fourier transform** of the autocorrelation function:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-i\omega\tau}\, d\tau \tag{4}$$

where ω is the angular frequency in radians per second.

This function gives the distribution of power per unit frequency in the signal.

Equivalently, if the signal is known in the time domain, it can be calculated as:

$$S_X(\omega) = \lim_{T\to\infty} \frac{1}{T} \left| \int_0^T X(t)\, e^{-i\omega t}\, dt \right|^2 \tag{5}$$

2. Discrete-Time Case

For a discrete-time signal $x[n]$, the **autocorrelation function** is:

$$R_x[k] = E[x[n]\, x[n+k]] \tag{6}$$

The **discrete-time spectral density** is given by:

$$S_x(\omega) = \sum_{k=-\infty}^{\infty} R_x[k]\, e^{-i\omega k} \tag{7}$$

where $\omega \in [-\pi, \pi]$.

3. Properties

- $S_X(\omega) \geq 0$ for all ω (power cannot be negative).
- Total power of the signal:

$$P = R_X(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega)\, d\omega \tag{8}$$

- $S_X(\omega)$ is real and even if $X(t)$ is real-valued:

$$S_X(-\omega) = S_X(\omega) \tag{9}$$