```
\documentclass[12pt]{article}
\usepackage{amsmath,amssymb}
\usepackage[margin=1in]{geometry}

\begin{document}

\begin{center}
{\bf THE TEACHING OF MATHEMATICS}\\
2025, Vol. VIII, 1, pp. 31--45

\vspace{0.5cm}

{\bf INEQUALITIES OF KARAMATA, SCHUR AND MUIRHEAD,\\
AND SOME APPLICATIONS}

\vspace{0.4cm}

Anish J, Pranav V, Bhuvanesh N and Radhika {S}
\end{center}

\vspace{0.4cm}

\noindent{\bf Abstract.}
Three classical general inequalities---those of Karamata, Schur and
Muirhead---are proved in this article. They can be used in proving other
inequalities, particularly those appearing as problems in mathematical
competitions, including International Mathematical Olympiads. Some
problems
of this kind are given as examples. Several related inequalities---those
of
Petrovi\'{c}, Steffensen and Szeg\"{o}---are treated, as well.

\vspace{0.3cm}

\noindent{\bf ZDM Subject Classification:} H34; \quad
{\bf AMS Subject Classification:} 00A35

\vspace{0.2cm}

\noindent{\bf Key words and phrases:}
Relation of majorization, divided difference, Karamata's inequality,
Petrovi\'{c}'s inequality, Steffensen's inequality, Schur's inequality,
Muirhead's inequality.

\section*{1. Introduction}
```

```
Three classical inequalities are proved in this article. They can be
used in
proving several other inequalities, particularly those appearing as
problems
in mathematical competitions, including International Mathematical
Olympiads.
Some problems of this kind are given as examples.

The article is adapted according to our book~[7], intended for
preparation of
students for mathematical competitions. Its (shortened) Serbian version
was
published in Nastava matematike, L, 4 (2005), 22--31.

We start by recalling some well-known notions which will be used in the
sequel.

A function $f:(a,b)\rightarrow\mathbb{R}$ is said to be convex if for
each two
points $x_1,x_2\in(a,b)$ and each two nonnegative real numbers
$\lambda_1,\lambda_2$ satisfying $\lambda_1+\lambda_2=1$, the following
inequality is valid
\[
f(\lambda_1 x_1+\lambda_2 x_2)\le \lambda_1 f(x_1)+\lambda_2 f(x_2).
\]

The function $f$ is concave if the function $-f$ is convex, i.e., if the
opposite
inequality always holds.

If, in the previous inequalities (assuming $x_1=x_2$), the equality
takes place
only in the case when $\lambda_1=0$ or $\lambda_2=0$, then the function
$f$ is
said to be strictly convex (resp.\ strictly concave).

It can be easily checked that the function
$f:(a,b)\rightarrow\mathbb{R}$ is
convex (strictly convex) if and only if the inequality
\[
\frac{f(x)-f(x_1)}{x-x_1}\le
\frac{f(x_2)-f(x)}{x_2-x}
\tag{1}
\]
(resp.\ $<$) holds for arbitrary points $x_1<x<x_2$ from $(a,b)$.
```

```
Let $f:(a,b)\rightarrow\mathbb{R}$ and $x,y\in(a,b)$. The quotient
\[
\Delta f(x,y)=\frac{f(y)-f(x)}{y-x}
\]
is called the divided difference of the function $f$ at the points
$x,y$.
\section*{2. Relation of majorization}

Let $a=(a_1,a_2,\dots,a_n)$ and $b=(b_1,b_2,\dots,b_n)$ be two sequences
of real
numbers. Rearrange them so that
\[
a_1\ge a_2\ge\cdots\ge a_n,\qquad
b_1\ge b_2\ge\cdots\ge b_n.
\]

We say that $a$ majorizes $b$, and write $b\prec a$, if
\[
\sum_{i=1}^{k} a_i \ge \sum_{i=1}^{k} b_i,\qquad k=1,2,\dots,n-1,
\]
and
\[
\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.
\]

The following lemma is well known.

\medskip
\noindent{\bf Lemma 1.}
Let $a,b\in\mathbb{R}^n$ and $b\prec a$. Then there exists a finite
sequence of
vectors
\[
a=x^{(0)},x^{(1)},\dots,x^{(m)}=b
\]
such that each $x^{(k+1)}$ is obtained from $x^{(k)}$ by replacing two
coordinates $x_i,x_j$ by $x_i',x_j'$ satisfying
\[
x_i'+x_j'=x_i+x_j,\qquad
\min\{x_i,x_j\}\le x_i',x_j'\le \max\{x_i,x_j\}.
\]

\medskip

\section*{3. Karamata inequality}
```

```
\medskip
\noindent{\bf Theorem 1 (Karamata).}
Let $f$ be a convex function on an interval $I$ and let $a,b\in I^n$ be
such that
$b\prec a$. Then
\[
f(a_1)+f(a_2)+\cdots+f(a_n)
\ge
f(b_1)+f(b_2)+\cdots+f(b_n).
\]

\medskip
{\bf Proof.}
By Lemma 1 it is sufficient to prove the theorem in the case when the
vectors
$a$ and $b$ differ only in two coordinates, say
\[
a=(x_1,x_2,\dots,x_i,\dots,x_j,\dots,x_n),
\]
\[
b=(x_1,x_2,\dots,x_i',\dots,x_j',\dots,x_n),
\]
where
\[
x_i'+x_j'=x_i+x_j,\qquad
\min\{x_i,x_j\}\le x_i',x_j'\le \max\{x_i,x_j\}.
\]

Then
\[
f(x_i)+f(x_j)
=
f(\lambda x_i'+(1-\lambda)x_j')+
f((1-\lambda)x_i'+\lambda x_j')
\]
for some $\lambda\in[0,1]$. By convexity,
\[
f(x_i)+f(x_j)\ge f(x_i')+f(x_j').
\]
Hence the assertion follows. \hfill$\square$

\medskip

\section*{4. Petrovi\'{c} and Steffensen inequalities}
```

```
\medskip
\noindent{\bf Theorem 2 (Petrovi\'{c}).}
Let $f$ be a convex function on $[0,a]$ and let
$x_1,x_2,\dots,x_n\in[0,a]$ satisfy
\[
x_1+x_2+\cdots+x_n\le a.
\]
Then
\[
f(x_1)+f(x_2)+\cdots+f(x_n)
\le
f(x_1+x_2+\cdots+x_n)+(n-1)f(0).
\]

\medskip
{\bf Proof.}
Since
\[
(x_1,x_2,\dots,x_n,0,0,\dots,0)\prec (x_1+x_2+\cdots+x_n,0,0,\dots,0),
\]
by Karamata inequality the assertion follows.

\medskip

\noindent{\bf Theorem 3 (Steffensen).}
Let $f$ be a convex function on $[0,a]$ and let
$x_1,x_2,\dots,x_n\in[0,a]$ satisfy
\[
x_1+x_2+\cdots+x_n\le a.
\]
Then
\[
\sum_{k=1}^n f(x_k)\le (n-1)f(0)+f\!\left(\sum_{k=1}^n x_k\right).
\]

\medskip
\section*{5. Schur's inequality}

\medskip
\noindent{\bf Theorem 4 (Schur).}
Let $x,y,z\ge 0$ and $r\ge 0$. Then
\[
x^r(x-y)(x-z)+y^r(y-x)(y-z)+z^r(z-x)(z-y)\ge 0.
\]

\medskip
```

```
{\bf Proof.}
Assume $x\ge y\ge z\ge 0$. Then
\[
x^r(x-y)(x-z)\ge x^r(y-x)(x-z)\ge 0,
\]
and similarly for the other terms. Summing gives the inequality.

\medskip

\section*{6. Muirhead's inequality}

\medskip
\noindent{\bf Theorem 5 (Muirhead).}
Let $\mathbf{p}=(p_1,p_2,\dots,p_n)$ and
$\mathbf{q}=(q_1,q_2,\dots,q_n)$ be two
non-increasing sequences of non-negative integers such that $\mathbf{p}$
majorizes
$\mathbf{q}$. Let $x_1,x_2,\dots,x_n$ be non-negative reals. Then
\[
\sum_{\text{sym}} x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}
\ge
\sum_{\text{sym}} x_1^{q_1}x_2^{q_2}\cdots x_n^{q_n},
\]
where the sums run over all distinct permutations of the exponents.

\medskip
{\bf Example 1.}
Prove that for non-negative $a,b,c$,
\[
a^3+b^3+c^3+3abc \ge ab^2+bc^2+ca^2+ba^2+cb^2+ac^2.
\]

\medskip
{\bf Solution.}
Consider vectors of exponents:
\[
\mathbf{p}=(3,0,0),\quad \mathbf{q}=(2,1,0).
\]
Since $(3,0,0)\succ (2,1,0)$, by Muirhead inequality,
\[
\sum_{\text{sym}} a^3 \ge \sum_{\text{sym}} a^2b.
\]
Adding $3abc$ to both sides preserves inequality.

\medskip
```

```
{\bf Example 2.}
Prove that for $a,b,c\ge 0$,
\[
a^4+b^4+c^4 \ge a^2b^2+b^2c^2+c^2a^2.
\]

\medskip
{\bf Solution.}
The exponent vectors are
\[
\mathbf{p}=(4,0,0),\quad \mathbf{q}=(2,2,0).
\]
Since $(4,0,0)\succ (2,2,0)$, Muirhead inequality gives the result
immediately.

\medskip

\section*{7. Applications to Olympiad inequalities}

\medskip
{\bf Problem 1.}
Let $a,b,c>0$ such that $a+b+c=1$. Prove that
\[
\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \le 1.
\]

\medskip
{\bf Solution.}
We rewrite
\[
\frac{a}{1+b} = \frac{a^2}{a(1+b)} \le \frac{a^2}{a(1+b)}.
\]
By Cauchy-Schwarz inequality,
\[
\sum_{cyc} \frac{a^2}{a(1+b)} \le \frac{(a+b+c)^2}{\sum a(1+b)} =
\frac{1}{1} =1.
\]

\medskip
{\bf Problem 2.}
For positive reals $x,y,z$ with $xyz=1$, prove that
\[
x^3+y^3+z^3 \ge x+y+z.
\]

\medskip
```

```
{\bf Solution.}
By AM-GM inequality,
\[
x^3+y^3+z^3 \ge 3(xyz) = 3 \ge x+y+z.
\]
Since $x+y+z \le 3$ for $xyz=1$, the inequality holds.

\medskip
\section*{8. Nesbitt's inequality}

\medskip
\noindent{\bf Theorem 6 (Nesbitt).}
For positive reals $a,b,c$,
\[
\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge \frac{3}{2}.
\]

\medskip
{\bf Proof.}
By AM-GM,
\[
\frac{a}{b+c}+\frac{b}{c+a}\ge \frac{(a+b)^2}{2(ab+bc+ca)}.
\]
Summing cyclically gives
\[
\sum \frac{a}{b+c} \ge \frac{(a+b+b+c+c+a)^2}{2\cdot 3(ab+bc+ca)} =
\frac{3}{2}.
\]

\medskip
{\bf Example 3.}
Prove that for positive $a,b,c$,
\[
\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge \frac{a+b+c}{2}.
\]

\medskip
{\bf Solution.}
By Cauchy-Schwarz:
\[
\left(\sum \frac{a^2}{b+c}\right)\left(\sum a(b+c)\right) \ge (a+b+c)^2.
\]
But $\sum a(b+c) = 2(ab+bc+ca) \le 2(a+b+c)^2/3$, so
\[
\sum \frac{a^2}{b+c} \ge \frac{(a+b+c)^2}{2(ab+bc+ca)} \ge
\frac{a+b+c}{2}.
```

```
\]

\medskip

\section*{9. Engel form of Cauchy inequality}

\medskip
\noindent{\bf Theorem 7.}
For positive reals $a_i,b_i$,
\[
\sum_{i=1}^n \frac{a_i^2}{b_i} \ge \frac{(\sum a_i)^2}{\sum b_i}.
\]

\medskip
{\bf Proof.}
Direct application of Cauchy-Schwarz in the form
\[
\left(\sum \frac{a_i^2}{b_i}\right)\left(\sum b_i\right) \ge \left(\sum
a_i\right)^2.
\]

\medskip
{\bf Example 4.}
Prove that for $a,b,c>0$,
\[
\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \ge \frac{a+b+c}{2}.
\]

\medskip
{\bf Solution.}
Use Engel form with $a_i = a,b,c$ and $b_i = b+c,c+a,a+b$. Then
\[
\sum \frac{a_i^2}{b_i} \ge \frac{(a+b+c)^2}{2(a+b+c)} = \frac{a+b+c}{2}.
\]

\medskip

\section*{10. Applications of AM-GM}

\medskip
{\bf Problem 3.}
For $a,b,c>0$, prove
\[
(a+b)(b+c)(c+a) \ge 8abc.
\]
```

```
\medskip
{\bf Solution.}
By AM-GM:
\[
a+b \ge 2\sqrt{ab},\quad b+c \ge 2\sqrt{bc},\quad c+a \ge 2\sqrt{ca}.
\]
Multiplying gives
\[
(a+b)(b+c)(c+a) \ge 8\sqrt{a^2b^2c^2} = 8abc.
\]

\medskip
{\bf Problem 4.}
Prove that for positive $x,y,z$,
\[
x+y+z \ge 3\sqrt[3]{xyz}.
\]

\medskip
{\bf Solution.}
AM-GM directly gives
\[
\frac{x+y+z}{3} \ge \sqrt[3]{xyz} \implies x+y+z \ge 3\sqrt[3]{xyz}.
\]

\medskip

\section*{11. Rearrangement inequality}

\medskip
\noindent{\bf Theorem 8 (Rearrangement).}
Let $a_1\le \cdots \le a_n$ and $b_1\le \cdots \le b_n$ be real numbers.
Then
\[
\sum_{i=1}^n a_i b_i \ge \sum_{i=1}^n a_i b_{\sigma(i)}
\]
for any permutation $\sigma$.

\medskip
{\bf Example 5.}
For positive $a,b,c$, prove that
\[
ab+bc+ca \le a^2+b^2+c^2
\]
if $a\le b\le c$.
```

```latex
\medskip
{\bf Solution.}
Rearrangement inequality implies pairing the largest $a_i$ with largest
$b_i$ maximizes the sum, hence
\[
ab+bc+ca \le a^2+b^2+c^2.
\]

\medskip

\section*{12. Cauchy-Schwarz examples continued}

\medskip
{\bf Problem 5.}
For $a,b,c>0$, prove
\[
(a+b+c)^2 \le 3(a^2+b^2+c^2).
\]

\medskip
{\bf Solution.}
By Cauchy-Schwarz:
\[
(1+1+1)(a^2+b^2+c^2) \ge (a+b+c)^2 \im 3(a^2+b^2+c^2) \ge (a+b+c)^2.
\]

\medskip
{\bf Problem 6.}
Prove that for positive reals $x,y$,
\[
x+y \le \sqrt{2(x^2+y^2)}.
\]

\medskip
{\bf Solution.}
Cauchy-Schwarz in 2D:
\[
(x^2+y^2)(1^2+1^2) \ge (x+y)^2 \im 2(x^2+y^2) \ge (x+y)^2.
\]

\medskip

\section*{13. Schur inequality applications}

\medskip
{\bf Problem 7.}
```

```
For $a,b,c\ge 0$ and $r=1$, prove
\[
a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) \ge 0.
\]

\medskip
{\bf Solution.}
Take cases: if $a\ge b\ge c$, then $a(a-b)(a-c) \ge 0$, $b(b-c)(b-a)\ge
0$, $c(c-a)(c-b)\ge 0$. Summing yields inequality.

\medskip
{\bf Problem 8.}
For $a,b,c\ge 0$, prove
\[
a^3+b^3+c^3+3abc \ge \sum a^2b.
\]

\medskip
{\bf Solution.}
Already covered via Muirhead in Example 1.

\medskip
\section*{14. Chebyshev's inequality}

\medskip
\noindent{\bf Theorem 9 (Chebyshev).}
If $a_1\le a_2\le \cdots \le a_n$ and $b_1\le b_2\le \cdots \le b_n$,
then
\[
\frac{1}{n}\sum_{i=1}^n a_i b_i \ge \left(\frac{1}{n}\sum_{i=1}^n
a_i\right)\left(\frac{1}{n}\sum_{i=1}^n b_i\right).
\]

\medskip
{\bf Example 6.}
For positive reals $a,b,c$, prove
\[
a^2+b^2+c^2 \ge ab+bc+ca.
\]

\medskip
{\bf Solution.}
Take sequences $a_1=a,b_1=b,c_1=c$ and $b_1=a,b_2=b,c_3=c$. Then
\[
\frac{a^2+b^2+c^2}{3} \ge \frac{(a+b+c)^2}{9} \im a^2+b^2+c^2 \ge
ab+bc+ca.
```

```latex
\]

\medskip

\section*{15. Muirhead inequality}

\medskip
\noindent{\bf Theorem 10 (Muirhead).}
If $(p_1,p_2,\dots,p_n) \succ (q_1,q_2,\dots,q_n)$, then for positive
$a_i$,
\[
\sum_{\text{sym}} a_1^{p_1} \dots a_n^{p_n} \ge \sum_{\text{sym}}
a_1^{q_1} \dots a_n^{q_n}.
\]

\medskip
{\bf Example 7.}
Prove that for $a,b,c>0$,
\[
a^3+b^3+c^3 \ge a^2b+a^2c+b^2a+b^2c+c^2a+c^2b.
\]

\medskip
{\bf Solution.}
Sequence $(3,0,0)\succ(2,1,0)$. By Muirhead, inequality holds.

\medskip

\section*{16. Titu's lemma}

\medskip
\noindent{\bf Lemma 1 (Titu).}
For positive $a_i,b_i$,
\[
\sum_{i=1}^n \frac{a_i^2}{b_i} \ge \frac{(\sum a_i)^2}{\sum b_i}.
\]

\medskip
{\bf Example 8.}
Prove for $a,b,c>0$,
\[
\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge a+b+c.
\]

\medskip
{\bf Solution.}
```

```
By Titu's lemma,
\[
\sum \frac{a^2}{b} \ge \frac{(a+b+c)^2}{a+b+c} = a+b+c.
\]

\medskip

\section*{17. AM-QM inequality}

\medskip
\noindent{\bf Theorem 11 (AM-QM).}
For positive $a_i$,
\[
\frac{a_1+\dots+a_n}{n} \le \sqrt{\frac{a_1^2+\dots+a_n^2}{n}}.
\]

\medskip
{\bf Example 9.}
Prove that for $x,y,z>0$,
\[
x+y+z \le \sqrt{3(x^2+y^2+z^2)}.
\]

\medskip
{\bf Solution.}
Direct application:
\[
\frac{x+y+z}{3} \le \sqrt{\frac{x^2+y^2+z^2}{3}} \im x+y+z \le
\sqrt{3(x^2+y^2+z^2)}.
\]

\medskip

\section*{18. Power mean inequality}

\medskip
\noindent{\bf Theorem 12.}
For positive $a_i$ and $r>s$,
\[
\left(\frac{a_1^r+\dots+a_n^r}{n}\right)^{1/r} \ge
\left(\frac{a_1^s+\dots+a_n^s}{n}\right)^{1/s}.
\]

\medskip
{\bf Example 10.}
Prove that for $a,b,c>0$,
```

```
\[
\sqrt{\frac{a^2+b^2+c^2}{3}} \ge \frac{a+b+c}{3}.
\]

\medskip
{\bf Solution.}
Take $r=2,s=1$, $n=3$. Power mean inequality gives exactly the
statement.

\medskip

\section*{19. Inequalities with symmetric sums}

\medskip
{\bf Problem 9.}
For positive $a,b,c$,
\[
a^2+b^2+c^2 \ge ab+bc+ca.
\]

\medskip
{\bf Solution.}
Use $(a-b)^2+(b-c)^2+(c-a)^2 \ge 0$:
\[
(a-b)^2+(b-c)^2+(c-a)^2 = 2(a^2+b^2+c^2-ab-bc-ca) \ge 0.
\]

\medskip

\section*{20. Ravi substitution for triangle inequalities}

\medskip
{\bf Problem 10.}
For a triangle with sides $a,b,c$, prove
\[
a^2+b^2+c^2 \ge 4\sqrt{3} S,
\]
where $S$ is area.

\medskip
{\bf Solution.}
Use Ravi substitution: $a=x+y$, $b=y+z$, $c=z+x$. Then
\[
a^2+b^2+c^2 = 2(x^2+y^2+z^2+xy+yz+zx) \ge 4\sqrt{3}S.
\]
```

```
\medskip

\section*{21. Nesbitt and cyclic sums Revised}

\medskip
{\bf Problem 11.}
Prove positive $a,b,c$:
\[
\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}.
\]

\medskip
{\bf Solution.}
Use Nesbitt's inequality as shown in Section 8. Cyclic sum approach:
\[
\sum_{\text{cyc}} \frac{a}{b+c} \ge \frac{(a+b+c)^2}{2(ab+bc+ca)} \ge
\frac{3}{2}.
\]

\medskip
\section*{22. Schur's inequality}

\medskip
\noindent{\bf Theorem 13 (Schur).}
For $a,b,c \ge 0$ and $r\ge 0$,
\[
a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \ge 0.
\]

\medskip
{\bf Example 12.}
Prove for positive $a,b,c$:
\[
a^3+b^3+c^3 + 3abc \ge ab(a+b)+bc(b+c)+ca(c+a).
\]

\medskip
{\bf Solution.}
Use $r=1$ in Schur's inequality. After expansion:
\[
a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) \ge 0
\]
leads to the required inequality.

\medskip
```

```latex
\section*{23. Jensen's inequality}

\medskip
\noindent{\bf Theorem 14 (Jensen).}
If $f$ is convex on $I$ and $\sum t_i = 1$ with $t_i \ge 0$,
\[
f\left(\sum t_i x_i\right) \le \sum t_i f(x_i).
\]

\medskip
{\bf Example 13.}
Prove for $x,y,z>0$:
\[
\ln(x)+\ln(y)+\ln(z) \le 3\ln\left(\frac{x+y+z}{3}\right).
\]

\medskip
{\bf Solution.}
Take $f(t)=\ln t$, which is concave. Using Jensen for concave function:
\[
\sum t_i f(x_i) \le f(\sum t_i x_i),
\]
with $t_i=\frac{1}{3}$, gives the inequality.

\medskip

\section*{24. Rearrangement inequality}

\medskip
\noindent{\bf Theorem 15.}
For $a_1\le a_2\le \cdots \le a_n$ and $b_1\le b_2\le \cdots \le b_n$,
\[
\sum a_ib_i \text{ is maximal when paired in order, minimal when paired
oppositely.}
\]

\medskip
{\bf Example 14.}
For positive $a,b,c$:
\[
ab+bc+ca \le a^2+b^2+c^2.
\]

\medskip
{\bf Solution.}
Take sequences $(a,b,c)$ and $(a,b,c)$, apply rearrangement. Maximal sum
```

```latex
is $a^2+b^2+c^2$.

\medskip

\section*{25. AM-GM for $n$ variables}

\medskip
\noindent{\bf Theorem 16 (AM-GM).}
For positive $a_i$,
\[
\frac{a_1+\dots+a_n}{n} \ge \sqrt[n]{a_1\dots a_n}.
\]

\medskip
{\bf Example 15.}
Prove for $a,b>0$:
\[
a+b \ge 2\sqrt{ab}.
\]

\medskip
{\bf Solution.}
Direct AM-GM:
\[
\frac{a+b}{2} \ge \sqrt{ab} \im a+b \ge 2\sqrt{ab}.
\]

\medskip

\section*{26. Cauchy-Schwarz in sum of squares}

\medskip
{\bf Problem 12.}
For positive $a,b,c$:
\[
(a+b+c)^2 \le 3(a^2+b^2+c^2).
\]

\medskip
{\bf Solution.}
Use Cauchy-Schwarz:
\[
(a+b+c)^2 = (1+1+1)(a^2+b^2+c^2) \le 3(a^2+b^2+c^2).
\]

\medskip
```

```latex
\section*{27. Weighted AM-GM inequality}

\medskip
\noindent{\bf Theorem 17.}
For positive $a_i$ and weights $w_i>0$,
\[
\frac{\sum w_i a_i}{\sum w_i} \ge \prod a_i^{w_i/\sum w_i}.
\]

\medskip
{\bf Example 16.}
Prove for $a,b>0$:
\[
\frac{2a+b}{3} \ge \sqrt[3]{a^2b}.
\]

\medskip
{\bf Solution.}
Take weights $w_1=2, w_2=1$, apply weighted AM-GM.

\medskip

\section*{28. Cauchy for fractions}

\medskip
{\bf Problem 13.}
For positive $a,b,c$:
\[
\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge a+b+c.
\]

\medskip
{\bf Solution.}
Apply Cauchy-Schwarz in form:
\[
\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)((b+c+a)) \ge
(a+b+c)^2 \im \sum \frac{a^2}{b} \ge a+b+c.
\]

\medskip

\section*{29. Engel's form of Cauchy}

\medskip
\noindent{\bf Lemma 2.}
```

```
For positive $a_i,b_i$:
\[
\sum \frac{a_i}{b_i} \ge \frac{(\sum a_i)^2}{\sum a_i b_i}.
\]

\medskip
{\bf Example 17.}
Prove for $x,y>0$:
\[
\frac{x}{y}+\frac{y}{x} \ge 2.
\]

\medskip
{\bf Solution.}
Apply Engel's lemma:
\[
\frac{x}{y}+\frac{y}{x} \ge \frac{(x+y)^2}{xy+xy} = 2.
\]

\medskip

\section*{30. Applications to cyclic inequalities}

\medskip
{\bf Problem 14.}
For positive $a,b,c$:
\[
\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \ge \frac{a+b+c}{2}.
\]

\medskip
{\bf Solution.}
Use Titu's lemma:
\[
\sum \frac{a^2}{b+c} \ge \frac{(a+b+c)^2}{2(a+b+c)} = \frac{a+b+c}{2}.
\]

\medskip
\documentclass[12pt]{article}
\usepackage{amsmath, amssymb, amsthm}

\title{Solutions to Selected Olympiad Inequalities}
\author{}
\date{}

\begin{document}
```

```
\maketitle

\section*{Problem 1 (Yugoslav International Selection Test 1969)}
\textbf{Problem:} Let real numbers $a_i, b_i$ $(i=1,2,\dots,n)$ satisfy
\[
a_1 \ge a_2 \ge \dots \ge a_n > 0, \quad b_1 \ge a_1, \quad b_1b_2 \ge
a_1a_2, \dots, b_1b_2\cdots b_n \ge a_1a_2\cdots a_n.
\]
Prove that
\[
b_1 + b_2 + \dots + b_n \ge a_1 + a_2 + \dots + a_n.
\]

\textbf{Solution:} Apply Karamata's inequality to the convex function
$f(x) = e^x$ and substitute $x_i = \ln a_i, y_i = \ln b_i$. Then the
given conditions
\[
b_1b_2\cdots b_k \ge a_1a_2\cdots a_k \im \sum_{i=1}^k \ln b_i \ge
\sum_{i=1}^k \ln a_i
\]
show that $(\ln b_1, \dots, \ln b_n)$ majorizes $(\ln a_1, \dots, \ln
a_n)$. By Karamata's inequality for convex $f$,
\[
\sum e^{\ln b_i} = \sum b_i \ge \sum e^{\ln a_i} = \sum a_i.
\]

\section*{Problem 2}
\textbf{Problem:} Prove that for positive $a_1, \dots, a_n$,
\[
\frac{a_1^3}{a_2} + \frac{a_2^3}{a_3} + \dots + \frac{a_n^3}{a_1} \ge
a_1^2 + \dots + a_n^2.
\]

\textbf{Solution:} Let $x_i = \ln a_i$. Then $\frac{a_i^3}{a_{i+1}} =
e^{3x_i - x_{i+1}}$. Consider $f(x) = e^x$, which is convex. By
Karamata, the sequence $(3x_1-x_2, 3x_2-x_3, \dots, 3x_n-x_1)$ majorizes
$(2x_1, 2x_2, \dots, 2x_n)$. Then
\[
\sum_{i=1}^n \frac{a_i^3}{a_{i+1}} = \sum e^{3x_i - x_{i+1}} \ge \sum
e^{2x_i} = \sum a_i^2.
\]

\section*{Problem 3}
\textbf{Problem:} Prove that for positive $a,b,c$,
\[
(a+b-c)(b+c-a)(c+a-b) \le abc.
```

```
\]

\textbf{Solution:} Using Ravi substitution, let
\[
a = y+z, \quad b = z+x, \quad c = x+y, \quad x,y,z>0.
\]
Then LHS becomes
\[
(x+y)(y+z)(z+x) \le (y+z)(z+x)(x+y) = \text{RHS}.
\]
Equality holds when $x=y=z$.

\section*{Problem 4}
\textbf{Problem:} For $a,b,c>0$ with $abc=1$, prove
\[
\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \le 1.
\]

\textbf{Solution:} Substitute $a = \frac{x}{y}, b = \frac{y}{z}, c =
\frac{z}{x}$ with $x,y,z>0$. Then
\[
\sum \frac{1}{a+1} = \sum \frac{1}{\frac{x}{y}+1} = \sum \frac{y}{x+y}
\le 1.
\]
Equality occurs if $x=y=z$.

\section*{Problem 5}
\textbf{Problem:} For positive $a,b,c$,
\[
\frac{a^3}{b^2-bc+c^2} + \frac{b^3}{c^2-ca+a^2} + \frac{c^3}{a^2-ab+b^2}
\ge \frac{3(ab+bc+ca)}{a+b+c}.
\]

\textbf{Solution:} Apply the Titu's lemma (Cauchy-Schwarz in Engel
form):
\[
\sum \frac{a_i^2}{b_i} \ge \frac{(a_1 + a_2 + a_3)^2}{b_1+b_2+b_3}.
\]
Let $a_i^2 = a^3,b^3,c^3$ and $b_i = b^2-bc+c^2,\dots$ to get the
desired inequality.

\section*{Problem 6 (IMO 1998 Shortlist)}
\textbf{Problem:} For $a,b,c>0$ with $abc=1$,
\[
\frac{a^3}{(1+b)(1+c)} + \frac{b^3}{(1+c)(1+a)} + \frac{c^3}{(1+a)(1+b)}
\ge \frac{3}{4}.
```

```
\]

\textbf{Solution:} Substitute $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c =
\frac{z}{x}$. Then
\[
\sum \frac{x^3}{(x+y)(x+z)} \ge \frac{3}{4}
\]
reduces to Nesbitt-type inequality.

\section*{Problem 7 (IMO 1984)}
\textbf{Problem:} Let $x,y,z \ge 0$ with $x+y+z=1$. Prove
\[
0 \le xy + yz + zx - 2xyz \le \frac{7}{27}.
\]

\textbf{Solution:} For the lower bound, $xy+yz+zx \ge 2xyz$ by AM-GM.
For the upper bound, by symmetry, assume $x=y$ and $z=1-2x$, then
\[
xy+yz+zx - 2xyz = 2x(1-x) - 2x^2(1-2x) = -2x^3 + 2x^2 - 2x^2 + 2x - 0?
\]
Compute derivative and maximize; maximum occurs at $x=y=z=1/3$, giving
$7/27$.

\section*{Problem 8 (IMO 1999)}
\textbf{Problem:} For $x_1,\dots,x_n \ge 0$, find smallest $C$ such that
\[
\sum_{1 \le i < j \le n} x_i x_j (x_i^2 + x_j^2) \le C
\left(\sum_{i=1}^n x_i \right)^4.
\]

\textbf{Solution:} By symmetry and testing $x_1 = x_2 = \dots = x_n$,
the best constant is
\[
C = \frac{1}{8}.
\]

\section*{Problem 9 (IMO 2000)}
\textbf{Problem:} For $x,y,z>0$ with $xyz=1$, prove
\[
\left(x-1+\frac{1}{y}\right)\left(y-1+\frac{1}{z}\right)\left(z-1+\frac{
1}{x}\right) \le 1.
\]

\textbf{Solution:} Substitute $x = \frac{a}{b}, y = \frac{b}{c}, z =
\frac{c}{a}$. Then
\[
```

```
\prod \left( \frac{a-b}{b} + \frac{c}{b} \right) = \prod \frac{a+c}{b} =
1,
\]using AM-GM and symmetry. Hence inequality holds.
\section*{The END.}
\textbf
Thank you for reading this article, it took 4 people and 150 hours of
Latex coding and additional research time for us to produce this paper,
we hope you gained valuable insight - Anish, Bhuvanesh, Radhika and
Pranav

\end{document}

\end{document}
```