The Ramanujan - Soldner Constant.

Very special constant, denoted by

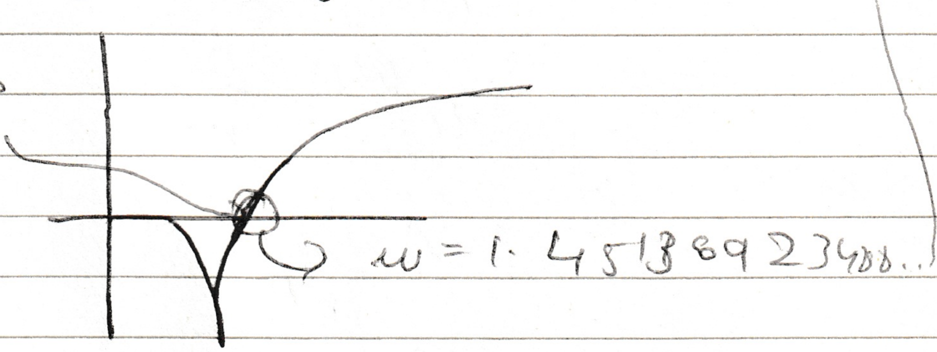
$$\mu \approx 1.45136923488338105 0 \ldots .85$$

Defined as the unique positive zeroconstant of the logarithmic integral function

$$li(\mu) = 0$$

It's where value of the log value crosses 0

Roughly looks like



The logarithmic integral

$$li(x) = \int_0^x \frac{d}{dx} \Rightarrow \frac{dt}{\ln t}$$

integral is improper at $t=1$

$$li(x) = PV \int_0^x \frac{dt}{\ln t}$$

This function grows slowly, has singularity, and behaves abnormaly around 1.

Appears in -

- Prime number distribution
- Zero crossing behaviour of analytic functions.
- Constants in analytic number theory

Ramanujan- Soldner constant is imp on a $li(x)$ graph.

Properties, Behaviour & Derivation

$$li(\mu) = 0$$

$$\frac{d}{dx} li(x) = \frac{1}{\ln x}$$

At $x = \mu$

$$li'(\mu) = \frac{1}{\ln \mu}$$

since $\ln \mu \approx 0.372$

$$li'(\mu) \approx 2.686$$

A numerical evaluation

$$x_{n+1} = x_n - \frac{li(x_n)}{1/\ln(x_n)}$$

Start with $x_0 = 1.5$, converges rapidly into constant

$li(x)$ is negative on $(0, \mu)$

Crosses 0 at $x = \mu$

Shoots faster as $x \to \infty$

Relation to prime number theory

Has a fun boundary!

for $x < 4$, $li(x) < 0$
$\because$ $li(x) \approx \pi(x) \rightarrow$ tells nothing much about prime numbers (primes)
- but it does define the point where the analytical approx climbes about zeros.

Series Expansion

$$li(x) = (x-\mu)\, li'(\mu) + \frac{(x-\mu)^2}{2!} li''(\mu) + \ldots$$

where - $li''(x) = \dfrac{1}{x(\ln x)^2}$

Fun Fact!

$$\ln x \approx \frac{1}{\ln x} \quad \boxed{!}$$

An example -

$li(x) = 0$

Where logarithmic integral is defined (Principal value) by

Step-1
$$li(x) = \int_0^x \frac{dt}{\ln t}$$

$$f(x) = li(x)$$
$$f'(x) = \frac{d}{dx} li(x) = \frac{1}{\ln(x)}$$

Step-2 Newtons method formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - li(x_n) \ln(x_n)$$

take
$$\boxed{x_0 = 1.5}$$

Manually aprox. iterations.

x	$li(x)$
1.50	0.213455
1.46	0.040250
1.452	0.00307
1.451	≈ 0

$$x_0 = 1.5$$
$$f(x_0) = li(1.5) = 0.213455$$
$$f'(x_0) = \frac{1}{\ln(1.5)} = \frac{1}{0.405465} = 2.465$$

$x_1 = 1.5 - 0.213455 \times 0.405465$
$x_1 = 1.5 - 0.08666$
$x_1 = \boxed{1.41334}$

Iteration 2
evaluate near 1.41

$$li(1.413) \approx -0.0572$$

$$f'(1.413) = \frac{1}{\ln(1.413)} = \frac{1}{0.344} \approx 2.907$$

$x_2 = 1.41334 - (-0.572) \cdot 0.344$
$x_2 = 1.41334 + 0.0197$
$x_2 \approx \boxed{1.4330}$

Iteration 3
$li(1.4330) \approx -0.01474$
$\ln(1.4330) = 0.359$
$f'(1.4330) = \frac{1}{0.359} = 2.786$

$\therefore x_3 = 1.4330 - (-0.01474)(0.359)$
$x_3 = 1.4330 + 0.00529$
$x_3 \approx 1.43829$

Iteration 4 ......
$li(1.4383) \approx -0.00488 \; l_n = 0.363$
$x_4 = 1.43829 + 0.00177$
$x_4 \approx 1.44006$

Iteration 5

$$\mathrm{li}(1.44006) \approx -0.00124$$
$$\ln(1.44006) = 0.3646$$
$$x_5 = 1.44006 - (-0.00124)\times(0.3646)$$
$$x_5 = 1.44006 + 0.000452$$
$$x_5 \approx 1.44051$$

Values will keep converging (and shrink fast)

Finally, you converge to

$$\mu \approx 1.451369234883381$$

Which, matches the known 20 digit constant:

$$\mu = 1.45136923488338105028.$$

Comes from the people -

- Johann Georg von Soldner (1809)
- Srinivasa Ramanujan

By Bhuvanesh Nallapati.

THANK YOU FOR READING