

INEQUALITIES OF KARAMATA, SCHUR AND MUIRHEAD,
AND SOME APPLICATIONS

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Abstract. Three classical general inequalities—those of Karamata, Schur and Muirhead—are proved in this article. They can be used in proving other inequalities, particularly those appearing as problems in mathematical competitions, including International Mathematical Olympiads. Some problems of this kind are given as examples. Several related inequalities—those of Petrović, Steffensen and Szegő—are treated, as well.

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1. Introduction

Three classical inequalities are proved in this article. They can be used in proving several other inequalities, particularly those appearing as problems in mathematical competitions, including International Mathematical Olympiads. Some problems of this kind are given as examples.

The article is adapted according to our book [7], intended for preparation of students for mathematical competitions. Its (shortened) Serbian version was published in *Nastava matematike*, L, 4 (2005), 22–31.

We start by recalling some well-known notions which will be used in the sequel.

A function $f : (a, b) \rightarrow \mathbb{R}$ is said to be convex if for each two points $x_1, x_2 \in (a, b)$ and each two nonnegative real numbers λ_1, λ_2 satisfying $\lambda_1 + \lambda_2 = 1$, the following inequality is valid

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

The function f is concave if the function $-f$ is convex, i.e., if the opposite inequality always holds.

If, in the previous inequalities (assuming $x_1 \neq x_2$), the equality takes place only in the case when $\lambda_1 = 0$ or $\lambda_2 = 0$, then the function f is said to be strictly convex (resp. strictly concave).

It can be easily checked that the function $f : (a, b) \rightarrow \mathbb{R}$ is convex (strictly convex) if and only if the inequality

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x} \quad (1)$$

(resp. $<$) holds for arbitrary points $x_1 < x < x_2$ from (a, b) .

Let $f : (a, b) \rightarrow \mathbb{R}$ and $x, y \in (a, b)$. The quotient

$$\Delta f(x, y) = \frac{f(y) - f(x)}{y - x}$$

is called the divided difference of the function f at the points x, y .

2. Relation of majorization

Let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ be two sequences of real numbers. Rearrange them so that

$$a_1 \geq a_2 \geq \dots \geq a_n, \quad b_1 \geq b_2 \geq \dots \geq b_n.$$

We say that a majorizes b , and write $b \prec a$, if

$$\sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i, \quad k = 1, 2, \dots, n-1,$$

and

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i.$$

The following lemma is well known.

Lemma 1. Let $a, b \in \mathbb{R}^n$ and $b \prec a$. Then there exists a finite sequence of vectors

$$a = x^{(0)}, x^{(1)}, \dots, x^{(m)} = b$$

such that each $x^{(k+1)}$ is obtained from $x^{(k)}$ by replacing two coordinates x_i, x_j by x'_i, x'_j satisfying

$$x'_i + x'_j = x_i + x_j, \quad \min\{x_i, x_j\} \leq x'_i, x'_j \leq \max\{x_i, x_j\}.$$

3. Karamata inequality

Theorem 1 (Karamata). Let f be a convex function on an interval I and let $a, b \in I^n$ be such that $b \prec a$. Then

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq f(b_1) + f(b_2) + \dots + f(b_n).$$

Proof. By Lemma 1 it is sufficient to prove the theorem in the case when the vectors a and b differ only in two coordinates, say

$$a = (x_1, x_2, \dots, x_i, \dots, x_j, \dots, x_n),$$

$$b = (x_1, x_2, \dots, x'_i, \dots, x'_j, \dots, x_n),$$

where

$$x'_i + x'_j = x_i + x_j, \quad \min\{x_i, x_j\} \leq x'_i, x'_j \leq \max\{x_i, x_j\}.$$

Then

$$f(x_i) + f(x_j) = f(\lambda x'_i + (1 - \lambda)x'_j) + f((1 - \lambda)x'_i + \lambda x'_j)$$

for some $\lambda \in [0, 1]$. By convexity,

$$f(x_i) + f(x_j) \geq f(x'_i) + f(x'_j).$$

Hence the assertion follows. □

4. Petrović and Steffensen inequalities

Theorem 2 (Petrović). Let f be a convex function on $[0, a]$ and let $x_1, x_2, \dots, x_n \in [0, a]$ satisfy

$$x_1 + x_2 + \dots + x_n \leq a.$$

Then

$$f(x_1) + f(x_2) + \dots + f(x_n) \leq f(x_1 + x_2 + \dots + x_n) + (n - 1)f(0).$$

Proof. Since

$$(x_1, x_2, \dots, x_n, 0, 0, \dots, 0) \prec (x_1 + x_2 + \dots + x_n, 0, 0, \dots, 0),$$

by Karamata inequality the assertion follows.

Theorem 3 (Steffensen). Let f be a convex function on $[0, a]$ and let $x_1, x_2, \dots, x_n \in [0, a]$ satisfy

$$x_1 + x_2 + \dots + x_n \leq a.$$

Then

$$\sum_{k=1}^n f(x_k) \leq (n - 1)f(0) + f\left(\sum_{k=1}^n x_k\right).$$

5. Schur's inequality

Theorem 4 (Schur). Let $x, y, z \geq 0$ and $r \geq 0$. Then

$$x^r(x - y)(x - z) + y^r(y - x)(y - z) + z^r(z - x)(z - y) \geq 0.$$

Proof. Assume $x \geq y \geq z \geq 0$. Then

$$x^r(x - y)(x - z) \geq x^r(y - x)(x - z) \geq 0,$$

and similarly for the other terms. Summing gives the inequality.

6. Muirhead's inequality

Theorem 5 (Muirhead). Let $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and $\mathbf{q} = (q_1, q_2, \dots, q_n)$ be two non-increasing sequences of non-negative integers such that \mathbf{p} majorizes \mathbf{q} . Let x_1, x_2, \dots, x_n be non-negative reals. Then

$$\sum_{\text{sym}} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \geq \sum_{\text{sym}} x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n},$$

where the sums run over all distinct permutations of the exponents.

Example 1. Prove that for non-negative a, b, c ,

$$a^3 + b^3 + c^3 + 3abc \geq ab^2 + bc^2 + ca^2 + ba^2 + cb^2 + ac^2.$$

Solution. Consider vectors of exponents:

$$\mathbf{p} = (3, 0, 0), \quad \mathbf{q} = (2, 1, 0).$$

Since $(3, 0, 0) \succ (2, 1, 0)$, by Muirhead inequality,

$$\sum_{\text{sym}} a^3 \geq \sum_{\text{sym}} a^2 b.$$

Adding $3abc$ to both sides preserves inequality.

Example 2. Prove that for $a, b, c \geq 0$,

$$a^4 + b^4 + c^4 \geq a^2 b^2 + b^2 c^2 + c^2 a^2.$$

Solution. The exponent vectors are

$$\mathbf{p} = (4, 0, 0), \quad \mathbf{q} = (2, 2, 0).$$

Since $(4, 0, 0) \succ (2, 2, 0)$, Muirhead inequality gives the result immediately.

7. Applications to Olympiad inequalities

Problem 1. Let $a, b, c > 0$ such that $a + b + c = 1$. Prove that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \leq 1.$$

Solution. We rewrite

$$\frac{a}{1+b} = \frac{a^2}{a(1+b)} \leq \frac{a^2}{a(1+b)}.$$

By Cauchy-Schwarz inequality,

$$\sum_{cyc} \frac{a^2}{a(1+b)} \leq \frac{(a+b+c)^2}{\sum a(1+b)} = \frac{1}{1} = 1.$$

Problem 2. For positive reals x, y, z with $xyz = 1$, prove that

$$x^3 + y^3 + z^3 \geq x + y + z.$$

Solution. By AM-GM inequality,

$$x^3 + y^3 + z^3 \geq 3(xyz) = 3 \geq x + y + z.$$

Since $x + y + z \leq 3$ for $xyz = 1$, the inequality holds.

8. Nesbitt's inequality

Theorem 6 (Nesbitt). For positive reals a, b, c ,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof. By AM-GM,

$$\frac{a}{b+c} + \frac{b}{c+a} \geq \frac{(a+b)^2}{2(ab+bc+ca)}.$$

Summing cyclically gives

$$\sum \frac{a}{b+c} \geq \frac{(a+b+b+c+c+a)^2}{2 \cdot 3(ab+bc+ca)} = \frac{3}{2}.$$

Example 3. Prove that for positive a, b, c ,

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}.$$

Solution. By Cauchy-Schwarz:

$$\left(\sum \frac{a^2}{b+c} \right) \left(\sum a(b+c) \right) \geq (a+b+c)^2.$$

But $\sum a(b+c) = 2(ab+bc+ca) \leq 2(a+b+c)^2/3$, so

$$\sum \frac{a^2}{b+c} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{a+b+c}{2}.$$

9. Engel form of Cauchy inequality

Theorem 7. For positive reals a_i, b_i ,

$$\sum_{i=1}^n \frac{a_i^2}{b_i} \geq \frac{(\sum a_i)^2}{\sum b_i}.$$

Proof. Direct application of Cauchy-Schwarz in the form

$$\left(\sum \frac{a_i^2}{b_i}\right) \left(\sum b_i\right) \geq \left(\sum a_i\right)^2.$$

Example 4. Prove that for $a, b, c > 0$,

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}.$$

Solution. Use Engel form with $a_i = a, b, c$ and $b_i = b+c, c+a, a+b$. Then

$$\sum \frac{a_i^2}{b_i} \geq \frac{(a+b+c)^2}{2(a+b+c)} = \frac{a+b+c}{2}.$$

10. Applications of AM-GM

Problem 3. For $a, b, c > 0$, prove

$$(a+b)(b+c)(c+a) \geq 8abc.$$

Solution. By AM-GM:

$$a+b \geq 2\sqrt{ab}, \quad b+c \geq 2\sqrt{bc}, \quad c+a \geq 2\sqrt{ca}.$$

Multiplying gives

$$(a+b)(b+c)(c+a) \geq 8\sqrt{a^2b^2c^2} = 8abc.$$

Problem 4. Prove that for positive x, y, z ,

$$x+y+z \geq 3\sqrt[3]{xyz}.$$

Solution. AM-GM directly gives

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz} \implies x+y+z \geq 3\sqrt[3]{xyz}.$$

11. Rearrangement inequality

Theorem 8 (Rearrangement). Let $a_1 \leq \dots \leq a_n$ and $b_1 \leq \dots \leq b_n$ be real numbers. Then

$$\sum_{i=1}^n a_i b_i \geq \sum_{i=1}^n a_i b_{\sigma(i)}$$

for any permutation σ .

Example 5. For positive a, b, c , prove that

$$ab + bc + ca \leq a^2 + b^2 + c^2$$

if $a \leq b \leq c$.

Solution. Rearrangement inequality implies pairing the largest a_i with largest b_i maximizes the sum, hence

$$ab + bc + ca \leq a^2 + b^2 + c^2.$$

12. Cauchy-Schwarz examples continued

Problem 5. For $a, b, c > 0$, prove

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2).$$

Solution. By Cauchy-Schwarz:

$$(1 + 1 + 1)(a^2 + b^2 + c^2) \geq (a + b + c)^2 \geq (a + b + c)^2.$$

Problem 6. Prove that for positive reals x, y ,

$$x + y \leq \sqrt{2(x^2 + y^2)}.$$

Solution. Cauchy-Schwarz in 2D:

$$(x^2 + y^2)(1^2 + 1^2) \geq (x + y)^2 \geq (x + y)^2.$$

13. Schur inequality applications

Problem 7. For $a, b, c \geq 0$ and $r = 1$, prove

$$a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b) \geq 0.$$

Solution. Take cases: if $a \geq b \geq c$, then $a(a - b)(a - c) \geq 0$, $b(b - c)(b - a) \geq 0$, $c(c - a)(c - b) \geq 0$. Summing yields inequality.

Problem 8. For $a, b, c \geq 0$, prove

$$a^3 + b^3 + c^3 + 3abc \geq \sum a^2 b.$$

Solution. Already covered via Muirhead in Example 1.

14. Chebyshev's inequality

Theorem 9 (Chebyshev). If $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$, then

$$\frac{1}{n} \sum_{i=1}^n a_i b_i \geq \left(\frac{1}{n} \sum_{i=1}^n a_i \right) \left(\frac{1}{n} \sum_{i=1}^n b_i \right).$$

Example 6. For positive reals a, b, c , prove

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Solution. Take sequences $a_1 = a, b_1 = b, c_1 = c$ and $b_1 = a, b_2 = b, c_3 = c$. Then

$$\frac{a^2 + b^2 + c^2}{3} \geq \frac{(a + b + c)^2}{9} a^2 + b^2 + c^2 \geq ab + bc + ca.$$

15. Muirhead inequality

Theorem 10 (Muirhead). If $(p_1, p_2, \dots, p_n) \succ (q_1, q_2, \dots, q_n)$, then for positive a_i ,

$$\sum_{\text{sym}} a_1^{p_1} \dots a_n^{p_n} \geq \sum_{\text{sym}} a_1^{q_1} \dots a_n^{q_n}.$$

Example 7. Prove that for $a, b, c > 0$,

$$a^3 + b^3 + c^3 \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b.$$

Solution. Sequence $(3, 0, 0) \succ (2, 1, 0)$. By Muirhead, inequality holds.

16. Titu's lemma

Lemma 1 (Titu). For positive a_i, b_i ,

$$\sum_{i=1}^n \frac{a_i^2}{b_i} \geq \frac{(\sum a_i)^2}{\sum b_i}.$$

Example 8. Prove for $a, b, c > 0$,

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c.$$

Solution. By Titu's lemma,

$$\sum \frac{a^2}{b} \geq \frac{(a + b + c)^2}{a + b + c} = a + b + c.$$

17. AM-QM inequality

Theorem 11 (AM-QM). For positive a_i ,

$$\frac{a_1 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}}.$$

Example 9. Prove that for $x, y, z > 0$,

$$x + y + z \leq \sqrt{3(x^2 + y^2 + z^2)}.$$

Solution. Direct application:

$$\frac{x + y + z}{3} \leq \sqrt{\frac{x^2 + y^2 + z^2}{3}} \implies x + y + z \leq \sqrt{3(x^2 + y^2 + z^2)}.$$

18. Power mean inequality

Theorem 12. For positive a_i and $r > s$,

$$\left(\frac{a_1^r + \cdots + a_n^r}{n} \right)^{1/r} \geq \left(\frac{a_1^s + \cdots + a_n^s}{n} \right)^{1/s}.$$

Example 10. Prove that for $a, b, c > 0$,

$$\sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq \frac{a + b + c}{3}.$$

Solution. Take $r = 2, s = 1, n = 3$. Power mean inequality gives exactly the statement.

19. Inequalities with symmetric sums

Problem 9. For positive a, b, c ,

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Solution. Use $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$:

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 2(a^2 + b^2 + c^2 - ab - bc - ca) \geq 0.$$

20. Ravi substitution for triangle inequalities

Problem 10. For a triangle with sides a, b, c , prove

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S,$$

where S is area.

Solution. Use Ravi substitution: $a = x + y$, $b = y + z$, $c = z + x$. Then

$$a^2 + b^2 + c^2 = 2(x^2 + y^2 + z^2 + xy + yz + zx) \geq 4\sqrt{3}S.$$

21. Nesbitt and cyclic sums Revised

Problem 11. Prove positive a, b, c :

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Solution. Use Nesbitt's inequality as shown in Section 8. Cyclic sum approach:

$$\sum_{\text{cyc}} \frac{a}{b+c} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3}{2}.$$

22. Schur's inequality

Theorem 13 (Schur). For $a, b, c \geq 0$ and $r \geq 0$,

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0.$$

Example 12. Prove for positive a, b, c :

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a).$$

Solution. Use $r = 1$ in Schur's inequality. After expansion:

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0$$

leads to the required inequality.

23. Jensen's inequality

Theorem 14 (Jensen). If f is convex on I and $\sum t_i = 1$ with $t_i \geq 0$,

$$f\left(\sum t_i x_i\right) \leq \sum t_i f(x_i).$$

Example 13. Prove for $x, y, z > 0$:

$$\ln(x) + \ln(y) + \ln(z) \leq 3 \ln\left(\frac{x+y+z}{3}\right).$$

Solution. Take $f(t) = \ln t$, which is concave. Using Jensen for concave function:

$$\sum t_i f(x_i) \leq f\left(\sum t_i x_i\right),$$

with $t_i = \frac{1}{3}$, gives the inequality.

24. Rearrangement inequality

Theorem 15. For $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$,

$\sum a_i b_i$ is maximal when paired in order, minimal when paired oppositely.

Example 14. For positive a, b, c :

$$ab + bc + ca \leq a^2 + b^2 + c^2.$$

Solution. Take sequences (a, b, c) and (a, b, c) , apply rearrangement. Maximal sum is $a^2 + b^2 + c^2$.

25. AM-GM for n variables

Theorem 16 (AM-GM). For positive a_i ,

$$\frac{a_1 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \dots a_n}.$$

Example 15. Prove for $a, b > 0$:

$$a + b \geq 2\sqrt{ab}.$$

Solution. Direct AM-GM:

$$\frac{a+b}{2} \geq \sqrt{ab} \Rightarrow a+b \geq 2\sqrt{ab}.$$

26. Cauchy-Schwarz in sum of squares

Problem 12. For positive a, b, c :

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2).$$

Solution. Use Cauchy-Schwarz:

$$(a + b + c)^2 = (1 + 1 + 1)(a^2 + b^2 + c^2) \leq 3(a^2 + b^2 + c^2).$$

27. Weighted AM-GM inequality

Theorem 17. For positive a_i and weights $w_i > 0$,

$$\frac{\sum w_i a_i}{\sum w_i} \geq \prod a_i^{w_i / \sum w_i}.$$

Example 16. Prove for $a, b > 0$:

$$\frac{2a + b}{3} \geq \sqrt[3]{a^2 b}.$$

Solution. Take weights $w_1 = 2, w_2 = 1$, apply weighted AM-GM.

28. Cauchy for fractions

Problem 13. For positive a, b, c :

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c.$$

Solution. Apply Cauchy-Schwarz in form:

$$\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \right) ((b + c + a)) \geq (a + b + c)^2 \sum \frac{a^2}{b} \geq a + b + c.$$

29. Engel's form of Cauchy

Lemma 2. For positive a_i, b_i :

$$\sum \frac{a_i}{b_i} \geq \frac{(\sum a_i)^2}{\sum a_i b_i}.$$

Example 17. Prove for $x, y > 0$:

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

Solution. Apply Engel's lemma:

$$\frac{x}{y} + \frac{y}{x} \geq \frac{(x+y)^2}{xy+xy} = 2.$$

30. Applications to cyclic inequalities

Problem 14. For positive a, b, c :

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}.$$

Solution. Use Titu's lemma:

$$\sum \frac{a^2}{b+c} \geq \frac{(a+b+c)^2}{2(a+b+c)} = \frac{a+b+c}{2}.$$

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Solutions to Selected Olympiad Inequalities

Problem 1 (Yugoslav International Selection Test 1969)

Problem: Let real numbers a_i, b_i ($i = 1, 2, \dots, n$) satisfy

$$a_1 \geq a_2 \geq \dots \geq a_n > 0, \quad b_1 \geq a_1, \quad b_1 b_2 \geq a_1 a_2, \dots, b_1 b_2 \dots b_n \geq a_1 a_2 \dots a_n.$$

Prove that

$$b_1 + b_2 + \dots + b_n \geq a_1 + a_2 + \dots + a_n.$$

Solution: Apply Karamata's inequality to the convex function $f(x) = e^x$ and substitute $x_i = \ln a_i, y_i = \ln b_i$. Then the given conditions

$$b_1 b_2 \dots b_k \geq a_1 a_2 \dots a_k \sum_{i=1}^k \ln b_i \geq \sum_{i=1}^k \ln a_i$$

show that $(\ln b_1, \dots, \ln b_n)$ majorizes $(\ln a_1, \dots, \ln a_n)$. By Karamata's inequality for convex f ,

$$\sum e^{\ln b_i} = \sum b_i \geq \sum e^{\ln a_i} = \sum a_i.$$

Problem 2

Problem: Prove that for positive a_1, \dots, a_n ,

$$\frac{a_1^3}{a_2} + \frac{a_2^3}{a_3} + \dots + \frac{a_n^3}{a_1} \geq a_1^2 + \dots + a_n^2.$$

Solution: Let $x_i = \ln a_i$. Then $\frac{a_i^3}{a_{i+1}} = e^{3x_i - x_{i+1}}$. Consider $f(x) = e^x$, which is convex. By Karamata, the sequence $(3x_1 - x_2, 3x_2 - x_3, \dots, 3x_n - x_1)$ majorizes $(2x_1, 2x_2, \dots, 2x_n)$. Then

$$\sum_{i=1}^n \frac{a_i^3}{a_{i+1}} = \sum e^{3x_i - x_{i+1}} \geq \sum e^{2x_i} = \sum a_i^2.$$

Problem 3

Problem: Prove that for positive a, b, c ,

$$(a + b - c)(b + c - a)(c + a - b) \leq abc.$$

Solution: Using Ravi substitution, let

$$a = y + z, \quad b = z + x, \quad c = x + y, \quad x, y, z > 0.$$

Then LHS becomes

$$(x + y)(y + z)(z + x) \leq (y + z)(z + x)(x + y) = \text{RHS}.$$

Equality holds when $x = y = z$.

Problem 4

Problem: For $a, b, c > 0$ with $abc = 1$, prove

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \leq 1.$$

Solution: Substitute $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$ with $x, y, z > 0$. Then

$$\sum \frac{1}{a+1} = \sum \frac{1}{\frac{x}{y}+1} = \sum \frac{y}{x+y} \leq 1.$$

Equality occurs if $x = y = z$.

Problem 5

Problem: For positive a, b, c ,

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq \frac{3(ab + bc + ca)}{a + b + c}.$$

Solution: Apply the Titu's lemma (Cauchy-Schwarz in Engel form):

$$\sum \frac{a_i^2}{b_i} \geq \frac{(a_1 + a_2 + a_3)^2}{b_1 + b_2 + b_3}.$$

Let $a_i^2 = a^3, b^3, c^3$ and $b_i = b^2 - bc + c^2, \dots$ to get the desired inequality.

Problem 6 (IMO 1998 Shortlist)

Problem: For $a, b, c > 0$ with $abc = 1$,

$$\frac{a^3}{(1+b)(1+c)} + \frac{b^3}{(1+c)(1+a)} + \frac{c^3}{(1+a)(1+b)} \geq \frac{3}{4}.$$

Solution: Substitute $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$. Then

$$\sum \frac{x^3}{(x+y)(x+z)} \geq \frac{3}{4}$$

reduces to Nesbitt-type inequality.

Problem 7 (IMO 1984)

Problem: Let $x, y, z \geq 0$ with $x + y + z = 1$. Prove

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Solution: For the lower bound, $xy + yz + zx \geq 2xyz$ by AM-GM. For the upper bound, by symmetry, assume $x = y$ and $z = 1 - 2x$, then

$$xy + yz + zx - 2xyz = 2x(1-x) - 2x^2(1-2x) = -2x^3 + 2x^2 - 2x^2 + 2x - 0?$$

Compute derivative and maximize; maximum occurs at $x = y = z = 1/3$, giving $7/27$.

Problem 8 (IMO 1999)

Problem: For $x_1, \dots, x_n \geq 0$, find smallest C such that

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{i=1}^n x_i \right)^4.$$

Solution: By symmetry and testing $x_1 = x_2 = \dots = x_n$, the best constant is

$$C = \frac{1}{8}.$$

Problem 9 (IMO 2000)

Problem: For $x, y, z > 0$ with $xyz = 1$, prove

$$\left(x - 1 + \frac{1}{y}\right) \left(y - 1 + \frac{1}{z}\right) \left(z - 1 + \frac{1}{x}\right) \leq 1.$$

Solution: Substitute $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$. Then

$$\prod \left(\frac{a-b}{b} + \frac{c}{b} \right) = \prod \frac{a+c}{b} = 1,$$

using AM-GM and symmetry. Hence inequality holds.

The END.

Thank you for reading this article, it took 4 people and 150 hours of Latex coding and additional research time for us to produce this paper, we hope you gained valuable insight - Anish, Bhuvanesh, Radhika and Pranav