84th William Lowell Putnam Mathematical Competition Solutions

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A1.

For a positive integer n, define

$$f_n(x) = \prod_{m=1}^{n} \cos(mx).$$

Using the product rule, $f_n''(x)$ is a sum of two types of terms: (1) terms where two distinct factors $\cos(m_1 x)$ and $\cos(m_2 x)$ are each differentiated once, and (2) terms where a single factor $\cos(mx)$ is differentiated twice.

Evaluating at $x = 0$, all terms of the first type vanish since $\sin(0) = 0$. For the second type,

$$(\cos(mx))''\Big|_{x=0} = -m^2.$$

Therefore,

$$f_n''(0) = -\sum_{m=1}^{n} m^2,$$

and hence

$$|f_n''(0)| = \sum_{m=1}^{n} m^2 = \frac{n(n+1)(2n+1)}{6}.$$

The function

$$g(n) = \frac{n(n+1)(2n+1)}{6}$$

is increasing for $n \in N$ and satisfies $g(17) = 1785$ and $g(18) = 2109$. Thus the smallest n for which $|f_n''(0)| > 2023$ is $n = 18$.

A2.

Let n be an even positive integer and let

$$p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$$

be a monic real polynomial of degree $2n$. Suppose

$$p(1/k) = k^2 \quad forallintegerskwith 1 \le |k| \le n.$$

Define

$$q(x) = x^{2n+2} - x^{2n}p(1/x).$$

For $x \neq 0$, the condition $p(1/x) = x^2$ is equivalent to $q(x) = 0$. Thus $\pm 1, \pm 2, \ldots, \pm n$ are roots of $q(x)$.

Hence we may write

$$q(x) = (x^2 + ax + b)(x^2 - 1)(x^2 - 4) \cdots (x^2 - n^2)$$

for some real numbers a, b.

Comparing coefficients of x^{2n+1} gives $a = 0$. Comparing constant terms gives

$$-1 = (-1)^n (n!)^2 b.$$

Since n is even, this implies

$$b = -\frac{1}{(n!)^2}.$$

Therefore,

$$x^2 + ax + b = x^2 - \frac{1}{(n!)^2},$$

and the remaining real roots of $q(x)$ are

$$x = \pm\frac{1}{n!}.$$

These are the only other real numbers satisfying $p(1/x) = x^2$.

A3.

The maximum possible value of r is

$$r = \frac{\pi}{2},$$

which is achieved by $f(x) = \cos x$ and $g(x) = \sin x$.

First solution. Assume for contradiction that $0 < r < \pi/2$. We may assume $f(x) \neq 0$ for $x \in [0, r)$. Define

$$k(x) = f(x)^2 + g(x)^2.$$

Then

$$|k'(x)| = 2|f(x)f'(x) + g(x)g'(x)| \leq 4|f(x)g(x)| \leq 2k(x).$$

Hence

$$\left|\frac{d}{dx} \log k(x)\right| \leq 2,$$

so $k(r) \neq 0$. In particular, if $f(r) = 0$ then $g(r) \neq 0$.

Define

$$h(x) = \tan^{-1}\left(\frac{g(x)}{f(x)}\right).$$

Then

$$h'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{f(x)^2 + g(x)^2},$$

so $|h'(x)| \leq 1$. Since $h(0) = 0$, we have $|h(x)| \leq x < r$. This implies $|g(x)/f(x)|$ is bounded on $[0, r)$, contradicting $\lim_{x\to r^-} g(x)/f(x) = \infty$. Hence $r < \pi/2$ is impossible.

A4.

All vertices of the icosahedron have equal length, so the center is at the origin. Scaling does not affect the conclusion, so we may take the vertices to be the cyclic permutations of

$$\left(\pm\frac{1}{2},\pm\frac{\phi}{2},0\right),$$

where $\phi = (1+\sqrt{5})/2$.

The subgroup of R^3 generated by these vectors contains $G \times G \times G$, where G is the subgroup of R generated by 1 and ϕ. Since ϕ is irrational, G is dense in R, and hence the generated subgroup is dense in R^3.

A5.

The complex numbers z with the stated property are

$$-\frac{3^{1010}-1}{2} \quad \textit{and} \quad -\frac{3^{1010}-1}{2} \pm \frac{\sqrt{9^{1010}-1}}{2}i.$$

For $n \geq 1$, we have the identity

$$\sum_{k=0}^{3^{n-1}-1} (-2)^{f(k)} x^k = \prod_{j=0}^{n-1} (x^{2\cdot 3^j} - 2x^{3^j} + 1),$$

which follows by induction using the relations $f(3^{n-1}+k) = f(k)+1$ and $f(2\cdot 3^{n-1}+k) = f(k)$.

Define the shift operator S on polynomials by $S(p(z)) = p(z+1)$. Let

$$p_n(z) = \sum_{k=0}^{3^{n-1}-1} (-2)^{f(k)} (z+k)^{2n+3}.$$

Then

$$p_n(z) = \prod_{j=0}^{n-1} (S^{2\cdot 3^j} - 2S^{3^j} + I) z^{2n+3}.$$

Repeated application of the operator $S^{\ell} - 2I + S^{-\ell}$ reduces the degree by 2, and after n steps we obtain a cubic polynomial whose roots are

$$0,\ \pm\frac{\sqrt{9^n-1}}{2}i.$$

Shifting back yields the stated roots.

A6.

For all n, Bob has a winning strategy.

The game corresponds to building a permutation of $\{1,\ldots,n\}$, where the number of times k is chosen on turn k equals the number of fixed points of the permutation.

If n is even, Bob chooses the goal "even". He partitions $\{1, \dots, n\}$ into pairs $\{1,2\}, \{3,4\}, \dots$. Each time Alice chooses one element of a pair, Bob chooses the other. Thus fixed points occur in pairs, so the total number is even.

If n is odd, Bob chooses the goal "odd". He makes an initial move to create exactly one fixed point and then mirrors Alice's strategy to maintain odd parity for the remainder of the game.

B1.

The number of reachable configurations is

$$m + n - 2m - 1.$$

Initially, the unoccupied squares form a path from $(1, n)$ to $(m, 1)$ with $m-1$ horizontal and $n-1$ vertical steps. Each move preserves this property, yielding an injective map from reachable configurations to such paths.

Conversely, working backwards, any such path can be reduced step by step to the initial configuration, proving surjectivity.

B2.

The minimum value of $k(n)$ is 3.

We factor $2023 = 7 \cdot 17^2$. If $k(n) = 1$, then $2023n = 2^a$, which is impossible. If $k(n) = 2$, then

$$2023n = 2^a + 2^b = 2^b(1 + 2^{a-b}),$$

but -1 is not a power of 2 modulo 7, so this is impossible.

To show $k(n) = 3$ is attainable, one constructs integers $a > b > 0$ such that

$$2^a + 2^b + 1 \equiv 0 \pmod{7} \quad \textit{and} \quad 2^a + 2^b + 1 \equiv 0 \pmod{17^2}.$$

The Chinese remainder theorem guarantees such a, b, yielding the desired n.

B3.

The expected value is

$$\frac{2n+2}{3}.$$

Partition the sequence into alternating increasing and decreasing segments; let there be N such segments. Then the longest zigzag subsequence has length $N + 1$.

For $n \geq 3$, the probability that X_1, X_2, X_3 is not monotone is $2/3$, so

$$E(a(X_1, \dots, X_n) - a(X_2, \dots, X_n)) = \frac{2}{3}.$$

Since $a(X_1, X_2) = 2$, induction yields the stated expectation.

B4.

The minimum possible value of T is 29.

Let $s_k = t_k - t_{k-1}$. Then

$$f(t_k) - f(t_{k-1}) = k^2 s_k^2,$$

and we seek to minimize

$$T = \sum s_k$$

subject to

$$\sum k s_k^2 = 4045 \quad \textit{and} \quad s_k \geq 1.$$

At the minimum, we must have $s_k = 1$ for $k \leq n$, yielding

$$4045 = \frac{n(n+1)}{2} + (n+1)(T-n)^2.$$

Solving shows the minimum occurs at $n = 9$ with $T = 29$.

B5.

The desired property holds if and only if

$$n = 1 \quad \textit{or} \quad n \equiv 2 \pmod 4.$$

A permutation has a square root if and only if it has an even number of cycles of each even length.

If n is odd or divisible by 4, one can choose m so that the permutation induced by multiplication has an odd number of 2-cycles. If $n \equiv 2 \pmod 4$, every cycle occurs in pairs, so a square root exists.

B6.

The determinant equals

$$(-1)^{\lceil n/2 \rceil - 1} 2^{\lfloor n/2 \rfloor}.$$

This follows either by systematic row and column elimination, reducing the matrix to a small block, or by writing the matrix as a product AB and applying the Cauchy–Binet formula.