Inequalities of Karamata, Schur and Muirhead

Selected Applications, Derivations, and Problems

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Three classical general inequalities—those of Karamata, Schur and Muirhead—. They can be used in proving other inequalities, particularly those appearing as problems in mathematical competitions, including International Mathematical Olympiads.

Inequalities of Karmata, Schur and Murihead. with Applications. 1

- By Bhuvanesh. N

3 classic inequalities are provided in these papers.

Lets start by recalling some well known notions which will be used in this sequel.

A function $f:(a,b)\to R$ is said to be convex if for each points $x_1, x_2 \in a, b$ and each two nonnegative real numbers λ_1, λ_2 Satisfying $\lambda_1+\lambda_2=1$, the following inequality is valid -

$$\Rightarrow f(\lambda_1 x_1+\lambda_2 x_2) \leq \lambda_1 f(x_1)+\lambda_2 (-f)(x_2)$$

The function f is concave if the function $-f$ is convex ie, the opposite inequality.

$$f(\lambda_1 x_1+\lambda_2 x_2) \geq \lambda_1 f(x_1)+\lambda_2 f(x_2).$$

The function f is concave if the previous inequalities (assuming $x_1 \neq x_2$), the inequality takes place only in the case when $\lambda_1=0$ or $\lambda_2=0$, then the function f is said to be strictly convex.

It can be easily checked that the function $f:(a,b)\to R$ is convex (strictly convex) if and only if the inequality

$$(1) \qquad \frac{f(x)-f(x_1)}{x-x_1} \leq \frac{f(x_2)-f(x)}{x_2-x}.$$

$\left(\text{resp. } \frac{f(x)-f(x_1)}{x-x_1} < \frac{f(x_2)-f(x)}{x_2-x}\right)$ holds for arbitrary points x_1, x_2, x from (a,b), such that $x_1 < x < x_2$, . an analogues criterion is valid for concave functions.

let $f:(a,b)\to R$ and $x, y \in (a,b)$. The quotient

$$\Delta_f(x,y) = \frac{f(y)-f(x)}{y-x}.$$

is called the divided difference of the function f at the points x, y. It is clear that the divided diff. is a symetric function in x, y i.e. $\Delta_f(x,y) = \Delta_f(y,x)$

$\Rightarrow$ Let the function f be convex and let $x_1, x_2 \in (a,b)$ so that $x_1 < x_2$
choose an arbitary $x \in (a,b)$ such that (for instence) $x_1 < x < x_2$ (for other values $x \in (a,b)$ the proof is similar.). Applying the assertions, the convexity of the function f implies that the inequality (1) is valid. In other words

(2) $$\Delta_f(x_1, x) = \Delta_f(x, x_1) \leq \Delta_f(x_2, x)$$

which means that the function Δ_f is increasing in its first argument. As far as it is symetric it is increasing in its second argument, as well
But, if Δ_f is increasingly in both arguments, then for $x_1 < x < x_2$ the inequality (2) holds, which implies (1), and so the function f is convex on $(a;b)$.

Majorization relation for finite sequences & Karmata's inequality.

Lets introduce a majorization relation for finite seq. of $\mathbb{R}$

Definition 1- let $a = (a_1, a_2 \ldots, a_n)$
$b = (b_1, b_2 \ldots, b_n)$, be two (finite) sequences of real numbers. We say that sequence a majorizes the sequence b and we say

$$a \succ b \text{ or } b \prec a.$$

$\prec$ is not a $\leq$ or $>$
it means
"more spread than"

if, after a possible renumeration the terms of the sequences a, b satisfy.

1. $a_1 \geq a_2 \ldots \geq a_n$ & $b_1 \geq b_2 \geq \ldots \geq b_n$;
2. $a_1 + a_2 \ldots a_k \geq b_1 + b_2 + \ldots + b_k$
3. $a_1 + a_2 \ldots a_n = b_1 + b_2 \ldots + b_n$.

Clearly, $a \succ a$ holds an arbitrary seq. a.

Example - If $a = (a_i)_{i=1}^{n}$ is an arbitrary seq of non negative numbers, having the sum = 1, then

$$(1, 0, \dots, 0) \succ (a_1, a_2, \dots, a_n) \succ \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$$

(b) The sequences $(4, 4, 1)$ & $(5, 5, 2)$ are incomparable. In the sequence & sense of the relation $\succ$, non of the two majorizes ~~are~~ more than the others.

$\boxed{\text{Theorem - I}}$ let $a = (a_i)_{i=1}^{n}$ & $(b_i)_{i=1}^{n}$ be (finite) sequences of real numbers from an interval (α, β) if the seq a majorizes b, $a \succ b$ and if $f: (\alpha, \beta) \to R$ is a convex function then the equality

$$\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i)$$

Use Abel's transformation [2]

$$C_i = \Delta_f (a_i, b_i) = \frac{f(b_i) - f(a_i)}{b_i - a_i}$$

further, denote.

$$A_k = \sum_{i=1}^{k} a_i, \quad B_k = \sum_{i=1}^{k} b_i; \quad k = (1, \dots, n);$$

$$A_0 = B_0 = 0$$

the assumption 3° implies that $A_n = B_n$. Now, we have

$$\sum_{i=1}^{n} f(a_i) - \sum_{i=1}^{n} f_{b_i} = \sum_{i=1}^{n} (f(a_i) - f(b_i))$$

$$= \sum_{i=1}^{n} c_i (a_i - b_i)$$

$$= \sum_{i=1}^{n} c_i (A_i - A_{i-1} - B_i + B_{i+1}) = \sum_{i=1}^{n} c_i (A_i - B_i)$$

$$- \sum_{i=1}^{n} c_i (A_{i-1} - B_{i-1}).$$

$$= \sum_{i=1}^{n-1} c_i (A_i - B_i) - \sum_{i=0}^{n-1} c_{i+1} (A_i - B_i) = \sum_{i=1}^{n-1} (c_i - c_{i+1}) (A_i - B_i).$$

It is $A_i \geq B_i$ for $i = 1, 2, \ldots, n-1$. Hence, the last sum and also the difference

$$\sum_{i=1}^{n} f(a_i) - \sum_{i=1}^{n} f(b_i) \qquad \psi_1(x)_2 = \frac{d^2}{dx^2}$$

is non-negative, which was to be proved.

2nd proof. In this proof, we use Stieltjes' integral

Let $\psi_1, \psi_2 : [\alpha, \beta) \to \mathbb{R}$ be two integrable functions. such that $\psi_1 \succ \psi_2$ in the sense that

$$\int_\alpha^x \psi_1\, dt \geq \int_\alpha^x \psi_2\, dt \text{ for } x \in [\alpha, \beta) \text{ and}$$

$$\int_\alpha^\beta \psi_1\, dt = \int_\alpha^\beta \psi_2\, dt.$$

Let $\varphi : [\alpha, \beta] \to \mathbb{R}$ be an increasing (integrable) function

$$\int_\alpha^\beta \varphi \psi_1\, dx \leq \int_\alpha^\beta \varphi\, \psi_2(t)\, dt.$$

Proof. Put $\psi(x) = \psi_1(x) - \psi_2(x)$ & $g(x)$

$= \int_a^x \psi(t)\,dt$. then, $g(x) \geq 0$ for $x \in [\alpha, \beta]$

& $g(\alpha) = g(\beta) = 0$. Using integration by parts ~~in~~ the integral, we get

$$\int_\alpha^\beta \varphi(t)\psi(t)\,dt = \int_\alpha^\beta \varphi(t)\,dg(t) = \varphi(t)g(t)\Big|_\alpha^\beta$$

$$- \int_\alpha^\beta g(t)\,d\varphi(t)$$

$$= - \int_\alpha^\beta g(t)\,d\varphi(t) \leq 0.$$

Proof of this theorem. The given function f, being convex is continuous, can be represented in the form of $f(x) = \int_\alpha^x \varphi\,dt$ for an increasing function φ. Introduce functions $A(x)$ & $B(x)$ by.

$$A(x) = \sum_{i=1}^{n} m\{[\alpha, x] \cap [\alpha, a_i]\},$$

$$B(x) = \sum_{i=1}^{n} m\{[\alpha, x] \cap [\alpha, b_i]\},$$

where mS denotes measure of the set S. It is easy to see that.

$$A(x) \leq B(x), \ A = (a_1) = B(a_1)$$

And let that $A'(x)$ & $B'(x)$ exist everywhere except a finite series of points, we conclude that:

$$(3) \qquad \int_{\alpha}^{a_1} f\,dA(x) \geq \int_{\alpha}^{a_1} f\,dB(x)$$

But,

$$\int_{\alpha}^{a_1} f\,dA(x) = n\int_{\alpha}^{a_n} \varphi\,dx + (n-1)\int_{a_n}^{a_{n-1}} \varphi\,dx + \cdots + \int_{a_2}^{a_1} \varphi\,dx.$$

$$= f(a_1) f(a_2) \cdots + f(a_n)$$

and the simmilar relation holds for the integral on the right hand side of (3). This proves Karmata's inequality.

Note - The condition that Karmata's inequality holds for every convex function f can be on (α, β) is not only necessary but also suffixient, for the relation $a \succ b$.

Jensen's inequality in the form,

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \cdots f(x_n)}{n}$$

for a Jensen-Convex function f, is obtained as a special case of Karmata's inequality, by putting $b_1 = b_2 = \cdots = b_n$ $= \frac{a_1 + \cdots + a_n}{n}$. The general form of this inequality follows from the weighted form of karmata's inequality

$$\sum_{i=1}^{n} \lambda_i f(a_i) \geq \sum_{i=1}^{n} \lambda_i f(b_i)$$

if $\lambda_i \in R$ & (a_i) & (b_i) satisfy condition no 1 of Definition 1

$$\sum_{i=1}^{k} \lambda_i a_i \geq \sum_{i=1}^{k} \lambda_i b_i$$

Corollary let $f, g: [0,a] \to R$, $0 \leq g(x) \leq 1$, f be decreasing on $[0,a]$, and let $F(x) = \int_0^x f\,dt$ then,

$$\int_0^a fg\,dx \leq F\left(\int_0^a g\,dx\right)$$

if we denote $C = \int_0^a g\,dx$, then $0 \leq C \leq a$

$$\text{let } \bar{g}(x) = \begin{cases} 1, & x \in [0,C] \\ 0, & x \in (C,a] \end{cases}$$

So by applying Lemma 2, we obtain Steffensen's inequality in the form

$$\int_0^C f\,dx = \int_0^a f\bar{g}\,dx \geq \int_0^a fg\,dx. \blacksquare$$

Example 2 prove that for arbitrary ~~position~~ positive numbers a, b, c the inequality

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} \quad \text{holds.}$$

Solution suppose that the numbers a, b, c are such that $a \geq b \geq c$ i.e, the sequence (a, b, c) is decreasing, which in my knowledge can be done without loss of generality. Then we have $(2a, 2b, 2c) \succ (a+b, b+c, c+a)$, and so, applying Karamata's inequality to the function $f(x) = \frac{1}{x}$, convex on the interval $(0, +\infty)$, we obtain the desired inequality.

Example 3 Prove that the inequality

$$\cos(2x_1 - x_2) + \cos(2x_2 - x_3) + \dots + \cos(2x_n - x_1)$$
$$\leq \cos x_1 + \cos x_2 + \dots + \cos x_n$$

holds for arbitrary numbers $x_1, x_2, \dots, x_n$ from the interval $[-\pi/6, \pi/6]$.

SOLUTION. The numbers $2x_i - x_{i+1}$, $i = 1, 2, \dots, n$ $(x_{n+1} = x_1)$, as well as the numbers x_i belong to the interval $[-\pi/2, \pi/2]$. The function $f(x) = \cos x$ is concave on this interval, and so the karmatas inequality holds with the opposite sign. Thus, it is sufficient to prove that the sequences $a = (2x_1 - x_2, 2x_2 - x_3, \dots, 2x_n - x_1)$ & $b = (x_1, x_2, \dots x_n)$, when arranged to be decreasing, satisfy the conditions of theorm-1.

Let indices $m_1, \dots m_n$ & $k_1, \dots, k_n$ be choosen so that

$$\{m_1, \dots, m_n\} = \{k_1, \dots, k_n\} = \{1, \dots, n\}$$

$$(4) \quad 2x_{m_1} - x_{m_1+1} \geq 2x_{m_2} - x_{m_2+1} \geq \dots \geq 2x_{m_n} - x_{m_n+1}$$

$$(5) \quad x_{k_1} \geq x_{k_2} \geq \dots \geq x_{k_n}.$$

Then

$$2x_{m_1} - x_{m_1+1} \geq 2x_{k_1} - x_{k_1+1} \geq x_{k_1}$$

the choice of numvers m_i, the greatest ~~inequality~~ numbers of the form $2x_{m_i} - x_{m_i+1}$; the second one follows by the choice of the numbers k_i, by reasons,

$$(2x_{m_1} - x_{m_1+1}) + (2x_{m_2} - x_{m_2-1}) \geq (2x_{k_1} - x_{k_1+1}) + (2x_{k_2} - x_{k_2+1})$$

$\geq x_{k_1} + x_{k_2}$, generally, the sum of the first l terms of sequence (4) is not less than the sum of the first l terms of the seq (5), for $l = 1, \dots, n-1$. For $l = 1$, the inequality is obtained & fullfilled.

Corollary 3.

let $f : [0, b_1] \to R$ be a convex function & $b_1 \geq b_2 \geq b_{2n+1} \geq 0$

Then,

$$f(b_1 - b_2 + \cdots + b_{2n+1}) \leq f(b_1) - f(b_2) + \cdots + f(b_{2n+1}).$$

holds.

Proof For $n=1$, the inequality reduces to (6) Assuming that it holds for $n-1$ where $n \geq 1$, let us prove that it holds also for n.

Note that $b' = b_1 - b_2 + \cdots + b_{2n-1} \geq b_{2n-1} \geq b_{2n} \geq b_{2n+1}$

$$f(b' - b_{2n} + b_{2n+1}) \leq f(b') - f(b_{2n}) + f(b_{2n+1})$$
$$\leq f(b_1) - f(b_2) + \cdots + f(b_{2n-1}) - f(b_{2n}) + f(b_{2n-1})$$

Inequalities of Schur & Muirhead. 3

Definition - 2.

Let $F(x_1, x_2, \ldots, x_n)$ be a function in n nonnegative real variables. Define $\sum^! F(x_1, x_2, \ldots, x_n)$ as the sum of $n!$ summands, obtained by the expression $F(x_1, x_2, \ldots, x_n)$ as all possible permutations of the sequence $x = (x_i)_{i=1}^n$.

Particularly, if some sequence of non-negative exponents $a = (a_i)_{i=1}^n$, the function F is of the form $F(x_1, x_2, \ldots, x_n)$ as all possible perm. $\sum^! F(x_1, x_2, \ldots, x_n)$ we shall write also,

$$T[a_1, a_2, \ldots, a_n](x_1, x_2, \ldots, x_n)$$

or just $T[a_1, a_2, \ldots, a_n]$ if it is clear which seq. x is used.

lets see an example where $T[1, 0, \ldots, 0] = (n-1)! \cdot (x_1 + x_2 + \cdots + x_n)$, & $T[\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}] = n!$.

Theorem 2. Let $f : [0, +\infty) \to R$ be a convex function, and $(x_i)_{i=1}^{n}$, be a sequence of positive numbers. Then the inequality

$$f(x_1) + \dots + f(x_n) \leq f(x_1 + \dots + x_n) + (n-1) f(0).$$

holds.

Denote $S = \sum_{i=1}^{n} x_i$ and $\lambda_i = \frac{x_i}{S}$. Then $\sum_{i=1}^{n} \lambda_i = 1$ & $x_i = (1-\lambda_i) \cdot 0 + \lambda_i s$, $i = 1, 2, \dots n$.

The convexity of the function f implies that

$$f(x_i) \leq (1-\lambda_i) f(0) + \lambda_i f(s) \quad i = 1, \dots, n,$$

$$\sum_{i=1}^{n} f(x_i) \leq (n-1) f(0) + f(s).$$

Corollary 2 (a) let $f : [a, b] \to R$ be a conv. fun. $0 \leq a \leq b$ & $(x_i)_{i=1}^{n}$ be a seq. +ve's. Such that $x_1 + \dots + x_n \leq b-a$. Then the inequality

$$f(a+x_1) + \dots + f(a+x_n) \leq f(a + x_1 + \dots + x_n) + (n-1)f(a)$$

let f $[0, b_i) \to R$ be a convex function, and $b_1 \geq b_2 \geq b_3 \geq 0$. Then,

$$f(b_1 - b_2 + b_3) \leq f(b_1) - f(b_2) + f(b_3).$$

Proof

apply petrovic's inequality to the convex function $\varphi : [0, b-a] \to R$ given by $\varphi(x) = f(a+x)$.

In the inequality (a) put $n=2$, $b_1 = a + x_1 + x_2$, $b_2 = a + x_1$, & $b_3 = a$.

$\sqrt[n]{x_1 x_2 \cdots x_n}$. Using this terminology, the arithmetic-geometric Mean Inequality can be written as

$$T[1,0,\ldots,0] \geq T\left[\frac{1}{n},\ldots,\frac{1}{n}\right]. \Delta$$

Lets prove now, the Schur's inequality.

Theorem 3.

"i.e." all cross terms.

$$T[a+2b,\underline{0,0}] + T[a,b,b] \geq 2T[a+b,b,0]$$

Let (x,y,z) be a sequence of positive numbers. Using elementary transformations, we obtain,

$$\frac{1}{2}T[a+2b,0,0] + \frac{1}{2}T[a,b,b] - T[a+b,b,0]$$
$$= x^a(x^b-y^b)(x^b-z^b) + y^a(y^b-x^b)(y^b-z^b)$$
$$+ z^a(z^b-x^b)(z^b-y^b).$$

without loss of generality, sufficient to prove that:

$$x^a(x^b-y^b)(x^b-z^b) + y^a(y^b-x^b)(y^b-z^b) \geq 0.$$

ie $(x^b-y^b)(x^a(x^b-z^b) - y^a(y^b-z^b)) \geq 0$. The inequality is eq. to

$$x^{a+b} - y^{a+b} - z^b(x^a - y^a) \geq 0.$$ However,

$$x^{a+b} - y^{a+b} - z^b(x^a - y^a) \geq x^{a+b} - y^{a+b} - y^b(x^a - y^a)$$
$$= x^a(x^b-y^b) \geq 0.$$ Which proves the theorem.

Corollary 4 if x, y, z are non-negative real numbers and $r \geq 0$, then the inequality

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0;$$ holds.

Proof Transforming,

$$T[r+2,0,0]+T[r,1,1] \geqslant 2T[r+1,1,0]$$

Which is an example special case of Schur's inequality for $a=r$, $b=1$.

Example-5. putting $a=b=1$ in Schur's inequality (or $r=1$ in corollary 4) we obtain

$$x^3+y^3+z^3+3xyz \geqslant x^2y+xy^2+y^2z+yz^2+z^2x + zx^2. \Delta$$

For arbitrary seq. a and b, $T[a]$ may be incomparable with $T[b]$, in the sense that it is not true that either $T[a](x) \leqslant T[b](x)$ holds for arbitrary values of the variable sequence $x=(x_i)$.

Theorm 4 The expression $T[a]$ is comparable with expression of $T[B]$ for the positive seq. x, if and only if one of the seq a and b majorizes the other one in the sense of the relation $\prec$. If $a \prec b$ then

$$T[a] \leqslant T[b]$$

$$c^{\sum a_i} \leqslant c^{\sum b_i}$$

Put now $x_1 = \cdots = x_k = c$ & $x_{k+1} = \cdots = x_n = 1$.

let $b_k = p+T$, $b_l = p-T$ $(0 < T \leqslant p)$.

Now, if $0 \leqslant \sigma \leqslant T \leqslant p$ def. the seq $a = L(b)$ as follows.

$$a_k = p+\sigma = \frac{T+\sigma}{2T} b_k + \frac{T-\sigma}{2T} b_l,$$

$$a_l = p-\sigma = \frac{T-\sigma}{2T} b_k + \frac{T+\sigma}{2T} b_l.$$

$a_v = b_v$, $(v \neq k, v \neq l)$.

Lemma 3. If $a = L(b)$, then $T[a] \leq T[b]$, while the equality holds only if the sequence x is ~~...~~ constant.

Proof: We can rearrange seq so that $k = 1$ & $l = 2$.

Then...

$$T[b] - T[a] = \sum^{!} x_3^{b_3} \cdots x_n^{b_n} \left(x_1^{p+r} x_2^{p-r} + x_1^{p+r} - x_1^{p+\sigma} x_2^{p-\sigma} - x_1^{p-\sigma} x_2^{p+\sigma} \right).$$

$$= \sum^{!} (x_1 x_2)^{p-\tau} x_3^{b_3} \cdots x_n^{b_n} \left(x_1^{r+\sigma} - x_2^{r+\sigma} \right) \left(x_1^{r-\sigma} - x_2^{r-\sigma} \right) \geq 0$$

The equality holds only if all the x_i's are equal to each other. ■

I have found a lot of problems which i will share on the next printed pages in LaTex format

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Inequalities of Karamata, Schur and Muirhead - Problems

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December 2025

Problems

Problem 1 (Yugoslav International Selection Test 1969)

Let real numbers a_i, b_i $(i = 1, 2, \dots, n)$ be given such that

$$a_1 \geq a_2 \geq \cdots \geq a_n > 0,$$

$$b_1 \geq a_1, \quad b_1 b_2 \geq a_1 a_2, \quad \dots, \quad b_1 b_2 \cdots b_n \geq a_1 a_2 \cdots a_n.$$

Prove that

$$b_1 + b_2 + \cdots + b_n \geq a_1 + a_2 + \cdots + a_n.$$

Hint. Apply a variant of Karamata's inequality to $f(x) = e^x$.

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Problem 2

Prove that

$$\frac{a_1^3}{a_2} + \frac{a_2^3}{a_3} + \cdots + \frac{a_n^3}{a_1} \geq a_1^2 + a_2^2 + \cdots + a_n^2$$

for all positive $a_1, a_2, \dots, a_n$.

Hint. Apply Karamata's inequality to $f(x) = e^x$ and substitute $x_i = \ln a_i$.

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Problem 3

Prove that

$$(a + b - c)(b + c - a)(c + a - b) \leq abc$$

for all positive a, b, c.

—

Problem 4

Let $a, b, c > 0$ with $abc = 1$. Prove that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

—

Problem 5

If $a, b, c > 0$, prove that

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq 3\frac{ab + bc + ca}{a + b + c}.$$

—

Problem 6 (IMO 1998 Shortlist)

Let $a, b, c > 0$ with $abc = 1$. Prove that

$$\frac{a^3}{(1+b)(1+c)} + \frac{b^3}{(1+c)(1+a)} + \frac{c^3}{(1+a)(1+b)} \geq \frac{3}{4}.$$

—

Problem 7 (IMO 1984)

Let $x, y, z \geq 0$ with $x + y + z = 1$. Prove that

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

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Problem 8 (IMO 1999)

Let $x_1, \ldots, x_n \geq 0$. Find the smallest constant C such that

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{i=1}^{n} x_i \right)^4.$$

The best constant is $C = \frac{1}{8}$.

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Problem 9 (IMO 2000)

Let $x, y, z > 0$ with $xyz = 1$. Prove that

$$\left(x - 1 + \frac{1}{y}\right)\left(y - 1 + \frac{1}{z}\right)\left(z - 1 + \frac{1}{x}\right) \leq 1.$$

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Inequalities of Karamata, Schur and Muirhead - Problems

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Problems

Problem 1 (Yugoslav International Selection Test 1969)

Let real numbers a_i, b_i $(i = 1, 2, \ldots, n)$ be given such that

$$a_1 \geq a_2 \geq \cdots \geq a_n > 0,$$

$$b_1 \geq a_1, \quad b_1 b_2 \geq a_1 a_2, \quad \ldots, \quad b_1 b_2 \cdots b_n \geq a_1 a_2 \cdots a_n.$$

Prove that

$$b_1 + b_2 + \cdots + b_n \geq a_1 + a_2 + \cdots + a_n.$$

Hint. Apply a variant of Karamata's inequality to $f(x) = e^x$.

—

Problem 2

Prove that

$$\frac{a_1^3}{a_2} + \frac{a_2^3}{a_3} + \cdots + \frac{a_n^3}{a_1} \geq a_1^2 + a_2^2 + \cdots + a_n^2$$

for all positive $a_1, a_2, \ldots, a_n$.

Hint. Apply Karamata's inequality to $f(x) = e^x$ and substitute $x_i = \ln a_i$.

—

Problem 3

Prove that

$$(a + b - c)(b + c - a)(c + a - b) \leq abc$$

for all positive a, b, c.

—

Problem 4

Let $a, b, c > 0$ with $abc = 1$. Prove that

$$\frac{1}{a + b + 1} + \frac{1}{b + c + 1} + \frac{1}{c + a + 1} \leq 1.$$

—

Problem 5

If $a, b, c > 0$, prove that

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq 3\frac{ab + bc + ca}{a + b + c}.$$

—

Problem 6 (IMO 1998 Shortlist)

Let $a, b, c > 0$ with $abc = 1$. Prove that

$$\frac{a^3}{(1 + b)(1 + c)} + \frac{b^3}{(1 + c)(1 + a)} + \frac{c^3}{(1 + a)(1 + b)} \geq \frac{3}{4}.$$

—

Problem 7 (IMO 1984)

Let $x, y, z \geq 0$ with $x + y + z = 1$. Prove that

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

—

Problem 8 (IMO 1999)

Let $x_1, \ldots, x_n \geq 0$. Find the smallest constant C such that

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{i=1}^{n} x_i \right)^4.$$

The best constant is $C = \frac{1}{8}$.

—

Problem 9 (IMO 2000)

Let $x, y, z > 0$ with $xyz = 1$. Prove that

$$\left(x - 1 + \frac{1}{y} \right) \left(y - 1 + \frac{1}{z} \right) \left(z - 1 + \frac{1}{x} \right) \leq 1.$$

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