Ramanujan's Pi Series (elliptic integrals)

① Ramanujan's famous

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}$$

Points to remember

→ Converges fast (8 digits of pi per term)

→ π at the end

→ Constants ≠ arbitrary.

② Structural autopsy of the series

$$\frac{(4n)!}{(n!)^4}$$

Central binomial coeff squared.

$$\binom{2n}{n}^2 \cdot \binom{4n}{2n}$$

which is a hypergeometric function.

More specifically -

$${}_2F_1\left(\frac{1}{2}; \frac{1}{2}; 1; z\right)$$ Note this

why 396^{4n}?

By factoring,

$$396 = 4 \cdot 99 = 2^2 \cdot 3^2 \cdot 11$$

this is an evaluation of mod functions at special imaginary points.

There is a CM point in this........
which we will get to.

what is this series saying?

Modular symmetry $\Rightarrow$ closed form period $\Rightarrow \pi$

(iii) Eliptic integral connection.

When we define a complete eliptic integral.

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

Euler's integral representation of F_1

$$_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

$$= \frac{2}{\pi} K(K)$$

Hence.

pi enters.

$$K(K) \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

(iv) Complementary modulus.

$$k'^2 = 1 - k^2$$

$$k' = \sqrt{1-k^2} \quad \text{(Pythagorean)}$$

$$\because \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$\theta \mapsto \frac{\pi}{2} - \theta$$

Then

$$\sin^2\theta \longleftrightarrow \cos^2\theta.$$

since

$$\cos^2\theta = 1 - \sin^2\theta.$$

this integrand becomes

$$\frac{1}{\sqrt{1-k^2\cos^2\theta}} = \frac{1}{\sqrt{1-(1-k^2)\sin^2\theta}} = \frac{1}{1-k'^2\sin^2\theta}.$$

limits stay the same and inconclusion –

$$K'(k) = K(k')$$

Function exists on a lattice in the complex plane, not a line.

(v) Period Ratio & Nome

$$\tau = i\frac{K'}{K'} \quad K' = K(k'),\ k'^2 = 1 - k^2$$

$$q = e^{i\pi\tau} = \exp\left(-\pi \frac{K'}{K}\right)$$

(q) - Turns periods into decay.
- linearizes eliptic expansions.

$K \leftrightarrow q$ is invertible

Alternatively, you could use series.

$$K = 4\sqrt{q}\left(1 + 2q + 15q^2 + 150q^3 + \ldots\ldots\right)$$

Conversely

$$q \exp\left(-\pi \frac{K'(k)}{K(k)}\right)$$

(vi) $\tau = i\frac{K'}{K}, \quad q = e^{i\pi\tau}$

$$\lambda(\tau) = k^2$$

$$\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} = 16q - 128q^2 + 704q^3 - 3072q^4$$

$$k^2 = \lambda(\tau)$$

$K(k), K'(k)$ Becomes period of a modular curve.

transformations like $k \leftrightarrow k'$

$$\tau \mapsto -\frac{1}{\tau}$$

in other words.
analytic identities are backbones of modular symetry!

(vii) Theta function representation.

$$q\, e^{i\pi\tau},$$

$$K = \frac{\pi}{2}\theta_3^2(q)$$

$$k = \frac{\theta_2^2(q)}{\theta_3^2(q)}$$

Elliptic integrals $\Rightarrow$ theta constants.

by - lambda function

$$k^2 = \lambda(\tau) = \frac{\theta_2^4}{\theta_3^4}$$

Asymptotics become trivial.

$q \to 0 \Rightarrow$ small-k regime
$q \to 1 \Rightarrow$ degeneration of the lattice.

So, you start with

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

You end with

$$K = \frac{\pi}{2}\theta_3^2(q), \quad k = \frac{\theta_2^2(q)}{\theta_3^2(q)}$$

(viii) with nome $q = e^{i\pi\tau}$

$$\theta_2(q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}$$

$$\theta_3(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$$

Elliptic theory = quadratically convergent q-series.

ix Complex multiplication (CM) Points.

$$\tau = -d, \; d \in \mathbb{Z}^+.$$

for $\tau = -d$.

$$k^2, \; j(q), \; K.$$

$K = \pi \cdot (\text{alg-no}),\; k^2 = \text{algebraic no.}$

x) Choice of Ramanujan.

$\tau \to$ I have been mentioning this 'τ' but its just a modular parameter or period ratio of an elliptic function.

He chooses $\tau = i\sqrt{58} \Rightarrow q = e^{i\pi\tau} = e^{-\pi\sqrt{58}}$

this is incredibly small!

$$|q| \approx e^{-182.212} \approx 10^{-79} \; !!$$

$$\therefore \quad \tau = \frac{\text{complex period}}{\text{real period}}.$$

(xi)

$$\Phi_{58}(k, k_{58}) = 0$$

$$k(\tau),\ k(58\tau)$$

(xii) $k^2 = \frac{1}{2} - \frac{3\sqrt{29}}{2}$

$\therefore\ k = \frac{\pi}{2}\,\theta_3^2(q)$

(xiii) $\frac{1}{\pi} = A \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{Bn + C}{D^{4n}}$

constant depends on the CM Value

(XiV) Plug in

$q = e^{-\pi\sqrt{58}}$.

Compute

$D = 396,\ B = 26390,\ C = 1103$.

from θ evaluations →

How D, B, C are computed →

evaluate θ constants at the chosen τ

$\theta_2(q), \theta_3(q), \theta_4(q),\ q = e^{i\pi\tau}$

Compute singular modulus

$$k = \frac{\theta_2^2(q)}{\theta_3^2(q)}$$

Compute C, B from series expansion.

C = theta combination → 1103.
B = derivative or linear coeff. → 26390.

This Gives us THE final series-

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}$$

$\therefore$ $q = e^{-\pi\sqrt{58}} \approx 10^{-8}$

each term adds 8 digits of π.

why so fast?

each error term:

$$O(q^{2n})$$

Conclusion-

why π appears?

$k = \frac{\pi}{2} \theta_3^2$ (Circular constant)

Ramanujan did all this mentally (did this backward)

(CM value → Modular expansion → theta values → series.

This is not a π identity.

Its a modular identity whose normalization constant is π.

$$\boxed{\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}}$$

12/12/25

Sources -

- Cantor's paradise → general idea
- The crooked pencil - wordpress.com → $\sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4}$ meaning.
- Reddit · r/thatsdamninteresting (3y ago) → discovered eq.

```python
import math
N_TERMS = 5
C1 = 2 * math.sqrt(2)
C2 = 9801
C = C1 / C2
factorials_4n = []
factorials_n = []
powers_396 = []
linear_terms = []
summand_list = []
for n in range(N_TERMS):
f4n = math.factorial(4*n)
factorials_4n.append(f4n)
fn4 = math.factorial(n)**4
factorials_n.append(fn4)
p396 = 396**(4*n)
powers_396.append(p396)
lin = 1103 + 26390 * n
linear_terms.append(lin)
    term = (f4n * lin) / (fn4 * p396)
    summand_list.append(term)
series_sum = sum(summand_list)
pi_inv = C * series_sum
pi_approx = 1 / pi_inv
print(factorials_4n)
print(factorials_n)
print(powers_396)
print(linear_terms)
print(summand_list)
print(series_sum)
print(C)
print(pi_approx)
```