NCERT 11.9.2.3

EE23BTECH11043 - BHUVANESH SUNIL NEHETE*

Question

In an A.P. the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that 20th term is -112

Solution

$$T_1 + T_2 + T_3 + T_4 + T_5 = \frac{1}{4}[T_6 + T_7 + T_8 + T_9 + T_{10}]$$

Let the first term a and the common difference d:

$$[a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d)] =$$

$$\frac{1}{4}[(a+5d)+(a+6d)+(a+7d)+(a+8d)+(a+9d)]$$

Simplifying:

$$(5a + 10d) = \frac{1}{4}(5a + 35d)$$

$$20a + 40d = 5a + 35d$$

$$15a + 5d = 0$$

$$3a+d = 0 \implies d = -3a \implies d = -6 \quad (\text{given } a = 2)$$

$$T_{20} = a + 19d = 2 + 19(-6) = -112$$

$$a = 2 \quad and \quad d = -6$$

$$t_n = a + (n-1)d$$

$$\implies t_n = 2 + (n-1)(-6)$$

$$\implies t_n = 8 - 6n$$

The Z-transform of a sequence t_n is given by:

$$T(z) = \sum_{n=0}^{\infty} t_n z^{-n}$$

For the sequence $t_n = 8 - 6n$ when $n > 0$, we can write:

$$T(z) = \sum_{n=1}^{\infty}(8 - 6n)z^{-n}$$

Now, let's manipulate this expression. We can split it into two parts:

$$T(z) = \sum_{n=1}^{\infty} 8z^{-n} - \sum_{n=1}^{\infty} 6nz^{-n}$$

1)

$$\begin{aligned}\sum_{n=1}^{\infty} 8z^{-n} &= 8\sum_{n=1}^{\infty} z^{-n} \\ &= 8(z^{-1} + z^{-2} + z^{-3} + \dots) \\ &= \frac{8z^{-1}}{1 - z^{-1}}\end{aligned}$$

2)

$$\begin{aligned}\sum_{n=1}^{\infty} 6nz^{-n} &= 6\sum_{n=1}^{\infty} nz^{-n} \\ &= 6(1\cdot z^{-1} + 2\cdot z^{-2} + 3\cdot z^{-3} + \dots)\end{aligned}$$

This is AGP.

$$\begin{aligned} S &= 1\cdot z^{-1} + 2\cdot z^{-2} + 3\cdot z^{-3} + \dots \\ - \quad z^{-1}S &= \qquad 1\cdot z^{-2} + 2\cdot z^{-3} + 3\cdot z^{-4} + \dots \end{aligned}$$

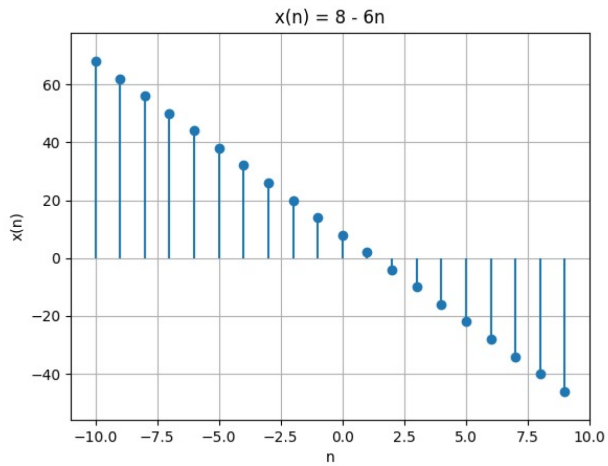
$$(1 - z^{-1}S) = z^{-1} + z^{-2} + z^{-3} + z^{-4} + \dots$$

$$(1 - z^{-1})S = \frac{z^{-1}}{1 - z^{-1}}$$

$$S = \frac{z^{-1}}{(1 - z^{-1})^2}$$

$$\implies \sum_{n=1}^{\infty} 6nz^{-n} = \frac{6z^{-1}}{(1 - z^{-1})^2}$$

$$T(z) = \frac{8z^{-1}}{1 - z^{-1}} + \frac{6z^{-1}}{(1 - z^{-1})^2}$$



$Fig.1\ \ x(n) = 8 - 6n$

Given that $n > 0$, let's analyze the ROC for the Z-transform of the sequence $x[n] = 8 - 6n$:

$$X(z) = \sum_{n=1}^{\infty}(8 - 6n) \cdot z^{-n}$$

For convergence, the series

$$\sum_{n=1}^{\infty}(8 - 6n) \cdot |z|^{-n}$$

must converge. This implies that $|z|$ should be greater than the magnitude of the terms $(8-6n)$. The terms $(8-6n)$ decrease in magnitude as n increases, so the ROC is the set of complex numbers z such that $|z|$ is greater than the magnitude of the terms for all n.

In this case, the ROC is the region outside the circle with radius equal to the magnitude of the first term $|8|$ in the series, i.e.; outside the circle $|z| > 8$.

So, the Region of Convergence (ROC) for the Z-transform of the sequence $x[n] = 8 - 6n$ with $n > 0$ is the exterior of the circle $|z| > 8$.