NCERT 11.9.2.3

EE23BTECH11043 - BHUVANESH SUNIL NEHETE*

Question

In an A.P. the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that 20^{th} term is -112

Solution

$$T_1 + T_2 + T_3 + T_4 + T_5 = \frac{1}{4}[T_6 + T_7 + T_8 + T_9 + T_{10}]$$

Let the first term a and the common difference d:

$$[a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d)] =$$

$$\frac{1}{4}[(a+5d)+(a+6d)+(a+7d)+(a+8d)+(a+9d)]$$

Simplifying:

$$(5a + 10d) = \frac{1}{4}(5a + 35d)$$

$$20a + 40d = 5a + 35d$$

$$15a + 5d = 0$$

$$3a+d = 0 \implies d = -3a \implies d = -6 \quad \text{(given } a = 2)$$

$$T_{20} = a + 19d = 2 + 19(-6) = -112$$

$$a = 2 \quad and \quad d = -6$$

$$t_n = a + (n-1)d$$

$$\implies t_n = 2 + (n-1)(-6)$$

$$\implies t_n = 8 - 6n$$

The Z-transform of a sequence t_n is given by:

$$T(z) = \sum_{n=0}^{\infty} t_n z^{-n}$$

For the sequence $t_n = 8 - 6n$ when $n > 0$, we can write:

$$T(z) = \sum_{n=1}^{\infty}(8 - 6n)z^{-n}$$

Now, let's manipulate this expression. We can split it into two parts:

$$T(z) = \sum_{n=1}^{\infty} 8z^{-n} - \sum_{n=1}^{\infty} 6nz^{-n}$$

1)

$$\begin{aligned}\sum_{n=1}^{\infty} 8z^{-n} &= 8\sum_{n=1}^{\infty} z^{-n}\\ &= 8(z^{-1} + z^{-2} + z^{-3} + \dots)\\ &= \frac{8z^{-1}}{1 - z^{-1}}\end{aligned}$$

2)

$$\begin{aligned}\sum_{n=1}^{\infty} 6nz^{-n} &= 6\sum_{n=1}^{\infty} nz^{-n}\\ &= 6(1\cdot z^{-1}+2\cdot z^{-2}+3\cdot z^{-3}+\dots)\end{aligned}$$

This is AGP.

$$\begin{aligned}S &= 1\cdot z^{-1} + 2\cdot z^{-2} + 3\cdot z^{-3} + \dots\\ z^{-1}S &= \qquad\quad 1\cdot z^{-2} + 2\cdot z^{-3} + 3\cdot z^{-4} + \dots\end{aligned}$$

$$(1 - z^{-1}S) = z^{-1} + z^{-2} + z^{-3} + z^{-4} + \dots$$

$$(1 - z^{-1})S = \frac{z^{-1}}{1 - z^{-1}}$$

$$S = \frac{z^{-1}}{(1 - z^{-1})^2}$$

$$\implies \sum_{n=1}^{\infty} 6nz^{-n} = \frac{6z^{-1}}{(1 - z^{-1})^2}$$

$$T(z) = \frac{8z^{-1}}{1 - z^{-1}} + \frac{6z^{-1}}{(1 - z^{-1})^2}$$