Jack BAI (Tester)

高顺 CQF Math Test PRE

Choices Card

DBDEE 1-5 DACCE 6-10
BBCEE 11-15 DABAC 16-20
BCABA 21-25 E 26

1. $f(x) = xe^{-x}$

$f^{(2)}_{(x)} = \dfrac{d^2 f}{dx^2} = f''(x)$

$f'(x) = 1 \cdot e^{-x} + x \cdot (-e^{-x}) = e^{-x} - xe^{-x}$

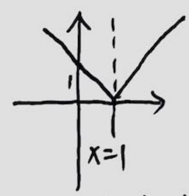
$= (1-x)e^{-x}$

$f''(x) = (1-x)'e^{-x} + (1-x)(e^{-x})' = -e^{-x} - (1-x)e^{-x}$

$= (-1-1+x)e^{-x} = (-2+x)e^{-x}$

$= -2e^{-x} + xe^{-x} = -2e^{-x} + f(x)$

$\therefore f^{(2)}_{(x)} = f(x) - 2e^{-x}$ (D)



2. $y = \ln(|1-x|) = \ln(\text{abs}(1-x))$

$y' = \dfrac{1}{\text{abs}(1-x)} \dfrac{d}{dx}(\text{abs}(1-x))$

$(|1-x|)' \to \begin{cases} 1, & x>1 \\ -1, & x\le 1 \end{cases}$

$= -\dfrac{1-x}{|1-x|} = -\dfrac{|1-x|}{1-x}$

$= \dfrac{1}{|1-x|} \times \left(-\dfrac{|1-x|}{1-x}\right)$

$= -\dfrac{1}{1-x}$

$y'' = (y')' = \left(-\dfrac{1}{1-x}\right)' = (1-x)' \cdot -\left(-\dfrac{1}{(1-x)^2}\right)$

$= -\dfrac{1}{(1-x)^2}$

$xy'' = -\dfrac{x}{(1-x)^2}$

$y' + xy'' = -\dfrac{1-x}{(1-x)^2} - \dfrac{x}{(1-x)^2} = -\dfrac{1}{(1-x)^2}$ (B)

3. $$\frac{(x+h)^3-x^3}{h} = \frac{\cancel{x^3}+3x^2h+3xh^2+h^3\cancel{-x^3}}{h} = 3x^2+3xh+h^2$$

$$\lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2 \qquad (D)$$

4. construct an l'hopital rule:

$$\cos(0)=1$$

$$\lim_{x\to 0}\frac{2x+\sin x}{x(x-1)} = \lim_{x\to 0}\frac{2+\frac{\sin x}{x}}{x-1} = \frac{\lim_{x\to 0} 2+\frac{\sin x}{x}}{\lim_{x\to 0} x-1} = \frac{2+\frac{\cos 0}{1}}{-1}$$

$$= -3 \quad (E)$$

5. $$\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} \Rightarrow \int_0^\infty xe^{-x^2}dx = \left|-\frac{1}{2}e^{-x^2}\right|_0^\infty = 0-\left(-\frac{1}{2}e^0\right) = \frac{1}{2}\ (E)$$

validation: $-\frac{1}{2}\left(e^{-x^2}\right)' = -\frac{1}{2}(-2x)\cdot e^{-x^2} = xe^{-x^2}$ (correct)

$$\therefore \int_0^\infty xe^{-x^2}dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^\infty = 0-\left(-\frac{1}{2}\right) = \frac{1}{2}$$

6. $$\int\frac{1}{x^2+3x+2}dx = \int\frac{1}{(x+\frac{3}{2})^2-\frac{1}{4}}dx \overset{u\text{-sub}}{=} ?$$

$$x^2+3x+2 = \left(x+\frac{3}{2}\right)^2-\frac{1}{4}$$

$$\int_0^1\frac{1}{(x+\frac{3}{2})^2-\frac{1}{4}}dx \overset{u=x+\frac{3}{2}}{=} \int_{\frac{3}{2}}^{\frac{5}{2}}\frac{1}{u^2-\frac{1}{4}}du = \int_{\frac{3}{2}}^{\frac{5}{2}}\frac{4}{4u^2-1}du$$

$$= \int_{\frac{3}{2}}^{\frac{5}{2}}\frac{4}{(2u+1)(2u-1)}du = 4\left(-\frac{1}{4}\right)\int_{\frac{3}{2}}^{\frac{5}{2}}\left(\frac{1}{2u+1}-\frac{1}{2u-1}\right)du$$

$$= -\left[\ln(2u+1)-\ln(2u-1)\right]_{\frac{3}{2}}^{\frac{5}{2}}$$

$$\ln(x^{-1}) = -\ln x$$

$$= -(\ln 6-\ln 4)-\left[-(\ln 4-\ln 2)\right]$$

$$= -\ln 6+\ln 4+\ln 4-\ln 2 = \ln\frac{4\times 4}{6\times 2} = \ln\frac{16}{12} = \ln\frac{4}{3} = -\ln\frac{3}{4}$$

$\therefore$ D.

7. $\text{taylor}(e^x, 3) = \underbrace{e^3}_{1} + \underbrace{e^3(x-3)}_{2} + \underbrace{\frac{e^3}{2}(x-3)^2}_{3} + \underbrace{\frac{e^3}{6}(x-3)^3}_{4} + \cdots$

$\text{taylor}(e^x, -3) = e^{-3} + e^{-3}(x+3) + \frac{1}{2}e^{-3}(x+3)^2 + \frac{1}{6}e^{-3}(x+3)^3 + \cdots$

$\text{taylor}^{(n)}(e^x, -3) = \frac{e^{-3}}{(n-1)!}(x+3)^{n-1}$

it seems that the question treats 0^{th} item as first item, so

$\text{taylor}^{(n)}(e^x, -3) = \frac{e^{-3}}{n!}(x+3)^n$ (A)

8. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n$

$\int \left(\frac{1}{1+x}\right)dx = \ln(1+x)$

$\cancel{= 0 - 1 + 2x - 3x^2 + \cdots = \sum_{n=0}^{\infty}(-1)^n n x^{n-1}}$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}x^{n+1}$ (C)

9. $\int_0^1 k(1-x^3)dx = 1$

$\int_0^1 (1-x^3)dx = \frac{1}{k}$

$\left[x - \frac{1}{4}x^4\right]_0^1 = \left[1 - \frac{1}{4}\right] - [0-0] = \frac{3}{4} = \frac{1}{k}$

$k = \frac{4}{3} \Rightarrow$ (C)

10. $\int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{3}{4}\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$

$= \frac{3}{4}\left(\frac{2}{3}+\frac{2}{3}\right) = \frac{3}{4} \times \frac{4}{3} = 1$ (success)

$\int_0^{\frac{1}{2}}(\cdot) = \frac{3}{4}\left[x - \frac{1}{3}x^3\right]_0^{\frac{1}{2}} = \frac{3}{4}\left[\left(\frac{1}{2} - \frac{1}{3}\times\frac{1}{8}\right) - (0-0)\right] = \frac{3}{4}\left(\frac{1}{2} - \frac{1}{24}\right) = \frac{\cancel{3}}{4}\left(\frac{11}{\cancel{24}_8}\right) = \frac{11}{32}$

$\therefore \int_0^{\frac{1}{2}} \frac{3}{4}(1-x^2)\,dx = \frac{11}{32}$ (E)

11. $V(X) = E(X^2) - E(X)^2$

$V(X) = \sigma^2$

$E(X) = \mu$

$\Rightarrow E(X^2) = \sigma^2 + \mu^2 = 4^2 + 2^2 = 20$ (B)

12. Uniform distribution has NO skew (B)

13. $\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = d(x)$ (pdf)

$\text{std} = \sigma$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(X-\mu)^2}{2\sigma^2}\right) = \frac{\nu}{\sqrt{2\pi\sigma_0}} \exp\left(-\frac{\nu^2(X-\mu_0)^2}{2\sigma_0}\right)$$

obviously, $\mu = \mu_0$

we obtain $\begin{cases} \frac{1}{\sigma} = \frac{\nu}{\sqrt{\sigma_0}} \\ \frac{1}{\sigma^2} = \frac{\nu^2}{\sigma_0} \end{cases}$ (same) $\Rightarrow \sigma = \frac{\sqrt{\sigma_0}}{\nu} = \text{std}$ (C)

14. $f_x(x,y) = 2x - 4y + \cancel{y^3}^{\,0} + \cancel{4y}^{\,0}$ $\quad f_y(x,y) = 0 - 4x + 3y^2 + 4 = 3y^2 - 4x + 4$

$f_{xx}(x,y) = 2$ $\quad f_{yy}(x,y) = 6y$

$\therefore f_{xx} + f_{yy} = 6y + 2$ (E)

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$X, V''(x)$

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15. separable ODE.

$\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \Rightarrow \frac{1}{1+y^2}\,dy = \frac{1}{1+x^2}\,dx.$ $\quad \int \frac{1}{1+y^2}\,dy = \int \frac{1}{1+x^2}\,dx$

$\arctan(y) + C_1 = \arctan(x) + C_2$

① $y = \tan[\arctan(x) + (C_2 - C_1)] = x + C$ (wrong) ⊗

arctan does not have comb law!

② $\arctan(y) - \arctan(x) = C \Rightarrow \tan(\arctan(y) - \arctan(x)) = C$

$\because \tan(A+B) = (\tan(A) + \tan(B)) / (1 - \tan A \tan B)$

$$\Rightarrow \frac{\tan(\arctan(y)) + \tan(\arctan(x))}{1 - \tan(\arctan(y))\tan(\arctan(x))} = \frac{y \mp x}{1 \pm xy} = C$$

$C(1 \pm xy) = y \mp x$

$C \pm Cxy = y \mp x$

$y(\cancel{Cx+1}\ 1 - Cx) \pm x = C$

$y = \frac{C \mp x}{\cancel{Cx+1}\ 1 \mp Cx}$

$\Rightarrow y = \frac{\cancel{C-x}\quad x-C}{\cancel{Cx+1}\quad Cx-1} = \frac{x+C}{1-Cx}$

(E)

16. separable ODE

$\frac{dy}{dx} - xy = x$, need to separate x & y: $\frac{dy}{dx} = x + xy = x(y+1)$

$\Rightarrow \frac{1}{y+1}\,dy = x\,dx \Rightarrow \int \frac{1}{y+1}\,dy = \int x\,dx \Rightarrow \ln(y+1) = \frac{1}{2}x^2 + C$

$y+1 = e^{\frac{1}{2}x^2 + C}$ ~~Ξ~~ $\Rightarrow y = e^{\frac{1}{2}x^2 + C} - 1 = Ae^{\frac{1}{2}x^2} - 1$ (D)

17. linear ODE (2-order) → homogeneous

solve the 2-order function: $m^2 - 4m + 13m = 0$

$\Delta = b^2 - 4ac = 16 - 4 \times 13 = 16 - 52 = -36$

$\frac{-b \pm \sqrt{\Delta}}{2a} = \frac{4 \pm 6i}{2} = 2 \pm 3i \Rightarrow \begin{cases} m_1 = 2+3i \\ m_2 = 2-3i \end{cases}$

$m_1 = 2+3i \Rightarrow y_1 = e^{2x} \cdot \cos(3x) \cdot C_1$

$m_2 = 2-3i \Rightarrow y_2 = e^{2x} \cdot \sin(3x) \cdot C_2$

$y = y_1 + y_2 = e^{2x}[A\cos(3x) + B\sin(3x)]$ (A)

18. ~~non~~-linear ODE (2-order) → non-homogeneous

assume we've got a solution $y_1(x) = u(x)v(x) \Rightarrow y_1'(x) = u'(x)v(x) + u(x)v'(x)$

$\Rightarrow x^2y'' - 4xy' + 6y = 0$

$y_1''(x) = u''(x)v(x) + u'(x)v'(x) + u'(x)v'(x) + u(x)v''(x)$
$= u''(x)v(x) + 2u'(x)v'(x) + u(x)v''(x)$

$\Downarrow$

$x^2(u''v + 2u'v' + uv'') - 4x(u'v + uv') + 6uv = 0$

divide by $x^2 \cdot u(x)v(x) \Rightarrow y = \frac{A}{x} + \frac{B}{x^3}$

homogeneous ODE! $\Rightarrow 2m^2 - 4m + 6 = 0 \Rightarrow m^2 - 2m + 3 = 0$

-3

$\begin{cases} m_1 = 3 \\ m_2 = -1 \end{cases}$ substitute u and v with ar^n we get $y = C_1x^2 + C_2x^3$

$\Rightarrow$ (B)

19. $\begin{vmatrix} k & k \\ 8 & 4k \end{vmatrix} = 0 \Rightarrow k(4k) - 8k = 0 \Rightarrow 4k^2 - 8k = 0 \Rightarrow k^2 - 2k = 0$

$k_1 = 0, k_2 = 2 \Rightarrow k = 0 \,\&\, 2$ (A)

20. $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 0 & -2 & -5 \\ 1 & -2 & 3 & 6 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 0 & -2 & -5 \\ 0 & -2 & 5 & 11 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 3 & 11 \\ 1 & 0 & -2 & -5 \\ 0 & -2 & 5 & 11 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 11 & 33 \end{array}\right] \Rightarrow \begin{cases} x = 1 \\ y = 2 \\ z = 3 \end{cases}$ (C)

21. $\begin{vmatrix} 2 & 0 & 1 \\ 3 & 2 & -3 \\ -1 & -3 & 5 \end{vmatrix} = +2 \begin{vmatrix} 2 & -3 \\ -3 & 5 \end{vmatrix} - 0 + 1 \begin{vmatrix} 3 & 2 \\ -1 & -3 \end{vmatrix}$

$= 2\,(10-9) + (-9+2) = 2 \times 1 + (-7)$

$= -5$ (B)

22. $\vec{u} \cdot \vec{v} = 0 \Rightarrow -6 + 0 + 2k = 0 \Rightarrow k = 3$ (C)

23. $|X| = |2-3i| = |(2,-3)| = \sqrt{2^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$ (A)

24. ~~if~~ $f(x,y) = x^2 + y^2 \Rightarrow f(\theta) = \sin^2(2\theta) + \cos^2(2\theta) = 1$

$\Rightarrow f'(\theta) = (1)' = 0$ (B)

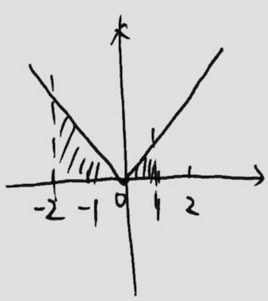
25. $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, $|\lambda I - A| \overset{\#}{=} 0 \Rightarrow \begin{vmatrix} \lambda - 2 & 2 \\ 1 & \lambda - 3 \end{vmatrix} = 0$

$(\lambda - 2)(\lambda - 3) - 2 = 0 \Rightarrow \lambda^2 - 5\lambda + 6 - 2 = 0 \Rightarrow \lambda^2 - 5\lambda + 4 = 0$

$\lambda_1 = 4, \lambda_2 = 1 \Rightarrow$ largest is 4 (A)

$\begin{matrix} 1 & -4 \\ 1 & -1 \end{matrix}$

26.



$\frac{2^2}{2} + \frac{1^2}{2} = 2 + \frac{1}{2} = \frac{5}{2}$ (E)