Grupa 341, Seminar 3, EDDP, 20.10.2020

Se cere integrarea ecuațiilor:

ec. omogene

1) $\dfrac{dx}{dt} = \dfrac{x+t}{x-t}$, $t \in (0,2)$, $x \in (2,\infty)$

2) $\dfrac{dx}{dt} = -\dfrac{3t^2x + x^3}{2t^3}$, $t>0$, $x \in \mathbb{R}$

3) $\dfrac{dx}{dt} = \dfrac{2xt}{x^2-t^2}$, $x \in (2,\infty)$, $t \in (0,2)$

4) $\dfrac{dx}{dt} = \dfrac{x}{t} + e^{\frac{x}{t}}$, $t>0$, $x \in \mathbb{R}$.

ec. afine

5) $x' = x(\operatorname{tg} t) + \sin t$, $t \in (0, \frac{\pi}{2})$, $x \in \mathbb{R}$

6) $x' = \dfrac{xe^{2t} + t}{e^{2t}}$, $t \in \mathbb{R}$, $x \in \mathbb{R}$

7) $x' + \dfrac{2x}{t^2-1} = 2t+2$, $t \in (1,+\infty)$, $x \in \mathbb{R}$

8) $\dfrac{dx}{dt} = \dfrac{2t-x+2}{4t-2x+1}$, $(t,x) \in D_1 \subset \{(t,x) \mid 4t-2x+1>0\}$

9) $\dfrac{dx}{dt} = \dfrac{3t+2x-7}{2t-x}$, $(t,x) \in D_1 \subset \{(t,x) \mid 2t-x<0\}$

10) $\dfrac{dx}{dt} = \dfrac{3t+3x-1}{t+x+1}$, $(t,x) \in D_1 \subset \{(t,x) \mid x+t+1>0\}$.

(2) $\dfrac{dx}{dt} = \underbrace{-\dfrac{3t^2x + x^3}{2t^3}}_{f(t,x)}$, $t>0$, $x \in \mathbb{R}$.

Avem $f(\alpha t, \alpha x) = -\dfrac{3\alpha^2 t^2 \cdot \alpha x + (\alpha x)^3}{2(\alpha t)^3} =$

$= -\dfrac{\cancel{\alpha^3}(3t^2x + x^3)}{2\cancel{\alpha^3}t^3} = f(t,x) \Rightarrow$ ec. este omogenă.

Se face s.v.: $(t,x) \xrightarrow{x=ty} (t,y)$.

Se obține:

$$(ty)' = -\frac{3t^2 ty + (ty)^3}{2t^3}$$

$$\underbrace{t'}_{=1} y + ty' = -\frac{\cancel{t^3}(3y+y^3)}{2\cancel{t^3}}$$

$$\boxed{(ty)' = \frac{d}{dt}(ty) = 1\cdot y + t\cdot y'}$$

$$y + ty' = \frac{-3y-y^3}{2}$$

$$y' = \frac{1}{t}\left(\frac{-3y-y^3-2y}{2}\right)$$

$$y' = \frac{1}{t}\cdot\left(\frac{-y^3-5y}{2}\right)$$ ec. cu var sep.

$a(t) = \frac{1}{t}$

$b(y) = \frac{-y^3-5y}{2}$.

- $b(y) = 0 \Rightarrow -y^3-5y = 0$.

$-y(y^2+5) = 0 \Rightarrow \boxed{y_1 = 0}$

$y^2+5=0$ nu sol reale.

$\Rightarrow y(t) = 0$ sol. staționară pt ec. în y

$\Rightarrow x(t) = t\cdot 0 = 0$.

- $b(y) \neq 0 \Rightarrow y \in \mathbb{R}\setminus\{0\}$. $\Rightarrow$ separăm variabilele

$$\frac{-2dy}{y(y^2+5)} = \frac{dt}{t}.$$

$$\int \frac{dt}{t} = \ln|t| + c \underset{t>0}{\Rightarrow} A(t) = \ln t.$$

$$\int \frac{-2dy}{y(y^2+5)} = I$$

$$\frac{-2}{y(y^2+5)} = \frac{\overset{y^2+5}{A}}{y} + \frac{\overset{y}{By}+C}{y^2+5}, \quad A, B, C \in \mathbb{R}$$

$$-2 = A(y^2+5) + By^2 + Cy \Rightarrow$$

$$\Rightarrow -2 = (A+B)y^2 + Cy + 5A \xrightarrow{\text{identif coef.}}$$

$$\begin{cases} A+B=0 \\ \boxed{C=0} \\ 5A=-2 \end{cases} \Rightarrow \boxed{A=-\frac{2}{5}} \Rightarrow \boxed{B=\frac{2}{5}}$$

$$J = -\frac{2}{5}\int \frac{1}{y}\,dy + \frac{2}{5}\int \frac{y}{y^2+5}\,dy =$$

$$= -\frac{2}{5}\ln|y| + \cancel{\frac{2}{5}\cdot\frac{1}{2}}\ln(y^2+5) + C$$

$$\Rightarrow \boxed{B(y) = -\frac{2}{5}\ln|y| + \frac{1}{5}\ln(y^2+5)}$$

$$B(y) = \frac{1}{5}\ln\left(\frac{y^2+5}{y^2}\right)$$

Sol. implicite:

$$\frac{1}{5}\ln\left(\frac{y^2+5}{y^2}\right) = \ln t + \ln C, \quad C>0$$

$$\ln\left(\frac{y^2+5}{y^2}\right) = \ln\left((tC)^5\right) \Rightarrow$$

$$\Rightarrow \frac{y^2+5}{y^2} = (tC)^5 \Rightarrow y^2 = \frac{-5}{1-(tC)^5} \Rightarrow$$

$$\Rightarrow \boxed{\left(\frac{x}{t}\right)^2 = \frac{-5}{1-t^5C^5}}, \quad C>0$$

(7) $x' = \underbrace{-\frac{2}{t^2-1}}_{a(t)}x + \underbrace{2t+2}_{b(t)}$. ec. afină (liniară neomogenă), $t\in(1,\infty)$, $x\in\mathbb{R}$.

• req. ec. liniară omogenă atașată:

$$\bar{x}' = \frac{-2}{t^2-1}\bar{x}. \Rightarrow \bar{x} = C e^{A(t)}$$

$$\int a(t)\,dt = \int \frac{-2}{t^2-1}\,dt = -2\cdot\int\frac{1}{t^2-1}\,dt =$$

$$= -\cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\ln\left|\frac{t-1}{t+1}\right| + C = -\ln\left(\frac{t-1}{t+1}\right) + C$$

$$\Rightarrow A(t) = -\ln\left(\frac{t-1}{t+1}\right) = \ln\left(\frac{t+1}{t-1}\right) \Rightarrow$$

$$\Rightarrow \bar{x}(t) = C e^{\ln\left(\frac{t+1}{t-1}\right)} = C \cdot \frac{t+1}{t-1}$$

• aplicăm metoda var. constantelor $\Rightarrow$

$\Rightarrow$ determinăm $C : (1, +\infty) \to \mathbb{R}$ a.î.

$x(t) = C(t) \cdot \frac{t+1}{t-1}$ să verifice ec. data afină $\Rightarrow$

$$\Rightarrow \left(C(t)\frac{t+1}{t-1}\right)' = \frac{-2}{(t-1)\cancel{(t+1)}} \cdot C(t)\frac{\cancel{t+1}}{t-1} + 2t+2 .$$

$$\Rightarrow C'(t) \cdot \frac{t+1}{t-1} + C(t) \cdot \frac{1\cdot(t-1) - 1\cdot(t+1)}{(t-1)^2} = \frac{-2C(t)}{(t-1)^2} + 2t+2$$

$$C'(t)\frac{t+1}{t-1} + \cancel{C(t) \cdot \frac{-2}{(t-1)^2}} = \cancel{\frac{-C(t)}{(t-1)^2}} + 2t+2$$

$$C'(t) = 2\cancel{(t+1)} \cdot \frac{t-1}{\cancel{t+1}} \; ; \; C'(t) = 2(t-1) \Rightarrow$$

$$\Rightarrow C(t) = \int (2t-2)\,dt = \cancel{2}\frac{t^2}{\cancel{2}} - 2t + K \Rightarrow$$

$$\Rightarrow x(t) = (t^2 - 2t + K)\frac{t+1}{t-1} \Rightarrow, K \in \mathbb{R}$$

$$\Rightarrow x(t) = \underbrace{\frac{(t^2-2t)(t+1)}{t-1}}_{\text{sol. particulară}} + \underbrace{K\frac{t+1}{t-1}}_{\text{sol. ec. omogene}} ,$$

(8) $\frac{dx}{dt} = \frac{2t - x + 2}{4t - 2x + 1}$

ec. de forma: $\frac{dx}{dt} = g\left(\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2}\right)$

unde $a_1 = 2$; $b_1 = -1$; $c_1 = 2$

$\underline{a_2 = 4 \; ; \; b_2 = -2}$; $c_2 = 1$

Calculăm $d = a_1 b_2 - a_2 b_1 = 2\cdot(-2) - 4\cdot(-1) =$

$= -4 + 4 = 0. \Rightarrow$

$\Rightarrow$ cum $b_1 \neq 0$, se face s.v.: $2t - x = y \Rightarrow$

$$\Rightarrow x = 2t - y$$

$$(t, x) \xrightarrow{x = 2t - y} (t, y)$$

Ec devine:

$$(2t-y)' = \frac{2t-(2t-y)+2}{4t-2(2t-y)+1}$$

$$\Rightarrow 2 - y' = \frac{\cancel{2t}-\cancel{2t}+y+2}{\cancel{4t}-\cancel{4t}+2y+1}$$

$$\overset{2y+1}{2} - \frac{y+2}{2y+1} = y' \Rightarrow y' = \frac{4y+\cancel{2}-y-\cancel{2}}{2y+1}$$

$$\frac{dy}{dt} = \underbrace{\frac{3y}{2y+1}}_{b(y)} \cdot \underbrace{1}_{a(t)} \qquad (*)$$

- $b(y) = 0 \Rightarrow 3y = 0 \Rightarrow y = 0 \Rightarrow \boxed{x(t) = 2t}$ o soluție particulară

soluție staționară.

- $y \neq 0 \Rightarrow$ separăm variabilele:

$$\frac{2y+1}{3y}\,dy = 1\,dt$$

$$\int \frac{2y+1}{3y}\,dy = \int \frac{2y}{3y}\,dy + \int \frac{1}{3y}\,dy =$$

$$= \frac{2}{3}y + \frac{1}{3}\ln|y| + C \Rightarrow$$

$$\Rightarrow B(y) = \frac{2}{3}y + \frac{1}{3}\ln|y|$$

$$\int 1\,dt = t + C \Rightarrow A(t) = t$$

Mult soluțiilor implicite pt ec. (*) este

$$\boxed{\frac{2}{3}y + \frac{1}{3}\ln|y| = t + C \quad C \in \mathbb{R}}$$

$y = 2t - x \Rightarrow$ Mult. sol. implicite pt ec în x:

$$\boxed{\frac{2}{3}\cdot(2t-x) + \frac{1}{3}\ln|2t-x| = t + C \quad C \in \mathbb{R}}$$

(9) $\dfrac{dx}{dt} = \dfrac{3t+2x-7}{2t-x}$, $(t,x) \in D_1 \subset \{(t,x) \mid 2t - x < 0\}$.

$a_1 = 3$; $b_1 = 2$; $c_1 = -7$

$a_2 = 2$; $b_2 = -1$; $c_2 = 0$

$d = 3(-1) - 2\cdot 2 = -3-4 = -7 \neq 0$.

- rezolvăm sistemul: $\begin{cases} 3t+2x=7 \\ 2t-x=0 \end{cases} \Big| \cdot 2 \Rightarrow \begin{cases} 3t+2x=7 \\ 4t-2x=0 \end{cases}$ (+)

$\Rightarrow 7t = 7 \Rightarrow \boxed{t_0 = 1} \Rightarrow \boxed{x_0 = 2}$

Se face schimbarea de variabilă:

$$(t,x) \longrightarrow (s,y)$$

$$\begin{cases} t = s+1 \\ x = y+2 \end{cases} \Leftrightarrow \begin{cases} s = t-1 \\ y = x-2 \end{cases}$$

$t(s) = s+1$

$s(t) = t-1 \Rightarrow s'(t) = 1$

$x(t) = y(s(t)) + 2$ sau $y(s) = x(t(s)) - 2$

$$\frac{dx}{dt} = \frac{dy}{ds}\cdot s'(t) = \frac{dy}{ds}\cdot 1$$

Ec devine:

$$\frac{dy}{ds} = \frac{3(s+1)+2(y+2)-7}{2(s+1)-(y+2)} \Rightarrow$$

$\Rightarrow \dfrac{dy}{ds} = \dfrac{3s+2y}{2s-y}$ ec. omogenă (se rezolvă la fel ca 1)

$$\frac{dy}{ds} = \frac{s\left(3+2\frac{y}{s}\right)}{s\left(2-\frac{y}{s}\right)} \Rightarrow \text{ec. omogenă } h\left(\frac{y}{s}\right)$$

Avem:

$$(t,x) \xrightarrow[\substack{t=s+1\\ x=y+2}]{} (s,y) \xrightarrow[y=sz]{} (s,z)$$

ec. omogena.

$$(sz)' = \frac{3+2\cdot z}{2-z}$$

$$\underset{\substack{\shortparallel\\ 1}}{s'}z + sz' = \frac{3+2z}{2-z} \Rightarrow z' = \underbrace{\frac{1}{s}}_{a(s)}\underbrace{\left(\frac{3+2z}{2-z} - \frac{2-z}{-z}\right)}_{b(z)}$$

(derivata în raport cu s)

$$\frac{dz}{ds} = \frac{1}{s}\cdot\frac{3+\cancel{2z}-\cancel{2z}+z^2}{2-z} \Rightarrow$$

$$\Rightarrow \frac{dz}{ds} = \frac{1}{s}\cdot\underbrace{\frac{z^2+3}{2-z}}_{b(z)}$$

- $b(z) = \frac{z^2+3}{2-z} = 0 \Rightarrow$ nu are sol. $\Rightarrow$ ec. în (s,z) nu are sol. staționare

- $b(z) \neq 0 \Rightarrow$ separăm variabilele:

$$\left(\frac{2-z}{z^2+3}\right)dz = \frac{1}{s}ds.$$

$$\int \frac{2-z}{z^2+3}dz = 2\int \frac{1}{z^2+(\sqrt{3})^2}dz - \int \frac{z}{z^2+3}dz =$$

$$= \underbrace{2\cdot\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{z}{\sqrt{3}} - \frac{1}{2}\ln(z^2+3)}_{B(z)} + C$$

$$\int \frac{1}{s}ds = \underbrace{\ln|s|}_{A(s)} + C$$

- Mulțimea sol. implicite ec. în (s,z) este:

$$\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{z}{\sqrt{3}}\right) - \frac{1}{2}\ln(z^2+3) = \ln|s| + C$$

dar $z = \frac{y}{s} \Rightarrow$ mult. sol. implicite pt ec. în (s,y) este:

$$\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{y}{s\sqrt{3}}\right)-\frac{1}{2}\ln\left(\frac{y^2}{s^2}+3\right)=\ln|s|+C \Rightarrow$$

$\Rightarrow$ mulț. sol. implicite pt ec în (t,x) este:

$$\boxed{\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{x-2}{(t-1)\sqrt{3}}\right)-\frac{1}{2}\ln\left(\frac{(x-2)^2}{(t-1)^2}+3\right)=\ln|t-1|+C.}$$

$C\in\mathbb{R}$

<u>Temă</u>: 1, 3, 4, 5, 6, 10.

(I). Fie ecuația $x'+p(t)x=q(t)$. $(*)$

a) Aflați p și q, dacă ecuația are soluțiile $\varphi_1(t)=t$ și $\varphi_2(t)=t\sin t$.

b) Aflați mulț. sol. ec $(*)$.

c) Determinați soluția particulară care verifică $x(\pi)=2\pi$.

(II) Fie ecuația:

$$x'-\frac{3t^2}{t^5-1}-\frac{t^4}{t^5-1}x+\frac{2t}{t^5-1}x^2=0. \quad (**)$$

a) Determinați $n, m\in\mathbb{R}$ a.î. ec $(**)$ să aibă sol. de forma $\varphi(t)=nt^m$.

b) Să se determine mulț. soluțiilor ec. $(**)$