Grupa 341, Seminar 12, EDDP, 05.01.2021

① Se cere forma generală a soluției pt. ec. cvasiliniare următoare:

a) $x_1^2 \partial_1 u + x_2^2 \partial_2 u = 2x_1 x_2$

$\boxed{n=2}$

forma generală a soluției este $f(\varphi_1(x_1,x_2,u), \varphi_2(x_1,x_2,u))=0$ unde φ_1, φ_2 sunt integrale prime independente ale sistemului caracteristic, iar f este o funcție arbitrară care are derivate parțiale de ordinul întâi.
$(f: D \subset \mathbb{R}^2 \to \mathbb{R})$

Sist. caracteristic:

$$\underbrace{\frac{dx_1}{x_1^2} = \frac{dx_2}{x_2^2}} = \frac{du}{2x_1x_2}$$

$$x_1^{-2}dx_1 = x_2^{-2}dx_2 \Rightarrow \int x_1^{-2}dx_1 = \int x_2^{-2}dx_2 \Rightarrow$$

$$\Rightarrow \frac{x_1^{-1}}{-1} = \frac{x_2^{-1}}{-1} + C_1 \Rightarrow \underbrace{\frac{1}{x_2} - \frac{1}{x_1}}_{\varphi_1(x_1,x_2,u)} = C_1 \Rightarrow$$

$$\Rightarrow \boxed{\varphi_1(x_1,x_2,u) = \frac{1}{x_2} - \frac{1}{x_1}}$$

Din sistemul caract $\Rightarrow$ $\dfrac{x_2 dx_1}{x_1^2 x_2} = \dfrac{x_1 dx_2}{x_2^2 x_1} = \dfrac{du}{2x_1x_2} =$

$$= \frac{x_2dx_1 + x_1dx_2}{x_1^2x_2 + x_2^2x_1} =$$

$$= \frac{d(x_1x_2)}{x_1x_2(x_1+x_2)} \Rightarrow$$

$$\Rightarrow \frac{d(x_1x_2)}{\cancel{x_1x_2}(x_1+x_2)} = \frac{du}{2\cancel{(x_1x_2)}} \Rightarrow \frac{d(x_1x_2)}{x_1+x_2} = \frac{du}{2} \Rightarrow$$

$$\Rightarrow \int \frac{d(x_1x_2)}{x_1+x_2} = \int \frac{du}{2} \Leftrightarrow \int \frac{d(x_1x_2)}{x_1+x_2} = \frac{u}{2} + C_2$$

Notăm $x_1x_2 = y$;

Sist. caract: $\dfrac{dx_1}{x_1^2} = \dfrac{dx_2}{x_2^2} = \dfrac{du}{2x_1x_2} = \dfrac{d(x_1+x_2+u)}{(x_1+x_2)^2}$

A doua integrală primă este:

$$\boxed{\varphi_2(x_1,x_2,u) = \int \frac{d(x_1x_2)}{x_1+x_2} - \frac{u}{2}}$$

a1) $x_1^2\,\partial_1 u + x_2^2\,\partial_2 u = 2(x_1+x_2)$

$$\frac{{}^{x_2/}dx_1}{x_1^2} = \frac{{}^{x_1/}dx_2}{x_2^2} = \frac{du}{2(x_1+x_2)}$$

$\Downarrow$

$$\varphi_1(x_1,x_2,u) = \frac{1}{x_2} - \frac{1}{x_1}$$

$$\frac{x_2\,dx_1}{x_1^2x_2} = \frac{x_1\,dx_2}{x_1x_2^2} = \frac{du}{2(x_1+x_2)} = \frac{d(x_1x_2)}{x_1x_2(x_1+x_2)} \Rightarrow$$

$$\Rightarrow \frac{du}{2} = \frac{d(x_1x_2)}{x_1x_2} \Rightarrow$$

$$\Rightarrow \int\frac{du}{2} = \int\frac{d(x_1x_2)}{x_1x_2} \Rightarrow$$

$$\Rightarrow \frac{u}{2} = \ln|x_1x_2| + C_2,\ C_2>0 \Rightarrow$$

$$\Rightarrow u = 2\ln|x_1x_2| + 2C_2 \Rightarrow$$

$$\Rightarrow \underbrace{u - \ln(x_1x_2)^2}_{\varphi_2(x_1,x_2,u)} = 2C_2$$

Forma generală a soluției:

$$\boxed{f\left(\frac{1}{x_2} - \frac{1}{x_1},\ u - \ln(x_1x_2)^2\right) = 0}$$

Exemple de soluții se pot da astfel:

luăm, de exemplu, $f(y_1,y_2) = y_1^2 + y_2 \Rightarrow$

$$\Rightarrow \left(\frac{1}{x_2} - \frac{1}{x_1}\right)^2 + u - \ln(x_1x_2)^2 = 0 \Rightarrow \underline{u(x_1,x_2) = \ln(x_1x_2)^2 - \left(\frac{1}{x_2} - \frac{1}{x_1}\right)^2}$$

(2) Să se determine soluția prob. Cauchy

(a) $\begin{cases} 2x_2\partial_1 u + (x_1+x_2)\partial_2 u = x_1^2 \\ u(x_1,x_2) = \dfrac{7x_1x_2}{4} \quad \text{pt } x \in S = \{x\in\mathbb{R}^2 \mid x_1 = 4x_2\} \end{cases}$

b) $\begin{cases} x_2\partial_1 u + (2x_1 - x_2)\partial_2 u = 4x_1(x_1+x_2) \\ u(x_1,x_2) = -\dfrac{1}{2}x_1x_2 \quad \text{pe } S = \{x\in\mathbb{R}^2 \mid x_1+2x_2=0,\ x_1>0\} \end{cases}$

(c) $\begin{cases} x_2\partial_1 u + x_1\partial_2 u = 2u \\ u(x_1,x_2) = \dfrac{x_1^2}{2} \quad \text{pe } S = \{x\in\mathbb{R}^2 \mid x_2 = 0\} \end{cases}$

(c) pt $x_2\partial_1 u + x_1\partial_2 u = 2u$

$$\underbrace{\frac{dx_1}{x_2} = \frac{dx_2}{x_1}}_{} = \frac{du}{2u}$$

$$\Downarrow$$

$$\left.\begin{aligned} x_1dx_1 &= x_2dx_2 \\ \int x_1dx_1 &= \int x_2dx_2 \end{aligned}\right\} \Rightarrow \frac{x_1^2}{2} = \frac{x_2^2}{2} + \frac{C_1}{2} \Rightarrow$$

$$\Rightarrow x_1^2 - x_2^2 = C_1 \Rightarrow$$

$$\Rightarrow \boxed{\varphi_1(x_1,x_2,u) = x_1^2 - x_2^2}$$

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_1} = \underbrace{\frac{du}{2u} = \frac{d(x_1+x_2)}{x_1+x_2}}_{} \Rightarrow \int\frac{du}{2u} = \int\frac{d(x_1+x_2)}{x_1+x_2}$$

$$\frac{1}{2}\ln|u| = \ln|x_1+x_2| + \ln C_2, \quad C_2>0$$

$$\ln|u| - \ln(x_1+x_2)^2 = \ln C_2^2$$

$$\ln\left(\frac{|u|}{(x_1+x_2)^2}\right) = \ln C_2^2 \Rightarrow \frac{u}{(x_1+x_2)^2} = \pm C_2^2$$

$$\Rightarrow \underbrace{\varphi_2(x_1,x_2,u) = \frac{u}{(x_1+x_2)^2}}_{}$$

Acum forma (implicită) generală a soluției : $f\left(x_1^2 - x_2^2, \dfrac{u}{(x_1+x_2)^2}\right) = 0$

Determinăm f ai să fie verificată cond:

$u(x_1, x_2) = \frac{x_1^2}{2}$ pt $x_2 = 0$.

Se obține:

$$f\left(x_1^2 - 0, \frac{\cancel{\frac{x_1^2}{2}}}{\cancel{x_1^2}}\right) = 0 \Rightarrow$$

$$\Rightarrow \underline{f\left(x_1^2, \frac{1}{2}\right) = 0}. \Rightarrow$$ este mai bine pt. prob. Cauchy să aplicăm alg. de rezolvare al prob. Cauchy folosind sist. caract cu cond inițiale.

• o param. pt S. $\left(\begin{array}{l} x_2 = 0 \\ h(x) = x_2 \end{array}\right)$ $\begin{cases} x_1 = s = \alpha_1(s) \\ x_2 = 0 = \alpha_2(s) \end{cases}$

• $\varphi(s) = u_0(\alpha_1(s), \alpha_2(s)) = u_0(s, 0) = \frac{s^2}{2}$.

$u_0(x) = \frac{x_1^2}{2}$

•

$$\begin{cases} \text{rang}\begin{pmatrix} \alpha_1'(s) \\ \alpha_2(s) \end{pmatrix} = \text{rang}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 = 2-1 \\ \begin{vmatrix} a_1(\alpha_1(s), \alpha_2(s), \varphi(s)) & \alpha_1'(s) \\ a_2(\alpha_1(s), \alpha_2(s), \varphi(s)) & \alpha_2(s) \end{vmatrix} = \begin{vmatrix} \alpha_2(s) & 1 \\ \alpha_1(s) & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0 \\ a_1(x, u) = x_2 \\ a_2(x, u) = x_1 \quad ; g(x, u) = 2u \end{cases}$$

• sist caracteristic:

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = x_1 \\ \frac{du}{dt} = 2u \\ x_1(0) = s \\ x_2(0) = 0 \\ u(0) = \frac{s^2}{2} \end{cases}$$

$\Rightarrow u(t) = C \cdot e^{2t}$, din $u(0) = \frac{s^2}{2}$ $\Rightarrow$

$\Rightarrow C = \frac{s^2}{2} \Rightarrow$

$\Rightarrow \boxed{\tilde{u}(t, s) = \frac{s^2}{2} e^{2t}}$

Pt x_1, x_2 se rezolvă sistemul $\begin{cases} x_1' = x_2 \\ x_2' = x_1 \end{cases}$ (vezi sem. 11) $\Rightarrow$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

x_1 verifică ec : $x_1'' = (\mathrm{tr}\,A)x_1' - (\det A)x_1$

$\mathrm{tr}\,A = 0$

$\det A = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$ $\Rightarrow$

$\Rightarrow x_1'' = x_1 \Rightarrow r^2 = 1$

$r_{1,2} = \pm 1$, $m_1 = m_2 = 1 \Rightarrow$

$\Rightarrow \begin{cases} \varphi_1(t) = e^t \\ \varphi_2(t) = e^{-t} \end{cases}$ sist. fundam. de soluții pt. ec. în $x_1 \Rightarrow$

$\Rightarrow x_1(t) = C_1 e^t + C_2 e^{-t}$

din sistem $x_2 = x_1' \Rightarrow x_2(t) = x_1'(t) = C_1 e^t - C_2 e^{-t}$

dar $\begin{array}{l} x_1(0) = s \\ x_2(0) = 0 \end{array} \Bigg| \Rightarrow \begin{cases} C_1 + C_2 = s \\ C_1 - C_2 = 0 \end{cases}$

$2C_1 \;/= s \Rightarrow \begin{cases} C_1 = \frac{s}{2} \\ C_2 = \frac{s}{2} \end{cases}$

$\Rightarrow \begin{cases} \tilde{x}_1(t,s) = \frac{s}{2}(e^t + e^{-t}) \\ \tilde{x}_2(t,s) = \frac{s}{2}(e^t - e^{-t}) \end{cases}$

Sol. param: $\begin{cases} x_1 = \frac{s}{2}(e^t + e^{-t}) \\ x_2 = \frac{s}{2}(e^t - e^{-t}) \\ u = \frac{s^2}{2} e^{2t} = \frac{1}{2}(se^t)^2 \end{cases}$

OBS: $\begin{cases} x_1 = s\cdot \mathrm{ch}(t) \\ x_2 = s\cdot \mathrm{sh}(t) \end{cases}$

Rezultă $x_1 + x_2 = \frac{s}{\cancel{2}}\cancel{2}e^t \Rightarrow se^t = x_1 + x_2 \Rightarrow$

$$\Rightarrow \boxed{u(x_1,x_2) = \frac{1}{2}(x_1+x_2)^2}$$

(a) $2x_2\,\partial_1 u + (x_1 + x_2)\partial_2 u = x_1^2$

$u(x_1,x_2) = \frac{7x_1x_2}{4}$ pt $x \in S = \{x \in \mathbb{R}^2;\ \underbrace{x_1 - 4x_2}_{h(x)} = 0\}$

$a_1(x,u) = 2x_2$

$a_2(x,u) = x_1 + x_2$

$g(x,u) = x_1^2$; $u_0(x_1,x_2) = \frac{7x_1x_2}{4}$; $h(x) = x_1 - 4x_2$.

• o parametrizare pt S: $\begin{cases} x_1 = 4s = \alpha_1(s) \\ x_2 = s = \alpha_2(s) \end{cases}$

• $\varphi(s) = u_0(\alpha_1(s), \alpha_2(s)) = \frac{7 \cdot \cancel{4} s^2}{\cancel{4}} = 7s^2$.

• cond:

$$\text{rang}\begin{pmatrix} \alpha_1'(s) \\ \alpha_2'(s) \end{pmatrix} = \text{rang}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 1 = 2-1$$

$$\begin{vmatrix} a_1(4s, s, 7s^2) & 4 \\ a_2(4s, s, 7s^2) & 1 \end{vmatrix} = \begin{vmatrix} 2s & 4 \\ 4s+s & 1 \end{vmatrix} = 2s - 20s = $$
$$= -18s \neq 0$$

$\boxed{s \neq 0}$

• sist. caract: $\begin{cases} \frac{dx_1}{dt} = 2x_2 \\ \frac{dx_2}{dt} = x_1 + x_2 \\ \frac{du}{dt} = x_1^2 \\ x_1(0) = 4s \\ x_2(0) = s \\ u(0) = 7s^2 \end{cases}$ $\Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow$

$\Rightarrow x_1$ verifică ec

$x_1'' = (\text{tr} A) x_1' - (\det A) x_1$

$\text{tr} A = 1$

$\det A = -2 \Rightarrow$

$\Rightarrow x_1'' = x_1' + 2x_1 \Rightarrow$ ec. caract: $r^2 = r + 2 \Rightarrow$

$\Rightarrow r^2 - r - 2 = 0$

$\Delta = 1 + 8 = 9 \Rightarrow r_{1,2} = \frac{1 \pm 3}{2}$ $\begin{cases} r_1 = 2, \ m_1 = 1 \\ r_2 = -1, \ m_2 = 1 \end{cases}$

$\Rightarrow \boxed{x_1(t) = C_1 e^{2t} + C_2 e^{-t}}$

din sistem (din prima ec.) $\Rightarrow x_2 = \frac{1}{2} x_1' = \frac{1}{2}\left(C_1 e^{2t} + C_2 e^{-t}\right)' \Rightarrow$

$\Rightarrow \boxed{x_2(t) = \frac{1}{2}\left(2C_1 e^{2t} - C_2 e^{-t}\right)}$

dar $\left.\begin{array}{l} x_1(0) = 4s \\ x_2(0) = s \end{array}\right| \Rightarrow \begin{cases} C_1 + C_2 = 4s \\ \frac{1}{2}(2C_1 - C_2) = s \quad | \cdot 2 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = 4s \\ 2C_1 - C_2 = 2s \end{cases}$ (+)

$3C_1 \ / = 6s$.

$\Rightarrow \boxed{C_1 = 2s} \Rightarrow \boxed{C_2 = 2s} \Longrightarrow$

$$\Rightarrow \begin{cases} \tilde{x}_1(t,s) = 2s\left(e^{2t}+e^{-t}\right) \\ \tilde{x}_2(t,s) = s\left(2e^{2t}-e^{-t}\right) \end{cases}$$

Ec pt u: $\dfrac{du}{dt} = x_1^2 \Rightarrow \dfrac{du}{dt} = 4s^2\left(e^{2t}+e^{-t}\right)^2$ ec. de tip primitivă $\Rightarrow$

$$\Rightarrow u(t) = 4s^2 \int \left(e^{2t}+e^{-t}\right)^2 dt =$$

$$= 4s^2 \int \left(e^{4t}+2e^{t}+e^{-2t}\right) dt =$$

$$= 4s^2\left(\frac{e^{4t}}{4} + 2e^{t} - \frac{e^{-2t}}{2}\right) + C_3.$$

dar $u(0) = 7s^2$ $\Rightarrow$

$$\Rightarrow 7s^2 = 4s^2\left(\frac{1}{4} + \overset{4}{\cancel{2}} - \frac{\overset{2}{\cancel{1}}}{2}\right) + C_3 \Rightarrow$$

$$\Rightarrow 7s^2 = \cancel{4}s^2 \cdot \frac{7}{\cancel{4}} + C_3 \Rightarrow C_3 = 0 \Rightarrow$$

$$\Rightarrow \tilde{u}(t,s) = 4s^2\left(\frac{e^{4t}}{4} + 2e^{t} - \frac{e^{-2t}}{2}\right)$$

Sol. parametrică:

$$\begin{cases} x_1 = 2s\left(e^{2t}+e^{-t}\right) \\ x_2 = s\left(2e^{2t}-e^{-t}\right) \\ u = 4s^2\left(\dfrac{e^{4t}}{4} + 2e^{t} - \dfrac{e^{-2t}}{2}\right) \end{cases}$$

$$u = \left(se^{2t}\right)^2 + 8\left(se^{2t}\right)\left(se^{-t}\right) - 2\left(se^{-t}\right)^2$$

Avem

$$x_1 = 2se^{2t} + 2se^{-t}$$
$$2x_2 = 4se^{2t} - 2se^{-t}$$

$(+)$ $x_1 + 2x_2 = 6se^{2t}$ $\Rightarrow$ $\boxed{se^{2t} = \dfrac{x_1+2x_2}{6}}$

$(-)$ $2x_1 - 2x_2 = 6se^{-t}$ $\Rightarrow$ $\boxed{se^{-t} = \dfrac{2x_1-2x_2}{6}}$

$\Rightarrow$

$$\boxed{u(x_1,x_2) = \frac{(x_1+2x_2)^2}{36} + \overset{4}{\cancel{8}}\,\frac{(x_1+2x_2)}{\underset{3}{\cancel{6}}}\cdot\frac{(2x_1-2x_2)}{6} - \overset{1}{\cancel{2}}\cdot\frac{(2x_1-2x_2)^2}{\underset{18}{\cancel{36}}}}$$

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$$\Rightarrow \boxed{u(x_1, x_2) = \frac{(x_1+2x_2)^2}{36} + \frac{4(x_1+2x_2)(2x_1-2x_2)}{18} - \frac{(2x_1-2x_2)^2}{18}}$$

<u>Verif:</u> $x_1 = 4s$, $x_2 = s$ $\Rightarrow$ $u = \frac{\cancel{36} s^2}{\cancel{36}} + \frac{4 \cdot \cancel{6}^{2} s \, \cancel{6}^{1} s}{\cancel{18}} - \frac{\cancel{36}^{2} s^2}{\cancel{18}} =$

$$= s^2 + 8s^2 - 2s^2 = 7s^2.$$

<u>Temă</u> (b) + ex. de la ec. neliniară la curs.