Grupa 341, Seminar 11, GDDP, 22.12.2020

① (⑥ din temă) $t^3 x'' - 2tx = 3\ln t,\ t>0. \quad |:t$

$t^2x'' - 2x = \frac{3\ln t}{t}$

$(t,x) \xrightarrow{t=e^s} (s,y)$

$s(t) = \ln t$

$\boxed{x(t) = y(s(t))}$

$\boxed{tx' = y'}$

$\boxed{t^2x'' = y''-y'}$

$y''-y'-2y = \frac{3s}{e^s} \Rightarrow \boxed{y''-y'-2y = 3se^{-s}}$

② (④ din temă) $t^3x''' + tx' - x = t^2$

$(t,x) \xrightarrow[t=e^s]{} (s,y)$

$s = \ln t$

$x = y(s(t))$

$\boxed{tx' = y'}$

$\boxed{t^2x'' = y''-y'} \Rightarrow x'' = \frac{1}{t^2}\cdot(y''(s)-y'(s)) \Rightarrow$

$\Rightarrow x''' = \left(\frac{1}{t^2}\right)'\cdot(y''(s)-y'(s)) + \frac{1}{t^2}(y'''(s)-y''(s))\cdot\underbrace{s'(t)}_{=\frac{1}{t}} \Rightarrow$

$\Rightarrow x''' = \frac{-2}{t^3}(y''-y') + \frac{1}{t^3}(y'''-y'') \quad |\cdot t^3$

$x''' = \left(\frac{y''-y'}{t^2}\right)' = \frac{(y'''-y'')\frac{1}{\cancel{t}}\cdot t^{\cancel{2}} - (y''-y')\cdot 2t}{(t^2)^2} = \frac{\cancel{t}(y'''-y''-2y''+2y')}{t^{\cancel{4}\,3}}$ ⇒

$\left(\frac{1}{t^2}\right)' = \frac{0\cdot t^2 - 1\cdot 2t}{t^4} = \frac{-2t}{t^4} = \frac{-2}{t^3}$; $\left(\frac{1}{t^2}\right)' = (t^{-2})' = -2t^{-3} = \frac{-2}{t^3}$.

⇒ $\boxed{t^3x''' = y''' - 3y'' + 2y'}$

③ (II din temă)

Fie ec:

$$x'' = a_1(t)x' + a_0(t)x \qquad (1)$$

$a_1, a_0 : I \subset \mathbb{R} \to \mathbb{R}$

Se dă $\varphi_1 : I \to \mathbb{R}$ sol. a ec.

a) $(t,x) \xrightarrow[y=\frac{x}{\varphi_1(t)}]{} (t,y)$

$$\boxed{x = \varphi_1(t)y} \Rightarrow \boxed{x' = \varphi_1'(t)y + \varphi_1(t)y'}$$

$$x'' = \varphi_1''(t)y + \varphi_1'(t)y' + \varphi_1'(t)y' + \varphi_1(t)y''$$

$$\boxed{x'' = \varphi_1''(t)y + 2\varphi_1'(t)y' + \varphi_1(t)y''}$$

Ec. (1) devine:

$$\varphi_1''(t)y + 2\varphi_1'(t)y' + \varphi_1(t)y'' = a_1(t)\varphi_1'(t)y + a_1(t)\varphi_1(t)y' + a_0(t)\varphi_1(t)y \Rightarrow$$

$$\overbrace{\left(\varphi_1''(t) - a_1(t)\varphi_1'(t) - a_0(t)\varphi_1(t)\right)}^{=0} y + (2\varphi_1'(t) - a_1(t)\varphi_1(t))y' + \varphi_1(t)y'' = 0.$$

pt că φ_1 sol. a ec $\Leftrightarrow \varphi_1''(t) = a_1(t)\varphi_1'(t) + a_0(t)\varphi_1(t)$ $\Rightarrow$

$$\Rightarrow (2\varphi_1'(t) - a_1(t)\varphi_1(t))y' + \varphi_1(t)y'' = 0$$

$$(t,y) \xrightarrow[z'(t)=y''(t)]{z(t)=y'(t)} (t,z) \quad \Rightarrow$$

$$\Rightarrow (2\varphi_1'(t) - a_1(t)\varphi_1(t))z + \varphi_1(t)z' = 0 \Rightarrow$$

$$\Rightarrow z' = \underbrace{\frac{a_1(t)\varphi_1(t) - 2\varphi_1'(t)}{\varphi_1(t)}}_{a(t)} z \Rightarrow \boxed{\frac{dz}{dt} = a(t)z}$$

b) Sol. generală pt. ec. (3) și φ_2 ai $\{\varphi_1, \varphi_2\}$ sistem fundam. de soluții pt. ec. (1).

din $\frac{dz}{dt} = a(t)z \Rightarrow z(t) = C_1 \cdot e^{A(t)}$

unde $A(t)$ primitivă pt $a(t) \Rightarrow$

$$\Rightarrow A(t) = \int_{t_0}^{t} a(s)\,ds = \int_{t_0}^{t} \frac{a_1(s)\varphi_1(s) - 2\varphi_1'(s)}{\varphi_1(s)}\,ds =$$

$$= \int_{t_0}^{t} a_1(s)\,ds - 2\int_{t_0}^{t} \frac{\varphi_1'(s)}{\varphi_1(s)}\,ds =$$

$$= \int_{t_0}^{t} a_1(s)\,ds - 2\ln|\varphi_1(s)|\Big|_{t_0}^{t} =$$

$$= \int_{t_0}^{t} a_1(s)\,ds - 2\left(\ln|\varphi_1(t)| - \ln|\varphi_1(t_0)|\right) =$$

$$= \int_{t_0}^{t} a_1(s)\,ds + \ln\left(\frac{\varphi_1(t)}{\varphi_1(t_0)}\right)^{-2} \Rightarrow$$

$$\Rightarrow z(t) = C_1 \cdot e^{\int_{t_0}^{t} a_1(s)\,ds} \cdot \left(\frac{\varphi_1(t_0)}{\varphi_1(t)}\right)^2 \Bigg| \Rightarrow$$

Din $\underline{y'(t) = z(t)}$, ec. de tip prim.

$$\Rightarrow y(t) = \int_{t_0}^{t} z(s)\,ds + C_2 = C_1\int_{t_0}^{t} e^{\int_{t_0}^{s} a_1(v)\,dv} \left(\frac{\varphi_1(t_0)}{\varphi_1(s)}\right)^2 ds + C_2$$

$\Rightarrow$ din $x(t) = \varphi_1(t)y(t)$

$$x(t) = C_1 \underbrace{\varphi_1(t)\int_{t_0}^{t} e^{\int_{t_0}^{s} a_1(v)\,dv} \left(\frac{\varphi_1(t_0)}{\varphi_1(s)}\right)^2 ds}_{\varphi_2(t)} + C_2\varphi_1(t).$$

$C_1, C_2 \in \mathbb{R}$.

<u>Aplicatie pt ex. 3</u>:

Fie ec: $(2t+1)x'' + 4tx' - 4x = 0$. (2)

$$(\Leftrightarrow) \quad x'' = \underbrace{\frac{-4t}{2t+1}}_{a_1(t)} x' + \underbrace{\frac{4}{2t+1}}_{a_0(t)} x$$

a) Să se determine α, β, γ a.î. $\varphi_1(t) = \alpha t^2 + \beta t + \gamma$ să fie sol. a ec. (2)

b) Determinați φ_2 a.î. $\{\varphi_1, \varphi_2\}$ sistem fundam de soluții pt (2), integrând ec prin s.v. succesive:

$$(t,x) \xrightarrow[x=\varphi_1(t)y]{} (t,y) \xrightarrow[y'(t)=z(t)]{} (t,z)$$

a) $\varphi_1(t) = \alpha t^2 + \beta t + \gamma$

$\varphi_1'(t) = 2\alpha t + \beta$

$\varphi_1''(t) = 2\alpha$

φ_1 sol a ec (2) $\Leftrightarrow (2t+1)\varphi_1''(t) + 4t\varphi_1'(t) - 4\varphi_1(t) = 0 \Leftrightarrow$

$\Leftrightarrow (2t+1)2\alpha + 4t(2\alpha t + \beta) - 4(\alpha t^2 + \beta t + \gamma) = 0$

$\Leftrightarrow 4t\alpha + 2\alpha + 8\alpha t^2 + \cancel{4\beta t} - 4\alpha t^2 - \cancel{4\beta t} - 4\gamma = 0.$

$\Leftrightarrow 4\alpha t^2 + 4\alpha t + 2\alpha - 4\gamma = 0 \Leftrightarrow \begin{cases} 4\alpha = 0 \\ 4\alpha = 0 \\ 2\alpha - 4\gamma = 0 \end{cases} \Rightarrow \begin{cases} \alpha = \gamma = 0 \\ \beta \in \mathbb{R} \end{cases} \Rightarrow$

$\Rightarrow \varphi_1(t) = \beta t, \ \forall \beta \in \mathbb{R}$

Luăm $\boxed{\beta = 1} \Rightarrow \boxed{\varphi_1(t) = t}$

S.v: $x = ty \Rightarrow \left.\begin{array}{l} x' = y + ty' \\ x'' = y' + y' + ty'' = 2y' + ty'' \end{array}\right\} \Rightarrow$

$\Rightarrow$ ec. în y: $(2t+1)(2y' + ty'') + 4t(y + ty') - 4ty = 0$

$\underline{(2t+1)2y'} + \cancel{t}y''\,{}^{(2t+1)} + \cancel{4ty} + \underline{4t^2y'} - \cancel{4ty} = 0.$

$\Rightarrow \underline{(4t^2 + 4t + 2)y' + t(2t+1)y'' = 0}$

S.v: $y'(t) = z(t) \Rightarrow y''(t) = z'(t)$

$\Rightarrow$ ec. în z: $(4t^2 + 4t + 2)z + t(2t+1)z' = 0 \Rightarrow$

$\Rightarrow z' = \underbrace{-\dfrac{(4t^2+4t+2)}{t(2t+1)}}_{a(t)} z \Rightarrow z(t) = C_2 e^{A(t)}$

~~$\int a(t)dt = \int(-4t - 4 - \frac{2}{t})dt = -4\frac{t^2}{2} - 4t - 2\ln|t| + C \Rightarrow$~~

~~$\Rightarrow A(t) = -2t^2 + t + \ln\left(\frac{1}{t^2}\right)$~~

$$\int a(t)\,dt = -\int \frac{4t^2+4t+2}{t(2t+1)}\,dt = -\int \frac{2t(2t+1)+2t+2}{t(2t+1)}\,dt =$$

$$= -2\int 1\,dt - \int \frac{2t+2}{t(2t+1)}\,dt = \qquad \frac{2t+2}{t(2t+1)} = \frac{m}{t} + \frac{n}{2t+1}$$

$$= -2\int 1\,dt - \int \frac{2(2t+1)-2t}{t(2t+1)}\,dt =$$

$$= -2t - 2\int \frac{1}{t}\,dt + 2\int \frac{1}{2t+1}\,dt = \qquad \int \frac{1}{at+b}\,dt = \frac{1}{a}\ln|at+b| + C$$

$$= \underbrace{-2t - 2\ln|t| + \cancel{2}\tfrac{1}{\cancel{2}}\ln|2t+1|} + C$$

$$A(t) = -2t + \ln\frac{|2t+1|}{t^2} \Rightarrow z(t) = C_2 \cdot e^{-2t+\ln\left(\frac{|2t+1|}{t^2}\right)} =$$

$$= C_2\, e^{-2t} \cdot \frac{(2t+1)}{t^2} \quad \xRightarrow{z(t) = y'(t)}$$

$$\Rightarrow y'(t) = C_2\, e^{-2t}\left(\frac{2}{t} + \frac{1}{t^2}\right) \quad \text{ec. de tip primitivă} \Rightarrow$$

$$\Rightarrow y(t) = C_2 \int e^{-2t}\left(\frac{2}{t} + \frac{1}{t^2}\right)dt = C_2 \int \left(-e^{-2t}\frac{1}{t}\right)'dt \Rightarrow$$

$$\left(-\frac{1}{t}e^{-2t}\right)' = \frac{1}{t^2}e^{-2t} - \frac{1}{t}\cdot e^{-2t}(-2) =$$
$$= e^{-2t}\left(\frac{1}{t^2} + \frac{2}{t}\right)$$

$$\Rightarrow \underbrace{y(t) = -C_2\, e^{-2t}\frac{1}{t} + C_1} \quad \xRightarrow{x = ty}$$

$$\Rightarrow x(t) = -C_2 \underbrace{e^{-2t}}_{\varphi_2(t)} + C_1 \underbrace{t}_{\varphi_1(t)} \Rightarrow \{\varphi_1(t) = t,\ \varphi_2(t) = e^{-2t}\}$$

sistem fundam de soluții pt (2)

(4) Fie sistemul $\underbrace{\text{liniar}}_{\text{diferențial}}$ în $\mathbb{R}^2$, cu coef constanți:

$$x' = Ax, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$$

Arătați că pt x_1 se poate obține o ec. dif. de ordin 2 cu coef. constanți.

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \end{cases}$$

derivăm prima ec. din sistem $\Rightarrow x_1'' = a_{11}x_1' + \cancel{a_{12}x_2'}$

(ec. 2) $\cdot a_{12}$ $\Longrightarrow \cancel{a_{12}x_2'} = a_{21}a_{12}x_1 + \cancel{a_{22}a_{12}x_2}$

(ec. 1) $\cdot (-a_{22})$ $\Longrightarrow -a_{22}x_1' = -a_{11}a_{22}x_1 - \cancel{a_{12}a_{22}x_2}$

$$x_1'' - a_{22}x_1' = a_{11}x_1' + x_1(a_{12}a_{21} - a_{11}a_{22}) \quad (+)$$

$$\Rightarrow \underbrace{x_1'' = \overbrace{(a_{11}+a_{22})}^{tr(A)}x_1' - \overbrace{(a_{22}a_{11} - a_{12}a_{21})}^{\det A}x_1}_{\text{ec. dif de ordin 2 pt } x_1} \Longrightarrow$$

$\Rightarrow$ pt sistemul $x' = Ax$, $A \in \mathcal{M}_2(\mathbb{R})$ rezultă că x_1 verifică ec.: $\boxed{x_1'' = (trA)x_1' - (\det A)x_1}$

<u>Temă</u>: Similar, pt un sistem $x' = Ax$ cu $A \in \mathcal{M}_3(\mathbb{R})$ determinați ec. dif de ordin 3, liniară, cu coef constanți pe care o verifică x_1.

<u>Ec. cu derivate parțiale</u>:

<u>Temă</u>: Să se găsească forma generală a soluției pt ec. cvasiliniare cu derivate parțiale de ordinul întâi:

temă
1) $x_1^2\partial_1 u + x_2^2\partial_2 u = 2x_1x_2$
2) $u\,\partial_1 u + x_2\partial_2 u = x_1$
3) $x_1\partial_1 u + x_2^2\partial_2 u = (x_1+x_2)u$

✓ 4) $\boxed{(x_2+u)\,\partial_1 u + x_2\partial_2 u = x_1 - x_2}$

4) $a_1(x,u) = x_2 + u$
$a_2(x,u) = x_2$
$g(x,u) = x_1 - x_2$
$\Rightarrow$ sist. caract:

$$\frac{dx_1}{x_2+u} = \frac{dx_2}{x_2} = \frac{du}{x_1 - x_2} \Longrightarrow$$

$$\Rightarrow \frac{dx_1}{x_2+u} = \frac{dx_2}{x_2} = \frac{du}{x_1-x_2} = \frac{dx_1 + du}{\cancel{x_2}+u+x_1-\cancel{x_2}} = \frac{d(x_1+u)}{x_1+u}$$

$$\frac{dx_2}{x_2} = \frac{d(x_1+u)}{x_1+u} \Rightarrow \ln|x_2| = \ln|x_1+u| + C \Rightarrow$$

$$\Rightarrow \ln\left|\frac{x_2}{x_1+u}\right| = C \Rightarrow$$

$$\Rightarrow \underbrace{\frac{x_2}{x_1+u}}_{\varphi_1(x,u)} = \pm e^C$$

$\varphi_1(x,u)$ integrala primă

$$\underbrace{\frac{dx_1}{x_2+u} = \frac{dx_2}{x_2}}_{} = \underbrace{\frac{du}{x_1-x_2} = \frac{dx_1 - dx_2}{\cancel{x_2}+u-\cancel{x_2}}}_{}$$

$$\frac{du}{x_1-x_2} = \frac{d(x_1-x_2)}{u} \Rightarrow u\,du = (x_1-x_2)\,d(x_1-x_2)$$

$$\Rightarrow \frac{u^2}{2} = \frac{(x_1-x_2)^2}{2} + C \Rightarrow \underbrace{u^2 - (x_1-x_2)^2}_{\varphi_2(x,u)} = 2C$$

Avem integralele prime:

$$\begin{cases} \varphi_1(x,u) = \dfrac{x_2}{x_1+u} \\ \varphi_2(x,u) = u^2 - (x_1-x_2)^2 \end{cases} \Rightarrow$$

$\Rightarrow$ forma generală a sol ec. este:

$$f\left(\frac{x_2}{x_1+u},\ u^2-(x_1-x_2)^2\right) = 0.$$

cu f funcție arbitrară, care are derivate part de ordin întâi.

$f: D_2 \subset \mathbb{R}^2 \to \mathbb{R}$

Se pot da exemple de soluții luând pt f cazuri particulare.

· de exemplu: $f(y_1, y_2) = y_1 - 1 \Rightarrow \dfrac{x_2}{x_1+u} - 1 = 0 \Rightarrow$

$$\Rightarrow x_2 - x_1 - u = 0 \Rightarrow \boxed{u(x_1, x_2) = x_2 - x_1}$$

<u>verificare</u> $\partial_1 u = -1$; $\partial_2 u = 1 \Rightarrow$

$$\Rightarrow (x_2+u)\partial_1 u + x_2\partial_2 u = (x_2+x_2-x_1)(-1) + x_2 \cdot 1 =$$

$$= -2x_2 + x_1 + x_2 = x_1 - x_2 \quad \underline{\text{Adev}}$$