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Ecuații diferențiale liniare și afine de ordin n

$$x^{(n)} = \sum_{k=0}^{n-1} a_k(t)\, x^{(k)} + g(t) \qquad (1)$$

$a_0, \dots, a_{n-1}, g : I \subset \mathbb{R} \to \mathbb{R}$

Dacă $a_0, \dots, a_{n-1}$ sunt constante atunci ec (1) s.n. cu coeficienți constanți.

(1) Determinați forma generală a soluției ecuației:

$$x^{(3)} = 3x^{(1)} - 2x + t^2, \quad t \in \mathbb{R}.$$

Este ec. afină (liniară neomogenă) de ordin 3 cu coef. constanți: $\begin{cases} a_2 = 0 \\ a_1 = 3 \\ a_0 = -2 \\ g(t) = t^2 \end{cases}$

• se rezolvă ec. liniară omogenă atașată:

$$\bar{x}^{(3)} = 3\bar{x}^{(1)} - 2\bar{x}$$

se scrie ec. caracteristică

$$r^3 = 3r - 2r^0$$

$$r^3 - 3r + 2 = 0 \Rightarrow r^3 - 2r - r + 2 = 0 \Rightarrow$$

$$\Rightarrow r(r^2-1) - 2(r-1) = 0 \Rightarrow$$

$$\Rightarrow r(r-1)(r+1) - 2(r-1) = 0 \Rightarrow$$

$$\Rightarrow (r-1)[r(r+1) - 2] = 0 \Rightarrow$$

$$\Rightarrow (r-1)(r^2 + r - 2) = 0 \Rightarrow$$

$$\Rightarrow (r-1)[(r^2 - r) + 2r - 2] = 0$$

$$\Rightarrow (r-1)[r(r-1) + 2(r-1)] = 0$$

$$\Rightarrow (r-1)^2(r+2) = 0 \Rightarrow r_1 = 1,\ m_1 = 2; \quad r_2 = -2,\ m_2 = 1$$

pt $\boxed{r_1 = 1,\ m_1 = 2} \Rightarrow \begin{cases} \varphi_1(t) = e^t \\ \varphi_2(t) = te^t \end{cases}$

pt $r_2 = -2$, $m_2 = 1$ $\Rightarrow$ $\varphi_3(t) = e^{-2t}$

$\Rightarrow$ Sistem fundamental de soluții pt ec. lin. omogenă atașată este $\{e^t, te^t, e^{-2t}\}$ $\Rightarrow$

$\Rightarrow$ $\underline{\bar{x}(t) = C_1 e^t + C_2 t e^t + C_3 e^{-2t}, \quad C_1, C_2, C_3 \in \mathbb{R}.}$

• aplicăm metoda variației constantelor:
determinăm $C_1, C_2, C_3 : \mathbb{R} \to \mathbb{R}$ a.î.

$$x(t) = C_1(t)e^t + C_2(t)te^t + C_3(t)e^{-2t}$$

să fie soluție a ec. afine.

Știm că C_1', C_2', C_3' verifică sistemul algebric liniar:

$$\begin{cases} C_1' \cdot e^t + C_2' te^t + C_3' e^{-2t} = 0 \\ C_1'(e^t)' + C_2'(te^t)' + C_3'(e^{-2t})(-2) = 0 \\ C_1'(e^t)'' + C_2'(te^t)'' + C_3' e^{-2t}(-2)^2 = t^2 \end{cases} \Rightarrow$$

$(te^t)' = e^t + te^t = e^t(t+1)$

$(te^t)'' = (e^t(t+1))' = e^t(t+1) + e^t \cdot 1 = e^t(t+2)$

$$\Rightarrow \begin{cases} C_1'e^t + C_2'te^t + C_3'e^{-2t} = 0 \\ C_1'e^t + C_2'e^t(t+1) + C_3'e^{-2t}(-2) = 0 \\ C_1'e^t + C_2'e^t(t+2) + C_3'e^{-2t} \cdot 4 = t^2 \end{cases} \xrightarrow{(-)} \begin{cases} C_2'e^t - 3C_3'e^{-2t} = 0 \\ C_2'e^t + 6C_3'e^{-2t} = t^2 \end{cases}$$

$$(-) \quad 9C_3'e^{-2t} = t^2$$

$\Rightarrow$ $\boxed{C_3' = \frac{1}{9}t^2 e^{2t}}$

$C_2'e^t - \cancel{3}\frac{1}{\cancel{9}_3}t^2e^{2t} \cdot e^{-2t} = 0$ $\Rightarrow$ $\boxed{C_2' = \frac{1}{3}t^2 e^{-t}}$

$C_1'e^t + \frac{1}{3}t^3e^{-t}e^t + \frac{1}{9}t^2e^{2t}e^{-2t} = 0.$

$\boxed{C_1' = -e^{-t}\left(\frac{1}{3}t^3 + \frac{1}{9}t^2\right)}$; $\underline{C_1' = -\frac{1}{3}t^3e^{-t} - \frac{1}{9}t^2e^{-t}}$

Ec pt C_1, C_2, C_3 sunt de tip primitivă.

$C_2 = \frac{1}{3}\int t^2e^{-t}\,dt = \frac{1}{3}\int t^2(-e^{-t})'\,dt = \frac{1}{3}\left(-t^2e^{-t} + 2\int te^{-t}dt\right)$

$= \frac{1}{3}\left(-t^2e^{-t} + 2\int t(-e^{-t})'dt\right) = \frac{1}{3}\left(-t^2e^{-t} - 2te^{-t} + 2\int e^{-t}dt\right)$

$= \frac{1}{3}\left(\underline{-t^2e^{-t} - 2te^{-t} - 2e^{-t}}\right) + k_2$

$$\boxed{c_2(t) = \frac{-e^{-t}}{3}(t^2+2t+2) + k_2}$$

$$c_1 = \frac{1}{3}\int t^3 (e^{-t})'\,dt - \frac{1}{3}\cdot\frac{1}{3}\underbrace{\int t^2 e^{-t}\,dt}_{c_2(t)} =$$

$$= -\frac{1}{3}\Big(t^3 e^{-t} - 3\underbrace{\int t^2 e^{-t}\,dt}_{3c_2(t)}\Big) - \frac{1}{3}c_2(t) =$$

$$= -\frac{1}{3}t^3 e^{-t} - \frac{1}{3}3\left(-\frac{e^{-t}}{3}\right)(t^2+2t+2) - \frac{1}{3}\cdot\left(-\frac{e^{-t}}{3}\right)(t^2+2t+2) + k_1 \Rightarrow$$

$$\boxed{c_1(t) = \frac{e^{-t}}{9}(3t^3+10t^2+20t+20) + k_1}$$

$$c_3(t) = \frac{1}{9}\int t^2 e^{2t}\,dt = \frac{1}{18}\int t^2 (e^{2t})'\,dt = \frac{1}{18}\Big(t^2 e^{2t} - 2\int t e^{2t}\,dt\Big) =$$

$$= \frac{1}{18}\Big(t^2 e^{2t} - \int t (e^{2t})'\,dt\Big) = \frac{1}{18}\Big(t^2 e^{2t} - t e^{2t} + \int e^{2t}\,dt\Big) \Rightarrow$$

$$\Rightarrow \boxed{c_3(t) = \frac{1}{18}\cdot e^{2t}\left(t^2 - t + \frac{1}{2}\right) + k_3}$$

Sol. ec. afine:

$$x(t) = \frac{\cancel{e^{-t}}}{9}(3t^3+10t^2+20t+20)\cancel{e^{t}} + k_1 e^{t} +$$

$$+ \left(-\frac{1}{3}\right)(t^2+2t+2)t + k_2 t e^{t} +$$

$$+ \frac{1}{18}e^{2t}\left(t^2 - t + \frac{1}{2}\right)e^{-2t} + k_3 e^{-2t} =$$

$$= \underbrace{\left(k_1 e^{t} + k_2 t e^{t} + k_3 e^{-2t}\right)}_{\bar{x}(t)} +$$

$$+ \underbrace{\frac{1}{18}\left(\cancel{6t^3} + 20t^2 + 40t + 40 - \cancel{6t^3} - 12t^2 - 12t + t^2 - t + \frac{1}{2}\right)}$$

$$\varphi_0(t) = \frac{1}{18}\left(9t^2 + 27t + \frac{81}{2}\right) = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{4}$$

$$\boxed{\varphi_0(t) = \frac{1}{4}(2t^2 + 6t + 9)}$$ soluție particulară

Verificare: φ_0 sol. a ec. afine inițiale:

$$\varphi_0'(t) = \frac{1}{4}(4t+6)\ ;\ \varphi_0''(t) = \frac{1}{4}\cdot 4 = 1\ ;\ \varphi_0'''(t) = 0.$$

$$3\varphi_0'(t) - 2\varphi_0(t) + t^2 = \frac{3}{4}(4t+6) - \frac{2}{4}(2t^2+6t+9) + t^2 =$$

$$= \frac{1}{4}\left(\cancel{12t} + \cancel{18} - \cancel{4t^2} - \cancel{12t} - \cancel{18} + \cancel{4t^2}\right) = 0 = \varphi_0'''(t).$$

I. Să se determine soluția generală a fiecăreia dintre ec.:

Temă
2) $x''' - x' = 2t \quad , t \in \mathbb{R}$

3) $x^{(4)} + x^{(2)} = 2\cos t \quad , t \in \mathbb{R}$

4) $t^3 x''' + t x' - x = t^2 \quad ; t > 0.$

✓5) $(2t+3)^3 x''' + 4(2t+3)^2 x'' + 4(2t+3)x' - 8x = 8(2t+3)^2$.

$t \neq -\frac{3}{2}$
$t > -\frac{3}{2}$

Temă
6) $t^3 x'' - 2t x = 3\ln t \quad , t > 0.$
(împărțiți cu $t \Rightarrow$ ec. Euler)

II Temă. Fie ecuația: $x'' = a_1(t)x' + a_0(t)x \qquad (2)$

unde $a_1, a_0 : I \subset \mathbb{R} \to \mathbb{R}$.

Se dă $\varphi_1 : I \to \mathbb{R}$ soluție a ec. (2)

a) Arătați că prin schimbările succesive de variabile:

$y = \frac{x}{\varphi_1(t)}$; $z = y' \quad \left(z(y(t)) = y'(t) \right)$

$\underset{(2)}{(t, x)} \longrightarrow (t, y) \longrightarrow (y, z)$

se ajunge la o ec. liniară scalară
$\left(\frac{dz}{dy} = a(y) \cdot z \right)$.

b) Să se determine soluția generală pentru ec. (2)
∄ o soluție φ_2 a.î. $\{\varphi_1, \varphi_2\}$ să fie sistem fundam de soluții pt (2)

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5) $\left(t > -\frac{3}{2}\right)$ $(t, x) \longrightarrow (s, y)$

$2t + 3 = e^s \Leftrightarrow s = \ln(2t+3) \Rightarrow s'(t) = \frac{1}{2t+3} \cdot 2$

$$\boxed{x(t) = y(s(t))}$$

$$x'(t) = y'(s(t)) \cdot s'(t) = \frac{2}{2t+3} \cdot y'(s) \Rightarrow$$

$$\Rightarrow \boxed{(2t+3)\, x'(t) = 2y'}$$

$$x''(t) = \left(\frac{2}{2t+3} y'(s)\right)' = 2\left(\frac{-1}{(2t+3)^2} \cdot 2y'(s) + \frac{1}{2t+3} \cdot y''(s) \cdot \frac{2}{2t+3}\right) \Rightarrow$$

$$\Rightarrow x''(t) = 4 \frac{1}{(2t+3)^2} \left(y''(s) - y'(s) \right) \Rightarrow \boxed{(2t+3)^2 x'' = 4(y'' - y')}$$

$$x'''(t) = 4\cdot\left(\frac{-2}{(2t+3)^3}\cdot 2\,(y''-y') + \frac{1}{(2t+3)^2}\cdot(y'''-y'')\cdot\frac{2}{(2t+3)}\right) =$$

$$= 4\cdot\frac{2}{(2t+3)^3}\left(y'''-y''-2y''+2y'\right) \Rightarrow$$

$$\Rightarrow \boxed{(2t+3)^3\, x''' = 8\left(y'''-3y''+2y'\right)}$$

Ec. din ex. 5) devine:

$$8\left(y'''-3y''+2y'\right) + 4\cdot 4\left(y''-y'\right) + 4\cdot 2y' - 8y = 8e^{3s} \quad \Big|:8$$

$$y'''-3y''+\cancel{2y'}+2y''-\cancel{2y'}+y'-y = e^{3s}.$$

$$\boxed{y'''-y''+y'-y = e^{3s}} \quad (3)$$

Cautam sol part de forma $\varphi_0(s) = a\cdot e^{3s} \Rightarrow$

$$\Rightarrow \varphi_0'(s) = 3ae^{3s};\quad \varphi_0''(s) = 9ae^{3s};\quad \varphi_0'''(s) = 27ae^{3s}.$$

φ_0 sol. pt ec (3) $\Leftrightarrow \varphi_0'''(s)-\varphi_0''(s)+\varphi_0'(s)-\varphi_0(s) = e^{3s} \Leftrightarrow$

$$\Leftrightarrow e^{3s}\left(27a-9a+3a-a\right) = e^{3s} \Rightarrow 20a = 1 \Rightarrow a = \frac{1}{20} \Rightarrow$$

$$\Rightarrow \boxed{\varphi_0(s) = \frac{1}{20}e^{3s}} \text{ sol. part. a ec. (3)} \Rightarrow$$

$$\Rightarrow \underbrace{y(s) = \varphi_0(s) + \bar{y}(s)}_{\text{forma gen. a sol. ec. (3)}}$$

unde $\bar{y}$ este sol. gen. a ec: $\bar{y}'''-\bar{y}''+\bar{y}'-\bar{y} = 0$

$$r^3 - r^2 + r - 1 = 0$$

$$r^2(r-1) + r - 1 = 0 \Rightarrow (r-1)(r^2+1) = 0 \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{ll} r_1 = 1, & m_1 = 1 \rightarrow \varphi_1(s) = e^s. \\ r_2 = i, & m_2 = 1 \\ r_3 = -i = \bar{r}_2, & m_3 = 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \begin{cases} \varphi_2(s) = \mathrm{Re}\left(e^{is}\right) \\ \varphi_3(s) = \mathrm{Im}\left(e^{is}\right) \end{cases} \Bigg| \rightarrow \begin{array}{l} \varphi_2(s) = \cos s \\ \varphi_3(s) = \sin s \end{array} \rightarrow \bar{y}(s) = C_1e^s + C_2\cos s + C_3\sin s$$

$$e^{is} = \cos s + i\sin s$$

$C_1, C_2, C_3 \in \mathbb{R}$.

Deci: sol. gen a ec. (3):

$$y(s) = \frac{1}{20}e^{3s} + C_1e^s + C_2\cos s + C_3\sin s,\quad C_1, C_2, C_3 \in \mathbb{R}.$$

$$\Rightarrow x(t) = y(\ln(2t+3)) = \frac{1}{20}(2t+3)^3 + C_1(2t+3) + C_2\cos(\ln(2t+3)) + C_3\sin(\ln(2t+3))$$

$C_1, C_2, C_3 \in \mathbb{R}$.