Gr. 341, Seminar 1, EDDP, 06.10.2020

Rezolvarea ec. de tip primitivă

• Determinarea primitivelor unei funcții $f : I \subset \mathbb{R} \to \mathbb{R}$.

$$\int f(x)dx$$

(D) $F : I \to \mathbb{R}$ este primitivă pt f dacă

1) F este derivabilă pe I.

2) $F'(x) = f(x)$, $\forall x \in I$.

(P) Două primitive ale unei funcții diferă printr-o constantă.

Se notează cu $\int f(x)dx$ = mulțimea primitivelor funcției f

Operații cu mulțimi de primitive:

1) $\int (f_1(x) \pm f_2(x))dx = \int f_1(x)dx \pm \int f_2(x)dx$.

2) $\int \alpha f(x)dx = \alpha \int f(x)dx$, $\alpha \in \mathbb{R}$.

Tabel de primitive:

1) $\int 1\, dx = x + C$

2) $\int x^r dx = \frac{x^{r+1}}{r+1} + C$, $r \in \mathbb{R} \setminus \{-1\}$.

3) $\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$

4) $\begin{cases} \int a^x dx = \frac{a^x}{\ln a} + C, & a \in (0,\infty) \setminus \{1\} \\ \int e^x dx = e^x + C, & e \simeq 2{,}71\ldots \end{cases}$

5) $\begin{cases} \int \sin x\, dx = -\cos x + C \\ \int \cos x\, dx = \sin x + C \\ \int tg\, x\, dx = -\ln|\cos x| + C \\ \int ctg\, x\, dx = \ln|\sin x| + C \\ \int \frac{1}{\cos^2 x} dx = \int (1 + tg^2 x) dx = tg\, x + C \\ \int \frac{1}{\sin^2 x} dx = \int (1 + ctg^2 x) dx = -ctg\, x + C. \end{cases}$

6) $\begin{cases} \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a} + C \\ \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \end{cases}$

7) $\begin{cases} \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right) + C \\ \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right| + C \\ \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C \end{cases}$

8) $\begin{cases} \int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2) + C \\ \int \frac{x}{x^2-a^2}\,dx = \frac{1}{2}\ln\left|x^2-a^2\right| + C \end{cases}$

9) $\begin{cases} \int \frac{x}{\sqrt{x^2+a^2}}\,dx = \sqrt{x^2+a^2} + C \\ \int \frac{x}{\sqrt{x^2-a^2}}\,dx = \sqrt{x^2-a^2} + C \\ \int \frac{x}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2} + C \end{cases}$

Metode de integrare

1) Reducerea funcției din integrală la formule din tabelul de primitive.

2) Metoda integrării prin părți:

$$\boxed{\int \underbrace{u(x)\,v'(x)}_{f(x)}\,dx = u(x)v(x) - \int u'(x)\,v(x)\,dx}$$

3) Prima metodă de schimbare de variabilă

$$\boxed{\int \underbrace{g(u(x))\cdot u'(x)}_{f(x)}\,dx = G(u(x)) + C}$$, unde G este o primitivă pt g.

$u(x) = y$

$u'(x)\,dx = dy \Rightarrow \int g(y)\,dy = G(y) + C$

4) A doua schimbare de variabilă

$$\boxed{\int \underbrace{g(u(x))}_{f(x)}\,dx = H(u(x)) + C}$$, unde H este o primitivă pt $g\cdot\varphi'$.

$$u(x) = y \Leftrightarrow x = u^{-1}(y) \overset{not}{=} \varphi(y)$$

$$x = \varphi(y) \Rightarrow dx = \varphi'(y)\,dy$$

$$\int \underbrace{g(y)\cdot\varphi'(y)}_{h(y)}dy = \int h(y)dy = H(y) + C$$

Ex. I Să se determine mulțimea primitivelor următoarelor funcții:

1) $f(x) = x^6 - 2x^3 + x + 2$

2) $f(x) = \sqrt[4]{x} - \sqrt[3]{x} + 1$

3) $f(x) = x^2\sqrt{x} + x\sqrt[3]{x} - 1$

4) $f(x) = \dfrac{(x+1)^2}{\sqrt{x}}$

5) $f(x) = 2^x \cdot 3^x$

6) $f(x) = \dfrac{(x-2)^3}{x^2}$

7) $f(x) = \dfrac{\cos 2x}{\sin^2 x \cdot \cos^2 x}$

8) $f(x) = tg^2 x$

9) $f(x) = \dfrac{1}{2x^2+8}$

10) $f(x) = \dfrac{1}{12-3x^2}$

11) $f(x) = \dfrac{1}{(x^2+3)(x^2-4)}$

12) $f(x) = \dfrac{\sqrt{x^2+1} - 2\sqrt{x^2-1}}{\sqrt{x^4-1}}$

13) $f(x) = \dfrac{x+2}{\sqrt{x^2+1}}$

14) $f(x) = (x+1)\,e^x$

15) $f(x) = (x^2+1)\ln x$

16) $f(x) = 2x\cdot\cos x$

17) $f(x) = e^x \sin x$

18) $f(x) = e^{2x}\cdot\cos 3x$

19) $f(x) = \sqrt{x^2-1}$

20) $f(x) = x\sqrt{x^2+4}$

21) $f(x) = \dfrac{x+1}{x^2+2x+5}$

22) $f(x) = 2x\cdot e^{x^2+1}$

23) $f(x) = \dfrac{\ln(\ln x)}{x\ln x}$

24) $f(x) = x(x+2)^{2020}$

25) $f(x) = (x^2+3)(x^2+2x+5)^{1000}$

26) $f(x) = \dfrac{1}{1+x^2}\,e^{arctg\,x}$

27) $f(x) = \dfrac{\sin 2x}{\sin^2 x + 4}$

28) $f(x) = \dfrac{\cos x}{\sin^2 x + 4}$

$$\Rightarrow \mathcal{I} = x\sqrt{x^2-1} - \int \frac{(\sqrt{x^2-1})^2}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx \Rightarrow$$

$$\Rightarrow \mathcal{I} = x\sqrt{x^2-1} - \mathcal{I} - \ln\left|x+\sqrt{x^2-1}\right| \Rightarrow$$

$$\Rightarrow 2\mathcal{I} = x\sqrt{x^2-1} - \ln\left|x+\sqrt{x^2-1}\right| \Rightarrow$$

$$\Rightarrow \mathcal{I} = \frac{1}{2}\left(x\sqrt{x^2-1} - \ln\left|x+\sqrt{x^2-1}\right|\right) + C.$$

(18) $\int e^{2x}\cdot \cos 3x\,dx = \mathcal{I}$

$u(x) = \cos 3x \qquad u'(x) = -(\sin 3x)\cdot(3x)' = -3\sin 3x$

$v'(x) = e^{2x} \qquad v(x) = \frac{e^{2x}}{2}$

$$\mathcal{I} = \frac{e^{2x}}{2}\cos 3x - \int(-3\sin 3x)\cdot\frac{e^{2x}}{2}\,dx =$$

$$= \frac{e^{2x}}{2}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx \Rightarrow$$

$u(x) = \sin 3x \qquad u'(x) = 3\cos 3x$

$v'(x) = e^{2x} \qquad v(x) = \frac{e^{2x}}{2}$

$$\Rightarrow \mathcal{I} = \frac{e^{2x}}{2}\cos 3x + \frac{3}{2}\left(\frac{e^{2x}}{2}\sin 3x - \int 3\cos 3x\cdot\frac{e^{2x}}{2}\,dx\right) \Rightarrow$$

$$\Rightarrow \mathcal{I} = \frac{e^{2x}\cos 3x}{2} + \frac{3e^{2x}\sin 3x}{4} - \frac{9}{4}\underbrace{\int e^{2x}\cos 3x\,dx}_{\mathcal{I}} \Rightarrow$$

$$\Rightarrow \mathcal{I} + \frac{9}{4}\mathcal{I} = \frac{e^{2x}}{4}(2\cos 3x + 3\sin 3x)$$

$$\mathcal{I} = \frac{\cancel{4}}{13}\cdot\frac{e^{2x}}{\cancel{4}}(2\cos 3x + 3\sin 3x) + C$$

$$\mathcal{I} = \frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x) + C.$$

(24) $\int x(x+2)^{2020}\,dx = \int(x+2-2)(x+2)^{2020}\,dx = \mathcal{I}$

$x+2 = y$

$dx = dy$

$$\int(y-2)y^{2020}\,dy = \int y^{2021}\,dy - 2\int y^{2020}\,dy =$$

$$= \frac{y^{2022}}{2022} - 2\frac{y^{2021}}{2021} + C \Rightarrow \mathcal{I} = \frac{(x+2)^{2022}}{2022} - 2\frac{(x+2)^{2021}}{2021} + C.$$

Temă: Integralele nerezolvate.

✓ 29) $f(x)=\frac{1}{\sin x}$; 30) $f(x)=\frac{1}{\cos x}$.

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(12) $\int \frac{\sqrt{x^2+1}-2\sqrt{x^2-1}}{\sqrt{x^4-1}}dx = \int \frac{\sqrt{x^2+1}-2\sqrt{x^2-1}}{\sqrt{(x^2-1)(x^2+1)}}dx =$

$x\in(-\infty,-1)$
sau $x\in(1,+\infty)$

$= \int \frac{\cancel{\sqrt{x^2+1}}}{\sqrt{x^2-1}\cdot\cancel{\sqrt{x^2+1}}}dx - 2\int\frac{\cancel{\sqrt{x^2-1}}}{\cancel{\sqrt{x^2-1}}\cdot\sqrt{x^2+1}}dx$

$= \int\frac{1}{\sqrt{x^2-1}}dx - 2\int\frac{1}{\sqrt{x^2+1}}dx =$

$= \ln\left|x+\sqrt{x^2-1}\right| - 2\ln\left(x+\sqrt{x^2+1}\right) + C$.

(6) $\int \frac{(x-2)^3}{x^2}dx = \int \frac{x^3-3x^2\cdot 2+3x\cdot 2^2-2^3}{x^2}dx =$

$= \int\frac{x^3}{x^2}dx - 6\int\frac{x^2}{x^2}dx + 12\int\frac{\cancel{x}}{x^{\cancel{2}}}dx - 8\int\frac{1}{x^2}dx =$

$= \int x\,dx - 6\int 1\,dx + 12\int\frac{1}{x}dx - 8\int x^{-2}dx =$

$= \frac{x^2}{2} - 6x + 12\ln|x| - 8\frac{x^{-1}}{-1} + C =$

$= \frac{x^2}{2} - 6x + 12\ln|x| + 8\frac{1}{x} + C$; $x\in(-\infty,0)\cup(0,+\infty)$

(5) $\int 2^x\cdot 3^x dx = \int(2\cdot 3)^x dx = \int 6^x dx = \frac{6^x}{\ln 6}+C$; $x\in\mathbb{R}$

(3) $\int\left(x^2\sqrt{x}+x\sqrt[3]{x}-1\right)dx = \int x^2\cdot x^{\frac{1}{2}}dx + \int x\cdot x^{\frac{1}{3}}dx - \int 1\,dx =$

$\boxed{\sqrt[n]{x}=x^{\frac{1}{n}}}$

$\boxed{\sqrt[n]{x^k}=x^{\frac{k}{n}}}$

$= \int x^{\frac{5}{2}}dx + \int x^{\frac{4}{3}}dx - x =$

$= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} - x + C =$

$= \frac{2}{7}x^{\frac{7}{2}} + \frac{3}{7}x^{\frac{7}{3}} - x + C =$

$= \frac{2}{7}\sqrt{x^7} + \frac{3}{7}\sqrt[3]{x^7} - x + C$, $x\geq 0$

$= \frac{2}{7}x^3\sqrt{x} + \frac{3}{7}x^2\sqrt[3]{x} - x + C$.

(29) $\int\frac{1}{\sin x}dx = \int\frac{\sin x}{\sin^2 x}dx = \frac{1}{-1}\int\frac{-\sin x}{1-\cos^2 x}dx$; $\cos x = y$
$-\sin x\,dx = dy$

$x\in\mathbb{R}\setminus\{k\pi \mid k\in\mathbb{Z}\}$

$$-\int \frac{dy}{1-y^2} = \int \frac{dy}{y^2-1} = \frac{1}{2\cdot 1}\ln\left|\frac{y-1}{y+1}\right| + C \Rightarrow$$

$$\left.\begin{array}{l}\Rightarrow I = \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| + C \\ \cos x \in [-1,1]\end{array}\right\} \Rightarrow I = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right) + C.$$

sau $\boxed{\sin x = \frac{2\,tg\frac{x}{2}}{1+tg^2\frac{x}{2}}}$ $\boxed{\cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}}$

$$I = \int \frac{1+tg^2\frac{x}{2}}{2\,tg\frac{x}{2}}\,dx$$

$tg\frac{x}{2} = y \Rightarrow \left(1+tg^2\frac{x}{2}\right)\left(\frac{x}{2}\right)'dx = dy \Rightarrow$

$\boxed{(tg\,x)' = \frac{1}{\cos^2 x} = 1+tg^2 x}$ $\Rightarrow \left(1+tg^2\frac{x}{2}\right)\frac{1}{2}dx = dy$

$\int \frac{1}{y}dy = \ln|y| + C \Rightarrow I = \ln\left|tg\frac{x}{2}\right| + C.$

(23) $\int \frac{\ln(\ln x)}{x\ln x}dx = I$

$\ln(\ln x) = y$

$\frac{1}{\ln x}\cdot(\ln x)'dx = dy$

$\frac{1}{\ln x}\cdot\frac{1}{x}dx = dy \Rightarrow \frac{1}{x\ln x}dx = dy$

$\int y\,dy = \frac{y^2}{2} + C \Rightarrow I = \frac{1}{2}(\ln(\ln x))^2 + C =$

$= \frac{1}{2}\ln^2(\ln x) + C, \quad , x > 0$

(19) $\int \sqrt{x^2-1}\,dx = \int x'\sqrt{x^2-1}\,dx = I$

$x \in (-\infty,-1) \cup (1,+\infty)$

$u(x) = \sqrt{x^2-1}$ $\quad u'(x) = \frac{1}{2\sqrt{x^2-1}}\cdot\underbrace{(x^2-1)'}_{2x} = \frac{x}{\sqrt{x^2-1}}$

$v'(x) = 1$ $\quad v(x) = x$

$$I = x\sqrt{x^2-1} - \int x\frac{x}{\sqrt{x^2-1}}dx = x\sqrt{x^2-1} - \int\frac{(x^2-1)+1}{\sqrt{x^2-1}}dx \Rightarrow$$