Grupa 341, Seminar 2, EDDP, 13.10.2020

25 (din tema de data trecută)

dacă $f(x) = (x+1)(x^2+2x+5)^{1000} \Rightarrow \int f(x)\,dx \in \mathcal{J}$.

$$\begin{cases} \int (x^2+3)(x^2+2x+5)^{1000}\,dx = \\ = \int \left[(x^2+2x+5) - \underbrace{2(x+1)}\right](x^2+2x+5)^{1000}\,dx = \\ = \int (x^2+2x+5)^{1001}\,dx - \int (x^2+2x+5)'(x^2+2x+5)^{1000}\,dx = \\ = \underbrace{\int (x^2+2x+5)^{1001}\,dx}_{?\ (*)} - \frac{(x^2+2x+5)^{1001}}{1001} \end{cases}$$

$$\mathcal{J} = \int \frac{1}{2}(x^2+2x+5)'(x^2+2x+5)^{1000}\,dx = \frac{1}{2}\,\frac{(x^2+2x+5)^{1001}}{1001} + C$$

8 $\int tg^2 x\,dx = \int \left((1+tg^2 x) - 1\right)dx = \int (1+tg^2 x)\,dx - \int 1\,dx =$
$= tg x - x + C$.

sau

$$\int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx - \int 1\,dx =$$
$$= tg x - x + C.$$

(*) $I_n = \int (x^2+2x+5)^n\,dx \Rightarrow$

$u(x) = (x^2+2x+5)^n \Rightarrow u'(x) = n(x^2+2x+5)^{n-1}(2x+2)$

$\boxed{v'(x) = 1 \Rightarrow v(x) = x+1}$

$$\Rightarrow I_n = (x+1)(x^2+2x+5)^n - \int 2n(\underbrace{x^2+2x}+5)^{n-1}(x+1)^2\,dx =$$

$$= (x+1)(x^2+2x+5)^n - 2n\int (x^2+2x+5)^{n-1}\left((x^2+2x+5)-4\right)dx =$$

$$= (x+1)(x^2+2x+5)^n - 2n\left(\underbrace{\int (x^2+2x+5)^n\,dx}_{I_n} - 4\underbrace{\int (x^2+2x+5)^{n-1}\,dx}_{I_{n-1}}\right)$$

$$\Rightarrow I_n = (x+1)(x^2+2x+5)^n - 2n I_n + 8n I_{n-1} \Rightarrow$$

$$\Rightarrow (2n+1)I_n = (x+1)(x^2+2x+5)^n + 8n I_{n-1} \quad \Big| : (2n+1)$$

$$\Rightarrow I_n = \frac{8n}{2n+1} I_{n-1} + \frac{1}{2n+1}(x+1)(x^2+2x+5)^n.$$

(I) Să se determine mulțimea soluțiilor ecuațiilor:

1) $\dfrac{dx}{dt} = \dfrac{1}{t}\,\dfrac{x+x^3}{x^2-1}$, $x \in (1,+\infty)$, $t \in (0,\infty)$

2) $\dfrac{dx}{dt} = \dfrac{x-1}{t^2+4t-5}$, $t \in (1,\infty)$, $x \in \mathbb{R}$.

3) $\dfrac{dx}{dt} = \dfrac{t\sqrt{x^2+1}}{x\sqrt{t^2+9}}$, $x \in (0,\infty)$, $t \in \mathbb{R}$

√ 4) $\dfrac{dx}{dt} = \dfrac{\sin x \cdot \cos t}{(\cos^2 x - 9)(\sin^2 t + 1)}$, $t \in [0,\pi]$, $x \in \mathbb{R}$.

√ 5) $\dfrac{dx}{dt} = \dfrac{(x^2-3x-4)(t+1)}{t^2+2t+3}$, $t \in \mathbb{R}$, $x \in \mathbb{R}$.

6) $\dfrac{dx}{dt} = \dfrac{(1+x^2)}{xt}$, $x, t > 0$

√ 7) $\dfrac{dx}{dt} = \dfrac{(x^3-1)(t-1)}{\sqrt{t^2+1}}$, $t, x \in \mathbb{R}$

√ 8) $\dfrac{dx}{dt} = \dfrac{e^{e^t+t}(x^2-2x+4)}{x-4}$, $x > 4$, $t \in \mathbb{R}$

9) $\dfrac{dx}{dt} = \dfrac{2tx(\ln x)}{\ln(\ln x)(t^2+1)}$, $x \in (3,+\infty)$, $t \in \mathbb{R}$.

10) $\dfrac{dx}{dt} = \dfrac{1}{(\operatorname{ctg}^2 x + 1)\sqrt{4-t^2}}$; $t \in (-2,2)$, $x \in \mathbb{R}$.

(8) $\dfrac{dx}{dt} = \dfrac{e^{e^t+t}(x^2-2x+4)}{x-4}$; $x>4$, $t\in\mathbb{R}$.

ec. cu var separabile : $a:\mathbb{R}\to\mathbb{R}$, $a(t)=e^{e^t+t}$

$b:(4,+\infty)\to\mathbb{R}$

$b(x)=\dfrac{x^2-2x+4}{x-4}$.

• $b(x)=0 \Rightarrow \dfrac{x^2-2x+4}{x-4}=0 \Rightarrow x^2-2x+4=0 \Rightarrow$

$\Rightarrow \Delta=(-2)^2-4\cdot1\cdot4 = 4-16=-12<0 \Rightarrow$

$\rightarrow$ ec. nu are rădăcini reale $\Rightarrow$

$\Rightarrow$ ec. diferențială nu are soluții staționare $\Rightarrow$

$\Rightarrow b(x)\neq 0,\ \forall x\in(4,+\infty)$

• $x\in(4,+\infty)$, $b(x)\neq0 \Rightarrow$ separăm variabilele $\Rightarrow$

$$\frac{(x-4)\,dx}{x^2-2x+4} = e^{e^t+t}\,dt$$

$$\int\frac{(x-4)\,dx}{x^2-2x+4} = \int\frac{(x-4)\,dx}{(x-1)^2+3} = \mathcal{I}$$

$x-1=t \Leftrightarrow x=t+1$

$dx=dt$

$$\int\frac{t+1-4}{t^2+3}dt = \int\frac{t}{t^2+3}dt - 3\int\frac{1}{t^2+(\sqrt3)^2}dt =$$

$$=\frac12\ln(t^2+3) - \cancel{3}^{\sqrt3}\frac{1}{\cancel{\sqrt3}}\operatorname{arctg}\frac{t}{\sqrt3}+C$$

$$\mathcal{I} = \underbrace{\frac12\ln(x^2-2x+4)-\sqrt3\operatorname{arctg}\frac{x-1}{\sqrt3}}_{B(x)}+C$$

$$\int e^{e^t+t}dt = \int e^{e^t}\cdot e^t\,dt = \mathcal{J} \quad ;\ e^t=y,\ e^t dt=dy.$$

$$\int e^y dy = e^y + C \Rightarrow \mathcal{J} = e^{e^t} + C \Rightarrow$$

$$\Rightarrow A(t) = e^{e^t}$$

Mulț. soluțiilor implicite:

$$\boxed{\frac{1}{2}\ln(x^2-2x+4) - \sqrt{3}\,\mathrm{arctg}\left(\frac{x-1}{\sqrt{3}}\right) = e^{e^t} + C \quad C \in \mathbb{R}.}$$

(7) $\frac{dx}{dt} = \frac{(x^3-1)(t-1)}{\sqrt{t^2+1}}$, $t, x \in \mathbb{R}$.

$a, b : \mathbb{R} \to \mathbb{R}$

ec. cu variabile separabile: $a(t) = \frac{t-1}{\sqrt{t^2+1}}$

$b(x) = x^3 - 1$

- $b(x) = 0 \Rightarrow x^3 - 1 = 0 \Rightarrow (x-1)(x^2+x+1) = 0 \Rightarrow$

$\Rightarrow x_1 = 1$

$x^2 + x + 1 = 0$

$\Delta = 1 - 4 = -3 < 0 \Rightarrow$ nu are sol. reale.

$\Rightarrow$ o soluție staționară: $\varphi_1 : \mathbb{R} \to \mathbb{R}$, $\varphi_1(t) = 1, \forall t \in \mathbb{R}$

- $b(x) \neq 0$, $x \in \mathbb{R} \setminus \{1\}$

$$\frac{dx}{x^3-1} = \frac{(t-1)}{\sqrt{t^2+1}}\, dt.$$

$$\int \frac{dx}{x^3-1} = \mathcal{I}.$$

Determinăm $A, B, C \in \mathbb{R}$ a.î. $\frac{1}{(x-1)(x^2+x+1)} = \frac{\overset{x^2+x+1}{A}}{x-1} + \frac{\overset{x-1}{Bx+C}}{x^2+x+1}$

$$1 = A(x^2+x+1) + (x-1)(Bx+C)$$

$$1 = \underline{Ax^2} + \underline{\underline{Ax}} + A + \underline{Bx^2} + \underline{\underline{Cx}} - \underline{\underline{Bx}} - C$$

$$0x^2 + 0\cdot x + 1 = x^2(A+B) + x(A-B+C) + (A-C)$$

Identificând coeficienții rezultă: $\left\{\begin{array}{l} A+B=0 \\ A-B+C=0 \\ A-C=1 \Rightarrow \boxed{C=A-1} \end{array}\right\} \Rightarrow$

$\Rightarrow \left\{\begin{array}{l} A+B=0 \\ 2A-B=1 \end{array}\right. \overset{(+)}{\Rightarrow} 3A = 1 \Rightarrow \boxed{A = \frac{1}{3}} \Rightarrow \boxed{B = -\frac{1}{3}} \Rightarrow \boxed{C = -\frac{2}{3}}.$

$$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1} \Rightarrow$$

$$\Rightarrow I = \frac{1}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\left(-\frac{1}{3}\right)\int \frac{x+2}{\underbrace{x^2+x+1}}dx =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{9}\underbrace{\int \frac{x+2}{\underbrace{x^2+2\cdot\frac{1}{2}x+\left(\frac{1}{2}\right)^2}-\left(\frac{1}{2}\right)^2+1}dx}_{J}$$

$$J = \int \frac{x+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$x+\frac{1}{2} = t \iff x = t - \frac{1}{2}$$

$$dx = dt$$

$$J\int \frac{t-\frac{1}{2}+2}{t^2+\frac{3}{4}}dt = \int \frac{t+\frac{3}{2}}{t^2+\frac{3}{4}}dx =$$

$$= \int \frac{t}{t^2+\frac{3}{4}}dt + \frac{3}{2}\int \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx =$$

$$= \frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) + \cancel{\frac{3}{2}}^{\sqrt{3}}\cdot\frac{\cancel{2}}{\cancel{\sqrt{3}}}\operatorname{arctg}\left(\frac{2t}{\sqrt{3}}\right) + C.$$

$$J = \frac{1}{2}\ln(x^2+x+1) + \sqrt{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C \Rightarrow$$

$$\Rightarrow \boxed{B(x) = \frac{1}{3}\ln|x-1| - \frac{1}{18}\ln(x^2+x+1) - \frac{\sqrt{3}}{9}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)}$$

$$\int \frac{t-1}{\sqrt{t^2+1}}dt = \int \frac{t}{\sqrt{t^2+1}}dt - \int \frac{1}{\sqrt{t^2+1}}dt =$$

$$= \underbrace{\sqrt{t^2+1} - \ln\left(t+\sqrt{t^2+1}\right)}_{A(t)} + C$$

Mult. de soluții implicite este:

$$\frac{1}{2}\ln|x-1| - \frac{1}{18}\ln(x^2+x+1) - \frac{\sqrt{3}}{9}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) =$$

$$= \sqrt{t^2+1} - \ln\left(t+\sqrt{t^2+1}\right) + C, \; t \in \mathbb{R}.$$

(4) $\dfrac{dx}{dt} = \dfrac{\sin x \cdot \cos t}{(\cos^2 x - 9)(\sin^2 t + 1)}$, $\quad t \in [0, \pi]$, $x \in \mathbb{R}$.

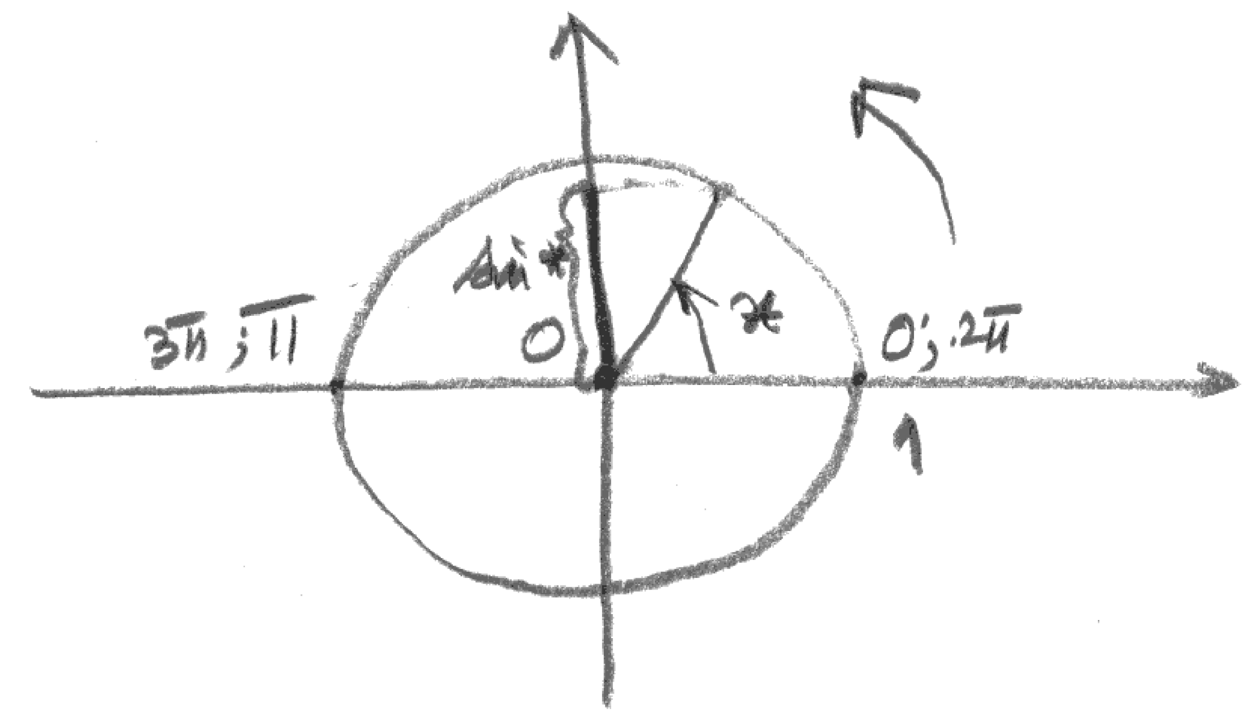
ec. cu var separabile:

$a : [0, \pi] \to \mathbb{R}$

$a(t) = \dfrac{\cos t}{\sin^2 t + 1}$

$b : \mathbb{R} \to \mathbb{R}$

$b(x) = \dfrac{\sin x}{\cos^2 x - 9}$



• $b(x) = 0 \Rightarrow \sin x = 0 \Rightarrow x = k\pi,\ k \in \mathbb{Z} \Rightarrow$

$\Rightarrow$ ec. are o infinitate de soluții staționare:

$$\varphi_k : [0, \pi] \to \mathbb{R}$$
$$\varphi_k(t) = k\pi, \quad k \in \mathbb{Z}$$

• pt x a.î. $b(x) \neq 0 \Rightarrow x \in \mathbb{R} \setminus \mathbb{Z} \Rightarrow$ separăm variabilele:

$$\frac{\cos^2 x - 9}{\sin x}\, dx = \frac{\cos t}{\sin^2 t + 1}\, dt.$$

$$\int \frac{\cos^2 x - 9}{\sin x}\, dx = \int \frac{1 - \sin^2 x - 9}{\sin x}\, dx = \int \frac{-8 - \sin^2 x}{\sin x}\, dx =$$

$$= -8\int \frac{1}{\sin x}\, dx - \int \cancel{\frac{\sin^2 x}{\sin x}}\, dx \underset{\text{(Seminarul 1)}}{=}$$

$$= -\cancel{8}^{4} \cdot \frac{1}{\cancel{2}} \ln\left(\frac{1 - \cos x}{1 + \cos x}\right) + \cos x + C \Rightarrow$$

$x \in \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$.

$$\Rightarrow B(x) = -4 \ln\left(\frac{1 - \cos x}{1 + \cos x}\right) + \cos x$$

$$\left.\begin{array}{l} \displaystyle\int \frac{\cos t}{\sin^2 t + 1}\, dt = \int \frac{(\sin t)'}{(\sin t)^2 + 1}\, dt = J \\ \sin t = y \\ (\sin t)'\, dt = dy \Rightarrow \displaystyle\int \frac{dy}{y^2 + 1} = \operatorname{arctg} y + C \end{array}\right\} \Rightarrow$$

$\Rightarrow y = \text{arctg}(\sin t) + c \Rightarrow A(t) = \text{arctg}(\sin t)$

Sol. în formă implicită:

$$\boxed{-4\ln\left(\frac{1-\cos x}{1+\cos x}\right) + \cos x = \text{arctg}(\sin t) + c,\quad c \in \mathbb{R}}$$

(5) $\frac{dx}{dt} = \frac{(x^2-3x-4)(t+1)}{t^2+2t+3}$, $t \in \mathbb{R}$, $x \in \mathbb{R}$.

ec. cu var. separabile:

$a : \mathbb{R} \to \mathbb{R}$; $a(t) = \frac{t+1}{t^2+2t+3}$

$b : \mathbb{R} \to \mathbb{R}$; $b(x) = x^2-3x-4$.

- $b(x) = 0 \Rightarrow x^2-3x-4=0$

$\Delta = (-3)^2 - 4\cdot 1(-4) = 9+16 = 25 = 5^2$

$x_{1,2} = \frac{3 \pm 5}{2} \begin{cases} x_1 = 4 \\ x_2 = -1 \end{cases} \Big| \Rightarrow$

$\Rightarrow$ sol. staționare: $\varphi_1 : \mathbb{R} \to \mathbb{R}$, $\varphi_1(t) = 4$; $\varphi_2 : \mathbb{R} \to \mathbb{R}$, $\varphi_2(t) = -1$

- $x \in \mathbb{R} \setminus \{4, -1\}$ $(b(x) \neq 0) \Rightarrow$ sep. variabilele:

$$\frac{dx}{x^2-3x-4} = \frac{(t+1)\,dt}{t^2+2t+3}$$

$$\int \frac{dx}{x^2-3x-4} = \int \frac{dx}{(x-4)(x+1)} = \frac{1}{5}\int \frac{(x+1)-(x-4)}{(x-4)(x+1)}\,dx$$

$$= \frac{1}{5}\left(\int \frac{1}{x-4}\,dx - \int \frac{1}{x+1}\,dx\right) =$$

$$= \frac{1}{5}\left(\ln|x-4| - \ln|x+1|\right) + C \Rightarrow$$

$$\Rightarrow \boxed{B(x) = \frac{4}{5} \ln \left| \frac{x-4}{x+1} \right|}$$

$$\int \frac{(t+1)\,dt}{t^2+2t+3} = \frac{1}{2} \int \frac{(t^2+2t+3)'}{t^2+2t+3}\,dt =$$

$$= \frac{1}{2} \ln |\underbrace{t^2+2t+3}_{>0,\ \forall t \in \mathbb{R}}| + C \Rightarrow$$

$$\Rightarrow \boxed{A(t) = \frac{1}{2} \ln (t^2+2t+3)}$$

Mult. soluțiilor implicite:

$$\frac{1}{5} \ln \left| \frac{x-4}{x+1} \right| = \frac{1}{2} \ln (t^2+2t+3) + \frac{1}{5} \ln C \quad \Big| \cdot 5$$

$$C > 0$$

$$\Rightarrow \ln \left| \frac{x-4}{x+1} \right| = \ln (t^2+2t+3)^{5/2} + \ln C$$

$$\left| \frac{x-4}{x+1} \right| = C \sqrt{(t^2+2t+3)^5}$$

$$\frac{x-4}{x+1} = \underbrace{\pm C}_{C_1 \in \mathbb{R}^*} \sqrt{(t^2+2t+3)^5} \Rightarrow$$

$$\Rightarrow x\left(1 - C_1 \sqrt{(t^2+2t+3)^5}\right) = C_1 \sqrt{(t^2+2t+3)^5} + 4.$$

$$\Rightarrow \boxed{x(t) = \frac{C_1 \sqrt{(t^2+2t+3)^5} + 4}{1 - C_1 \sqrt{(t^2+2t+3)^5}}, \quad C_1 \in \mathbb{R}^*}$$

Temă: 1, 2, 3, 6, 9, 10.

2) dacă $\boxed{d \neq 0}$, atunci prin schimbarea de variabile: $\boxed{\begin{cases} t = s + t_0 \\ x = y + x_0 \end{cases}}$; $(t, x) \longrightarrow (s, y)$

unde (t_0, x_0) este soluția sistemului liniar:

$$\begin{cases} a_1 t + b_1 x + c_1 = 0 \\ a_2 t + b_2 x + c_2 = 0 \end{cases}$$

adică:

$$\begin{cases} a_1 t_0 + b_1 x_0 + c_1 = 0 \\ a_2 t_0 + b_2 x_0 + c_2 = 0 \end{cases}$$

ec. (12) devine o ecuație omogenă în (s, y).