Grupa 341, EDDP, Seminar 4, 27.10.2020

I $\quad x' + p(t)x = q(t)$; $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

a) $\left.\begin{array}{l}\varphi_1(t) = t\\ \varphi_2(t) = t\sin t\end{array}\right\}$ soluţii $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} t' + p(t)\cdot t = q(t) \\ (t\sin t)' + p(t)\cdot t\sin t = q(t) \end{cases} \Rightarrow$$

$$\Rightarrow 1 + p(t)t = \sin t + t\cos t + p(t)\cdot t\sin t \Rightarrow$$

$$\Rightarrow p(t)(t - t\sin t) = \sin t + t\cos t - 1$$

$$\boxed{p(t) = \frac{\sin t + t\cos t - 1}{t - t\sin t}} = \frac{-(t - t\sin t)'}{t - t\sin t}$$

$$q(t) = \overset{1-\sin t}{1} + \frac{\sin t + t\cos t - 1}{\cancel{t}(1-\sin t)}\cdot\cancel{t} = \frac{\cancel{1} - \cancel{\sin t} + \cancel{\sin t} + t\cos t - \cancel{1}}{1 - \sin t}$$

$$\boxed{q(t) = \frac{t\cos t}{1 - \sin t}}$$

b) $x' = \underbrace{-p(t)}_{a(t)}x + \underbrace{q(t)}_{b(t)}$

ec. afină (liniară neomogenă). $\Rightarrow$

$$\Rightarrow x(t) = \bar{x}(t) + \varphi_0(t)$$

$\bar{x}(t)$: ec. liniară omogenă ataşată; $\varphi_0(t)$: o soluţie particulară (φ_1 sau φ_2)

$$\frac{d\bar{x}}{dt} = \underbrace{\frac{(t - t\sin t)'}{t - t\sin t}}_{a(t)}\bar{x} \Rightarrow \bar{x}(t) = c\,e^{A(t)}$$

$$\int \frac{(t - t\sin t)'}{t - t\sin t}\,dt = \underbrace{\ln|t - t\sin t|}_{A(t)} + c$$

$\Rightarrow$

$$\Rightarrow \bar{x}(t) = c\,|t(1-\sin t)| \underset{t\in(0,\frac{\pi}{2})}{=} c\,t(1-\sin t)$$

Luăm $\varphi_0(t) = t \Rightarrow \boxed{x(t) = c\,t(1-\sin t) + t,\ c \in \mathbb{R}}$

(II) $x' = -\frac{2t}{t^5-1}x^2 + \frac{t^4}{t^5-1}x + \frac{3t^2}{t^5-1}$, $t \in (1,+\infty)$, $x \in \mathbb{R}$.

ec. Riccati

a) $\varphi_0(t) = n t^m$, $n, m \in \mathbb{R}$.

$(nt^m)' = -\frac{2t}{t^5-1}\cdot n^2 t^{2m} + \frac{t^4}{t^5-1}\cdot n t^m + \frac{3t^2}{t^5-1}$

$(t^5-1)(n m t^{m-1}) = -2n^2 t^{2m+1} + n t^{m+4} + 3t^2$

$\boxed{n m t^{m+4} - n m t^{m-1} = -2n^2 t^{2m+1} + n t^{m+4} + 3t^2}$

Dacă $m+4 \neq 2$ şi $m-1 \neq 2$ şi $2m+1 \neq 2$, atunci identif coef $\Rightarrow 0 = 3$ fals.

c1) $\boxed{m+4=2} \Rightarrow \boxed{m=-2}$

$-2n t^2 + 2n t^{-3} = -2n^2 t^{-3} + n t^2 + 3t^2$

Identif coef $\Rightarrow -2n = n+3 \Rightarrow -3n = 3 \Rightarrow \boxed{n=-1}$

$2n = -2n^2$

$n=-1$ verifica ✓

$\Rightarrow \boxed{\varphi_0(t) = -t^{-2} = -\frac{1}{t^2}}$

c2) $\boxed{2m+1=2} \Rightarrow \boxed{m=\frac{1}{2}} \Rightarrow$

$\Rightarrow \frac{n}{2} t^{\frac{9}{2}} - \frac{n}{2} t^{-\frac{1}{2}} = -2n^2 t^2 + n t^{\frac{9}{2}} + 3t^2 \Rightarrow$

$\Rightarrow \begin{cases} \frac{n}{2} = n \\ -\frac{n}{2} = 0 \\ -2n^2+3 = 0 \end{cases}$ $\left.\begin{matrix}\frac{n}{2}=n\\-\frac{n}{2}=0\end{matrix}\right\} \Rightarrow n=0$ $\Big\} \Rightarrow 3=0$ Fals!

c3) $\boxed{2 = m-1} \Rightarrow \boxed{m=3}$

$3n t^7 - 3n t^2 = -2n^2 t^7 + n t^7 + 3t^2$

$\begin{cases} 3n = -2n^2 + n \\ -3n = 3 \Rightarrow \boxed{n=-1} \end{cases}$ $-3 = -2 + (-1)$. Adev

$\boxed{\varphi_0(t) = -t^3}$

b) se face schimbarea de variabila.

$(t,x) \xrightarrow[x = y + (-t^3)]{} (t,y)$

$x(t) = y(t) - t^3$

(I) c) $x(\pi) = 2\pi$

$C\pi(1 - \underbrace{\sin\pi}_{0}) + \pi = 2\pi$

$C\pi = \pi \Rightarrow C = 1$

$\Rightarrow x(t) = 2t - t\sin t$

$$(y-t^3)' = \frac{-2t}{t^5-1}(y-t^3)^2 + \frac{t^4}{t^5-1}(y-t^3) + \frac{3t^2}{t^5-1} \quad \Big| \cdot t^5-1$$

$$(t^5-1)y' - 3t^2(t^5-1) = -2t(y^2-2yt^3+t^6) + t^4(y-t^3)+3t^2$$

$$(t^5-1)y' - \cancel{3t^7} + \cancel{3t^2} = -2ty^2+4yt^4-\cancel{2t^7}+yt^4-\cancel{t^7}+\cancel{3t^2}$$

$$(t^5-1)y' = 5t^4y - 2ty^2$$

$$y' = \underbrace{\frac{5t^4}{t^5-1}}_{a(t)}y - \underbrace{\frac{2t}{t^5-1}}_{b_1(t)}y^2 \quad \text{ec. Bernoulli} \quad \alpha=2$$

Var 1 <u>cu met. var. constantei</u>

• ec. lin. omogenă atașată

$$\frac{d\bar{y}}{dt} = \frac{5t^4}{t^5-1}\bar{y} \Rightarrow \bar{y}(t) = C\cdot e^{A_1(t)}$$

$$\int \frac{5t^4}{t^5-1}dt = \int \frac{(t^5-1)'}{t^5-1}dt = \ln|t^5-1| + C \underset{t>1}{=}$$

$$= \ln(t^5-1)+C \Rightarrow$$

$$\Rightarrow A_1(t) = \ln(t^5-1) \Rightarrow$$

$$\Rightarrow \boxed{\bar{y}(t) = C(t^5-1)}$$

• det. $C:(1,+\infty)\to\mathbb{R}$ a.î. $\boxed{y(t) = C(t)\cdot(t^5-1)}$ să fie sol a ec. Bernoulli:

$$(C(t)(t^5-1))' = \frac{5t^4}{\cancel{t^5-1}}\cdot C(t)\cancel{(t^5-1)} - \frac{2t}{\cancel{t^5-1}}\cdot C^2(t)(t^5-1)^2$$

$$C'(t)\cancel{(t^5-1)} + \cancel{C(t)\cdot 5t^4} = \cancel{5t^4C(t)} - 2tC^2(t)\cancel{(t^5-1)}$$

$$\frac{dC}{dt} = \underbrace{-2t}_{a_2(t)}\underbrace{C^2}_{b_2(C)}$$

• $b_2(C)=0 \Rightarrow C=0 \Rightarrow y(t)=0 \Rightarrow x(t) = -t^3$

• $C\neq 0 \Rightarrow C^{-2}dC = -2t\,dt$

$$\frac{C^{-1}}{-1} = -t^2+k \Rightarrow C(t) = \frac{1}{t^2-k} \Rightarrow$$

$$k\in\mathbb{R}$$

$$\Rightarrow y(t) = \frac{t^5-1}{t^2-k} \Rightarrow \boxed{x(t) = \frac{t^5-1}{t^2-k} - t^3, \quad k\in\mathbb{R}}$$

Obs: Pt $k=0 \Rightarrow x(t) = \cancel{t^3} - \frac{1}{t^2} - \cancel{t^3} = -\frac{1}{t^2}$.

var 2

$$\frac{dy}{dt} = \frac{5t^4}{t^5-1}y - \frac{2t}{t^5-1}y^2$$

cu schimbarea de variab: $y = z^{\frac{1}{1-\alpha}}$.

$y = z^{-1}$

$$(t, y) \xrightarrow{y=\frac{1}{z}} (t, z)$$

$$\left(\frac{1}{z}\right)' = \frac{5t^4}{t^5-1}\cdot\frac{1}{z} - \frac{2t}{t^5-1}\frac{1}{z^2}.$$

$$-\frac{1}{z^2}\cdot z' = \frac{5t^4}{t^5-1}\cdot\frac{1}{z} - \frac{2t}{t^5-1}\cdot\frac{1}{z^2} \quad \Big| : z^{-2}$$

$$-z' = \frac{5t^4}{t^5-1}\cdot z - \frac{2t}{t^5-1} \Rightarrow \boxed{\frac{dz}{dt} = \frac{-5t^4}{t^5-1}z + \frac{2t}{t^5-1}}$$

ec. afină

cu sol. particulară

cu var. const.

$$(t, x) \xrightarrow{x=y-t^3} (t, y) \xrightarrow{y=\frac{1}{z}} (t, z)$$

$\varphi_0(t) = -\frac{1}{t^2}$

$\psi_0(t) = \varphi_0(t) + t^3$
$= -\frac{1}{t^2} + t^3 = \frac{t^5-1}{t^2}$

$\theta_0(t) = \frac{t^2}{t^5-1}$

ec. liniară omogenă atașată:

$$\frac{d\bar{z}}{dt} = -\frac{5t^4}{t^5-1}\bar{z}$$

$$\bar{z}(t) = C\cdot e^{-\ln(t^5-1)} = C e^{\ln(t^5-1)^{-1}} = \frac{C}{t^5-1}$$

$$\Rightarrow \quad z(t) = \bar{z}(t) + \theta_0(t) = \frac{C}{t^5-1} + \frac{t^2}{t^5-1} \Rightarrow$$

$$\Rightarrow \quad y(t) = \frac{1}{z(t)} = \frac{t^5-1}{t^2+C} \Rightarrow x(t) = \frac{t^5-1}{t^2+C} - t^3.$$

$C \in \mathbb{R}$.

<u>Aplicații (exerciții)</u>

I) Se cere mulț. sol. ecuațiilor:

a) $x' - \frac{4x}{t} = t\sqrt{x}$, $t, x > 0$

b) $x' = \frac{x}{t} + \frac{1}{x^2t^2}$, $t, x > 0$

c) $x' + xt = x^2 \sin t$, $t \in (0, \frac{\pi}{2})$, $x \in \mathbb{R}$.

II). Fie ec: $2(t - t^2\sqrt{t})x' + 2\sqrt{t}\, x^2 - x - t = 0$, $t > 0$, $x \in \mathbb{R}$. (1)

a) Determinați $m \in \mathbb{R}$ a.î. $\varphi_0(t) = mt$ să fie soluție a ecuației.

b) Determinați mulțimea soluțiilor ecuației (1).

III). Să se determine mulțimea sol. ec. dif. implicite:

1) $t + x = \left(\frac{x'+1}{x'-1}\right)^2$

V 2) $x = 2tx' - (x')^2$

3) $x = t(1 + x') + (x')^2$

V 4) $x = tx' + \frac{1}{(x')^2}$.

5) $x = tx' - 2(1 + (x')^2)$.

III 2) $x = 2tx' - (x')^2$; $x = g(t, x')$

ec. Lagrange: $x = t \cdot \varphi(x') + \psi(x')$

$\varphi, \psi : I \subset \mathbb{R} \to \mathbb{R}$ derivabile.

Notăm: $p = x' \overset{ec}{\Rightarrow} x = 2tp - p^2$

Derivăm ecuația în raport:

$$x' = 2t' \cdot x' + 2t(x')' - 2x' \cdot (x')'$$

Cum $x' = p \Rightarrow p = 2p + \underbrace{2t\, p' - 2p \cdot p'}$ $\Rightarrow$

$$\Rightarrow 2p'(p - t) = p \Rightarrow \frac{dp}{dt} = \frac{p}{2(p-t)} \Rightarrow$$

$\Rightarrow$ ec. răsturnată: $\frac{dt}{dp} = \frac{2(p-t)}{p}$

$$\frac{dt}{dp} = -\frac{2}{p}t + 2$$

Căutăm sol. part. de forma $\varphi_0(p) = \alpha p$ $\Big\} \Rightarrow$

$\Rightarrow (\alpha p)' = -\frac{2}{\cancel{p}} \cdot \alpha \cancel{p} + 2 \Rightarrow \alpha = -2\alpha + 2$

$3\alpha = 2 \Rightarrow \alpha = \frac{2}{3} \Rightarrow$

$$\Rightarrow \boxed{\varphi_0(p) = \frac{2}{3}p}$$

• ec. liniară omog. atașată: $\frac{d\bar{t}}{dp} = -\frac{2}{p}\bar{t} \Rightarrow \bar{t}(p) = c\, e^{-2\ln|p|} \Rightarrow$

$$\bar{t}(p) = C e^{\ln\left(\frac{1}{p^2}\right)} = \frac{C}{p^2} \Rightarrow t(p) = \frac{C}{p^2} + \frac{2}{3}p, \quad C \in \mathbb{R}$$

Sol. parametrice ale ec:

$$\begin{cases} x = 2tp - p^2 \\ t = \frac{C}{p^2} + \frac{2}{3}p \end{cases}, \quad C \in \mathbb{R} \quad (p = \text{parametru})$$

3) $x = tx' + \frac{1}{(x')^2}$

$\varphi(x') = x'$ (ec. Clairaut)

$\psi(x') = \frac{1}{(x')^2} = (x')^{-2}$

$p = x' \Rightarrow x = tp + \frac{1}{p^2}$

derivăm ec în funcție de t $\Rightarrow$ $\cancel{x'} = \cancel{x'} + t(x')' + (-2)(x')^{-3}(x')'$

$$0 = tp' - \frac{2}{p^3} \cdot p' \Rightarrow p'\left(t - \frac{2}{p^3}\right) = 0.$$

$x' = p$

c1) $p' = 0 \Rightarrow p = C_1 \Rightarrow x' = C_1 \Rightarrow x = \int C_1 \, dt = C_1 t + C_2$

$\frac{dx}{dt} = C_1$

$C_1, C_2 \in \mathbb{R}$.

x înloc în ec $\Rightarrow$ $\cancel{C_1 t} + C_2 = \cancel{t \cdot C_1} + \frac{1}{C_1^2} \Rightarrow$

$\Rightarrow C_2 = \frac{1}{C_1^2} \ (= \psi(C_1))$

$$\Rightarrow \boxed{x = C_1 t + \frac{1}{C_1^2}, \quad C_1 \in \mathbb{R}^*}$$

c2) $t - \frac{2}{p^3} = 0 \Rightarrow t = \frac{2}{p^3} \Rightarrow$ o sol. parametrică $\Rightarrow$

$$\Rightarrow \boxed{\begin{cases} x = tp + \frac{1}{p^2} \\ t = \frac{2}{p^3} \end{cases}} \Rightarrow p = \sqrt[3]{\frac{2}{t}} \Bigg\} \Rightarrow$$

$$\Rightarrow \boxed{x(t) = t\sqrt[3]{\frac{2}{t}} + \sqrt[3]{\frac{t^2}{4}}}$$

Temă: I, II, III (1, 3, 5)