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① Se cer soluțiile problemelor Cauchy:

a) $\begin{cases} (\partial_1 u)^2 - 2(\partial_1 u)(\partial_2 u) + 2(\partial_2 u)^2 - 4u = 0 \\ u(x_1, x_2) = \frac{x_2^2}{2} \text{ pe } S = \{x \in \mathbb{R}^2 \mid x_1 = 0\} \end{cases}$

b) $\begin{cases} (\partial_1 u)^2 + (\partial_2 u)^2 + (\partial_1 u)(\partial_2 u) - x_1(\partial_2 u) - x_1(\partial_1 u) + x_1 x_2 - 2u = 0. \\ u(x_1, x_2) = \frac{x_2^2}{2} \text{ pe } S = \{x \in \mathbb{R}^2 \mid x_1 = -1\}. \end{cases}$

✓ c) $\begin{cases} x_1^2 + x_2^2 + \frac{1}{2}(\partial_1 u)^2 + \frac{1}{2}(\partial_2 u)^2 - 3u = 0. \\ u(\underset{\alpha_1(s)}{s}, \underset{\alpha_2(s)}{2}) = \underbrace{s^2+4}_{\varphi(s)}, \ s > 0. \end{cases}$

d) $\begin{cases} x_2(\partial_2 u) - (\partial_1 u)(\partial_2 u) + x_1 x_2 - u = 0 \\ u(1, s) = s, \ s \in \mathbb{R} \end{cases}$

e) $\begin{cases} x_2 \partial_1 u + (2x_1 - x_2)\partial_2 u = 4x_1(x_1 + x_2) \\ u(2s, -s) = s^2, \ s > 0 \end{cases}$

c) $F(x_1, x_2, u, \partial_1 u, \partial_2 u) = x_1^2 + x_2^2 + \frac{1}{2}(\partial_1 u)^2 + \frac{1}{2}(\partial_2 u)^2 - 3u.$

din cond inițială: $\begin{cases} \alpha_1(s) = s \\ \alpha_2(s) = 2 \end{cases}$; $\varphi(s) = s^2 + 4$, $s > 0$.

$p_1 = \partial_1 u$; $p_2 = \partial_2 u$

Se determină γ_1, γ_2, valorile inițiale pentru p_1 și p_2 pe S, rezolvând sistemul:

$$\begin{cases} F(\alpha_1(s), \alpha_2(s), \varphi(s), \gamma_1, \gamma_2) = 0 \\ \gamma_1 \alpha_1'(s) + \gamma_2 \cdot \alpha_2'(s) = \varphi'(s) \end{cases} \Rightarrow$$

$$\Rightarrow \left.\begin{cases} s^2 + 4 + \frac{1}{2}\gamma_1^2 + \frac{1}{2}\gamma_2^2 - 3(s^2+4) = 0 \\ \gamma_1 \cdot 1 + \gamma_2 \cdot 0 = 2s. \Rightarrow \boxed{\gamma_1 = 2s} \end{cases}\right\} \Rightarrow$$

$$\Rightarrow \cancel{s^2} + 4 + \frac{1}{\cancel{2}} \cdot \overset{2}{\cancel{4}} \cancel{s^2} + \frac{1}{2}\gamma_2^2 - \cancel{3s^2} - 12 = 0.$$

$$\frac{1}{2}\gamma_2^2 - 8 = 0 \Rightarrow \gamma_2^2 = 16 \Rightarrow \boxed{\gamma_2 = \pm 4}$$

Avem 2 cazuri: I) $\begin{cases}\gamma_1 = 21\\ \gamma_2 = 4\end{cases}$; II) $\begin{cases}\gamma_1 = 21\\ \gamma_2 = -4\end{cases}$

Pt. cele 2 cazuri ec din sistemul caracteristic sunt aceleași, doar condițiile inițiale se schimbă.

Considerăm I).

Sistemul caracteristic presupune să calculăm întâi derivatele parțiale pt $F(x_1,x_2,u,p_1,p_2) = x_1^2 + x_2^2 + \frac{1}{2}(p_1)^2 + \frac{1}{2}(p_2)^2 - 3u$:

$$\frac{\partial F}{\partial x_1} = 2x_1;\quad \frac{\partial F}{\partial x_2} = 2x_2;\quad \frac{\partial F}{\partial p_1} = \frac{1}{\cancel{2}}\cdot\cancel{2}p_1;\quad \frac{\partial F}{\partial p_2} = p_2;\quad \frac{\partial F}{\partial u} = -3$$

Sistemul caracteristic este:

$$\begin{cases}\dfrac{dx_1}{dt} = p_1 \\ \boxed{\dfrac{dx_2}{dt} = p_2} \\ \dfrac{dp_1}{dt} = -2x_1 - p_1(-3) \\ \boxed{\dfrac{dp_2}{dt} = -2x_2 - p_2(-3)} \\ \dfrac{du}{dt} = p_1\cdot p_1 + p_2\cdot p_2 \\ x_1(0) = 1 \\ x_2(0) = 2 \\ p_1(0) = 21 \\ p_2(0) = 4 \\ u(0) = 1^2 + 4.\end{cases}$$

(1) (2)

t = variab. independentă

x_1, x_2, p_1, p_2, u = variab dependente.

Sistemul, de obicei, se descompune în subsisteme care conțin doar o parte dintre variab x_1, x_2, u, p_1, p_2. Pt. sistemul alăturat avem de rezolvat sistemele liniare cu coef const:

$$(1)\ \begin{cases} x_1' = p_1 \\ p_1' = -2x_1 + 3p_1 \\ x_1(0) = 1 \\ p_1(0) = 21\end{cases}$$

$$(2)\ \begin{cases} x_2' = p_2 \\ p_2' = -2x_2 + 3p_2 \\ x_2(0) = 2 \\ p_2(0) = 4.\end{cases}$$

și apoi ec $\frac{du}{dt} = p_1^2 + p_2^2$ cu cond $u(0) = 1^2 + 4$.

$$(1):\ \begin{cases} x_1' = p_1 \\ p_1' = -2x_1 + 3p_1\end{cases} \Rightarrow \begin{pmatrix} x_1' \\ p_1'\end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & 3\end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ p_1\end{pmatrix}$$

A, sistem liniar cu coef const.

Sist liniar $\begin{pmatrix} x_1' \\ p_1' \end{pmatrix} = A \begin{pmatrix} x_1 \\ p_1 \end{pmatrix}$ în $\mathbb{R}^2$ i se asociază ec. de ord 2 cu coef const: $\underbrace{x_1'' = (\operatorname{tr} A) x_1' - (\det A) x_1}$

$$\left.\begin{aligned} \operatorname{tr} A &= 0+3=3 \\ \det A &= 0\cdot 3 - (-2)\cdot 1 = 2 \end{aligned}\right| \Rightarrow x_1'' = 3x_1' - 2x_1$$

se scrie ec. caracteristică:

$$r^2 = 3r - 2 \Rightarrow$$

$$\Rightarrow r^2 - 3r + 2 = 0$$

$$\Delta = 9 - 8 = 1$$

$$r_{1,2} = \frac{3 \pm 1}{2} \Big\langle \begin{matrix} r_1 = 2,\ m_1 = 1 \Rightarrow \varphi_1(t) = e^{2t} \\ r_2 = 1,\ m_2 = 1 \Rightarrow \varphi_2(t) = e^{t} \end{matrix}$$

Deci: $\underline{|x_1(t) = C_1 e^{2t} + C_2 e^{t},\ C_1, C_2 \in \mathbb{R}.|}$

p_1 se calculează din ec. în care nu se află $p_1' \Rightarrow$

$$\Rightarrow p_1 = x_1' = C_1 e^{2t} \cdot 2 + C_2 e^{t}$$

$$\underline{|p_1(t) = 2C_1 e^{2t} + C_2 e^{t}|}$$

Din $\left.\begin{aligned} x_1(0) &= s \\ p_1(0) &= 2s \end{aligned}\right| \Rightarrow \begin{cases} C_1 \cdot e^0 + C_2 e^0 = s \\ 2C_1 e^0 + C_2 e^0 = 2s \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = s \ \ |(-1) \\ 2C_1 + C_2 = 2s \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} -C_1 - C_2 = -s \\ 2C_1 + C_2 = 2s \end{cases} \quad (+)$$

$$C_1 \quad / \quad = s \Rightarrow \boxed{C_1 = s} \Rightarrow \cancel{s} + C_2 = \cancel{s}$$

$$\boxed{C_2 = 0} \Rightarrow$$

$$\Rightarrow \boxed{\begin{aligned} \tilde{x}_1(t, s) &= s e^{2t} \\ \tilde{p}_1(t, s) &= 2s e^{2t} \end{aligned}}$$

Pt. sistemul (2): $\begin{cases} x_2' = p_2 \\ p_2' = -2x_2 + 3p_2 \\ x_2(0) = 2 \\ p_2(0) = 4 \end{cases} \Rightarrow \begin{pmatrix} x_2' \\ p_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x_2 \\ p_2 \end{pmatrix} \Rightarrow$

aceeași matrice ca la sist (1)

$$\Rightarrow \underline{\begin{cases} x_2(t) = C_3 e^{2t} + C_4 e^{t} \\ p_2(t) = 2C_3 e^{2t} + C_4 e^{t} \end{cases}}$$

dar $\left.\begin{aligned} x_2(0) &= 2 \\ p_2(0) &= 4 \end{aligned}\right\} \Rightarrow \begin{cases} C_3 + C_4 = 2 \ |(-1) \\ 2C_3 + C_4 = 4 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} -C_3 - C_4 = -2 \\ 2C_3 + C_4 = 4 \end{cases} \quad (+)$$

$$C_3 \quad / \quad = 2$$

Deci $\boxed{C_3=2} \Rightarrow 2+C_4=2 \Rightarrow \boxed{C_4=0} \Rightarrow$

$$\Rightarrow \boxed{\begin{aligned}\tilde{x}_2(t,s) &= 2e^{2t}\\ \tilde{p}_2(t,s) &= 4e^{2t}\end{aligned}}$$

Ec. pt u: $\frac{du}{dt} = p_1^2+p_2^2$ devine. $\frac{du}{dt} = (2se^{2t})^2+(4e^{2t})^2$

$$\frac{du}{dt} = 4s^2e^{4t}+4^2e^{4t}$$

$$\frac{du}{dt} = 4(s^2+4)e^{4t}$$ este ec. de tip primitivă

$$u(t) = 4(s^2+4)\int e^{4t}dt) = \cancel{4}(s^2+4)\frac{e^{4t}}{\cancel{4}}+C_5 \Rightarrow$$

dar $u(0) = s^2+4$

$$\Rightarrow \cancel{4}(s^2+4)e^0+C_5 = s^2+4.$$

$$\Rightarrow \cancel{s^2+4}+C_5 = \cancel{s^2+4} \Rightarrow \boxed{C_5=0} \Rightarrow$$

$$\Rightarrow \boxed{\tilde{u}(t,s) = (s^2+4)e^{4t}}$$

Soluția parametrică a problemei este:

$$\begin{cases} x_1 = se^{2t}\\ x_2 = 2e^{2t}\\ u = (s^2+4)e^{4t}\end{cases}$$

Obs că $x_1^2+x_2^2 = s^2e^{4t}+4e^{4t} = (s^2+4)e^{4t}$ $\Rightarrow$

$$\Rightarrow \boxed{u(x_1,x_2) = x_1^2+x_2^2.}$$

<u>Temă</u>: a, b, d, e.

Verificare în prob:

$u(s,2) = s^2+2^2 = s^2+4$ (A)

în ec: $\partial_1 u = 2x_1$; $\partial_2 u = 2x_2$

Calc: $x_1^2+x_2^2+\frac{1}{\cancel{2}}\cancel{4}x_1^2+\frac{1}{\cancel{2}}\cancel{4}x_2^2-3x_1^2-3x_2^2 =$

$= 3x_1^2+3x_2^2-3x_1^2-3x_2^2 = 0$ Adev

La fel rezolvăm cazul II.

În cazul II) avem aceleași ec. în sistem, doar că s-au schimbat condițiile inițiale.

• pt $\left.\begin{aligned} x_1(t) &= C_1' e^{2t} + C_2' e^{t} \\ p_1(t) &= 2C_1 e^{2t} + C_2 e^{t} \\ \text{avem } &\begin{cases} x_1(0) = s \\ p_1(0) = 2s \end{cases} \end{aligned}\right\} \Rightarrow \boxed{\begin{cases} \tilde{x}_1(t,s) = s e^{2t} \\ \tilde{p}_1(t,s) = 2s e^{2t} \end{cases}}$

• pt $\left.\begin{aligned} &\begin{cases} x_2(t) = C_3 e^{2t} + C_4 e^{t} \\ p_2(t) = 2C_3 e^{2t} + C_4 e^{t} \end{cases} \\ &\text{avem } \begin{cases} x_2(0) = 2 \\ p_2(0) = -4 \end{cases} \end{aligned}\right| \Rightarrow \begin{cases} C_3 + C_4 = 2 \\ 2C_3 + C_4 = -4 \end{cases} \; (-)$

$C_3 \;/ = -6$

$\boxed{C_3 = -6} \Rightarrow$

$-6 + C_4 = 2$

$\boxed{C_4 = 8}$

$\Rightarrow \boxed{\begin{aligned} \tilde{x}_2(t,s) &= -6e^{2t} + 8e^{t} \\ \tilde{p}_2(t,s) &= -12e^{2t} + 8e^{t} \end{aligned}}$

ec. pt u: $\dfrac{du}{dt} = p_1^2 + p_2^2$

$\dfrac{du}{dt} = 4s^2 e^{4t} + (-12e^{2t} + 8e^{t})^2 \Rightarrow$

$\Rightarrow \dfrac{du}{dt} = 4s^2 e^{4t} + 144e^{4t} - 192e^{3t} + 64e^{2t}$

$u(t) = \cancel{4}s^2 \dfrac{e^{4t}}{\cancel{4}} + \cancel{144}^{36} \dfrac{e^{4t}}{\cancel{4}} - \cancel{192}^{64} \dfrac{e^{3t}}{\cancel{3}} + \cancel{64}^{32} \dfrac{e^{2t}}{\cancel{2}} + C_5$

$u(t) = s^2 e^{4t} + 36e^{4t} - 64e^{3t} + 32e^{2t} + C_5$

dar $u(0) = s^2 + 4 \Rightarrow \cancel{s^2} + 36 - 64 + 32 + C_5 = \cancel{s^2} + 4$

$\cancel{4} + C_5 = \cancel{4} \Rightarrow \boxed{C_5 = 0} \Rightarrow$

$\Rightarrow \boxed{\tilde{u}(t,s) = s^2 e^{4t} + 36e^{4t} - 64e^{3t} + 32e^{2t}}$

Ec. param.: $\begin{cases} x_1 = s e^{2t} \\ x_2 = 2(4e^{t} - 3e^{2t}) \\ u = s^2 e^{4t} + 4(9e^{4t} - 16e^{3t} + 8e^{2t}) \end{cases}$ (*)

$$\begin{cases} x_1 = s e^{2t} \\ x_2 = s(4e^t - 3e^{2t}) \end{cases} \Rightarrow x_2 = s e^t (4 - 3e^t)$$

$$\begin{cases} +6x_1 = 6s e^{2t} \\ s x_2 = 8s e^t - 6s e^{2t} \end{cases}$$

$$\frac{x_1}{x_2} = \frac{s \cancel{e^{2t}} e^t}{s \cancel{e^t}(4-3e^t)} \Rightarrow \frac{x_1}{x_2} = \frac{s e^t}{4-3e^t} \Rightarrow$$

$$\Rightarrow \boxed{s = \frac{x_1(4-3e^t)}{x_2 e^t}}$$

pentru cazul II, rămâne soluția parametrică (*).