Grupa 341, EDDP, Seminar 5, 03.11.2020

1) $t + x = \left(\frac{x'+1}{x'-1}\right)^2$

$x = -t + \left(\frac{x'+1}{x'-1}\right)^2$

derivăm $\Rightarrow x' = -1 + 2\,\frac{x'+1}{x'-1}\cdot\frac{(x'+1)'(x'-1)-(x'+1)(x'-1)'}{(x'-1)^2}$

notăm $x' = p \Rightarrow p = -1 + 2\,\frac{p+1}{p-1}\cdot\frac{p'(p-1)-(p+1)p'}{(p-1)^2} \Rightarrow$

$\Rightarrow p+1 = 2\,\frac{p+1}{p-1}\cdot\frac{p'(\cancel{p}-1-\cancel{p}-1)}{(p-1)^2}$

$p+1 = -\frac{4(p+1)}{(p-1)^3}\cdot p'$

I) $\boxed{p+1=0} \Rightarrow p=-1 \Rightarrow x'=-1$

$\frac{dx}{dt} = -1 \Rightarrow x = \int(-1)\,dt = -t+c$

ec. de tip primitivă

$\Rightarrow x(t) = -t+c,\ c\in\mathbb{R}$

verificăm ec: $-\cancel{t}+c = -\cancel{t} + \left(\frac{-1+1}{-1-1}\right)^2$

$c = 0. \Rightarrow$

$\Rightarrow \boxed{x(t) = -t}$ sol. particulară (1)

II) $\boxed{p+1\neq 0} \Rightarrow p' = \frac{(p-1)^3\cancel{(p+1)}}{-4\cancel{(p+1)}} \Rightarrow \frac{dp}{dt} = \frac{(p-1)^3}{-4}.$ $\Rightarrow$

ec. răsturnată

$\Rightarrow \frac{dt}{dp} = \frac{-4}{(p-1)^3}$

ec. de tip primitivă $\Rightarrow t = \int\frac{-4}{(p-1)^3}\,dp =$

$= -4\int(p-1)^{-3}\,dp =$

$= -\cancel{4}^{\,2}\,\frac{(p-1)^{-2}}{-\cancel{2}} + C \Rightarrow$

$\Rightarrow$ sol parametrică: $\begin{cases} t = \frac{2}{(p-1)^2} + c \\ x = -t + \left(\frac{p+1}{p-1}\right)^2 \end{cases}$ $c\in\mathbb{R}$ (2)

Mult. sol. ec (1) ∪ (2).

$$\begin{cases} (x'+1)' = x'' \\ \left((x'+1)^2\right)' = 2(x'+1)\cdot(x'+1)' = 2(x'+1)x'' \\ \left((x')^2\right)' = 2x'\cdot x'' \end{cases}$$

② Să se rezolve următoarele ecuații folosind reducerea ordinului:

a) $[1+(x')^2]\,x''' = 3x'(x'')^2$

b) $2x^{(2)}x^{(4)} - 3\left(x^{(3)}\right)^3 = 0$

c) $x^2 + (x')^2 - 2xx'' = 0$

d) $t^2x'' - 2txx' + tx' = 0$

e) $\left(\frac{x}{t}\right)^2 + (x')^2 = 3tx'' + \frac{2xx'}{t}$.

a) $\underbrace{[1+(x')^2]\,x''' - 3x'(x'')^2}=0$

$F(\cancel{t}, \cancel{x}, x', x'', x''')=0$. lipsesc derivatele până la ordinul $m=1$. ordinul ec este $k=3$

$$(t,x) \xrightarrow[(x'(t)=y(t))]{x'=y} (t,y)$$

$x''(t) = y'(t)$

$x'''(t) = y''(t)$

ec. în (t,y): $\underbrace{(1+y^2)y'' - 3y(y')^2 = 0}$,

$F_1(\cancel{t}, y, y', y'')=0$

$$(t,y) \xrightarrow[y'(t)=z(y(t))]{y'=z} (y,z)$$

$y''(t) = \frac{d}{dt}\left(z(y(t))\right) = \frac{dz}{dy}(y(t))\cdot y'(t) \Rightarrow$

$\Rightarrow \boxed{y'' = z'\cdot z}$

ec. în (y,z) este: $\boxed{(1+y^2)z'z - 3y\cdot z^2 = 0.}$

1) $\boxed{z=0} \Rightarrow y'=0 \Rightarrow y = C_1 \Rightarrow x' = C_1 \Rightarrow \boxed{x = C_1t + C_2,\ C_1, C_2 \in R}$ (3)

$x''' = 0$

$(x'')' = 0$

$x'' = C_1$

$(x')' = C_1$

$x' = C_1t + C_2$

$x = C_1\frac{t^2}{2} + C_2t + C_3$.

$C_1, C_2, C_3 \in R$.

2) $\boxed{z \neq 0} \Rightarrow (1+y^2)z' - 3yz = 0 \Rightarrow \frac{dz}{dy} = \frac{3yz}{1+y^2}$

$z' = \frac{dz}{dy}$

$\frac{dz}{dy} = \underbrace{\frac{3y}{1+y^2}}_{=a(y)} \cdot z$ ec. liniara omogena $\Rightarrow$

$\Rightarrow z(y) = C_1 \cdot e^{A(y)}$

$\int \frac{3y}{1+y^2} dy = 3 \cdot \frac{1}{2} \ln(1+y^2) + k =$

$= \underbrace{\ln\left((1+y^2)^{3/2}\right)}_{A(y)} + k \Rightarrow$

$\Rightarrow z(y) = C_1 \cdot (1+y^2)^{3/2}$
dar $z = y'$ $\Rightarrow y' = C_1 (1+y^2)^{3/2}$

ec. cu var sep.

$y' = \frac{dy}{dt}$; $a_1(t) = C_1$

$b_1(y) = (1+y^2)^{3/2}$

- $b_1(y) = 0 \Rightarrow 1+y^2 = 0 \Rightarrow$ nu are sol. reale.

$\Rightarrow b_1(y) \neq 0$, sep variab:

$$\frac{dy}{(1+y^2)^{3/2}} = C_1 dt.$$

$$\int \frac{dy}{\sqrt{(1+y^2)^3}} = I$$

$y = tg\, u \Leftrightarrow u = arctg\, y$

$dy = \frac{1}{\cos^2 u} du$

$1 + tg^2 u = \frac{1}{\cos^2 u}$

$$\int \frac{1}{\sqrt{\left(\frac{1}{\cos^2 u}\right)^3}} \frac{1}{\cos^2 u} du = \int \cos^3 u \frac{1}{\cancel{\cos^2 u}} du = \int \cos u\, du =$$

presupunem că: $\cos u > 0$

$= \sin u + k \Rightarrow$

$$\Rightarrow J = \underbrace{\sin(\operatorname{arctg} y)}_{B_1(y)} + k$$

$$\int C_1\, dt = C_1 t + C_2$$

Mult sol implicite pt ec. in y : $\sin(\operatorname{arctg} y) = C_1 t + C_2$

$$\operatorname{arctg} y = \arcsin(C_1 t + C_2)$$

$$\left.\begin{array}{l} y = \operatorname{tg}(\arcsin(C_1 t + C_2)),\ C_1, C_2 \in \mathbb{R} \\ \text{dar } x' = y \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} x' = \operatorname{tg}(\arcsin(C_1 t + C_2)) \\ \text{ec. de tip primitivă} \end{array}\right\} \Rightarrow$$

$$\Rightarrow x = \int \operatorname{tg}(\arcsin(C_1 t + C_2))\, dt = J.$$

$$\arcsin(C_1 t + C_2) = u$$

$$C_1 t + C_2 = \sin u$$

$$t = \frac{\sin u - C_2}{C_1} \quad ; C_1 \neq 0.$$

$$\boxed{dt = \frac{1}{C_1} \cos u\, du}$$

$$J = \int \operatorname{tg} u \cdot \frac{\cos u}{C_1} du = \frac{1}{C_1} \int \sin u\, du = -\frac{1}{C_1} \cos u + C_3$$

$$\Rightarrow x(t) = -\frac{1}{C_1} \cos(\arcsin(C_1 t + C_2)) + C_3, \quad C_1 \neq 0,\ C_1, C_2, C_3 \in \mathbb{R} \qquad (4)$$

Mult. sol ec: (3) $\cup$ (4).

---

e) $\left(\frac{x}{t}\right)^2 + (x')^2 - 3 t x'' - \frac{2 x x'}{t} = 0 \quad \rightarrow 2 \frac{x}{t} \cdot x'$

$F\left(\frac{x}{t}, x', t x''\right) = 0$. ec. omogenă de ordin 2.

$$(t, x) \xrightarrow[x(t) = t y(t)]{\frac{x}{t} = y} (t, y)$$

Scanned with CamScanner

$$\boxed{x' = y + ty'}$$

$$x'' = y' + y' + ty'' \quad | \cdot t \Rightarrow \boxed{tx'' = 2ty' + t^2 y''}$$

ec. în $(t,y) \Rightarrow \quad y^2 + (y+ty')^2 - 3(2ty' + t^2y'') - 2y(y+ty') = 0.$

$$\cancel{y^2} + \cancel{y^2} + \cancel{2yty'} + t^2(y')^2 - 6ty' - 3t^2y'' \cancel{- 2y^2} \cancel{- 2tyy'} = 0.$$

$$(ty')^2 - 6ty' - 3t^2y'' = 0.$$

$F_1(\cancel{y}, ty', t^2y'') = 0$ ec. Euler de ordin 2 neliniară.

$$(t,y) \xrightarrow{|t| = e^s} (s,z)$$

$$\Updownarrow$$

$$s = \ln|t|$$

$$s'(t) = \frac{1}{t}$$

$$y(t) = z(s(t))$$

$$y'(t) = z'(s(t)) \cdot s'(t) = z'(s(t)) \cdot \frac{1}{t} \Rightarrow \boxed{ty' = z'}$$

$$y''(t) = \left(z' \cdot \frac{1}{t}\right)' = z'' \cdot s'(t) \cdot \frac{1}{t} + z' \cdot \left(\frac{-1}{t^2}\right) \Rightarrow$$

$$y'' = z'' \cdot \frac{1}{t} \cdot \frac{1}{t} - z' \frac{1}{t^2} \quad | \cdot t^2 \Rightarrow \boxed{t^2 y'' = z'' - z'}$$

Ec. în (s,z): $(z')^2 - 6z' - 3(z'' - z') = 0$

$$(z')^2 - 6z' - 3z'' + 3z' = 0.$$

$$\underbrace{(z')^2 - 3z' - 3z'' = 0}_{F_2(\cancel{s}, \cancel{z}, z', z'') = 0.}$$

$$(s,z) \xrightarrow[z'(s) = w(s)]{z' = w} (s,w)$$

$$z''(s) = w'(s)$$

ec în (s,w): $\quad w^2 - 3w - 3w' = 0, \Rightarrow w' = \dfrac{w^2 - 3w}{3}.$

$$\frac{dw}{ds} = \frac{w^2 - 3w}{3}$$

ec. cu var separabile $a_2(s) = 1$; $b_2(w) = \dfrac{w^2 - 3w}{3}.$

• $b_2(w) = 0 \Rightarrow w(w-3) = 0 \begin{cases} w_1 = 0 \\ w_2 = 3 \end{cases}$

• $w = 0 \xRightarrow{z' = w} z' = 0 \Rightarrow z = C_1 \xRightarrow[s(t) = \ln|t|]{y(t) = z(s(t))} y = C_1 \xRightarrow{x = ty}$

$\Rightarrow \boxed{x(t) = C_1 t, \ C_1 \in \mathbb{R}}$ (5)

• $w = 3 \Rightarrow z' = 3 \Rightarrow z = \int 3\,ds = 3s + C_1 \Rightarrow$

$\Rightarrow y(t) = z(\ln|t|) = 3\ln|t| + C_1 \Rightarrow$

$\Rightarrow \boxed{x(t) = t(3\ln|t| + C_1), \ C_1 \in \mathbb{R}}$ (6)

• $b_2(w) \neq 0 \Rightarrow$ sep. variabilele $\dfrac{3}{w(w-3)}\,dw = ds$.

$$\int \frac{3}{w(w-3)}\,dw = \int \frac{w-(w-3)}{w(w-3)}\,dw = \int \frac{\cancel{w}}{\cancel{w}(w-3)}\,dw - \int \frac{\cancel{w-3}}{w\cancel{(w-3)}}\,dw = \int \frac{1}{w-3}\,dw - \int \frac{1}{w}\,dw =$$

$$= \ln|w-3| - \ln|w| + C =$$

$$= \underbrace{\ln\left|\frac{w-3}{w}\right|}_{B_2(w)} + C$$

$$\int ds = \underbrace{s}_{A_2(s)} + C$$

Forma implicită a sol: $\ln\left|\dfrac{w-3}{w}\right| = s + C \Rightarrow$

$\Rightarrow \left|\dfrac{w-3}{w}\right| = e^{s+C} \Rightarrow \dfrac{w-3}{w} = \underbrace{\pm e^{C}}_{C_1} e^{s} \Rightarrow$

$\Rightarrow w - 3 = C_1 w e^{s} \Rightarrow w(1 - C_1 e^{s}) = 3 \Rightarrow$

$w(s) = \dfrac{3}{1 - C_1 e^{s}} \xRightarrow[z' = w]{} \dfrac{dz}{ds} = \dfrac{3}{1 - C_1 e^{s}}$ $\Bigg\} \Rightarrow$

ec. de tip primitivă

$\Rightarrow z(s) = \displaystyle\int \frac{3}{1 - C_1 e^{s}}\,ds = 3\int \frac{(1 - C_1 e^{s}) + C_1 e^{s}}{1 - C_1 e^{s}}\,ds = 3\left(\int 1\,ds - \int \frac{(1 - C_1 e^{s})'}{1 - C_1 e^{s}}\,ds\right) \Rightarrow$

$$\Rightarrow \quad z(s) = 3\left(s - \ln|1 - C_1 e^s|\right) + C_2 =$$

$$= 3\left(\ln e^s - \ln|1 - C_1 e^s|\right) + C_2 \Rightarrow$$

$$\Rightarrow z(s) = 3 \ln\left(\frac{e^s}{|1 - C_1 e^s|}\right) + C_2 .$$

Dar $y(t) = z(\ln|t|)$, $|t| = e^s$ $\Rightarrow$ $y(t) = 3 \ln\left(\frac{|t|}{|1 - C_1|t||}\right) + C_2 \Rightarrow$

$$\Rightarrow x(t) = t\left(3 \ln\left|\frac{t}{1 - C_1|t|}\right| + C_2\right), \quad C_1, C_2 \in \mathbb{R}. \qquad (7)$$

Mulț. sol. ec.: $(5) \cup (6) \cup (7)$.

Temă: ② (b, c, d).

II (ex. 4)

$\varphi_0(t) = t$

$x = y + t$

$$2(t - t^2\sqrt{t})\cdot(y' + 1) + 2\sqrt{t}\,(y + t)^2 - y - t - t = 0 .$$

$$2(t - t^2\sqrt{t})y' + \cancel{2t} - \cancel{2t^2\sqrt{t}} + 2\sqrt{t}\,y^2 + 2ty\sqrt{t} + \cancel{2t^2\sqrt{t}} - y - \cancel{2t} = 0$$

$$2(t - t^2\sqrt{t})y' = (-2t\sqrt{t})\,y - 2\sqrt{t}\,y^2$$

ec. Bernoulli cu $\alpha = 2$