Grupa 341, Seminar 6, EDDP, 10.11.2020

① Se dă problema Cauchy: $\begin{cases} \frac{dx}{dt} = x + t \\ x(0) = 1 \end{cases}$ $\left(\begin{cases} \frac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}\right)$

Să se determine șirul aproximațiilor succesive.

$(\varphi_n)_{n\geq 0}: \begin{cases} \varphi_0(t) = x_0 \\ \varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\,ds \end{cases}, \forall n \in \mathbb{N}.$

Pt. problema: $f(t,x) = t + x$
$t_0 = 0$
$x_0 = 1$

$\varphi_0(t) = 1$

$\varphi_1(t) = 1 + \int_0^t f(s, \varphi_0(s))\,ds = 1 + \int_0^t (s + \varphi_0(s))\,ds =$

$= 1 + \int_0^t (s+1)\,ds = 1 + \frac{s^2}{2}\Big|_0^t + s\Big|_0^t \Rightarrow$

$$\boxed{\varphi_1(t) = 1 + \frac{t^2}{2} + t = (1+t) + \frac{t^2}{2}}$$

$\varphi_2(t) = 1 + \int_0^t f(s, \varphi_1(s))\,ds = 1 + \int_0^t (s + \varphi_1(s))\,ds =$

$= 1 + \int_0^t \left(s + (1+s) + \frac{s^2}{2}\right) ds =$

$= 1 + \left(s + \frac{s^2}{2}\right)\Big|_0^t + \left(\frac{s^2}{2} + \frac{s^3}{2\cdot 3}\right)\Big|_0^t \Rightarrow$

$$\boxed{\varphi_2(t) = \left(1 + t + \frac{t^2}{2!}\right) + \left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)}$$

$\varphi_3(t) = 1 + \int_0^t f(s, \varphi_2(s))\,ds =$

$= 1 + \int_0^t (s + \varphi_2(s))\,ds = 1 + \int_0^t \left[\left(s + \frac{s^2}{2!} + \frac{s^3}{3!}\right) + \left(1 + s + \frac{s^2}{2!}\right)\right] ds$

$= 1 + \left(\frac{s^2}{2} + \frac{s^3}{2!\cdot 3} + \frac{s^4}{3!\,4}\right)\Big|_0^t + \left(s + \frac{s^2}{2} + \frac{s^3}{2!\cdot 3}\right)\Big|_0^t =$

$= \left(\frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right) + \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right)$

Arătam prin inducție matematică:

$\varphi_n(t) = \left(\frac{t^2}{2!} + \frac{t^3}{3!} + \dots + \frac{t^{n+1}}{(n+1)!}\right) + \left(1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!}\right) \quad (1)$

Presupunem φ_n de forma (1) și dem că

$$\varphi_{n+1}(t)=\left(\frac{t^2}{2!}+\frac{t^3}{3!}+\dots+\frac{t^{n+2}}{(n+2)!}\right)+\left(1+t+\frac{t^2}{2}+\dots+\frac{t^{n+1}}{(n+1)!}\right)$$

Din recurență avem:

$$\varphi_{n+1}(t)=1+\int_0^t f(s,\varphi_n(s))\,ds=1+\int_0^t (s+\varphi_n(s))\,ds=$$

$$=1+\int_0^t\left[\left(s+\frac{s^2}{2!}+\frac{s^3}{3!}+\dots+\frac{s^{n+1}}{(n+1)!}\right)+\left(1+s+\frac{s^2}{2!}+\dots+\frac{s^n}{n!}\right)\right]ds$$

$$=1+\left(\frac{s^2}{2}+\frac{s^3}{2!\cdot 3}+\frac{s^4}{3!\cdot 4}+\dots+\frac{s^{n+2}}{(n+1)!(n+2)}\right)\Bigg|_0^t+\left(s+\frac{s^2}{2}+\frac{s^3}{2!3}+\dots+\frac{s^{n+1}}{n!(n+1)}\right)\Bigg|_0^t$$

$$=\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots+\frac{t^{n+1}}{(n+1)!}\right)+\left(\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\dots+\frac{t^{n+2}}{(n+2)!}\right)$$

<u>Temă</u>: aceeași cerință pt $\begin{cases} x'=xt \\ x(0)=1 \end{cases}$

② Fie prob. Cauchy: $\begin{cases} \dfrac{dx}{dt}=2t\cos x \\ x(0)=\dfrac{\pi}{4} \end{cases}$, $(t,x)\in[-1,1]\times\left[0,\frac{\pi}{2}\right]$.

a) Verif. ip TEU.

$f:\underbrace{[-1,1]\times\left[0,\frac{\pi}{2}\right]}_{D}\to\mathbb{R}$

$f(t,x)=2t\cos x$

$t_0=0$, $x_0=\frac{\pi}{4}$

$\left(0,\frac{\pi}{4}\right)\in D$

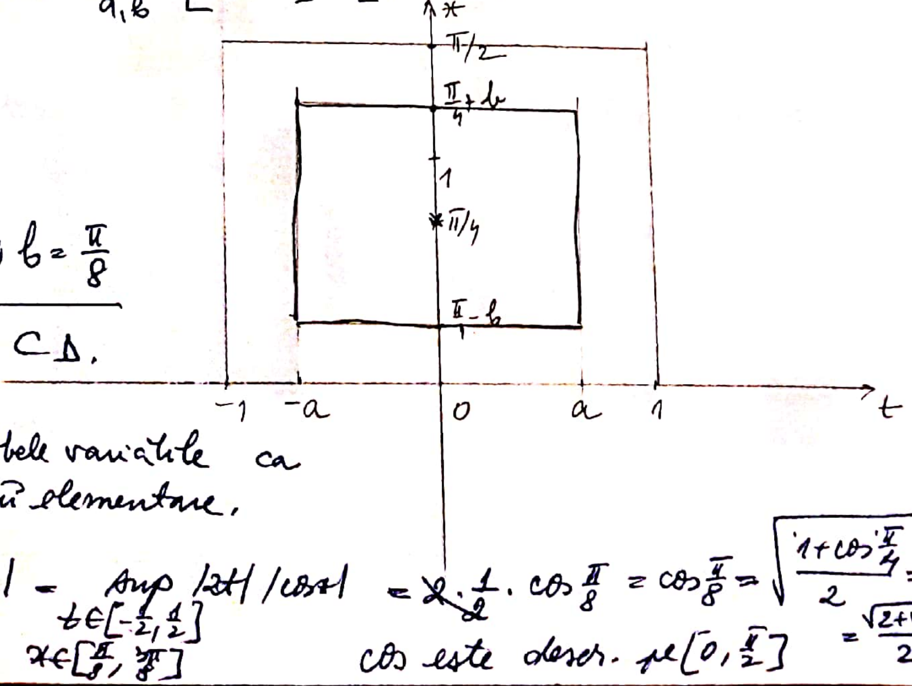
1) $\exists\, a,b>0$ a.î. $D_{a,b}=[-a,a]\times\left[\frac{\pi}{4}-b,\frac{\pi}{4}+b\right]\subset D$.

Pt $\begin{cases} a\in(0,1) \\ b\in\left(0,\frac{\pi}{4}\right) \end{cases}$

$D_{a,b}\subset D$.

De exemplu: $a=\frac{1}{2}$; $b=\frac{\pi}{8}$

$D_{\frac{1}{2},\frac{\pi}{8}}=\left[-\frac{1}{2},\frac{1}{2}\right]\times\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\subset D$.



2) f continuă în ambele variabile ca produs de funcții elementare.

$$M=\sup_{(t,x)\in D_{\frac{1}{2},\frac{\pi}{8}}}|f(t,x)|=\sup_{\substack{t\in[-\frac{1}{2},\frac{1}{2}]\\ x\in[\frac{\pi}{8},\frac{3\pi}{8}]}}|2t|\,|\cos x|=\cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\cos\frac{\pi}{8}=\cos\frac{\pi}{8}=\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}=\frac{\sqrt{2+\sqrt{2}}}{2}$$

cos este descr. pe $\left[0,\frac{\pi}{2}\right]$

sau pt $a\in(0,1)$, $b\in(0,\frac{\pi}{4})$ $\Rightarrow$ $M=\sup_{\substack{t\in[-a,a]\\ x\in[\frac{\pi}{4}-b,\,\frac{\pi}{4}+b]}} |2t|\,|\cos x| = \underline{2a\cos(\frac{\pi}{4}-b)}$

3) $\frac{\partial f}{\partial x}(t,x) = 2t(-\sin x)$ este continuă ca produs de funcții continue

$L=\sup_{(t,x)\in D_{\frac{1}{2},\frac{\pi}{8}}} \left|\frac{\partial f}{\partial x}(t,x)\right| = \sup_{\substack{t\in[-\frac{1}{2},\frac{1}{2}]\\ x\in[\frac{\pi}{8},\frac{3\pi}{8}]}} |2t|\,|\sin x| = \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\sin\frac{3\pi}{8} =$

sin este cresc. pe $[0,\frac{\pi}{2}]$

$=\cos\left(\frac{\pi}{2}-\frac{3\pi}{8}\right) = \cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$.

<u>Sau</u>:

$L = 2\cdot a\,\sin\left(\frac{\pi}{4}+b\right) = 2a\cos\left(\frac{\pi}{2}-\frac{\pi}{4}-b\right) = \underline{2a\cos\left(\frac{\pi}{4}-b\right)}$

Din 1), 2), 3) $\xrightarrow{TEU}$ $\forall\alpha\in\left(0,\min\left(a,\frac{b}{M}\right)\right)$, $\exists!\,\varphi:[-\alpha,\alpha]\to[\frac{\pi}{4}-b,\frac{\pi}{4}+b]$ soluție a prob. Cauchy.

b) $\varphi_0(t) = x_0 = \frac{\pi}{4}$; $f(t,x) = 2t\cos x$

$$\varphi_{n+1}(t) = \frac{\pi}{4} + \int_0^t \overbrace{2s\cos(\varphi_n(s))}^{f(s,\varphi_n(s))}\,ds,\quad n\in\mathbb{N}$$

$$\varphi_1(t) = \frac{\pi}{4} + \int_0^t 2s\cos\varphi_0(s)\,ds = \frac{\pi}{4} + \int_0^t 2s\cdot\cos\frac{\pi}{4}\,ds =$$
$$= \frac{\pi}{4} + \cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}\int_0^t s\,ds = \frac{\pi}{4} + \sqrt{2}\,\frac{s^2}{2}\Big|_0^t \Rightarrow$$

$$\boxed{\varphi_1(t) = \frac{\pi}{4} + \sqrt{2}\,\frac{t^2}{2}}$$

$$\varphi_2(t) = \frac{\pi}{4} + \int_0^t 2s\cos\varphi_1(s)\,ds = \frac{\pi}{4} + \int_0^t 2s\cos\left(\frac{\pi}{4}+\frac{\sqrt{2}}{2}s^2\right)ds =$$

$\sqrt{2}\cdot\sqrt{2}s$

$$= \frac{\pi}{4} + \int_{\frac{\pi}{4}}^{\frac{\pi}{4}+\frac{\sqrt{2}}{2}t^2} \sqrt{2}\cdot\cos u\,du = \frac{\pi}{4} + \sqrt{2}\sin u\Big|_{\frac{\pi}{4}}^{\frac{\pi}{4}+\frac{\sqrt{2}}{2}t^2} \Rightarrow$$

$\frac{\pi}{4}+\frac{\sqrt{2}}{2}s^2 = u$

$du = \frac{\sqrt{2}}{\cancel{2}}\cdot\cancel{2}s\,ds = \sqrt{2}s\,ds$

s	0	t
u	$\frac{\pi}{4}$	$\frac{\pi}{4}+\frac{\sqrt{2}}{2}t^2$

$$\Rightarrow \varphi_2(t) = \frac{\pi}{4} + \sqrt{2}\Big(\sin\Big(\frac{\pi}{4}+\frac{\sqrt{2}}{2}t^2\Big) - \underbrace{\sin\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}}\Big) \Rightarrow$$

$$\Rightarrow \boxed{\varphi_2(t) = \frac{\pi}{4} - 1 + \sqrt{2}\sin\Big(\frac{\pi}{4}+\frac{\sqrt{2}}{2}t^2\Big)}$$

$$\varphi_3(t) = \frac{\pi}{4} + \int_0^t 2s\cos\underbrace{\varphi_2(s)}\,ds =$$

$$= \frac{\pi}{4} + \int_0^t 2s\cos\Big(\frac{\pi}{4} - 1 + \sqrt{2}\sin\Big(\frac{\pi}{4}+\frac{\sqrt{2}}{2}s^2\Big)\Big)\,ds.$$

c) Soluția problemei Cauchy:

Integrăm ec.: $\frac{dx}{dt} = \underbrace{2t}_{a(t)}\,\underbrace{\cos x}_{b(x)}$

- $b(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2} \Rightarrow x(t) = \frac{\pi}{2}$, $x \in [0, \frac{\pi}{2}]$

  dar nu e sol. a prob. Cauchy pt că $x(0) = \frac{\pi}{2} \neq 1$

- $b(x) \neq 0 \Rightarrow x \in [0, \frac{\pi}{2})$

  separăm variabilele:

$$\frac{dx}{\cos x} = 2t\,dt$$

$$J = \int \frac{dx}{\cos x} \underset{\uparrow}{=} \int \frac{1+tg^2\frac{x}{2}}{1-tg^2\frac{x}{2}}\,dx = -\int \frac{1+tg^2\frac{x}{2}}{tg^2\frac{x}{2}-1}\,dx = -2\int \frac{\frac{1}{2\cos^2\frac{x}{2}}}{tg^2\frac{x}{2}-1}\,dx$$

$$\boxed{\cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}}$$

$$tg\frac{x}{2} = u$$

$$\frac{1}{\cos^2\frac{x}{2}}\Big(\frac{x}{2}\Big)'dx = du \Rightarrow \frac{1}{2\cos^2\frac{x}{2}}dx = du$$

$$1 + tg^2\frac{x}{2} = 1 + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1}{\cos^2\frac{x}{2}}$$

$$-2\int\frac{du}{u^2-1} = -\cancel{2}\cdot\frac{1}{\cancel{2}\cdot 1}\ln\Big|\frac{u-1}{u+1}\Big| + c = \ln\Big|\frac{u+1}{u-1}\Big| + c$$

$$J = \underbrace{\ln\Big|\frac{tg\frac{x}{2}+1}{tg\frac{x}{2}-1}\Big|}_{B(x)} + c$$

$$\int 2t\,dt = t^2 + C \Rightarrow A(t) = t^2$$

Soluțiile implicite: $\ln\Big|\frac{tg\frac{x}{2}+1}{tg\frac{x}{2}-1}\Big| = t^2 + C \quad (2) \Rightarrow$

→ pt $t=0$, $x=\frac{\pi}{4}$ $\Big|$ $\Rightarrow \ln\left|\frac{tg\frac{\pi}{8}+1}{tg\frac{\pi}{8}-1}\right| = 0^2+c \Rightarrow c = \ln\left(\frac{1+tg\frac{\pi}{8}}{1-tg\frac{\pi}{8}}\right)$

$\Downarrow$

prob. Cauchy are sol în forma implicită (2)

$$\left|\frac{tg\frac{x}{2}+1}{tg\frac{x}{2}-1}\right| = e^{t^2}\cdot e^{c} \Rightarrow \frac{tg\frac{x}{2}+1}{tg\frac{x}{2}-1} = \underbrace{\pm e^{c}}_{c_1\in\mathbb{R}^*}\cdot e^{t^2} \Rightarrow$$

$$\Rightarrow tg\frac{x}{2}+1 = c_1 e^{t^2}\left(tg\frac{x}{2}-1\right)$$

$$tg\frac{x}{2}\left(c_1e^{t^2}-1\right) = 1+c_1e^{t^2}$$

$$tg\frac{x}{2} = \frac{1+c_1e^{t^2}}{c_1e^{t^2}-1}$$

$$x\in\left[0,\frac{\pi}{2}\right) \Rightarrow \frac{x}{2}\in\left[0,\frac{\pi}{4}\right)\subset\underbrace{\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}_{\text{intervalul pe care se inversează } tg}$$

$\Bigg| \Rightarrow x = 2\,arctg\left(\frac{1+c_1e^{t^2}}{c_1e^{t^2}-1}\right)$, $c_1\in\mathbb{R}^2$

Din cond $x(0)=\frac{\pi}{4} \Rightarrow \frac{\pi}{4} = 2\,arctg\left(\frac{1+c_1\cdot e^0}{c_1e^0-1}\right) \Rightarrow$

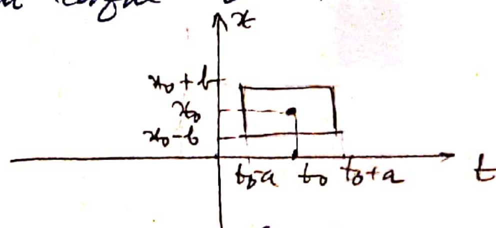
$$\Rightarrow \frac{\pi}{8} = arctg\left(\frac{1+c_1}{c_1-1}\right) \Rightarrow \frac{1+c_1}{c_1-1} = tg\frac{\pi}{8} \Rightarrow$$

$$\Rightarrow c_1 = \frac{1+tg\frac{\pi}{8}}{tg\frac{\pi}{8}-1} \qquad \pm e^{c} = c_1$$

$$c_1<0 \Rightarrow -e^{c} = c_1 \Rightarrow c = \ln(-c_1) = \ln\left(\frac{1+tg\frac{\pi}{8}}{1-tg\frac{\pi}{8}}\right)$$

③ Fie problema Cauchy: $\begin{cases} x' = 3\sqrt[3]{x^2} \\ x(t_0)=x_0 \end{cases}$, $(t,x)\in\mathbb{R}^2$

a) Verificați dacă sunt îndeplinite ip. TEU pt $x_0\neq 0$.
b) Câte soluții are problema în cazul $x_0=0$?



a) $f(t,x) = 3\sqrt[3]{x^2}$; $f:\mathbb{R}^2\to\mathbb{R}$
$(t_0,x_0)\in\mathbb{R}^2$
$x_0\neq 0$

$\frac{\partial f}{\partial x}(t,x) = \cancel{3}\cdot\frac{2}{\cancel{3}}\,x^{\frac{2}{3}-1} = 2\,\frac{1}{\sqrt[3]{x}}$ nu e cont în punctele $\underline{(t,0)}$

$x_0 > 0$ $\quad \left.\begin{array}{l} b>0 \\ x_0 - b > 0 \end{array}\right| \Rightarrow b \in (0, x_0)$

$a > 0$

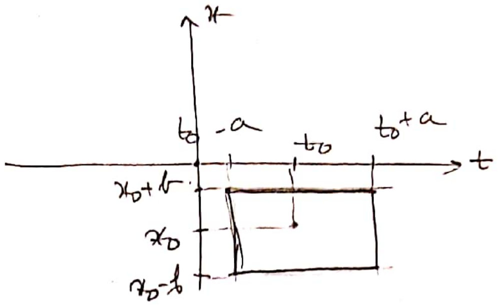
$x_0 < 0$

$b > 0$

$x_0 + b < 0$

$b < -x_0$

$b \in (0, -x_0)$



b) $\underline{x_0 = 0}$ $\Rightarrow \begin{cases} x' = 3\sqrt[3]{x^2} \\ x(t_0) = 0 . \end{cases}$

$$\frac{dx}{dt} = \underbrace{3\sqrt[3]{x^2}}_{b(x)} \cdot \underbrace{1}_{\overset{\|}{a(t)}}$$

cu var sep:

$\bullet\ b(x) = 0 \Rightarrow 3\sqrt[3]{x^2} = 0 \Rightarrow \boxed{x = 0} \Rightarrow \boxed{x(t) = 0}$.

verifică cond $x(t_0) = 0$.

$\bullet\ b(x) \neq 0 \Rightarrow \dfrac{dx}{3\sqrt[3]{x^2}} = 1\, dt$

$$\cancel{\frac{1}{3}} \frac{x^{-\frac{2}{3}+1}}{\cancel{-\frac{2}{3}+1}} = t + C$$

$$x^{\frac{1}{3}} = t + C \Rightarrow \boxed{\begin{array}{c} x(t) = (t + C)^3 \\ C \in \mathbb{R} \end{array}}$$

Cond $x(t_0) = 0 \to (t_0 + C)^3 = 0 \Rightarrow C = -t_0 \Rightarrow$

$$\Rightarrow \boxed{x(t) = (t - t_0)^3}$$

Temă: Sunt îndeplinite cond TEU pt $\begin{cases} x' = 3\sqrt[3]{x^2} \\ x(t_0) = 0 \end{cases}$?