Grupa 341, Seminar 9, EDDP, 08.12.2020

1) (6 din temă)

$$\begin{cases} x_1' = -x_1 + x_2 - 2x_3 \\ x_2' = 4x_1 + x_2 + e^{-t} \\ x_3' = 2x_1 + x_2 - x_3 \end{cases}$$

$$x' = Ax + b(t) \;;\; A = \begin{pmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix} \;;\; b(t) = \begin{pmatrix} 0 \\ e^{-t} \\ 0 \end{pmatrix}$$

• rezolvăm $\bar{x}' = A\bar{x}$:

$$\det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} -1-\lambda & 1 & -2 \\ 4 & 1-\lambda & 0 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow (-1-\lambda)^2(1-\lambda) - 8 + 0 + 4(1-\lambda) - 0 - 4(-1-\lambda) = 0$$

$$(1+\lambda)^2(1-\lambda) \cancel{-8} + \cancel{4} - \cancel{4\lambda} + \cancel{4} + \cancel{4\lambda} = 0 \Rightarrow \lambda_1 = -1,\ m_1 = 2$$
$$\lambda_2 = 1,\ m_2 = 1$$

$$\boxed{\lambda_1 = -1,\ m_1 = 2}$$

Obs: Dacă multiplicitatea coincide cu dimensiunea spațiului, adică, avem doar o valoare proprie λ_1, $m_1 = n$, atunci $(A - \lambda_1 I_n)^{m_1} = O_n$.

$p_0, p_1 \in \mathbb{R}^3$, nu amândoi nuli a.î. $\varphi(t) = (p_0 + p_1 t)e^{-t}$ soluție a sistemului $\bar{x}' = A\bar{x}$: $\varphi'(t) = A\varphi(t) \Rightarrow \begin{cases} -p_1 = Ap_1 \\ p_1 - p_0 = Ap_0 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} 0_{\mathbb{R}^3} = (A + I_3)p_1 \\ p_1 = (A + I_3)p_0 \end{cases} \Rightarrow (A + I_3)^2 p_0 = 0_{\mathbb{R}^3} \Rightarrow$$

$$\Rightarrow p_0 \in \operatorname{Ker}\left((A + I_3)^2\right)$$

$$(A + I_3)^2 = \begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 8 & 8 & -8 \\ 4 & 4 & -4 \end{pmatrix}$$

Determinăm $v \in \mathbb{R}$ a.î. $(A + I_3)^2 v = 0_{\mathbb{R}^3}$ $\Rightarrow$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 8 & 8 & -8 \\ 4 & 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 0 = 0 \\ 8(v_1 + v_2 - v_3) = 0 \\ 4(v_1 + v_2 - v_3) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow v_1 + v_2 - v_3 = 0 \Rightarrow v_3 = v_1 + v_2 \Rightarrow v = \begin{pmatrix} v_1 \\ v_2 \\ v_1 + v_2 \end{pmatrix} \Rightarrow$$

$$\Rightarrow v = \begin{pmatrix} v_1 \\ v_2 \\ v_1+v_2 \end{pmatrix} = v_1\begin{pmatrix}1\\0\\1\end{pmatrix} + v_2\begin{pmatrix}0\\1\\1\end{pmatrix} \Rightarrow \ker\left((A+I_3)^2\right) = \mathrm{Span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$$

$\Rightarrow$ • $p_0 = \begin{pmatrix}1\\0\\1\end{pmatrix} \Rightarrow p_1 = (A+I_3)p_0 = \begin{pmatrix}0 & 1 & -2\\ 4 & 2 & 0\\ 2 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\4\\2\end{pmatrix} \Rightarrow$

$$\Rightarrow \boxed{\varphi_1(t) = \left(\begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}-2\\4\\2\end{pmatrix}t\right)e^{-t}} = \boxed{\begin{pmatrix}(1-2t)e^{-t}\\ 4te^{-t}\\ (1+2t)e^{-t}\end{pmatrix}}$$

• $p_0 = \begin{pmatrix}0\\1\\1\end{pmatrix} \Rightarrow p_1 = \ldots$

(temă de continuat)

② Fie sistemul:
$$\begin{cases} x_1' = \frac{1}{t}(x_1 - 2x_2) + \ln t \\ x_2' = \frac{1}{t}(2x_1 - 3x_2) \end{cases}, \quad t>0 \qquad (1)$$

a) Arătați că prin s.v. $\boxed{t = e^s}$ se obține un sistem cu matrice constantă, pentru care determinați mulțimea soluțiilor.

b) Determinați mulțimea soluțiilor sist. (1).

a) $$\begin{pmatrix}x_1'\\x_2'\end{pmatrix} = \underbrace{\begin{pmatrix}\frac{1}{t} & -\frac{2}{t}\\ \frac{2}{t} & -\frac{3}{t}\end{pmatrix}}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \underbrace{\begin{pmatrix}\ln t\\0\end{pmatrix}}_{b(t)}$$

$$A(t) = \frac{1}{t}B, \quad B = \begin{pmatrix}1 & -2\\ 2 & -3\end{pmatrix}$$

$$\boxed{x' = \frac{1}{t}Bx + b(t)}$$

$$(t,x) \xrightarrow[t=e^s \Leftrightarrow s=\ln t,\ t>0]{} (s,y)$$

$x(t) = y(s(t))$

$s'(t) = \frac{1}{t}$

$x'(t) = y'(s(t))\cdot s'(t) = \frac{1}{t}y' \Rightarrow \boxed{x' = \frac{1}{e^s}y'}$

Sist: $\frac{1}{e^s}y' = \frac{1}{e^s}By + \begin{pmatrix}\ln(e^s)\\0\end{pmatrix} \Big| \cdot e^s$

$$\boxed{y' = By + \begin{pmatrix}se^s\\0\end{pmatrix}}$$

$\bar{y}' = B\bar{y}$; $\det(B - \lambda I_2) = 0$, $n=2$

$$\begin{vmatrix}1-\lambda & -2\\ 2 & -3-\lambda\end{vmatrix} = 0 \Rightarrow$$

$\Rightarrow -3-\lambda+3\lambda+\lambda^2+4=0 \Rightarrow \lambda^2+2\lambda+1=0 \Rightarrow (\lambda+1)^2=0$

$\lambda_1=-1,\ m_1=2=n \Rightarrow$

$\Rightarrow p_0, p_1 \in \mathbb{R}^2$, nu amândoi nuli a.î. $\varphi(t)=(p_0+p_1 s)e^{-s}$ sol. a sistemului $\bar{y}'=B\bar{y} \Rightarrow 0_{\mathbb{R}^2}=(B+I_2)p_1$

$p_1=\underbrace{(B+I_2)}_{(A-\lambda_1 I_2)}p_0 \Rightarrow$

$\Rightarrow (B+I_2)^2 p_0=0_{\mathbb{R}^2} \Rightarrow p_0 \in \ker((B+I_2)^2)$

$(B+I_2)^2=\begin{pmatrix}2&-2\\2&-2\end{pmatrix}\begin{pmatrix}2&-2\\2&-2\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}=0_2 \Rightarrow$

$\Rightarrow \ker((B+I_2)^2)=\mathbb{R}^2=\text{Sp gen}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\} \Rightarrow$

$\Rightarrow p_0=\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow p_1=\begin{pmatrix}2&-2\\2&-2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix} \Rightarrow$

$\Rightarrow \boxed{\varphi_1(s)=(p_0+p_1 s)e^{-s}=\begin{pmatrix}(1+2s)e^{-s}\\2se^{-s}\end{pmatrix}}$

$p_0=\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow p_1=\begin{pmatrix}2&-2\\2&-2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-2\\-2\end{pmatrix} \Rightarrow$

$\Rightarrow \boxed{\varphi_2(s)=\left(\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}-2\\-2\end{pmatrix}s\right)e^{-s}=\begin{pmatrix}-2se^{-s}\\(1-2s)e^{-s}\end{pmatrix}}$

$\phi(s)=\begin{pmatrix}(1+2s)e^{-s} & -2se^{-s}\\ 2se^{-s} & (1-2s)e^{-s}\end{pmatrix} \Rightarrow \bar{y}(s)=\phi(s)C,\ C\in\mathbb{R}^2.$

Aplicăm variația constantelor: determinăm $C:(0,\infty)\to\mathbb{R}^2$ a.î. $y(s)=\phi(s)C(s)$ sol. a sistemului $y'=By+\underbrace{\begin{pmatrix}se^{s}\\0\end{pmatrix}}_{\tilde{b}(s)}$

$\Rightarrow \phi(s)C'(s)=\tilde{b}(s) \Rightarrow$

$\Rightarrow \begin{pmatrix}(1+2s) & -2s\\ 2s & (1-2s)\end{pmatrix}\begin{pmatrix}C_1'\\C_2'\end{pmatrix}e^{-s}=\begin{pmatrix}se^{s}\\0\end{pmatrix} \Big| \cdot e^{s}$

$\Rightarrow \begin{cases}(1+2s)C_1'-2sC_2'=se^{2s}\\ 2sC_1'+(1-2s)C_2'=0\end{cases}\ (-)$

$C_1'-C_2'=se^{2s} \Rightarrow \boxed{C_1'=se^{2s}+C_2'}$

$2s(se^{2s}+\cancel{C_2'})+(1-\cancel{2s})C_2'=0$

$2s^2e^{2s}+C_2'=0 \Rightarrow \boxed{C_2'=-2s^2e^{2s}} \Rightarrow \boxed{C_1'=(s-2s^2)e^{2s}}$

ec. de tip primitivă $\Rightarrow$

$\Rightarrow C_2(s)=\int(-2s^2e^{2s})ds=-\int s^2(e^{2s})'ds \Rightarrow$

$$\Rightarrow C_2(s) = -s^2e^{2s} + 2\int s e^{2s}\,ds = -s^2e^{2s} + \int s\,(e^{2s})'\,ds =$$

$$= -s^2e^{2s} + se^{2s} - \int e^{2s}\,ds \Rightarrow$$

$$\boxed{C_2(s) = -s^2e^{2s} + se^{2s} - \frac{e^{2s}}{2} + K_1}$$

$$C_1(s) = \int (s-2s^2)\left(\frac{e^{2s}}{2}\right)'ds = (s-2s^2)\frac{e^{2s}}{2} - \frac{1}{2}\int (1-4s)\,e^{2s}\,ds$$

$$= (s-2s^2)\frac{e^{2s}}{2} - \frac{1}{2}\int (1-4s)\left(\frac{e^{2s}}{2}\right)'ds =$$

$$= (s-2s^2)\frac{e^{2s}}{2} - \frac{1}{2}\left((1-4s)\frac{e^{2s}}{2} + \frac{4}{2}\int e^{2s}\,ds\right) \Rightarrow$$

$$\Rightarrow \boxed{C_1(s) = (s-2s^2)\frac{e^{2s}}{2} - \frac{1}{4}(1-4s)e^{2s} - \frac{e^{2s}}{2} + K_1}$$

Deci:

$$y(s) = \phi(s)\cdot\begin{pmatrix} C_1(s)\\ C_2(s)\end{pmatrix}$$

b) $$x(t) = y(\ln t) = \phi(\ln t)\begin{pmatrix} C_1(\ln t)\\ C_2(\ln t)\end{pmatrix}$$

$$\phi(\ln t) = \begin{pmatrix} \frac{1+2\ln t}{t} & \frac{-2\ln t}{t}\\ \frac{2\ln t}{t} & \frac{1-2\ln t}{t}\end{pmatrix}$$

$$e^{-s} = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1} = \frac{1}{t}$$

$$C_2(s) = \frac{e^{2s}}{2}(-2s^2+2s-1) + K_2$$

$$C_1(s) = \frac{e^{2s}}{4}(2s-4s^2-1+4s-2) + K_1 = \frac{e^{2s}}{4}(-4s^2+6s-3) + K_1 \Rightarrow$$

$$\Rightarrow \begin{cases} C_1(\ln t) = \frac{t^2}{4}(-4\ln^2 t + 6\ln t - 3) + K_1\\ C_2(\ln t) = \frac{t^2}{2}(-2\ln^2 t + 2\ln t - 1) + K_2\end{cases}$$

Avem

$$x(t) = \begin{pmatrix} \frac{1+2\ln t}{t} & \frac{-2\ln t}{t}\\ \frac{2\ln t}{t} & \frac{1-2\ln t}{t}\end{pmatrix}\left(\begin{pmatrix} \frac{t^2}{4}(-4\ln^2 t + 6\ln t - 3)\\ \frac{t^2}{2}(-2\ln^2 t + 2\ln t - 1)\end{pmatrix} + \begin{pmatrix} K_1\\ K_2\end{pmatrix}\right)$$

c) Se cere soluția sist. (1) care verifică $\begin{cases} x_1(1) = -1\\ x_2(1) = 2\end{cases} \Rightarrow$

$$\Rightarrow x(1) = \begin{pmatrix} -1\\ 2\end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\left(\begin{pmatrix} \frac{1}{4}(-3)\\ \frac{1}{2}(-1)\end{pmatrix} + \begin{pmatrix} K_1\\ K_2\end{pmatrix}\right) = \begin{pmatrix} -1\\ 2\end{pmatrix} \Rightarrow \begin{cases} -\frac{3}{4} + K_1 = -1\\ -\frac{1}{2} + K_2 = 2\end{cases}$$

$\Rightarrow K_1 = -\frac{1}{4}$; $K_2 = \frac{5}{2}$

OBS: $x(t) = \varphi_0(t) + \phi(t)\begin{pmatrix}K_1\\K_2\end{pmatrix}$

$$(\varphi_0(t))_1 = \frac{(1+2\ln t)}{\cancel{t}}(-4\ln^2 t + 6\ln t - 3)\frac{t^{\cancel{2}}}{4} - \frac{\cancel{8}^{4}\ln t}{\cancel{t}}\cdot\frac{t^{\cancel{2}}}{\cancel{2}}(-2\ln^2 t + 2\ln t - 1)$$

$$= \frac{t}{4}\left(-\cancel{4\ln^2 t} + \cancel{6\ln t} - 3 - \cancel{8\ln^3 t} + \cancel{12\ln^2 t} - \cancel{6\ln t} + \cancel{8\ln^3 t} - \cancel{8\ln^2 t} + 4\ln t\right) \Rightarrow$$

$$\boxed{(\varphi_0(t))_1 = \frac{t}{4}(4\ln t - 3)}$$

$$\boxed{(\varphi_0(t))_2} = \frac{\cancel{2}\ln t}{\cancel{t}}\cdot\frac{t^{\cancel{2}}}{2\cancel{4}}(-4\ln^2 t + 6\ln t - 3) + \frac{(1-2\ln t)}{\cancel{t}}\cdot\frac{t^{\cancel{2}}}{2}(-2\ln^2 t + 2\ln t - 1) =$$

$$= \frac{t}{2}\left(-\cancel{4\ln^3 t} + \cancel{6\ln^2 t} - 3\ln t - \cancel{2\ln^2 t} + 2\ln t - 1 + \cancel{4\ln^3 t} - \cancel{4\ln^2 t} + 2\ln t\right) = \boxed{\frac{t}{2}(\ln t - 1)}$$

Deci:

$$\boxed{x(t) = \underbrace{\frac{t}{4}\begin{pmatrix}4\ln t - 3\\ 2\ln t - 2\end{pmatrix}}_{\varphi_0(t)} + \phi(t)\begin{pmatrix}K_1\\K_2\end{pmatrix}}$$

$K_1, K_2 \in \mathbb{R}$.

(3) Fie sistemul: $\begin{cases} x_1' = 3t^2 x_2 \\ x_2' = 3t^2 x_1 \end{cases}$, $t \in \mathbb{R}$ (2)

a) Arătați că $\varphi_1(t) = \begin{pmatrix} e^{-t^3} \\ -e^{-t^3}\end{pmatrix}$ este soluție a sistemului.

b) Determinați, folosind reducerea dimensiunii o soluție φ_2 a.î. $\{\varphi_1, \varphi_2\}$ să fie sistem fundam. de soluții pt. (2).

a) $x' = A(t)x$; $A(t) = \begin{pmatrix} 0 & 3t^2 \\ 3t^2 & 0 \end{pmatrix}$, $t \in \mathbb{R}$

φ_1 soluție $\Leftrightarrow \varphi_1'(t) = A(t)\cdot\varphi_1(t)$, $\forall t \in \mathbb{R} \Leftrightarrow$ Adev.

$$\Leftrightarrow \begin{pmatrix} e^{-t^3} \\ -e^{-t^3}\end{pmatrix}' = \begin{pmatrix} 0 & 3t^2 \\ 3t^2 & 0 \end{pmatrix}\begin{pmatrix} e^{-t^3} \\ -e^{-t^3}\end{pmatrix} \Leftrightarrow \begin{pmatrix} e^{-t^3}(-3t^2) \\ -e^{-t^3}(-3t^2)\end{pmatrix} = \begin{pmatrix} -3t^2e^{-t^3} \\ 3t^2e^{-t^3}\end{pmatrix}$$

b) $n = 2$

$\varphi_1 = \text{soluție} \Rightarrow m = 1$

$\det\left(\left(\varphi_1(t)\right)_1\right) = e^{-t^3} \neq 0,\ \forall t \in \mathbb{R}$

$$Z(t) = \begin{pmatrix} e^{-t^3} & 0 \\ -e^{-t^3} & 1 \end{pmatrix}$$

$$\det Z(t) = e^{-t^3} \neq 0$$

Întrebare:

$\boxed{\text{dacă } \operatorname{rang}(\varphi_1, \dots \varphi_m) = m \text{ și } \det\left(\varphi_{ij}(t)\right)_{i,j=\overline{1,m}} = 0 \text{ atunci cum se poate face reducerea dimensiunii?}}$

$$(t, x) \xrightarrow[x = Z(t)y]{} (t, y)$$

Sist în y: $\quad y' = B(t)\, y$

$$B(t) = (Z(t))^{-1}\left[A(t) Z(t) - Z'(t)\right]$$

$$(Z(t))^{-1} = \frac{1}{\det(Z(t))} \cdot (Z(t))^{*} \quad ; \quad (Z(t))^{*} = \begin{pmatrix} 1 & 0 \\ e^{-t^3} & e^{-t^3} \end{pmatrix}$$

$$(Z(t))^{-1} = \begin{pmatrix} e^{t^3} & 0 \\ 1 & 1 \end{pmatrix} \quad ; \quad Z'(t) = \begin{pmatrix} -3t^2 e^{-t^3} & 0 \\ 3t^2 e^{-t^3} & 0 \end{pmatrix}$$

$$B(t) = \begin{pmatrix} e^{t^3} & 0 \\ 1 & 1 \end{pmatrix} \left[\begin{pmatrix} 0 & 3t^2 \\ 3t^2 & 0 \end{pmatrix} \begin{pmatrix} e^{-t^3} & 0 \\ -e^{-t^3} & 1 \end{pmatrix} - \begin{pmatrix} -3t^2 e^{-t^3} & 0 \\ 3t^2 e^{-t^3} & 0 \end{pmatrix} \right] =$$

$$= \begin{pmatrix} e^{t^3} & 0 \\ 1 & 1 \end{pmatrix} \left[\begin{pmatrix} -3t^2 e^{-t^3} & 3t^2 \\ 3t^2 e^{-t^3} & 0 \end{pmatrix} - \begin{pmatrix} -3t^2 e^{-t^3} & 0 \\ 3t^2 e^{-t^3} & 0 \end{pmatrix} \right] =$$

$$= \begin{pmatrix} e^{t^3} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3t^2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3t^2 e^{t^3} \\ 0 & 3t^2 \end{pmatrix}$$

primele $m=1$ coloane sunt zero.

Sistemul în y: $\quad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 3t^2 e^{t^3} \\ 0 & 3t^2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow$

$$\Rightarrow \begin{cases} y_1' = 3t^2 e^{t^3} y_2 \\ \underbrace{y_2' = 3t^2 y_2}_{\text{ec. liniară în } y_2} \end{cases}$$

ultimele $\underset{1}{n}$ ec în var $\underbrace{y_{m+1}, \dots, y_n}_{y_2}$

$$\boxed{y_2(t) = C_2 \cdot e^{t^3}}\ , \quad C_2 \in \mathbb{R}.$$

$y_1' = 3t^2 e^{2t^3} \cdot C_2 \Rightarrow y_1 = 3C_2 \int t^2 e^{2t^3} dt =$

ec. de tip primitivă

$= \frac{3C_2}{6} \int \left(e^{2t^3}\right)' dt \Rightarrow$

$$\Rightarrow \underbrace{y_1(t) = \frac{C_2}{2} e^{2t^3} + C_1}, \quad C_1, C_2 \in \mathbb{R}$$

Se obține:

$$x(t) = \begin{pmatrix} e^{-t^3} & 0 \\ -e^{-t^3} & 1 \end{pmatrix} \begin{pmatrix} C_1 + \frac{C_2}{2} e^{2t^3} \\ C_2 e^{t^3} \end{pmatrix} =$$

$$= \begin{pmatrix} C_1 e^{-t^3} + \frac{C_2}{2} e^{t^3} \\ -C_1 e^{-t^3} \underbrace{- \frac{C_2}{2} e^{t^3} + \overset{2}{C_2} e^{t^3}} \end{pmatrix} = C_1 \underbrace{\begin{pmatrix} e^{-t^3} \\ -e^{-t^3} \end{pmatrix}}_{\varphi_1(t)} + C_2 \underbrace{\begin{pmatrix} \frac{1}{2} e^{t^3} \\ \frac{1}{2} e^{t^3} \end{pmatrix}}_{\varphi_2(t)}$$

$\Rightarrow \{\varphi_1, \varphi_2\}$ sistem fundam. de soluții pt. (2)

<u>Temă</u>: – prob. la reducerea din în curs 8.

(4) Fie sistemul:

$$\begin{cases} x_1' = \frac{1}{t} x_2 \\ x_2' = \frac{1}{t} x_1 \end{cases}, \quad t > 0 \quad (3)$$

a) Arătați că $\varphi_1(t) = \begin{pmatrix} \frac{1}{t} \\ -\frac{1}{t} \end{pmatrix}$ este soluție a sist. (3).

b) Determinați φ_2 ai $\{\varphi_1, \varphi_2\}$ să fie sistem fundam. de soluții pt (3).