Grupa 341, Seminar 7, EDDP, 17.11.2020

① Fie sistemul
$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 2x_1^3 \end{cases} \quad (1)$$

a) Arătați că $F(t,(x_1,x_2)) = x_1^4 + x_1^2 + x_2^2$ este integrală primă pt (1)

b) Precizați cum se poate reduce dimensiunea sistemului folosind integrala primă.

②
$$\begin{cases} x_1' = (x_2)^2 \\ x_2' = -2(x_1)^3 x_2 \end{cases}$$

a) $F(t,x_1,x_2) = x_1^4 + x_2^2$ este integrală primă

b) Reducerea dimensiunii sistemului.

③
$$\begin{cases} x' = \dfrac{t+x+y}{x-t} \\ y' = \dfrac{t-x}{x-t} \\ z' = \dfrac{t-y}{x-t} \end{cases}$$

a) $F(t,(x,y,z)) = t+y$ este integrală primă

b) Se poate reduce dimensiunea sistemului.

④
$$(4)\begin{cases} x' = 1 - \dfrac{1}{y} \\ y' = \dfrac{1}{x-t} \end{cases}$$

a) $F(t,(x,y)) = y(x-t)$ integrală primă

b) Soluția sistemului.

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④ a) Verificăm că $\dfrac{\partial F}{\partial t}(t,(x,y)) + \dfrac{\partial F}{\partial x}(t,(x,y))\cdot\underbrace{\left(1-\dfrac{1}{y}\right)}_{f_1(t,(x,y))} + \dfrac{\partial F}{\partial y}(t,(x,y))\cdot\dfrac{1}{x-t} = 0$

$F(t,(x,y)) = xy - yt$

$\dfrac{\partial F}{\partial t}(t,(x,y)) = -y$

$\dfrac{\partial F}{\partial x}(t,(x,y)) = y$; $\dfrac{\partial F}{\partial y}(t,(x,y)) = x-t$

Avem:
$$-y + y\left(1-\frac{1}{y}\right) + \cancel{(x-t)}\cdot\frac{1}{\cancel{x-t}} =$$
$$= -y + y - 1 + 1 = 0 \Rightarrow F \text{ integrală primă pt sistemul } (4)$$

b) F integrală primă $\Rightarrow F(t,(x,y)) = c_1$, $c_1 \in \mathbb{R}$.

$\Rightarrow y(x-t) = c_1 \Rightarrow \boxed{y = \dfrac{c_1}{x-t}} \Rightarrow$ în ec. 1 se obține: $x' = 1 - \dfrac{x-t}{c_1} \Rightarrow x' = \left(-\dfrac{1}{c_1}\right)x + \left(\dfrac{t}{c_1}+1\right)$; $c_1 \in \mathbb{R}^*$

$$\frac{dx}{dt} = \left(-\frac{1}{C_1}\right)x + \left(\frac{t}{C_1} + 1\right) \quad \text{ec. afină.}$$

Obs că $x = t$ verifică: $1 = -\frac{1}{C_1}t + \cancel{\frac{t}{C_1}} + 1 \Leftrightarrow 0 = 0$

$\Rightarrow \boxed{\varphi_0(t) = t}$ pt ec. în x.

$a(t) = -\frac{1}{C_1}$; $b(t) = \frac{t}{C_1} + 1$;

Rez ec. liniară omogenă atașată:

$$\frac{d\bar{x}}{dt} = -\frac{1}{C_1}\bar{x} \Rightarrow \bar{x}(t) = C_2 e^{A(t)} \Rightarrow$$

$$\int a(t)\,dt = -\int \frac{1}{C_1}\,dt = \underbrace{\left(-\frac{1}{C_1}t\right)}_{A(t)} + K$$

$$\Rightarrow \boxed{\bar{x}(t) = C_2 e^{-\frac{t}{C_1}},\ C_2 \in \mathbb{R}}$$

Cum cunoaștem $\varphi_0 \Rightarrow$ sol. ec. afine:

$$\boxed{x(t) = C_2 e^{-\frac{t}{C_1}} + t},\ C_1 \in \mathbb{R}^*,\ C_2 \in \mathbb{R} \Rightarrow$$

$$\Rightarrow \boxed{y(t) = \frac{C_1}{x - t} = \frac{C_1}{C_2 e^{-\frac{t}{C_1}}}}$$

(5) Să se determine soluția generală a următoarelor sisteme liniare:

V a) $\begin{cases} x' = 2x + 4y \\ y' = 4x + 2y \end{cases}$; b) $\begin{cases} x' = 2x + y \\ y' = 4y - x \end{cases}$ (a')

V c) $\begin{cases} x' = x + y \\ y' = 3y - 2x \end{cases}$; d) $\begin{cases} x' = 2x - y + 2z \\ y' = x + 2z \\ z' = -2x + y - z \end{cases}$

e) $\begin{cases} x' = 2x - y - z \\ y' = 3x - 2y - 3z \\ z' = -x + y + 2z \end{cases}$; f) $\begin{cases} x' = 4x - y \\ y' = 3x + y - z \\ z' = x + z \end{cases}$

<u>OBS</u>: $(x, y) = (x_1, x_2)$

$(x, y, z) = (x_1, x_2, x_3)$.

a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$, $n=2$.

. determinăm val. proprii pt A:

$$\det(A - \lambda I_2) = 0$$

$$\det\left(\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (2-\lambda)^2 - 16 = 0.$

$(2-\lambda)^2 = 16 \Rightarrow 2-\lambda = \pm 4 \begin{cases} 2-\lambda = 4 \Rightarrow \lambda_1 = -2 \\ 2-\lambda = -4 \Rightarrow \lambda_2 = 6 \end{cases}$

$p_A(\lambda) = (2-\lambda)^2 - 16 = (2-\lambda-4)(2-\lambda+4) = (-\lambda-2)(-\lambda+6) =$
$= (\lambda+2)(\lambda-6)$

$\Rightarrow k=2$: $\lambda_1 = -2$, $m_1 = 1$; $\lambda_2 = 6$, $m_2 = 1$

• pt $\boxed{\lambda_1 = -2}\ \boxed{m_1 = 1}$ determinăm $u \in \mathbb{R}^2$, $u \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ a.î.

$$Au = \lambda_1 u \Rightarrow \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow$$

$\Rightarrow \begin{cases} 2u_1 + 4u_2 = -2u_1 \\ 4u_1 + 2u_2 = -2u_2 \end{cases} \Rightarrow \begin{cases} 4u_1 + 4u_2 = 0 \,|:4 \\ 4u_1 + 4u_2 = 0 \end{cases} \Rightarrow u_1 + u_2 = 0$
$\Rightarrow u_2 = -u_1 \Rightarrow$

$\Rightarrow u = \begin{pmatrix} u_1 \\ -u_1 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \boxed{\varphi_1(t) = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

• pt $\boxed{\lambda_2 = 6, m_1 = 1}$; $u \in \mathbb{R}^2$, $u \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ a.î. $Au = \lambda_2 u \Rightarrow$

$\Rightarrow \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 6 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \begin{cases} 2u_1 + 4u_2 = 6u_1 \\ 4u_1 + 2u_2 = 6u_2 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} -4u_1 + 4u_2 = 0 \\ 4u_1 - 4u_2 = 0 \end{cases} \Rightarrow \boxed{u_1 = u_2} \Rightarrow u = u_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow \boxed{\varphi_2(t) = e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Avem $\{\varphi_1, \varphi_2\}$ formează sistem fundam. de soluții

$\Rightarrow S_A = \{C_1\varphi_1 + C_2\varphi_2 \mid C_1, C_2 \in \mathbb{R}\} \Rightarrow$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}(t) = C_1 \cdot e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$

$$\Rightarrow \begin{cases} x(t) = C_1 e^{-2t} + C_2 e^{6t} \\ y(t) = -C_1 e^{-2t} + C_2 e^{6t} \end{cases}, \quad \underline{C_1, C_2 \in \mathbb{R}}$$

Dacă ne cere soluția care verifică $\begin{cases} x(0) = 1 \\ y(0) = 3 \end{cases} \Rightarrow$

$\boxed{t=0} \Rightarrow$

$$\begin{cases} 1 = C_1 e^0 + C_2 e^0 \\ 3 = -C_1 e^0 + C_2 e^0 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = 1 \\ -C_1 + C_2 = 3 \end{cases}$$

$$/ \quad 2C_2 = 4 \Rightarrow \boxed{C_2 = 2}$$

$$\boxed{C_1 = -1}$$

$$\Rightarrow \boxed{\begin{cases} x(t) = -e^{-2t} + 2e^{6t} \\ y(t) = e^{-2t} + 2e^{6t} \end{cases}}$$

a') $\begin{cases} x_1' = 2x_1 - x_2 \\ x_2' = x_1 + 4x_2 \end{cases} \Longrightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\det\left(\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(4-\lambda) + 1 = 0$$

$$8 - 6\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 3 \Rightarrow$$

$$m_1 = 2$$

$\Rightarrow \begin{cases} n = 2 \\ k = 1 \\ \boxed{\begin{matrix} \lambda_1 = 3 \\ m_1 = 2 \end{matrix}} \end{cases} \Rightarrow$ Căutăm $p_0, p_1 \in \mathbb{R}^2$ nu amândoi nuli a.î.

$$\varphi(t) = (p_0 + p_1 t) e^{3t}$$

să verifice sistemul: $x' = A x \Rightarrow$

$$\Rightarrow \left((p_0 + p_1 t) e^{3t}\right)' = e^{3t} A \cdot (p_0 + p_1 t)$$

$$p_1 e^{3t} + 3e^{3t}(p_0 + p_1 t) = e^{3t} A (p_0 + p_1 t) \quad \Big| : e^{3t}$$

$$p_1 + 3p_0 + 3p_1 t = A p_0 + A p_1 \cdot t$$

identif. coef puterilor lui t $\Longrightarrow \begin{cases} A p_1 = 3 p_1 \\ A p_0 = p_1 + 3 p_0 \end{cases} \Rightarrow \begin{cases} A p_1 - 3 p_1 = O_{\mathbb{R}^2} \\ A p_0 - 3 p_0 = p_1 \end{cases}$

$$\begin{cases}(A-3I_2)p_1 = 0_{\mathbb{R}^2} \\ (A-3I_2)p_0 = p_1 \quad \Big| \cdot (A-3I_2) \text{ la stg} \Rightarrow\end{cases}$$

$$\Rightarrow (A-3I_2)^2 p_0 = \underbrace{(A-3I_2)p_1}_{0_{\mathbb{R}^2}} \Rightarrow \boxed{(A-3I_2)^2 p_0 = 0}$$

dacă considerăm apl liniară $f:\mathbb{R}^2\to\mathbb{R}^2$

$f(v) = (A-3I_2)^2 v$,

atunci $p_0 \in \underbrace{\ker(f)}_{f(p_0)=0} = \{v\in\mathbb{R}^2 \mid f(v) = 0_{\mathbb{R}^2}\}$

$\ker((A-3I_2)^2)$

<u>OBS</u>: În general, pt λ_j cu multiplicitate $m_j \geq 1 \Rightarrow$

$\Rightarrow p_0 \in \ker((A-\lambda_j I_n)^{m_j})$

$$(A-3I_2)^2 = \begin{pmatrix}2-3 & -1 \\ 1 & 4-3\end{pmatrix}\begin{pmatrix}2-3 & -1 \\ 1 & 4-3\end{pmatrix} = \begin{pmatrix}-1 & -1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}-1 & -1 \\ 1 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}$$

$\Rightarrow \ker((A-3I_2)^2) = \mathbb{R}^2 \Rightarrow$ pt p_0 alegem vectorii unei baze din $\mathbb{R}^2 \Rightarrow$ alegem vectorii bazei canonice

$\ker(O_2)$

$\Rightarrow p_0 \in \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

I) $p_0 = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow p_1 = (A-3I_2)p_0 = \begin{pmatrix}-1 & -1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \Rightarrow$

$\Rightarrow \varphi_1(t) = \left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix}t\right)e^{3t} \Rightarrow \boxed{\varphi_1(t) = \begin{pmatrix}1-t\\t\end{pmatrix}e^{3t}}$

II) $p_0 = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow p_1 = \begin{pmatrix}-1 & -1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \Rightarrow$

$\Rightarrow \varphi_2(t) = \left(\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix}t\right)e^{3t} \Rightarrow \boxed{\varphi_2(t) = \begin{pmatrix}-t\\1+t\end{pmatrix}e^{3t}}$

$\Rightarrow S_A = \{c_1\varphi_1 + c_2\varphi_2 \mid c_1, c_2 \in \mathbb{R}\}$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}(t) = c_1\begin{pmatrix}(1-t)e^{3t}\\te^{3t}\end{pmatrix} + c_2\begin{pmatrix}-te^{3t}\\(1+t)e^{3t}\end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases}x_1(t) = (c_1(1-t) - c_2 t)e^{3t} \\ x_2(t) = (c_1 t + c_2(1+t))e^{3t}\end{cases}, \ c_1, c_2 \in \mathbb{R}$$

b) $\begin{cases} x' = 2x + y \\ y' = -x + 4y \end{cases} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}, \; n = 2$

$\det\left((A - \lambda I_2)\right) = 0 \Rightarrow \det\left(\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 \Rightarrow$

$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(4-\lambda) + 1 = 0 \Rightarrow$ temă!

$\boxed{\text{analog } a')}$

c) $\begin{cases} x' = x + y \\ y' = -2x + 3y \end{cases} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}, \; n = 2.$

$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0.$

$(1-\lambda)(3-\lambda) + 2 = 0 \Rightarrow 3 - \lambda - 3\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 4\lambda + 5 = 0$

$\Delta = 16 - 20 = -4.$

$\lambda_{1,2} = \frac{4 \pm 2i}{2} = 2 \pm i$

$\underline{k=2}$: $\begin{cases} \lambda_1 = 2 + i, & m_1 = 1 \\ \lambda_2 = 2 - i = \overline{\lambda_1}, & m_2 = 1 \end{cases}$

$\Downarrow$ determinăm $u \in \mathbb{C}^2$, $u \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ a.î. $Au = \lambda_1 u \Rightarrow$

$\Rightarrow \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (2+i) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \begin{cases} u_1 + u_2 = (2+i) u_1 \\ -2u_1 + 3u_2 = (2+i) u_2 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} (1-2-i) u_1 + u_2 = 0 \\ -2u_1 + (3-2-i) u_2 = 0 \end{cases} \Rightarrow \begin{cases} u_2 = (1+i) u_1 \\ u_2 \overset{1+i/2}{=} \frac{2}{1-i} u_1 = \frac{\cancel{2}(1+i)}{\cancel{1+1}} u_1 \end{cases} \Rightarrow$

$\Rightarrow \boxed{u_2 = (1+i) u_1} \Rightarrow u = u_1 \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \Rightarrow$

$e^{(2+i)t} = e^{2t+it} = e^{2t} \cdot e^{it} = e^{2t} (\cos t + i \sin t)$

$\boxed{\cos t + i \sin t = e^{it}}$

$\Rightarrow \varphi(t) = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(2+i)t} = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) e^{2t} \underbrace{(\cos t + i \sin t)}_{e^{it}} \Rightarrow$

$\Rightarrow \varphi_1(t) = Re(\varphi(t)) = e^{2t} \left((\cos t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - (\sin t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

$\boxed{\varphi_1(t) = e^{2t} \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix}}$

$\varphi_2(t) = Im(\varphi(t)) = e^{2t} \left((\sin t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (\cos t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \Rightarrow \boxed{\varphi_2(t) = e^{2t} \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix}}$

$S_A = \{ c_1 \varphi_1 + c_2 \varphi_2 \mid c_1, c_2 \in \mathbb{R} \}$. $\underline{\text{Temă}}$: b, d, e, f