The Small Angle Approximation

1 INTRODUCTION

A Simple harmonic motion (SHM) describes systems that oscillate predictably with time. One system that exhibits SHM is an oscillating mass attached to a spring. Another system that *approximately* exhibits SHM is a simple pendulum.

When solving the equation of motion for a pendulum an assumption is typically made to simplify the math. The assumption – called the "small angle approximation" – limits the amplitude of the pendulum to small angles. With this assumption, the equations that describe the motion of a pendulum are identical to SHM. The equations of motion for a pendulum are often accompanied by the disclaimer that the solution is good for "sufficiently small angles".

This lab explores the limitations of the small angle approximation in a simple pendulum.

2 LEARNING OBJECTIVES

At the conclusion of this activity you should be able to:

- Explain what is meant by the *small angle approximation.*
- Design an experiment to measure the angle at which the small angle approximation is no longer valid.
- Observe the role that uncertainty plays in making a measurement.

3 BACKGROUND

3.1 MATHEMATICS: PLOTTING A SINE FUNCTION

Sine and cosine functions are often used to model periodic systems[1][2]. A generalized sine wave can be written as:

$$f(x) = A\sin(Bx + C) + D, \tag{3.1}$$

where the parameters A, B, C, and D specify the shape and position of the function (see Figure 3.1). A detailed description of each of the parameters follows:

A = **Amplitude:** The amplitude refers to the swing of the sine function. The amplitude is typically measured as the "peak amplitude" or the maximum absolute value of the function.

B = **Angular Frequency:** The angular frequency describes the number of cycles of the function in a given time period. The period T – time required for a single oscillation – is related to the parameter B by:

$$T = \frac{2\pi}{B}. \tag{3.2}$$

For a system changing in time, B is typically expressed in radians per unit time and is often labeled ω.

C = **Phase:** The phase describes a horizontal translation of the signal. The phase shifts the signal a horizontal distance of C/B. Positive values of C result in *negative* translation along the x axis. The phase is often labeled ϕ.

D = **Offset:** The offset parameter translates the function in the vertical direction. A positive D moves the signal in the positive y direction.

$$f(x) = A\sin(Bx + C) + D$$

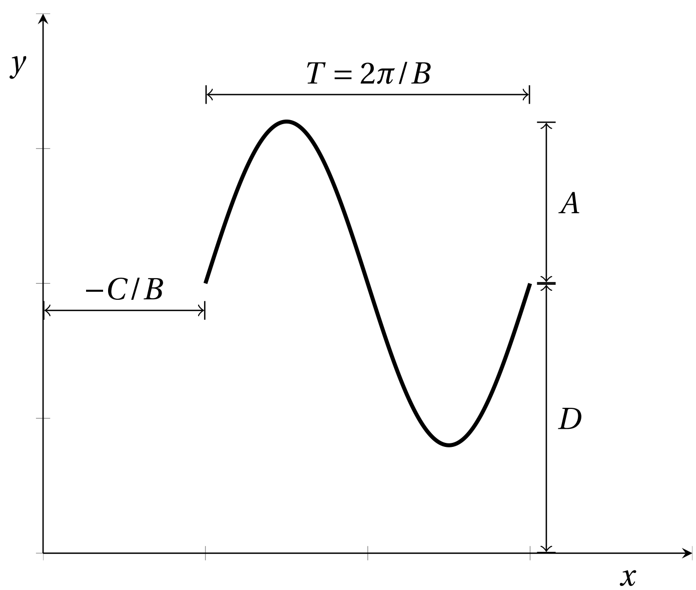

Figure 3.1: A cartoon of one cycle of a sine function showing the effect of each of the parameters: A, B, C, and D on the shape and position of the graph.

3.2 Taylor Series

A Taylor series is a mathematical tool to represent a function near a point in terms of an infinite sum[3][4]. The Taylor series for sine and cosine are:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{3.3}$$

and

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \tag{3.4}$$

The more terms that are included in a Taylor polynomial, the closer the series approximates the original function (see Figure 3.2).

In physics, we often estimate the solution for a complicated system by including only the first term (*i.e.* "first order") of the Taylor series. In this case sine and cosine are approximated by:

$$\sin(x) \approx x \tag{3.5}$$

$$\cos(x) \approx 1. \tag{3.6}$$

For sufficiently small values of x, this is often an acceptable approximation.

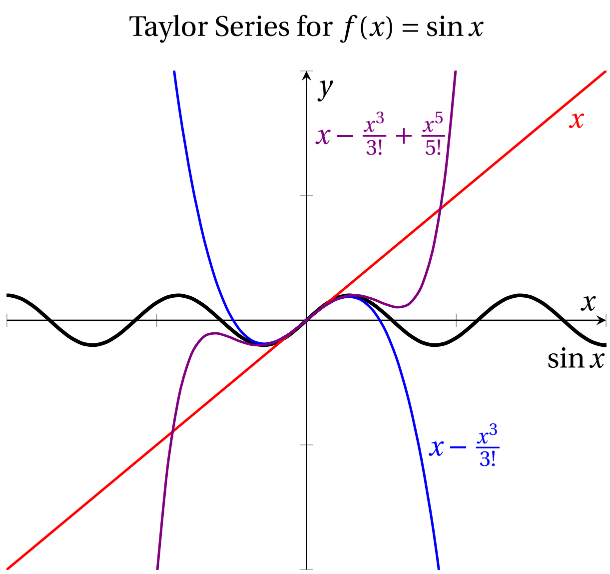

Figure 3.2: The first three terms of the Taylor series for $\sin x$. For small values of x it is clear that even the first order (red line) is a good approximation of the function.

3.3 The Motion of a Pendulum

One way to analyze the forces acting on a pendulum is in terms of torque[5]. Recall that torque $\vec{\tau}$ is given by:

$$\vec{\tau} = \vec{r} \times \vec{F} = I\vec{\alpha}, \tag{3.7}$$

where $\vec{r}$ is the vector from the point of rotation to the force $\vec{F}$, I is the moment of inertia, and α is the angular acceleration. Recall that the angular acceleration is the second time derivative of the displacement angle, θ. That is:

$$\alpha = \frac{d^2\theta}{dt^2}. \tag{3.8}$$

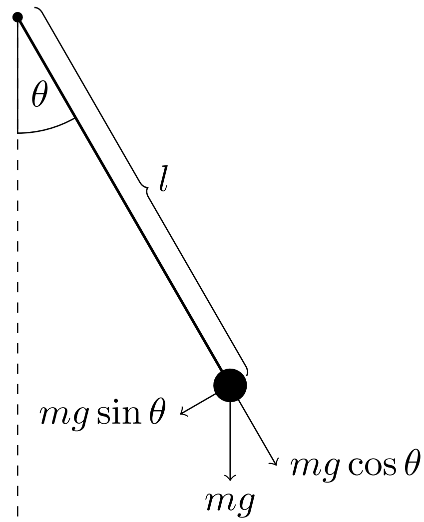

Figure 3.3: A simple pendulum

Consider a simple pendulum with a bob of mass m and a massless chord of length l as shown in Figure 3.3. The moment of inertia for this system is given by:

$$I = ml^2. \tag{3.9}$$

The restoring force of the pendulum is given by;

$$|\vec{F}_r| = -mg\sin\theta. \tag{3.10}$$

Note that the restoring force is perpendicular to the chord and opposes the displacement.

Starting from Equation 3.7, with $\vec{r} = l$ and substituting Equation 3.9 and 3.10 we obtain:

$$(ml^2)\alpha = l \times (-mg\sin\theta) \tag{3.11}$$

Simplifying and expressing α in terms of the second time derivative of θ (Equation 3.8).

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta. \tag{3.12}$$

Obtaining an exact solution for Equation 3.12 is difficult. In order to solve the equation more easily, we make the assumption that the amplitude of the pendulum oscillations is be small – this is the *small angle approximation*. Under this condition, we can use the first-order Taylor approximation described in Equation 3.5.

With this assumption, Equation 3.12 simplifies to equation:

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{l}\theta, \tag{3.13}$$

We define the angular frequency ω in terms of g and l:

$$\omega \equiv \sqrt{g/l} \tag{3.14}$$

and re-write the equation of motion as:

$$\frac{d^2\theta}{dt^2} \approx -\omega^2\theta. \tag{3.15}$$

Equation 3.15 is the general form for simple harmonic motion. Masses oscillating on springs and simple pendulums are two examples of systems that exhibit simple harmonic motion.

The solutions to Equation 3.15 are given by:

$$\theta(t) = A\sin(\omega t + \phi). \tag{3.16}$$

For a pendulum, A is the amplitude, or maximum opening angle θ_{max}, that the pendulum makes with respect to vertical. ω is the angular frequency as defined in Equation 3.14 and ϕ is known as the phase angle. Note the similarities between Equation 3.16 and 3.1.

4 Procedure

To obtain Equation 3.15 we limit ourselves to small angles. What exactly qualifies as a small angle? The goal of this lab is to study the motion of a pendulum to determine when the small angle approximation breaks down.

For several amplitudes, measure the angular frequencies of a pendulum.

- A Vernier rotary motion sensor[6] will be used to collect data from an oscillating pendulum. The data are to be analyzed using Logger Pro. Open the file: ``LoggerPro_Pendulum.cmbl'' (available from Blackboard), to begin your measurement.
- Fit your data with a sine function using the ``Analyze'' → ``Curve Fit...'' menu option. One of the built-in fitting functions that Logger Pro offers is a sine function of the form `A*sin(B*x+C)+D`. Based on the parameters obtained from the fit, determine the observed angular frequency.
- Compare your observations with the predicted angular frequency using Equation 3.14 and the acceleration due to gravity*.

5 Lab Notebook (In the Form of a Poster)

Create a poster to summarize and present your work. Your poster should be clean, easy to read, and organized. Your poster should address the following general ideas in an order that seems the most logical for you.

- Poster Design (6 points)
  - The poster draws attention to the most important and most interesting pieces of the analysis.

*According to the NGS Surface Gravity model, the value of the acceleration due to gravity near the Bloomberg building is $9.80095 \pm 0.00002\ m/s^2$. See https://www.ngs.noaa.gov/TOOLS/Gravity/gravcon.html for more information.

  - The poster is well organized and easy to read.
  - Relevant information (*e.g.* your name, your lab partner's name, date, *etc.*) is present.
- Data Analysis & Plots (6 points)
  - The poster tells a scientific story; it is an accurate record of the work that you did.
  - The notebook should show evidence of trial and error. Keep a good record of your work – recording mistakes is useful.
  - Final versions of plots should be well formatted and meet the plotting guidelines for the course.
  - Use models to identify trends that your data exhibit or other apparent relationships between your independent and dependent variables.
- Results and Comparison (6 points)
  - Clearly state the final result(s) of your experiment. Remember to quote your result with units and appropriate significant digits.
  - Final result plots are well formatted and meet the standards described in the Figure Formatting reference.
  - A useful comparison is made to a known/expected value or another similar result.
  - Choose the best available tools for your comparison (*e.g.* plots, pictures, discrepancy, significance of discrepancy, etc).
- Uncertainty and Error Propagation (6 points)
  - Identify the dominant source(s) of error in your experiment.
  - Support your conclusions with appropriate error estimates and error propagation calculations.
- Physical Interpretation (6 points)
  - Throughout the poster, interpret the data, , and final results in terms of the underlying physics.
  - What are you able to conclude from your data? Clearly explain how you arrived at your conclusions from your experimental observations.
  - Reflect on how your experiments connects with the physics concepts you are studying.

6 How to Create a Poster

Microsoft Powerpoint makes it easy to create a scientific poster. Here are some general instructions:

1. Open a "Blank Presentation" in Powerpoint.
2. Edit the size of the page using the File > Page Setup... menu. Your poster can be either landscape or portrait, but should be 24" x 36".
3. Your poster will be a single Powerpoint slide. Add text, plots, photographs, equations, etc. to your poster.

4. When you are finished, save the file as a .PDF using either the File > Export... or File > Print... > Save as PDF... menu options.

5. **Printing your poster:** The poster can be set to print over multiple pages that can be taped together. This is called "tiled printing". **Be sure to print single-sided pages.**

   One way to print a large version of your poster uses Adobe's free Acrobat Reader DC PDF reader:

   (a) Use Acrobat Reader DC to open the .PDF file you created using Powerpoint.

   (b) Under File > Print..., set the "Page Sizing & Handling" to "Poster".

   (c) Adjust the "Tile Scale:" value until the poster is the size that you want. A preview of the printed poster should appear in the Print dialogue box showing the overall size and number of sheets of paper across which your poster will be tiled.

REFERENCES

[1] Swartz, C. E. (1993) *Used Math: For the First Two Years of College Science*. College Park, MD: American Association of Physics Teachers. See Chapter 4.2 (pp. 39-53).

[2] Resnick, R., Halliday, D., Krane, K. S. (2002) *Physics, Vol. 1, 5th Edition*. Danvers, MA: John Wiley & Sons, Inc.. See Chapter 17-3 (pp. 376-378).

[3] Swartz, C. E. (1993) *Used Math: For the First Two Years of College Science*. College Park, MD: American Association of Physics Teachers. See Chapter 14.3 (pp. 193-195).

[4] See the Wikipedia entry on Taylor Series: https://en.wikipedia.org/wiki/Taylor_series.

[5] Resnick, R., Halliday, D., Krane, K. S. (2002) *Physics, Vol. 1, 5th Edition*. Danvers, MA: John Wiley & Sons, Inc.. See Chapter 17-5 (pp. 380-384).

[6] The Vernier rotary motion sensor manual:
http://www.vernier.com/files/manuals/rmv-btd/rmv-btd.pdf.