Projectile Motion: Hitting a Target

1 INTRODUCTION

In this lab, you will use a calibrated cannon and your knowledge of projectile motion to aim a cannon to hit several targets.

2 LEARNING GOALS

- Hit a target with a calibrated projectile launcher.
- Experiment with the physics of projectile motion.

3 BACKGROUND

3.1 PROJECTILE MOTION

Recall that the general kinematic equation of motion is given by;

$$\vec{x}(t) = \vec{x_o} + \vec{v_o}t + \frac{1}{2}\vec{a}t^2. \tag{3.1}$$

Which, for a projectile near the surface of the earth, can be decomposed into independent equations of motion in the horizontal

$$x(t) = x_o + v_{ox}t, \tag{3.2}$$

and vertical

$$y(t) = y_o + v_{oy}t - \frac{1}{2}gt^2, \tag{3.3}$$

components.

3.2 TRIGONOMETRIC IDENTITIES

Trigonometric identities are often useful for simplifying mathematical expressions. It is typically more straight-forward to interpret the physical predictions of a simplified expression.

A list of useful trigonometric identities follows:

Converting angles between radians and degrees

$$\theta_{\text{degrees}} = \theta_{\text{radians}} \cdot \frac{180^\circ}{\pi} \qquad \theta_{\text{radians}} = \theta_{\text{degrees}} \cdot \frac{\pi}{180^\circ}$$

Cofunction identities (expressed in degrees):

$$\begin{aligned} \sin(90^\circ - x) &= \cos x & \cos(90^\circ - x) &= \sin x \\ \tan(90^\circ - x) &= \cot x & \cot(90^\circ - x) &= \tan x \\ \sec(90^\circ - x) &= \csc x & \csc(90^\circ - x) &= \sec x \end{aligned}$$

Double-angle identities:

$$\sin 2x = 2 \sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x.$$

The Pythagorean identities:

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \cot^2 x = \frac{1}{\sin^2 x} \qquad \tan^2 x + 1 = \frac{1}{\cos^2 x}.$$

3.3 The Angle Required to Hit a Point

Suppose that our goal is to hit a target that is located some horizontal distance x and at some height y above the cannon. Given the muzzle velocity v_o it is possible to calculate the angle θ required to hit the target.

We again start from the equations of motion for a projectile given in Equations 3.2 and 3.3. At the time t when the projectile hits the intended target, the vertical equation of motion will be:

$$y = v_o \sin\theta\, t - \frac{1}{2} g t^2 \tag{3.4}$$

Note that this equation is quadratic in t and can be written in the standard form:

$$0 = -\frac{1}{2} g t^2 + v_o \sin\theta\, t - y \tag{3.5}$$

At the same time t of impact, the horizontal equation of motion is:

$$x = v_o \cos\theta\, t, \tag{3.6}$$

which can be solved for time to obtain:

$$t = \frac{x}{v_o \cos\theta}. \tag{3.7}$$

Substituting 3.7 into 3.5, we get an expression that is independent of time:

$$0 = -\frac{1}{2} g \left(\frac{x}{v_o \cos\theta}\right)^2 + v_o \sin\theta \left(\frac{x}{v_o \cos\theta}\right) - y. \tag{3.8}$$

After using some trigonometric identities, the quadratic equation, and a fair amount of algebra, it can be shown that the expression for the angle θ required to hit a point (x, y) is given by:

$$\theta = \arctan\left(\frac{v_o^2 \pm \sqrt{v_o^4 - g(2yv_o^2 + gx^2)}}{gx}\right). \tag{3.9}$$

3.4 Uncertainty in the Horizontal Distance

Using Equation 3.9 we can determine the angle required to hit a target at any point (x, y) from a projectile launcher. A natural question to ask next is "How reliably will we hit the prescribed target?"

Considering a projectile launcher with a muzzle velocity of $v_o \pm \delta v_o$ set at an angle of $\theta \pm \delta\theta$, the uncertainty in the horizontal distance can be expressed as:

$$\delta x = t\sqrt{(\cos\theta)^2(\delta v_o)^2 + (v_o \sin\theta)^2(\delta\theta)^2}. \tag{3.10}$$

Where t is the time that the projectile is in the air. The expression above has the desired behavior – longer projectile flights result in larger horizontal uncertainty.

In some specific cases, Equation 3.10 can be expressed in terms of the horizontal distance x that a projectile travels. The uncertainty then becomes:

$$\delta x = \frac{x}{v_o \cos\theta}\sqrt{(\cos\theta)^2(\delta v_o)^2 + (v_o \sin\theta)^2(\delta\theta)^2}. \tag{3.11}$$

Remember, the uncertainty of the angle **must** be expressed in radians to obtain the correct result. Although not completely general, Equation 3.11 can be used to estimate the uncertainty of all of the shots that are described in the Procedure (Section 4).

4 Procedure

You will be using a Pasco ME-6800 short range projectile launcher to hit a series of targets. The muzzle velocity, $v_o \pm \delta v_o$ (measured previously), will be used to calculate the angle required to put shots into a basket.

Use the equations of projectile motion and your value for $v_o \pm \delta v_o$ to calculate all possible angles θ_i that will result in the projectile hitting a basket located at the following positions:

1. The maximum range of the projectile launcher (with the same vertical height as the launcher).
2. 1.0 meter away horizontally and 0.2 meters above the height of the launcher (see Figure 3.2).

How accurately will you need to measure θ_i in order to hit each target? How many solutions for θ_i do you find in each case?

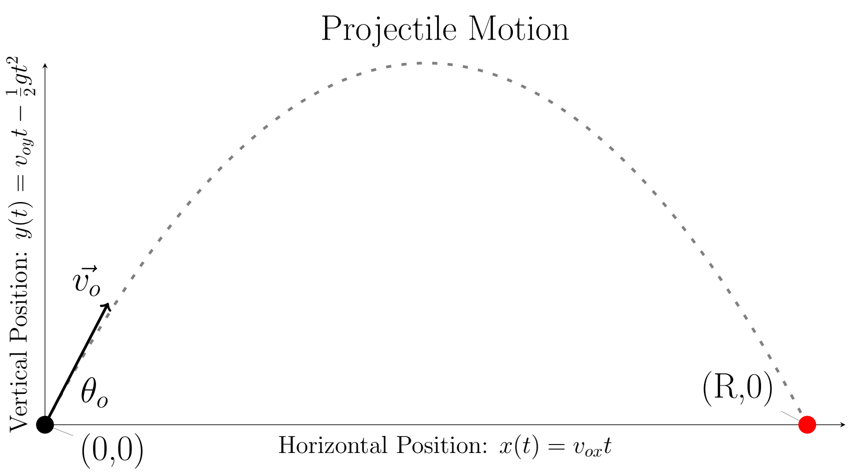

Figure 3.1: A projectile is fired with some velocity $\vec{v}_o$ above the horizon. The projectile travels a horizontal distance R before hitting the ground.

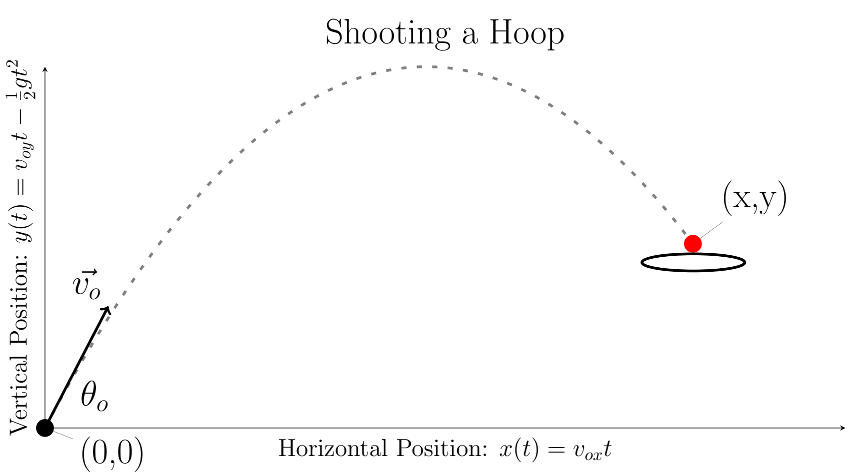

Figure 3.2: A projectile is fired with some velocity $\vec{v}_o$ above the horizon. The projectile travels a horizontal distance x before passing through a hoop at vertical elevation y.

4.1 Maximum Range: Verify the Muzzle Velocity

The goal for this section is to verify the muzzle velocity $v_o \pm \delta v_o$.

Given the muzzle velocity of a cannon v_o and the angle at which it is fired θ, it is possible to calculate the maximum horizontal distance, or maximum *range*, that a projectile will travel using the equations of projectile motion. This exercise is left for you to do in the Pre-lab Quiz.

Fire a projectile into a basket positioned at the maximum range of your launcher.

- What is the angle required to achieve maximum range?
- If the calculated angle does not put the projectile into the basket, is your measured v_o too large or too small?
- Adjust the position of the basket until you are able to consistently make baskets. Based on this new position, what is the revised value of the muzzle velocity, $\widetilde{v_o}$?
- Explicitly compare $v_o \pm \delta v_o$ and $\widetilde{v_o}$.
- What systematic effect(s) could explain the difference between v_o and $\widetilde{v_o}$ that you observe?

4.2 Ranged Shot: Statistics

The goal of this section is to make strong statistical statements regarding the number of shots made in a basket.

Remember: Your result will not be assessed based on how successfully you hit each target. If the shot is a miss, try to explain the possible reasons. For example, is the miss the result of a statistical fluctuation? Is there a systematic uncertainty that has not been considered? *It is the process, not the result, that is important.*

Position the basket at the position $(x, y) = (1.0\text{m}, 0.2\text{m})$ relative to where the projectile exits the launcher.

- Calculate the angle(s) required to hit the target at the specified location.
- Based on the precision of the apparatus, how accurately can the angle of the launcher be set? That is, for a given angle θ what is $\delta\theta$?
- Using your values of $\theta \pm \delta\theta$ and $v_o \pm \delta v_o$ (or revised muzzle velocity), calculate the horizontal uncertainty δx when the projectile reaches the target. See Equation 3.11.
- Based on the measured radius of the basket and the uncertainty associated with the horizontal distance δx, what fraction of shots do you *predict* will land inside the basket?
- Fire several shots (at least 20). What fraction of shots do you *observe* landing in the basket?
- Your result is a quantitative, statistical, comparison of the *predicted* number of baskets and the *observed* number of baskets. Discuss and justify your result(s).

5 Lab Report

Before you leave the lab, submit to your TA, via Blackboard, a lab report. Be sure to address the following items:

- Results: State your final result(s). Be sure to include a label, appropriate uncertainty estimates, measurement units, and significant figures.

- Uncertainty: What are some reasonable sources of random uncertainty?
  What are some reasonable sources of systematic uncertainty?
  For each identified source of systematic uncertainty, clearly state how it affects your result.
- Discussion: Discuss your findings.
  What physical principles can be used to explain your observations?
  Do you observe what you expect?
  Do your data and observations make sense?
  What changes could be made to improve the accuracy and precision of your measurement?