Ballistic Pendulum

1 INTRODUCTION

The ballistic pendulum was developed in 1742 by the English mathematician Benjamin Robins. It was the first accurate way to determine the muzzle velocity of a bullet.

A ballistic pendulum is a mechanical device that captures a fired projectile and is allowed to swing (see Figure 3.1). Based on the amplitude of the resulting pendulum swing, the initial velocity of the projectile can be determine by using the conservation of momentum and energy.

2 EXPERIMENT GOALS

- Measure the spring constant k of the projectile launcher.
- Reinforce the concepts of momentum and energy conservation.
- Measure the velocity of a fired projectile.

3 BACKGROUND

3.1 MOMENT OF INERTIA

The moment of inertia describes a body's resistance to change in rotation about a particular axis. For all but the most symmetric shapes, the moment of inertia is difficult to calculate analytically. However, *measuring* the moment of inertia is a task that is straightforward in the laboratory.

For any rigid body – or physical – pendulum the moment of inertia can be determined using Newton's laws.

The torque $\vec{\tau}$ on a pendulum rotating about a pivot point is given by:

$$\vec{\tau} = -\vec{R_{cm}} \times M\vec{g}\sin\theta \tag{3.1}$$

where R_{cm} is the distance from the pivot point to the center of mass of the object, M is the total mass of the pendulum, g is the acceleration due to gravity*, and θ is the angle measured at the pivot point between vertical and the center of mass of the pendulum.

*According to the NGS Surface Gravity model, the value of the acceleration due to gravity near the Bloomberg building is $9.80095 \pm 0.00002\ m/s^2$. See https://www.ngs.noaa.gov/TOOLS/Gravity/gravcon.html for more information.

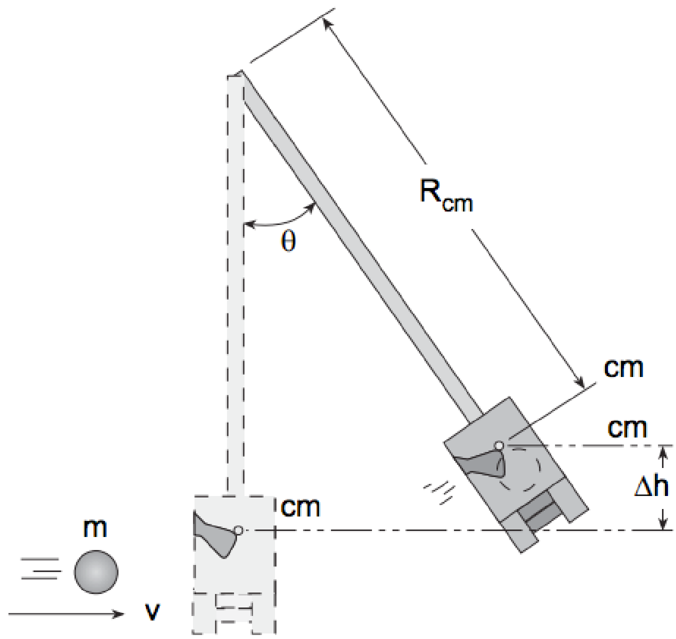

Figure 3.1: A ballistic pendulum consists of a rigid-body pendulum that is able to capture a moving projectile. After the capture, the pendulum swings to a maximum opening angle of θ. Using conservation of momentum and energy, the initial velocity v of the projectile can be determine based on the opening angle θ. The center of mass of the pendulum is labeled "cm".

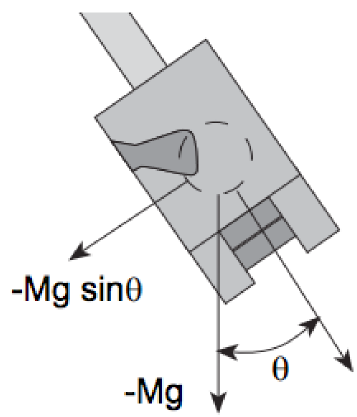

Figure 3.2: The free-body diagram for the ballistic pendulum. The pendulum experiences a restoring force $-Mg\sin\theta$ that is a component of the force due to gravity $-Mg$.

Torque is related to the angular acceleration of an object by:

$$\vec{\tau} = I\vec{\alpha} = I\frac{d^2\theta}{dt^2} \tag{3.2}$$

where I is the moment of inertia of the object about the rotation axis and $\vec{\alpha}$ is the angular acceleration (the second time derivative of the angle θ).

The equation of motion for the physical pendulum is then found by combining Equations 3.1 and 3.2 to obtain:

$$I\frac{d^2\theta}{dt^2} = -R_{cm}Mg\sin\theta. \tag{3.3}$$

Solving for the angular acceleration we obtain:

$$\frac{d^2\theta}{dt^2} = -\frac{R_{cm}Mg}{I}\sin\theta \tag{3.4}$$

If we limit the amplitude of oscillation to small angles, where $\sin\theta \approx \theta$ we can rewrite the equation as:

$$\frac{d^2\theta}{dt^2} \approx -\frac{R_{cm}Mg}{I}\theta \tag{3.5}$$

which is the general form of the equation of motion for a simple harmonic oscillator.

We define ω to be:

$$\omega^2 \equiv \frac{R_{cm}Mg}{I}. \tag{3.6}$$

The solutions to Equation 3.5 are given by:

$$\theta(t) = A\sin(\omega t + \phi). \tag{3.7}$$

For an oscillating pendulum, A is the amplitude, or maximum opening angle $\theta_{\max}$, that the center of mass of the pendulum makes with respect to vertical. ω is the angular velocity as defined in Equation 3.6 and ϕ is known as the phase angle.

Recall from trigonometry that the angular velocity (measured in radians per second) is related to the frequency f (oscillations per second) and period T (seconds) of oscillation by:

$$\omega = 2\pi f = \frac{2\pi}{T}. \tag{3.8}$$

Combining Equations 3.6 and 3.8 we obtain:

$$\left(\frac{2\pi}{T}\right)^2 = \frac{R_{cm}Mg}{I} \tag{3.9}$$

which can be solved to obtain the moment of inertia:

$$I = \frac{R_{cm}MgT^2}{4\pi^2}. \tag{3.10}$$

Equation 3.10 demonstrates that the moment of inertia for a physical pendulum can be obtained by measuring the distance from the pivot to the center of mass R_{cm}, the total mass M, and the period of small-angle oscillations T.

3.2 Conservation of Energy

The total energy of a pendulum is conserved. That is the total energy $\mathbb{T}$ is a constant. We define the potential energy to be zero when the pendulum is hanging vertically, with an opening angle of $\theta = 0$. The potential energy is largest when the kinetic energy is zero and opening angle is maximized *i.e.* at the top of the pendulum's swing.

The potential energy $\mathbb{U}$ of a pendulum as a function of opening angle θ is given by:

$$\mathbb{U} = MgR_{cm}(1-\cos\theta). \tag{3.11}$$

The kinetic energy $\mathbb{E}$ is given by:

$$\mathbb{E} = \frac{1}{2}I\omega^2. \tag{3.12}$$

The kinetic energy is maximized when the pendulum is swinging with the greatest angular velocity *i.e.* at the bottom of the pendulum's swing; where the potential energy is zero.

The conservation of energy requires that the energy at the bottom of the swing is equal to the potential energy at the top of the swing:

$$\mathbb{E}_{\theta=0} = \mathbb{U}_{\theta=\theta_{\max}}, \tag{3.13}$$

$$\mathbb{E}_{\theta=0} = MgR_{cm}(1-\cos\theta_{\max}). \tag{3.14}$$

3.3 Conservation of Angular Momentum

Angular momentum $\vec{L}$ about a point is given by:

$$\vec{L} = \vec{r} \times \vec{P} \tag{3.15}$$

where $\vec{r}$ is the vector from the reference point to a particle moving with momentum $\vec{P} = m\vec{v}$.

For a rigid body rotating about some axis the angular momentum depends on the moment of inertia about that axis and the angular velocity:

$$\vec{L} = I\vec{\omega}. \tag{3.16}$$

Consider a ball that is fired into a ballistic pendulum. At the instant before the ball hits the pendulum, the angular momentum is given by:

$$\vec{L}_{\text{ball}} = \vec{L_b} = \vec{R_b} \times m\vec{v} \tag{3.17}$$

where $\vec{R_b}$ is the vector from the pivot point of the pendulum to the center of mass of the ball, m is the mass of the ball, and $\vec{v}$ is its velocity.

The ball then collides inelastically[†] with the pendulum. Immediately after the collision the angular momentum is given by:

$$\vec{L}_{\text{pendulum + ball}} = \vec{L}_{pb} = I\vec{\omega} \tag{3.18}$$

where I is the moment of inertial for the, now combined, bullet and pendulum system and $\vec{\omega}$ is the angular velocity.

Since momentum is conserved we know that:

$$\vec{L_b} = \vec{L_{pb}} \tag{3.19}$$

[†]Recall that in an inelastic collision energy is not conserved but linear and angular momentum is.

3.4 Muzzle Velocity

To compute the muzzle velocity of a projectile using a ballistic pendulum, we first rewrite Equation 3.12 in terms of the angular momentum of the pendulum given in Equation 3.18:

$$\mathbb{E} = \frac{1}{2}\frac{|\vec{L_{pb}}|^2}{I}. \tag{3.20}$$

Solving for $\vec{L_{pb}}$ we obtain:

$$|\vec{L_{pb}}| = \sqrt{2I\mathbb{E}}. \tag{3.21}$$

We can rewrite this expression using the conservation of energy of the pendulum as expressed in Equation 3.14. The expression then becomes:

$$|\vec{L_{pb}}| = \sqrt{2IMgR_{cm}(1-\cos\theta)}. \tag{3.22}$$

Since momentum is conserved (as shown in Equation 3.19) we can equate the angular momentum of the pendulum with the angular momentum of the bullet (Equation 3.17) to obtain:

$$R_b m v = \sqrt{2IMgR_{cm}(1-\cos\theta)} \tag{3.23}$$

where m is the mass of the bullet and M is the combined mass of the pendulum and bullet.

Solving for the muzzle velocity of the bullet v we obtain:

$$v = \frac{1}{mR_b}\sqrt{2IMgR_{cm}(1-\cos\theta)}. \tag{3.24}$$

The muzzle velocity of the bullet is now expressed in terms of the measurable quantities: m, M, R_b, R_{cm}, and the maximum opening angle θ. The moment of inertia I of the bullet and pendulum system is measured using Equation 3.10.

4 Procedure

The primary goal of the experiment is to measure the spring constant k of the projectile launcher. This measurement will be achieved by applying the conservation of momentum and energy.

4.1 Measure the Moment of Inertia of the Pendulum and Ball

The first step is to measure the moment of inertia of the pendulum and the captured ball.

Remove the projectile launcher from the base so that the pendulum can swing freely. The pendulum can also be removed from the base by unscrewing the pivot axle from which it hangs. Remove the pendulum to measure distances and the mass. When replacing the pendulum, ensure that the angle indicator is to the right of the pendulum rod.

Equation 3.10 will be used to compute the moment of inertia. Refer to your results from the Pre-lab Quiz as you measure the moment of inertia.

To find the moment of inertia, make the following measurements (with an associated uncertainty estimate).

R_{cm}: the distance from the pendulum pivot to the center of mass of the combined pendulum ball system. Try balancing the pendulum on the edge of a ruler to determine the location of the center of mass.

M: the combined mass of the pendulum and ball.

T: the period of one oscillation of the pendulum and ball system. In order to measure the period accurately, small angle oscillations should be used (perhaps $5 - 10°$). It is difficult to measure a single oscillation accurately; there are large errors associated with starting and stopping the timing device accurately. To minimize the impact of this error, the oscillation of several (perhaps 10) oscillations should be timed.

Present the measured moment of inertia with its associated uncertainty (and appropriate units).

As a simple cross check, compare your result to a simplified model where the entire mass of the pendulum is assumed to be located a distance R_{cm} from the axis of rotation. That is:

$$I_{\text{simple}} = MR_{cm}^2. \tag{4.1}$$

The two results should be reasonably close.

4.2 Fire the Projectile Into the Pendulum

Attach the projectile launcher to the base so that the barrel of the cannon is aimed into the catching end of the pendulum. Hold the pendulum up out of the way while loading the projectile into the launcher.

When the launcher is loaded, let the pendulum hang in front of the projectile launcher. Be sure to lower the opening angle indicator on the pendulum support before each shot.

Fire the launcher several (~ 5) times and record the maximum angle reached by the pendulum. Set the angle indicator a couple of degrees below the previous angle reached by the pendulum each time to reduce the drag on the pendulum. Calculate the average opening angle and the associated uncertainty in the angle ($\bar{\theta} \pm \sigma_{\bar{\theta}}$).

Repeat this process for each setting of the projectile launcher: short, medium, and long range.

4.3 Compute the Muzzle Velocity

Using $\bar{\theta} \pm \sigma_{\bar{\theta}}$, compute the muzzle velocity of the projectile launcher for each setting.

To compute v using Equation 3.24, the following additional measurements should be made (with associated uncertainty estimates):

m: the mass of the projectile.

R_b: the distance between the pendulum pivot and the center of the ball when it is captured.

I: the moment of inertia computed in Section 4.1.

4.4 Determine the Spring Constant k of the Projectile Launcher

Use your data to determine the spring constant k of the projectile launcher.

5 Lab Notebook

Your submission will be evaluated using the following rubric:

Lab Notebook Practices

- Lab Notebook Mechanics (4 points)
  - Relevant information *e.g.*: your name, your lab partner's name, date, *etc.* is present.
  - The notebook is organized and easy to read. Markdown cells are used for narrative text. Code cells are clearly organized and commented.
  - The ZIP file of the notebook is healthy and runs correctly.
- Experiment Purpose (2 points)
  - In your own words, state the purpose of this experiment.
  - The work you record in your notebook should specifically address the stated purpose.
- Calculations & Error Propagation (6 points)
  - Present the equations used to analyze your collected data.
  - Present the equations used to account for and propagate error.
- Data Collection & Analysis (6 points)
  - The notebook tells a scientific story; it is an accurate record of the work that you did.
  - Record the informal observations you made during the experiment.
  - The notebook should show evidence of trial and error. Keep a good record of your work – recording mistakes is useful.
  - *Briefly* describe the methods you used to collect your data.
  - Measured quantities – that do not appear directly in your plot(s) – are clearly recorded.
  - Record rough data and plots that you used to verify that the analysis was on the right track.

Results and Interpretation

- Results (5 points)
  - Clearly state the final result(s) of your experiment. Remember to quote your result with units and appropriate significant digits.
  - Final result plots are well formatted and meet the standards described in the Figure Formatting reference.

- Physical Interpretation (4 points)
  - Throughout the notebook, interpret the data, rough plots, and final results in terms of the underlying physics.
- Significance (5 points)
  - Compare measured results to each other and/or to a known/expected value.
  - Choose the best available tools for your comparison (*e.g.* plots, pictures, discrepancy, significance, etc).
- Confidence (8 points)
  - Communicate to your audience how seriously your result should be taken. How confident you are in your result?
  - Discuss factors that may be affecting the *accuracy* and *precision* of your result.
  - Suggest improvements to your experiment to address your confidence, accuracy, and precision.