Moment of Inertia

1 LEARNING OBJECTIVES

At the conclusion of this activity you should be able to:

- Explain the relationship between linear motion and rotational motion.
- Design an experiment, using the conservation of energy, to measure the moment of inertia of a rolling object.
- Measure the moment of inertia for several balls.

2 BACKGROUND

2.1 CONSERVATION OF ENERGY

In an isolated system, the total amount of energy is constant[1]. That is the total initial energy E_i is equal to the total final energy E_f.

$$E_i = E_f \tag{2.1}$$

We often simplify the energy book-keeping by separating the total energy of a system into two distinct parts: kinetic and potential energy.

2.2 POTENTIAL ENERGY

Potential energy is energy that is due to the configuration of objects that exert forces on one another. One example of potential energy is due to the force of gravity. Gravitational potential energy is determined by the separation of an object from the surface of the earth[2]. The greater the separation, the greater the gravitational potential energy.

Gravitational potential energy U is given by:

$$U = mgh \tag{2.2}$$

where m is the mass of the object, g is the acceleration due to the force of gravity, and h is the distance above the surface of the earth.

2.3 KINETIC ENERGY

Kinetic energy is associated with the motion of an object[3].

An object of mass m and velocity $\vec{v}$ has a translational kinetic energy K_t given by:

$$K_t = \frac{1}{2} m \vec{v}^2 \tag{2.3}$$

When an object is dropped from a height near the surface of the earth, the potential energy of the system is converted into kinetic energy as the object falls towards the ground. Given the original height of the object, one can calculate the final velocity by using the conservation of energy:

$$\begin{aligned} E_i &= E_f && (2.4) \\ U &= K_t && (2.5) \\ mgh &= \frac{1}{2} m v^2 && (2.6) \end{aligned}$$

2.4 KINETIC ENERGY OF ROTATION

When an object rotates, it also has kinetic energy that is determined by the mass and shape of the object as well as the rate of rotation[4]. Rotational kinetic energy is given by:

$$K_r = \frac{1}{2} I \omega^2, \tag{2.7}$$

where ω is the angular speed and I is the moment of inertia. The moment of inertia is analogous to mass in many ways. The moment of inertia describes an object's resistance to changes in rotational motion.

2.5 LINEAR AND ANGULAR MOTION

When a wheel is rolling across a surface, the rate at which it rotates is given by the angular velocity ω. The angular velocity is the same for all points on the wheel[5].

The speed at which the center of the wheel moves depends on the radius r of the wheel. Two wheels spinning with the same angular velocity will have different speeds if the radii differ.

For a rotating body, for example a wheel, with radius r that is rolling without slipping with an angular velocity of ω, the speed at some distance r from the axis of rotation is given by:

$$v = \omega r. \tag{2.8}$$

2.6 MOMENT OF INERTIA

The moment of inertia is defined by the mass of an object and how the mass is distributed[6]. The moment of inertia is defined to be:

$$I = \int r^2 \, dm \tag{2.9}$$

where r is the distance to the mass element dm from the axis of rotation.

For solid objects with uniform density, calculating the moment of inertia is dictated by the geometry of the object and the axis about which it is rotated. Conveniently, the integrals prescribed in Equation 2.9 have already been done for many different shapes. Table 2.1 lists moments of inertia that have been

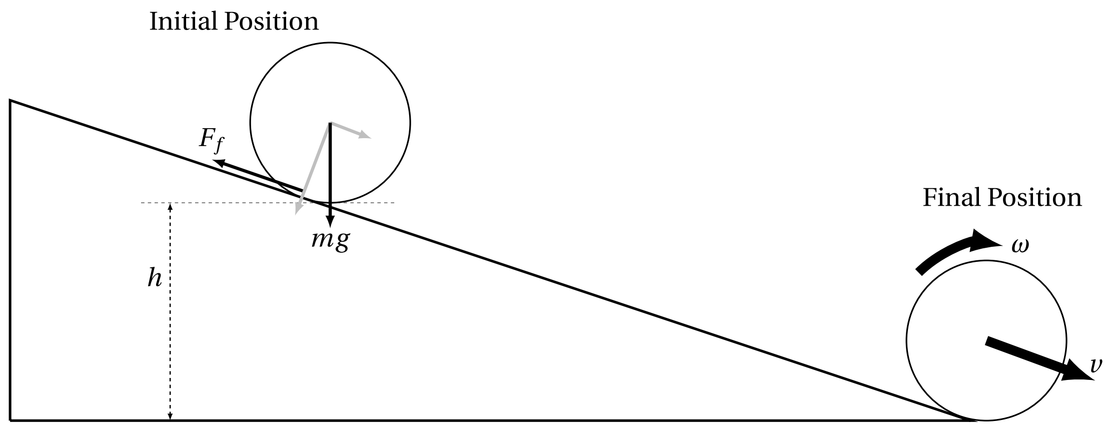

Figure 2.1: A ball with potential energy $U = mgh$ is released from rest and allowed to roll down a ramp. The force of gravity causes the ball to accelerate. The friction F_f between the ramp and the ball also causes the ball to rotate. At the bottom of the ramp, the ball is rotating with angular velocity ω and traveling with linear velocity v; all of the gravitational potential energy has been changed into kinetic energy. Assuming that the ball is not slipping, the linear velocity v and angular velocity ω are related by $v = \omega r$.

calculated for several objects that are relevant for this lab. Note that the general form of the moment of inertia for an object rotating around a given axis is:

$$I = C\left(mr^2\right) \tag{2.10}$$

where C is simply a number. Large values for the parameter C, indicate large moments of inertia.

Type of Sphere with radius r	**Moment of Inertia** I	C
Solid	$\frac{2}{5}mr^2$	0.40
Thin Hollow Shell	$\frac{2}{3}mr^2$	~ 0.67

Table 2.1: A list of several theoretical moments of inertia that will be useful for this lab. In all cases the sphere has mass m and radius r. Since spheres are symmetric, the moment of inertia is the same for rotation about any diameter.

2.7 Measuring the Moment of Inertia

Suppose we are interested in measuring the moment of inertia I of a ball with mass m and radius r.

One way to make the measurement is to use the conservation of energy.

Suppose that the ball is released from height h and allowed to roll down a ramp as shown in Figure 2.1. Before the ball is released, the ball is stationary and has only potential energy. The total initial energy of the system can be written as:

$$E_i = mgh. \tag{2.11}$$

At the bottom of the ramp, all of the gravitational potential energy has been converted to kinetic energy. The total final energy of the system can be written as:

$$E_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \tag{2.12}$$

where the first term is the kinetic energy due to the motion of the center of mass and the second term is additional kinetic energy due to the *rotation* of the ball.

Using the conservation of energy (Equation 2.1) we can equate the total initial energy to the total final energy to obtain:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2. \tag{2.13}$$

Assuming that the ramp is shallow enough that the ball rolls without slipping, we can use Equation 2.8 to write Equation 2.13 in terms of the linear velocity v only:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{r^2}. \tag{2.14}$$

Equation 2.14 can be solved for the moment of inertia I. It can be shown that the moment of inertia is given by:

$$I = \frac{2mr^2}{v^2}\left(gh - \frac{1}{2}v^2\right) \tag{2.15}$$

where the acceleration due to gravity g is known and the variables m, r, v, and h are all observable quantities.

2.8 Measuring Velocity with Photogates

Photogates[7] are used to measure the velocity of a rolling ball. In Logger Pro, open the file: ‘‘`File`’’ → ‘‘`Open...`’’ → ‘‘`Probes & Sensors`’’ → ‘‘`Photogates`’’ → ‘‘`One Gate Timer.cmbl`’’. The parameter `PhotogateDistance1` sets the length of the object that will obstruct the photogate. Logger Pro computes velocity by dividing the distance set in `PhotogateDistance1` by the time that the photogate is obstructed[8]:

$$v_{\text{measured}} = \frac{d}{t} = \frac{\text{PhotogateDistance1}}{\text{obstruction time}}. \tag{2.16}$$

In order to accurately measure the velocity, it is critical that the photogate is positioned so that it is obscured by the diameter of the ball (*i.e.* `PhotogateDistance1`). If the photogate is positioned too high or too low, the reported velocity will not be accurate (see Figure 2.2).

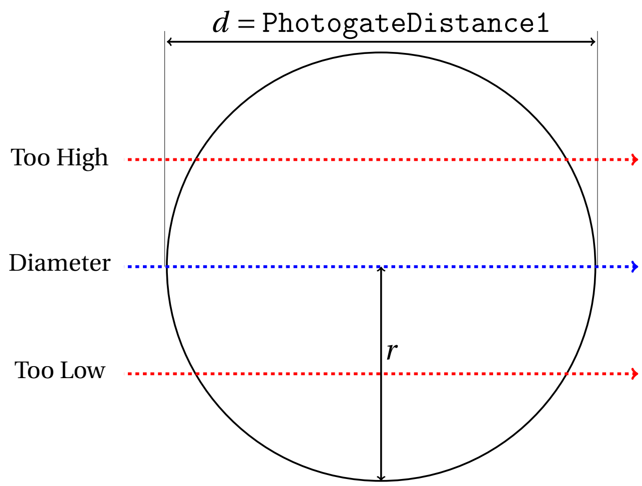

Figure 2.2: Photogates can be used to measure the velocity of an object. When the photogate is blocked, a timer starts. When it is unblocked, the timer stops. The velocity is computed by using the measured time and a user-defined distance. The distance, is the size of the object that blocks the photogate. In the case of a ball, if the input distance is the diameter of the ball, the photogate must be positioned such that the sensor is blocked by the diameter of the ball. If the photogate is positioned too high or too low, the obscured distance will be less than expected resulting in systematically biased velocity measurements.

3 PROCEDURE

There are three spheres: One solid, one hollow, and another with an unknown interior structure. The goal of this lab is to make a data-informed statement about the internal structure of the unknown sphere.

3.1 EXPERIMENT

- Setup an experiment to measure C for three different balls. Recall that C is related to the moment of inertia I by Equation 2.10.
- From the Prelab Quiz, we see that the observable quantities h (the change in height down a ramp) and v (the final velocity) can be used to experimentally determine C (see Figure 2.1). Take data to generate linear plots from which you will determine the parameter C for each of the balls.
- Vary the height h through which the ball falls by varying the angle of the ramp. Multiple trials at each h may be required to account for random fluctuations in the measured final velocity v.

3.2 SUGGESTIONS FOR ANALYSIS

- Fit the data with a linear model.
- Propagate the uncertainty in the model parameters to estimate the uncertainty associated with C.
- Compare the measured C values for the solid and hollow balls to the theoretical predictions listed in Table 2.1.
- Is the ball with unknown internal structure more solid-like or hollow-like?
- Are you able to statistically distinguish the moment of inertia between the various balls?

4 LAB NOTEBOOK

Your submission will be evaluated using the following rubric:

- Lab Notebook Mechanics (6 points)
  - Relevant information *e.g.*: your name, your lab partner's name, date, *etc.* is present.
  - The notebook is organized and easy to read. Markdown cells are used for narrative text. Code cells are clearly organized and commented.
  - The ZIP file of the notebook is healthy and runs correctly.
- Data Analysis & Plots (6 points)
  - The notebook tells a scientific story; it is an accurate record of the work that you did.
  - The notebook should show evidence of trial and error. Keep a good record of your work – recording mistakes is useful.
  - Record rough data and plots that you used to verify that the analysis was on the right track. Final versions of plots should be well formatted and meet the plotting guidelines for the course.
  - Use models to identify trends that your data exhibit or other apparent relationships between your independent and dependent variables.
- Results and Comparison (6 points)
  - Clearly state the final result(s) of your experiment. Remember to quote your result with units and appropriate significant digits.
  - Final result plots are well formatted and meet the standards described in the Figure Formatting reference.
  - A useful comparison is made to a known/expected value or another similar result.
  - Choose the best available tools for your comparison (*e.g.* plots, pictures, discrepancy, significance of discrepancy, etc).
- Uncertainty and Error Propagation (6 points)
  - Identify the dominant source(s) of error in your experiment.
  - Support your conclusions with appropriate error estimates and error propagation calculations.
- Physical Interpretation (6 points)
  - Throughout the notebook, interpret the data, rough plots, and final results in terms of the underlying physics.
  - What are you able to conclude from your data? Clearly explain how you arrived at your conclusions from your experimental observations.
  - Reflect on how your experiments connects with the physics concepts you are studying.

REFERENCES

[1] Resnick, R., Halliday, D., Krane, K. S. (2002) *Physics, Vol. 1, 5th Edition.* Danvers, MA: John Wiley & Sons, Inc.. See Chapter 13-1 (pg. 279).

[2] Ibid. See Chapter 12-2 (pg. 259).

[3] Ibid. See Chapter 11-6 (pg. 239).

[4] Ibid. See Chapter 11-7 (pg. 243).

[5] Ibid. See Chapter 8-2 (pg. 160).

[6] Ibid. See Chapter 9-3 (pg. 183).

[7] `https://www.pasco.com/file_downloads/Downloads_Manuals/Photogate-Head-Manual-ME-9498A.pdf`

[8] `http://www2.vernier.com/manuals/Logger_Pro_Introduction_to_the_Vernier_Photogate.pdf`. See "Part I Gate Timing" (pp. 2-3).