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SUBMODULAR SET FUNCTIONS, MATROIDS AND THE GREEDY ALGORITHM: TIGHT WORST-CASE BOUNDS AND SOME GENERALIZATIONS OF THE RADO-EDMONDS THEOREM

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For the problem max$\{Z(S)$: S is an independent set in the matroid $X\}$, it is well-known that the greedy algorithm finds an optimal solution when Z is an additive set function (Rado-Edmonds theorem). Fisher, Nemhauser and Wolsey have shown that, when Z is a nondecreasing submodular set function satisfying $Z(\emptyset)=0$, the greedy algorithm finds a solution with value at least half the optimum value. In this paper we show that it finds a solution with value at least $1/(1+\alpha)$ times the optimum value, where α is a parameter which represents the 'total curvature' of Z. This parameter satisfies $0\leq\alpha\leq 1$ and $\alpha=0$ if and only if the set function Z is additive. Thus the theorems of Rado-Edmonds and Fisher-Nemhauser-Wolsey are both contained in the bound $1/(1+\alpha)$. We show that this bound is best possible in terms of α. Another bound which generalizes the Rado-Edmonds theorem is given in terms of a 'greedy curvature' of the set function. Unlike the first bound, this bound can prove the optimality of the greedy algorithm even in instances where Z is not additive. A third bound, in terms of the rank and the girth of X, unifies and generalizes the bounds $(e-1)/e$ known for uniform matroids and $\frac{1}{2}$ for general matroids. We also analyze the performance of the greedy algorithm when X is an independence system instead of a matroid. Then we derive two bounds, both tight: The first one is $[1-(1-\alpha/K)^k]/\alpha$ where K and k are the sizes of the largest and smallest maximal independent sets in X respectively; the second one is $1/(p+\alpha)$ where p is the minimum number of matroids that must be intersected to obtain X.

1. Introduction

Many problems in combinatorial optimization can be written in a natural way as

$$\max\{Z(S): S\in X\} \tag{1.1}$$

where X is a family of subsets of a finite set N and Z is a set function defined on $\{S\subseteq N\}$. The family X is called an *independence system* if

$$S\in X \text{ and } T\subseteq S \quad\Rightarrow\quad T\in X. \tag{1.2}$$

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The sets in X are often called *independent sets*. If furthermore

$$S, T \in X \text{ and } |T|+1=|S| \quad \Rightarrow \quad \exists j \in S-T \text{ such that } T \cup \{j\} \in X, \tag{1.3}$$

then the family X is called a *matroid*. See [12], [10].

For a set function Z, we define the *discrete derivative* at $S \subset N$ in direction $j \in N$ as $\varrho_j(S) = Z(S \cup \{j\}) - Z(S)$. The set function Z is said to be *submodular* if

$$T \subseteq S \subset N \quad \Rightarrow \quad \varrho_j(T) \geq \varrho_j(S) \quad \text{for all } j \in N-S. \tag{1.4}$$

A *greedy* (or steepest ascent) algorithm comes naturally to mind when X is a matroid (or an independence system) and when Z is submodular.

Greedy Algorithm. Start with the empty set. Then recursively add to the current solution set S an element j with the largest discrete derivative $\varrho_j(S)$ among all $j \in N-S$ such that $S \cup \{j\} \in X$ and $\varrho_j(S) \geq 0$. Stop when no such element exists.

Well-known examples of problems fitting the framework (1.1) include:

(1.5) The problem of finding a maximum weight independent set in a matroid: X is a matroid and Z is *additive* (i.e., $\varrho_j(S) = \varrho_j$, a constant independent of S). Then the greedy algorithm finds an optimal solution (Rado–Edmonds theorem) [3]; a common application occurs when the independent sets are the forests of a graph [9], [10].

(1.6) A simple plant location problem [2]: X is a *uniform* matroid (i.e. $X = \{S \subseteq N: |S| \leq K\}$) and Z is a nondecreasing submodular set function with $Z(\emptyset) = 0$. Then the greedy algorithm finds a solution with a value which is guaranteed to be at least $(e-1)/e$ times the optimum value [2], [11], where e is the base of the natural logarithms.

(1.7) The problem of finding a set of maximum weight in the intersection of two matroids: X is one of the two matroids and, for all $S \subseteq N$, $Z(S)$ is the maximum weight over all sets $T \subseteq S$ which are independent in the second matroid. Then the greedy algorithm guarantees a solution within 50% of the optimum [5].

These are three examples where the feasible set is a matroid. Although the bound guaranteed by the greedy algorithm is different in each case we believe that these results can be unified. For example, we will show that the bounds (1.6) and (1.7) are the two extreme values of a bound expressed in terms of the cardinalities of the smallest infeasible and largest feasible sets. These parameters are called the *girth* and the *rank* of the matroid X respectively. We will also show that the bounds (1.5) and (1.7) are the two extreme values of a bound expressed in terms of a parameter reflecting the 'total curvature' of the function Z (see definition below).

It will be convenient to assume that, in (1.1), the objective function is nondecreasing and satisfies $Z(\emptyset)=0$. (As in [11], general submodular set functions can be handled by using an appropriate performance measure; however, with the above assumption, the greedy performance will simply be given as a percentage of the optimum value.) Nondecreasing submodular set functions such that $Z(\emptyset)=0$ are *subadditive* (i.e. $Z(S)+Z(T)\geq Z(S\cup T)$ $\forall S, T\subseteq N$). They arise in location theory and more generally in economic problems where the marginal profit $\varrho_j(S)$ of performing a new action j once a set S of actions is already undertaken is nonincreasing with respect to S. They have also been used to measure consumer satisfaction [8]. In the maximum weight forest problem [see (1.5)] it is sometimes more realistic to assume that the objective function is submodular rather than just additive. Three other examples from the mathematical programming literature can be found in [11]. An example which may be little known occurs in network flow theory. Given a network with edge capacities, a source s and a set N of sinks, let $Z(S)$ be the maximum flow from s to a subset S of sinks. Obviously the set function Z is nondecreasing and $Z(\emptyset)=0$. Fulkerson liked to ask whether Z is submodular in his course on network flows. It is left here as an exercise.

The *total curvature* of a nondecreasing submodular set function is

$$\alpha=\max_{j\in N^*}\left\{\frac{\varrho_j(\emptyset)-\varrho_j(N-\{j\})}{\varrho_j(\emptyset)}\right\}$$

where $N^*=\{j\in N\colon \varrho_j(\emptyset)>0\}$. Note that α can vary between 0 and 1 and that $\alpha=0$ if and only if Z is additive. In Section 2 we prove that the greedy algorithm finds a solution with a value which is guaranteed to be at least $1/(1+\alpha)$ times the optimum value for problem (1.1) when X is a matroid and Z is a nondecreasing submodular set function with $Z(\emptyset)=0$ and total curvature α. This bound generalizes the Rado–Edmonds theorem, see (1.5), as well as the bound (1.7), obtained when $\alpha=0$ and $\alpha=1$ respectively. We show also that the bound $1/(1+\alpha)$ is best possible in terms of α.

Let $S^0=\emptyset\subseteq S^1\subseteq\cdots\subseteq S^K$ be the sets which are successively constructed in the course of the greedy algorithm (S^K is the greedy solution). We define the *greedy curvature* of Z as

$$\alpha_G=\max_{0\leq i\leq K-1}\ \max_{j\in N^i}\left\{\frac{\varrho_j(\emptyset)-\varrho_j(S^i)}{\varrho_j(\emptyset)}\right\}$$

where $N^i=N^*\cap\{j\in N-S^i\colon S^i\cup\{j\}\in X\}$. Note that $\alpha_G\leq\alpha$, the total curvature of Z. Note also that α_G can equal 0 even when Z is not additive. In Section 3 we prove that the greedy algorithm finds a solution with a value which is guaranteed to be at least $(1-\alpha_G)$ times the optimum value of problem (1.1), again with the assumptions that X is a matroid and that Z is nondecreasing, submodular and $Z(\emptyset)=0$. Note that when $\alpha_G=0$ we can guarantee the optimality of the greedy algorithm even though the objective function may not be additive.

In Section 4 we give a bound which depends only on the matroid X. Let K be the *rank* of X, i.e. the common cardinality of the maximal independent sets and let $(h+1)$ be its *girth*, i.e. the cardinality of a smallest dependent set. We prove the following tight bound. The value of a greedy solution is at least half the optimum value if $K \geq 2h$ and at least

$$\left[1-\frac{h}{K}\left(\frac{K-1}{K}\right)^{2h-K}\right] \quad \text{times the optimum value}$$

if $K<2h$. Our bounding method is based on the weak duality theorem of linear programming in the same spirit as [1, 11]. More precisely we decompose the greedy solution $Z_G = \varrho_1 + \varrho_2 + \cdots + \varrho_K$ where $\varrho_i \geq 0$. Then we find inequalities relating the optimum value Z^* to this decomposition, say $A\varrho \geq Z^*$, where ϱ is the column vector of ϱ_i's and A is a matrix. Now find $\pi \geq 0$ such that $\pi A \leq e$ where e is a row vector with K ones. We have

$$Z_G = \sum_{i=1}^{K} \varrho_i \geq \pi A \varrho \geq \left(\sum_{i=1}^{m} \pi_i\right) Z^*$$

providing a bound on the greedy solution. To show that the bound is tight we give a family of examples which achieve it. The originality of our system $A\varrho \geq Z^*$ is that it incorporates simultaneously information on the objective function Z and on the matroidal structure of the feasible set.

Examples of independence systems are quite common in 0, 1 programming. In fact, given a nonnegative matrix A, the family of 0, 1 vectors that satisfy $Ax \leq b$ is an independence system (here we identify a set S and its incidence vector $x_j = 1$ if $j \in S$, 0 otherwise). Conversely any independence system is the solution set of such a 0, 1 program.

Two bounds were proven in [5] regarding the greedy algorithm for problem (1.1) when X is an independence system and Z is a nondecreasing submodular set function with $Z(\emptyset)=0$. First it was shown that the greedy algorithm guarantees a solution value at least $[1-((K-1)/K)^k]$ times the optimum value, where K and k are respectively the maximum and minimum cardinalities of a maximal independent set in X. The second bound is $1/(p+1)$ where p is the minimum number of matroids that one needs to intersect in order to obtain the independence system X. (The fact that any independence system can be expressed as the intersection of matroids is proved in [7].) When the set function Z is additive these two bounds can be sharpened to k/K and $1/p$ respectively ([6], [7]).

In Section 5 we show that, in terms of K, k and the total curvature of Z, the greedy algorithm guarantees a solution value at least equal to

$$\frac{1}{\alpha}\left[1-\left(\frac{K-\alpha}{K}\right)^{k}\right] \quad \text{times the optimum value.}$$

This bound is tight for all $0<\alpha\leq 1$. Note that when we set $\alpha=1$ and $\alpha \to 0$ we get the bounds $[1-((K-1)/K)^k]$ and k/K respectively. Note also that, when $k=K$, we get a result for the uniform matroid, namely the bound $(1-e^{-\alpha})/\alpha$; furthermore this bound is best possible in terms of α. In particular it is easy to see that it dominates the bound $1/(1+\alpha)$ for any $0<\alpha<1$ since

$$(1-e^{-\alpha})/\alpha > 1-\tfrac{1}{2}\alpha > 1/(1+\alpha).$$

In Section 6 we give a bound in terms of p and α, namely the bound $1/(p+\alpha)$. This generalizes the bounds $1/(p+1)$, $1/p$ and $1/(1+\alpha)$ mentioned earlier in this introduction for different variations of problem (1.1). In fact the result is proved in the more general context $\max\{Z(v): v \in X\}$ where Z is a nondecreasing submodular vector function and X is the intersection of p polymatroids.

2. The bound $1/(1+\alpha)$

Let N be a finite set and $Z: 2^N \to \mathbb{R}^+$ a nondecreasing submodular set function with $Z(\emptyset)=0$. Given a set $\Omega \subseteq N$ and an ordered set $S=\{j_1,\dots,j_t\} \subseteq N$, we define $S^0=\emptyset$, $S^i=\{j_1,\dots,j_i\}$ for $1 \le i \le t$, and

$$\alpha_0 = \max_{i: j_i \in S^*} \left\{ \frac{\varrho_{j_i}(S^{i-1}) - \varrho_{j_i}(S^{i-1} \cup \Omega)}{\varrho_{j_i}(S^{i-1})} \right\}$$

where $S^* = \{j_i \in S-\Omega: \varrho_{j_i}(S^{i-1})>0\}$. Note that $\alpha_0 \le \alpha$, the total curvature of Z. Denote $\varrho_i = \varrho_{j_i}(S^{i-1})$, $i=1,\dots,t$.

Lemma 2.1.

$$Z(\Omega) \le \alpha_0 \sum_{i: j_i \in S-\Omega} \varrho_i + \sum_{i: j_i \in \Omega \cap S} \varrho_i + \sum_{\omega \in \Omega - S} \varrho_\omega(S).$$

Proof. A simple consequence of the definition (1.4) is

$$Z(\Omega \cup S) \le Z(S) + \sum_{\omega \in \Omega - S} \varrho_\omega(S). \tag{2.1}$$

By the definition of α_0

$$\begin{aligned} Z(\Omega \cup S) &= Z(\Omega) + \sum_{i: j_i \in S-\Omega} \varrho_{j_i}(\Omega \cup S^{i-1}) \\ &\ge Z(\Omega) + (1-\alpha_0) \sum_{i: j_i \in S-\Omega} \varrho_i. \qquad \square \end{aligned}$$

In this section and the next two we assume that X is a matroid. We also assume that $S^K=\{j_1,\dots,j_K\}$ is the sequence chosen by the greedy algorithm. Note that S^K is a *base* (i.e. a maximal set in X). A consequence of axiom (1.3) is that all the bases of X have the same cardinality. Recall the notation $S^i=\{j_1,\dots,j_i\}$ and $\varrho_i = \varrho_{j_i}(S^{i-1})$, $i=1,\dots,K$.

Lemma 2.2. *The elements of any base $\Omega^K=\{\omega_1,\dots,\omega_K\}$ can be ordered so that $\varrho_{\omega_i}(S^{i-1}) \le \varrho_i$, $i=1,\dots,K$. Furthermore, if $\omega_i \in \Omega^K \cap S^K$, then $\omega_i \equiv j_i$.*

Proof. The lemma is proved by induction on i, for $i=K,\dots,1$. Assume that the elements ω_l satisfy the inequality $\varrho_{\omega_l}(S^{l-1})\le\varrho_l$ for $l>i$, and let $\Omega^i=\Omega^K-\{\omega_l : l>i\}$. Consider the sets S^{i-1} and Ω^i. By the matroid axiom (1.3), $\exists\omega_i\in\Omega^i-S^{i-1}$ such that $S^{i-1}\cup\{\omega_i\}\in X$. Since j_i is the element chosen by the greedy algorithm, $\varrho_{\omega_i}(S^{i-1})\le\varrho_{j_i}(S^{i-1})$. Furthermore if $j_i\in\Omega^i$ we can set $\omega_i=j_i$. □

Let Z^{G} be the value of a greedy solution and Z^* the optimal value of problem (1.1).

Theorem 2.3. *If X is a matroid and Z is a nondecreasing submodular set function with $Z(\emptyset)=0$ and total curvature α, then*

$$Z^{\mathrm{G}}\ge\frac{1}{1+\alpha}Z^*.$$

Proof. Let Ω^K be an optimal solution and S^K the greedy solution. By Lemma 2.1

$$Z^*\le\alpha_0 Z^{\mathrm{G}}+\sum_{i:j_i\in\Omega^K\cap S^K}\varrho_i+\sum_{i:\omega_i\in\Omega^K-S^K}\varrho_{\omega_i}(S^K).$$

By Lemma 2.2, $\varrho_{\omega_i}(S^K)\le\varrho_{\omega_i}(S^{i-1})\le\varrho_i$. Therefore,

$$Z^*\le\alpha_0 Z^{\mathrm{G}}+Z^{\mathrm{G}}\le(1+\alpha)Z^{\mathrm{G}}.\qquad\square$$

Corollary 2.4. *The proof actually shows the stronger bound $Z^{\mathrm{G}}\ge Z^*/(1+\alpha_0)$. When $\alpha_0=0$, the greedy algorithm finds an optimal solution.*

Corollary 2.5. *When Z is additive* (*equivalently when $\alpha=0$*), *the greedy algorithm finds an optimal solution* (Rado–Edmonds Theorem).

Corollary 2.6. $Z^{\mathrm{G}}\ge\frac{1}{2}Z^*$. (See [5].)

Corollary 2.7. *Any two maximal sets in the intersection of two matroids have cardinalities which are within a factor of 2 of each other.*

Remark 2.8. It is worth stressing the combinatorial spirit of the derivation of the bound $1/(1+\alpha)$. This derivation is based on two observations, namely Lemma 2.2 and the inequality (2.1). It is close to the classical proof of the Rado–Edmonds Theorem, where the additive version of Lemma 2.2 is used implicitly.

Remark 2.9. The proof of Theorem 2.3 can be modified to yield a stronger bound: instead of Ω^K and S^K, consider Ω^K and S^{K-1}. Then, by Lemma 2.1

$$Z^*\le\alpha_0(Z^{\mathrm{G}}-\varrho_K)+\sum_{i:j_i\in\Omega^K\cap S^{K-1}}\varrho_i+\sum_{i:\omega_i\in\Omega^K-S^{K-1}}\varrho_{\omega_i}(S^{K-1}).$$

By Lemma 2.2, $\varrho_{\omega_i}(S^{K-1}) \le \varrho_{\omega_i}(S^{i-1}) \le \varrho_i$ for all i. Therefore $Z^* \le (1+\alpha_0)Z^{\mathrm{G}} - \alpha_0 \varrho_K$. This proves the bound

$$Z^{\mathrm{G}} \ge \frac{1}{1+\alpha_0} Z^* + \frac{\alpha_0}{1+\alpha_0} \varrho_K. \tag{2.2}$$

Corollary 2.10. *If $Z^* \ne 0$ and $\alpha_0 \ne 0$ or 1, then $Z^{\mathrm{G}} > Z^*/(1+\alpha_0)$ (a strict inequality).*

Proof. Assume that the inequality is not strict; then $\varrho_K = 0$ as a consequence of (2.2). Then in every base there exists an element ω_K such that $\varrho_{\omega_K}(S^{K-1}) = 0$. Therefore $\varrho_{\omega_K}(\emptyset) = 0$, since $\alpha_0 < 1$. The greedy and optimal solution values are not changed if we intersect the matroid X by the uniform matroid $X^{K-1} = \{T : |T| \le K-1\}$. The bound (2.2) becomes

$$\frac{1}{1+\alpha_0} Z^* + \frac{\alpha_0}{1+\alpha_0} \varrho_{K-1}.$$

Again by our assumption we must have $\varrho_{K-1} = 0$ and, by induction, $\varrho_i = 0$ $\forall i = 1, \dots, K$. This would imply $Z^* = 0$, a contradiction. □

Corollary 2.11. *If $Z^* \ne 0$ and $\alpha \ne 0$ or 1, then $Z^{\mathrm{G}} > Z^*/(1+\alpha)$ (a strict inequality).*

Next we show that the bound $1/(1+\alpha)$ is best possible in terms of α. In turn this implies that the bound $1/(1+\alpha_0)$ is best possible in terms of α_0.

Theorem 2.12. *There exists an infinite family of problems such that*

$$Z_K^{\mathrm{G}} \to \frac{1}{1+\alpha} Z_K^* \quad \text{as } K \to \infty$$

where Z_K^{G} and $Z_K^* \ne 0$ are respectively the greedy and optimal values of the Kth problem.

Proof. When $\alpha = 0$ the bound is always tight, so there is nothing to prove. When $\alpha = 1$, the result is already known [5]. So assume $0 < \alpha < 1$.

Let $N = \{j_1, \dots, j_K, \omega_1, \dots, \omega_K\}$ and $N^t = \{j_1, \dots, j_t, \omega_1, \dots, \omega_t\}$ for $t = 1, \dots, K$. We define X as the family of all the subsets $S \subseteq N$ such that

$$|S \cap N^t| \le t \quad \text{for } t = 1, \dots, K. \tag{2.3}$$

It is clear that X is an independence system, i.e. axiom (1.2) is verified. So to prove that X is a matroid it remains to show that axiom (1.3) holds. Let $S, T \in X$ be such that $|T| + 1 = |S|$. Then $S - T \ne \emptyset$. Let $e_i \in S - T$ be an element with largest index i, $1 \le i \le K$, where e_i denotes either j_i or ω_i. We will show that $T \cup \{e_i\} \in X$, namely that $|(T \cup \{e_i\}) \cap N^t| \le t$ for $t = 1, \dots, K$. By the choice of e_i, $|S \cap N^t| \ge |(T \cup \{e_i\}) \cap N^t|$ for $t \ge i$. This implies $|(T \cup \{e_i\}) \cap N^t| \le t$ for $t \ge i$, using the fact that S is independent and (2.3). When $t<i$, $(T\cup\{e_i\})\cap N^t=T\cap N^t$, so the inequality $|(T\cup\{e_i\})\cap N^t|\le t$ for $t<i$ follows from the fact that T is independent. This shows $T\cup\{e_i\}\in X$ as announced.

Define the set function Z, for any $S\subseteq N$, as

$$Z(S)=\sum\{\varrho_i : j_i\in S\}+\sum\{\varrho_i:\omega_i\in S \text{ such that } i=1 \text{ or } j_{i-1}\in S\}$$
$$+\frac{1}{1-\alpha}\sum\{\varrho_i:\omega_i\in S \text{ such that } i\ge 2 \text{ and } j_{i-1}\notin S\}$$

where

$$\varrho_i=\frac{(1-\alpha)^{i-1}\alpha}{1+\alpha}\quad \text{for } i=1,\dots,K.$$

This function is submodular, nondecreasing with total curvature α since

$$\varrho_{\omega_i}(S)=\begin{cases}\dfrac{1}{1-\alpha}\varrho_i & \text{if } i\ge 2 \text{ and } j_{i-1}\notin S,\\ \varrho_i & \text{if } i=1 \text{ or } j_{i-1}\in S,\end{cases}$$

and

$$\varrho_{j_i}(S)=\begin{cases}\varrho_i & \text{if } \omega_{i+1}\notin S,\\ \varrho_i-\dfrac{\alpha}{1-\alpha}\varrho_{i+1}=(1-\alpha)\varrho_i & \text{if } \omega_{i+1}\in S.\end{cases}$$

Now we compute the value of a greedy solution. The largest discrete derivative at $\emptyset$ is $\varrho_{j_1}(\emptyset)=\varrho_{\omega_1}(\emptyset)=\varrho_{\omega_2}(\emptyset)=\alpha/(1+\alpha)$. So the greedy algorithm can choose j_1 in the first iteration. Assume $S^t=\{j_1,\dots,j_t\}$ has been chosen. $\varrho_{j_{t+1}}(S^t)=\varrho_{t+1}$, $S^t\cup\{\omega_i\}\notin X$ for $i\le t$ and $\varrho_{\omega_i}(S^t)\le\varrho_{t+1}$ for $i\ge t+1$. So j_{t+1} can be chosen next. The greedy solution $S^K=\{j_1,\dots,j_K\}$ has the value

$$Z_K^G=\sum_{i=1}^{K}\varrho_i=\frac{1}{1+\alpha}[1-(1-\alpha)^K]. \tag{2.4}$$

The optimal set is $\{\omega_1,\omega_2,\dots,\omega_K\}$ and has value

$$Z_K^*=1-\frac{1}{1+\alpha}(1-\alpha)^{K-1}. \tag{2.5}$$

This completes the proof. □

3. The bound $(1-\alpha_G)$

Let $S^0=\emptyset\subseteq S^1\subseteq\cdots\subseteq S^K$ be the sets which are successively constructed in the course of the greedy algorithm. Let $N^i=\{j\in N-S^i\colon S^i\cup\{j\}\in X \text{ and } \varrho_j(\emptyset)>0\}$. The greedy curvature of Z is

$$\alpha_{\mathrm{G}}=\max_{1\le i\le K-1}\max_{j\in N^i}\left\{\frac{\varrho_j(\emptyset)-\varrho_j(S^i)}{\varrho_j(\emptyset)}\right\}.$$

Note that the greedy curvature α_{G} of Z is defined with respect to X and that $\alpha_{\mathrm{G}}\le\alpha$, the total curvature of Z.

Theorem 3.1. *If X is a matroid and Z a nondecreasing submodular set function with $Z(\emptyset)=0$, then*

$$\frac{Z^{\mathrm{G}}}{Z^*}\ge 1-\alpha_{\mathrm{G}}\frac{K-1}{K} \tag{3.1}$$

and this bound is tight for $0\le\alpha_{\mathrm{G}}\le 1/K$.

Proof. By Lemma 2.2, $\varrho_{\omega_i}(S^{i-1})\le\varrho_i$, $i=1,\dots,K$. By the definition of α_{G}, $\varrho_{\omega_i}(S^{i-1})\ge(1-\alpha_{\mathrm{G}})\varrho_{\omega_i}(\emptyset)$ for $i=2,\dots,K$. So the optimal value Z^* satisfies

$$Z^*\le\sum_{i=1}^{K}\varrho_{\omega_i}(\emptyset)\le\varrho_1+\frac{1}{(1-\alpha_{\mathrm{G}})}\sum_{i=2}^{K}\varrho_i.$$

Therefore $Z^{\mathrm{G}}/Z^*\ge 1-\alpha_{\mathrm{G}}+\alpha_{\mathrm{G}}(\varrho_1/Z^*)$. Since $Z^*\le K\varrho_1$, the validity of the bound (3.1) is proved.

The fact that the bound can be achieved is shown by the following example. The matroid has $K+1$ elements and only one set is infeasible, namely the full set. The set function is defined by

$$Z(S)=\begin{cases}|S| & \text{if } x_1\notin S,\\ \beta+|S|(1-\beta) & \text{if } x_1\in S.\end{cases}$$

It is easy to check that Z is nondecreasing and submodular for $0\le\beta\le 1/K$, that $\alpha_{\mathrm{G}}=\beta$ and that $Z^{\mathrm{G}}/Z^*=1-\alpha_{\mathrm{G}}(K-1)/K$ if the greedy algorithm chooses x_1 in the first iteration. □

Remark 3.2. The bound $(1-\alpha_{\mathrm{G}}(K-1)/K)$ can easily be computed in the course of the algorithm. It gives an a posteriori bound on the quality of the greedy solution which can be tighter than the a priori bound $1/1(+\alpha)$. In fact it proves the optimality of the greedy algorithm when $\alpha_{\mathrm{G}}=0$, which occurs when Z is additive but may also occur for more general set functions.

Corollary 3.3. *If X is a matroid, Z a submodular set function and the greedy algorithm is such that $\varrho_j(S^i)=\varrho_j(\emptyset)$ for $i=1,\dots,K-1$ and all $j\in N^i$, then the greedy solution is optimum.*

Remark 3.4. Theorem 3.1 remains true if α_{G} is replaced by the parameter

$$\alpha_{\mathrm{G}}'=\max_{j\in N-S^{K-1}:\,\varrho_j(\emptyset)>0}\left\{\frac{\varrho_j(\emptyset)-\varrho_j(S^{K-1})}{\varrho_j(\emptyset)}\right\}$$

or

$$\alpha''_G = \max_{j \in N - \{j_1\}} \left\{ \frac{\varrho_j(\emptyset) - \varrho_j(N - \{j\})}{\varrho_j(\emptyset)} \right\}.$$

Note that the parameters α'_G and α_G do not dominate each other. So the two vesions of Theorem 3.1 are interesting. It is also worth noting that

$$\alpha = \max \left\{ \alpha''_G, \frac{\varrho_{j_1}(\emptyset) - \varrho_{j_1}(N - \{j_1\})}{\varrho_{j_1}(\emptyset)} \right\}.$$

However, the worst-case example of Theorem 3.1 works for α''_G but not for α.

4. A bound in terms of the rank and the girth of the matroid

Let $S^0 = \emptyset$, $S^1, \ldots, S^K$ be the sequence of sets chosen by the greedy algorithm, and define $\varrho_j = Z(S^j) - Z(S^{j-1})$. Note that the greedy solution has the value $Z^G = \varrho_1 + \varrho_2 + \cdots + \varrho_K$.

It has been shown in [11] that Z is submodular and nondecreasing if and only if

$$Z(\Omega) \leq Z(S) + \sum_{j \in \Omega - S} \varrho_j(S) \quad \text{for all } \Omega, S \subseteq N.$$

Let Ω be an optimal solution.

$$Z^* \equiv Z(\Omega) \leq Z(S^t) + \sum_{j \in \Omega - S^t} \varrho_j(S^t), \quad 0 \leq t \leq K. \tag{4.1}$$

Let $h+1$ be the girth of the matroid. For all $t < h$ and $j \in \Omega - S^t$, $S^t \cup \{j\} \in X$, and therefore $\varrho_j(S^t) \leq \varrho_{t+1}$. Since $|\Omega - S^t| \leq K$, we obtain that Z^* must satisfy the following relationship:

$$Z^* \leq \sum_{i=1}^{t} \varrho_i + K\varrho_{t+1}, \quad t = 0, \ldots, h-1. \tag{4.2}$$

For independence systems, it turns out that these constraints are the only essential ones in the analysis of the greedy heuristic, see [5]. For a matroid, however, the optimal solution must satisfy another family of inequalities.

Proposition 4.1. *The elements of any basis of a matroid can be ordered* $\{\omega_1, \omega_2, \ldots, \omega_K\}$ *so that, for any* $h \leq t \leq K$,

$$\varrho_{\omega_i}(S^t) \leq \begin{cases} \varrho_h & \text{if } i \leq h, \\ \varrho_i & \text{if } h < i \leq t, \\ \varrho_{t+1} & \text{if } i > t. \end{cases}$$

Proof. Consider the order defined in the proof of Lemma 2.2. Since $S^{i-1} \cup \{\omega_i\}$ is independent, so is $S^t \cup \{\omega_i\}$ for every $t < i$. Therefore, by the choice made in the greedy algorithm $\varrho_{\omega_i}(S^t) \leq \varrho_{t+1}$ for $i > t$.

For $h < i \leq t$, $\varrho_{\omega_i}(S^{i-1}) \leq \varrho_i$ by Lemma 2.2; this implies $\varrho_{\omega_i}(S^t) \leq \varrho_i$ since $S^{i-1} \subset S^t$.

Finally, for $i \leq h$, $\varrho_{\omega_i}(S^{h-1}) \leq \varrho_h$ since $S^{h-1} \cup \{\omega_i\}$ is independent. Therefore

$\varrho_{\omega_i}(S^t) \le \varrho_h$, as a consequence of the hypothesis $h \le t$. □

Proposition 4.1 allows us to write the inequality

$$\sum_{j \in \Omega - S^t} \varrho_j(S^t) \le h\varrho_h + \varrho_{h+1} + \cdots + \varrho_t + (K-t)\varrho_{t+1}, \quad h \le t \le K-1. \tag{4.3}$$

Combining the inequalities 4.1 and 4.3 we get:

$$Z^* \le \varrho_1 + \cdots + \varrho_{h-1} + (h+1)\varrho_h + 2\varrho_{h+1} + \cdots + 2\varrho_t + (K-t)\varrho_{t+1} \tag{4.4}$$

for $h \le t \le K-1$. We have just proved:

Theorem 4.2. *The following inequalities are valid:*

$$Z^* \le \sum_{i=1}^{t} \varrho_i + K\varrho_{t+1}, \quad 0 \le t \le h-1. \tag{4.5}$$

$$Z^* \le \sum_{i=1}^{h-1} \varrho_i + (h+1)\varrho_h + \sum_{i=h+1}^{t} 2\varrho_i + (K-t)\varrho_{t+1}, \quad h \le t \le K-1. \tag{4.6}$$

$$0 \le \varrho_t - \varrho_{t+1}, \quad 0 \le t \le K-1. \tag{4.7}$$

The inequality 4.6 for $t = K-1$ is always dominated by the one corresponding to $t = K-2$ and will be removed from the system.

Now we use the bounding technique presented in the introduction. Thus, any $\pi \ge 0$ which is a solution of the following system yields a bound $\sum_{i=1}^{K-1} \pi_i$ for the performance of the greedy algorithm.

$$\pi \begin{bmatrix}
K & & & & & & & & & & \\
1 & K & & & & & & & & & \\
\cdot & 1 & & & & & & \mathbf{0} & & & \\
\cdot & \cdot & & & & & & & & & \\
\cdot & \cdot & & K & & & & & & & \\
\cdot & \cdot & & 1 & K & & & & & & \\
1 & 1 & \cdots & 1 & h+1 & K-h & & & & & \\
1 & 1 & \cdots & 1 & h+1 & 2 & K-h-1 & & & & \\
\cdot & \cdot & & \cdot & \cdot & \cdot & 2 & & & & \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & & & \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & & & \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & 3 & & \\
1 & 1 & \cdots & 1 & h+1 & 2 & 2 & \cdots & 2 & 2 & 0 \\
1 & -1 & & & & & & & & & \\
 & 1 & & -1 & & & & & \mathbf{0} & & \\
 & & & & \cdots & & & & & & \\
 & & & & & & \cdots & & & & \\
 & & & & & & & & 1 & -1 & \\
 & & & \mathbf{0} & & & & & & 1 & -1
\end{bmatrix}
\begin{array}{l}
\\ \\ \\ \\ \\
h\text{th row} \\
\\ \\ \\ \\
\le e \\
\\
(K-1)\text{st row} \\
\\ \\ \\ \\ \\ \\
\end{array}$$

We decompose π into two vectors $\pi=(u,v)$, where $u=(u_i : i=1,\dots,K-1)$ is associated with the first $(K-1)$ rows of the above matrix and $v=(v_i : i=1,\dots,K-1)$ is associated with the remaining rows.

$$Ku_1+\sum_{i=2}^{K-1} u_i+v_1\le 1. \tag{4.8}$$

$$Ku_t+\sum_{i=t+1}^{K-1} u_i+v_t-v_{t-1}\le 1, \quad 2\le t\le h-1. \tag{4.9}$$

$$Ku_h+\sum_{i=h+1}^{K-1} (h+1)u_i+v_h-v_{h-1}\le 1. \tag{4.10}$$

$$(K-t+1)u_t+\sum_{i=t+1}^{K-1} 2u_i+v_t-v_{t-1}\le 1, \quad h+1\le t\le K-1. \tag{4.11}$$

To compute an analytic solution of this system, we consider two separate cases: $K\ge 2h$ and $K<2h$.

When $K\ge 2h$, setting $u_i=0$ for $1\le i\le K-2$, $u_{K-1}=\frac{1}{2}$, $v_i=\frac{1}{2}i$ for $1\le i\le h-1$ and $v_i=0$ for $h\le i\le K-1$, we get a bound of value $\frac{1}{2}$.

When $K<2h$, we set $u_i=0$ for $2h-K+1\le i\le K-2$ and $v_i=0$ for $1\le i\le 2h-K$ and $h\le i\le K-1$. (Note that, in the cases $h=K$ or $K-1$, the whole vector v is set equal to 0 whereas no u_i is. In these two cases the remaining system is triangular.) When $h\le K-2$, the remaining system is

$$\begin{aligned}
&Ku_t+u_{t+1}+\cdots+u_{2h-K}+u_{K-1}\le 1, && 1\le t\le 2h-K,\\
&u_{K-1}\le 1-v_{t-1}-v_t, && 2h-K+1\le t\le h-1,\\
&(h+1)u_{K-1}\le 1+v_{h-1}, && t=h,\\
&2u_{K-1}\le 1, && h+1\le t\le K-1.
\end{aligned} \tag{4.12.t}$$

We consider the solution

$$u_i=\frac{h}{K^2}\left(\frac{K-1}{K}\right)^{2h-K-i}, \quad 1\le i\le 2h-K,$$

$$u_{K-1}=\frac{K-h}{K} \qquad \text{(when } h\le K-2)$$

and

$$v_i=\frac{h(K-2h+i)}{K}, \quad 2h-K\le i\le h-1.$$

We now prove that this solution is feasible. When $h\le K-2$, the inequalities (4.12.t) are verified for $t\ge h+1$ as a consequence of the assumption $K\le 2h$. For $2h-K+1\le t\le h$, the inequalities are satisfied with equality, i.e. when $t=h$

$$(h+1)u_{K-1}=\frac{(h+1)(K-h)}{K}=1+\frac{h(K-h-1)}{K}=1+v_{h-1}$$

and when $2h-K+1 \le t \le h-1$

$$u_{K-1} = \frac{K-h}{K} = 1 - \frac{h}{K} = 1 - v_{t-1} - v_t.$$

The values u_i, $1 \le i \le 2h-K$, are obtained by solving at equality the triangular system (4.12.t), $1 \le t \le 2h-K$:

$$Ku_t + u_{t-1} + \cdots + u_{2h-K} = (h/K)(=1-u_{K-1}).$$

(The cases $h=K$ or $K-1$ are also obtained by solving the corresponding triangular systems at equality.)

Since (4.8) holds with equality, the value of the bound is

$$\sum_{i=1}^{K-1} u_i = 1 - v_1 - (K-1)u_1 = 1 - \frac{h}{K}\left(\frac{K-1}{K}\right)^{2h-K}.$$

Note that the best bound which can be obtained from the system (4.8)-(4.11) is the optimal value of the linear program

$$\max \sum_{i=1}^{K-1} u_i \tag{4.13}$$

subject to (4.8)-(4.11) and $u \ge 0$, $v \ge 0$.

We claim that the solution derived in this section is indeed an optimal solution of (4.13). To check it, it suffices to exhibit a feasible solution of the dual linear program with the same objective value:

$$\min \sum_{i=1}^{K} \varrho_i \tag{4.14}$$

subject to (4.5)-(4.7) with Z^* set equal to 1 and $\varrho \ge 0$.

We propose the following solutions: When $K \ge 2h$, take $\varrho_i = 1/2h$ for $1 \le i \le h$ and $\varrho_i = 0$ for $h+1 \le i \le K$. When $K < 2h$, take

$$\begin{aligned} \varrho_i &= \frac{(K-1)^{i-1}}{K^i} && \text{for } 1 \le i \le 2h-K, \\ \varrho_i &= \frac{(K-1)^{h-K}}{K^{2h-K+1}} && \text{for } 2h-K+1 \le i \le h, \\ \varrho_i &= 0 && \text{for } h+1 \le i \le K. \end{aligned} \tag{4.15}$$

The interested reader can verify for himself the feasibility of these solutions. The fact that $\sum_{i=1}^{K} \varrho_i = \frac{1}{2}$ when $K \ge 2h$ is obvious. When $K < 2h$,

$$\sum_{i=1}^{2h-K} \varrho_i = 1 - \left(\frac{K-1}{K}\right)^{2h-K} \qquad \text{(geometric series)}$$

and

$$\sum_{i=2h-K+1}^{K} \varrho_i = \frac{K-h}{K}\left(\frac{K-1}{K}\right)^{2h-K}.$$

Therefore

$$\sum_{i=1}^{K} \varrho_i = 1 - \frac{h}{K}\left(\frac{K-1}{K}\right)^{2h-K}. \tag{4.16}$$

Now we show that the bounds

$$\frac{Z^{\mathrm{G}}}{Z^*} \geq \tfrac{1}{2}, \qquad K \geq 2h,$$

$$\frac{Z^{\mathrm{G}}}{Z^*} \geq 1 - \frac{h}{K}\left(\frac{K-1}{K}\right)^{2h-K}, \quad K \leq 2h$$

obtained above are tight; that is, we exhibit families of matroids and submodular nondecreasing set functions for which the greedy performance satisfies the above bounds with equality. We define a matroid on the set of elements denoted by $B \cup A \cup T$, where $|B| = h$, $|A| = h|$, $|T| = K - h$. The elements in B will be the first elements chosen by the greedy algorithm, the elements in A will belong only to the optimal solution and the elements in T will be common to the greedy and the optimal solution. Let's define an independence system in the following way. The independent sets are all the sets of size at most K not containing more than h elements in the set $B \cup A$. The sets of h elements in $B \cup A$ are called *critical sets*.

Proposition 4.3. *The independence system is a matroid.*

Proof. It is the direct sum of two uniform matroids, [12]. It is also easy to check the matroid axioms (1.2) and (1.3). □

We now examine the case $K \geq 2h$ and define a nondecreasing submodular set function Z which gives the worst case of $\frac{1}{2}$. The subsets of A of a given cardinality will be indistinguishable, as far as the value of Z is concerned. So we will denote by A^j any subset of cardinality j. Similarly the subsets of $B \cup T$ of a given cardinality will be indistinguishable, so we will denote $W^i \subseteq B \cup T$ any subset of cardinality i.

$$Z(\emptyset) = 0,$$

$$Z(W^i \cup A^j) = \frac{i+j}{2h}, \qquad i \leq h,$$

$$Z(W^i \cup A^j) = \frac{1}{2} + \frac{j}{2h}, \quad i > h.$$

Proposition 4.4. *The function Z is submodular and nondecreasing.*

The proof is very easy. It is left to the reader.

The set B can be chosen first by the greedy algorithm because the increment given by any element $x \in B$ is not smaller than the increment given by any other element, when the set of elements has cardinality less than or equal to h (ties are broken arbitrarily). At stage h, the only elements which give a positive increment are the elements $a \in A$, but they form circuits with the set B, because it is a critical set; therefore

$$Z^{G} = Z(B \cup T) = \tfrac{1}{2}.$$

Since $|T| = K-h \geq h$, $Z(T \cup A) = \frac{1}{2} + \frac{1}{2} = 1$.

For the case $K < 2h$, we use the matroid defined earlier. However, we need to partition the set B in two subsets, one will still be denoted by B, the other by Y. Namely, let B be the set of the first $2h-K$ elements in the greedy solution, Y the set of the next $K-h$ elements in the greedy solution, T the set of $K-h$ elements common to the greedy and the optimal set, and A the set of h elements only in the optimal solution. We denote by $B^i \subseteq B$, $W^j \subseteq Y \cup T$ and $A^m \subseteq A$ subsets of cardinality i, j and m, respectively. We consider the following function:

$$Z(B^i \cup W^j \cup A^m) = \sum_{t=1}^{i+r} \varrho_t + (q+m)\varrho_{i+r+1} \qquad i+j \leq h,$$

$$Z(B^i \cup W^j \cup A^m) = \sum_{t=1}^{2h-K} \varrho_t + (K-h+m)\varrho_{2h-K+1}, \quad i+j \geq h$$

where

$$q = \min[j, K-h], \qquad r = \max[0, j-(K-h)]$$

and ϱ_t, $1 \leq t \leq K$, is given in (4.15).

Note that the function is doubly defined when $i+j=h$. It is easy to verify that the two expressions are then identical since $j \geq K-h$ (a consequence of the fact that $i \leq 2h-K$). For the proof of our next theorem, we will find it useful to have both expressions available.

Theorem 4.5. *The function Z defined above is submodular and nondecreasing.*

The proof is straightforward, though somewhat long. Anyone interested can find the proof in the appendix.

Again, the set B can be chosen first by greedy, because when $B^i \subset B$ and $l \in B - B^i$, we have $\varrho_l(B^i) = \varrho_{i+1}$, which is equal to the increment given by elements in $Y \cup T$ or in A. When all the elements in B have been added to the greedy solution, we have $i = 2h-K$ and the elements in Y give increments ϱ_{2h-K+1} because r is equal to zero. After that, elements in A give a positive increment, but $B \cup Y$ is a critical set and the addition of an element in A would create a circuit. Therefore, only elements in T can be added, but since $i+j=h$, they give null increments. Therefore, we obtain

$$Z^{G} = \varrho_1 + \cdots + \varrho_{2h-K} + (K-h)\varrho_{2h-K+1}.$$

If we consider the solution $T\cup A$, the elements in T give increments of ϱ_1, as well as the element in A. Therefore,

$$Z^* = \frac{(K-h)}{K} + \frac{h}{K} = 1.$$

And, by 4.16,

$$\frac{Z^{\mathrm{G}}}{Z^*} = 1 - \frac{h}{K}\left(\frac{K-1}{K}\right)^{2h-K}.$$

Theorem 4.6. *If X is a matroid with rank K and girth $h+1$ and Z is a nondecreasing submodular set function with $Z(\emptyset)=0$, then*

$$\frac{Z^{\mathrm{G}}}{Z^*} \geq \frac{1}{2} \qquad \textit{for } K \geq 2h,$$

$$\frac{Z^{\mathrm{G}}}{Z^*} \geq 1 - \frac{h}{K}\left(\frac{K-1}{K}\right)^{2h-K} \qquad \textit{for } K \leq 2h,$$

and these bounds are tight.

5. A tight bound for independence systems

In this section we consider instances of problem (1.1) where X is an independence system. As earlier we assume that Z is a nondecreasing submodular set function with $Z(\emptyset)=0$. Let $S^0=\emptyset$ and $S^t=\{j_1,\dots,j_t\}$, $t=1,\dots,l$, be the successive sets chosen by the greedy algorithm. Note that $k\leq l\leq K$, where K and k are respectively the maximum and minimum cardinality of a maximal set in X (K and k are sometimes called respectively the *upper* and *lower ranks* of X.) Recall that $\varrho_i=\varrho_{j_i}(S^{i-1})$ and that α denotes the total curvature of Z. (In this section α could be replaced by α_0 defined as in Section 2 with Ω being an optimal solution and S being the set S^k.)

Lemma 5.1. *For any independent set Ω and $t=0,\dots,k-1$,*

$$Z(\Omega) \leq \alpha \sum_{i:j_i\in S^t-\Omega} \varrho_i + \sum_{i:j_i\in\Omega\cap S^t} \varrho_i + (K-s)\varrho_{t+1}$$

where $s=|\Omega\cap S^t|$.

Proof. Follows from Lemma 2.1 and the observation that $S^t\cup\{\omega\}$ is independent as a consequence of the assumption $t\leq k-1$. $\square$

We will only use Lemma 5.1 when Ω is an optimal solution.

Consider the family $\mathscr{F}_{K,k,\alpha}$ of all instances of problem (1.1) where X and Z have the given parameters K, k and α as defined above. For simplicity of notation we write $\mathscr{F}=\mathscr{F}_{K,k,\alpha}$. For $0\le s\le k$, let $1\le i_1<\cdots<i_s\le k$ be a sequence of integers and let $\mathscr{F}(i_1,\dots,i_s)\subset\mathscr{F}$ be the family of problems such that a greedy solution $S^l=\{j_1,\dots,j_l\}$ has the elements $j_{i_1},\dots,j_{i_s}$ in common with an optimal solution Ω. Note that when $s=0$ the set of common elements is empty.

Let Z^{G} and Z^* be the values of a greedy and optimal solution respectively. As a consequence of Lemma 5.1, for any problem in $\mathscr{F}(i_1,\dots,i_s)$, $Z^{\mathrm{G}}\ge B(i_1,\dots,i_s)Z^*$ where

$$B(i_1,\dots,i_s)=\operatorname{Min}\sum_{i=1}^{k}\varrho_i \quad \text{subject to } \varrho_i\ge 0,\ i=1,\dots,k \tag{5.1}$$

and

$$\begin{bmatrix}1\\1\\ \\ \\ \\ \\ \\ \\ \\ \\ \vdots\\1\end{bmatrix}\le
\begin{array}{c}
\begin{array}{cccccccccccc}1 & & & i_1 & & & & i_s & & & & k\end{array}\\
\begin{bmatrix}
K & & & & & & & & & & & \\
\alpha & & & & & & & & & & & \\
 & & K & & & & & & & & & \\
 & & \alpha & K & & & & & & & & \\
 & & \alpha & 1 & K-1 & & & & \mathbf{0} & & & \\
 & & \alpha & 1 & \alpha & & & & & & & \\
 & & \vdots & \vdots & \vdots & & & & & & & \\
 & & & & & & & K-s+1 & & & & \\
 & & & & & & & 1 & K-s & & & \\
 & & & & & & & & \alpha & & & \\
\vdots & & \vdots & \vdots & \vdots & & & \vdots & \vdots & & & \\
\alpha & \cdots & \alpha & 1 & \alpha & \cdots & \alpha & 1 & \alpha & \cdots & \alpha & K-s
\end{bmatrix}\end{array}
\begin{bmatrix}\varrho_1\\ \varrho_2\\ \\ \\ \\ \\ \\ \\ \\ \\ \vdots\\ \varrho_k\end{bmatrix}.$$

Lemma 5.2. $B(i_1,\dots,i_s)\ge B(\emptyset)$ *for any integer sequence* $1\le i_1<\cdots<i_s\le k$.

Proof. Assume $s\ge 1$ and consider i_r for $1\le r\le s$. For simplicity of notation we denote $q=i_r$. First we show that $\varrho_q\le\varrho_{q+1}$ in some optimal solution of the linear program (5.1) associated with $B(i_1,\dots,i_s)$. Assume not, i.e., assume that $\varrho_q>\varrho_{q+1}$. The inequalities q and $q+1$ of the system are

$$\alpha\varrho_1+\cdots+\alpha\varrho_{q-1}+(K-r+1)\varrho_q\ge 1$$

and

$$\alpha\varrho_1+\cdots+\alpha\varrho_{q-1}+\varrho_q+(K-r)\varrho_{q+1}\ge 1.$$

Note that the first of these two constraints is not tight. Decrease the value of ϱ_q by $\varepsilon>0$ small enough so that the inequality remains feasible, and add $\varepsilon/(K-r)$ to ϱ_i for $q+1\le i\le k$. It is clear that this new solution is feasible. The objective value of the linear program is modified by $(k-q)\varepsilon/(K-r)-\varepsilon\le 0$ since $k\le K$ and $q\ge r$. Therefore $\varrho_q\ge\varrho_{q+1}$ in some optimal solution of (5.1).

Now assume $q=i_r<i_{r+1}-1$. Denote by A the constraint matrix of the linear program (5.1) associated with $B(i_1,\dots,i_{r-1},i_r,i_{r+1},\dots,i_s)$ and by A' the constraint matrix associated with $B(i_1,\dots,i_{r-1},i_r+1,i_{r+1},\dots,i_s)$. A and A' only differ by their

columns q and $q+1$. Thus any vector ϱ which satisfies $1 \leq A\varrho$ and $\varrho_q \leq \varrho_{q+1}$ also satisfies $1 \leq A'\varrho$. This implies

$$B(i_1, \dots, i_{r-1}, i_r, i_{r+1}, \dots, i_s) \geq B(i_1, \dots, i_r - 1, i_r + 1, i_{r+1}, \dots, i_s).$$

Repeating iteratively this argument for all $1 \leq r \leq s$ such that $i_r < i_{r+1} - 1$, we obtain

$$B(i_1, \dots, i_s) \geq B(k-s+1, k-s+2, \dots, k).$$

Now let A_s be the constraint matrix associated with $B(k-s+1, k-s+2, \dots, k)$. Any vector ϱ which satisfies $1 \leq A_s\varrho$ and $\varrho_{k-s+1} \leq \varrho_{k-s+2} \leq \cdots \leq \varrho_k$ also satisfies $1 \leq A_{s+1}\varrho$. This shows

$$B(k-s+1, k-s+2, \dots, k) \geq B(k-s+2, \dots, k) \geq \cdots \geq B(k) = B(\emptyset). \qquad \square$$

Lemma 5.3.

$$B(\emptyset) \geq \frac{1}{\alpha}\left[1 - \left(\frac{K-\alpha}{K}\right)^k\right].$$

Proof. Consider the linear program (5.1) associated with $B(\emptyset)$. Multiply the tth constraint by

$$\frac{1}{K}\left(\frac{K-\alpha}{K}\right)^{k-t}$$

and add all the constraints.

$$\sum_{t=1}^{k} \frac{1}{K}\left(\frac{K-\alpha}{K}\right)^{k-t} \leq \sum_{i=1}^{k} \varrho_i \left[\left(\frac{K-\alpha}{K}\right)^{k-i} + \alpha \sum_{t=i+1}^{k} \frac{1}{K}\left(\frac{K-\alpha}{K}\right)^{k-t}\right].$$

Summing the geometric series we observe that the coefficients of ϱ_i equal 1 for every $i = 1, \dots, k$ and that the left-hand side of the inequality equals

$$\frac{1}{\alpha}\left[1 - \left(\frac{K-\alpha}{K}\right)^k\right]$$

as required. $\square$

Theorem 5.4. *If X is an independence system with upper rank K, lower rank k, and if Z is a nondecreasing submodular set function with $Z(\emptyset) = 0$ and total curvature α, then*

$$Z^{\mathrm{G}} \geq \frac{1}{\alpha}\left[1 - \left(\frac{K-\alpha}{K}\right)^k\right] Z^*$$

and this bound is tight for all $0 \leq \alpha \leq 1$ and $k \leq K$.

Proof. The bound is valid as a consequence of Lemmas 5.2 and 5.3 and the fact that $\mathscr{F} = \bigcup\{\mathscr{F}(i_1, \dots, i_s)\colon 1 \leq i_1 < \cdots < i_s \leq k$ is a (possibly empty) integer sequence$\}$.

The fact that the bound is tight is shown by the following worst-case examples.

Let $N=\{j_1,j_2,\dots,j_{K-1},\omega_1,\dots,\omega_K\}$ and let X be the family of all the subsets $S\subset N$ which contain at most k elements if $j_1\in S$ and at most K if $j_1\notin S$, where $k\le K$. Define

$$\varrho_i=\frac{1}{K}\left(\frac{K-\alpha}{K}\right)^{i-1}\quad\text{for } i=1,\dots,K,$$

and consider the set function defined on the subsets of N as

$$Z(j_{i_1},j_{i_2},\dots,j_{i_t},\omega_{r_1},\dots,\omega_{r_u})=\frac{K-u\alpha}{K}\sum_{h=1}^{t}\varrho_{j_h}+\frac{u}{K}.$$

In this formula we allow t or u to take the value 0. The summation is taken to be 0 if $t=0$. Therefore $Z(\emptyset)=0$. Note that

$$\varrho_{j_i}(j_{i_1},\dots,j_{i_t},\omega_{r_1},\dots,\omega_{r_u})=\left(1-\alpha\frac{u}{K}\right)\varrho_i\ge(1-\alpha)\varrho_i,$$

$$\varrho_{\omega_r}(j_{i_1},\dots,j_{i_t},\omega_{r_1},\dots,\omega_{r_u})=\frac{1}{K}-\frac{\alpha}{K}\sum_{h=1}^{t}\varrho_{j_h}\ge(1-\alpha)\frac{1}{K}.$$

This shows that the set function Z is submodular, nondecreasing and has total curvature α.

The optimal solution of problem (1.1) is $\{\omega_1,\dots,\omega_K\}$ with value 1. Since $\varrho_1=1/K$, the greedy algorithm can choose j_1 in the first iteration. Assume it has chosen $S^{i-1}=\{j_1,\dots,j_{i-1}\}$. Then

$$\varrho_{j_i}(S^{i-1})=\varrho_i=\frac{1}{K}\left(\frac{K-\alpha}{K}\right)^{i-1}$$

whereas

$$\varrho_{\omega_r}(S^{i-1})=\frac{1}{K}-\frac{\alpha}{K}\sum_{h=1}^{i-1}\varrho_h=\frac{1}{K}-\frac{1}{K}\left[1-\left(\frac{K-\alpha}{K}\right)^{i-1}\right]=\frac{1}{K}\left(\frac{K-\alpha}{K}\right)^{i-1}.$$

So the greedy algorithm can choose the element j_i in the ith iteration. The greedy solution has the value

$$\sum_{h=1}^{k}\varrho_h=\frac{1}{\alpha}\left[1-\left(\frac{K-\alpha}{K}\right)^{k}\right]$$

as required. □

Corollary 5.5 [5].

$$Z^{\mathrm{G}}\ge\left[1-\left(\frac{K-1}{K}\right)^{k}\right]Z^*.$$

Proof. Set $\alpha=1$ in the bound of Theorem 5.4. □

Corollary 5.6 [6], [7]. *If X is an independence system and Z is additive, then $Z^{\mathrm{G}}\ge(k/K)Z^*$.*

Proof. Let $\alpha \to 0$ in the bound of Theorem 5.4. The fact that $(1-\alpha/K)^k \to 1-\alpha k/K$ implies the result. $\square$

Corollary 5.7. *If X is a uniform matroid and Z has total curvature α, then*

$$Z^{\mathrm{G}} \geq \frac{1}{\alpha}\left[1-\left(\frac{K-\alpha}{K}\right)^K\right] Z^*.$$

Corollary 5.8 [2], [11]. *If X is a uniform matroid, then*

$$Z^{\mathrm{G}} \geq \left[1-\left(\frac{K-1}{K}\right)^K\right] Z^*.$$

Proof. Set $\alpha=1$ in Corollary 5.7. $\square$

Corollary 5.9. *If X is a uniform matroid and Z has total curvature α, then*

$$Z^{\mathrm{G}} \geq \frac{1-\mathrm{e}^{-\alpha}}{\alpha} Z^* \geq \left(1-\frac{\alpha}{2}\right) Z^*.$$

Proof. For any integer K, $((K-\alpha)/K)^K \leq \mathrm{e}^{-\alpha}$. Therefore the bound follows from Corollary 5.7. Furthermore $(1-\mathrm{e}^{-\alpha})/\alpha > 1-\alpha/2$ for all $0<\alpha<1$. $\square$

6. The bound $1/(p+\alpha)$

The last result that we shall prove concerning problem (1.1) is the following. Let X be an independence system, p the minimum number of matroids that one needs to intersect in order to obtain X and Z a nondecreasing submodular set function with $Z(\emptyset)=0$ and total curvature α. Then the greedy algorithm finds a solution with value $Z^{\mathrm{G}} \geq Z^*/(p+\alpha)$ where Z^* is the optimal value.

However we derive the bound $1/(p+\alpha)$ for a more general model than (1.1), as an example of a possible extension of the results in this paper.

Given a nonnegative vector $u=(u_j : j\in N)$, let $|u| = \sum_{j\in N} u_j$. Given two vectors $u, v \in \mathbb{R}^N$ we define $w = u \vee v$ as the vector of $\mathbb{R}^N$ with components $w_j = \max(u_j, v_j)$ for all $j \in N$. An *integral polymatroid* is a pair (N, P) where N is a nonempty finite set and $P \subseteq \mathbb{R}_+^N$ is a finite family of integral vectors such that

(i) $v \in P, u \leq v$ and u is a nonnegative integral vector $\Rightarrow u \in P$, and

(ii) $u, v \in P$ and $|u|+1=|v| \Rightarrow \exists w \in P$ such that $u < w \leq u \vee v$.

The vectors in P are called *independent vectors*. The concept of integral polymatroid was introduced by Edmonds [4] as a generalization of matroids (obtained when P contains only 0, 1 vectors. Then the independent sets of the matroid are precisely the subsets of N whose 0, 1 incidence vectors belong to P.) An introduction to integral polymatroids can be found in [12]. A known property is that P can be written as

$$P=\left\{x\geq 0 \text{ and integral: } \sum_{j\in S} x_j \leq r(S),\ \forall S\subseteq N\right\} \tag{6.1}$$

where r is a nondecreasing integral submodular set function with $r(\emptyset)=0$.

A *vector function* $Z:\mathbb{R}^N\to\mathbb{R}_+$ is submodular and nondecreasing if

$$\varrho_i(v)\geq\varrho_i(u)\geq 0 \quad \text{for all } i\in N \text{ and } v\leq u\in\mathbb{R}^N, \tag{6.2}$$

where $\varrho_i(v)=Z(v+e_i)-Z(v)$ and e_i is the unit vector whose component indexed by $i\in N$ is equal to 1.

Given a nondecreasing submodular vector function Z, a generalization of problem (1.1) is

$$\max\{Z(v)\colon v\in X\} \tag{6.3}$$

where X is the intersection of p integral polymatroids. Note that, as a consequence of (6.1), the problem (6.3) can be written as

$$\max Z(x),$$
$$Ax\leq b,$$
$$x\geq 0 \text{ and integral,}$$

where A is a 0,1 matrix. Conversely, for any 0, 1 matrix A, the problem (6.4) is equivalent to (6.3) where X is the intersection of a number of integral polymatroids. For example $X=\bigcap_i P_i$ where $P_i=\{x\geq 0$ and integral: $a_i x\leq b_i\}$, a_i is the ith row of A and b_i is the ith component of b.

A steepest ascent (or greedy) algorithm for solving problem (6.3) or (6.4) would be

Greedy Algorithm

Initialization: Set $v^0=0$ and $t=1$.

Step t: Find $j_t\in N$ such that $\varrho_{j_t}(v^{t-1})=\max\{\varrho_j(v^{t-1})\colon v^{t-1}+e_j\in X\}$. If no such j_t exists, stop. Otherwise set $v^t=v^{t-1}+e_{j_t}$, increment t by 1 and repeat Step t.

In this section we assume $Z(0)=0$. If we define $\varrho_t=\varrho_{j_t}(v^{t-1})$, the value of the greedy solution v^k is $Z^G=\varrho_1+\cdots+\varrho_k$, where k is the value of the parameter t when the greedy algorithm stops. In fact, $k=r(N)$ as defined in (6.1). Note that the greedy algorithm defined above is not polynomial in $|N|$.

Let $m=(m_i : i\in N)$ where m_i is the largest integer λ such that the vector $\lambda e_i\in X$. Define the total curvature of Z with respect to X as

$$\alpha=\max_{j\in N^*}\left\{\frac{\varrho_j(0)-\varrho_j(m-e_j)}{\varrho_j(0)}\right\}$$

where $N^*=\{j\in N\colon \varrho_j(0)>0\}$.

Theorem 6.1. *Let X be the intersection of p integral polymatroids P_i, $i=1,\ldots,p$*

and Z a nondecreasing submodular vector function with $Z(0)=0$ and total curvature α. Then a greedy solution to problem (6.3) *has a value $Z^G \geq Z^*/(p+\alpha)$ where Z^* is the optimal value.*

Proof. Consider an optimal solution ω. We will write $\omega=\sum_{l=1}^{|\omega|} e^{(l)}$ where the $e^{(l)}$'s are unit vectors, i.e. $e^{(l)}=e_{i(l)}$ for some $i(l)\in N$. Note that the same unit vector e_i may appear several times in the summation, indexed by different values of l.

Let s^t be the vector obtained at iteration t of the greedy algorithm, $t=1,\dots,k$. If $|\omega|>t$, then we claim that, for all i,

$$s^t+e^{(l)}\in P_i \quad \text{for at least } |\omega|-t \text{ of the vectors } e^{(l)}. \tag{6.5}$$

This is proved by repeated use of axiom (ii) of the definition of integral polymatroids: consider $\omega'\leq\omega$ such that $|\omega'|=t+1$. By (ii), $\exists e^{(l)}$ such that $s^t<s^t+e^{(l)}\leq \omega'\vee s^t$. Now replace ω by $\omega-e^{(l)}$ and repeat the argument. Since it can be repeated $|\omega|-t$ times, the proof of the claim (6.5) is complete. A consequence of property (6.5) is that

$$\text{if } |\omega|>pt, \quad \text{then } s^t+e^{(l)}\in\bigcap_{i=1}^{p} P_i \quad \text{for at least } |\omega|-pt \text{ of the vectors } e^{(l)}. \tag{6.6}$$

For any such l, $\varrho_{i(l)}(s^t)\leq\varrho_{t+1}$, as a consequence of the choice made by the greedy algorithm. So the $e^{(l)}$'s can be ordered so that

$$\varrho_{i(l)}(s^t)\leq\varrho_{t+1} \quad \text{for } pt<l\leq p(t+1) \text{ and } t=0,\dots,k-1. \tag{6.7}$$

Note that $k\geq|\omega|/p$ as a consequence of (6.6).

Let $\omega\wedge s^k$ be the vector whose jth component is $\min(\omega_j, s_j^k)$ and let $L\subseteq\{1,\dots,|\omega|\}$ be a set of indices such that $\omega\wedge s^k=\sum_{l\in L} e^{(l)}$. Then

$$Z(\omega)+(1-\alpha)\sum_{t=1}^{k}\varrho_t\leq Z(\omega\vee s^k)+\sum_{l\in L}\varrho_{i(l)}(s^k)\leq Z(s^k)+\sum_{l=1}^{|\omega|}\varrho_{i(l)}(s^k).$$

As a consequence of (6.7)

$$Z(\omega)+(1-\alpha)\sum_{t=1}^{k}\varrho_t\leq(1+p)\sum_{t=1}^{k}\varrho_t.$$

Therefore $Z^*\leq(p+\alpha)Z^G$ as required. □

Appendix. Proof of Theorem 4.5

Proposition. *Let ϱ_n, $1\leq n\leq K$, be given in* (4.15). *The difference $\lambda\varrho_{n+1}-(\lambda-1)\varrho_n$, $1\leq n\leq 2h-K$, is equal to $((K-\lambda)/K)\varrho_n$.*

Proof.

$$\lambda\varrho_{n+1}-(\lambda-1)\varrho_n=\frac{\lambda(K-1)^n-(\lambda-1)K(K-1)^{n-1}}{K^{n+1}}$$

$$= \frac{\lambda(K-1)-K(\lambda-1)}{K}\varrho_n = \frac{K-\lambda}{K}\varrho_n. \qquad \square$$

We are now ready for the proof of the theorem: We prove

$$\varrho_l(S) \geq \varrho_l(R) \geq 0, \quad \text{for each } S \subseteq R \subset E, \quad \text{for each } l \in E-R.$$

Case 1: $l \in B$.

(1.a) $\quad i+j<h, \quad j<K-h \quad$ (note that $r=0$, $q=j$),

$$\varrho_l(B^i \cup W^j \cup A^m) = \varrho_{i+1} + (j+m)\varrho_{i+2} - (j+m)\varrho_{i+1},$$

$$\varrho_l = (j+m)\varrho_{i+2} - (j+m-1)\varrho_{i+1}.$$

From the above proposition, by substituting $j+m$ for λ, we get

$$\varrho_l = \frac{K-(j+m)}{K}\varrho_{i+1}.$$

Since $j<K-h$, $m \leq h$, $j+m<K$, we have $\varrho_l>0$. Moreover, by lowering j or m or i, ϱ_l cannot decrease.

(1.b) $\quad i+j<h, \quad j \geq K-h \quad$ (note that $q=K-h$, $r=j+(K-h) \geq 0$),

$$\varrho_l(B^i \cup W^j \cup A^m) = \varrho_{i+r+1} + (K-h+m)\varrho_{i+r+2} - (K-h+m)\varrho_{i+r+1}$$
$$= (K-h+m)\varrho_{i+r+2} - (K-h-1+m)\varrho_{i+r+1}.$$

Again, from the preceding proposition, by setting $\lambda = (K-h+m)$, we get:

$$\varrho_l = \frac{h-m}{K}\varrho_{i+r+1}.$$

Since $\varrho_{i+r+1} \geq 0$ and $m \leq h$, we get $\varrho_l \geq 0$. Moreover, if i or r or m decreases, ϱ_l increases and

$$\frac{h-m}{K}\varrho_{i+r+1} \leq \frac{K-(j+m)}{K}\varrho_{i+1} \quad \text{when } j<K-h.$$

(1.c) $\quad i+j \geq h, \quad \varrho_l(B^i \cup W^j \cup A^m) = 0.$

Case 2: $l \in A$.

(2.a) $\quad i+j<h, \quad \varrho_l(B^i \cup W^j \cup A^m) = \varrho_{i+r+1}.$

(2.b) $\quad i+j \geq h, \quad \varrho_l(B^i \cup W^j \cup A^m) = \varrho_{2h-K+1}.$

ϱ_{i+r+1} is greater than or equal to ϱ_{2h-K+1} because when $r=0$, $i \leq 2h-K$; when $r>0$, $r=j-(K-h)$, but $i+r=i+j-(K-h)<h-(K-h)=2h-K$.

Case 3: $l \in Y \cup T$.

(3.a) $\quad i+j<h, \quad j<K-h \quad$ (note that $r=0$, $q=j$),

$$\varrho_l(B^i \cup W^j \cup A^m) = (j+1+m)\varrho_{i+1} - (j+m)\varrho_{i+1} = \varrho_{i+1}.$$

(3.b) $\quad i+j<h, \quad j \geq K-h \quad$ (note that $q=K-h$, $r \geq 0$),

$$\begin{aligned}\varrho_l(B^i \cup W^j \cup A^m) &= \varrho_{i+r+1} + (K-h+m)\varrho_{i+r+2} - (K-h+m)\varrho_{i+r+1} \\ &= (K-h+m)\varrho_{i+r+2} - (K-h-1+m)\varrho_{i+r+1}.\end{aligned}$$

Again, by applying the proposition, we get

$$\varrho_l = \frac{h-m}{K}\varrho_{i+r+1}.$$

Since $(h-m)/K \leq 1$ and $\varrho_{i+r+1} \leq \varrho_{i+1}$ the condition $\varrho_{i+1} \geq ((h-m)/K)\varrho_{i+r+1}$ is always satisfied.

(3.c) $\quad i+j \geq h, \quad \varrho_l(B^i \cup W^j \cup A^m) = 0.$

Therefore, the function is submodular and nondecreasing.

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