Chapter 1

Sets and Relations

1.1 Cantor's Concept of a Set

A **set** S is any collection of definite, distinguishable objects of our intuition or of our intellect to be conceived as a whole. The objects are called the **elements** or **members** of S.

1.2 The Basis of Intuitive Set Theory

Membership relation: $x \in A$ if the object x is a member of the set A. If x is not a member of A then $x \notin A$. $x_1, x_2, \ldots, x_n \in A$ is shorthand for $x_1 \in A \wedge x_2 \in A \ldots x_n \in A$.
The intuitive principle of extension: Two sets are equal iff they have the same members.
Set equality: The equality of two sets X and Y will be denoted by $X = Y$ and inequality of X and Y by $X \neq Y$. Among the basic properties of this relation are:

$$\begin{gathered} X = X, \\ X = Y \Rightarrow Y = X, \\ X = Y \wedge Y = Z \Rightarrow X = Z, \end{gathered}$$

for all sets X, Y, and Z.
unit set: a set $\{x\}$ whose sole member is x.
collection of sets: a set whose members are sets.
The intuitive principle of abstraction: A formula $P(x)$ defines a set A by the convention that the members of A are exactly those objects a such that $P(a)$ is a true statement, denoted by $A = \{x \mid P(x)\}$.
Note: $\{x \in A \mid P(x)\} := \{x \mid x \in A \wedge P(x)\}$. For a property P and function f we can write $\{f(x) \mid P(x)\} := \{y \mid \exists x \colon P(x) \wedge y = f(x)\}$.

1.3 Inclusion

If A and B are sets, then A is **included in** B iff each member of A is a member of B. Symbolized: $A \subseteq B$. We also say that A is a **subset** of B. Equivalently, B **includes** A, symbolized by $B \supseteq A$.
The set A is **properly included in** B (A is a **proper subset** of B / B **properly includes** A) iff $A \subseteq B$ and $A \neq B$.
Among the basic properties of the inclusion relation are

$$X \subseteq X;$$
$$X \subseteq Y \wedge Y \subseteq Z \Rightarrow X \subseteq Z;$$
$$X \subseteq Y \wedge Y \subseteq X \Rightarrow X = Y.$$

empty set: $\{x \in A \mid x \neq x\}$ for any set A is the set with no elements, symbolized by $\emptyset$.
power set: the set of all subsets of a given set. $\mathcal{P}(A) = \{B \mid B \subseteq A\}$ for a given set A.

1.4 Operations for Sets

union: for sets A and B, the set of all objects which are members of either A or B. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. (**sum**/**join**)
intersection: for sets A and B, the set of all objects which are members of both A and B. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. (**product**/**meet**)
disjoint: $A \cap B = \emptyset$ for sets A and B.
intersect: $A \cap B \neq \emptyset$ for sets A and B.
disjoint collection: for a collection of sets, each distinct pair of its member sets is disjoint.
partition: for a set X, a disjoint collection $\mathcal{A}$ of nonempty and distinct subsets of X such that each member of X is a members of some (exactly one) member of $\mathcal{A}$.
absolute complement of A: $\overline{A} = \{x \mid x \notin A\}$, the set of all members which are not in A.
relative complement of A with respect to X: $X - A = X \cap \overline{A} = \{x \in X \mid x \notin A\}$, the set of those members of X which are not members of A.
symmetric difference of A and B: $A + B = (A - B) \cup (B - A)$.
universal set: the set U such that all sets under consideration in a certain discussion are subsets of U.

1.5 The Algebra of Sets

identities: equations which are true whatever the universal set U and no matter what particular subsets the letters (other than U and $\emptyset$) represent.

<u>THEOREM 5.1</u>: For any subsets A, B, C of a set U the following equations are identities. Here

	1. $A \cup (B \cup C) = (A \cup B) \cup C$.	1'. $A \cap (B \cap C) = (A \cap B) \cap C$.
	2. $A \cup B = B \cup A$.	2'. $A \cap B = B \cap A$.
$\overline{A}$ is an abbreviation for $U - A$.	3. $A \cup (B \cap C) = (A \cup B) \cap (B \cup C)$.	3'. $A \cap (B \cup C) = (A \cap B) \cup (A \cap$
	4. $A \cup \emptyset = A$.	4'. $A \cap U = A$.
	5. $A \cup \overline{A} = U$	5'. $A \cap \overline{A} = \emptyset$.

<u>PROOF</u>:
<u>Lemma</u>: Let X, Y be subsets of U. Then $X \subseteq X \cup Y$ and $X \subseteq Y \cup X$.
<u>Proof</u>: Assume $x \in X$ and $x \notin X \cup Y$. Then $x \notin \{z \mid z \in X \text{ or } z \in Y\} \Rightarrow x \notin \{z \mid z \in X\} = X$ which is a contradiction. Now assume $x \in X$ and $x \notin Y \cup X$. Then $x \notin \{z \mid z \in Y \text{ or } z \in X\} \Rightarrow x \notin \{z \mid z \in X\} = X$ which is a contradiction.
1. Assume $x \in A \cup (B \cup C)$. Then $x \in A$ or $x \in B \cup C$. If $x \in A$ then $x \in A \cup B$ and so $x \in (A \cup B) \cup C$. Otherwise if $x \in B \cup C$ then $x \in B$ or $x \in C$. If $x \in B$ then $x \in (A \cup B)$ and so $x \in (A \cup B) \cup C$. If $x \in C$ then $x \in (A \cup B) \cup C$. Therefore $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.
Now assume $x \in (A \cup B) \cup C$. Then $x \in A \cup B$ or $x \in C$. If $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \cup C)$. If $x \in B$ then $x \in (B \cup C)$ and so $x \in A \cup (B \cup C)$. Otherwise if $x \in C$ then $x \in B \cup C$ and so $x \in A \cup (B \cup C)$. Therefore $(A \cup B) \cup C \subseteq A \cup (B \cup C)$. Hence $A \cup (B \cup C) = (A \cup B) \cup C$.
1'. Assume $x \in A \cap (B \cap C)$. Then $x \in A$ and $x \in B \cap C$. Since $x \in B \cap C$ we have $x \in B$ and $x \in C$. Then since $x \in A$ and $x \in B$ we have $x \in A \cap B$. Since $x \in C$ we have $x \in (A \cap B) \cap C$. Therefore $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.
Now assume $x \in (A \cap B) \cap C$. Then $x \in A \cap B$ and $x \in C$. Since $x \in A \cap B$ we have $x \in A$ and $x \in B$. Then since $x \in B$ and $x \in C$ we have $x \in B \cap C$. Since $x \in A$ we have $x \in A \cap (B \cap C)$. Therefore $(A \cap B) \cap C \subseteq A \cap (B \cap C)$. Hence $A \cap (B \cap C) = (A \cap B) \cap C$.
2. Assume $x \in A \cup B$. Then $x \in A$ or $x \in B$. In either case $x \in B \cup A$ and so $A \cup B \subseteq B \cup A$. Now assume $x \in B \cup A$. Then $x \in B$ or $x \in A$. In either case $x \in A \cup B$ and so $B \cup A \subseteq A \cup B$. Hence $A \cup B = B \cup A$.
2'. Assume $x \in A \cap B$. Then $x \in A$ and $x \in B$ and so $x \in B \cap A$. Therefore $A \cap B \subseteq B \cap A$. Now assume $x \in B \cap A$. Then $x \in B$ and $x \in A$ and so $x \in A \cap B$. Therefore $B \cap A \subseteq B \cap A$. Hence $A \cap B = B \cap A$.
3. Assume $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ and so $x \in (A \cup B) \cap (A \cup C)$. Otherwise if $x \in B \cap C$ then $x \in B$ and $x \in C$. Since $x \in B$ we have $x \in A \cup B$. Since $x \in C$ we have $x \in A \cup C$. Then $x \in (A \cup B) \cap (A \cup C)$ and therefore $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
Now assume $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. Since $x \in A \cup B$ we have $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \cap C)$. Otherwise if $x \in B$ then $x \in A \cup B$. Since $x \in A \cup C$ we also have that $x \in A$ or $x \in C$. If $x \in A$ then $x \in A \cup (B \cap C)$. Otherwise if $x \in C$ then since $x \in B$ we have $x \in B \cap C$ and so $x \in A \cup (B \cap C)$. Therefore

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
3'. Assume $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. If $x \in B$ then since $x \in A$ we have $x \in A \cap B$ and so $x \in (A \cap B) \cup (A \cap C)$. Otherwise if $x \in C$ then since $x \in A$ we have $x \in A \cap C$ and so $x \in (A \cap B) \cup (A \cap C)$. Therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Now assume $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$ then $x \in A$ and $x \in B$. Since $x \in B$ we have $x \in B \cup C$. Since we also have $x \in A$ then $x \in A \cap (B \cup C)$. Otherwise if $x \in A \cap C$ then $x \in A$ and $x \in C$. Since $x \in C$ we have $x \in B \cup C$. Since we also have $x \in A$ then $x \in A \cap (B \cup C)$. Therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. Assume $x \in A \cup \emptyset$. Then $x \in A$ or $x \in \emptyset$. Since $x \in \emptyset$ is impossible, we must have the $x \in A$ and so $A \cup \emptyset \subseteq A$. Now assume $x \in A$ then $x \in A \cup \emptyset$ and so $A \subseteq A \cup \emptyset$. Hence $A \cup \emptyset = A$.
4'. Assume $x \in A \cap U$. Then $x \in A$ and $x \in U$. Therefore $A \cap U \subseteq A$. Now assume $x \in A$. Then since $A \subseteq U$ we have $x \in U$ and so $x \in A \cap U$. Therefore $A \subseteq A \cap U$. Hence $A \cap U = A$.
5. Assume $x \in A \cup \overline{A}$. Then $x \in A$ or $x \in \overline{A}$. Since $A \subseteq U$ and $\overline{A} \subseteq U$ in either case we have $x \in U$ and so $A \cup \overline{A} \subseteq U$. Now assume $x \in U$. Then $x \in A$ or $X \notin A$ for any set A. Thus $x \in A$ or $x \in \overline{A}$ and so $x \in A \cup \overline{A}$. Therefore $U \subseteq A \cup \overline{A}$. Hence $A \cup \overline{A} = U$.
5'. Assume $x \in A \cap \overline{A}$. Then $x \in A$ and $x \in \overline{A}$. Since $x \in \overline{A}$ we have $x \notin A$. Since $x \in A$ and $x \notin A$ we have $x \in \emptyset$. Therefore $A \cap \overline{A} \subseteq \emptyset$. Since $\emptyset \subseteq X$ for any set X we have $\emptyset \subseteq A \cap \overline{A}$. Hence $A \cap \overline{A} = \emptyset$. ■

General associative law for set union: The sets obtainable from given sets $A_1, A_2, \ldots, A_n$ in that order, by use of the operation of union are all equal to one another. The set defined by $A_1, A_2, \ldots, A_n$ in this way will be written as

$$A_1 \cup A_2 \cup \cdots \cup A_n.$$

General associative law for set intersection: The sets obtainable from given sets $A_1, A_2, \ldots, A_n$ in that order, by use of the operation of intersection are all equal to one another. The set defined by $A_1, A_2, \ldots, A_n$ in this way will be written as

$$A_1 \cap A_2 \cap \cdots \cap A_n.$$

General commutative law for set union: If $1', 2', \ldots, n'$ are $1, 2, \ldots, n$ in any order, then

$$A_1 \cup A_2 \cup \cdots \cup A_n = A_{1'} \cup A_{2'} \cup \cdots \cup A_{n'}.$$

General commutative law for set intersection: If $1', 2', \ldots, n'$ are $1, 2, \ldots, n$ in any order, then

$$A_1 \cap A_2 \cap \cdots \cap A_n = A_{1'} \cap A_{2'} \cap \cdots \cap A_{n'}.$$

General distributive law for set union:

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n).$$

General distributive law for set intersection:

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n).$$

dual: An equation, or an expression, or a statement within the framework of the algebra of sets obtained from another by interchanging $\cup$ and $\cap$ along with $\emptyset$ and U.

principle of duality for the algebra of sets: If T is any theorem expressed in terms of $\cup$, $\cap$, and $\overline{}$, then the dual of T is also a theorem.

THEOREM 5.2: For all subsets A and B of a set U, the following statements are valid. Here $\overline{A}$ is an abbreviation for $U - A$.

6. If, for all A, $A \cup B = A$, then $B = \emptyset$. | 6'. If, for all A, $A \cap B = A$ then $B = U$.

7,7'. If $A \cup B = U$ and $A \cap B = \emptyset$, then $B = \overline{A}$.

8,8'. $\overline{\overline{A}} = A$.

9. $\overline{\emptyset} = U$. | 9'. $\overline{U} = \emptyset$.

10. $A \cup A = A$ | 10'. $A \cap A = A$.

11. $A \cup U = U$ | 11'. $A \cap \emptyset = \emptyset$.

12. $A \cup (A \cap B) = A$. | 12'. $A \cap (A \cup B) = A$.

13. $\overline{A \cup B} = \overline{A} \cap \overline{B}$ | 13'. $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

PROOF: