Chapter 1

Sets and Relations

2.1 Explain why $2 \in \{1, 2, 3\}$.
Solution: $2 \in \{1, 2, 3\}$ by definition: 2 is an element of $\{1, 2, 3\}$.

2.2 Is $\{1, 2\} \in \{\{1, 2, 3\}, \{1, 3\}, 1, 2\}$? Justify your answer.
Solution: $\{1, 2\} \notin \{\{1, 2, 3\}, \{1, 3\}, 1, 2\}$ since it does not appear as a member of the given set.

2.3 Try to devise a set which is a member of itself.
Solution: The set of all sets.

2.4 Give an example of sets A, B, and C such that $A \in B$, $B \in C$, and $A \notin C$.
Solution: A = $\{1\}$. B = $\{1, \{1\}\}$. C = $\{1, \{1, \{1\}\}\}$. Then A $\in$ B, B $\in$ C, but A $\notin$ C.

2.5 Describe in prose each of the following sets.
Solution:

(a) $\{x \in \mathbb{Z} \mid x \text{ is divisible by 2 and } x \text{ is divisible by 3}\}$
Solution: All integer multiples of 6.

(b) $\{x \mid x \in A \text{ and } x \in B\}$
Solution: All elements common to sets A and B.

(c) $\{x \mid x \in A \text{ or } x \in B\}$
Solution: All elements from set A and from set B.

(d) $\{x \in \mathbb{Z}^+ \mid x \in \{x \in \mathbb{Z} \mid \text{for some integer } y,\ x = 2y\} \text{ and } x \in \{x \in \mathbb{Z} \mid \text{for some integer } y,\ x = 3y\}\}$
Solution: All positive integer multiples of 6.

(e) $\{x^2 \mid x \text{ is a prime}\}$
Solution: The squares of all prime numbers.

(f) $\{a/b \in \mathbb{Q} \mid a + b = 1 \text{ and } a, b \in \mathbb{Q}\}$
Solution: All ratios of rational numbers whose numerator and denominator sum to 1; i.e. all rational numbers.

(g) $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
Solution: All points on the unit circle.

(h) $\{(x,y) \in \mathbb{R}^2 \mid y = 2x \text{ and } y = 3x\}$
Solution: The single point (0,0).

2.6 Prove that if a, b, c, and d are any objects, not necessarily distinct from one another, then $\{\{a\}, \{a,b\}\} = \{\{c\}, \{c,d\}\}$ iff both $a = c$ and $b = d$.
Solution: ($\Rightarrow$) Let $\{\{a\}, \{a,b\}\} = \{\{c\}, \{c,d\}\}$. Then $\{\{a\}, \{a,b\}\} \subseteq \{\{c\}, \{c,d\}\}$ and so in particular $\{a\} \in \{\{c\}, \{c,d\}\}$. Suppose $c = d$. Then $\{\{c\}, \{c,d\}\} = \{\{c\}, \{c,c\}\} = \{\{c\}, \{c\}\} = \{\{c\}\}$ and therefore $\{a\} \in \{\{c\}\} \Rightarrow \{a\} = \{c\} \Rightarrow \{a\} \subseteq \{c\} \Rightarrow a \in \{c\} \Rightarrow a = c$. Then since $\{\{a\}, \{a,b\}\} \subseteq \{\{c\}, \{c,d\}\} = \{\{c\}\} = \{\{a\}\}$ we also have that $\{a,b\} \in \{\{a\}\} \Rightarrow \{a,b\} = \{a\} \Rightarrow \{a,b\} \subseteq \{a\} \Rightarrow b \in \{a\} \Rightarrow a = b$. Now suppose $c \neq d$. Since $\{a\} \in \{\{c\}, \{c,d\}\} \Rightarrow \{a\} = \{c\} \Rightarrow a = c$. By $\{\{a\}, \{a,b\}\} \subseteq \{\{c\}, \{c,d\}\}$ we also have that $\{a,b\} = \{c,b\} \in \{\{c\}, \{c,d\}\}$ and so $\{c,b\} = \{c,d\} \Rightarrow b = d$.
($\Leftarrow$) Let $a = c$ and $b = d$. Then $\{\{a\}, \{a,b\}\} = \{\{c\}, \{c,d\}\}$. ■

3.1 Prove each of the following, using any properties of numbers that may be needed.

(a) $\{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 6y\} = \{x \in \mathbb{Z} \mid \text{for integers } u \text{ and } v,\ x = 2u \text{ and } x = 3v\}$.
Solution: Let $a \in \{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 6y\}$. Then $a = 6b$ for an integer b. Then $a = 2 \cdot 3b$ and $a = 3 \cdot 2b$. Since b is an integer, $3b$ and $2b$ are also integers. Therefore $a \in \{x \in \mathbb{Z} \mid \text{for integers } u \text{ and } v,\ x = 2u \text{ and } x = 3v\}$ and so $\{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 6y\} \subseteq \{x \in \mathbb{Z} \mid \text{for integers } u \text{ and } v,\ x = 2u \text{ and } x = 3v\}$.
Now let $a \in \{x \in \mathbb{Z} \mid \text{for integers } u \text{ and } v,\ x = 2u \text{ and } x = 3v\}$. Then there is an m and n such that $a = 2m$ and $a = 3n$. Then $m - n = \frac{a}{6}$ and so $a = 6(m-n)$. Since m and n are integers, $m - n$ is also an integer and so $a \in \{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 6y\}$. Therefore $\{x \in \mathbb{Z} \mid \text{for integers } u \text{ and } v,\ x = 2u \text{ and } x = 3v\} \subseteq \{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 6y\}$.
Thus $\{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 6y\} = \{x \in \mathbb{Z} \mid \text{for integers } u \text{ and } v,\ x = 2u \text{ and } x = 3v\}$. ■

(b) $\{x \in \mathbb{R} \mid \text{for a real number } y,\ x = y^2\} = \{x \in \mathbb{R} \mid x \geq 0\}$
Solution: Let $a \in \{x \in \mathbb{R} \mid \text{for a real number } y,\ x = y^2\}$. Then $a = b^2$ for some $b \in \mathbb{R}$. If $b = 0$ then $a = 0$. Otherwise, $a > 0$ since the square of a nonzero real number is positive. Thus $a \in \{x \in \mathbb{R} \mid x \geq 0\}$ and therefore $\{x \in \mathbb{R} \mid \text{for a real number } y,\ x = y^2\} \subseteq \{x \in \mathbb{R} \mid x \geq 0\}$.
Now let $a \in \{x \in \mathbb{R} \mid x \geq 0\}$. Then $a \geq 0$. Then we may find a $b \in \mathbb{R}$ such that $a = b^2$: simply pick $b = \sqrt{a} \in \mathbb{R}$. (The square root of any nonnegative real number is again a real number.) Then $a \in \{x \in \mathbb{R} \mid \text{for a real number } y,\ x = y^2\}$ and therefore $\{x \in \mathbb{R} \mid x \geq 0\} \subseteq \{x \in \mathbb{R} \mid \text{for a real number } y,\ x = y^2\}$.
Thus $\{x \in \mathbb{R} \mid \text{for a real number } y,\ x = y^2\} = \{x \in \mathbb{R} \mid x \geq 0\}$. ■

(c) $\{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 6y\} \subseteq \{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 2y\}$.
Solution: Let $a \in \{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 6y\}$. Then $a = 6b$ for some integer b. Then $a = 2 \cdot 3b$. Since b is an integer, $3b$ is also an integer. Therefore $a \in \{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 2y\}$. Thus $\{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 6y\} \subseteq \{x \in \mathbb{Z} \mid \text{for an integer } y,\ x = 2y\}$. ■

3.2 Prove each of the following for sets A, B, and C.

(a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
Solution: Suppose $A \subseteq B$ and $B \subseteq C$. Then for any $a \in A$ we have that $a \in B$ and since $a \in B$ we have that $a \in C$. Thus $a \in C$ whenever $a \in A$ and therefore $A \subseteq C$.

(b) If $A \subseteq B$ and $B \subset C$, then $A \subset C$.
Solution: Suppose $A \subseteq B$ and $B \subset C$. Take the case when $A = B$. Since $B \subset C$ we have that $A \subset C$. Now take the case when $A \subset B$. For any $a \in A$ we have that $a \in B$. Since $B \subset C$ we have that $a \in C$. Then $A \subseteq C$. Since $B \subset C$, we know that there is some element $c \in C$ such that $c \notin B$ and since $A \subset B$ we know that $c \notin A$ and so $A \neq C$. Therefore $A \subset C$.

(c) If $A \subset B$ and $B \subseteq C$, then $A \subset C$.
Solution: Suppose $A \subset B$ and $B \subseteq C$. Take the case when $B = C$. Since $A \subset B$ we have that $A \subset C$. Now take the case when $B \subset C$. For any $a \in A$ we have that $a \in B$. Since $B \subset C$ we have that $a \in C$. Then $A \subseteq C$. Since $B \subset C$, we also know that there is some element $c \in C$ such that $c \notin B$ and since $A \subset B$ we know that $c \notin A$ and so $A \neq C$. Therefore $A \subset C$.

(d) If $A \subset B$ and $B \subset C$, then $A \subset C$.
Solution: Suppose $A \subset B$ and $B \subset C$. For any $a \in A$ we have that $a \in B$. Since $B \subset C$ we have that $a \in C$. Then $A \subset C$. Since $B \subset C$, we know that there is some element $c \in C$ such that $c \notin B$ and since $A \subset B$ we know that $c \notin A$ and so $A \neq C$. Therefore $A \subset C$.

3.3 Give an example of sets A, B, C, D, and E which satisfy the following conditions simultaneously: $A \subset B$, $B \in C$, $C \subset D$, and $D \subset E$.
Solution: Let $A = \emptyset$, $B = \{\emptyset\}$, $C = \{\{\emptyset\}\}$, $D = \{\emptyset, \{\emptyset\}\}$, and $E = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$.

3.4 Which of the following are true for all sets A, B, and C?

(a) If $A \notin B$ and $B \notin C$, then $A \notin C$.
Solution: False: Let $A = \emptyset$, $B = \{0\}$ and $C = \{\emptyset\}$.

(b) If $A \neq B$ and $B \neq C$, then $A \neq C$.
Solution: False: Let $A = \mathbb{R}$, $B = \mathbb{Z}$ and $C = \mathbb{R}$.

(c) If $A \in B$ and $B \nsubseteq C$, then $A \notin C$.
Solution: False: Let $A = \emptyset$, $B = \{\emptyset, 0\}$ and $C = \{\emptyset, 1\}$.

(d) If $A \subset B$ and $B \subseteq C$, then $C \not\subseteq A$.
Solution: True: Suppose $A \subset B$ and $B \subseteq C$. Since $A \subset B$, there is some $b \in B$ such that $b \notin A$. Since $B \subseteq C$ we have that this $b \in C$ and thus there is a $c \in C$ such that $c \notin A$. Therefore $C \not\subseteq A$.

(e) If $A \subset B$ and $B \subset C$, then $A \subset C$.
Solution: True: Suppose $A \subset B$ and $B \subset C$. Then for any $a \in A$ we have that $a \in B$. Since $a \in B$ we have that $a \in C$. Therefore $A \subseteq C$. Since $B \subset C$ we have that there is some $c \in C$ such that $c \notin B$. Since $A \subset B$ we have that $c \notin A$. Therefore $A \neq C$ and so $A \subset C$.

3.5 Show that for every set A, $A \subseteq \emptyset$ iff $A = \emptyset$.
Solution: ($\Rightarrow$) Suppose $A \subseteq \emptyset$. Then for any $a \in A$ we have that $a \in \emptyset$. But since the empty set has no members, no such a can exist. Therefore A has no members and so $A = \emptyset$. ($\Leftarrow$) Suppose $A = \emptyset$. Then A has no members and so, certainly, for all $a \in A$ we have that $a \in B$, for any set B. Therefore $A \subseteq B$. Letting $B = \emptyset$, we conclude that $A \subseteq \emptyset$. ■

3.6 Let $A_1, A_2, \dots, A_n$ be n sets. Show that

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_1 \text{ iff } A_1 = A_2 = \cdots = A_n.$$

Solution: ($\Leftarrow$) Suppose sets $A_1 = A_2 = \cdots = A_n$. Then clearly $A_i \subseteq A_{i+1}$ for all $1 \leq i \leq n-1$ and $A_n \subseteq A_1$. Therefore $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_1$.
($\Rightarrow$) Suppose $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_1$. Then for any $a \in A_1$ we have that $a \in A_2$, $a \in A_3$, ..., $a \in A_n$ and so $A_1 \subseteq A_n$. But since $A_n \subseteq A_1$ we must have that $A_1 = A_n$. Therefore $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{n-1} \subseteq A_1$. Repeating this argument $n-2$ times for $j = n-1, n-2, \dots, 2$ we find that for any $a \in A_1$ we have that $a \in A_j$ and therefore $A_1 \subseteq A_j$. But we also have that $A_j \subseteq A_1$ and so $A_1 = A_j$. Finally, we conclude that $A_1 = A_2 = \cdots = A_n$. ■

3.7 Give several examples of a set X such that each element of X is a subset of X.
Solution: $X_1 = \emptyset$. $X_2 = \{\emptyset\}$. $X_3 =?$

3.8 List the members of $\mathcal{P}(A)$ if $A = \{\{1,2\},\{3\},1\}$.
Solution: $\mathcal{P}(A) = \{A, \{\{1,2\},\{3\}\}, \{\{1,2\},1\}, \{\{3\},1\}, \{\{1,2\}\}, \{\{3\}\}, \{1\}, \emptyset\}$

3.9 For each positive integer n, give an example of a set A_n of n elements such that for each pair of elements of A_n, one member is an element of the other.
Solution: Let $A_0 = \emptyset$ and $A_1 = \{A_0\}$. For $n \geq 2$, let $A_n = \{A_0, A_1, \dots, A_{n-1}\}$.

4.1 Prove that for all sets A and B, $\emptyset \subseteq A \cap B \subseteq A \cup B$.
Solution: Let A and B be any sets. Since the empty set has no members, it's clear

that for all $x \in \emptyset$ we have that $x \in C$ for any set C. Since $A \cap B$ is a set, we have that $\emptyset \subseteq A \cap B$. Now take any $x \in A \cap B$. Then $x \in A$ and $x \in B$ and so, clearly, $x \in A$ or $x \in B$. Therefore $x \in A \cup B$ and so $A \cap B \subseteq A \cup B$. Thus $\emptyset \subseteq A \cap B \subseteq A \cup B$.

4.2 Let $\mathbb{Z}$ be the universal set, and let

$$\begin{aligned} A &= \{x \in \mathbb{Z} \mid \text{for some positive integer } y,\ x = 2y\}, \\ B &= \{x \in \mathbb{Z} \mid \text{for some positive integer } y,\ x = 2y - 1\}, \\ C &= \{x \in \mathbb{Z} \mid x < 10\}. \end{aligned}$$

Describe $\overline{A}, \overline{A \cup B}, A - \overline{C}$, and $C - (A \cup B)$, either in prose or by a defining property.
Solution:

- $\overline{A} = \{x \in \mathbb{Z} \mid x < 1\} \cup B$
- $\overline{A \cup B} = \{x \in \mathbb{Z} \mid x < 1\}$
- $A - \overline{C} = \{2, 3, 6, 8\}$
- $C - (A \cup B) = \{x \in \mathbb{Z} \mid x < 1\}$

4.3 Consider the following subsets of $\mathbb{Z}^+$, the set of positive integers:

$$\begin{aligned} A &= \{x \in \mathbb{Z}^+ \mid \text{for some integer } y,\ x = 2y\}, \\ B &= \{x \in \mathbb{Z}^+ \mid \text{for some integer } y,\ x = 2y + 1\}, \\ C &= \{x \in \mathbb{Z}^+ \mid \text{for some integer } y,\ x = 3y\}. \end{aligned}$$

(a) Describe $A \cap C, B \cup C$, and $B - C$.
Solution:

- $A \cap C = \{x \in \mathbb{Z}^+ \mid \text{for some integer } y,\ x = 6y\}$. This is the set of all positive multiples of both 2 and 3 (i.e. all positive multiples of 6).
- $B \cup C = B \cup \{x \in \mathbb{Z}^+ \mid \text{for some integer } y,\ x = 6y\}$. This is the set of all positive odd integers along with all even positive multiples of 3 (i.e. all positive multiples of 6).
- $B - C = \{x \in \mathbb{Z}^+ \mid \text{for some integer } y,\ x = 3y + 1 \text{ or } x = 3y + 2\}$. This is the set of all positive integers which are not divisible by 3.

(b) Verify that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
Solution: In this example:

- $A \cap (B \cup C) = A \cap C$
- $A \cap B = \emptyset \Rightarrow (A \cap B) \cup (A \cap C) = A \cap C$

A general proof:
Assume $x \in A \cap (B \cup C)$. Then by definition of set intersection, $x \in A$ and $x \in B \cup C$. By the definition of set union, $x \in B$ or $x \in C$. If $x \in B$ then we have that $x \in A \cap B$ since $x \in A$. Otherwise, if $x \in C$ then we have that

$x \in A \cap C$ since $x \in A$. In both cases we know that $x \in (A \cap B) \cup (A \cap C)$ since both $A \cap B \subseteq (A \cap B) \cup (A \cap C)$ and $A \cap C \subseteq (A \cap B) \cup (A \cap C)$. Therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Now assume $x \in (A \cap B) \cup (A \cap C)$. By the definition of set union, $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$ then by the definition of set intersection, we have that $x \in A$ and $x \in B$. Since $B \subseteq B \cup C$ we know that $x \in B \cup C$. Thus $x \in A \cap (B \cup C)$. Otherwise if $x \in A \cap C$ then by the definition of set intersection, we have that $x \in A$ and $x \in C$. Since $C \subseteq B \cup C$ we know that $x \in B \cup C$. Thus $x \in A \cap (B \cup C)$. Therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Therefore $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$. ■

4.4 If A is any set, what are each of the following sets? $A \cap \emptyset$, $A \cup \emptyset$, $A - \emptyset$, $A - A$, $\emptyset - A$.
Solution:

- $A \cap \emptyset = \{x \mid x \in A \text{ and } x \in \emptyset\} = \emptyset$.
- $A \cup \emptyset = \{x \mid x \in A \text{ or } x \in \emptyset\} = A$.
- $A - \emptyset = \{x \mid x \in A \text{ and } x \notin \emptyset\} = A$.
- $A - A = \{x \mid x \in A \text{ and } x \notin A\} = \emptyset$.
- $\emptyset - A = \{x \mid x \in \emptyset \text{ and } x \notin A\} = \emptyset$.

4.5 Determine $\emptyset \cap \{\emptyset\}, \{\emptyset\} \cap \{\emptyset\}, \{\emptyset, \{\emptyset\}\} - \emptyset, \{\emptyset, \{\emptyset\}\} - \{\emptyset\}, \{\emptyset, \{\emptyset\}\} - \{\{\emptyset\}\}$
Solution:

- $\emptyset \cap \{\emptyset\} = \{x \mid x \in \emptyset \text{ and } x \in \{\emptyset\}\} = \emptyset$.
- $\{\emptyset\} \cap \{\emptyset\} = \{x \mid x \in \{\emptyset\} \text{ and } x \in \{\emptyset\}\} = \{x \mid x \in \{\emptyset\}\} = \{\emptyset\}$.
- $\{\emptyset, \{\emptyset\}\} - \emptyset = \{x \mid x \in \{\emptyset, \{\emptyset\}\} \text{ and } x \notin \emptyset\} = \{\emptyset, \{\emptyset\}\}$.
- $\{\emptyset, \{\emptyset\}\} - \{\emptyset\} = \{x \mid x \in \{\emptyset, \{\emptyset\}\} \text{ and } x \notin \{\emptyset\}\} = \{\{\emptyset\}\}$.
- $\{\emptyset, \{\emptyset\}\} - \{\{\emptyset\}\} = \{x \mid x \in \{\emptyset, \{\emptyset\}\} \text{ and } x \notin \{\{\emptyset\}\}\} = \{\emptyset\}$.

4.6 Suppose A and B are subsets of U. Show that in each of (a), (b), and (c) below, if any one of the relations stated holds, then each of the others holds.

(a) $A \subseteq B, \overline{A} \supseteq \overline{B}, A \cup B = B, A \cap B = A$.

(b) $A \cap B = \emptyset, A \subseteq \overline{B}, B \subseteq \overline{A}$.

(c) $A \cup B = U, \overline{A} \subseteq B, \overline{B} \subseteq A$.
Solution:

(a) i. Assume $A \subseteq B$.

- Take $x \notin B$. Assume $x \in A$. Since $A \subseteq B$ we have that $x \in B$. Then by contradiction we must have that $x \notin A$. Therefore $\overline{B} \subseteq \overline{A}$ or $\overline{A} \supseteq \overline{B}$.
- Take $x \in A \cup B$. Then $x \in A$ or $x \in B$. Assume $x \in A$. Then since $A \subseteq B$ we have that $x \in B$. Therefore $A \cup B \subseteq B$. But since $B \subseteq A \cup B$ we have that $A \cup B = B$.

- Take $x \in A$. Because $A \subseteq B$ we have that $x \in B$. Since $x \in A$ and $x \in B$ we have that $x \in A \cap B$ and therefore $A \subseteq A \cap B$. But since $A \cap B \subseteq A$, we have $A \cap B = A$.

ii. Assume $\overline{A} \supseteq \overline{B}$.

- Take $x \in A$. Assume $x \notin B$. Then because $\overline{A} \supseteq \overline{B}$ we have that $x \notin A$. Then by contradiction we must have that $x \in B$. Therefore $A \subseteq B$.
- Take $x \in A \cup B$. Then by the definition of set union, $x \in A$ or $x \in B$. Assume $x \notin B$. Then because $\overline{A} \supseteq \overline{B}$ we have that $x \notin A$. But then since $x \notin A$ and $x \notin B$ we have that $x \notin A \cup B$. By contradiction we must have that $x \in B$ and thus $A \cup B \subseteq B$. But since $B \subseteq A \cup B$ we have $A \cup B = B$.
- Take $x \in A$. Assume $x \notin B$. Then because $\overline{A} \supseteq \overline{B}$ we have that $x \notin A$. By contradiction, $x \in B$. Thus $A \subseteq A \cap B$. But since $A \cap B \subseteq A$ we have $A \cap B = A$.

iii. Assume $A \cup B = B$.

- Take $x \in A$. Then $x \in A \cup B = B$ and therefore $A \subseteq B$.
- Take $x \notin B$. Then $x \notin A \cup B$ and so $x \notin A$. Therefore $\overline{A} \supseteq \overline{B}$.
- Take $x \in A$. Then $x \in A \cup B = B$. Since $x \in A$ and $x \in B$ then $x \in A \cap B$ and so $A \subseteq A \cap B$. But since $A \cap B \subseteq A$ we have $A \cap B = A$.

iv. Assume $A \cap B = A$.

- Take $x \in A$. Then $x \in A \cap B$ and so $x \in B$. Therefore $A \subseteq B$.
- Take $x \notin B$. Then $x \notin A \cap B$ and so $x \notin A$. Therefore $\overline{A} \supseteq \overline{B}$.
- Take $x \in A \cup B$. Then $x \in A$ or $x \in B$. Assume $x \notin B$. Then $x \notin A \cap B = A$ and so $x \notin A \cup B$. By contradiction, we must have the $x \in B$. Then $A \cup B \subseteq B$. But since $B \subseteq A \cup B$ we have that $A \cup B = B$.

(b) i. Assume $A \cap B = \emptyset$.

- Take $x \in A$. Since $A \cap B = \emptyset$, $x \notin B$. Then $A \subseteq \overline{B}$.
- Take $x \in B$. Since $A \cap B = \emptyset$, $x \notin A$. Then $B \subseteq \overline{A}$.

ii. Assume $A \subseteq \overline{B}$.

- Take $x \in A$. Then since $A \subseteq \overline{B}$ we have $x \notin B$. Now take $x \in B$. Assume $x \in A$. But then since $A \subseteq \overline{B}$ we have $x \notin B$. By contradiction we must have that $x \notin A$. Thus $x \in A \Rightarrow x \notin B$ and $x \in B \Rightarrow x \notin A$ and so $A \cap B = \emptyset$.
- Take $x \in B$. Assume $x \in A$. Then since $A \subseteq \overline{B}$ we have that $x \notin B$. Then by contradiction, we must have $x \notin A$ and so $B \subseteq \overline{A}$.

iii. Assume $B \subseteq \overline{A}$.

- Take $x \in B$. Then since $B \subseteq \overline{A}$ we have $x \notin A$. Now take $x \in A$ and assume $x \in B$. But since $B \subseteq \overline{A}$ we have $x \notin A$. Then by contradiction, we must have that $x \notin B$. Thus $x \in A \Rightarrow x \notin B$ and $x \in B \Rightarrow x \notin A$ and so $A \cap B = \emptyset$.

- Take $x \in A$. Assume $x \in B$. Then since $B \subseteq \overline{A}$ we have $x \notin A$. Then by contradiction we must have that $x \notin B$. Therefore $A \subseteq \overline{B}$.

(c) i. Assume $A \cup B = U$.

- Take $x \notin A$. Assume $x \notin B$. Then since $x \notin A$ and $x \notin B$, we have that $x \notin A \cup B = U$. But since $A \subseteq U$ this is a contradiction. So we must have that $x \in B$. Therefore $\overline{A} \subseteq B$.
- Take $x \notin B$. Assume $x \notin A$. Then since $x \notin A$ and $x \notin B$, we have that $x \notin A \cup B = U$. But since $B \subseteq U$ this is a contradiction. So we must have that $x \in A$. Therefore $\overline{B} \subseteq A$.

ii. Assume $\overline{A} \subseteq B$.

- Take $x \in U$. Then either $x \in A$ or $x \notin A$. Assume that $x \in A$. Then $x \in A \cup B$. Now assume that $x \notin A$. Then since $\overline{A} \subseteq B$ we have that $x \in B$. Then $x \in A \cup B$. Therefore $U \subseteq A \cup B$. But since $A \cup B \subseteq U$ we must have that $A \cup B = U$.
- Take $x \notin B$. Assume$\overline{B} \subseteq A$

iii. Assume $\overline{B} \subseteq A$.

- Take $x \in U$. Then $x \in B$ or $x \notin B$. If $x \in B$ then $x \in A \cup B$. If $x \notin B$ then since $\overline{B} \subseteq A$ we have that $x \in A$ and so $x \in A \cup B$. Therefore $U \subseteq A \cup B$. But since $A \cup B \subseteq U$ we must have that $A \cup B = U$.
- Take $x \notin A$ and assume $x \notin B$. Then since $\overline{B} \subseteq A$ we have that $x \in A$. Then by contradiction we must have that $x \in B$ and so $\overline{A} \subseteq B$.

4.7 Prove that for all sets A, B, and C,

$$(A \cap B) \cup C = A \cap (B \cup C) \text{ iff } C \subseteq A.$$

Solution: ($\Rightarrow$) Assume $(A \cap B) \cup C = A \cap (B \cup C)$. Let $x \in C$. Then $x \in (A \cap B) \cup C$. Since $(A \cap B) \cup C = A \cap (B \cup C)$ we have that $x \in A \cap (B \cup C)$ and thus $x \in A$. Therefore $C \subseteq A$.
($\Leftarrow$) Assume $C \subseteq A$. Let $x \in (A \cap B) \cup C$. Then $x \in A \cap B$ or $x \in C$. Assume $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $x \in B$ then $x \in B \cup C$. Since $x \in A$ and $x \in B \cup C$ we have that $x \in A \cap (B \cup C)$. Now assume $x \in C$. Then $x \in B \cup C$. Since $C \subseteq A$ we also have that $x \in A$. Thus $x \in A \cap (B \cup C)$. Therefore $(A \cap B) \cup C \subseteq A \cap (B \cup C)$. Now let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Then $x \in B$ or $x \in C$. Assume $x \in B$. Then since $x \in A$ and $x \in B$ we have that $x \in A \cap B$. Therefore $x \in (A \cap B) \cup C$. Now assume $x \in C$. Then $x \in (A \cap B) \cup C$. Therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup C$ and so we can conclude that $(A \cap B) \cup C = A \cap (B \cup C)$. ■

4.8 Prove that for all sets A, B, and C,

$$(A - B) - C = (A - C) - (B - C).$$

Solution: Let $x \in (A - B) - C$. Then $x \in A - B$ and $x \notin C$. Since $x \in A - B$ we have that $x \in A$ and $x \notin B$. Then since $x \in A$ and $x \notin C$ we have that $x \in A - C$. Since $x \notin B$ we have that $x \notin B - C$. Then since $x \in A - C$ and $x \notin B - C$ we have that $x \in (A - C) - (B - C)$. Now let $x \in (A - C) - (B - C)$. Then $x \in A - C$ and $x \notin B - C$. Then since $x \in A - C$ we have that $x \in A$ and $x \notin C$. Since $x \notin B - C$ we have that either $x \notin B$ or $x \in C$. But $x \in C$ is a contradiction because $x \notin C$. Therefore $x \notin B$. Then since $x \in A$ and $x \notin B$ we have that $x \in A - B$ and since $x \notin C$ we have that $x \in (A - B) - C$. Therefore $(A - C) - (B - C) \subseteq (A - B) - C$ and thus $(A - B) - C = (A - C) - (B - C)$. $\blacksquare$