Chapter 1

Sets and Relations

1.1 Cantor's Concept of a Set

A **set** S is any collection of definite, distinguishable objects of our intuition or of our intellect to be conceived as a whole. The objects are called the **elements** or **members** of S.

1.2 The Basis of Intuitive Set Theory

Membership relation: $x \in A$ if the object x is a member of the set A. If x is not a member of A then $x \notin A$. $x_1, x_2, \ldots, x_n \in A$ is shorthand for $x_1 \in A \wedge x_2 \in A \wedge \cdots \wedge x_n \in A$.
The intuitive principle of extension: Two sets are equal iff they have the same members.
Set equality: The equality of two sets X and Y will be denoted by $X = Y$ and inequality of X and Y by $X \neq Y$. Among the basic properties of this relation are:

$$X = X,$$
$$X = Y \Rightarrow Y = X,$$
$$X = Y \wedge Y = Z \Rightarrow X = Z,$$

for all sets X, Y, and Z.
unit set: a set $\{x\}$ whose sole member is x.
collection of sets: a set whose members are sets.
The intuitive principle of abstraction: A formula $P(x)$ defines a set A by the convention that the members of A are exactly those objects a such that $P(a)$ is a true statement, denoted by $A = \{x \mid P(x)\}$.
Note: $\{x \in A \mid P(x)\} := \{x \mid x \in A \wedge P(x)\}$. For a property P and function f we can write $\{f(x) \mid P(x)\} := \{y \mid \exists x \colon P(x) \wedge y = f(x)\}$.

1.3 Inclusion

If A and B are sets, then A is **included in** B iff each member of A is a member of B. Symbolized: $A \subseteq B$. We also say that A is a **subset** of B. Equivalently, B **includes** A, symbolized by $B \supseteq A$.
The set A is **properly included in** B (A is a **proper subset** of B / B **properly includes** A) iff $A \subseteq B$ and $A \neq B$.
Among the basic properties of the inclusion relation are

$$X \subseteq X;$$
$$X \subseteq Y \wedge Y \subseteq Z \Rightarrow X \subseteq Z;$$
$$X \subseteq Y \wedge Y \subseteq X \Rightarrow X = Y.$$

empty set: $\{x \in A \mid x \neq x\}$ for any set A is the set with no elements, symbolized by $\emptyset$.
power set: the set of all subsets of a given set. $\mathcal{P}(A) = \{B \mid B \subseteq A\}$ for a given set A.

1.4 Operations for Sets

union: for sets A and B, the set of all objects which are members of either A or B. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. (**sum**/**join**)
intersection: for sets A and B, the set of all objects which are members of both A and B. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. (**product**/**meet**)

<u>Lemma</u>: For every pair of sets A and B the following inclusions hold:

$$\emptyset \subseteq A \cap B \subseteq A \subseteq A \cup B.$$

<u>Proof</u>: Take $x \in \emptyset$. Since this is false, we can conclude $x \in A \cap B$ and so $\emptyset \subseteq A \cap B$. Now take $x \in A \cap B$. Then $x \in \{y \mid y \in A \text{ and } y \in B\}$ and so $x \in \{y \mid y \in A\} = A$ and thus $A \cap B \subseteq A$. Now take $x \in A$. Then we must have $x \in \{y \mid y \in A \text{ or } y \in B\} = A \cup B$. Then $A \subseteq A \cup B$. ■

disjoint: $A \cap B = \emptyset$ for sets A and B.
intersect: $A \cap B \neq \emptyset$ for sets A and B.
disjoint collection: for a collection of sets, each distinct pair of its member sets is disjoint.
partition: for a set X, a disjoint collection $\mathcal{A}$ of nonempty and distinct subsets of X such that each member of X is a member of some (exactly one) member of $\mathcal{A}$.
absolute complement of A: $\overline{A} = \{x \mid x \notin A\}$, the set of all members which are not in A.
relative complement of A with respect to X: $X - A = X \cap \overline{A} = \{x \in X \mid x \notin A\}$, the set of those members of X which are not members of A.
symmetric difference of A and B: $A + B = (A - B) \cup (B - A)$.
universal set: the set U such that all sets under consideration in a certain discussion are subsets of U.

1.5 The Algebra of Sets

identities: equations which are true whatever the universal set U and no matter what particular subsets the letters (other than U and $\emptyset$) represent.

<u>THEOREM 5.1</u>: For any subsets A, B, C of a set U the following equations are identities. Here $\overline{A}$ is an abbreviation for $U - A$.

1. $A \cup (B \cup C) = (A \cup B) \cup C$.	1'. $A \cap (B \cap C) = (A \cap B) \cap C$.
2. $A \cup B = B \cup A$.	2'. $A \cap B = B \cap A$.
3. $A \cup (B \cap C) = (A \cup B) \cap (B \cup C)$.	3'. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. $A \cup \emptyset = A$.	4'. $A \cap U = A$.
5. $A \cup \overline{A} = U$	5'. $A \cap \overline{A} = \emptyset$.

<u>PROOF</u>:
<u>Lemma</u>: Let X, Y be subsets of U. Then $X \subseteq X \cup Y$ and $X \subseteq Y \cup X$.
<u>Proof</u>: Assume $x \in X$ and $x \notin X \cup Y$. Then $x \notin \{z \mid z \in X \text{ or } z \in Y\} \Rightarrow x \notin \{z \mid z \in X\} = X$ which is a contradiction. Now assume $x \in X$ and $x \notin Y \cup X$. Then $x \notin \{z \mid z \in Y \text{ or } z \in X\} \Rightarrow x \notin \{z \mid z \in X\} = X$ which is a contradiction.

1. Assume $x \in A \cup (B \cup C)$. Then $x \in A$ or $x \in B \cup C$. If $x \in A$ then $x \in A \cup B$ and so $x \in (A \cup B) \cup C$. Otherwise if $x \in B \cup C$ then $x \in B$ or $x \in C$. If $x \in B$ then $x \in (A \cup B)$ and so $x \in (A \cup B) \cup C$. If $x \in C$ then $x \in (A \cup B) \cup C$. Therefore $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.
   Now assume $x \in (A \cup B) \cup C$. Then $x \in A \cup B$ or $x \in C$. If $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \cup C)$. If $x \in B$ then $x \in (B \cup C)$ and so $x \in A \cup (B \cup C)$. Otherwise if $x \in C$ then $x \in B \cup C$ and so $x \in A \cup (B \cup C)$. Therefore $(A \cup B) \cup C \subseteq A \cup (B \cup C)$. Hence $A \cup (B \cup C) = (A \cup B) \cup C$.

1'. Assume $x \in A \cap (B \cap C)$. Then $x \in A$ and $x \in B \cap C$. Since $x \in B \cap C$ we have $x \in B$ and $x \in C$. Then since $x \in A$ and $x \in B$ we have $x \in A \cap B$. Since $x \in C$ we have $x \in (A \cap B) \cap C$. Therefore $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.
   Now assume $x \in (A \cap B) \cap C$. Then $x \in A \cap B$ and $x \in C$. Since $x \in A \cap B$ we have $x \in A$ and $x \in B$. Then since $x \in B$ and $x \in C$ we have $x \in B \cap C$. Since $x \in A$ we have $x \in A \cap (B \cap C)$. Therefore $(A \cap B) \cap C \subseteq A \cap (B \cap C)$. Hence $A \cap (B \cap C) = (A \cap B) \cap C$.

2. Assume $x \in A \cup B$. Then $x \in A$ or $x \in B$. In either case $x \in B \cup A$ and so $A \cup B \subseteq B \cup A$. Now assume $x \in B \cup A$. Then $x \in B$ or $x \in A$. In either case $x \in A \cup B$ and so $B \cup A \subseteq A \cup B$. Hence $A \cup B = B \cup A$.

2'. Assume $x \in A \cap B$. Then $x \in A$ and $x \in B$ and so $x \in B \cap A$. Therefore $A \cap B \subseteq B \cap A$. Now assume $x \in B \cap A$. Then $x \in B$ and $x \in A$ and so $x \in A \cap B$. Therefore $B \cap A \subseteq B \cap A$. Hence $A \cap B = B \cap A$.

3. Assume $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ and so $x \in (A \cup B) \cap (A \cup C)$. Otherwise if $x \in B \cap C$ then $x \in B$ and

$x \in C$. Since $x \in B$ we have $x \in A \cup B$. Since $x \in C$ we have $x \in A \cup C$. Then $x \in (A \cup B) \cap (A \cup C)$ and therefore $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
Now assume $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. Since $x \in A \cup B$ we have $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \cap C)$. Otherwise if $x \in B$ then $x \in A \cup B$. Since $x \in A \cup C$ we also have that $x \in A$ or $x \in C$. If $x \in A$ then $x \in A \cup (B \cap C)$. Otherwise if $x \in C$ then since $x \in B$ we have $x \in B \cap C$ and so $x \in A \cup (B \cap C)$. Therefore $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

3'. Assume $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. If $x \in B$ then since $x \in A$ we have $x \in A \cap B$ and so $x \in (A \cap B) \cup (A \cap C)$. Otherwise if $x \in C$ then since $x \in A$ we have $x \in A \cap C$ and so $x \in (A \cap B) \cup (A \cap C)$. Therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Now assume $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$ then $x \in A$ and $x \in B$. Since $x \in B$ we have $x \in B \cup C$. Since we also have $x \in A$ then $x \in A \cap (B \cup C)$. Otherwise if $x \in A \cap C$ then $x \in A$ and $x \in C$. Since $x \in C$ we have $x \in B \cup C$. Since we also have $x \in A$ then $x \in A \cap (B \cup C)$. Therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. Assume $x \in A \cup \emptyset$. Then $x \in A$ or $x \in \emptyset$. Since $x \in \emptyset$ is impossible, we must have the $x \in A$ and so $A \cup \emptyset \subseteq A$. Now assume $x \in A$ then $x \in A \cup \emptyset$ and so $A \subseteq A \cup \emptyset$. Hence $A \cup \emptyset = A$.

4'. Assume $x \in A \cap U$. Then $x \in A$ and $x \in U$. Therefore $A \cap U \subseteq A$. Now assume $x \in A$. Then since $A \subseteq U$ we have $x \in U$ and so $x \in A \cap U$. Therefore $A \subseteq A \cap U$. Hence $A \cap U = A$.

5. Assume $x \in A \cup \overline{A}$. Then $x \in A$ or $x \in \overline{A}$. Since $A \subseteq U$ and $\overline{A} \subseteq U$ in either case we have $x \in U$ and so $A \cup \overline{A} \subseteq U$. Now assume $x \in U$. Then $x \in A$ or $X \notin A$ for any set A. Thus $x \in A$ or $x \in \overline{A}$ and so $x \in A \cup \overline{A}$. Therefore $U \subseteq A \cup \overline{A}$. Hence $A \cup \overline{A} = U$.

5'. Assume $x \in A \cap \overline{A}$. Then $x \in A$ and $x \in \overline{A}$. Since $x \in \overline{A}$ we have $x \notin A$. Since $x \in A$ and $x \notin A$ we have $x \in \emptyset$. Therefore $A \cap \overline{A} \subseteq \emptyset$. Since $\emptyset \subseteq X$ for any set X we have $\emptyset \subseteq A \cap \overline{A}$. Hence $A \cap \overline{A} = \emptyset$. ■

General associative law for set union: The sets obtainable from given sets $A_1, A_2, \ldots, A_n$ in that order, by use of the operation of union are all equal to one another. The set defined by $A_1, A_2, \ldots, A_n$ in this way will be written as

$$A_1 \cup A_2 \cup \cdots \cup A_n.$$

General associative law for set intersection: The sets obtainable from given sets $A_1, A_2, \ldots, A_n$ in that order, by use of the operation of intersection are all equal to one another. The set defined by $A_1, A_2, \ldots, A_n$ in this way will be written as

$$A_1 \cap A_2 \cap \cdots \cap A_n.$$

General commutative law for set union: If $1', 2', \ldots, n'$ are $1, 2, \ldots, n$ in any order, then

$$A_1 \cup A_2 \cup \cdots \cup A_n = A_{1'} \cup A_{2'} \cup \cdots \cup A_{n'}.$$

General commutative law for set intersection: If $1', 2', \ldots, n'$ are $1, 2, \ldots, n$ in any order, then

$$A_1 \cap A_2 \cap \cdots \cap A_n = A_{1'} \cap A_{2'} \cap \cdots \cap A_{n'}.$$

General distributive law for set union:

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n).$$

General distributive law for set intersection:

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n).$$

dual: An equation, or an expression, or a statement within the framework of the algebra of sets obtained from another by interchanging $\cup$ and $\cap$ along with $\emptyset$ and U.

principle of duality for the algebra of sets: If T is any theorem expressed in terms of $\cup$, $\cap$, and $\overline{}$, then the dual of T is also a theorem.

Theorem 5.2: For all subsets A and B of a set U, the following statements are valid. Here $\overline{A}$ is an abbreviation for $U - A$.

6. If, for all A, $A \cup B = A$, then $B = \emptyset$.
6'. If, for all A, $A \cap B = A$ then $B = U$.
7,7'. If $A \cup B = U$ and $A \cap B = \emptyset$, then $B = \overline{A}$.
8,8'. $\overline{\overline{A}} = A$.
9. $\overline{\emptyset} = U$.
9'. $\overline{U} = \emptyset$.
10. $A \cup A = A$
10'. $A \cap A = A$.
11. $A \cup U = U$
11'. $A \cap \emptyset = \emptyset$.
12. $A \cup (A \cap B) = A$.
12'. $A \cap (A \cup B) = A$.
13. $\overline{A \cup B} = \overline{A} \cap \overline{B}$
13'. $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proof:

6. Assume $A \cup B = A$ for all A. Take $A = \emptyset$. Then $\emptyset \cup B = \emptyset$. Then if $x \in B$ we have $x \in \emptyset \cup B = \emptyset$ and so $B \subseteq \emptyset$. Since $\emptyset \subseteq B$ we have $B = \emptyset$.

6'. Assume $A \cap B = A$ for all A. Take $A = U$. Then $U \cap B = U$. Then if $x \in U$ we have $x \in U \cap B$ and so $x \in B$. Therefore $U \subseteq B$. Since $B \subseteq U$ we have $B = U$.

7,7'. Assume $A \cup B = U$ and $A \cap B = \emptyset$ for sets A and B. Take $x \in B$. Assume $x \in A$. Then $x \in A \cap B = \emptyset$. By contradiction we have $x \in \overline{A}$. Then $B \subseteq \overline{A}$. Now take $x \in \overline{A}$. Then $x \notin A$. Assume $x \notin B$. Then $x \notin A \cup B = U$. By contradiction we have $x \in B$. Therefore $\overline{A} \subseteq B$ and so $B = \overline{A}$.

8,8'. Take $x \in \overline{\overline{A}}$ for a set A. Then $x \notin \overline{A}$ and so $x \in A$. Therefore $\overline{\overline{A}} \subseteq A$. Now take $x \in A$. Then $x \notin \overline{A}$ and so $x \in \overline{\overline{A}}$. Therefore $A \subseteq \overline{\overline{A}}$ and so $\overline{\overline{A}} = A$.

9. Take $x \in \overline{\emptyset}$. Then $x \in U \cap \overline{\emptyset}$ and so $x \in U$. Then $\overline{\emptyset} \subseteq U$. Now take $x \in U$. Then $x \notin \emptyset$ and so $x \in U \cap \overline{\emptyset} = U - \emptyset = \overline{\emptyset}$. Therefore $U \subseteq \overline{\emptyset}$ and so $\overline{\emptyset} = U$.

9'. Take $x \in \overline{U}$. Then $x \in U \cap \overline{U}$ so $x \in U$ and $x \notin U$. By contradiction $x \in \emptyset$ and so $\overline{U} \subseteq \emptyset$. Now take $x \in \emptyset$. Then $x \notin U$ and so $x \in \overline{U}$. Therefore $\emptyset \subseteq \overline{U}$ and so $\overline{U} = \emptyset$.

10. Take $x \in A \cup A$. Then $x \in A$ or $x \in A$ and so $x \in A$. Thus $A \cup A \subseteq A$. Now take $x \in A$. Then $x \in A \cup A$ and so $A \subseteq A \cup A$. Therefore $A \cup A = A$.

10'. Take $x \in A \cap A$. Then $x \in A$ and so $A \cap A \subseteq A$. Now take $x \in A$. Then $x \in A \cap A$ and so $A \subseteq A \cap A$. Therefore $A \cap A = A$.

11. Take $x \in A \cup U$. Then $x \in A$ or $x \in U$. If $x \in A$ then $x \in U$ since $A \subseteq U$. Therefore $A \cup U \subseteq U$. Now take $x \in U$. Then $x \in A \cup U$ and so $U \subseteq A \cup U$. Therefore $A \cup U = U$.

11'. Take $x \in A \cap \emptyset$. Then $x \in \emptyset$ and so $A \cap \emptyset \subseteq \emptyset$. Now take $x \in \emptyset$. Then $x \in A$ (ex falso quodlibet). Thus $x \in A \cap \emptyset$ and so $\emptyset \subseteq A \cap \emptyset$. Therefore $A \cap \emptyset = \emptyset$.

12. Take $x \in A \cup (A \cap B)$. Then $x \in A$ or $x \in A \cap B$. If $x \in A \cap B$ then $x \in A$. Therefore $A \cup (A \cap B) \subseteq A$. Now take $x \in A$. Then $x \in A \cup (A \cap B)$. Thus $A \subseteq A \cup (A \cap B)$ and so $A \cup (A \cap B) = A$.

12'. Take $x \in A \cap (A \cup B)$. Then $x \in A$ and so $A \cap (A \cup B) \subseteq A$. Now take $x \in A$. Then $x \in A \cup B$ and so $x \in A \cap (A \cup B)$. Therefore $A \subseteq A \cap (A \cup B)$ and so $A \cap (A \cup B) = A$.

13. Take $x \in \overline{A \cup B}$. Then $x \notin A \cup B$ and so $x \notin A$ and $x \notin B$. Then $x \in \overline{A}$ and $x \in \overline{B}$ and so $x \in \overline{A} \cap \overline{B}$. Therefore $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. Now take $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$ and so $x \notin A$ and $x \notin B$. Then $x \notin A \cup B$ and so $x \in \overline{A \cup B}$. Therefore $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$ and so $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

13'. Take $x \in \overline{A \cap B}$. Then $x \notin A \cap B$. Then $x \notin A$ or $x \notin B$. Then $x \in \overline{A}$ or $x \in \overline{B}$ and so $x \in \overline{A} \cup \overline{B}$. Therefore $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Now take $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$ and so $x \notin A$ or $x \notin B$. Then $x \notin A \cap B$ and so $x \in \overline{A \cap B}$. Therefore $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ and so $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

<u>THEOREM 5.3</u>: The following statements about sets A and B are equivalent to one another.

(I) $A \subseteq B$

(II) $A \cap B = A$

(III) $A \cup B = B$

PROOF:
(I) implies (II). Assume $A \subseteq B$. Since, for all A and B, $A \cap B \subseteq A$, it is sufficient to prove that $A \subseteq A \cap B$. But if $x \in A$, then $x \in B$ and, hence, $x \in A \cap B$. Hence $A \subseteq A \cap B$.
(II) implies (III). Assume $A \cap B = A$. Then $A \cup B = (A \cap B) \cup B = (A \cup B) \cap (B \cup B) = (A \cup B) \cap B = B$.
(III) implies (I). Assume $A \cup B = B$. Then this and the identity $A \subseteq A \cup B$ imply $A \subseteq B$.

NOTE: The principle of duality does not apply directly to expressions in which $-$ or $\subseteq$ appears. Replace $A - B$ with $A \cap \overline{B}$. Replace $A \subseteq B$ with $A \cap B = A$ or $A \cup B = B$. The dual of $A \cap B = A$ is $A \cup B = A \Leftrightarrow A \supseteq B$. So we can extend the principle of duality to include the inclusion symbol: swap $\subseteq$ with $\supseteq$ (inclusion signs are reversed).

THEORY OF EQUATIONS FOR THE ALGEBRA OF SETS: For an equation formed using $\cup$, $\cap$, and $\overline{}$ on symbols $A_1, A_2, \ldots, A_n$ and X where the A's denote fixed subsets of some universal set U and X denotes a subset of U which is constrained only by the equation in which it appears, determine under what conditions such an equation has a solution and then, assuming these are satisfied, obtain all solutions.
Step I. Two sets are equal iff their symmetric difference is equal to $\emptyset$. Hence, an equation in X is equivalent to one whose righthand side is $\emptyset$.
Step II. An equation in X with righthand side $\emptyset$ is equivalent to one of the form

$$(A \cap X) \cup (B \cap \overline{X}) = \emptyset,$$

where A and B are free of X.
Step III. The union of two sets is equal to $\emptyset$ iff each set is equal to $\emptyset$. Hence the equation in Step II is equivalent to the pair of simultaneous equations

$$A \cap X = \emptyset, B \cap \overline{X} = \emptyset.$$

Step IV. The above pair of equations, and hence the original equation, has a solution iff $B \subseteq \overline{A}$. In this event, any X, such that $B \subseteq X \subseteq \overline{A}$, is a solution. [See exercise 5.7]