Chapter 1

Sets and Relations

1.1 Cantor's Concept of a Set

A **set** S is any collection of definite, distinguishable objects of our intuition or of our intellect to be conceived as a whole. The objects are called the **elements** or **members** of S.

1.2 The Basis of Intuitive Set Theory

Membership relation: $x \in A$ if the object x is a member of the set A. If x is not a member of A then $x \notin A$. $x_1, x_2, \ldots, x_n \in A$ is shorthand for $x_1 \in A \land x_2 \in A \land \cdots \land x_n \in A$.
The intuitive principle of extension: Two sets are equal iff they have the same members.
Set equality: The equality of two sets X and Y will be denoted by $X = Y$ and inequality of X and Y by $X \neq Y$. Among the basic properties of this relation are:

$$\begin{gathered} X = X, \\ X = Y \Rightarrow Y = X, \\ X = Y \land Y = Z \Rightarrow X = Z, \end{gathered}$$

for all sets X, Y, and Z.
unit set: a set $\{x\}$ whose sole member is x.
collection of sets: a set whose members are sets.
The intuitive principle of abstraction: A formula $P(x)$ defines a set A by the convention that the members of A are exactly those objects a such that $P(a)$ is a true statement, denoted by $A = \{x \mid P(x)\}$.
Note: $\{x \in A \mid P(x)\} := \{x \mid x \in A \land P(x)\}$. For a property P and function f we can write $\{f(x) \mid P(x)\} := \{y \mid \exists x\colon P(x) \land y = f(x)\}$.

1.3 Inclusion

If A and B are sets, then A is **included in** B iff each member of A is a member of B. Symbolized: $A \subseteq B$. We also say that A is a **subset** of B. Equivalently, B **includes** A, symbolized by $B \supseteq A$.
The set A is **properly included in** B (A is a **proper subset** of B / B **properly includes** A) iff $A \subseteq B$ and $A \neq B$.
Among the basic properties of the inclusion relation are

$$X \subseteq X;$$
$$X \subseteq Y \wedge Y \subseteq Z \Rightarrow X \subseteq Z;$$
$$X \subseteq Y \wedge Y \subseteq X \Rightarrow X = Y.$$

empty set: $\{x \in A \mid x \neq x\}$ for any set A is the set with no elements, symbolized by $\emptyset$.
power set: the set of all subsets of a given set. $\mathcal{P}(A) = \{B \mid B \subseteq A\}$ for a given set A.

1.4 Operations for Sets

union: for sets A and B, the set of all objects which are members of either A or B. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. (**sum**/**join**)
intersection: for sets A and B, the set of all objects which are members of both A and B. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. (**product**/**meet**)

<u>Lemma</u>: For every pair of sets A and B the following inclusions hold:

$$\emptyset \subseteq A \cap B \subseteq A \subseteq A \cup B.$$

<u>Proof</u>: Take $x \in \emptyset$. Since this is false, we can conclude $x \in A \cap B$ and so $\emptyset \subseteq A \cap B$. Now take $x \in A \cap B$. Then $x \in \{y \mid y \in A \text{ and } y \in B\}$ and so $x \in \{y \mid y \in A\} = A$ and thus $A \cap B \subseteq A$. Now take $x \in A$. Then we must have $x \in \{y \mid y \in A \text{ or } y \in B\} = A \cup B$. Then $A \subseteq A \cup B$. ■

disjoint: $A \cap B = \emptyset$ for sets A and B.
intersect: $A \cap B \neq \emptyset$ for sets A and B.
disjoint collection: for a collection of sets, each distinct pair of its member sets is disjoint.
partition: for a set X, a disjoint collection $\mathcal{A}$ of nonempty and distinct subsets of X such that each member of X is a member of some (exactly one) member of $\mathcal{A}$.
absolute complement of A: $\overline{A} = \{x \mid x \notin A\}$, the set of all members which are not in A.
relative complement of A with respect to X: $X - A = X \cap \overline{A} = \{x \in X \mid x \notin A\}$, the set of those members of X which are not members of A.
symmetric difference of A and B: $A + B = (A - B) \cup (B - A)$.
universal set: the set U such that all sets under consideration in a certain discussion are subsets of U.

1.5 The Algebra of Sets

identities: equations which are true whatever the universal set U and no matter what particular subsets the letters (other than U and $\emptyset$) represent.

<u>Theorem 5.1</u>: For any subsets A, B, C of a set U the following equations are identities. Here $\overline{A}$ is an abbreviation for $U - A$.

1. $A \cup (B \cup C) = (A \cup B) \cup C$.	1'. $A \cap (B \cap C) = (A \cap B) \cap C$.
2. $A \cup B = B \cup A$.	2'. $A \cap B = B \cap A$.
3. $A \cup (B \cap C) = (A \cup B) \cap (B \cup C)$.	3'. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. $A \cup \emptyset = A$.	4'. $A \cap U = A$.
5. $A \cup \overline{A} = U$	5'. $A \cap \overline{A} = \emptyset$.

<u>Proof</u>:
<u>Lemma</u>: Let X, Y be subsets of U. Then $X \subseteq X \cup Y$ and $X \subseteq Y \cup X$.
<u>Proof</u>: Assume $x \in X$ and $x \notin X \cup Y$. Then $x \notin \{z \mid z \in X \text{ or } z \in Y\} \Rightarrow x \notin \{z \mid z \in X\} = X$ which is a contradiction. Now assume $x \in X$ and $x \notin Y \cup X$. Then $x \notin \{z \mid z \in Y \text{ or } z \in X\} \Rightarrow x \notin \{z \mid z \in X\} = X$ which is a contradiction.

1. Assume $x \in A \cup (B \cup C)$. Then $x \in A$ or $x \in B \cup C$. If $x \in A$ then $x \in A \cup B$ and so $x \in (A \cup B) \cup C$. Otherwise if $x \in B \cup C$ then $x \in B$ or $x \in C$. If $x \in B$ then $x \in (A \cup B)$ and so $x \in (A \cup B) \cup C$. If $x \in C$ then $x \in (A \cup B) \cup C$. Therefore $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.
   Now assume $x \in (A \cup B) \cup C$. Then $x \in A \cup B$ or $x \in C$. If $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \cup C)$. If $x \in B$ then $x \in (B \cup C)$ and so $x \in A \cup (B \cup C)$. Otherwise if $x \in C$ then $x \in B \cup C$ and so $x \in A \cup (B \cup C)$. Therefore $(A \cup B) \cup C \subseteq A \cup (B \cup C)$. Hence $A \cup (B \cup C) = (A \cup B) \cup C$.

1'. Assume $x \in A \cap (B \cap C)$. Then $x \in A$ and $x \in B \cap C$. Since $x \in B \cap C$ we have $x \in B$ and $x \in C$. Then since $x \in A$ and $x \in B$ we have $x \in A \cap B$. Since $x \in C$ we have $x \in (A \cap B) \cap C$. Therefore $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.
   Now assume $x \in (A \cap B) \cap C$. Then $x \in A \cap B$ and $x \in C$. Since $x \in A \cap B$ we have $x \in A$ and $x \in B$. Then since $x \in B$ and $x \in C$ we have $x \in B \cap C$. Since $x \in A$ we have $x \in A \cap (B \cap C)$. Therefore $(A \cap B) \cap C \subseteq A \cap (B \cap C)$. Hence $A \cap (B \cap C) = (A \cap B) \cap C$.

2. Assume $x \in A \cup B$. Then $x \in A$ or $x \in B$. In either case $x \in B \cup A$ and so $A \cup B \subseteq B \cup A$. Now assume $x \in B \cup A$. Then $x \in B$ or $x \in A$. In either case $x \in A \cup B$ and so $B \cup A \subseteq A \cup B$. Hence $A \cup B = B \cup A$.

2'. Assume $x \in A \cap B$. Then $x \in A$ and $x \in B$ and so $x \in B \cap A$. Therefore $A \cap B \subseteq B \cap A$. Now assume $x \in B \cap A$. Then $x \in B$ and $x \in A$ and so $x \in A \cap B$. Therefore $B \cap A \subseteq B \cap A$. Hence $A \cap B = B \cap A$.

3. Assume $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ and so $x \in (A \cup B) \cap (A \cup C)$. Otherwise if $x \in B \cap C$ then $x \in B$ and

$x \in C$. Since $x \in B$ we have $x \in A \cup B$. Since $x \in C$ we have $x \in A \cup C$. Then $x \in (A \cup B) \cap (A \cup C)$ and therefore $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
Now assume $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. Since $x \in A \cup B$ we have $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \cap C)$. Otherwise if $x \in B$ then $x \in A \cup B$. Since $x \in A \cup C$ we also have that $x \in A$ or $x \in C$. If $x \in A$ then $x \in A \cup (B \cap C)$. Otherwise if $x \in C$ then since $x \in B$ we have $x \in B \cap C$ and so $x \in A \cup (B \cap C)$. Therefore $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

3'. Assume $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. If $x \in B$ then since $x \in A$ we have $x \in A \cap B$ and so $x \in (A \cap B) \cup (A \cap C)$. Otherwise if $x \in C$ then since $x \in A$ we have $x \in A \cap C$ and so $x \in (A \cap B) \cup (A \cap C)$. Therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Now assume $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$ then $x \in A$ and $x \in B$. Since $x \in B$ we have $x \in B \cup C$. Since we also have $x \in A$ then $x \in A \cap (B \cup C)$. Otherwise if $x \in A \cap C$ then $x \in A$ and $x \in C$. Since $x \in C$ we have $x \in B \cup C$. Since we also have $x \in A$ then $x \in A \cap (B \cup C)$. Therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. Assume $x \in A \cup \emptyset$. Then $x \in A$ or $x \in \emptyset$. Since $x \in \emptyset$ is impossible, we must have the $x \in A$ and so $A \cup \emptyset \subseteq A$. Now assume $x \in A$ then $x \in A \cup \emptyset$ and so $A \subseteq A \cup \emptyset$. Hence $A \cup \emptyset = A$.

4'. Assume $x \in A \cap U$. Then $x \in A$ and $x \in U$. Therefore $A \cap U \subseteq A$. Now assume $x \in A$. Then since $A \subseteq U$ we have $x \in U$ and so $x \in A \cap U$. Therefore $A \subseteq A \cap U$. Hence $A \cap U = A$.

5. Assume $x \in A \cup \overline{A}$. Then $x \in A$ or $x \in \overline{A}$. Since $A \subseteq U$ and $\overline{A} \subseteq U$ in either case we have $x \in U$ and so $A \cup \overline{A} \subseteq U$. Now assume $x \in U$. Then $x \in A$ or $X \notin A$ for any set A. Thus $x \in A$ or $x \in \overline{A}$ and so $x \in A \cup \overline{A}$. Therefore $U \subseteq A \cup \overline{A}$. Hence $A \cup \overline{A} = U$.

5'. Assume $x \in A \cap \overline{A}$. Then $x \in A$ and $x \in \overline{A}$. Since $x \in \overline{A}$ we have $x \notin A$. Since $x \in A$ and $x \notin A$ we have $x \in \emptyset$. Therefore $A \cap \overline{A} \subseteq \emptyset$. Since $\emptyset \subseteq X$ for any set X we have $\emptyset \subseteq A \cap \overline{A}$. Hence $A \cap \overline{A} = \emptyset$. ■

<u>General associative law for set union</u>: The sets obtainable from given sets $A_1, A_2, \dots, A_n$ in that order, by use of the operation of union are all equal to one another. The set defined by $A_1, A_2, \dots, A_n$ in this way will be written as

$$A_1 \cup A_2 \cup \cdots \cup A_n.$$

<u>General associative law for set intersection</u>: The sets obtainable from given sets $A_1, A_2, \dots, A_n$ in that order, by use of the operation of intersection are all equal to one another. The set defined by $A_1, A_2, \dots, A_n$ in this way will be written as

$$A_1 \cap A_2 \cap \cdots \cap A_n.$$

General commutative law for set union: If $1', 2', \ldots, n'$ are $1, 2, \ldots, n$ in any order, then

$$A_1 \cup A_2 \cup \cdots \cup A_n = A_{1'} \cup A_{2'} \cup \cdots \cup A_{n'}.$$

General commutative law for set intersection: If $1', 2', \ldots, n'$ are $1, 2, \ldots, n$ in any order, then

$$A_1 \cap A_2 \cap \cdots \cap A_n = A_{1'} \cap A_{2'} \cap \cdots \cap A_{n'}.$$

General distributive law for set union:

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n).$$

General distributive law for set intersection:

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n).$$

dual: An equation, or an expression, or a statement within the framework of the algebra of sets obtained from another by interchanging $\cup$ and $\cap$ along with $\emptyset$ and U.

principle of duality for the algebra of sets: If T is any theorem expressed in terms of $\cup$, $\cap$, and $\overline{}$, then the dual of T is also a theorem.

THEOREM 5.2: For all subsets A and B of a set U, the following statements are valid. Here $\overline{A}$ is an abbreviation for $U - A$.

6. If, for all A, $A \cup B = A$, then $B = \emptyset$.

6'. If, for all A, $A \cap B = A$ then $B = U$.

7,7'. If $A \cup B = U$ and $A \cap B = \emptyset$, then $B = \overline{A}$.

8,8'. $\overline{\overline{A}} = A$.

9. $\overline{\emptyset} = U$.

9'. $\overline{U} = \emptyset$.

10. $A \cup A = A$

10'. $A \cap A = A$.

11. $A \cup U = U$

11'. $A \cap \emptyset = \emptyset$.

12. $A \cup (A \cap B) = A$.

12'. $A \cap (A \cup B) = A$.

13. $\overline{A \cup B} = \overline{A} \cap \overline{B}$

13'. $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

PROOF:

6. Assume $A \cup B = A$ for all A. Take $A = \emptyset$. Then $\emptyset \cup B = \emptyset$. Then if $x \in B$ we have $x \in \emptyset \cup B = \emptyset$ and so $B \subseteq \emptyset$. Since $\emptyset \subseteq B$ we have $B = \emptyset$.

6'. Assume $A \cap B = A$ for all A. Take $A = U$. Then $U \cap B = U$. Then if $x \in U$ we have $x \in U \cap B$ and so $x \in B$. Therefore $U \subseteq B$. Since $B \subseteq U$ we have $B = U$.

7,7'. Assume $A \cup B = U$ and $A \cap B = \emptyset$ for sets A and B. Take $x \in B$. Assume $x \in A$. Then $x \in A \cap B = \emptyset$. By contradiction we have $x \in \overline{A}$. Then $B \subseteq \overline{A}$. Now take $x \in \overline{A}$. Then $x \notin A$. Assume $x \notin B$. Then $x \notin A \cup B = U$. By contradiction we have $x \in B$. Therefore $\overline{A} \subseteq B$ and so $B = \overline{A}$.

8,8'. Take $x \in \overline{\overline{A}}$ for a set A. Then $x \notin \overline{A}$ and so $x \in A$. Therefore $\overline{\overline{A}} \subseteq A$. Now take $x \in A$. Then $x \notin \overline{A}$ and so $x \in \overline{\overline{A}}$. Therefore $A \subseteq \overline{\overline{A}}$ and so $\overline{\overline{A}} = A$.

9. Take $x \in \overline{\emptyset}$. Then $x \in U \cap \overline{\emptyset}$ and so $x \in U$. Then $\overline{\emptyset} \subseteq U$. Now take $x \in U$. Then $x \notin \emptyset$ and so $x \in U \cap \overline{\emptyset} = U - \emptyset = \overline{\emptyset}$. Therefore $U \subseteq \overline{\emptyset}$ and so $\overline{\emptyset} = U$.

9'. Take $x \in \overline{U}$. Then $x \in U \cap \overline{U}$ so $x \in U$ and $x \notin U$. By contradiction $x \in \emptyset$ and so $\overline{U} \subseteq \emptyset$. Now take $x \in \emptyset$. Then $x \notin U$ and so $x \in \overline{U}$. Therefore $\emptyset \subseteq \overline{U}$ and so $\overline{U} = \emptyset$.

10. Take $x \in A \cup A$. Then $x \in A$ or $x \in A$ and so $x \in A$. Thus $A \cup A \subseteq A$. Now take $x \in A$. Then $x \in A \cup A$ and so $A \subseteq A \cup A$. Therefore $A \cup A = A$.

10'. Take $x \in A \cap A$. Then $x \in A$ and so $A \cap A \subseteq A$. Now take $x \in A$. Then $x \in A \cap A$ and so $A \subseteq A \cap A$. Therefore $A \cap A = A$.

11. Take $x \in A \cup U$. Then $x \in A$ or $x \in U$. If $x \in A$ then $x \in U$ since $A \subseteq U$. Therefore $A \cup U \subseteq U$. Now take $x \in U$. Then $x \in A \cup U$ and so $U \subseteq A \cup U$. Therefore $A \cup U = U$.

11'. Take $x \in A \cap \emptyset$. Then $x \in \emptyset$ and so $A \cap \emptyset \subseteq \emptyset$. Now take $x \in \emptyset$. Then $x \in A$ (ex falso quodlibet). Thus $x \in A \cap \emptyset$ and so $\emptyset \subseteq A \cap \emptyset$. Therefore $A \cap \emptyset = \emptyset$.

12. Take $x \in A \cup (A \cap B)$. Then $x \in A$ or $x \in A \cap B$. If $x \in A \cap B$ then $x \in A$. Therefore $A \cup (A \cap B) \subseteq A$. Now take $x \in A$. Then $x \in A \cup (A \cap B)$. Thus $A \subseteq A \cup (A \cap B)$ and so $A \cup (A \cap B) = A$.

12'. Take $x \in A \cap (A \cup B)$. Then $x \in A$ and so $A \cap (A \cup B) \subseteq A$. Now take $x \in A$. Then $x \in A \cup B$ and so $x \in A \cap (A \cup B)$. Therefore $A \subseteq A \cap (A \cup B)$ and so $A \cap (A \cup B) = A$.

13. Take $x \in \overline{A \cup B}$. Then $x \notin A \cup B$ and so $x \notin A$ and $x \notin B$. Then $x \in \overline{A}$ and $x \in \overline{B}$ and so $x \in \overline{A} \cap \overline{B}$. Therefore $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. Now take $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$ and so $x \notin A$ and $x \notin B$. Then $x \notin A \cup B$ and so $x \in \overline{A \cup B}$. Therefore $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$ and so $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

13'. Take $x \in \overline{A \cap B}$. Then $x \notin A \cap B$. Then $x \notin A$ or $x \notin B$. Then $x \in \overline{A}$ or $x \in \overline{B}$ and so $x \in \overline{A} \cup \overline{B}$. Therefore $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Now take $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$ and so $x \notin A$ or $x \notin B$. Then $x \notin A \cap B$ and so $x \in \overline{A \cap B}$. Therefore $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ and so $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

<u>Theorem 5.3</u>: The following statements about sets A and B are equivalent to one another.

(I) $A \subseteq B$

(II) $A \cap B = A$

(III) $A \cup B = B$

PROOF:
(I) implies (II). Assume $A \subseteq B$. Since, for all A and B, $A \cap B \subseteq A$, it is sufficient to prove that $A \subseteq A \cap B$. But if $x \in A$, then $x \in B$ and, hence, $x \in A \cap B$. Hence $A \subseteq A \cap B$.
(II) implies (III). Assume $A \cap B = A$. Then $A \cup B = (A \cap B) \cup B = (A \cup B) \cap (B \cup B) = (A \cup B) \cap B = B$.
(III) implies (I). Assume $A \cup B = B$. Then this and the identity $A \subseteq A \cup B$ imply $A \subseteq B$.

NOTE: The principle of duality does not apply directly to expressions in which $-$ or $\subseteq$ appears. Replace $A - B$ with $A \cap \overline{B}$. Replace $A \subseteq B$ with $A \cap B = A$ or $A \cup B = B$. The dual of $A \cap B = A$ is $A \cup B = A \Leftrightarrow A \supseteq B$. So we can extend the principle of duality to include the inclusion symbol: swap $\subseteq$ with $\supseteq$ (inclusion signs are reversed).

THEORY OF EQUATIONS FOR THE ALGEBRA OF SETS: For an equation formed using $\cup$, $\cap$, and $\overline{}$ on symbols $A_1, A_2, \ldots, A_n$ and X where the A's denote fixed subsets of some universal set U and X denotes a subset of U which is constrained only by the equation in which it appears, determine under what conditions such an equation has a solution and then, assuming these are satisfied, obtain all solutions.
Step I. Two sets are equal iff their symmetric difference is equal to $\emptyset$. Hence, an equation in X is equivalent to one whose righthand side is $\emptyset$.
Step II. An equation in X with righthand side $\emptyset$ is equivalent to one of the form

$$(A \cap X) \cup (B \cap \overline{X}) = \emptyset,$$

where A and B are free of X.
Step III. The union of two sets is equal to $\emptyset$ iff each set is equal to $\emptyset$. Hence the equation in Step II is equivalent to the pair of simultaneous equations

$$A \cap X = \emptyset, B \cap \overline{X} = \emptyset.$$

Step IV. The above pair of equations, and hence the original equation, has a solution iff $B \subseteq \overline{A}$. In this event, any X, such that $B \subseteq X \subseteq \overline{A}$, is a solution. [See exercise 5.7]

1.6 Relations

ordered pair: $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$.

THEOREM 6.1: The ordered pair of x and y is uniquely determined by x and y. Moreover, if $\langle x, y\rangle = \langle u, v\rangle$ then $x = u$ and $y = v$.
PROOF:
That x and y uniquely determine $\langle x, y\rangle$ follows from our assumption that a set is uniquely determined by its members. Now assume $\langle x, y\rangle = \langle u, v\rangle$.
(Case I) $u = v$: Then $\langle u, v\rangle = \{\{u\}, \{u, v\}\} = \{\{u\}\}$. Hence $\{\{x\}, \{x, y\}\} = \{\{u\}\} \Rightarrow \{x\} = \{\{x, y\}\} = \{u\}$ and so $x = u$ and $y = v$.

(Case II) $u \neq v$: Then $\{u\} \neq \{\{u\},\{u,v\}\}$ and $\{x\} \neq \{\{u\},\{u,v\}\}$. Then $\{x\} \in \{\{u\},\{u,v\}\} \Rightarrow \{x\} = \{u\} \Rightarrow x = u$ and $\{x,y\} \in \{\{u\},\{u,v\}\} \Rightarrow \{x,y\} = \{u,v\}$. Then $\{x,y\} \neq \{u\}$ and so $x \neq y$ and $y \neq u$. Therefore $y = v$.

first coordinate: x in $\langle x, y\rangle$.
second coordinate: y in $\langle x, y\rangle$.
ordered triple: $\langle x, y, z\rangle = \langle\langle x, y\rangle, z\rangle$.
ordered n-tuple: $\langle x_1, x_2, \ldots, x_n\rangle = \langle\langle x_1, x_2, \ldots, x_{n-1}\rangle, x_n\rangle$.
binary relation: a set of ordered pairs. Given relation ρ and $\langle x, y\rangle \in \rho$ we write $x\rho y$.
ρ-related: x is ρ-related to y iff $x\rho y$.
n-ary relation: a set of ordered n-tuples.
domain: $D_\rho = \{x \mid \text{for some } y, \langle x, y\rangle \in \rho\}$.
range: $R_\rho = \{y \mid \text{for some } x, \langle x, y\rangle \in \rho\}$.
cartesian product: $X \times Y = \{\langle x, y\rangle \mid x \in X \wedge y \in Y\}$.
relation from X to Y: $\rho \subseteq X \times Y$.
relation in Z: $\rho \subseteq Z \times Z$.
universal relation in X: $\rho = X \times X$.
void relation in X: $\rho = \emptyset$.
identity relation in X: $\iota_X = \{\langle x, x\rangle \mid x \in X\}$.
ρ-relatives of A: $\rho[A] = \{y \mid x\rho y \text{ for some } x \in A\}$. Then we have $\rho(D_\rho) = R_\rho$, and, for any set A, $\rho[A] \subseteq R_\rho$.

1.7 Equivalence Relations

reflexive: a relation ρ in a set X is reflexive (in X) iff $x\rho x$ for each $x \in X$.
symmetric: a relation ρ is symmetric if $x\rho y \Rightarrow y\rho x$.
transitive: a relation ρ is transitive iff $x\rho y \wedge y\rho z \Rightarrow x\rho z$.
equivalence relation: a relation which is reflexive, symmetric, and transitive. Any equivalence relation in X is an equivalence relation on X since $D_\rho = X$ for any equivalence relation ρ in X.
equivalence class: if ρ is an equivalence relation on X, then $A \subseteq X$ is an equivalence class (ρ-equivalence class) iff there is some $x \in A$ such that $A = \{y \mid x\rho y\}$ iff there is some $x \in X$ such that $A = \rho[\{x\}]$. The equivalence class generated by x is denoted $[x]$. Two basic properties follow from this definition: (I) $x \in [x]$ and (II) if $x\rho y$, then $[x] = [y]$.

<u>Theorem 7.1</u>: Let ρ be an equivalence relation on X. Then the collection of distinct ρ-equivalence classes is a partition of X. Conversely, if $\mathcal{P}$ is a partition of X, and a relation ρ defined by $a\rho b$ iff there exists A in $\mathcal{P}$ such that $a, b \in A$, then ρ is an equivalence relation on X. Moreover, if an equivalence relation ρ determines the partition $\mathcal{P}$ of X, then the equivalence relation defined by $\mathcal{P}$ is equal to ρ. Conversely, if a partition $\mathcal{P}$ of X determines the equivalence relation ρ, then the partition of X defined by ρ is equal to $\mathcal{P}$.
<u>Proof</u>: From (II) above, we have that two equivalence classes are either disjoint or equal,

since $z \in [x]$ and $z \in [y]$ then $[x] = [z]$ and $[y] = [z]$ and so $[x] = [y]$. Therefore the collection of distinct ρ-equivalence classes determines a partition $\mathcal{P}$ of X. To show the converse, let $\mathcal{P}$ be a partition of X and let relation ρ on X be defined such that $a\rho b$ iff there exists $A \in \mathcal{P}$ such that $a, b \in A$. Then ρ is symmetric by its definition. For all $a \in X$, there exists some $A \in \mathcal{P}$ such that $a \in A$ and so ρ is reflexive. To show the transitivity of ρ, assume $a\rho b$ and $b\rho c$. Then there exist $A \in \mathcal{P}$ such that $a, b \in A$ and $B \in \mathcal{P}$ such that $b, c \in B$. Then $b \in A$ and $b \in B$ but since $\mathcal{P}$ is a partition, we must have that $A = B$, which means $c \in A$ and so $a\rho c$. Therefore ρ is an equivalence relation on X.

Now assume that an equivalence relation ρ on X is given, that it determines the partition $\mathcal{P}$ of X and that $\mathcal{P}$ determines the equivalence relation ρ^*. We show $\rho = \rho^*$. Assume $\langle x, y\rangle \in \rho$. Then $x, y \in [x]$ and $[x] \in \mathcal{P}$. By the definition of ρ^* it follows that $x\rho^* y$ or $\langle x, y\rangle \in \rho^*$. Conversely, given $\langle x, y\rangle \in \rho^*$, there exists A in $\mathcal{P}$ with $x, y \in A$. But A is a ρ-equivalence class, and hence $x\rho y$ or $\langle x, y\rangle \in \rho$. Thus $\rho = \rho^*$.

For the converse, assume that $\mathcal{P}$ is a partition of X, that it determines the equivalence relation ρ on X, and that ρ determines the partition $\mathcal{P}^*$ of X. We will show $\mathcal{P} = \mathcal{P}^*$. Take any $A \in \mathcal{P}$. Then for any $x, y \in A$ we have $\langle x, y\rangle \in \rho$ and so $A = [x] = [y]$. Then, since ρ determines the partition $\mathcal{P}^*$, we must have $A \in \mathcal{P}^*$. Conversely, take any $A^* \in \mathcal{P}^*$. Then for any $x, y \in A^*$ we have $\langle x, y\rangle \in \rho$ since $\mathcal{P}^*$ is determined by ρ and thus $A^* = [x]$. Then we must have $A^* \in \mathcal{P}$ since ρ is determined by $\mathcal{P}$. Therefore $\mathcal{P} = \mathcal{P}^*$.

congruence mod n in $\mathbb{Z}$: x is congruent to y mod n in $\mathbb{Z}$, symbolized $x \equiv y (\text{mod } n)$, iff n divides $x - y$ for some nonzero $n \in \mathbb{Z}$.

residue class modulo n: congruence class mod n - $[a]$ consists of all numbers $a + kn$ for $k \in \mathbb{Z}$. The residue class mod n are $[0], [1], \ldots, [n-1]$. The collection of residue classes mod n is denoted $\mathbb{Z}_n$.

quotient set of X by ρ: the partition of X induced by an equivalence relation ρ on X, denoted by X/ρ.

<u>Theorem 7.2</u>: A relation ρ is an equivalence relation iff there exists a disjoint collection $\mathcal{P}$ of nonempty sets such that

$$\rho = \{\langle x, y\rangle \mid \text{for some } C \in \mathcal{P}, \langle x, y\rangle \in C \times C\}.$$

<u>Proof</u>: Let $R = \{\langle x, y\rangle \mid \text{for some } C \in \mathcal{P}, \langle x, y\rangle \in C \times C\}$.

($\Rightarrow$) Assume that ρ is an equivalence relation on X. Then the collection of distinct ρ-equivalence classes is disjoint, and we contend that with this choice for $\mathcal{P}$, ρ has the structure described in the theorem. Assume $\langle x, y\rangle \in R$. Then there exists an equivalence class $[z]$ with $x, y \in [z]$. Then $z\rho x$ and $z\rho y$ and so $x\rho y$ and thus $\langle x, y\rangle \in \rho$. Therefore $R \subseteq \rho$. Now assume $\langle x, y\rangle \in \rho$. Then $x, y \in [x]$ and so $\langle x, y\rangle \in [x] \times [x]$. Therefore $D\rho \subseteq R$ and hence $\rho = R$.

($\Leftarrow$) Assume ρ is a relation and that there exists a disjoint collection $\mathcal{P}$ of nonempty sets such that $\rho = R$. Then we must show that ρ is an equivalence relation. ρ is reflexive: given any $C \in \mathcal{P}$, for all $x \in C$ we have $\langle x, x\rangle \in C \times C$ and so $\langle x, x\rangle \in \rho$. ρ is symmetric: assume $x\rho y$. Then we have $\langle x, y\rangle \in C \times C$ and so $x, y \in C$. Then $\langle y, x\rangle \in C \times C$ and therefore

$\langle y,x\rangle \in \rho$. ρ is transitive: assume $x\rho y$ and $y\rho z$ then $\langle x,y\rangle \in C\times C$ for some $C \in \mathcal{P}$ and $\langle y,z\rangle \in D\times D$ for some $D\in\mathcal{P}$. Then we have $x,y\in C$ and $y,z\in D$. But since $\mathcal{P}$ is a partition and $y\in C$ and $y\in D$ we must have that $C=D$. Therefore $z\in C$ and so $\langle x,z\rangle \in C\times C$ and hence $x\rho z$.

1.8 Functions

function: a relation such that no two distinct members have the same first coordinate.
f is a function $\Leftrightarrow f\subseteq A\times B \wedge \langle x,y\rangle, \langle x,z\rangle \in f \Rightarrow y=z$.

synonyms for **function**: transformation, map, mapping, correspondence, operator.

If f is a function and $\langle x,y\rangle\in f$, so that xfy, then x is an **argument** of f.
y is the **value** of f at x, the **image** of x under f, the element into which f **carries** x.
Symbols for y: $xf, f(x), fx, x^f$.
$f(x)$ is the name for the sole member of $f[\{x\}]$, the set of f-relatives of x.
The characteristic feature of a function among relations is that each member of the domain of a function has a single relative.

into: f is into $Y \Leftrightarrow R_f\subseteq Y$.

onto: f is onto $Y\Leftrightarrow R_f = Y$.

on: f is on $X\Leftrightarrow D_f = X$.

$f:X\to Y$ **or** $X\xrightarrow{f} Y$: f is a function on the set X into the set Y.

Y^X: the set of all functions on X into Y. $Y^X\subseteq\mathcal{P}(X\times Y)$. $Y^\emptyset=\{\emptyset\}$ and $\emptyset^X=\emptyset$ if $X\neq\emptyset$.

restriction of f **to** A: $f\cap(A\times Y)$ where $f:X\to Y$ and $A\subseteq X$. Denoted $f|A$.
$f|A : A\to Y$ such that $(f|A)(a)=f(a)$ for $a\in A$. We have $(f|A)\subseteq f$.

extension of g **to** f : $g\subseteq f$.

identity map on X: $i_X(x)=x$ for all $x\in X$.

injection mapping on A **into** X: $i_X|A = i_A$.

one-to-one: f maps distinct elements onto distinct elements.
f is one-to-one $\Leftrightarrow x_1\neq x_2\Rightarrow f(x_1)\neq f(x_2) \Leftrightarrow f(x_1)=f(x_2)\Rightarrow x_1=x_2$.

one-to-one correspondence between X and Y: f is a one-to-one function on X onto Y.

n^X: The set of all functions on X into a set of n elements.

characteristic function of A: $\chi_A(x) = 1$ if $x \in A$ else $\chi_A(x) = 0$ for $A \subseteq X$. $\chi_A \in 2^X$. $\mathcal{P}(X)$ is in one-to-one correspondence with 2^X via the function $f : \mathcal{P}(X) \to 2^X$ by $f(A \subseteq X) = \chi_A$.

n-ary operation in X: a function f such that $D_f = X^n$ and $R_f \subseteq X$ where X^n is the set of all n-tuples $\langle x_1, x_2, \ldots, x_n \rangle$ for $x_i \in X$. This is a function of n variables.

1.9 Composition and Inversion for Functions

composite of f and g: $g \circ f = \{\langle x, z \rangle \mid \exists y : xfy \wedge ygz\}$.

functional composition: the operation of computing $f \circ g$ from f and g. As a special case, if $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ and $(g \circ f)(x) = g(f(x))$.

<u>ASSOCIATIVE LAW FOR COMPOSITION</u>: $f \circ (g \circ h) = (f \circ g) \circ h$.
<u>PROOF</u>: Assume $\langle x, u \rangle \in f \circ (g \circ h)$. Then there is a z such that $\langle x, z \rangle \in g \circ h$ and $\langle z, u \rangle \in f$. Since $\langle x, z \rangle \in g \circ h$, there is a y such that $\langle x, y \rangle \in h$ and $\langle y, z \rangle \in g$. Now $\langle y, z \rangle \in g$ and $\langle z, u \rangle \in f$ imply that $\langle y, u \rangle \in f \circ g$. Further, $\langle x, y \rangle \in h$ and $\langle y, u \rangle \in f \circ g$ imply that $\langle x, u \rangle \in (f \circ g) \circ h$. Then $f \circ (g \circ h) \subseteq (f \circ g) \circ h$. Reversing the foregoing steps yields the reverse inclusion and hence equality. ■

canonical/natural mapping on X onto X/ρ: $j : X \to X/\rho$ with $j(x) = [x]$ where ρ is an equivalence relation with domain X. Then j is onto the quotient set X/ρ.

If f is a mapping on X into Y, then the relation $x_1 \rho x_2$ iff $f(x_1) = f(x_2)$ is an equivalence relation on X. Let j be the canonical map on X onto X/ρ. Define $g : X/\rho \to f[X]$ by $g([x]) = f(x)$. Define $i : f[X] \to Y$ by $i(y) = y$, the injection of $f[X]$ into Y. We have that j is onto, i is one-to-one, and g is one-to-one and onto. Then we may write $f = i \circ g \circ j$.

Let $f : A \to B$. Define $\text{Inj}(f) \equiv f$ is one-to-one. Define $\text{Surj}(f) \equiv f$ is onto. Define $\text{Bij}(f) \equiv \text{Inj}(f) \wedge \text{Surj}(f)$.

<u>A characterization of one-to-oneness</u>:
Let $f : X \to Y$. Then $\text{Inj}(f) \Leftrightarrow \forall g, h : Z \to X : f \circ g = f \circ h \Rightarrow g = h$.

<u>A characterization of ontoness</u>:
Let $f : X \to Y$. Then $\text{Surj}(f) \Leftrightarrow \forall g, h : Y \to Z : g \circ f = h \circ f \Rightarrow g = h$.

inverse function f^{-1} of f: the function resulting from f by interchanging the coordinates of members of f, given that f is one-to-one.
$\forall f : \exists f^{-1} \Rightarrow D_{f^{-1}} = R_f \wedge R_{f^{-1}} = D_f \wedge (x = f^{-1}(y) \Leftrightarrow y = f(x))$.
$\mathrm{Inj}\,(f^{-1}) \wedge (f^{-1})^{-1} = f$.
$[f : X \to Y \wedge \mathrm{Bij}\,(f)] \Rightarrow [f^{-1} : Y \to X \wedge \mathrm{Bij}\,(f^{-1})]$.
$f^{-1} \circ f = i_X \wedge f \circ f^{-1} = i_X$.

functional inversion: the operation of computing f^{-1} from f.

$\mathrm{Inj}\,(f) \wedge \mathrm{Inj}\,(g) \Rightarrow \mathrm{Inj}\,(g \circ f) \wedge [(g \circ f)^{-1} = f^{-1} \circ g^{-1}]$.

inverse or **counter image** of A under f: $f^{-1}[A]$.