# Understanding Graphs with Analysis

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## Final Report

The project kicked off with the basic concept of Simple Graphs. A Graph is basically a structure $(V, E)$, where $V$ is the set of vertices and $E \subseteq V \times V$ is the set of edges. A simple Graph has an additional condition that E be antireflexive and symmetric. Graphs as characteristic functions was also seen. The Characteristic Function of a Graph G is $1_G : V \times V \rightarrow \{0, 1\}$ such that $1_G(a, b) = 1$ iff $(a, b) \in E$. The handshake lemma was covered which is a very basic theorem regarding the sum of all degree of all vertices in terms of the count of undirected edges in the Graph. Then, homorphisms and isomorphism of graphs were seen, which led to Graphs being studied as an Algebraic Structure. All of this was pretty intuitive and similiar to concepts of homomorphism in Group, Rings etc, But one extra condition of the function being injective is required for homomorphisms(hence the term homorphism).
With the concept of homorphism, the concept of a Graph(say G) having a copy of another Graph (say H) in it was defined. The number of H in the Graph was expressed in terms of the Characteristic function of the Graph.It is easy to see that this will be

$$\#H_{/G} = \frac{1}{/Aut(H)/} \Sigma_{(a_1, a_2, \ldots a_{/H/}) \in V^{/H/}} \Pi_{(i,j) \in E_{/H}} 1_G(a_i, a_j)$$

where H has been modified to have the vertex set $[/H/]$ and $Aut(H)$ denotes the number of self isomorphisms in the Graph H. Paul Erdos and the likes of him investigated the questions as to how many edges must a Graph have which necessitates the Graph G having a copy of H in it.
The extremal number, denotes ex(n,H), was defined to be as the smallest number such that any Graph G with n vertices having greater than that many edges must have a copy of H in it.
The set of extremal Graph, denoted Ex(n,H), was defined as the set of Graphs with n vertices without a copy of H in it, such that adding any edge in G leads to the Graph having a copy of H in it.
The problem of actually bounding ex(n,H) within an upper and a lower bound was covered rather than completely determining it.
We consider right now the case that H is $K_r$ namely the Graph on r vertices

with all edges. A very interesting technique used in proving upper bounds is Zykov Symmetrization. Basically, what this says is that given two vertices x and y which are not neighbours, on a Graph that is $K_r$ free, we can actually disconnect all of y's neighbours from y and join y with every neighbour of x. This new Graph is also $K_r$ free.
Using the process of Zykov Symmetrisation we can prove Turan's Theorem which basically says that the extremal graph on Ex(n,$K_r$) with the greatest number of edges is actually unique upto isomorphism. This Graph is an r-1 partite complete Graph.
The Erdos Stone Theorem, dubbed the fundamental theorem of Extremal Graph Theory in a sense generalises Turan's Theorem to Graph H that are not necessarily Complete. This is done in terms of the Chromatic Number, which is the minimum number of colors required to color a Graph such that no adjacent vertices have the same color.
How can we say that L$\leq$ ex(n,H)? This can be done by actually giving a Graph with n vertices having L vertices which is $H$ free.
We will now look at a probabilistic way of getting such Graphs. We actually studied how we can make up Graphs randomly. We do this by actually choosing every edge to be present in the Graph with a probability of $p$. We then calculated the expected number of copies of H in the Probability distribution on the set of Graph induced by this procedure. Then, we developed a heuristic to estimate a Lower Bound.
Finally, the concept of Psuedorandom Graphs was discussed extensively. A Psuedorandom Graph (or basically a family of graphs indexed by the natural numbers and with number of vertices equal to the index given to that graph) is used to describe Graphs that are "like" random Graphs described in the previous paragraphs with $p = 1/2$. A more analytical definition of the Psuedorandom Graphs was given and later shown to be equivalent to lots of other properties that psuedorandomness notion must imply. How did the properties correspond to Random Graphs? Well, quantities that had an expected value of 0 in Random Graphs described earlier with an additional error term in Landau Little o notation led to these properties. The paper that we studied can be found here.