FREQUENCY RESPONSE OF CIRCUITS

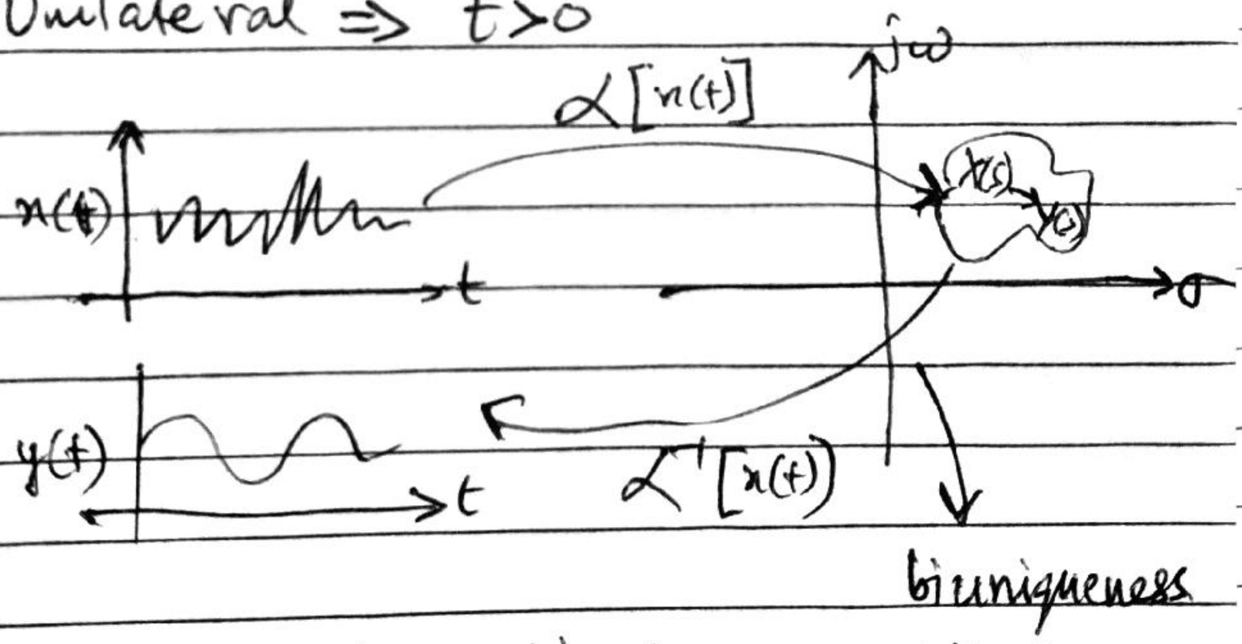
THE UNILATERAL LAPLACE TRANSFORM

The unilateral Laplace transform ($\mathcal{L}$-transform) is itself an operator that maps a function of time into a function of time into a function of a complex variable $s = \sigma + j\omega$ according to formula:

$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty x(t) e^{-st} \, dt$$

Note:

Unilateral $\Rightarrow t > 0$



The property of biuniqueness allows the $x(t)$ or $y(t)$ to be recovered uniquely from $X(s)$ & $Y(s)$ respectively

Initial condition

The operator $Y(s) = F[X(s), \{\lambda_i(0)\}]$ is thus said to characterize the system in frequency domain whereas

$y(t) = f[x(t), \{\lambda_i(0)\}]$ is time domain characterization.

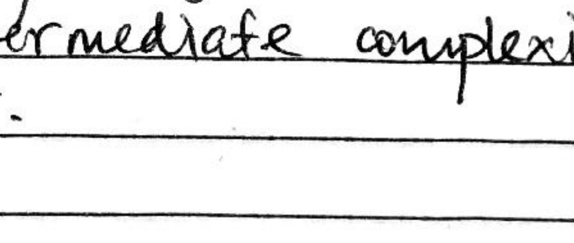
$$
\begin{array}{lccc}
 & \text{Input} & & \text{Output} \\
 & \downarrow & & \downarrow \\
\text{time domain:} & x(t) & \Rightarrow & y(t) = f[x(t), \{\lambda_i(0)\}] \\
\text{biunique} \leftarrow & \Updownarrow & & \Updownarrow \\
\text{Frequency domain:} & X(s) & \Rightarrow & Y(s) = F[X(s), \{\lambda_i(0)\}]
\end{array}
$$

It is primarily useful for LTI systems of intermediate complexity eg. 3rd order.

Example:

(i) step input

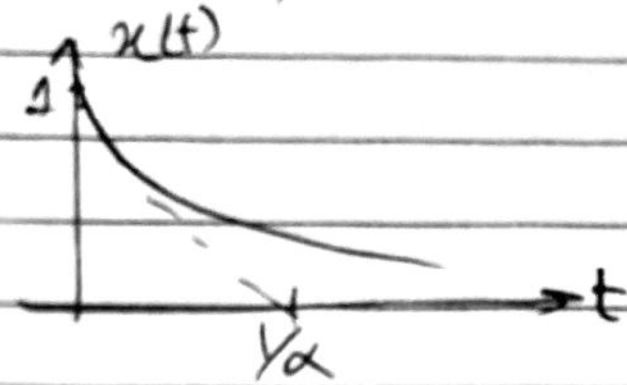
$x(t) = 1, \; t > 0$



$$X(s) = \int_0^{\infty} x(t) e^{-st} dt = \int_0^{\infty} e^{-st} dt = \frac{1}{s}$$

② Exponential decay

$x(t) = e^{-\alpha t}, \; t > 0$



$$X(s) = \int_0^{\infty} e^{-\alpha t} \cdot e^{-st} \, dt = \frac{1}{s+\alpha}$$

provided $R[s] > -\alpha$ so $e^{-(s+\alpha)t} \to 0$.
when $t \to \infty$

<u>Note:</u> $x(t)$ for $t<0$ do not influence $X(s)$ ∴ cannot be recovered from $X(s)$.

⊛ Rational functions (ratios of polynomials of s) of $X(s)$ mostly interest us for circuit analysis

⊛ Roots of denominator → <u>poles</u>
⊛ Roots of numerator → <u>zeros</u>.

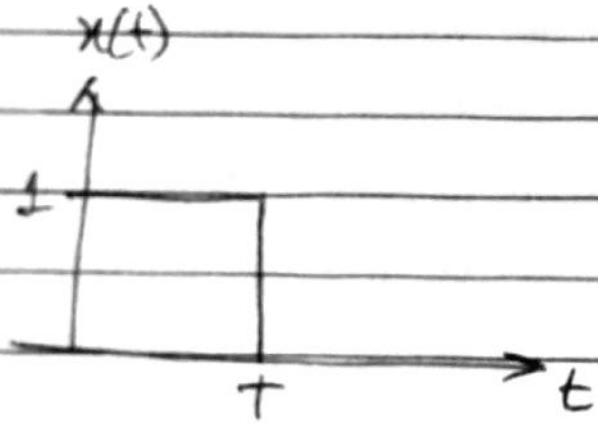
DELAY THEOREM

Let $\mathcal{L}[x(t)] = X(s)$,
Then for $T > 0$

$$\mathcal{L}[\underbrace{x(t-T)\,u(t-T)}_{x(t) \text{ delayed by } T}] = X(s)\,e^{-sT}$$

Example:

$$x(t) = \begin{cases} 1 & 0 < t < T \\ 0 & t > T \end{cases}$$



$$X(s) = \int_0^T 1 e^{-st}\,dt = \frac{1}{s}\left(1 - e^{-sT}\right) \quad \forall s$$

Note $x(t)$ can be written as
$$x(t) = 1 - u(t-T), \quad t > 0,\ T > 0$$

$$\& \ \mathcal{L}[x(t)] = \frac{1}{s}\left(1 - e^{-sT}\right)$$

LINEARITY THEOREM

Let $\mathcal{L}[x_1(t)] = X_1(s)$ & $\mathcal{L}[x_2(t)] = X_2(s)$

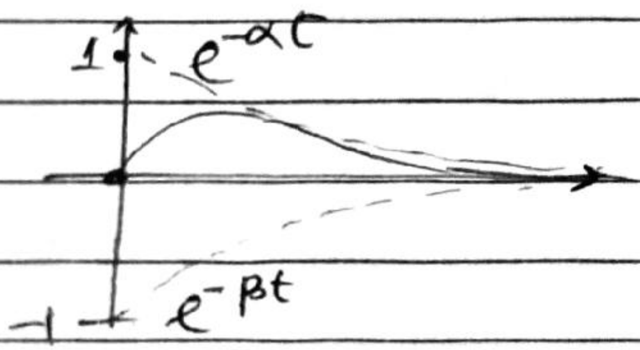
Then,

$$\mathcal{L}[a x_1(t) + b x_2(t)] = a X_1(s) + b X_2(s)$$

(Proof of linearity follows straight from Defn)

Example:

$x(t) = e^{-\alpha t} - e^{-\beta t}, \; t \geq 0$



$$X(s) = \frac{1}{s+\alpha} - \frac{1}{s+\beta} = \frac{\beta - \alpha}{(s+\alpha)(s+\beta)}$$

THE INVERSE LAPLACE TRANSFORM

In order to exploit Laplace transform to analyze circuits & systems, it requires that we be able to reverse the process through Inverse Laplace Transform

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi j} \int_C X(s)\, e^{st}\, ds$$

where the integral is a line integral along an appropriate contour C in the complex plane.

Example: Inverse Laplace using partial fraction.

$$X(s) = \frac{s+3}{s^2+s} = \frac{s+3}{s(s+1)} = \frac{a}{s} + \frac{b}{s+1}$$

$$= \frac{3}{s} - \frac{2}{s+1}$$

$$\Rightarrow x(t) = 3 - 2e^{-t}$$

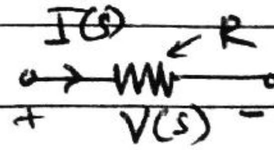
CIRCUIT ANALYSIS WITH LAPLACE TRANSFORM

Previously we solved a 1^{st}-order dynamic circuit by solving the formulated differential equation.

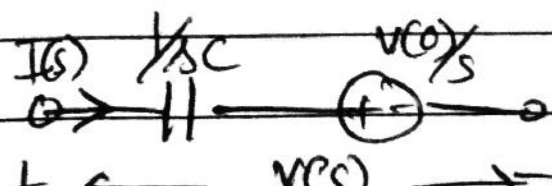
Alternatively, we could describe the circuit in its equivalent Laplace transform that result in a LTI circuit with a set of algebraic equations that are much easier than ~~...~~ solving differential equation.

By replacing the two-terminal lumped elements (R, L, C) by its equivalent Laplace transform

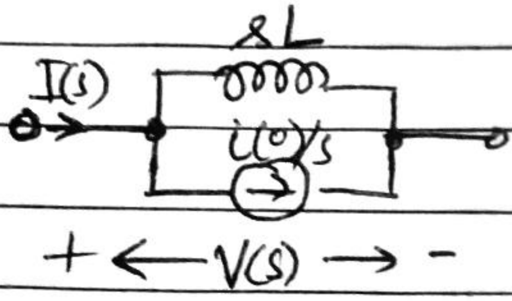
Resistor: $V(s) = R \cdot I(s)$



Capacitor: $V(s) = \frac{1}{sC} I(s) + \frac{v(0)}{s}$



Inductor: $I(s) = \frac{1}{sL} V(s) + \frac{i(0)}{s}$

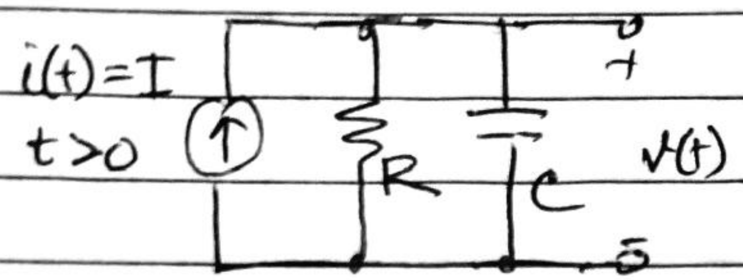


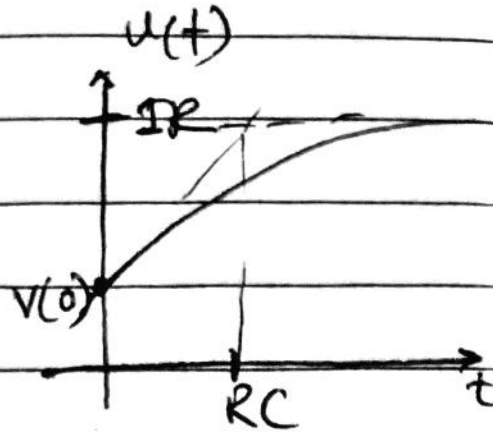
* As a result of the Linearity Theorem, the constraints imposed on the time-domain branch voltages & currents by Kirchoff's Laws carry over without alteration unto frequency domain:

time-domain	frequency domain
$\sum i_j(t) = 0$ ⟺	$\sum I_j(s) = 0$

$$\sum_{loop} v_j(t) = 0 \iff \sum_{Loop} V_j(s) = 0$$

Example:

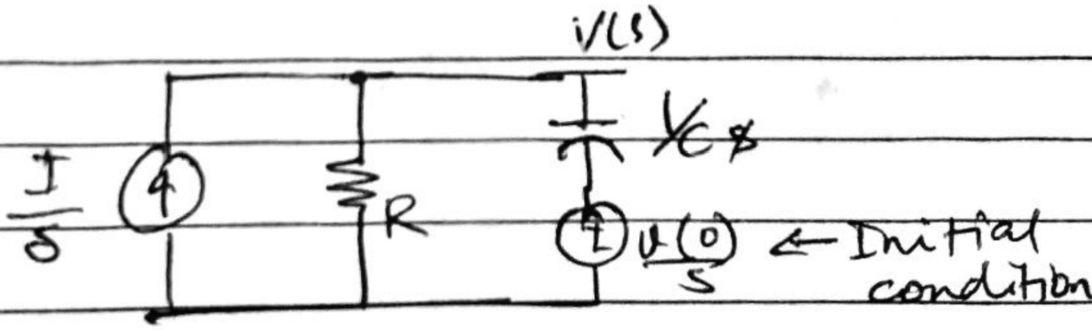




Previously we computed the output response v(t) in time-domain ~~to~~ resulting in

$$v(t) = IR + (v(0) - IR) \cdot e^{-t/RC}$$

The frequency domain circuit is as shown below.



Treating V(s) as a node voltage, we can use elementary resistive circuit theory to derive the node equation

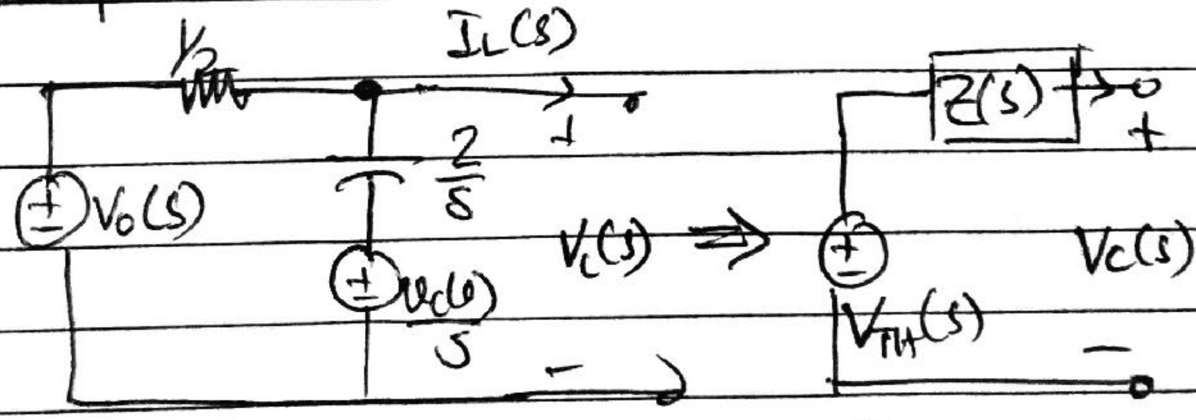
$$\frac{V(s)}{R} + \left(V(s) - \frac{v(0)}{s}\right)sC - \frac{I}{s} = 0$$

$$\text{or } V(s) = \underbrace{\frac{IR}{s}}_{\text{step function}} + \underbrace{\frac{v(0) - IR}{s + 1/RC}}_{\text{exponential}}$$

$$v(t) = IR + (v(0) - IR)e^{-t/RC}, \quad t > 0$$

Also, same simplification techniques like in resistor ckts: equivalent resistances, Thevenin, Norton, etc, be apply to the s-domain ckt. as well.

Example:



$$V_{TH}(s) = \frac{4V_0(s) + v_c(0)}{s+4} \qquad Z(s) = \frac{2}{s+4}.$$