Formulas for Computing Edges and Vertices of Graceful Symmetric Stars

Velina V Veleva

Definition 1. A **graceful labeling** is a labeling of the vertices of a graph with distinct integers from the set $\{0, 1, 2, \ldots, n\}$, where n represents the number of edges such that if $f(v)$ denotes the label at vertex v, then the edge uv is given the value $\left|f(u) - f(v)\right|$, and all the edges are distinctly labeled $\{1, 2, \ldots, n\}$.

Definition 2. Let $S(m, n)$ be a **symmetric star** with m legs, each of length n.

The center vertex is the vertex which connects all the legs of the star $S(m, n)$, and we will always label it 0. Let l_i be defined as the set of vertices that lay i edges away from the center vertex. For example, a graceful labeling for $S(5, 4)$ is

$$\begin{array}{cccccl}
 & & 0 & & & \\
/ & / & | & \backslash & \backslash & \\
20 & 16 & 12 & 8 & 4 & = l_1 \\
| & | & | & | & | & \\
1 & 5 & 9 & 13 & 17 & = l_2 \\
| & | & | & | & | & \\
19 & 15 & 11 & 7 & 3 & = l_3 \\
| & | & | & | & | & \\
2 & 6 & 10 & 14 & 18 & = l_4
\end{array}$$

where $l_1 = \{20, 16, 12, 8, 4\}$, $l_2 = \{1, 5, 9, 13, 17\}$, etc. Note that if we take the numbers from 1 through 20 and arrange them in a 4×5 matrix starting from top left down in a decreasing order, we get

$$\begin{array}{cccccl}
20 & 16 & 12 & 8 & 4 & = r_1 \\
19 & 15 & 11 & 7 & 3 & = r_2 \\
18 & 14 & 10 & 6 & 2 & = r_3 \\
17 & 13 & 9 & 5 & 1 & = r_4
\end{array}$$

Then the graceful labeling of $S(5, 4)$ can be expressed as $\begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_4^{-1} \\ r_2 \\ r_3^{-1} \end{bmatrix}$ where r_i^{-1} is r_i in reverse order.

We can factor l_1 through l_4 as follows:

$l_1 = \{20, 16, 12, 8, 4\} = 4 \times \big(5 - \{0, 1, 2, 3, 4\}\big)$

$l_2 = \{1, 5, 9, 13, 17\} \ = 4 \times \{0, 1, 2, 3, 4\} + 1$

$l_3 = \{19, 15, 11, 7, 3\} = 4 \times \big(5 - \{0, 1, 2, 3, 4\}\big) - 1$

$l_4 = \{2, 6, 10, 14, 18\} = 4 \times \{0, 1, 2, 3, 4\} + 2$

More generally, for any $m, n \in N$ and $i \in [1, n]$, we label $S(m, n)$ in the following way:

$$\begin{array}{ccccccc}
nm & n(m-1) & n(m-2) & n(m-3) & \dots & n & = l_1 \\
1 & n+1 & 2n+1 & 3n+1 & \dots & (m-1)n+1 & = l_2 \\
nm-1 & n(m-1)-1 & n(m-2)-1 & n(m-3)-1 & \dots & n-1 & = l_3 \\
2 & n+2 & 2n+2 & 3n+2 & \dots & (m-1)n+2 & = l_4 \\
nm-1 & n(m-1)-2 & n(m-2)-2 & n(m-3)-2 & \dots & n-2 & = l_5 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
\frac{i}{2} & n+\frac{i}{2} & 2n+\frac{i}{2} & 3n+\frac{i}{2} & \dots & n(m-1)+\frac{i}{2} & = l_i,\ i \text{ even} \\
nm-\frac{i-1}{2} & n(m-1)-\frac{i-1}{2} & n(m-2)-\frac{i-1}{2} & n(m-3)-\frac{i-1}{2} & \dots & n-\frac{i-1}{2} & = l_i,\ i \text{ odd}
\end{array}$$

Thus, for $j \in \left[1, m\right]$ and $i \in \left[1, n\right]$, each vertex v_{ij} can be written as follows:

$$v_{ij} = \begin{cases} n(j-1) + \frac{i}{2} & \text{if } i \text{ is even} \\ n(m-j) + \left(n - \frac{i-1}{2}\right) & \text{if } i \text{ is odd} \end{cases}$$

Lemma. *Let $q \in [1, nm]$. Then $\exists i \in [1, n]$ and $\exists j \in [1, m]$ such that $f_v(i, j) = q$ for*

$$f_v(i, j) = \begin{cases} n(j-1) + \frac{i}{2} & \textit{if } i \textit{ is even} \\ n(m-j) + \left(n - \frac{i-1}{2}\right) & \textit{if } i \textit{ is odd} \end{cases}$$

Proof. Pick $q \in [1, nm]$. Pick $k \in [0, m-1]$, $r \in [1, n]$ so that $q = nk + r$. Now we want to show $\exists i, j$ such that $f_v(i, j) = q$.

Case 1. $1 \leq r \leq \frac{n}{2}$

Let $j = k + 1$ and $i = 2r$. Clearly, $j \in [1, m]$, $i \in [1, n]$ and i is even. When we substitute for k, r in q we get

$$q = nk + r = n(j-1) + \frac{i}{2} = f_v(i, j)$$

Case 2. $\frac{n}{2} < r \leq n$

Let $j = m - k$ and $i = 2(n - r) + 1$. Clearly, $j \in [1, m]$, $i \in [1, n]$ and i is odd. When we substitute for k, r in q we get

$$q = nk + r = n(m-j) + \left(n - \frac{i-1}{2}\right) = f_v(i, j)$$

□

For $i \in [1, n]$ and $j \in [1, m]$, an edge e_{ij} in the graph $S(m, n)$ is defined as

$$e_{ij} = \begin{cases} v_{1j} & \text{if } i = 1 \\ \left|v_{ij} - v_{(i-1)j}\right| & \text{if } i > 1 \end{cases}$$

When $i = 1$,

$$e_{1j} = v_{1j} = n(m-j) + \left(n - \frac{1-1}{2}\right) = n(m-j) + n = n(m-j+1)$$

When i is odd,

$$e_{ij} = \left|v_{ij} - v_{(i-1)j}\right| = \left|\left[n(m-j) + \left(n - \frac{i-1}{2}\right)\right] - \left[n(j-1) + \frac{i-1}{2}\right]\right| =$$

$$= \left| nm - nj + n - \frac{i-1}{2} - n(j-1) - \frac{i-1}{2} \right| =$$
$$= \left| n(m - 2j + 2) - \frac{2(i-1)}{2} \right| =$$
$$= \left| n(m - 2j + 2) - i + 1 \right|$$

When i is even,

$$e_{ij} = \left| v_{ij} - v_{(i-1)j} \right| = \left| \left[n(j-1) + \frac{i}{2} \right] - \left[n(m - j + 1) - \frac{i-2}{2} \right] \right| =$$
$$= \left| \left[nj - n + \frac{i}{2} \right] - \left[nm - nj + n - \frac{i}{2} + \frac{2}{2} \right] \right| =$$
$$= \left| nj - n + \frac{i}{2} - nm + nj - n + \frac{i}{2} - 1 \right| =$$
$$= \left| -nm + 2nj - 2n + i - 1 \right| =$$
$$= \left| n(m - 2j + 2) - i + 1 \right|$$

Therefore, we can write

$$e_{ij} = \begin{cases} n(m - j + 1) & \text{if } i = 1 \\ \left| n(m - 2j + 2) - (i - 1) \right| & \text{if } i > 1 \end{cases}$$

Lemma. *Let $q \in [1, nm]$. Then $\exists i \in [1, n]$ and $\exists j \in [1, m]$ such that $f_e(i, j) = q$ for*

$$f_e(i, j) = \begin{cases} n(m - j + 1) & \textit{if } i = 1 \\ \left| n(m - 2j + 2) - (i - 1) \right| & \textit{if } i > 1 \end{cases}$$

Proof. Pick $q \in [1, nm]$. Pick $k \in [1, m]$, $r \in [1, n]$ so that $q = n(m - k) + r$. Now we want to show $\exists i, j$ such that $f_e(i, j) = q$.

Case 1. $r = n$

Pick $j = k$ and $i = 1$. Clearly, $j \in [1, m]$ and $i \in [1, n]$. Then

$$q = n(m - k) + r = n(m - j) + n = n(m - j + 1) = f_e(1, j)$$

Case 2. $r < n$, k even

Pick $j = \frac{2m-k+2}{2}$ and since k is even, then j is an integer. Pick $i = r + 1$. Clearly, $j \in [1, m]$ and $i \in [2, n]$. Then

$$q = n(m - k) + r = n(m - (2m - 2j + 2)) + i - 1 =$$
$$= n(-m + 2j - 2) + (i - 1) = \left| n(-m + 2j - 2) + (i - 1) \right| =$$
$$= \left| n(m - 2j + 2) - (i - 1) \right| = f_e(i, j)$$

Case 3. $r < n$, k odd

Pick $j = \frac{k+1}{2}$ and since k is odd, then j is an integer. Pick $i = n - r + 1$. Clearly, $j \in [1, m]$ and $i \in [2, n]$. Then

$$q = n(m - k) + r = n(m - (2j - 1)) + n - i + 1 =$$
$$= n(m - 2j + 1) + n - i + 1 = n(m - 2j + 2) - (i - 1) =$$
$$= \left| n(m - 2j + 2) - (i - 1) \right| = f_e(i, j)$$

$\square$