Appendix for

"Productivity Shocks, Financial Frictions, and Business Cycles in Emerging Market Economies"

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Appendix A: Data

A.1 World Real Interest Rate

The world real interest rate measures the interest rate at which EMEs borrow at the international bond market. The world real interest rate data is not readily available. I follow Neumeyer and Perri (2005), Uribe and Yue (2006), and Akinci (2013) to constructed the world real interest rate as

$$\text{world real interest rate} = \text{U.S. treasury bill rate} - \text{U.S. inflation rate} + \text{ EMBI+ spread},$$

where EMBI+ is the emerging market bond index plus. The quarterly EMBI+ spread data is from Global Economic Monitor (GEM) database of the World Bank, and is recorded as basis points from 3-month U.S. treasury bill rate.

The 3-month U.S. treasury bill rate data is collected from Federal Reserve Economic Data (FRED) database and is categorized as the 'constant maturity rate' at monthly frequency. I use the last monthly rate of a quarter as the quarterly 3-month U.S. treasury bill rate. U.S. GDP deflator data is also from FRED database and is categorized as the 'implicit price deflator' at quarterly frequency. Inflation is calculated as the percentage change rate between two consecutive quarters of U.S. GDP deflator.

A.2 Tobin's Q

Tobin's Q data is constructed. It is a ratio of the market value of capital to the replacement cost of capital. As a proxy, I measure the Tobin's Q as

$$\text{Tobin's Q} = \frac{\text{Equity Price Index}}{\text{Producer Price Index}}.$$

The equity price index is average and end of month indices of daily share prices in Sao Paulo security exchange. The producer price index is prices of a sample of merchandise at the wholesale level in business to business transactions in Brazil. The quarterly equity price index and producer price index data are drawn from the international financial statistics (IFS) database.

Appendix B: The SOE-RBC Model

B.1 Optimality and Equilibrium Conditions

I document the detailed math derivations of the SOE-RBC model in this section. The representative household maximizes its life time utility, subject to the budget constraint, law of motion of capital, and Kiyotaki and Moore (1997) (KM) collateral constraint. The

Lagrangian of this optimization problem is

$$
\begin{aligned}
L =& E_t \sum_{t=0}^{\infty} \beta^t \Big\{ [\phi ln C_t + (1-\phi) ln(1-N_t)] + \lambda_{1,t} \Big[(a_t N_t)^{1-\theta} (K_t)^{\theta} + D_{t+1} - C_t - I_t - G - (1+\omega_t) D_t \\
& - \varphi \frac{D_t^2}{(a_t N_t)^{1-\theta}(K_t)^{\theta}} \Big] + \lambda_{2,t} \Big[(1-\delta+\alpha_3) K_t + \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} (\frac{I_t}{K_t})^{1-\alpha_2} \nu_t K_t - K_{t+1} \Big] \\
& + \lambda_{3,t} (\kappa_t q_t K_{t+1} - D_{t+1}) \Big\}.
\end{aligned}
$$

The associated first order necessary conditions with respect to C_t, N_t, I_t, D_{t+1} and K_{t+1} are

$$\lambda_{1,t} = \frac{\phi}{C_t}, \tag{FOC.1}$$

$$\frac{1-\phi}{1-N_t} = \lambda_{1,t} (1-\theta) \frac{Y_t}{N_t} [1 + \varphi (\frac{D_t}{Y_t})^2], \tag{FOC.2}$$

$$\frac{\lambda_{2,t}}{\lambda_{1,t}} = \frac{1}{\alpha_1 \gamma^{*\alpha_2} \nu_t} (\frac{I_t}{K_t})^{\alpha_2}, \tag{FOC.3}$$

$$\lambda_{1,t} - \lambda_{3,t} = \beta E_t [\lambda_{1,t+1} (1 + \omega_{t+1} + 2\varphi \frac{D_{t+1}}{Y_{t+1}})], \tag{FOC.4}$$

$$
\begin{aligned}
\lambda_{2,t} - \lambda_{3,t} \kappa_t q_t = \beta E_t \Big\{ & \lambda_{1,t+1} \theta \frac{Y_{t+1}}{K_{t+1}} [1 + \varphi (\frac{D_{t+1}}{Y_{t+1}})^2] \\
& + \lambda_{2,t+1} [(1-\delta+\alpha_3) + \frac{\alpha_1 \alpha_2 \gamma^{*\alpha_2}}{1-\alpha_2} (\frac{I_{t+1}}{K_{t+1}})^{1-\alpha_2} \nu_{t+1}] \Big\},
\end{aligned} \tag{FOC.5}
$$

where I assume the KM collateral constraint binds and the household internalize the endogenous risk premium of the world real interest rate. Equations (FOC.1), (FOC.2), and (FOC.3) are intratemporal optimality conditions of consumption, labor, and investment, respectively. The intertemporal optimality conditions of foreign debt and capital are (FOC.4) and (FOC.5). Euler equations for capital and foreign debt are also (FOC.4) and (FOC.5).

The equilibrium conditions are

$$K_{t+1} = (1-\delta)K_t + \Big[\frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I_t}{K_t})^{1-\alpha_2}\nu_t + \alpha_3\Big]K_t, \tag{EC.1}$$

$$q_t = \frac{1}{\alpha_1 \gamma^{*\alpha_2}\nu_t}(\frac{I_t}{K_t})^{\alpha_2}, \tag{EC.2}$$

$$D_{t+1} = \kappa_t q_t K_{t+1}, \tag{EC.3}$$

$$r_t = \omega_t + \varphi(\frac{D_t}{Y_t}), \tag{EC.4}$$

$$Y_t + D_{t+1} = C_t + I_t + (1+r_t)D_t + G. \tag{EC.5}$$

The production technology and exogenous shock process are given as

$$Y_t = (a_t N_t)^{1-\theta} K_t^{\theta}, \tag{G.1}$$

$$a_t = a_{t-1} exp(\mu + \varepsilon_t), \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2), \tag{G.2}$$

$$\nu_t = \nu_{t-1}^{\rho_\nu} exp(\eta_t), \quad \eta_t \sim N(0, \sigma_\eta^2), \tag{G.3}$$

$$\omega_t = \omega^{*(1-\rho_\omega)} \omega_{t-1}^{\rho_\omega} exp(\gamma_t), \quad \gamma_t \sim N(0, \sigma_\gamma^2), \tag{G.4}$$

$$\kappa_t = (\kappa^*)^{(1-\rho_\kappa)} \kappa_{t-1}^{\rho_\kappa} exp(\zeta_t), \quad \zeta_t \sim N(0, \sigma_\zeta^2). \tag{G.5}$$

Transversality conditions are $\lim_{j\to\infty} \beta^j E_t[\lambda_{2,t+j} K_{t+1+j}] = 0$ and $\lim_{j\to\infty} \beta^j E_t[\lambda_{3,t+j} D_{t+1+j}] = 0$, which when satisfied provide sufficient conditions for an equilibrium. Brock (1982, 1-46) shows when optimality, equilibrium, and transversality conditions are satisfied, the solution of the SOE-RBC model is unique.

In order to facilitate the model solution method, I record a system of equations, which are

$$\frac{(1-\phi)}{\phi}\frac{C_t}{(1-N_t)} = (1-\theta)\frac{Y_t}{N_t}[1+\varphi(\frac{D_t}{Y_t})^2], \tag{SE.1}$$

$$\begin{aligned} q_t - \kappa_t q_t = \beta E_t \frac{C_t}{C_{t+1}} \Big\{ \theta \frac{Y_{t+1}}{K_{t+1}}[1+\varphi(\frac{D_{t+1}}{Y_{t+1}})^2] + q_{t+1}\Big[(1-\delta+\alpha_3) + \frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I_{t+1}}{K_{t+1}})^{1-\alpha_2}\nu_{t+1}\Big] \\ - \kappa_t q_t(1+\omega_{t+1}+2\varphi\frac{D_{t+1}}{Y_{t+1}})\Big\}, \end{aligned} \tag{SE.2}$$

$$Y_t = (a_t N_t)^{1-\theta} K_t^{\theta}, \tag{SE.3}$$

$$q_t = \frac{1}{\alpha_1\gamma^{*\alpha_2}\nu_t}(\frac{I_t}{K_t})^{\alpha_2}, \tag{SE.4}$$

$$D_{t+1} = \kappa_t q_t K_{t+1}, \tag{SE.5}$$

$$K_{t+1} = (1-\delta)K_t + \Big[\frac{\alpha_1\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I_t}{K_t})^{1-\alpha_2}\nu_t + \alpha_3\Big]K_t, \tag{SE.6}$$

$$r_t = \omega_t + \varphi(\frac{D_t}{Y_t}), \tag{SE.7}$$

$$Y_t + D_{t+1} = C_t + I_t + (1+r_t)D_t + G, \tag{SE.8}$$

plus exogenous processes

$$a_t = a_{t-1}exp(\mu+\varepsilon_t), \quad \varepsilon_t \sim N(0,\sigma_\varepsilon^2). \tag{SE.9}$$

$$\nu_t = \nu_{t-1}^{\rho_\nu} exp(\eta_t), \quad \eta_t \sim N(0, \sigma_\eta^2), \tag{SE.10}$$

$$\omega_t = \omega^{*(1-\rho_\omega)} \omega_{t-1}^{\rho_\omega} exp(\gamma_t), \quad \gamma_t \sim N(0, \sigma_\gamma^2). \tag{SE.11}$$

$$\kappa_t = (\kappa^*)^{(1-\rho_\kappa)} \kappa_{t-1}^{\rho_\kappa} exp(\zeta_t), \quad \zeta_t \sim N(0, \sigma_\zeta^2). \tag{SE.12}$$

B.2 Stochastically Detrending

Along a balanced growth path that follows the random walk of TFP shock, stationary variables after detrending are denoted with hats

$$\hat{Y}_t = \frac{Y_t}{a_t},\ \hat{C}_t = \frac{C_t}{a_t},\ \hat{I}_t = \frac{I_t}{a_t},\ \hat{D}_{t+1} = \frac{D_{t+1}}{a_t},\ \hat{K}_{t+1} = \frac{K_{t+1}}{a_t}.$$

Labor N_t follows a unit process $N_t = 1 - L_t$ and is not detrended. Detrend the production function

$$\hat{Y}_t = N_t^{1-\theta} \hat{K}_t^{\theta} exp(-\theta\mu - \theta\varepsilon_t).$$

Detrend the law of motion of capital

$$\frac{K_{t+1}}{a_t} = \frac{(1-\delta+\alpha_3)K_t}{a_t} + \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} (\frac{I_t}{K_t})^{1-\alpha_2} \frac{\nu_t K_t}{a_t} \Rightarrow$$
$$\hat{K}_{t+1} = (1-\delta+\alpha_3)\hat{K}_t exp(-\mu - \varepsilon_t) + \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} \hat{I}_t^{1-\alpha_2} \hat{K}_t^{\alpha_2} \nu_t exp(-\alpha_2\mu - \alpha_2\varepsilon_t).$$

Tobin's Q with detrended variables is

$$\hat{q}_t = q_t = \frac{1}{\alpha_1 \gamma^{*\alpha_2} \nu_t} (\frac{\hat{I}_t}{\hat{K}_t})^{\alpha_2} exp(\alpha_2\mu + \alpha_2\varepsilon_t).$$

The world real interest rate with detrended variables is

$$\hat{r}_t = r_t = \omega_t + \varphi(\frac{\hat{D}_t}{\hat{Y}_t})exp(-\mu - \varepsilon_t).$$

The collateral constraint with detrended variables

$$\hat{D}_{t+1} = \kappa_t \hat{q}_t \hat{K}_{t+1}.$$

Detrend the resource constraint

$$\hat{Y}_t + \hat{D}_{t+1} = \hat{C}_t + \hat{I}_t + \hat{G} + (1+\hat{r}_t)\hat{D}_t exp(-\mu - \varepsilon_t).$$

Detrend the labor Euler equation

$$\frac{1-\phi}{\phi}\frac{\hat{C}_t}{1-N_t} = (1-\theta)\frac{\hat{Y}_t}{N_t}\Big[1 + \varphi(\frac{\hat{D}_t}{\hat{Y}_t})^2 exp(-2\mu - 2\varepsilon_t)\Big].$$

Detrend the capital Euler equation

$$\begin{aligned}
\hat{q}_t - \kappa_t \hat{q}_t = \beta E_t \frac{\hat{C}_t}{\hat{C}_{t+1}} \Big\{ \theta \frac{\hat{Y}_{t+1}}{\hat{K}_{t+1}} \Big[1 + \varphi(\frac{\hat{D}_{t+1}}{\hat{Y}_{t+1}})^2 exp(-2\mu - 2\varepsilon_{t+1}) \Big] \\
+ \hat{q}_{t+1} exp(-\mu - \varepsilon_{t+1}) \Big[1 - \delta + \alpha_3 + \frac{\alpha_1 \alpha_2 \gamma^{*\alpha_2}}{1-\alpha_2} (\frac{\hat{I}_{t+1}}{\hat{K}_{t+1}})^{1-\alpha_2} \nu_{t+1} exp[(\mu + \varepsilon_{t+1})(1-\alpha_2)] \Big] \\
- \kappa_t \hat{q}_t exp(-\mu - \varepsilon_{t+1})[1 + \omega_{t+1} + 2\varphi \frac{\hat{D}_{t+1}}{\hat{Y}_{t+1}} exp(-\mu - \varepsilon_{t+1})] \Big\}.
\end{aligned}$$

Finally, I obtain the system of equations that characterize a stationary competitive equilibrium

$$\frac{1-\phi}{\phi}\frac{\hat{C}_t}{1-N_t} = (1-\theta)\frac{\hat{Y}_t}{N_t}\Big[1+\varphi(\frac{\hat{D}_t}{\hat{Y}_t})^2 exp(-2\mu-2\varepsilon_t)\Big], \tag{SD.1}$$

$$\begin{aligned}\hat{q}_t - \kappa_t\hat{q}_t = \beta E_t\frac{\hat{C}_t}{\hat{C}_{t+1}}\Big\{\theta\frac{\hat{Y}_{t+1}}{\hat{K}_{t+1}}\Big[1+\varphi(\frac{\hat{D}_{t+1}}{\hat{Y}_{t+1}})^2 exp(-2\mu-2\varepsilon_{t+1})\Big]\\ + \hat{q}_{t+1}exp(-\mu-\varepsilon_{t+1})\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{\hat{I}_{t+1}}{\hat{K}_{t+1}})^{1-\alpha_2}\nu_{t+1}exp[(\mu+\varepsilon_{t+1})(1-\alpha_2)]\Big]\\ - \kappa_t\hat{q}_t exp(-\mu-\varepsilon_{t+1})[1+\omega_{t+1}+2\varphi\frac{\hat{D}_{t+1}}{\hat{Y}_{t+1}}exp(-\mu-\varepsilon_{t+1})]\Big\},\end{aligned} \tag{SD.2}$$

$$\hat{Y}_t = N_t^{1-\theta}\hat{K}_t^{\theta} exp(-\theta\mu-\theta\varepsilon_t), \tag{SD.3}$$

$$\hat{K}_{t+1} = (1-\delta+\alpha_3)\hat{K}_t exp(-\mu-\varepsilon_t) + \frac{\alpha_1\gamma^{*\alpha_2}}{1-\alpha_2}\hat{I}_t^{1-\alpha_2}\hat{K}_t^{\alpha_2}\nu_t exp(-\alpha_2\mu-\alpha_2\varepsilon_t), \tag{SD.4}$$

$$\hat{q}_t = \frac{1}{\alpha_1\gamma^{*\alpha_2}\nu_t}(\frac{\hat{I}_t}{\hat{K}_t})^{\alpha_2} exp(\alpha_2\mu+\alpha_2\varepsilon_t), \tag{SD.5}$$

$$\hat{r}_t = \omega_t + \varphi(\frac{\hat{D}_t}{\hat{Y}_t})exp(-\mu-\varepsilon_t), \tag{SD.6}$$

$$\hat{D}_{t+1} = \kappa_t\hat{q}_t\hat{K}_{t+1}, \tag{SD.7}$$

$$\hat{Y}_t + \hat{D}_{t+1} = \hat{C}_t + \hat{I}_t + \hat{G} + (1+\hat{r}_t)\hat{D}_t exp(-\mu-\varepsilon_t), \tag{SD.8}$$

plus exogenous shock processes

$$\varepsilon_t \sim N(0,\sigma_\varepsilon^2), \tag{SD.9}$$

$$\nu_t = \nu_{t-1}^{\rho_\nu} exp(\eta_t), \quad \eta_t \sim N(0, \sigma_\eta^2), \tag{SD.10}$$

$$\kappa_t = (\kappa^*)^{(1-\rho_\kappa)} \kappa_{t-1}^{\rho_\kappa} exp(\zeta_t), \quad \zeta_t \sim N(0, \sigma_\zeta^2), \tag{SD.11}$$

$$\omega_t = \omega^{*(1-\rho_\omega)} \omega_{t-1}^{\rho_\omega} exp(\gamma_t), \quad \gamma_t \sim N(0, \sigma_\gamma^2). \tag{SD.12}$$

B.3 Steady State

The steady states of the stationary system of equations are

$$\frac{1-\phi}{\phi}\frac{C^*}{1-N^*} = (1-\theta)\frac{Y^*}{N^*}\Big[1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)\Big], \tag{SS.1}$$

$$\begin{aligned} q^*(1-\kappa^*) = \beta\Big\{&\theta\frac{Y^*}{K^*}\Big[1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)\Big] \\ &+ q^* exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2} exp(\mu-\mu\alpha_2)\Big] \\ &- \kappa^* q^* exp(-\mu)\Big[1+\omega^*+2\varphi\frac{D^*}{Y^*} exp(-\mu)\Big]\Big\}, \end{aligned} \tag{SS.2}$$

$$Y^* = N^{*1-\theta} K^{*\theta} exp(-\theta\mu), \tag{SS.3}$$

$$\frac{I^*}{K^*} = \Big(\frac{(1-\alpha_2)\Big[1-(1-\delta+\alpha_3)exp(-\mu)\Big]}{\alpha_1\gamma^{*\alpha_2} exp(-\alpha_2\mu)}\Big)^{\frac{1}{1-\alpha_2}}, \tag{SS.4}$$

$$q^* = \frac{1}{\alpha_1\gamma^{*\alpha_2}}(\frac{I^*}{K^*})^{\alpha_2} exp(\alpha_2\mu), \tag{SS.5}$$

where $\gamma^* = exp(\mu)$, and $\alpha_3 = (1-\frac{\alpha_1^{\frac{1}{\alpha_2}} q^{*\frac{1-\alpha_2}{\alpha_2}}}{1-\alpha_2})\gamma^* + \delta - 1$,

$$r^* = \omega^* + \varphi(\frac{D^*}{Y^*}) exp(-\mu), \tag{SS.6}$$

$$D^* = \kappa^* q^* K^*, \tag{SS.7}$$

$$Y^* + D^* = C^* + I^* + G^* + (1 + r^*)D^* exp(-\mu). \tag{SS.8}$$

Substitute equation (SS.4), (SS.5), and (SS.7) into equation (SS.2) to obtain the steady state capital to output ratio, $\frac{K^*}{Y^*}$. Then the rest of the steady state ratios $[\frac{D^*}{Y^*}, \frac{I^*}{Y^*}, \frac{C^*}{Y^*}, \frac{N^*}{Y^*}]$ are easily found.

Appendix C: Linear Solution

C.1 Taylor Expansion

Consider a real or complex-valued function $f(x)$, Taylor's theorem tells us that this can be expressed as a power series about a particular point x^* (real or complex), where x^* belongs to the set of possible x values:

$$f(x) = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + \frac{f'''(x^*)}{3!}(x - x^*)^3 + ...,$$

where $f'(x^*)$ is the first derivative of f with respect to x evaluated at the point x^*, $f''(x^*)$ is the second derivative evaluated at the same point, $f'''(x^*)$ is the third derivative, and so on.

For a function whose higher order derivatives are small, this function is smooth and it can be well approximated (at least in the neighborhood of x^*) linearly as:

$$f(x) = f(x^*) + f'(x^*)(x - x^*).$$

For a multivariate function $f(x, y)$, Taylor's expansion applies as:

$$f(x, y) = f(x^*, y^*) + f'_x(x^*, y^*)(x - x^*) + f'_y(x^*, y^*)(y - y^*).$$

C.2 Linearization

I take the first order Taylor expansions around the log-steady state of equations (SD.1)-(SD.8). I define $\tilde{Y}_t = ln\hat{Y}_t - lnY^* \simeq \frac{\hat{Y}_t - Y^*}{Y^*}$.

Linearize the production function (SD.3) by first taking logs on both sides

$$ln\hat{Y}_t = (1 - \theta)lnN_t + \theta ln\hat{K}_t - \theta\mu - \theta\varepsilon_t.$$

then take the Taylor expansions for $ln\hat{Y}_t$, lnN_t, and $\hat{K}_t$

$$lnY^* + \frac{\hat{Y}_t - Y^*}{Y^*} = (1 - \theta)lnN^* + (1 - \theta)\frac{N_t - N^*}{N^*} + \theta lnK^* + \theta\frac{\hat{K}_t - K^*}{K^*} - \theta\mu - \theta\varepsilon_t.$$

Now cancel terms on both sides to get

$$\tilde{Y}_t = (1 - \theta)\tilde{N}_t + \theta\tilde{K}_t - \theta\varepsilon_t. \tag{LE.1}$$

Linearize the KM collateral constraint (SD.7) by taking logs on both sides

$$ln\hat{D}_{t+1} = ln\kappa_t + ln\hat{q}_t + ln\hat{K}_{t+1},$$

then take the Taylor expansions for $ln\hat{D}_{t+1}$, $ln\hat{q}_t$, $ln\kappa_t$ and $ln\hat{K}_{t+1}$

$$lnD^* + \frac{\hat{D}_{t+1} - D^*}{D^*} = ln\kappa^* + \tilde{\kappa}_t + lnq^* + \frac{\hat{q}_t - q^*}{q^*} + lnK^* + \frac{\hat{K}_{t+1} - K^*}{K^*}.$$

Now cancel terms on both sides to get

$$\tilde{D}_{t+1} = \tilde{\kappa}_t + \tilde{q}_t + \tilde{K}_{t+1}. \tag{LE.2}$$

Linearize the price of capital (SD.5) by taking logs on both sides

$$ln\hat{q}_t = ln\frac{1}{\alpha_1\gamma^{*\alpha_2}} - ln\nu_t + \alpha_2 ln\hat{I}_t - \alpha_2 ln\hat{K}_t + \alpha_2\mu + \alpha_2\varepsilon_t,$$

then take the Taylor expansions for $ln\hat{q}_t$, $ln\nu_t$, $ln\hat{I}_t$, and $ln\hat{K}_t$

$$lnq^* + \frac{\hat{q}_t - q^*}{q^*} = ln\frac{1}{\alpha_1\gamma^{*\alpha_2}} - \tilde{\nu}_t + \alpha_2(lnI^* + \frac{\hat{I}_t - I^*}{I^*}) - \alpha_2(lnK^* + \frac{\hat{K}_t - K^*}{K^*}) + \alpha_2\mu + \alpha_2\varepsilon_t.$$

Now cancel terms on both sides to get

$$\tilde{q}_t = -\tilde{\nu}_t + \alpha_2\tilde{I}_t - \alpha_2\tilde{K}_t + \alpha_2\varepsilon_t. \tag{LE.3}$$

Linearize the world real interest rate (SD.6) by taking logs on both sides

$$ln\hat{r}_t = ln[\omega_t + \varphi\frac{\hat{D}_t}{\hat{Y}_t}exp(-\mu - \varepsilon_t)],$$

then take the Taylor expansions for $ln\hat{r}_t$, and $ln[\omega_t + \varphi\frac{\hat{D}_t}{\hat{Y}_t}exp(-\mu - \varepsilon_t)]$ w.r.t ω_t, $\hat{D}_t$, $\hat{Y}_t$, ε_t

$$lnr^* + \frac{\hat{r}_t - r^*}{r^*} = ln[\omega^* + \varphi\frac{D^*}{Y^*}exp(-\mu)] + \frac{\omega^*}{[\omega^* + \varphi\frac{D^*}{Y^*}exp(-\mu)]}\frac{\omega_t - \omega^*}{\omega^*} + \frac{\varphi\frac{D^*}{Y^*}exp(-\mu)}{[\omega^* + \varphi\frac{D^*}{Y^*}exp(-\mu)]}\frac{\hat{D}_t - D^*}{D^*}$$
$$- \frac{\varphi\frac{D^*}{Y^*}exp(-\mu)}{[\omega^* + \varphi\frac{D^*}{Y^*}exp(-\mu)]}\frac{\hat{Y}_t - Y^*}{Y^*} - \frac{\varphi\frac{D^*}{Y^*}exp(-\mu)}{[\omega^* + \varphi\frac{D^*}{Y^*}exp(-\mu)]}\varepsilon_t.$$

Now cancel terms on both sides to get

$$r^*\tilde{r}_t = \omega^*\tilde{\omega}_t + \varphi\frac{D^*}{Y^*}exp(-\mu)\tilde{D}_t - \varphi\frac{D^*}{Y^*}exp(-\mu)\tilde{Y}_t - \varphi\frac{D^*}{Y^*}exp(-\mu)\varepsilon_t. \qquad \text{(LE.4)}$$

Linearize the resource constraint (SD.8) by taking logs on both sides

$$ln(\hat{Y}_t + \hat{D}_{t+1}) = ln[\hat{C}_t + \hat{I}_t + \hat{G} + (1 + \hat{r}_t)\hat{D}_t exp(-\mu - \varepsilon_t)],$$

then take the Taylor expansions w.r.t $\hat{Y}_t$, $\hat{D}_{t+1}$, $\hat{C}_t$, $\hat{I}_t$, $\hat{r}_t$, $\hat{D}_t$, ε_t

$$
\begin{aligned}
&ln(Y^* + D^*) + \frac{Y^*}{Y^* + D^*}\frac{\hat{Y}_t - Y^*}{Y^*} + \frac{D^*}{Y^* + D^*}\frac{\hat{D}_{t+1} - D^*}{D^*} = ln[C^* + I^* + G^* + (1 + r^*)D^*exp(-\mu)] \\
&+ \frac{C^*}{C^* + I^* + G^* + (1 + r^*)D^*exp(-\mu)}\frac{\hat{C}_t - C^*}{C^*} + \frac{I^*}{C^* + I^* + G^* + (1 + r^*)D^*exp(-\mu)}\frac{\hat{I}_t - I^*}{I^*} \\
&+ \frac{D^*exp(-\mu)r^*}{C^* + I^* + G^* + (1 + r^*)D^*exp(-\mu)}\frac{\hat{r}_t - r^*}{r^*} + \frac{(1 + r^*)D^*exp(-\mu)}{C^* + I^* + G^* + (1 + r^*)D^*exp(-\mu)}\frac{\hat{D}_t - D^*}{D^*} \\
&- \frac{(1 + r^*)D^*exp(-\mu)}{C^* + I^* + G^* + (1 + r^*)D^*exp(-\mu)}\varepsilon_t.
\end{aligned}
$$

Now cancel terms on both sides to get

$$
Y^*\tilde{Y}_t + D^*\tilde{D}_{t+1} = C^*\tilde{C}_t + I^*\tilde{I}_t + D^*exp(-\mu)r^*\tilde{r}_t + (1 + r^*)D^*exp(-\mu)\tilde{D}_t - (1 + r^*)D^*exp(-\mu)\varepsilon_t. \tag{LE.5}
$$

Linearize the law of motion of capital (SD.4) by taking logs on both sides

$$
ln\hat{K}_{t+1} = ln[(1 - \delta + \alpha_3)\hat{K}_t exp(-\mu - \varepsilon_t) + \frac{\alpha_1\gamma^{*\alpha_2}}{1 - \alpha_2}\hat{I}_t^{1-\alpha_2}\hat{K}_t^{\alpha_2}\nu_t exp(-\alpha_2\mu - \alpha_2\varepsilon_t)],
$$

then take the Taylor expansions w.r.t $\hat{K}_{t+1}$, ν_t, $\hat{K}_t$, $\hat{I}_t$, ε_t,

$$
\begin{aligned}
ln(K^*) + \frac{\hat{K}_{t+1} - K^*}{K^*} &= ln[(1-\delta+\alpha_3)K^* exp(-\mu) + \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)] \\
&+ \frac{K^*(1-\delta+\alpha_3)exp(-\mu)}{(1-\delta+\alpha_3)K^* exp(-\mu) + \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)} \frac{\hat{K}_t - K^*}{K^*} \\
&- \frac{K^*(1-\delta+\alpha_3)exp(-\mu)}{(1-\delta+\alpha_3)K^* exp(-\mu) + \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)} \varepsilon_t \\
&+ \frac{\alpha_1 \gamma^{*\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)}{(1-\delta+\alpha_3)K^* exp(-\mu) + \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)} \frac{\hat{I}_t - I^*}{I^*} \\
&+ \frac{\frac{\alpha_1 \alpha_2 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)}{(1-\delta+\alpha_3)K^* exp(-\mu) + \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)} \frac{\hat{K}_t - K^*}{K^*} \\
&+ \frac{\frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)}{(1-\delta+\alpha_3)K^* exp(-\mu) + \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)} \tilde{\nu}_t \\
&- \frac{\frac{\alpha_1 \alpha_2 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)}{(1-\delta+\alpha_3)K^* exp(-\mu) + \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2} exp(-\alpha_2 \mu)} \varepsilon_t.
\end{aligned}
$$

Now cancel terms on both sides to get

$$
\begin{aligned}
&\tilde{K}_{t+1} = (1-\delta+\alpha_3)exp(-\mu)\tilde{K}_t - (1-\delta+\alpha_3)exp(-\mu)\varepsilon_t \\
&+ \alpha_1 \gamma^{*\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2 - 1} exp(-\alpha_2 \mu)\tilde{I}_t + \frac{\alpha_1 \alpha_2 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2 - 1} exp(-\alpha_2 \mu)\tilde{K}_t \qquad \text{(LE.6)} \\
&+ \frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2 - 1} exp(-\alpha_2 \mu)\tilde{\nu}_t - \frac{\alpha_1 \alpha_2 \gamma^{*\alpha_2}}{1-\alpha_2} I^{*1-\alpha_2} K^{*\alpha_2 - 1} exp(-\alpha_2 \mu)\varepsilon_t.
\end{aligned}
$$

Linearize the intratemporal Euler equation of labor (SD.1) by taking logs on both sides

$$
ln\frac{1-\phi}{\phi} + ln\hat{C}_t - ln(1-N_t) = ln(1-\theta) + ln\hat{Y}_t - lnN_t + ln[1 + \varphi(\frac{\hat{D}_t}{\hat{Y}_t})^2 exp(-2\mu - 2\varepsilon_t)],
$$

then take the Taylor expansions for $ln\hat{C}_t$, lnN_t, $ln\hat{Y}_t$, $ln\hat{D}_t$, and ε_t

$$
ln\frac{1-\phi}{\phi} + lnC^* + \frac{\hat{C}_t - C^*}{C^*} - [ln(1-N^*) - \frac{N^*}{1-N^*}\frac{N_t - N^*}{N^*}] = ln(1-\theta) + lnY^* + \frac{\hat{Y}_t - Y^*}{Y^*}
$$
$$
- lnN^* - \frac{N_t - N^*}{N^*} + ln[1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)] + \frac{2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}\frac{\hat{D}_t - D^*}{D^*}
$$
$$
- \frac{2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}\frac{\hat{Y}_t - Y^*}{Y^*} - \frac{2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}\varepsilon_t.
$$

Now cancel terms on both sides to get

$$
\begin{aligned}
&\tilde{C}_t + \frac{N^*}{1-N^*}\tilde{N}_t = \tilde{Y}_t - \tilde{N}_t + \frac{2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}\tilde{D}_t - \frac{2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}\tilde{Y}_t \\
&- \frac{2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}\varepsilon_t.
\end{aligned}
\tag{LE.7}
$$

Linearize the intertemporal Euler equation of capital (SD.2) by taking logs on both sides

$$
\begin{aligned}
&ln\hat{q}_t + ln(1-\kappa_t) = ln\beta + ln\hat{C}_t - ln\hat{C}_{t+1} + ln\Big\{\theta\frac{\hat{Y}_{t+1}}{\hat{K}_{t+1}}\Big[1+\varphi(\frac{\hat{D}_{t+1}}{\hat{Y}_{t+1}})^2 exp(-2\mu - 2\varepsilon_{t+1})\Big] \\
&+ \hat{q}_{t+1} exp(-\mu - \varepsilon_{t+1})\Big[1 - \delta + \alpha_3 + \frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{\hat{I}_{t+1}}{\hat{K}_{t+1}})^{1-\alpha_2}\nu_{t+1} exp[(\mu + \varepsilon_{t+1})(1-\alpha_2)]\Big] \\
&- \kappa_t\hat{q}_t exp(-\mu - \varepsilon_{t+1})[1 + \omega_{t+1} + 2\varphi\frac{\hat{D}_{t+1}}{\hat{Y}_{t+1}} exp(-\mu - \varepsilon_{t+1})]\Big\},
\end{aligned}
$$

then take the Taylor expansions for $ln\hat{q}_t$, $ln\hat{q}_{t+1}$, $ln\kappa_t$, $ln\hat{C}_t$, $ln\hat{C}_{t+1}$, $ln\hat{Y}_{t+1}$, $ln\hat{K}_{t+1}$, $ln\hat{D}_{t+1}$,

$ln\hat{I}_{t+1}$, ν_{t+1}, ω_{t+1}, and ε_{t+1}

$$
\begin{aligned}
& lnq^* + \frac{\hat{q}_t - q^*}{q^*} + ln(1-\kappa^*) - \frac{\kappa_t - \kappa^*}{1-\kappa^*} = ln\beta + lnC^* + \frac{\hat{C}_t - C^*}{C^*} - lnC^* - \frac{\hat{C}_{t+1} - C^*}{C^*} \\
& + lnA + \frac{\theta\frac{Y^*}{K^*}\Big[1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)\Big]}{A}\frac{\hat{Y}_{t+1} - Y^*}{Y^*} - \frac{\theta\frac{Y^*}{K^*}\Big[1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)\Big]}{A}\frac{\hat{K}_{t+1} - K^*}{K^*} \\
& + \frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{A}\frac{\hat{D}_{t+1} - D^*}{D^*} - \frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{A}\frac{\hat{Y}_{t+1} - Y^*}{Y^*} - \frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{A}\varepsilon_{t+1} \\
& + \frac{q^* exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp[(\mu)(1-\alpha_2)]\Big]}{A}\frac{\hat{q}_{t+1} - q^*}{q^*} \\
& - \frac{q^* exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp[(\mu)(1-\alpha_2)]\Big]}{A}\varepsilon_{t+1} \\
& + \frac{q^*\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A}\frac{\hat{I}_{t+1} - I^*}{I^*} - \frac{q^*\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A}\frac{\hat{K}_{t+1} - K^*}{K^*} \\
& + \frac{q^*\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A}\tilde{\nu}_{t+1} + \frac{q^*\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A}\varepsilon_{t+1} \\
& - \frac{\kappa^* q^* exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A}\frac{\kappa_t - \kappa^*}{\kappa^*} - \frac{\kappa^* q^* exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A}\frac{\hat{q}_t - q^*}{q^*} \\
& + \frac{\kappa^* q^* exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A}\varepsilon_{t+1} - \frac{\kappa^* q^* exp(-\mu)\omega^*}{A}\frac{\omega_{t+1} - \omega^*}{\omega^*} \\
& - \frac{\kappa^* q^* exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A}\frac{\hat{D}_{t+1} - D^*}{D^*} + \frac{\kappa^* q^* exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A}\frac{\hat{Y}_{t+1} - Y^*}{Y^*} + \frac{\kappa^* q^* exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A}\varepsilon_{t+1},
\end{aligned}
$$

where

$$
\begin{aligned}
& A = \theta\frac{Y^*}{K^*}\Big[1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)\Big] + q^* exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp[(\mu)(1-\alpha_2)]\Big] \\
& - \kappa^* q^* exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)].
\end{aligned}
$$

Now cancel terms on both sides to get

$$\begin{aligned}
\tilde{q}_t - \frac{\kappa^*}{1-\kappa^*}\tilde{\kappa}_t = \tilde{C}_t - \tilde{C}_{t+1} + \frac{\theta\frac{Y^*}{K^*}\Big[1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)\Big]}{A}\tilde{Y}_{t+1} \\
- \frac{\theta\frac{Y^*}{K^*}\Big[1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)\Big]}{A}\tilde{K}_{t+1} + \frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{A}\tilde{D}_{t+1} - \frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{A}\tilde{Y}_{t+1} \\
- \frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{A}\varepsilon_{t+1} + \frac{q^* exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2} exp[(\mu)(1-\alpha_2)]\Big]}{A}\tilde{q}_{t+1} \\
- \frac{q^* exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2} exp[(\mu)(1-\alpha_2)]\Big]}{A}\varepsilon_{t+1} \\
+ \frac{q^*\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2} exp(-\mu\alpha_2)}{A}\tilde{I}_{t+1} - \frac{q^*\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2} exp(-\mu\alpha_2)}{A}\tilde{K}_{t+1} \\
+ \frac{q^*\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2} exp(-\mu\alpha_2)}{A}\tilde{\nu}_{t+1} + \frac{q^*\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2} exp(-\mu\alpha_2)}{A}\varepsilon_{t+1} \\
- \frac{\kappa^* q^* exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A}\tilde{\kappa}_t - \frac{\kappa^* q^* exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A}\tilde{q}_t \\
+ \frac{\kappa^* q^* exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A}\varepsilon_{t+1} - \frac{\kappa^* q^* exp(-\mu)\omega^*}{A}\tilde{\omega}_{t+1} \\
- \frac{\kappa^* q^* exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A}\tilde{D}_{t+1} + \frac{\kappa^* q^* exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A}\tilde{Y}_{t+1} + \frac{\kappa^* q^* exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A}\varepsilon_{t+1}.
\end{aligned} \tag{LE.8}$$

The system of log-linearized equations are

$$\tilde{Y}_t = (1-\theta)\tilde{N}_t + \theta\tilde{K}_t - \theta\varepsilon_t, \tag{LE.1}$$

$$\tilde{D}_{t+1} = \tilde{\kappa}_t + \tilde{q}_t + \tilde{K}_{t+1}, \tag{LE.2}$$

$$\tilde{q}_t = -\tilde{\nu}_t + \alpha_2\tilde{I}_t - \alpha_2\tilde{K}_t + \alpha_2\varepsilon_t, \tag{LE.3}$$

$$r^*\tilde{r}_t = \omega^*\tilde{\omega}_t + \varphi\frac{D^*}{Y^*}exp(-\mu)\tilde{D}_t - \varphi\frac{D^*}{Y^*}exp(-\mu)\tilde{Y}_t - \varphi\frac{D^*}{Y^*}exp(-\mu)\varepsilon_t, \tag{LE.4}$$

$$\begin{aligned} &Y^*\tilde{Y}_t + D^*\tilde{D}_{t+1} = C^*\tilde{C}_t + I^*\tilde{I}_t + D^*exp(-\mu)r^*\tilde{r}_t + (1+r^*)D^*exp(-\mu)\tilde{D}_t \\ &- (1+r^*)D^*exp(-\mu)\varepsilon_t, \end{aligned} \tag{LE.5}$$

$$\begin{aligned} &\tilde{K}_{t+1} = [(1-\delta+\alpha_3)exp(-\mu) + \frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}I^{*1-\alpha_2}K^{*\alpha_2-1}exp(-\alpha_2\mu)]\tilde{K}_t \\ &+ \alpha_1\gamma^{*\alpha_2}I^{*1-\alpha_2}K^{*\alpha_2-1}exp(-\alpha_2\mu)\tilde{I}_t + \frac{\alpha_1\gamma^{*\alpha_2}}{1-\alpha_2}I^{*1-\alpha_2}K^{*\alpha_2-1}exp(-\alpha_2\mu)\tilde{\nu}_t \\ &- [(1-\delta+\alpha_3)exp(-\mu) + \frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}I^{*1-\alpha_2}K^{*\alpha_2-1}exp(-\alpha_2\mu)]\varepsilon_t, \end{aligned} \tag{LE.6}$$

$$\begin{aligned} &\tilde{C}_t + \frac{1}{1-N^*}\tilde{N}_t = \frac{2\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}\tilde{D}_t + \frac{1-\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}\tilde{Y}_t \\ &- \frac{2\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}\varepsilon_t, \end{aligned} \tag{LE.7}$$

$$\begin{aligned} &\tilde{q}_t[1 + \frac{\kappa^*q^*exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A}] + [\frac{\kappa^*q^*exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A} - \frac{\kappa^*}{1-\kappa^*}]\tilde{\kappa}_t \\ &- \tilde{C}_t = E_t\Big\{\Big[\frac{\theta\frac{Y^*}{K^*}[1+\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)]}{A} - \frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}{A} + \frac{\kappa^*q^*exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A}\Big]\tilde{Y}_{t+1} \\ &- \Big[\frac{\theta\frac{Y^*}{K^*}\Big[1+\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)\Big]}{A} + \frac{q^*\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A}\Big]\tilde{K}_{t+1} \\ &+ \frac{q^*\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A}\tilde{I}_{t+1} + \Big[\frac{q^*\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A} - \frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}{A} \\ &- \frac{q^*exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp[(\mu)(1-\alpha_2)]\Big]}{A} + \frac{\kappa^*q^*exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A} \\ &+ \frac{\kappa^*q^*exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A}\Big]\varepsilon_{t+1} + \frac{q^*\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A}\tilde{\nu}_{t+1} \\ &+ \frac{q^*exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp[(\mu)(1-\alpha_2)]\Big]}{A}\tilde{q}_{t+1} - \frac{\kappa^*q^*exp(-\mu)\omega^*}{A}\tilde{\omega}_{t+1} \\ &+ \Big[\frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}{A} - \frac{\kappa^*q^*exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A}\Big]\tilde{D}_{t+1} - \tilde{C}_{t+1}\Big\}, \end{aligned} \tag{LE.8}$$

where

$$A = \theta\frac{Y^*}{K^*}\Big[1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)\Big] + q^* exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2} exp[(\mu)(1-\alpha_2)]\Big]$$
$$-\kappa^* q^* exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)].$$

Plus exogenous shock processes

$$\varepsilon_t \sim N(0,\sigma_\varepsilon^2), \tag{LE.9}$$

$$\tilde{\nu}_t = \rho_\nu \tilde{\nu}_{t-1} + \eta_t, \quad \eta_t \sim N(0,\sigma_\eta^2), \tag{LE.10}$$

$$\tilde{\kappa}_t = \rho_\kappa \tilde{\kappa}_{t-1} + \zeta_t, \quad \zeta_t \sim N(0,\sigma_\zeta^2), \tag{LE.11}$$

$$\tilde{\omega}_t = \rho_\omega \tilde{\omega}_{t-1} + \gamma_t, \quad \gamma_t \sim N(0,\sigma_\gamma^2). \tag{LE.12}$$

C.3 Linear Solution

I adapt Sims (2002)'s method to solve my linearized SOE-RBC model. I cast the system of linear equations into the form

$$\Gamma_0 \boldsymbol{y_t} = \Gamma_1 \boldsymbol{y_{t-1}} + C + \Psi \boldsymbol{z_t} + \Pi \boldsymbol{\tau_t}, \tag{1}$$

where $y_t = E_t y_t + \tau_t^y$. Shown by (LE.8), $\{C_t, I_t, q_t, Y_t\}$ have expectational errors. Therefore

$\boldsymbol{y_t}$, $\boldsymbol{z_t}$, and $\boldsymbol{\tau_t}$ are organized as

$$\boldsymbol{y_t} = \begin{bmatrix} \tilde{C}_t \\ \tilde{I}_t \\ \tilde{q}_t \\ \tilde{Y}_t \\ \tilde{r}_t \\ \tilde{N}_t \\ \tilde{K}_{t+1} \\ \tilde{D}_{t+1} \\ \tilde{\nu}_t \\ \tilde{\kappa}_t \\ \tilde{\omega}_t \end{bmatrix}, \quad \boldsymbol{z_t} = \begin{bmatrix} \varepsilon_t \\ \eta_t \\ \zeta_t \\ \xi_t \end{bmatrix}, \quad \boldsymbol{\tau_t} = \begin{bmatrix} \tau_t^c \\ \tau_t^i \\ \tau_t^q \\ \tau_t^y \end{bmatrix}.$$

Following equations (LE.1-12), I specify Γ_0 as

$$\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & -(1-\theta) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \\
0 & -\alpha_2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \varphi \frac{D^*}{Y^*} exp(-\mu) & r^* & 0 & 0 & 0 & 0 & 0 & -\omega^* \\
-C^* & -I^* & 0 & Y^* & -D^* exp(-\mu) r^* & 0 & 0 & D^* & 0 & 0 & 0 \\
0 & -\alpha_1 \gamma^{*\alpha_2} (\frac{I^*}{K^*})^{1-\alpha_2} exp(-\alpha_2 \mu) & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{\alpha_1 \gamma^{*\alpha_2}}{1-\alpha_2} (\frac{I^*}{K^*})^{1-\alpha_2} exp(-\alpha_2 \mu) & 0 & 0 \\
1 & 0 & 0 & -\frac{1-\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2 exp(-2\mu)} & 0 & \frac{1}{1-N^*} & 0 & 0 & 0 & 0 & 0 \\
-1 & \Gamma_{8,2} & \Gamma_{8,3} & \Gamma_{8,4} & 0 & 0 & 0 & 0 & \Gamma_{8,9} & 0 & \Gamma_{8,11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

where

$$
\begin{aligned}
\Gamma_{8,2} &= \frac{q^{*}\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A},\\
\Gamma_{8,3} &= \frac{q^{*}exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp[(\mu)(1-\alpha_2)]\Big]}{A},\\
\Gamma_{8,4} &= \Big[\frac{\theta\frac{Y^*}{K^*}[1+\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)]}{A} - \frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}{A} + \frac{\kappa^{*}q^{*}exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A}\Big],\\
\Gamma_{8,9} &= \frac{q^{*}\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A},\quad \Gamma_{8,11} = -\frac{\kappa^{*}q^{*}exp(-\mu)\omega^{*}}{A}.
\end{aligned}
$$

The Γ_1 is

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \theta & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\alpha_2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \varphi\frac{D^*}{Y^*}exp(-\mu) & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (1+r^*)D^*exp(-\mu) & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & G_{6,7} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}{1+\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)} & 0 & 0 & 0\\
-1 & 0 & 1+\frac{\kappa^{*}q^{*}exp(-\mu)[1+\omega^{*}+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A} & 0 & 0 & 0 & G_{8,7} & G_{8,8} & 0 & G_{8,10} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_\nu & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_\kappa & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_\omega
\end{bmatrix},
$$

where

$$
\begin{aligned}
G_{6,7} &= (1-\delta+\alpha_3)exp(-\mu) + \frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\alpha_2\mu)\\
G_{8,7} &= \frac{\theta\frac{Y^*}{K^*}\Big[1+\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)\Big]}{A} + \frac{q^{*}\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A}\\
G_{8,8} &= -\frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}{A} + \frac{\kappa^{*}q^{*}exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A}\\
G_{8,10} &= \frac{\kappa^{*}q^{*}exp(-\mu)[1+\omega^{*}+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A} - \frac{\kappa^{*}}{1-\kappa^{*}}
\end{aligned}
$$

The C is a 11×1 zero vector. The Ψ is

$$\begin{bmatrix} -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ -\varphi\frac{D^*}{Y^*}exp(-\mu) & 0 & 0 & 0 \\ -(1+r^*)D^*exp(-\mu) & 0 & 0 & 0 \\ -[(1-\delta+\alpha_3)exp(-\mu)+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}I^{*1-\alpha_2}K^{*\alpha_2-1}exp(-\alpha_2\mu)] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Z_{8,1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$\begin{aligned} Z_{8,1} = -\Big[&\frac{q^*\alpha_1\alpha_2\gamma^{*\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp(-\mu\alpha_2)}{A} - \frac{\theta\frac{Y^*}{K^*}2\varphi(\frac{D^*}{Y^*})^2exp(-2\mu)}{A} \\ &- \frac{q^*exp(-\mu)\Big[1-\delta+\alpha_3+\frac{\alpha_1\alpha_2\gamma^{*\alpha_2}}{1-\alpha_2}(\frac{I^*}{K^*})^{1-\alpha_2}exp[(\mu)(1-\alpha_2)]\Big]}{A} + \frac{\kappa^*q^*exp(-2\mu)2\varphi\frac{D^*}{Y^*}}{A} \\ &+ \frac{\kappa^*q^*exp(-\mu)[1+\omega^*+2\varphi\frac{D^*}{Y^*}exp(-\mu)]}{A}\Big]. \end{aligned}$$

Finally, Π is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & \Gamma_{8,2} & \Gamma_{8,3} & \Gamma_{8,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Appendix D: SVARs

D.1 Likelihood Function of VAR

Since the VAR is Gaussian, $y \sim N\big((I_n \otimes \boldsymbol{X})c, \Sigma_u \otimes I_T\big)$, this gives the likelihood of y as

$$\mathcal{L}(y|c,\Sigma_u) = (2\pi)^{-\frac{nT}{2}} |\Sigma_u \otimes I_T|^{-\frac{1}{2}} exp\Big(-\frac{1}{2}[y - (I_n \otimes \boldsymbol{X})c]'(\Sigma_u \otimes I_T)^{-1}[y - (I_n \otimes \boldsymbol{X})c]\Big).$$

The first term in the likelihood is the constant factor of proportional $(2\pi)^{-\frac{nT}{2}}$. The implication for the likelihood is

$$\mathcal{L}(y|c,\Sigma_u) \propto |\Sigma_u \otimes I_T|^{-\frac{1}{2}} exp\big(-\frac{1}{2}[y-(I_n\otimes \boldsymbol{X})c]'(\Sigma_u\otimes I_T)^{-1}[y-(I_n\otimes \boldsymbol{X})c]\big).$$

I follow Dieppe et al. (2016) and Doan (2010) to start at the likelihood function of y

$$\mathcal{L}(y|c,\Sigma_u) \propto |\Sigma_u \otimes I_T|^{-\frac{1}{2}} exp\big(-\frac{1}{2}[y-(I_n\otimes \boldsymbol{X})c]'(\Sigma_u\otimes I_T)^{-1}[y-(I_n\otimes \boldsymbol{X})c]\big). \tag{A1}$$

Next, develop the $[y-(I_n\otimes \boldsymbol{X})c]'(\Sigma_u\otimes I_T)^{-1}[y-(I_n\otimes \boldsymbol{X})c]$ term in equation (A1)

$$\begin{aligned}
&[y-(I_n\otimes \boldsymbol{X})c]'(\Sigma_u\otimes I_T)^{-1}[y-(I_n\otimes \boldsymbol{X})c]\\
&= y'(\Sigma_u^{-1}\otimes I_T)y - 2c'(I_n\otimes \boldsymbol{X})'(\Sigma_u^{-1}\otimes I_T)y + c'(I_n\otimes \boldsymbol{X})'(\Sigma_u^{-1}\otimes I_T)(I_n\otimes \boldsymbol{X})c\\
&= y'(\Sigma_u^{-1}\otimes I_T)y - 2c'(\Sigma_u^{-1}\otimes \boldsymbol{X}')y + c'(\Sigma_u^{-1}\otimes \boldsymbol{X}'\boldsymbol{X})c,
\end{aligned}$$

then add and subtract $2\hat{c}'(\Sigma_u^{-1}\otimes \boldsymbol{X}')y$ to continue

$$\begin{aligned}
&= y'(\Sigma_u^{-1}\otimes I_T)y - 2\hat{c}'(\Sigma_u^{-1}\otimes \boldsymbol{X}')y + 2\hat{c}'(\Sigma_u^{-1}\otimes \boldsymbol{X}')y - 2c'(\Sigma_u^{-1}\otimes \boldsymbol{X}')y\\
&\quad + c'(\Sigma_u^{-1}\otimes \boldsymbol{X}'\boldsymbol{X})c\\
&= y'(\Sigma_u^{-1}\otimes I_T)y - 2\hat{c}'(\Sigma_u^{-1}\otimes \boldsymbol{X}')y + 2\hat{c}'(\Sigma_u^{-1}\otimes \boldsymbol{X}'\boldsymbol{X})(\Sigma_u^{-1}\otimes \boldsymbol{X}'\boldsymbol{X})^{-1}(\Sigma_u^{-1}\otimes \boldsymbol{X}')y\\
&\quad - 2c'(\Sigma_u^{-1}\otimes \boldsymbol{X}'\boldsymbol{X})(\Sigma_u^{-1}\otimes \boldsymbol{X}'\boldsymbol{X})^{-1}(\Sigma_u^{-1}\otimes \boldsymbol{X}')y + c'(\Sigma_u^{-1}\otimes \boldsymbol{X}'\boldsymbol{X})c,
\end{aligned} \tag{A2}$$

where $\hat{c} = (\Sigma_u^{-1} \otimes \boldsymbol{X}'\boldsymbol{X})^{-1}(\Sigma_u^{-1} \otimes \boldsymbol{X}')y$. Hence, equation (A2) becomes

$$\begin{aligned}
&[y - (I_n \otimes \boldsymbol{X})c]'(\Sigma_u \otimes I_T)^{-1}[y - (I_n \otimes \boldsymbol{X})c] \\
&= y'(\Sigma_u^{-1} \otimes I_T)y - 2\hat{c}'(\Sigma_u^{-1} \otimes \boldsymbol{X}')y + 2\hat{c}'(\Sigma_u^{-1} \otimes \boldsymbol{X}'\boldsymbol{X})\hat{c} - 2c'(\Sigma_u^{-1} \otimes \boldsymbol{X}'\boldsymbol{X})\hat{c} \\
&\quad + c'(\Sigma_u^{-1} \otimes \boldsymbol{X}'\boldsymbol{X})c \qquad\qquad \text{(A3)} \\
&= y'(\Sigma_u^{-1} \otimes I_T)y - 2\hat{c}'(\Sigma_u^{-1} \otimes \boldsymbol{X}')y + \hat{c}'(\Sigma_u^{-1} \otimes \boldsymbol{X}'\boldsymbol{X})\hat{c} \\
&\quad + (c - \hat{c})'(\Sigma_u \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1})^{-1}(c - \hat{c}).
\end{aligned}$$

In equation (A3), the $(c - \hat{c})'(\Sigma_u \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1})^{-1}(c - \hat{c})$ component is in a squared form, but the $y'(\Sigma_u^{-1} \otimes I_T)y - 2\hat{c}'(\Sigma_u^{-1} \otimes \boldsymbol{X}')y + \hat{c}'(\Sigma_u^{-1} \otimes \boldsymbol{X}'\boldsymbol{X})\hat{c}$ part needs to be factored into a recognizable kernel of a distribution.

According to Dieppe et al. (2016), the $y'(\Sigma_u^{-1} \otimes I_T)y - 2\hat{c}'(\Sigma_u^{-1} \otimes \boldsymbol{X}')y + \hat{c}'(\Sigma_u^{-1} \otimes \boldsymbol{X}'\boldsymbol{X})\hat{c}$ term becomes

$$\begin{aligned}
&y'(\Sigma_u^{-1} \otimes I_T)y - 2\hat{c}'(\Sigma_u^{-1} \otimes \boldsymbol{X}')y + \hat{c}'(\Sigma_u^{-1} \otimes \boldsymbol{X}'\boldsymbol{X})\hat{c} \\
&= y'(\Sigma_u \otimes I_T)^{-1}y - 2[(I_n \otimes \boldsymbol{X})\hat{c}]'(\Sigma_u \otimes I_T)^{-1}y + \hat{c}'(I_n \otimes \boldsymbol{X})'(\Sigma_u \otimes I_T)^{-1}(I_n \otimes \boldsymbol{X})\hat{c}.
\end{aligned}$$

Dieppe et al. (2016) shows a formula, $tr\{V^{-1}(X-M)'U^{-1}(Y-N)\} = \big(vec(X)-vec(M)\big)'(V\otimes$

$U)^{-1}\big(vec(X) - vec(N)\big)$. Now apply this formula to continue from the above equation

$$
\begin{aligned}
& y'(\Sigma_u \otimes I_T)^{-1} y - 2[(I_n \otimes \boldsymbol{X})\hat{c}]'(\Sigma_u \otimes I_T)^{-1} y + \hat{c}'(I_n \otimes \boldsymbol{X})'(\Sigma_u \otimes I_T)^{-1}(I_n \otimes \boldsymbol{X})\hat{c} \\
&= tr\{\boldsymbol{Y}\Sigma_u^{-1}\boldsymbol{Y}'\} - 2tr\{\boldsymbol{Y}\Sigma_u^{-1}\hat{\mathbb{C}}'\boldsymbol{X}'\} + tr\{\boldsymbol{X}\hat{\mathbb{C}}\Sigma_u^{-1}\hat{\mathbb{C}}'\boldsymbol{X}'\} \\
&= tr\{\boldsymbol{Y}\Sigma_u^{-1}\boldsymbol{Y}' - 2\boldsymbol{Y}\Sigma_u^{-1}\hat{\mathbb{C}}'\boldsymbol{X}' + \boldsymbol{X}\hat{\mathbb{C}}\Sigma_u^{-1}\hat{\mathbb{C}}'\boldsymbol{X}'\} \\
&= tr\{(\boldsymbol{Y} - \boldsymbol{X}\hat{\mathbb{C}})\Sigma_u^{-1}(\boldsymbol{Y} - \boldsymbol{X}\hat{\mathbb{C}})'\},
\end{aligned}
\tag{A4}
$$

where $\hat{\mathbb{C}}$ is the OLS estimate of $\mathbb{C}$ from

$$
\boldsymbol{Y} = \boldsymbol{X}\mathbb{C} + \boldsymbol{U}.
$$

Finally, combine equation (A3) and (A4) to get

$$
\begin{aligned}
& [y - (I_n \otimes \boldsymbol{X})c]'(\Sigma_u \otimes I_T)^{-1}[y - (I_n \otimes \boldsymbol{X})c] \\
&= tr\{(\boldsymbol{Y} - \boldsymbol{X}\hat{\mathbb{C}})\Sigma_u^{-1}(\boldsymbol{Y} - \boldsymbol{X}\hat{\mathbb{C}})'\} + (c - \hat{c})'(\Sigma_u \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1})^{-1}(c - \hat{c}),
\end{aligned}
\tag{A5}
$$

and recall that the $[y - (I_n \otimes \boldsymbol{X})c]'(\Sigma_u \otimes I_T)^{-1}[y - (I_n \otimes \boldsymbol{X})c]$ term is from the likelihood function (A1).

Now I return to equation (A1). Note that

$$
|\Sigma_u \otimes I_T|^{-\frac{1}{2}} = |\Sigma_u|^{-\frac{T}{2}} = |\Sigma_u|^{-\frac{k}{2}}|\Sigma_u|^{-\frac{T-k}{2}} = |\Sigma_u|^{-\frac{k}{2}}|\Sigma_u|^{-\frac{(T-k-n-1)+n+1}{2}}, \tag{A6}
$$

where $k = np + 1$, n is the number of endogenous variables of the VAR, and p is the lag length. Substitute equation (A5) and (A6) into (A1)

$$
\begin{aligned}
\mathcal{L}(y|c,\Sigma_u) &\propto |\Sigma_u \otimes I_T|^{-\frac{1}{2}} exp\Big[-\frac{1}{2}[y-(I_n\otimes \boldsymbol{X})c]'(\Sigma_u\otimes I_T)^{-1}[y-(I_n\otimes \boldsymbol{X})c]\Big] \\
&\propto |\Sigma_u|^{-\frac{k}{2}}|\Sigma_u|^{-\frac{(T-k-n-1)+n+1}{2}} exp\Big[-\frac{1}{2}\Big\{tr\{(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})\Sigma_u^{-1}(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})'\} \\
&\quad + (c-\hat{c})'(\Sigma_u\otimes(\boldsymbol{X}'\boldsymbol{X})^{-1})^{-1}(c-\hat{c})\Big\}\Big],
\end{aligned}
$$

and finally I achieve

$$
\begin{aligned}
\mathcal{L}(y|c,\Sigma_u) &\propto |\Sigma_u|^{-\frac{k}{2}} exp\Big[-\frac{1}{2}(c-\hat{c})'(\Sigma_u\otimes(\boldsymbol{X}'\boldsymbol{X})^{-1})^{-1}(c-\hat{c})\Big] \\
&\times |\Sigma_u|^{-\frac{(T-k-n-1)+n+1}{2}} exp\Big[-\frac{1}{2}tr\{(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})\Sigma_u^{-1}(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})'\}\Big].
\end{aligned}
\tag{A7}
$$

D.2 Priors and Posteriors

The non-informative prior density is proportional to

$$
\pi(c,\Sigma_u) \propto |\Sigma_u|^{-\frac{n+1}{2}}.
$$

Given the prior density and likelihood (A7), posterior density of c and Σ_u are derived by applying the Bayes rule

$$
\begin{aligned}
\Pi(c,\Sigma_u|y) &\propto \mathcal{L}(y|c,\Sigma_u)\times\pi(c,\Sigma_u) \\
&\propto |\Sigma_u|^{-\frac{k}{2}} exp\Big[-\frac{1}{2}(c-\hat{c})'(\Sigma_u\otimes(\boldsymbol{X}'\boldsymbol{X})^{-1})^{-1}(c-\hat{c})\Big]\times|\Sigma_u|^{-\frac{(T-k-n-1)+n+1}{2}} \\
&\quad exp\Big[-\frac{1}{2}tr\{(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})'\Sigma_u^{-1}(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})\}\Big]\times|\Sigma_u|^{-\frac{n+1}{2}}.
\end{aligned}
$$

Then I obtain the posterior density

$$\Pi(c, \Sigma_u|y) \propto |\Sigma_u|^{-\frac{k}{2}} exp\big[-\frac{1}{2}(c-\hat{c})'(\Sigma_u \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1})^{-1}(c-\hat{c})\big] \times |\Sigma_u|^{-\frac{(T-k)+n+1}{2}}$$
$$exp\big[-\frac{1}{2}tr\{(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})'\Sigma_u^{-1}(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})\}\big].$$

The kernel, $|\Sigma_u|^{-\frac{k}{2}} exp\big[-\frac{1}{2}(c-\hat{c})'(\Sigma_u \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1})^{-1}(c-\hat{c})\big]$, in the posterior density indicates the posterior distribution of c follows

$$\underline{c} \sim N\big(\hat{c}, \Sigma_u \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}\big),$$

which is the same as the distribution of c. The $|\Sigma_u|^{-\frac{(T-k)+n+1}{2}} exp\big[-\frac{1}{2}tr\{(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})'\Sigma_u^{-1}(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})\}\big]$ component allows me to read off the posterior for Σ_u^{-1} as

$$\underline{\Sigma}_u \sim IW\big((\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})'(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}}), T-k\big),$$

conditional on the assumption that the errors of the VAR is Gaussian.

D.3 Bayesian Monte Carlo Integration under Short-Run Restrictions

First, use OLS to get estimates of the coefficient matrix, $\hat{\mathbb{C}}$, and the covariance matrix, $\hat{\Sigma}_u$. The next objective is to acquire posterior distributions of IRFs and FEVDs. The algorithm of Doan (2010) consists of the following steps.

i) Draw a matrix, $\underline{\Sigma}_k$, from $IW\big((\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}})'(\boldsymbol{Y}-\boldsymbol{X}\hat{\mathbb{C}}), T-k\big)$.

ii) Generate $\underline{c}_k = \hat{c} + \underline{\nu}_k$, where $\underline{\nu}_k$ is a vector drawn from $N\big(0, \underline{\Sigma}_k \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}\big)$.

iii) Compute $\underline{\boldsymbol{U}}_k = \boldsymbol{Y} - \boldsymbol{X}\underline{\mathbb{C}}_k$, where $\underline{\mathbb{C}}_k$ is a matrix reshaped from $\underline{c}_k$.

iv) Take the Cholesky decomposition $\underline{D}_k = \underline{\Sigma}_u^{0.5}$, where $\underline{\Sigma}_u = \dfrac{\underline{\boldsymbol{U}}_k'\underline{\boldsymbol{U}}_k}{T}$.

v) Calculate IRFs and FEVDs using $\underline{\mathbb{C}}_k$ and $\underline{D}_k$.[1]

vi) Repeat the above steps for $J = 5000$ times to acquire the sampling distributions of IRFs and FEVDs.

This Bayesian Monte Carlo integration algorithm is valid only when the short-run restrictions are recursive. The short-run identification specifies D as lower triangular, which does not change the distribution properties of $\underline{c}$ and $\underline{\Sigma_u}$ of the SVAR.

D.4 Bayesian Monte Carlo Integration under Long-Run Restrictions

Continue with the OLS estimates of $\hat{\mathbb{C}}$ and $\hat{\Sigma}_u$. The long-run Bayesian Monte Carlo integration algorithm is following steps.

i) Draw a matrix, $\underline{\Sigma}_k$, from $IW\big((\boldsymbol{Y} - \boldsymbol{X}\hat{\mathbb{C}})'(\boldsymbol{Y} - \boldsymbol{X}\hat{\mathbb{C}}), T - k\big)$.

ii) Generate $\underline{c}_k = \hat{c} + \underline{\nu}_k$, where $\underline{\nu}_k$ is a vector drawn from $N\big(0, \underline{\Sigma}_k \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}\big)$.

iii) Compute $\underline{\boldsymbol{U}}_k = \boldsymbol{Y} - \boldsymbol{X}\underline{\mathbb{C}}_k$, where $\underline{\mathbb{C}}_k$ is a matrix reshaped from $\underline{c}_k$.

[1] I follow the standard algorithm of IRF and FEVD computations from the VAR literature. See, for example, Hamilton (1994).

iv) Get $\underline{\Sigma}_u = \dfrac{\underline{\boldsymbol{U}}_k'\underline{\boldsymbol{U}}_k}{T}$.

v) Calculate $\underline{\Gamma}_k(1) = \left([I_{4\times4} - \underline{C}_k(1)]^{-1}\underline{\Sigma}_u([I_{4\times4} - \underline{C}_k(1)]^{-1})'\right)^{0.5}$, where $\underline{C}_k(1) = \underline{C}_1 + \underline{C}_2$ and they are reshaped from $\underline{c}_k$.

vi) Recover $\underline{D}_k = [I_{4\times4} - \underline{C}_k(1)]\underline{\Gamma}_k(1)$.

vii) Calculate IRFs and FEVDs using $\underline{\mathbb{C}}_k$ and $\underline{D}_k$.

viii) Repeat the above steps for $J = 5000$ times to acquire the sampling distributions of IRFs and FEVDs.

There is a caveat to the long-run Bayesian Monte Carlo integration algorithm. The Blanchard and Quah (1989) decomposition rests on non-linear computations to specify D, which could change the distribution property of Σ_u. This may result in badly drawn D_k.

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