MATH 503: Mathematical Statistics
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Homework 3 Solutions

1. Given $f(x;\theta) = \frac{1}{\theta}$, $0 < x < \theta$, zero elsewhere, with $\theta > 0$, formally compute the reciprocal of $nE\left\{\left[\frac{\partial \log f(X;\theta)}{\partial \theta}\right]^2\right\}$. Compare this with the variance of $(n+1)Y_n/n$, where Y_n is the largest observation of a random sample of size n from this distribution. Comment.

Solution:

$$\begin{aligned}
f(x;\theta) &= \frac{1}{\theta}, 0 < x < \theta \\
&= \frac{1}{\theta} I_{(0,\theta)}(x) \\
\log f(x;\theta) &= -\log\theta \\
\frac{\partial}{\partial\theta}\log f(x;\theta) &= \frac{-1}{\theta} \\
nE\left[\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)^2\right] &= nE\left(\frac{1}{\theta^2}\right) = \frac{n}{\theta^2} \\
\therefore \frac{1}{nE\left[\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)^2\right]} &= \frac{\theta^2}{n}
\end{aligned}$$

Meanwhile,

$$\mathrm{Var}\left(\frac{(n+1)Y_n}{n}\right) = \left(\frac{n+1}{n}\right)^2 \mathrm{Var}(Y_n) = \left(\frac{n+1}{n}\right)^2 \mathrm{Var}(X_{(n)}) = \left(\frac{n+1}{n}\right)^2 \left(E(X_{(n)}^2) - E^2(X_{(n)})\right),$$

where $f_{X_{(n)}}(x) = nF^{n-1}(x)f(x) = n\left(\frac{x}{\theta}\right)^{n-1}\frac{1}{\theta} = \frac{nx^{n-1}}{\theta^n}$, so

$$E(X_{(n)}^k) = \int_0^\theta x^k \frac{nx^{n-1}}{\theta^n}dx = \frac{n}{\theta^n}\int_0^\theta x^{n+k-1}dx = \frac{n}{\theta^n}\left(\frac{1}{n+k}x^{n+k}\,|_0^\theta\right) = \frac{n\theta^{n+k}}{(n+k)\theta^n} = \frac{n\theta^k}{n+k}, \quad k = 1, 2, \ldots,$$

so $E(X_{(n)}) = \frac{n\theta}{n+1}$ and $E(X_{(n)}^2) = \frac{n\theta^2}{n+2}$. Note that this implies that $\frac{(n+1)X_{(n)}}{n}$ is unbiased for θ. Meanwhile,

$$\begin{aligned}
\mathrm{Var}\left(\frac{(n+1)X_{(n)}}{n}\right) &= \left(\frac{n+1}{n}\right)^2\left(\frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2\right) = \left(\frac{n+1}{n}\right)^2 n\theta^2\left(\frac{1}{n+2} - \frac{n}{(n+1)^2}\right) \\
&= \frac{(n+1)^2\theta^2}{n}\left(\frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)^2}\right) = \frac{\theta^2}{n(n+2)}.
\end{aligned}$$

The variance for $\frac{(n+1)X_n}{n}$ is smaller than the reciprocal of $nE\left(\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)^2\right)$. While it would appear that this is a counter example to the Cramér Rao Lower Bound (CRLB) theorem, note that the regularity conditions aren't satisfied (particularly the condition that the pdfs have common support for all θ, i.e. that the support does not depend on θ), thus we cannot apply the CRLB theorem here.

2. Let X be $N(0,\theta)$, $0<\theta<\infty$, where $\mathrm{Var}(X)=\theta$.

   (a) Find the Fisher information $I(\theta)$.

   (b) If $X_1, X_2, \ldots, X_n$ is a random sample from this distribution, show that the MLE of θ is an efficient estimator of θ.

Solution:

(a)

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi\theta}}e^{\frac{-1}{2\theta}(x-0)^2} = (2\pi\theta)^{-1/2}e^{\frac{-1}{2\theta}x^2} \\
\ln f(x) &= \frac{-1}{2}\ln(2\pi) - \frac{1}{2}\ln\theta - \frac{1}{2\theta}x^2 \\
\frac{\partial}{\partial\theta}\ln f(x) &= -\frac{1}{2\theta} + \frac{1}{2\theta^2}x^2 = \frac{-1}{2}\theta^{-1} + \frac{x^2}{2}\theta^{-2} \\
\frac{\partial^2}{\partial\theta^2}\ln f(x) &= \frac{1}{2}\theta^{-2} - x^2\theta^{-3} = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3} \\
I(\theta) &= -E\left(\frac{\partial^2}{\partial\theta^2}\ln f(x)\right) = -\frac{1}{2\theta^2} + \frac{E(X^2)}{\theta^3},
\end{aligned}$$

where $E(X^2) = \mathrm{Var}(X) + E^2(X) = \theta + 0^2 = \theta$, therefore $I(\theta) = \frac{-1}{2\theta^2} + \frac{\theta}{\theta^2} = \frac{-1+2}{2\theta^2} = \frac{1}{2\theta^2}$.

(b) Consider random sample $X_1, \ldots, X_n \sim N(0,\theta)$. From (a), we have that $nI(\theta) = \frac{n}{2\theta^2}$. Meanwhile,

$$\begin{aligned}
L(\theta;\boldsymbol{x}) &= \prod_{i=1}^n f(x_i) = (2\pi\theta)^{-n/2}e^{\frac{-1}{2\theta}\sum x_i^2} \\
\ln L(\theta;\boldsymbol{x}) &= \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln\theta - \frac{1}{2\theta}\sum x_i^2 \\
\frac{\partial}{\partial\theta}\ln L(\theta;\boldsymbol{x}) &= -\frac{n}{2\theta} + \frac{\sum x_i^2}{2\theta^2} = 0 \Rightarrow \hat{\theta} = \frac{\sum x_i^2}{n} = \bar{x^2}.
\end{aligned}$$

$$\mathrm{Var}(\bar{X^2}) = \mathrm{Var}\left(\frac{\sum X_i^2}{n}\right) = \mathrm{Var}\left(\frac{\theta}{n}\frac{\sum X_i^2}{\theta}\right) = \frac{\theta^2}{n^2}\mathrm{Var}\left(\frac{\sum X_i^2}{\theta}\right),$$

where $\frac{\sum X_i^2}{\theta} \sim \chi_n^2$ (claim proven below) $\therefore \mathrm{Var}\left(\bar{X^2}\right) = \frac{\theta^2}{n^2}(2n) = \frac{2\theta^2}{n}$, which equals the CRLB $= \frac{1}{nI(\theta)} = \frac{2\theta^2}{n}$, therefore $\hat{\theta} = \bar{X^2}$ is an efficient estimator of θ.

Proof to claim: $X_1, \ldots, X_n \sim N(0,\theta)$ iid implies that $\frac{X_1}{\sqrt{\theta}}, \ldots, \frac{X_n}{\sqrt{\theta}} \sim N(0,1)$ iid, which further implies that $\frac{X_1^2}{\theta}, \ldots, \frac{X_n^2}{\theta} \sim \chi_1^2$ iid. Thus, $\sum_{i=1}^n \frac{X_i^2}{\theta} = \frac{\sum_{i=1}^n X_i^2}{\theta} \sim \chi_n^2$.

3. Let $\bar{X}$ be the mean of a random sample of size n from a $N(\theta, \sigma^2)$ distribution, $-\infty < \theta < \infty$, $\sigma^2 > 0$. Assume that σ^2 is known. Show that $\bar{X}^2 - \frac{\sigma^2}{n}$ is an unbiased estimator of θ^2 and find its efficiency.

Solution:

$$\begin{aligned} E\left(\bar{X}^2 - \frac{\sigma^2}{n}\right) &= E(\bar{X}^2) - \frac{\sigma^2}{n} = \left[\mathrm{Var}(\bar{X}) + E^2(\bar{X})\right] - \frac{\sigma^2}{n}, \text{ where } \bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right) \\ &= \frac{\sigma^2}{n} + \theta^2 - \frac{\sigma^2}{n} = \theta^2, \end{aligned}$$

so $\bar{X}^2 - \frac{\sigma^2}{n}$ is unbiased for θ^2. Meanwhile,

$$\begin{aligned} \mathrm{Var}\left(\bar{X}^2 - \frac{\sigma^2}{n}\right) &= \mathrm{Var}\left(\bar{X}^2\right) \text{ because } \frac{\sigma^2}{n} \text{ is additive constant} \\ &= 2\frac{\sigma^4}{n^2} + 4\frac{\sigma^2}{n}\theta^2 \text{ (see details below).} \end{aligned}$$

To compute the Cramér-Rao Lower Bound (CRLB),

$$\begin{aligned} f(x;\theta) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} = (\sqrt{2\pi}\theta)^{-1} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \\ \ln f(x;\theta) &= -\ln(\sqrt{2\pi}\sigma) - \frac{1}{2\sigma^2}(x-\theta)^2 \\ \frac{\partial}{\partial\theta}\ln f(x;\theta) &= \frac{1}{\sigma^2}(x-\theta) \\ \frac{\partial^2}{\partial\theta^2}\ln f(x;\theta) &= -\frac{1}{\sigma^2} \\ \Rightarrow nI(\theta) &= -nE\left(\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right) = \frac{n}{\sigma^2} \\ \therefore CRLB &= \frac{[k'(\theta)]^2}{nI(\theta)} = \frac{(2\theta)^2}{n/\sigma^2} = \frac{4\theta^2\sigma^2}{n} \end{aligned}$$

$$EFF\left(\bar{X}^2 - \frac{\sigma^2}{n}\right) = \frac{2\frac{\sigma^4}{n^2} + 4\frac{\sigma^2}{n}\theta^2}{\frac{4\theta^2\sigma^2}{n}} = 1 + \frac{\sigma^2}{2n\theta^2}.$$

Derivation of $\mathrm{Var}\left(\bar{X}^2 - \frac{\sigma^2}{n}\right)$ result:

$$\mathrm{Var}\left(\bar{X}^2 - \frac{\sigma^2}{n}\right) = \mathrm{Var}(\bar{X}^2) = E(\bar{X}^4) - E^2(\bar{X}^2)$$

where $\bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right)$ thus its moment generating function (mgf) is $M_{\bar{X}}(t) = \exp\left[\theta t + \frac{1}{2} \cdot \frac{\sigma^2}{n} t\right]$. Computing the necessary number of derivatives and evaluating each at $t = 0$ will determine $E(\bar{X}^2)$ and $E(\bar{X}^4)$.

$$
\begin{aligned}
M'_{\bar{X}}(t) &= \left(\theta + \frac{\sigma^2}{n}t\right)\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] \\
M''_{\bar{X}}(t) &= \frac{\sigma^2}{n}\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] + \left(\theta + \frac{\sigma^2}{n}t\right)\left(\theta + \frac{\sigma^2}{n}t\right)\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] \\
&= \frac{\sigma^2}{n}\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] + \left(\theta + \frac{\sigma^2}{n}t\right)^2\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] \\
M'''_{\bar{X}}(t) &= \frac{\sigma^2}{n}\left(\theta + \frac{\sigma^2}{n}t\right)\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] + 2\left(\theta + \frac{\sigma^2}{n}t\right)\frac{\sigma^2}{n}\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] \\
&\quad + \left(\theta + \frac{\sigma^2}{n}t\right)^3\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] \\
M^{(4)}_{\bar{X}}(t) &= 3\left(\frac{\sigma^2}{n}\right)\left(\frac{\sigma^2}{n}\right)\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] + 3\frac{\sigma^2}{n}\left(\theta + \frac{\sigma^2}{n}t\right)\left(\theta + \frac{\sigma^2}{n}t\right)\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] \\
&\quad + 3\frac{\sigma^2}{n}\left(\theta + \frac{\sigma^2}{n}t\right)\left(\theta + \frac{\sigma^2}{n}t\right)\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right] + \left(\theta + \frac{\sigma^2}{n}t\right)^3\left(\theta + \frac{\sigma^2}{n}t\right)\exp\left[\theta t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right]
\end{aligned}
$$

thus

$$
\begin{aligned}
E(\bar{X}^2) &= M''_{\bar{X}} \mid_{t=0} = \frac{\sigma^2}{n} + \theta^2 \\
E(\bar{X}^4) &= M^{(4)}_{\bar{X}} \mid_{t=0} = 3\frac{\sigma^4}{n^2} + 6\frac{\sigma^2}{n}\theta^2 + \theta^4 \\
\therefore \operatorname{Var}(\bar{X}^2) &= 3\frac{\sigma^4}{n^2} + 6\frac{\sigma^2}{n}\theta^2 + \theta^4 - \left(\frac{\sigma^2}{n} + \theta^2\right)^2 = 2\frac{\sigma^4}{n^2} + 4\frac{\sigma^2}{n}\theta^2
\end{aligned}
$$

4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a geometric distribution that has pmf $f(x;\theta) = (1-\theta)^x\theta$, $x = 0, 1, 2, \ldots$, $0 < \theta < 1$, zero elsewhere. Show that $\sum_{i=1}^n X_i$ is a sufficient statistic for θ.

   Solution:

$$\prod_{i=1}^n f(x_i;\theta) = (1-\theta)^{\sum_{i=1}^n x_i} \cdot \theta^n = (1-\theta)^{\sum_{i=1}^n x_i}\theta^n \cdot 1.$$

   Applying the Neyman-Fisher Factorization Theorem (NFFT) with $k_1(\sum_{i=1}^n x_i;\theta) = \theta^n(1-\theta)^{\sum_{i=1}^n x_i}$ and $k_2(\boldsymbol{x}) = 1$, we find that $\sum_{i=1}^n x_i$ is a sufficient statistic for θ.

5. Show that the sum of the observations of a random sample of size n from a gamma distribution has pdf $f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, is a sufficient statistic for θ.

   Solution:

$$\prod_{i=1}^n f(x_i;\theta) = \frac{1}{\theta^n}e^{-\frac{\sum_{i=1}^n x_i}{\theta}} = \frac{1}{\theta^n}e^{-\frac{\sum_{i=1}^n x_i}{\theta}} \cdot 1.$$

   Applying the Neyman-Fisher Factorization Theorem (NFFT) with $k_1(\sum_{i=1}^n x_i;\theta) = \frac{1}{\theta^n}e^{-\frac{\sum_{i=1}^n x_i}{\theta}}$ and $k_2(\boldsymbol{x}) = 1$, we find that $\sum_{i=1}^n x_i$ is a sufficient statistic for θ.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a beta distribution with parameters $\alpha = \theta$ and $\beta = 2$. Show that the product $X_1 X_2 \ldots X_n$ is a sufficient statistic for θ.

   Solution: $X_1, X_2, \ldots, X_n \sim \text{Beta}(\theta, 2) \Rightarrow f(x;\theta) = \frac{\Gamma(\theta+2)}{\Gamma(\theta)\Gamma(2)} x^{\theta-1}(1-x)^{2-1} = (\theta+1)\theta x^{\theta-1}(1-x)$, thus

$$\prod_{i=1}^{n} f(x_i;\theta) = [(\theta+1)\theta]^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1} \left[\prod_{i=1}^{n}(1-x_i)\right].$$

   Applying the Neyman-Fisher Factorization Theorem (NFFT) with $k_1(\prod_{i=1}^n x_i;\theta) = [(\theta+1)\theta]^n \left(\prod_{i=1}^n x_i\right)^{\theta-1}$ and $k_2(\boldsymbol{x}) = [\prod_{i=1}^n (1-x_i)]$, we find that $\prod_{i=1}^n x_i$ is a sufficient statistic for θ.

7. Show that the product of the sample observations is a sufficient statistic for $\theta > 0$ if the random sample is taken from a gamma distribution with parameters $\alpha = \theta$ and $\beta = 6$.

   Solution: $X_1, X_2, \ldots, X_n \sim \text{Gamma}(\alpha = \theta, \beta = 6) \Rightarrow f(x;\theta) = \frac{1}{\Gamma(\theta)6^{-\theta}} x^{\theta-1} e^{-x/6}$, thus

$$\prod_{i=1}^{n} f(x_i;\theta) = \frac{1}{\Gamma^n(\theta)6^{\theta n}} \left(\prod_{i=1}^{n} x_i\right)^{\theta-1} e^{-\sum_{i=1}^n x_i/6}.$$

   Applying the Neyman-Fisher Factorization Theorem (NFFT) with $k_1(\prod_{i=1}^n x_i;\theta) = \frac{1}{\Gamma^n(\theta)6^{\theta n}} \left(\prod_{i=1}^n x_i\right)^{\theta-1}$ and $k_2(\boldsymbol{x}) = e^{-\sum_{i=1}^n x_i/6}$, we find that $\prod_{i=1}^n x_i$ is a sufficient statistic for θ.

8. What is the sufficient statistic for θ if the sample arises from a beta distribution in which $\alpha = \beta = \theta > 0$?

   Solution: $X_1, X_2, \ldots, X_n \sim \text{Beta}(\theta, \theta) \Rightarrow f(x;\theta) = \frac{\Gamma(2\theta)}{\Gamma(\theta)\Gamma(\theta)} x^{\theta-1}(1-x)^{\theta-1} = \frac{\Gamma(2\theta)}{\Gamma^2(\theta)}[x(1-x)]^{\theta-1}$, thus

$$\prod_{i=1}^{n} f(x_i;\theta) = \frac{\Gamma^n(2\theta)}{\Gamma^{2n}(\theta)} \left[\prod_{i=1}^{n} x_i(1-x_i)\right]^{\theta-1} = \frac{\Gamma^n(2\theta)}{\Gamma^{2n}(\theta)} \left[\prod_{i=1}^{n} x_i(1-x_i)\right]^{\theta-1} \cdot 1.$$

   Applying the Neyman-Fisher Factorization Theorem (NFFT) with

$$k_1(\prod_{i=1}^{n} x_i(1-x_i);\theta) = \frac{\Gamma^n(2\theta)}{\Gamma^{2n}(\theta)} \left[\prod_{i=1}^{n} x_i(1-x_i)\right]^{\theta-1} \quad \text{and } k_2(\boldsymbol{x}) = 1,$$

   we find that $\prod_{i=1}^n x_i(1-x_i)$ is a sufficient statistic for θ.