Homework 3

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Problem 1

Let

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix}, \ b = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(a) Use the LU factorization to express $A = LU$ where L is a lower triangular and U is an upper triangular matrix.

To solve this, we choose matrices L_i that represent elementary operations lower triangular matrixcs to convert A to an upper triangular matrix, as in $L_{n-1}...L_1 A = U$.

We first select L_1 as follow:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 2/3 & 13/3 \end{pmatrix}$$

Then we select L_2 as follow:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 2/3 & 13/3 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

We know that to find L we use $L = (L_2 L_1)^{-1} = L_1^{-1} L_2^{-1}$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 1/3 & 1 \end{pmatrix}, U = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

Thus we arrive at our L and U matrices.

(b) Obtain the solution x^* of the system$Ax = b$ using LU factorization of A together with forward and backward substitution.

We can solve a system using LU factorization by $L(Ux) = b \Leftrightarrow Ly = b$, and then solving via backward substitution. It is then easy to solve $Ux = y$ again via backward substitution.

First we solve

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 1/3 & 1 \end{pmatrix} y = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

This gives

$$y = \begin{pmatrix} -1 \\ 0 \\ 2/3 \end{pmatrix}$$

Then we solve

$$\begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix} x = \begin{pmatrix} -1 \\ 0 \\ 2/3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} -5/12 \\ -1/12 \\ 1/6 \end{pmatrix}$$

(c) Use Jacobi method twice, starting with $x^{(0)} = [1,1,0]^T$ to find an approximate solution. Report the error $\|x^{(k)} - x^*\|_\infty$, $k = 1,2$.

The Jacobi method tries to solve using $x^{(k)} = x^{(k-1)} + D^{-1}(b - Ax^{(k-1)})$, where D is a diagonal matrix and is arrived at using $A = L + U + D$, where L and U are lower and upper triangular matrices, respectively.

We know that

$$D = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}, \; x^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We solve

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \right)$$

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \left(\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix}$$

For the second iterate we repeat

$$x^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} \right)$$

$$x^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/4 \\ 1/4 \\ 1 \end{pmatrix} \right)$$

$$x^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \left(\begin{pmatrix} -5/4 \\ -1/4 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} + \begin{pmatrix} -5/12 \\ -1/8 \\ 0 \end{pmatrix} = \begin{pmatrix} -5/12 \\ -1/8 \\ 1/4 \end{pmatrix}$$

This is approaching the correct solution.

(d) Write code for the Gauss-Siedel method, and apply it to find the solution of the system $Ax = b$ with 10^{-5} digits of accuracy. That is $\|x^k - x^*\| \leq 10^{-5}$ where x^* is the solution of the system $x^* = A^{-1}b$ and x^k is the kth iterate of Gauss-Seidel.

The formula for Gauss-Seidel is given as $Dx^{(k+1)} = -Lx^{(k+1)} - Ux^{(k)} + b, k = 0, 1, 2, ...$ and for $n = 3$ we have the following clear form:

$$\begin{aligned} a_{11}x_1^{(k+1)} &= -a_{12}x_2^{(k)} - a_{13}x_3^{(k)} + b_1 \\ a_{22}x_2^{(k+1)} &= -a_{21}x_1^{(k)} - a_{23}x_3^{(k)} + b_2 \\ a_{33}x_3^{(k+1)} &= -a_{31}x_1^{(k)} - a_{32}x_2^{(k)} + b_3 \end{aligned}$$

So we implement the above in a functoin and then call it iteratively.

```
import math

# function to do Gauss Seidel
def gauss_seidel(a, x ,b):
    #Finding length of a, here it's 3
    n = len(a)
    # for loop for 3 times as to calculate x, y , z
    for j in range(0, n):
        # temp variable d to store b[j]
        d = b[j]

        # to calculate  xi, yi, zi
        for i in range(0, n):
            if(j != i):
                d-=a[j][i] * x[i]
        x[j] = d / a[j][j]
    return x
```

We have defined the function, so now we apply it to our data.

```
x_star = [-5/12,-1/12,1/6]
x = [1, 1, 0]
a = [[3, -1, 1],[0, 2, 1],[-1, 1, 4]]
b = [-1,0,1]


#loop i times
for i in range(0, 10):
    x = gauss_seidel(a, x, b)
    print(x)
```

```
## [0.0, 0.0, 0.25]
## [-0.4166666666666667, -0.125, 0.17708333333333331]
## [-0.43402777777777773, -0.08854166666666666, 0.16362847222222224]
## [-0.41739004629629634, -0.08181423611111112, 0.16610604745370372]
## [-0.4159734278549383, -0.08305302372685186, 0.1667698989679784]
## [-0.4166076408982768, -0.0833849494839892, 0.1666943271464281]
## [-0.41669309221013906, -0.08334716357321405, 0.16666351784076874]
## [-0.4166702271379943, -0.08333175892038437, 0.16666538294559752]
## [-0.4166657139553273, -0.08333269147279876, 0.16666674437936785]
```

```
## [-0.41666647861738887, -0.08333337218968392, 0.16666672339307378]
```

We see that the results are similar to that of the Jacobi method, which is also similar to the direct solution.