Homework 2

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2022-09-17

Problem 1

Consider the quadratic function

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 5x_3^2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3 + x_1 - x_2$$

(a) Choose a matrix A and vector b so that $x = (x_1, x_2, x_3)$, and $f(x) = x^T Ax + b \cdot x$
(b) Choose another matrix B such that $A \neq B$ and $B = B^T$ so that $f(x) = x^T B + b^T x$

Response

(a) We choose such that

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix}$$

and $b = (1, -1, 0)^T$

(b) We choose such that

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

and $b = (1, -1, 0)^T$. We see that $B = B^T$.

Problem 2

Use the Spectral Decomposition Theorem and determine the eigenvalue decomposition of the matrix A given by

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

Response

The spectral decomposition gives us $A = QDQ^T$ where Q is a matrix of orthonormal eigenvectors and D is a diagonal matrix of eigenvalues, given that A is symmetric real $n \times n$ matrix.

We first solve for the eigenvalues, so we solve $det(A - \lambda I) = 0$.

$$\begin{vmatrix} 3-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

Which gives us

$$(3-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} - (2-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(2-\lambda) - 1 - (2-\lambda)(3-\lambda)^2 = 0$$

Solving for this, gives us three eigenvalues: $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 1$.

To solve for the eigenvectors, we solve the $(A - \lambda I)v = 0$.

For $\lambda_1 = 4$ (and then for the others), we have the following matrix for which we find the nullspace.

$$\begin{pmatrix} 3-4 & 1 & 0 \\ 1 & 2-4 & 1 \\ 0 & 1 & 3-4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3-3 & 1 & 0 \\ 1 & 2-3 & 1 \\ 0 & 1 & 3-3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3-1 & 1 & 0 \\ 1 & 2-1 & 1 \\ 0 & 1 & 3-1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow v = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

So the decomposition is

$$A = QDQ^T \Leftrightarrow \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}^T$$

Problem 3

Find the linear and quadratic approximation of the following function $f(x) = exp(x_1^2+x_2^2+x_3^2)$ at $\bar{x} = (0,0,0)^T$.

Response

The local linear approximation is given by

$$f(x) \approx f(x_0) + \nabla f(x_0)^T(x - x_0)$$

and the local quadratic approximation is given by

$$f(x) \approx f(x_0) + \nabla f(x_0)^T(x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0)$$

First we calculate the derivatives. We see that $\nabla exp(x_1^2 + x_2^2 + x_3^2) = exp(x_1^2 + x_2^2 + x_3^2) * (2x_1, 2x_2, 2x_3)^T$

Then we calculate $\nabla^2 exp(x_1^2 + x_2^2 + x_3^2)$ which gives the below:

$$\begin{pmatrix} (4e^(x_1^2+x_2^2+x_3^2)x_1^2+2e^(x_1^2+x_2^2+x_3^2) & 4e^(x_1^2+x_2^2+x_3^2)x_1x_2 & 4e^(x_1^2+x_2^2+x_3^2)x_1x_3 \\ 4e^(x_1^2+x_2^2+x_3^2)x_1x_2 & 4e^(x_1^2+x_2^2+x_3^2)x_2^2+2e^(x_1^2+x_2^2+x_3^2) & 4e^(x_1^2+x_2^2+x_3^2)x_2x_3 \\ 4e^(x_1^2+x_2^2+x_3^2)x_1x_3 & 4e^(x_1^2+x_2^2+x_3^2)x_2x_3 & 4e^(x_1^2+x_2^2+x_3^2)x_3^2+2e^(x_1^2+x_2^2+x_3^2)) \end{pmatrix}$$

Which we can evaluate both of the above at $(0, 0, 0)$ to get the linear and the quadratic approximations.

We get

$$\nabla f = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\nabla^2 f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Therefore the linear approximation is

$$L = 1$$

and the quadratic approximation is

$$Q = 1 + \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1 + 2 \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1 + 2x_1^2 + 2x_2^2 + 2x_3^2$$

Problem 4

Determine whether the following quadratic function has a min, max, or saddle point. Explain why.

$$f(x_1, x_2) = 2x_1 - x_2^2 - x_1x_2 + 5x_2 - 1$$

Response

To determine this, we consider the matrix representation of this function. This is given by

$$f(x) = x^T Ax + b^T x$$

with

$$A = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}$$

In the case where there is no cross term, we simply inspect the sign of the elements on the diagonal. If these elements are both positive, then the function is convex/opens up and has a min. If these are negative, the function is concave/opens down and has a max. If one is positive and one is negative, then it is a saddle point.

In this case we do have a cross term, so we must determine if the matrix A is positive definite, negative definite, or neither. If A is positive definite, then the function is convex, and if A is negative definite, the function is concave. This requires us to check the sign of the eigenvalues of A.

To find the eigenvalues, we solve $det(A - \lambda I) = 0$.

$$\begin{vmatrix} 2-\lambda & -1 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

This gives us: $(2-\lambda)(-1-\lambda)=0$, which we solve to get $\lambda=-1$ and $\lambda=2$. We see that the eigenvalues are of different signs, so the case is a saddle point.