Homework #1

Nathan Bick

1. Reading: Sections 1.1 – 1.6

2. Exercises: 1.17, 1.19, 1.22, 1.34

Exercise 1.17: Let $p_1, p_2, \dots, p_n$ be non-negative numbers such that $p_1 + p_2 + \dots + p_n = 1$, and let $\Omega = \{w_1, w_2, \dots, w_n\}$, with $\mathcal{F}$ the power set of Ω. Show that the function Q given by $Q(A) = \sum_{i: w_i \in A} p_i$ for $A \in \mathcal{F}$ is a probability measure on $(\Omega, \mathcal{F})$. Is Q a probability measure on $(\Omega, \mathcal{F})$ if $\mathcal{F}$ is not the power set of Ω but merely some event space of subsets of Ω?

Answer: (a) For Q to be a probability measure it must satisfy the Axioms of Probability. Let's check each in turn.

1. $Q(A) \geq 0$ for all A, $A \in \mathcal{F}$.

$Q(A) = \sum_{i; w_i \in A} p_i \geq 0$ because all p_i are non-negative and so any sum of p_i is non-negative.

2. $Q(\emptyset) = 0$, $Q(\Omega) = 1$

$Q(\Omega) = Q(\{w_1, w_2, \dots, w_n\}) = \sum_{i=1}^{n} p_i = 1$

$Q(\emptyset) = \sum 0 = 0$.

3. $A_1, \dots, A_n \in \mathcal{F}$ and $A_i \cap A_j \neq \emptyset$ then

$$Q(A_1 \cup \dots \cup A_n) = P(A_1) + \dots + P(A_n).$$

Let $A_1, \dots, A_n$ be disjoint s.t. $A_1, \dots, A_n \in \mathcal{F}$.

$Q(A_1 \cup \dots \cup A_n) = Q\left(\bigcup_{i=1}^{n} A_i\right)$. If $B = \bigcup_{i=1}^{n} A_i$,

then $Q(B) = \sum_{i: w_i \in B} p_i = \sum_{j: w_j \in \bigcup_{i=1}^{n} A_i} p_i$

Because $A_1, \dots, A_n$ are disjoint, then each w_j is in one A_i and only one. so $w_j \in \bigcup_{i=1}^{n} A_i \Rightarrow w_j \in A_i$ for unique i.

so $\sum_{j: w_j \in \bigcup_{i=1}^{n} A_i} p_i = \sum_{i=1}^{n} \left(\sum_{j: w_j \in A_i} p_i \right) = \sum_{i=1}^{n} Q(A_i)$

so $Q\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} Q A_i$.

Q satisfies all 3 components of definition of a probability measure.

(b). Take as example $\mathcal{F} = \{\emptyset, A, \Omega - A, \Omega\}$.

Then $Q(\emptyset) = \sum 0 = 0$. $Q(A) = \sum_{w_i \in A} p_i \geq 0$.

$Q(\Omega) = \sum_{i=1}^{n} p_i = 1$. $Q(\Omega - A) = Q(\Omega) - Q(\Omega \cap A)$

$= Q(\Omega) - Q(A)$

$= 1 - \sum_{i: w_i \in A} p_i$.

Also, $Q(\Omega - A \cup \Omega) = Q(\Omega)$

$Q(\emptyset \cup A) = Q(A)$

$Q(\emptyset \cup (\Omega - A)) = Q(\Omega - A)$

$Q(\emptyset \cup \Omega) = Q(\Omega)$

$Q(\Omega - A \cup A) = Q(\Omega)$

$Q(A \cup \Omega) = Q(\Omega)$

$\Rightarrow Q(\cup A_i) = \sum Q_i$.

Exercise 1.19: If $A, B \in \mathcal{F}$, show that $P(A-B) = P(A) - P(A \cap B)$

Proof: A is the ~~disjoint~~ union of disjoint sets $A-B$ and $A \cap B$.

$$A = (A-B) \cup (A \cap B) \Rightarrow P(A) = P((A-B) \cup (A \cap B))$$

$$\Rightarrow P(A) = P(A-B) + P(A \cap B)$$

$$\Rightarrow P(A-B) = P(A) - P(A \cap B).$$

Exercise 1.22: A fair coin is tossed 10 times. Describe the appropriate probability space in detail for the two cases when (a) the outcome of every toss is of interest, (b) only the total number of tails is of interest.

In the first case your event space should have $2^{2^{10}}$ events, but in the second case it need only have 2^{11} events.

Answer:

(a) A probability space is defined as $(\Omega, \mathcal{F}, P)$. Here,

$\Omega = \{(t_1, t_2, \ldots, t_{10}) : t_i \in \{T, H\}\}$, or the set of ordered 10-tuples of Ts and Hs.

$\mathcal{F} = \mathcal{P}(\Omega)$ = the power set of Ω = the set of all subsets of Ω.

P = each 10-tuple in Ω is of equal probability, so

$$P((t_1, \ldots, t_{10})) = \frac{1}{2^{10}}$$

(b) A probability space is defined as $(\Omega, \mathcal{F}, P)$, and here

$\Omega = \{0, 1, 2, \ldots, 10\}$.

$\mathcal{F} = \mathcal{P}(\Omega)$ = the power set of $\{0, 1, 2, \ldots, 10\}$.

P. There are $\binom{10}{n}$ ways to throw n tails on 10 coins.
So each $n \in \{0, 1, \ldots, 10\}$ is of probability $\frac{1}{2^{10}}\binom{10}{n}$.

Exercise 1.34: If $(\Omega, \mathcal{F}, P)$ is a probability space and A, B, C are events, show that $P(A \cap B \cap C) = P(A|B \cap C) \cdot P(B|C) \cdot P(C)$ so long as $P(B \cap C) > 0$.

Proof: Note that $P(X|Y) = \dfrac{P(X \cap Y)}{P(Y)}$ if $X, Y \in \mathcal{F}$, $P(Y) > 0$.

Also $P(B \cap C) > 0 \Rightarrow P(B) > 0$ and $P(C) > 0$.

$$P(A|B\cap C)P(B|C)P(C) = P(A|B\cap C)\frac{P(B\cap C)}{P(C)}P(C)$$

$$= P(A|B\cap C)P(B\cap C) = \frac{P(A\cap B\cap C)}{P(B\cap C)}P(B\cap C)$$

$$= P(A\cap B\cap C).$$

Problem 3: Prove Boole's inequality:

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Proof: By induction.

1 $P(A_1) \leq P(A_1)$ by definition.

2 $P(A_1 \cup A_2)$. If A_1, A_2 disjoint, then by third axiom of probability, $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ $\Rightarrow P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$. If A_1, A_2 are not disjoint, then $A_1 \cap A_2 \neq \emptyset$. Since $P(A_1 \cap A_2) \geq 0$ by first axiom of probability, and since $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$, then $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$

Assume $n-1$ case, so $P\left(\bigcup_{i=1}^{n-1} A_i\right) \leq \sum_{i=1}^{n-1} P(A_i)$.

n $P\left(\bigcup_{i=1}^{n-1} A_i \cup A_n\right)$. If $\bigcup_{i=1}^{n-1} A_i$, A_n are disjoint then by third axiom of probability, $P\left(\bigcup_{i=1}^{n-1} A_i \cup A_n\right) = P\left(\bigcup_{i=1}^{n-1} A_i\right) + P(A_n)$ $\leq \sum_{i=1}^{n-1} P(A_i) + P(A_n) = \sum_{i=1}^{n} A_n$.

If $\bigcup_{i=1}^{n-1} A_i$, A_n are not disjoint, then

$P\left(\bigcup_{i=1}^{n-1} A_i \cup A_n\right) = P\left(\bigcup_{i=1}^{n-1} A_i\right) + P(A_n) - P\left(\bigcup_{i=1}^{n-1} A_i \cap A_n\right)$

and since $\bigcup_{i=1}^{n-1} A_i \cap A_n \neq \emptyset$, its probability ≥ 0.

So $P\left(\bigcup_{i=1}^{n-1} A_i \cup A_n\right) \leq P\left(\bigcup_{i=1}^{n-1} A_i\right) + P(A_n) =$
$\leq \sum_{i=1}^{n-1} P(A_i) + P(A_n) = \sum_{i=1}^{n} P(A_i)$