MATH 503: Mathematical Statistics

Lecture 7: Hypothesis Testing II
Reading: C&B Chp. 8, HMC Sec. 6.3

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Today's Topics

- Maximum likelihood tests
  - Likelihood Ratio Test
  - Wald Test
  - (Rao's) Score Test
- Uniformly most powerful (UMP) tests
  - Monotone Likelihood Ratio

Likelihood Ratio Test (LRT)

- Let $X_1, \dots, X_n$ be iid with pdf $f(x;\theta)$
- Consider $H_0: \theta \in \Theta_0$ vs. $H_1: \theta \in \Theta_0'$
- Let $\hat{\theta}$ denote the MLE of θ
- Consider the ratio of two likelihoods, namely

$$\Lambda = \frac{\sup_{\Theta_0} L(\theta \mid x)}{\sup_{\Theta} L(\theta \mid x)}$$

largest likelihood value under H_0

largest likelihood value in general

- Question: what are the bounds for Λ?

$[0,1]$

Likelihood Ratio Test (cont.)

- By definition,

  Λ close to 1 if H_0 true

  Λ "small" if H_1 true

- For specified significance level α, decision rule says to reject H_0 in favor of H_1 if $\Lambda \le c$, where c chosen st. $\alpha = P_{\theta_0}[\Lambda \le c]$

ie if the likelihood ratio test statistic is statistically significantly small

Steps to Performing LRT

- Identify hypotheses
- Determine likelihood function, $L(\theta; \boldsymbol{x})$
- Find associated MLEs $\hat{\theta} \in \Theta$, and $\hat{\theta}_0 \in \Theta_0$
- Determine likelihood ratio Λ (simplify as necessary)
- Determine appropriate decision rule based on $\Lambda \leq c$

Example 1

Let $X_1, \ldots, X_n$ be iid Exponential(θ). Determine the appropriate LRT for testing $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$.

$f_{X_i}(x) = \frac{1}{\theta} e^{-x/\theta} \Rightarrow L(\theta; \underline{x}) = \frac{1}{\theta^n} e^{-\sum x_i/\theta}$

Under $H_0: \theta = \theta_0$, $L(\theta_0; \underline{x}) = \frac{1}{\theta_0^n} e^{-\sum x_i/\theta_0}$

In general, the likelihood is maximized at MLE, $\hat{\theta}$:

$\log L(\theta; \underline{x}) = -n \log \theta - \frac{\sum x_i}{\theta}$

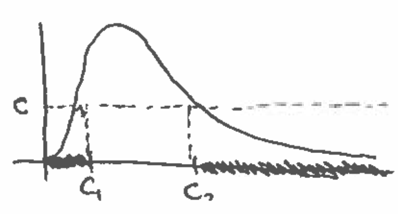
$\frac{\partial \log L(\theta; \underline{x})}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2} = 0 \Rightarrow \sum x_i = n\theta$

$\hat{\theta} = \bar{X}$

$$\Lambda = \frac{\frac{1}{\theta_0^n} e^{-\sum x_i/\theta_0}}{\frac{1}{\bar{x}^n} e^{-\sum x_i/\bar{x}}} = \left(\frac{\bar{x}}{\theta_0}\right)^n \exp\left(-\frac{\sum x_i}{\theta_0} + n\right) = \left(\frac{\bar{x}}{\theta_0}\right)^n e^{-n\left(\bar{x}/\theta_0 - 1\right)} = t^n e^{-nt} e^n$$

where $t = \bar{x}/\theta_0$

$\Lambda = t^n e^{-nt} e^n$



$\Rightarrow \Lambda \leq c$ when $\bar{X} \leq \theta_0 c_1$ or $\bar{X} \geq \theta_0 c_2$

Example 2

- Let $X_1, \ldots, X_n$ be iid with pdf
  $$f(x \mid \theta) = e^{-(x-\theta)}, \quad x \geq \theta, \quad -\infty < \theta < \infty$$
- Consider $H_0: \theta \leq \theta_0$ vs. $H_1: \theta > \theta_0$. Determine the appropriate LRT.

SEE ATTACHED

Example 3

Let $X_1, \ldots, X_n$ be iid Normal(θ, σ^2), $-\infty < \theta < \infty$ unknown, $\sigma^2 > 0$ known. Consider $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$. Determine the appropriate LRT.

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Example 2

$X_1, \ldots, X_n \quad f(x;\theta) = e^{-(x-\theta)}, \quad x \geq \theta$

$$\mathcal{L}(\theta;\underline{x}) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)} I_{[\theta,\infty)}(x_i)$$

$$= e^{-\sum_{i=1}^{n} x_i + n\theta} \; I_{[\theta,\infty)}(x_{(1)})$$

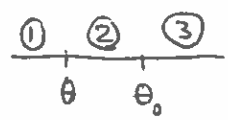
$$= \begin{cases} e^{-\sum_1^n x_i} e^{n\theta} & x_{(1)} \geq \theta \\ 0 & \text{o/w.} \end{cases}$$

$\Rightarrow \hat{\theta} = X_{(1)}$ is MLE.

Considering $H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$,

$$\Lambda = \frac{\sup_{\Theta_0} \mathcal{L}(\theta;\underline{x})}{\sup_{\Theta} \mathcal{L}(\theta;\underline{x})} = \frac{\sup_{\Theta_0} \mathcal{L}(\theta;\underline{x})}{e^{-\sum_1^n x_i} e^{n x_{(1)}}}.$$

What is $\sup_{\Theta_0} \mathcal{L}(\theta;\underline{x})$?

$H_0: \theta \leq \theta_0 \Rightarrow$ consider cases  where outcome sample can lie

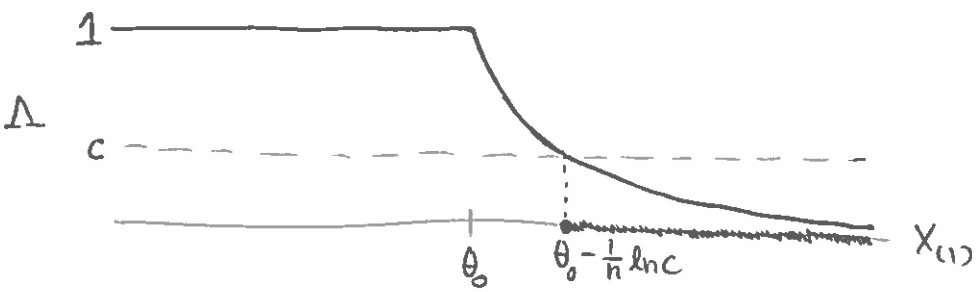
① $\Rightarrow \underline{x} < \theta < \theta_0$. Then $\mathcal{L}(\theta;\underline{x}) = 0$ because x's are less than θ.

② $\Rightarrow \theta < \underline{x} < \theta_0$. Then $\mathcal{L}(\theta;\underline{x})$ maximized at $x_{(1)}$

③ $\Rightarrow \theta < \theta_0 < \underline{x}$. We are only interested in the likelihood under $H_0: \theta < \theta_0$. Since $\mathcal{L}(\theta;\underline{x})$ is increasing function wrt. θ, under H_0, $\mathcal{L}(\theta;\underline{x})$ is maximized at θ_0.

$$\therefore \Lambda = \begin{cases} \dfrac{\cancel{e^{-\sum x_i}}\,\cancel{e^{n x_{(1)}}}}{\cancel{e^{-\sum x_i}}\,\cancel{e^{n x_{(1)}}}} = 1, & \theta \leq x_{(1)} \leq \theta_0 \\[2ex] \dfrac{\cancel{e^{-\sum x_i}}\, e^{n\theta_0}}{\cancel{e^{-\sum x_i}}\, e^{n x_{(1)}}} = e^{n(\theta_0 - x_{(1)})}, & \theta \leq \theta_0 \leq x_{(1)} \end{cases}$$

Example 2 (cont.)



$$\Lambda \le c$$
$$e^{n(\theta_0 - X_{(1)})} \le c$$
$$n(\theta_0 - X_{(1)}) \le \ln c$$
$$\theta_0 - X_{(1)} \le \frac{1}{n}\ln c$$
$$X_{(1)} \ge \theta_0 - \frac{1}{n}\ln c$$

Example 3

$X_1, \ldots, X_n \sim N(\theta, \sigma^2)$ where σ^2 known

$$f(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(x_i-\theta)^2}$$

$$\mathcal{L}(\theta; \underline{x}) = \prod_{i=1}^{n} f(x_i; \theta) = (2\pi\sigma^2)^{-n/2} \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \theta)^2\right)$$

$$\ln \mathcal{L}(\theta; \underline{x}) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \theta)^2$$

$$\frac{\partial \ln \mathcal{L}}{\partial \theta} = +\frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \theta) = 0$$

$$\sum x_i = n\theta$$

$\hat{\theta} = \bar{x}$ is the MLE

$H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$

$$\Rightarrow \Lambda = \frac{\cancel{(2\pi\sigma^2)^{-n/2}} \exp\left(\frac{-1}{2\sigma^2} \sum (x_i - \theta_0)^2\right)}{\cancel{(2\pi\sigma^2)^{-n/2}} \exp\left(\frac{-1}{2\sigma^2} \sum (x_i - \bar{x})^2\right)} = e^{\frac{-1}{2\sigma^2}\left[\sum_1^n (x_i-\theta_0)^2 - \sum_1^n (x_i - \bar{x})^2\right]}$$

where $\sum_{i=1}^{n} (x_i - \theta_0)^2 - \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} \left[\cancel{x_i^2} - 2x_i\theta_0 + \theta_0^2 - (\cancel{x_i^2} - 2x_i\bar{x} + \bar{x}^2)\right]$

$$= -2n\bar{x}\theta_0 + n\theta_0^2 + 2n\bar{x}^2 - n\bar{x}^2$$

$$= n\bar{x}^2 - 2n\bar{x}\theta_0 + n\theta_0^2$$

$$= n(\bar{x} - \theta_0)^2$$

$$\therefore \Lambda = \exp\left(\frac{-1}{2\sigma^2}\left\{n(\bar{x}-\theta_0)^2\right\}\right) = \exp\left(\frac{-n(\bar{x}-\theta_0)^2}{2\sigma^2}\right) = \exp\left(\frac{-1}{2}\left(\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}}\right)^2\right)$$

Example 3 (cont.)

$$\Rightarrow \Lambda = \exp\left[\frac{-1}{2} \left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \right)^2 \right] \le c$$

$$\ln \Lambda = \frac{-1}{2} \left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \right)^2 \le \ln c$$

$$-2 \ln \Lambda = \underbrace{\left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \right)^2}_{\chi^2_1 \text{ under } H_0} \ge -2 \ln c \doteq k \quad \text{where } k \text{ chosen so that } P_{\theta_0}(-2\ln\Lambda \ge k) = \alpha$$

$$\therefore k = \chi^2_1(\alpha)$$

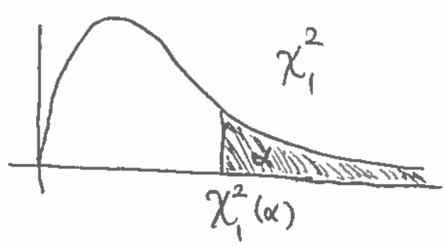
Theorem

- Assume that the appropriate regularity conditions hold:
  - Pdfs are distinct, i.e. $\theta \neq \theta' \Rightarrow f(x_i;\theta) \neq f(x_i;\theta')$
  - Pdfs have common support for all θ
  - The point θ_0 is an interior point in Ω
  - Pdf is twice differentiable as a function of θ
  - Integral $\int f(x;\theta)dx$ can be differentiated twice under the integral sign as a function of θ
  - The pdf $f(x;\theta)$ is three times differentiable as a function of θ. Further, for all $\theta \in \Omega$, there exists a constant c and function $M(x)$ s.t. $\left|\frac{\partial^3}{\partial\theta^3}\log f(x;\theta)\right| \leq M(x)$, with $E_{\theta_0}[M(x)] < \infty$, for all $\theta_0 - c < \theta < \theta_0 + c$ and all x in the support of X.

Theorem (cont.)

- Under H_0, $-2\log\Lambda \rightarrow \chi_1^2$ in distribution.
- Decision rule: Reject H_0 in favor of H_1 if

$$\chi_L^2 = -2\log\Lambda \geq \chi_1^2(\alpha)$$



Use chi-square chart to find critical value.

Wald Test

- The Wald Test statistic: $\chi_W^2 = \left[\sqrt{nI(\hat{\theta})}(\hat{\theta} - \theta_0)\right]^2$
- Taylor expansion implies
  $-2\log\Lambda = 2\left(l(\hat{\theta}) - l(\theta_0)\right) = \left[\sqrt{nI(\theta_0)}(\hat{\theta} - \theta_0)\right]^2 + \underbrace{R_n^*}_{\xrightarrow{p} 0}$

  where $I(\hat{\theta}) \xrightarrow{p} I(\theta_0)$ $\therefore$ $\chi_W^2 = \left[\sqrt{nI(\hat{\theta})}(\hat{\theta} - \theta_0)\right]^2 \xrightarrow{d} \chi_1^2$
- Decision rule: Reject H_0 in favor of H_1 if $\chi_W^2 \geq \chi_1^2(\alpha)$
- Under H_0, $\chi_W^2 - \chi_L^2 \xrightarrow{p} 0$

Rao's Score Test

- $\chi_R^2 = \left(\frac{l'(\theta_0)}{\sqrt{nI(\theta_0)}}\right)^2$, where scores are
  $S(\theta) = \left(\frac{\partial \log f(X_1;\theta)}{\partial\theta}, \ldots, \frac{\partial \log f(X_n;\theta)}{\partial\theta}\right)'$ and
  $l'(\theta_0) = \sum_{i=1}^{n} \frac{\partial \log f(X_i;\theta)}{\partial\theta}$
- Decision rule: Reject H_0 in favor of H_1 if
  $$\chi_R^2 \geq \chi_1^2(\alpha)$$

Example 4

- Let $X_1, \dots, X_n$ be a random sample from Poisson(θ), $\theta > 0$. Test $H_0: \theta = 2$ vs. $H_1: \theta \neq 2$ using (a) $-2 \log \Lambda$, (b) a Wald-test statistic, (c) Rao's score statistic.

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Uniformly Most Powerful Tests

- The critical region C is a uniformly most powerful (UMP) critical region of size α for testing the simple hypothesis H_0 against an alternative composite hypothesis H_1 if the set C is a best critical region of size α for testing H_0 against each simple hypothesis in H_1.
- A test defined by this critical region C is called a uniformly most power (UMP) test, with significance level α, for testing the simple hypothesis H_0 against the alternative composite hypothesis H_1.

Example 4

$X_1, \dots, X_n \sim \text{Poisson}(\theta) \therefore f(x;\theta) = \dfrac{e^{-\theta}\theta^{x}}{x!}$

$$\mathcal{L}(\theta;\underline{x}) = \frac{e^{-n\theta}\,\theta^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!}$$

$$\ln \mathcal{L}(\theta;\underline{x}) = -n\theta + (\Sigma x_i)\ln\theta - \sum_{i=1}^{n}\ln(x_i!)$$

$$\frac{\partial \ln \mathcal{L}}{\partial\theta} = -n + \frac{\Sigma x_i}{\theta} = 0$$

$$\frac{\Sigma x_i}{\theta} = n$$

$$\hat{\theta} = \frac{\Sigma x_i}{n} = \bar{x} \text{ is MLE.}$$

Consider $H_0: \theta = 2$ vs. $H_1: \theta \neq 2$.

(A) $$\Lambda = \frac{\mathcal{L}(2;\underline{x})}{\mathcal{L}(\bar{x};\underline{x})} = \frac{e^{-2n}\,2^{\Sigma x_i}}{\cancel{\prod x_i!}} \cdot \frac{\cancel{\prod x_i!}}{e^{-n\bar{x}}\,\bar{x}^{\Sigma x_i}} = e^{-n(2-\bar{x})}\left(\frac{2}{\bar{x}}\right)^{\Sigma x_i} \le c$$

$$\log\Lambda = -n(2-\bar{x}) + (\Sigma x_i)(\log 2 - \log\bar{x}) \le \log c$$

$$\boxed{-2\log\Lambda = 2n(2-\bar{x}) - 2(\Sigma x_i)(\log 2 - \log\bar{x}) \ge -2\log c = \chi^2_1(\alpha)}$$

(B) $$\log f(x_i;\theta) = -\theta + x_i\log\theta - \log(x_i!)$$

$$\frac{\partial \log f(x_i;\theta)}{\partial\theta} = -1 + \frac{x_i}{\theta}$$

$$\frac{\partial^2 \log f(x_i;\theta)}{\partial\theta^2} = \frac{-x_i}{\theta^2} \Rightarrow I(\theta) = -E\left(\frac{\partial^2 \log f}{\partial\theta^2}\right) = \frac{E(x_i)}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

$$\Rightarrow \boxed{\chi^2_W = \left[\sqrt{nI(\hat{\theta})}\,(\hat{\theta}-\theta_0)\right]^2 = \left[\sqrt{\frac{n}{\hat{\theta}}}\,(\hat{\theta}-\theta_0)\right]^2 = \frac{n(\bar{x}-2)^2}{\bar{x}} = \left(\frac{\bar{x}-2}{\sqrt{\bar{x}/n}}\right)^2}$$

Example 4 (cont.)

ⓒ $$\chi_R^2 = \left(\frac{\ell'(\theta_0)}{\sqrt{nI(\theta_0)}}\right)^2 \text{ where } \ell'(\theta_0) = \sum_{i=1}^{n} \frac{\partial \log f(x_i;\theta_0)}{\partial\theta}$$

$$= \sum_{i=1}^{n}\left(-1 + \frac{x_i}{\theta_0}\right)$$

$$= -n + \frac{\Sigma x_i}{\theta_0} = -n + \frac{\Sigma x_i}{2}$$

$$I(\theta_0) = \frac{1}{\theta_0} = \frac{1}{2}$$

$$\therefore \chi_R^2 = \left(\frac{-n + \frac{\Sigma x_i}{2}}{\sqrt{n/2}}\right)^2 = \frac{2}{n}\left(\frac{\Sigma x_i}{2} - n\right)^2 = \frac{2}{n}\cdot\left(\frac{\Sigma x_i - 2n}{2}\right)^2$$

$$= \frac{\cancel{2}}{n}\cdot\frac{(\Sigma x_i - 2n)^2}{\cancel{4}\,2}$$

$$= \frac{[n(\bar{x}-2)]^2}{2n} = \frac{n^{\cancel{2}}(\bar{x}-2)^2}{2\cancel{n}}$$

$$= \frac{n(\bar{x}-2)^2}{2}$$

Notes re. UMP Tests

- UMP tests don't always exist
- When they do exist, Neymann-Pearson can help determine them.
- UMP tests are based on sufficient statistics

How do we determine the best critical region?

Neymann-Pearson Thm.: Let $X_1, \dots, X_n$ (n, *a* positive fixed integer) denote a random sample from a distribution that has pdf/pmf $f(x;\theta)$. Then the likelihood of $X_1, \dots, X_n$ is

$$L(\theta; \boldsymbol{x}) = \prod_{i=1}^{n} f(x_i;\theta).$$

Let θ' and θ'' be distinct fixed values of θ s.t. $\Omega = \{\theta: \theta = \theta', \theta''\}$, and let k be a positive number.

$H_0: \theta = \theta'$ vs. $H_1: \theta = \theta''$

Neymann-Pearson Thm. (cont.)

Let C be a subset of the sample space s.t.

a) $\frac{L(\theta';x)}{L(\theta'';x)} \leq k$, for each point $x \in C$

b) $\frac{L(\theta';x)}{L(\theta'';x)} \geq k$, for each point $x \in C^c$

c) $\alpha = P_{H_0}[X \in C]$

Then C is a best critical region of size α for testing the simple hypothesis $H_0: \theta = \theta'$ vs. $H_1: \theta = \theta''$.

Example 5

Let $X_1, \ldots, X_n$ be a random sample from a distribution that is $N(0, \theta)$, where the variance θ is an unknown positive number. Show that there exists a UMP test with significance level α for testing $H_0: \theta = \theta'$ vs. $H_1: \theta > \theta'$.

$f(x_i;\theta) = (2\pi\theta)^{-\frac{1}{2}} e^{-\frac{1}{2\theta}x_i^2} \Rightarrow \mathcal{L}(\theta;\underline{x}) = (2\pi\theta)^{-\frac{n}{2}} e^{\frac{-1}{2\theta}\sum_1^n x_i^2}$

Consider $H_0: \theta = \theta'$ vs. $H_1: \theta = \theta''$ (where $\theta'' > \theta'$)

$$\frac{\mathcal{L}(\theta';\underline{x})}{\mathcal{L}(\theta'';\underline{x})} = \frac{(2\pi\theta')^{-\frac{n}{2}} \exp\left(\frac{-1}{2\theta'}\sum x_i^2\right)}{(2\pi\theta'')^{-\frac{n}{2}} \exp\left(\frac{-1}{2\theta''}\sum x_i^2\right)} = \left(\frac{\theta''}{\theta'}\right)^{\frac{n}{2}} \exp\left[\frac{\sum x_i^2}{2\theta''} - \frac{\sum x_i^2}{2\theta'}\right] \leq k$$

$$\therefore \exp\left(\frac{\theta'\sum x_i^2 - \theta''\sum x_i^2}{2\theta'\theta''}\right) \leq k_1$$

$$\left(\sum x_i^2\right)\underbrace{\left(\theta' - \theta''\right)}_{<0} \leq k_2$$

$\sum x_i^2 \geq k_3$ where $\mathbb{P}(\sum x_i^2 \geq k_3) = \alpha$ for some k_3

$X_i \sim N(0,\theta) \Rightarrow \sum_{i=1}^n \frac{X_i^2}{\theta} \sim \chi^2_n \Rightarrow \mathbb{P}_{H_0}(\sum X_i^2 \geq k_3) = \mathbb{P}\left(\sum \frac{X_i^2}{\theta'} \geq \frac{k_3}{\theta'}\right) = \alpha$

$\therefore \frac{k_3}{\theta'} = \chi^2_n(\alpha)$ or $k_3 = \theta' \chi^2_n(\alpha)$

Example 6

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\theta, 1)$, where θ is unknown. Does a UMP test of $H_0: \theta = \theta'$ vs. $H_0: \theta \neq \theta'$ exist?

SEE ATTACHED

Example 7

Let $X_1, \ldots, X_{10}$ be a random sample of size 10 from a Poisson(θ) distribution. Find a best critical region for testing $H_0: \theta = 0.1$ vs. $H_1: \theta = 0.5$. Is this region uniformly most powerful for $H_0: \theta = 0.1$ vs. $H_1: \theta > 0.1$?

$$f(x) = \frac{e^{-\theta}\theta^x}{x!} \Rightarrow \mathcal{L}(\theta; \underline{x}) = \frac{e^{-n\theta}\theta^{\sum_1^n x_i}}{\prod_1 x_i!}$$

Consider general test $H_0: \theta = \theta'$ vs. $H_1: \theta = \theta''$ where $\theta'' > \theta'$.

By Neymann–Pearson Theorem,

$$\frac{\mathcal{L}(\theta'; \underline{x})}{\mathcal{L}(\theta''; \underline{x})} = \frac{e^{-n\theta'}\theta'^{\sum x_i}}{\cancel{\prod x_i!}} \cdot \frac{\cancel{\prod x_i!}}{e^{-n\theta''}\theta''^{\sum x_i}} = \left(\frac{\theta'}{\theta''}\right)^{\sum x_i} e^{n(\theta'' - \theta')} \leq k$$

(taking ln of both sides)

$$\therefore (\sum x_i)\underbrace{(\ln\theta' - \ln\theta'')}_{<0} + n\underbrace{(\theta'' - \theta')}_{>0} \leq \ln k \doteq k_1$$

$$\Rightarrow \sum_1^n x_i \geq k_2$$

Because this critical region was derived for any $\theta'' > \theta'$, the region is UMP, i.e. this is a UMP test for $H_0: \theta = \theta'$ vs $H_1: \theta > \theta'$ where k_2 satisfies $P_{\theta'}(\sum_1^n x_i \geq k_2) = \alpha$.

Example 6

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}$$

$$\mathcal{L}(\theta; \underline{x}) = (2\pi)^{-n/2} \exp\left(\frac{-1}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right)$$

For $\theta'' > \theta'$,
$$\frac{\mathcal{L}(\theta'; \underline{x})}{\mathcal{L}(\theta''; \underline{x})} = \frac{\cancel{(2\pi)^{-n/2}} \exp\left(\frac{-1}{2}\sum(x_i-\theta')^2\right)}{\cancel{(2\pi)^{-n/2}} \exp\left(\frac{-1}{2}\sum(x_i-\theta'')^2\right)}$$

$$= \exp\left[\frac{-1}{2}\left(\sum_1^n (x_i-\theta')^2 - \sum_1^n (x_i-\theta'')^2\right)\right] \le k$$

$$\Rightarrow \sum (x_i-\theta')^2 - \sum (x_i-\theta'')^2 \ge k_1$$

$$\sum_1^n (\cancel{x_i^2} - 2x_i\theta' + \theta'^2) - \sum_1^n (\cancel{x_i^2} - 2x_i\theta'' + \theta''^2) \ge k_1$$

$$2\left(\sum_1^n x_i\right)\underbrace{(\theta''-\theta')}_{>0} + n\underbrace{(\theta'^2-\theta''^2)}_{<0} \ge k_1$$

$$\boxed{\sum_1^n x_i \ge k_2 \quad \text{if } \theta'' > \theta'}$$

For $\theta'' < \theta'$, $\frac{\mathcal{L}(\theta'; \underline{x})}{\mathcal{L}(\theta''; \underline{x})} \le k \Rightarrow 2\left(\sum x_i\right)\underbrace{(\theta''-\theta')}_{<0} + n\underbrace{(\theta'^2-\theta''^2)}_{>0} \ge k_1$

$$\boxed{\sum_1^n x_i \le k_2 \quad \text{if } \theta'' < \theta'}$$

Because the resulting critical region is not the same under both conditions, this test is <u>NOT UMP</u>.

For One-sided Hypotheses...

- Consider $H_0: \theta \leq \theta'$ vs. $H_1: \theta > \theta'$
  [or, analogously, $H_0: \theta \geq \theta'$ vs. $H_1: \theta < \theta'$]
- For some families of pdfs and hypotheses, we can obtain general forms of UMP tests
- Introduce the monotone likelihood ratio....

Monotone Likelihood Ratio

[In CB] A family of pdfs or pmfs $\{g(t \mid \theta): \theta \in \Theta\}$ for a univariate random variable T with real-valued parameter θ has a monotone likelihood ratio (MLR) if, for every $\theta_2 > \theta_1$, $g(t \mid \theta_2)/g(t \mid \theta_1)$ is a monotone (nonincreasing or nondecreasing) function of t on $\{t: g(t \mid \theta_1) > 0 \text{ or } g(t \mid \theta_2) > 0\}$

[In HMC] The likelihood $L(\theta; \boldsymbol{x})$ has monotome likelihood ratio (MLR) in the statistic $y = u(\boldsymbol{x})$, if for $\theta_1 < \theta_2$, the ratio $L(\theta_1; \boldsymbol{x})/L(\theta_2; \boldsymbol{x})$ is a monotone function of $y = u(\boldsymbol{x})$.

Example 8

Let $X_1, \dots, X_n$ be a random sample from a Bernoulli distribution with parameter $p = \theta$, where $0 < \theta < 1$. Show that this distribution satisfies the MLR property.

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Karlin-Rubin Theorem

Consider testing $H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$. Suppose that T is a sufficient statistic for θ and the family of pdfs or pmfs $\{g(t \mid \theta): \theta \in \Theta\}$ of T has a nondecreasing MLR*. Then for any t_0, the test that rejects H_0 if and only if $T > t_0$ is a UMP level α test, where $\alpha = P_{\theta_0}(T > t_0)$.

* as defined in CB.

Example 8

$X_1, \ldots, X_n \sim \text{Bernoulli}(\theta)$ where $0<\theta<1$

$$\Rightarrow f(x) = \theta^{x}(1-\theta)^{1-x}$$

$$\mathcal{L}(\theta;\underline{x}) = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i} = \theta^{t}(1-\theta)^{n-t} \text{ where } T=\Sigma X_i$$
$$= g(t|\theta)$$

Let $\theta_1 < \theta_2$.

$$\frac{g(t|\theta_2)}{g(t|\theta_1)} = \frac{\theta_2^{t}(1-\theta_2)^{n-t}}{\theta_1^{t}(1-\theta_1)^{n-t}} = \left(\frac{\theta_2(1-\theta_1)}{\theta_1(1-\theta_2)}\right)^{t}\left(\frac{1-\theta_2}{1-\theta_1}\right)^{n}$$

Because $\theta_1<\theta_2$, $1-\theta_1 > 1-\theta_2$. Thus $\frac{1-\theta_2}{1-\theta_1} < 1$ and $\frac{1-\theta_1}{1-\theta_2} > 1$

and $\frac{\theta_2}{\theta_1} > 1 \Rightarrow \frac{\theta_2}{\theta_1}\cdot\frac{(1-\theta_1)}{(1-\theta_2)} > 1$

$\therefore \frac{g(t|\theta_2)}{g(t|\theta_1)}$ is monotonically increasing as t increases.

Alternatively, we can show as follows:

$$\frac{g(t|\theta_2)}{g(t|\theta_1)} = \underbrace{\left(\frac{\theta_2(1-\theta_1)}{\theta_1(1-\theta_2)}\right)}_{c_1}^{t}\underbrace{\left(\frac{1-\theta_2}{1-\theta_1}\right)}_{c_2}^{n} \doteq c_1^{t}c_2 \text{ where } c_1>1 \text{ and } 0<c_2<1$$

Consider $h(t) = c_1^{t}c_2 = c_2 e^{t\ln c_1}$

$$h'(t) = c_2(\ln c_1)e^{t\ln c_1} = \underbrace{c_2}_{>0}\underbrace{(\ln c_1)}_{>0}\underbrace{c_1^{t}}_{>0} > 0$$

$\therefore h(t) = \frac{g(t|\theta_2)}{g(t|\theta_1)}$ monotonically increasing as t increases

Example 9

Let $X_1, \ldots, X_n$ be a random sample whose distribution can be represented as an exponential family. Show that this distribution satisfies the MLR property, if $p(\theta)$ is monotone.

$f(x) = \exp[p(\theta)K(x) + S(x) + q(\theta)]$

$\mathcal{L}(\theta; \underline{x}) = \exp[p(\theta)\sum_{i=1}^{n} K(x_i) + \sum_{i=1}^{n} S(x_i) + nq(\theta)]$

Let $\theta_1 < \theta_2$.

$$\frac{\mathcal{L}(\theta_2; \underline{x})}{\mathcal{L}(\theta_1; \underline{x})} = \exp\left[p(\theta_2)\sum_{i=1}^{n} K(x_i) + \cancel{\sum_{i=1}^{n} S(x_i)} + nq(\theta_2) - \left\{p(\theta_1)\sum_{i=1}^{n} K(x_i) + \cancel{\sum_{i=1}^{n} S(x_i)} + nq(\theta_1)\right\}\right]$$

$$= \exp\left[(p(\theta_2) - p(\theta_1))\sum_{i=1}^{n} K(x_i) + n(q(\theta_2) - q(\theta_1))\right]$$

$\therefore$ If $p(\theta)$ is monotone, then $\frac{\mathcal{L}(\theta_2; \underline{x})}{\mathcal{L}(\theta_1; \underline{x})}$ is monotone in $T = \sum_{i=1}^{n} K(x_i)$.

Note: if $p(\theta)$ is monotone increasing, then Karlin-Rubin Thm. implies UMP level α test $T > t_0$ so that $P(T > t_0) = \alpha$.

Unbiased Tests

- A test is <u>unbiased</u> if its power never falls below the significance level
- Examples:
  - MP test of simple H_0 vs. simple H_1
  - One-sided tests based on MLR pdfs