MATH 503 Midterm Exam I Solutions

1. $X \sim \text{Poisson}(\lambda)$, therefore

$$\begin{aligned} E(X) &= \lambda \\ E(X^2) &= V(X) + E^2(X) = \lambda + \lambda^2 = E(X) + \lambda^2. \end{aligned}$$

   Thus, let $\hat{\theta} = X^2 - X$. Then, $E(X^2 - X) = E(X^2) - E(X) = (\lambda + \lambda^2) - (\lambda) = \lambda^2$.

2. To find the MLE,

$$\begin{aligned} \log f &= n \log \beta + n\beta \log \alpha - (\beta + 1) \sum_{i=1}^{n} \log x_i \\ \frac{\partial}{\partial \beta} \log f &= \frac{n}{\beta} + n \log \alpha - \sum_{i=1}^{n} \log x_i = 0, \end{aligned}$$

   which implies $\hat{\beta} = \frac{n}{\sum_{i=1}^{n} \log x_i - n \log \alpha}$. To compute the MOM estimator, note that X is Pareto distributed (or compute the mean directly) such that $E(X) = \frac{\beta\alpha}{\beta - 1} = \bar{X}$. Because α is known, we can solve for β directly to get $\hat{\beta} = \frac{\bar{X}}{\bar{X} - \alpha}$.

3. The pdf of $Y_{(1)}$ equals $f_{Y_{(1)}}(x) = n[1 - F(x)]^{n-1} f(x)$, where

$$\begin{aligned} f(x) &= \lambda e^{-\lambda x} \\ F(x) &= \int_0^x f(t)dt = -e^{-\lambda t} \mid_{t=0}^{x} = 1 - e^{-\lambda x}, \end{aligned}$$

   therefore $f_{Y_{(1)}}(x) = n(e^{-\lambda x})^{n-1}(\lambda e^{-\lambda x}) = n\lambda e^{-n\lambda x}$, i.e. $Y_{(1)} \sim \text{Exp}(\frac{1}{n\lambda}) = \text{Gamma}(1, \frac{1}{n\lambda})$. Thus, the associated mgf of $Y_{(1)}$ if $M_{Y_{(1)}} = \frac{1}{1 - \frac{t}{n\lambda}} \to 1 = e^0$ as $n \to \infty$, thus the limiting distribution is degenerate at zero.

4.

$$\begin{aligned} f(x_i) &= \frac{1}{\Gamma(4)\theta^4} x^{4-1} e^{-x/\theta} \\ L(\mathbf{x}) = \prod_{i=1}^{n} f(x_i) &= \frac{1}{\Gamma^n(4)\theta^{4n}} (\prod_{i=1}^{n} x_i)^3 e^{-\sum_{i=1}^{n} x_i/\theta} \\ &= \underbrace{\frac{1}{\Gamma^n(4)\theta^{4n}} e^{-\sum_{i=1}^{n} x_i/\theta}}_{k_1(\sum_{i=1}^{n} x_i;\theta)} \underbrace{(\prod_{i=1}^{n} x_i)^3}_{k_2(\mathbf{x})}, \end{aligned}$$

   thus by the Neymann-Fisher Factorization Thm, $Y = \sum_{i=1}^{n} X_i$ is sufficient for θ.

   To find an efficient estimator, we want to find a function $\phi(Y)$ that is unbiased and satisfies the Cramer-Rao lower bound, namely $\frac{1}{nI(\theta)}$.

$$E(Y) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = 4n\theta \doteq \theta$$

therefore $\phi(Y) = \frac{Y}{4n} = \frac{\bar{X}}{4}$ is unbiased. Meanwhile,

$$V\left(\frac{\bar{X}}{4}\right) = \frac{1}{16n^2}\sum_{i=1}^{n} V(X_i) = \frac{4n\theta^2}{16n^2} = \frac{\theta^2}{4n},$$

and

$$\begin{aligned}
\log f &= -\log(\Gamma(4)) - 4\log\theta + 3\log x - \frac{x}{\theta} \\
\frac{\partial}{\partial\theta}\log f &= \frac{-4}{\theta} + \frac{x}{\theta^2} \\
\frac{\partial^2}{\partial\theta^2}\log f &= \frac{4}{\theta^2} + \frac{2x}{\theta^3}.
\end{aligned}$$

Therefore, $I(\theta) = -E(\frac{\partial^2}{\partial\theta^2}\log f) = \frac{-4}{\theta^2} + \frac{2E(X)}{\theta^3} = \frac{8-4}{\theta^2} = \frac{4}{\theta^2}$, and the CRLB $= \frac{1}{nI(\theta)} = \frac{\theta^2}{4n}$. Thus $\phi(Y) = \frac{\bar{X}}{4}$ is efficient for θ.