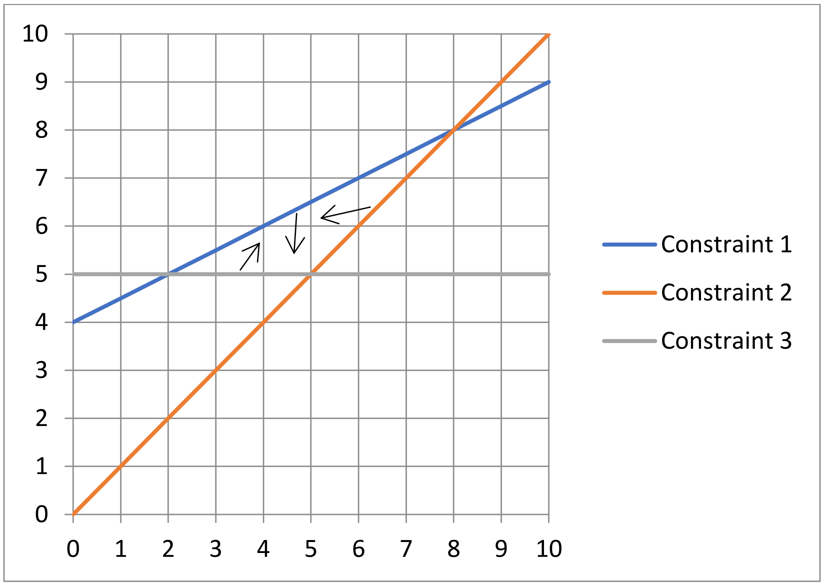
Homework 1 Solutions

1. Graph the feasible region and identify the extreme points for the following constraints.

$$\begin{aligned} -A + 2B &\leq 8 \\ A - B &\leq 0 \\ B &\geq 5 \\ A, B &\geq 0 \end{aligned}$$



The feasible region is the interior of the triangle formed by the three constraints. To find the extreme points we must see where each pair of constraints intersect.

Constraints 1 and 2 intersect when:

$$\begin{aligned} -A + 2B &= 8 \\ A - B &= 0 \end{aligned}$$

We can solve for B by adding these two equations together giving us B = 8 at the point of intersection. Then plugging this back into either equation, when B = 8, then A = 8. We could also first rewrite the second equation as A = B, then substitute B for A in the first equation giving us:

$$-A + 2B = -B + 2B = B = 8$$

And since A = B we have our first extreme point (8,8).

Constraints 1 and 3 intersect when:

$$\begin{aligned} -A + 2B &= 8 \\ B &= 5 \end{aligned}$$

Since we know B=5 we can solve for A by substituting 5 for B in the first equation giving us A=2. Thus we have an extreme point at (2,5).

Constraints 2 and 3 intersect when:

$$\begin{aligned} A - B &= 0 \\ B &= 5 \end{aligned}$$

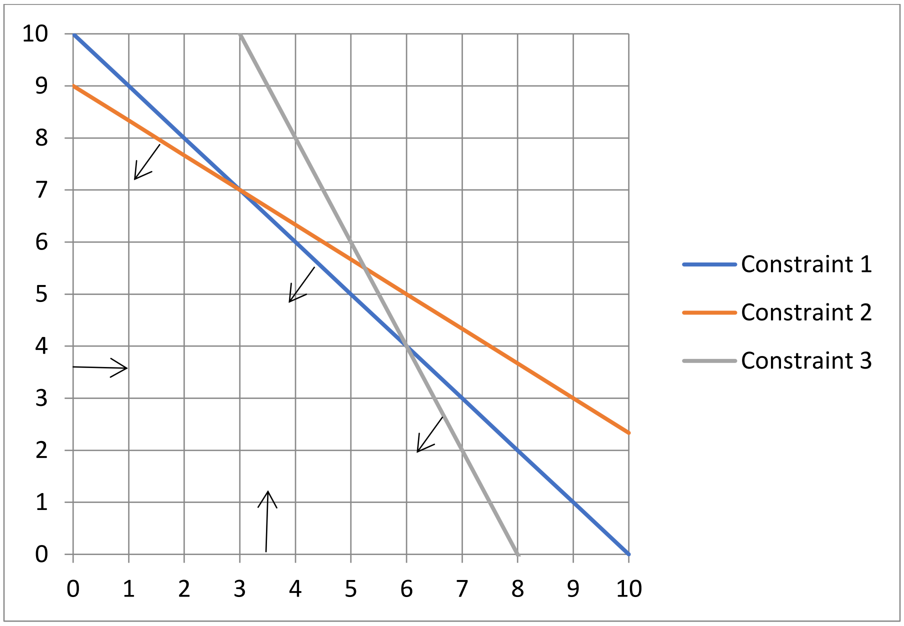
Since B = 5 and A = B we know that we have and extreme point at (5,5).

2. For the following Linear Program:

$$\text{Max } 20X + 25Y$$

$$\begin{aligned} X + Y &\leq 10 \\ 2X + 3Y &\leq 27 \\ 2X + Y &\leq 16 \\ X,Y &\geq 0 \end{aligned}$$

a. Graph the feasible region.
b. Identify the extreme points of the feasible region.
c. Find the optimal solution.



The feasible region is the area beneath all three of the lines and in the first quadrant (i.e. above the x and to the right of the y axis). There are 5 extreme points. The first is formed by the x and y axis at (0,0).

The next extreme point is where constraint 2 intersects the y-axis. We know at the y-axis X=0 so plugging X=0 into constraint 2 we can solve (0,9).

The next extreme point is where constraints 1 and 2 are binding. Thus:

$$\begin{aligned} X + Y &= 10 \\ 2X + 3Y &= 27 \end{aligned}$$

Solving the first equation to see that X = 10 – Y we substitute this into the second equation so that:

$$2X + 3Y = 2(10\text{-}Y) + 3Y = 20 - 2Y + 3Y = 20 + Y = 27$$

Solving for Y we see that Y=7, thus X=10-7=3, thus we have an extreme point at (3,7).

Next we have an extreme point where constraints 1 and 3 are binding. Thus:

$$\begin{aligned} X + Y &= 10 \\ 2X + Y &= 16 \end{aligned}$$

Subtracting the first equation from the second we can solve that X = 6, which from the first equation implies Y=4. Thus (6,4) is an extreme point.

Finally the X intercept of constraint 3 is an extreme point. Thus since Y=0 at the intercept we know 2X=16 from equation 3 implying the final extreme point is (8,0).

To find the optimal solution we must evaluate the objective function at each of the extreme points and identify the maximum objective value.

(0,0)	20*0 + 25*0 =	0
(0,9)	20*0 + 25*9 =	225
(3,7)	20*3 + 25*7 =	235
(6,4)	20*6 + 25*4 =	220
(8,0)	20*8 + 25*0 =	160

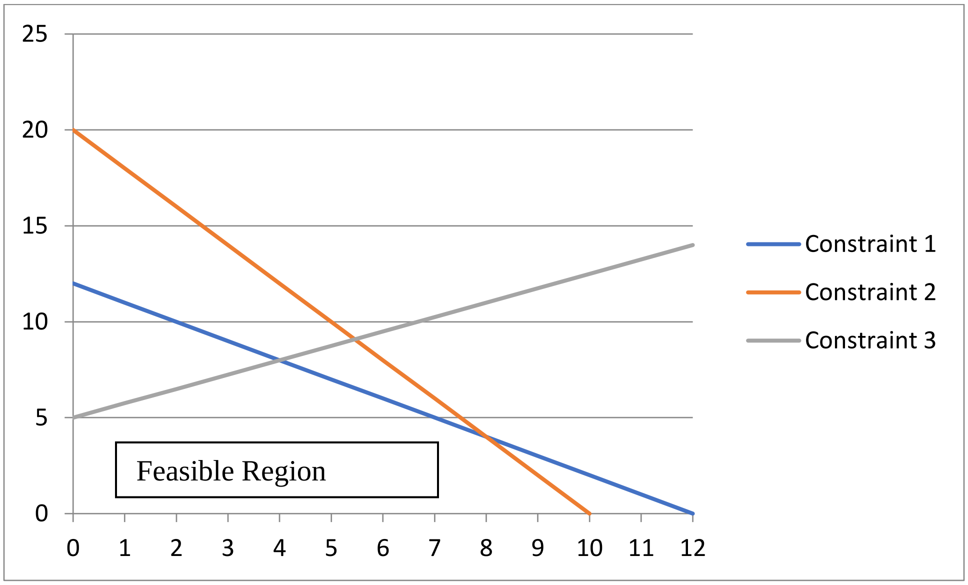
235 is clearly the largest objective value thus the maximum occurs at extreme point (3,7).

3. For the following Linear Program:

$$\text{Max } 80X + 65Y$$

$$\begin{aligned} X + \quad Y &\leq 12 \\ 2X + \quad Y &\leq 20 \\ -3X + 4Y &\leq 20 \\ X,Y &\geq 0 \end{aligned}$$

a. Graph the feasible region.
b. Identify the extreme points of the feasible region.
c. Find the optimal solution.
d. Write the problem in standard form.
e. What are the values of the slack variables at the optimal solution?



We can identify the feasible region as beneath each of the lines by checking that (0,0) satisfies each of the three constraints.

We can see that there are five extreme points to the feasible region. (0,0) is created by the non-negativity constraints.

Constraint 3 intercepts the Y axis (when X=0) at Y=5 so (0,5) is an extreme point.

Constraint 1 and 3 intercept when both X+Y=12 and -3X+4Y=20. Note that I can add 3 times the first equation to the second and get

$$\begin{array}{rl} -3X+4Y & = 20 \\ 3X+3Y & = 36 \\ \hline 7Y & = 56 \end{array}$$

This then simplifies to Y=8. We can then plug this back into constraint one and solve X=4, so (4,8) is an extreme point.

Next constraints 1 and 2 intersect when X+Y=12 thus X=12-Y. We can plug this into constraint 2 giving us 2X+Y=2(12-Y)+Y=24-Y=20. Thus Y=4 and (8,4) is an extreme point.

Finally we have where constraint 2 intersects the X-axis (Y=0) This we can solve for (10,0).

Plugging all the extreme points into the objective function we see:

(0,0)	80*0+65*0 = 0
(0,5)	80*0+65*5 = 325
(4,8)	80*4+65*8 = 840
(8,4)	80*8+65*4 = 900
(10,0)	80*10+65*0 = 800

Thus the optimal solution occurs at (8,4).

To write the problem in standard form we want all the constraints to be equalities (excluding the non-negativity constraints). To do this we add a slack variable to each less than or equal constraint and subtract a surplus variable from each greater than or equal constraint.

$$\text{Max } 80X + 65Y$$

$$\begin{aligned} X + Y + S_1 &= 12 \\ 2X + Y + S_2 &= 20 \\ -3X + 4Y + S_3 &= 20 \\ X, Y, S_1, S_2, S_3 &\geq 0 \end{aligned}$$

Finally we know that since the linear program in standard form is equivalent to the original linear program that the optimal solution (8,4) remains the same. Thus we can solve for the slack variables at the optimal solution:

$S_1 = 12-8-4=0$
$S_2 = 20 -2*8 – 4= 20 – 16- 4= 0$
$S_3 =20 +3*8-4*4=20+24-16=28$

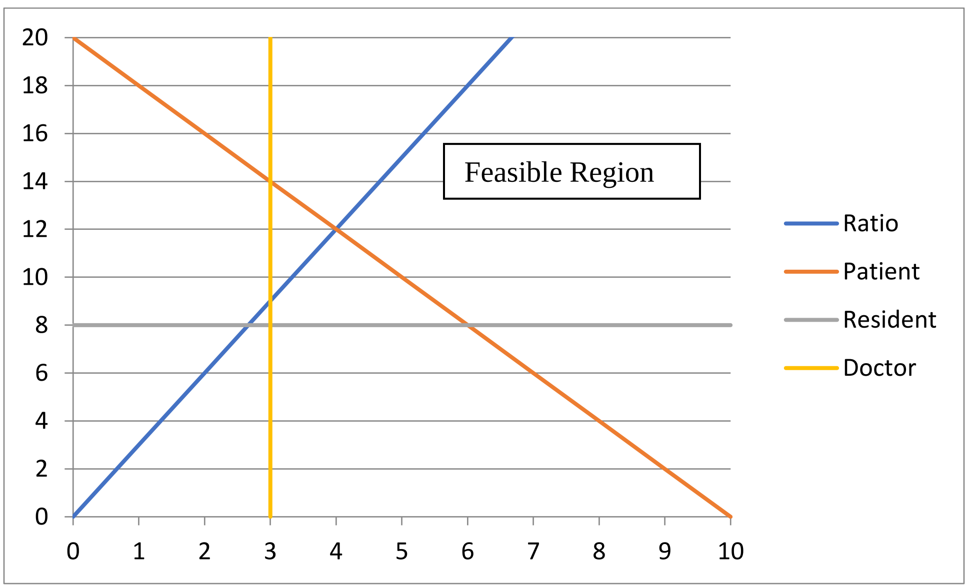
4. You are in charge of hiring for a major teaching hospital. The hospital has recently expanded the number of beds available in the Intensive Care Unit (ICU) from 12 to 20. The hospital currently assigns 3 teaching doctors and 8 residents to the ICU, for each of three shifts per day. Due to their experience teaching doctors can oversee up to 2 patients per shift while residents are only able to treat one. Experienced doctors are paid $280,000 annually compared to only $90,000 for residents. Because it is a teaching hospital a ratio of no more than three residents per teaching doctor must be maintained. The hospital does not wish to fire any of its current employees and is only looking to hire new ones.

a.) What are your decision variables? Constraints? Graph the feasible region.
b.) How many new doctors and residents should the hospital hire to minimize the total annual salary of the ICU? What is the salary budget needed for the ICU?
c.) Are any constraints redundant? If so which and why.
d.) Is the problem infeasible? Explain.
e.) Is the problem unbounded? If so, what real world constraint should be added?

D= the number of doctors on each shift

R=the number of residents on each shift

Staffing constraint: $2D+R \geq 20$ (Note that it is greater than or equal to because we want to at least be able to treat all the patients in the ICU)
Ratio Constraint: $3 \geq R/D$ which is not linear so we get $3D \geq R$
No Firing: $D \geq 3$, $R \geq 8$



Testing each of the constraints we see that the feasible region is unbounded and has two extreme points (4,12) and (6,8). Given the salary data we know we spend 280000D+90000R. From this we see that 280000*4+90000*12 gives us the minimum budget of 2.2 Million Dollars. Since the current staffing is 3 doctors and 8 residents we need to hire 1 doctor and 4 residents per shift.

We can see from the graph that the doctors constraint is redundant because it does not border the feasible region.

The problem is not infeasible (meaning it is feasible) because there exists solutions that satisfy all the constraints (for example 10 doctors and 10 residents).

The problem is unbounded because the feasible region is infinite, but since the objective function does not improve indefinitely we do have an optimal solution and it occurs at one of the two extreme points. In the real world the hospital would have budget constraints and a finite number of doctors and residents that it could hire, thus bounding the feasible region.

5. Prove that a local minima of a convex function on a convex set is globally optimal on that set.
Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $S \subset \mathbb{R}$ be a convex set. Let x^* be an element of S. Suppose that x^* is a local minima for f, that is, there exists $\varepsilon \geq 0$ such that $f(x^*) \leq f(x)$ for all $x \in S$ for which $\|x - x^*\| \leq \varepsilon$. Suppose there exists a solution x_{opt} such that $f(x_{opt}) < f(x^*)$. Because the set is convex, every point on the line segment connecting x_{opt} and $x^*$is in S. Therefore using the definition of a convex function $f(tx^* + (1-t)x_{opt}) \leq tf(x^*) + (1-t)f(x_{opt})\ \forall t \in [0,1]$.

But since $f(x_{opt}) < f(x^*)$ we know for $0 \leq t < 1$:

$f(tx^* + (1-t)x_{opt}) \leq tf(x^*) + (1-t)f(x_{opt}) < tf(x^*) + (1-t)f(x^*) \to$
$f(tx^* + (1-t)x_{opt}) < f(x^*)$ next choosing
$1 > t > 1 - \frac{\varepsilon}{\|x_{opt} - x^*\|}$ implies $0 < (1-t)\|x_{opt} - x^*\| < \varepsilon$

$$f(tx^* + (1-t)x_{opt}) = f(x^* + (1-t)(x_{opt} - x^*)) < f(x^*)$$

This now contradicts that $x^*$is a local minimum since

$$\|x^* + (1-t)(x_{opt} - x^*) - x^*\| \leq \varepsilon$$

Therefore there cannot exist a solution $f(x_{opt}) < f(x^*)$, making $f(x^*)$ a global minimum.