LECTURE 5 - 2/23/2021

Today's Topics:

1. Distribution, part 2
2. functions of continuous r.v.
3. expectations of continuous r.v.
4. joint distributions

From last time

X is a r.v.

$F(x) = P(X \leq x)$, F is distribution of X, cumulative distribution function, cdf.

Recall that in general we had $(\Omega, \mathcal{F}, \mathbb{P})$

How does this connect?

$\Omega = \text{Im}(X) = \{\text{all possible values of } x\}$.

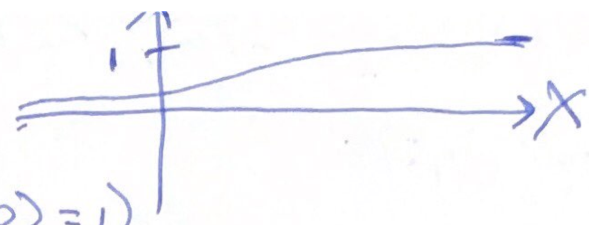
$= \text{sample space of } X$.

$A = \{X \leq x\} \in \mathcal{F}$.

$F(z) = P(X \leq z) = P(\{X \leq z\})$

Keep in mind that we still have $(\Omega, \mathcal{F}, P)$.

Properties of $F(x)$.

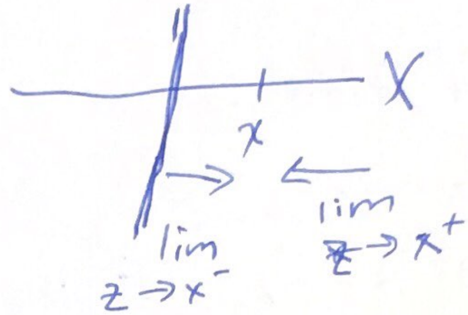


① $\lim\limits_{x \to \infty} F(x) = 1 \quad (P(X \le \infty) = 1)$

② $\lim\limits_{x \to -\infty} F(x) = 0 \quad (P(X \le -\infty) = 0)$

③ $x, y \in \mathbb{R},\ x < y \Rightarrow F(x) \le F(y)$

④ $x \in \mathbb{R}$

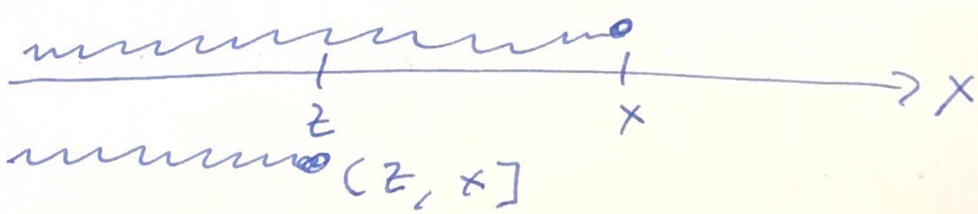


$\lim\limits_{z \to x^-} F(z)$ exists.

$\lim\limits_{z \to x^+} F(z) = F(x)$.

if $z < x$ $\quad F(x) - F(z) = P(X \le x) - P(X \le z)$

$$\{X \le x\} - \{X \le z\} = \{X \in (z, x]\}.$$



$$\lim_{z \to x^-} F(x) - F(z) = \lim_{z \to x^-} P(X \in (z, x])$$

$$\Rightarrow F(x) - \lim_{z \to x^-} F(z) = P(X = x)$$

$$\lim_{z \to x^-} F(z) = F(x) - P(X = x)$$

Note:

- if $P(X=x)=0$, then $\lim\limits_{z\to x^-} F(x) = F(x)$ and F continuous.
- if $P(X=x)\neq 0$, then $\lim\limits_{z\to x^-} F(x)$ exists but $\neq F(x)$.

$z > x$

$$\lim_{z\to x^+} F(z) - F(x) = \lim_{z\to x^+} P(X\in(x,z])$$

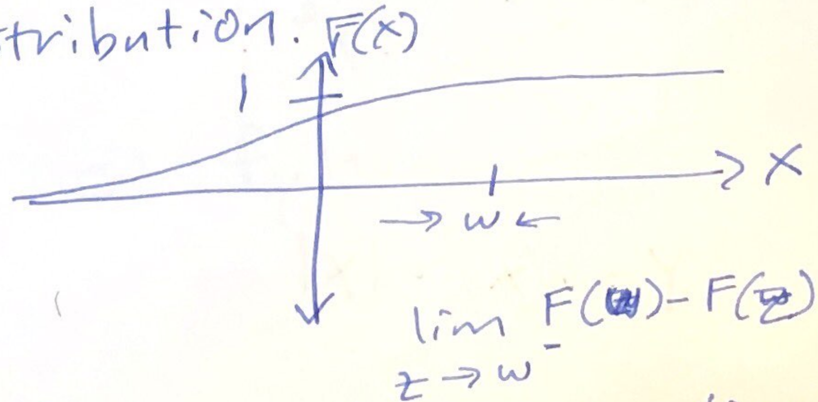
$$= P(X\in\emptyset)$$

$$= 0$$

$$\Rightarrow \lim_{z\to x^+} F(z) = F(x)$$

Now we move to (2).

A continuous r.v. is a r.v. with a continuous distribution.



$$\lim_{z\to w^-} F(w) - F(z) = P(X=w)$$

If F continuous, then $\lim\limits_{z\to w} F(z) = F(w)$, so

Prop: $F(x)$ is continuous iff $P(X=\omega)=0$, for all $\omega \in \mathbb{R}$.

X continuous r.v.

$F(x) \to$ cdf

$f(x) = F'(x) \to$ pdf $\Rightarrow F(x)$ has to be differentiable.

$F(z) = P(X \leq z)$

$$
\begin{aligned}
F(b) - F(a) &= P(X \in (a,b]) \\
&= P(X \in [a,b]) \quad \text{since } F \text{ continuous.} \\
&= \int_a^b f(w)\,dw.
\end{aligned}
$$

$$\Rightarrow \int_a^b f(w)\,dw = P(X \in [a,b]).$$

Given $X \sim f(x)$, maybe we want X^2, $|X|$, e^X, etc

$g(x) : \mathbb{R} \to \mathbb{R}$

$Y = g(X)$

Ex: $g(x) = x^2$

$Y = g(X) = X^2$.

$h(x) = |x|$

$Y = h(X) = |X|$

How do we compute the pdf of Y given the pdf of X?

<u>Recall</u>: $X \sim \text{Poisson}(\mu)$

$Y = |\sin(\pi/2\, X)|$ $\rightarrow$

$$P(Y=0) = P(X \text{ even}) = \sum P(X=2k) = \sum_{k=0}^{\infty} \left(e^{-\mu} \mu^{2k} / (2k)!\right)$$

$$P(Y=1) = \sum_{k=0} \left(e^{-\mu} \mu^{2k-1} / (2k-1)!\right)$$

<u>Ex</u>: $X \sim \text{Exp}(\lambda)$ $\quad f_X(x) = \lambda e^{-\lambda x}$

$Y = X^2$ $\quad f_Y(y) = ?$ $\quad , \; a, b \geq 0$.

$$P(Y \in [a,b]) = \int_a^b f_Y(w)\,dw$$

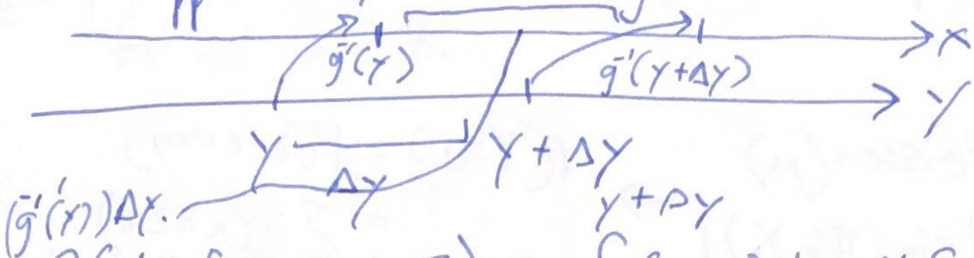
$$\Rightarrow P(X^2 \in [a,b])$$

$$\Rightarrow P(X \in [\sqrt{a}, \sqrt{b}]) = \int_{\sqrt{a}}^{\sqrt{b}} \lambda e^{-\lambda v}\,dv \qquad \begin{aligned} u &= v^2 \\ du &= 2v\,dv \\ dv &= \frac{du}{2v} = \frac{du}{2\sqrt{u}}. \end{aligned}$$

$$\Rightarrow P(Y \in [a,b]) = \int_a^b \frac{\lambda e^{-\lambda \sqrt{u}}}{2\sqrt{u}}\,du.$$

$$\Rightarrow f_Y(u) = \frac{\lambda e^{-\lambda \sqrt{u}}}{2\sqrt{u}}$$

Another approach, assume g is invertible, g^{-1} exists



$$P(Y \in [y, y+\Delta y]) = \int_y^{y+\Delta y} f_Y(w)\,dw \approx f_Y(y)\,\Delta y$$

$$\parallel$$

$$P(X \in [g^{-1}(y), g^{-1}(y+\Delta y)])$$

$$g^{-1}(y+\Delta y) = g^{-1}(y) + ((g^{-1})'(y))\Delta y$$

Taylor series, base point y.

$$\simeq P(X \in [g^{-1}(y), g^{-1}(y) + (g^{-1})'(y)\Delta y])$$

$$= \int_{g^{-1}(y)}^{g^{-1}(y)+(g^{-1})'(y)\Delta y} f_X(w)\,dw = f_X(g^{-1}(y))(g^{-1})'(y)\Delta y.$$

$$\Rightarrow f_Y(y)\Delta y = f_X(g^{-1}(y))(g^{-1})'(y)\Delta y.$$

$$g(y) = y^2$$

$$g^{-1}(y) = \sqrt{y} \Rightarrow (g^{-1})'(y) = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} = \frac{\lambda e^{-\sqrt{y}}}{2\sqrt{y}}.$$

<u>Ex</u>: $X \sim N(0, \sigma^2)$, $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$

Need to memorize $X \sim N(\mu, \sigma^2)$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

$Y = |x|$ $\quad g(x) = |x|$

$Y = g(X)$, what is pdf of Y?

$$P(Y \in [a,b]) = \int_a^b f_Y(w)\,dw$$

$$\Rightarrow P(|X| \in [a,b]) \Rightarrow P(X \in [a,b] \cup [-b,-a]).$$

$$= P(X \in [a,b]) + P(X \in [-b,-a]).$$

$$= \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\,dx + \int_{-b}^{-a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$= \cancel{2} \int_a^b \underbrace{\frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{\sigma^2}}}_{f_Y(y).}\,dx$$

Expected Values for continuous r.v. with pdf.

(differentiable).

$X \sim f(x)$.

$$E[X] = \int_{-\infty}^{\infty} f(z)\, z\, dz$$

$$E[g(X)] = \int_{-\infty}^{\infty} f(z) g(z)\, dz.$$

Recall in discrete case

$$E[X] = \sum_{z \in \Omega} P(X=z)\, z.$$

Recall

$$E[g(X)] = \sum_{z \in \Omega} P(X=z) g(z)$$

Ex 1. $X \sim N(\mu, \sigma^2)$ Note: $N(0,1) \to$ standard normal.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow E[X] = \int_{-\infty}^{\infty} f(z)\, z\, dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-\mu)^2/2\sigma^2}\, z\, dz.$$

$w = \frac{z-\mu}{\sigma}$. $\frac{(z-\mu)^2}{\sqrt{2}^2\sigma^2}$

$z = \mu + \sigma w$

$dw = \frac{1}{\sigma} dz$.

$\cancel{\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}} e$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-w^2/2} (\mu + w\sigma)\, \sigma\, dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} (\mu + w\sigma)\, dw.$$

$$= \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw}_{} + \sigma \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, w\, dw}_{},$$

$$= \mu \cdot 1 + \sigma \cdot 0$$

$$= \mu.$$

<u>A useful technique.</u>

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx = \int_{-\infty}^{\infty} \underbrace{x}_{v} \underbrace{\left(\frac{x e^{-x^2/2}}{\sqrt{2\pi}}\right)}_{du} dx$$

$$= -\int_{-\infty}^{\infty} dv\, u\, dx + vu \Big|_{-\infty}^{\infty}$$

$$= \int_{-\infty}^{\infty} 1 \cdot \frac{e^{-x^2/2}}{\sqrt{2\pi}} (-1)(-x)\, dx + x \left(\frac{-e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right) \Big|_{-\infty}^{\infty}$$

$$= 1 + 0.$$