Homework 3 Solution :: MATH 504

Your homework submission must be a single pdf called "LASTNAME-hw1.pdf" with your solutions to all theory problem to receive full credit. All answers must be typed in Latex.

Let

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

a) Use the LU factorization to express $A = LU$ where L is a lower triangular and U is an upper triangular matrices.

Solution.

$$L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & \frac{2}{3} & \frac{13}{3} \end{bmatrix}$$

$$L_2(L_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & \frac{2}{3} & \frac{13}{3} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} = U$$

Now $L = (L_2 L_1)^{-1} = L_1^{-1} L_2^{-1}$ with

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \quad L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}$$

hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix}$$

Therefore,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

b) Obtain the solution x^* of the system $Ax = b$ using LU factorization of A together with forward and backward substitution.

Solution.

$$Ly = b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{-1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{array}{c} y_1 = -1 \\ y_2 = 0 \\ \frac{y_1}{3} - \frac{y_2}{3} + y_3 = 1 \end{array}$$

With this we get $y = \begin{bmatrix} -1 \\ 0 \\ \frac{2}{3} \end{bmatrix}$ and we go to $Ux = y$ to get x^*

$$Ux = y = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \frac{2}{3} \end{bmatrix} \longrightarrow \begin{array}{c} 3x_1 - x_2 + x_3 = -1 \\ 2x_2 + x_3 = 0 \\ 4x_3 = \frac{2}{3} \end{array}$$

From this we get $x^* = \begin{bmatrix} \frac{-5}{12} \\ \frac{-1}{12} \\ \frac{1}{6} \end{bmatrix}$

c) Use Jacobi method twice, starting with $x^{(0)} = [1, 1, 0]^T$ to find an approximate to the solution. Report the error $\|x^{(k)} - x^*\|_\infty$, $k = 1, 2$.

Solution.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad D^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\begin{aligned} x^{(1)} &= x^{(0)} + D^{-1}(b - Ax^{(0)}) \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{4} \end{bmatrix} \end{aligned}$$

$$\|x^{(1)} - x^*\|_\infty = \| \begin{bmatrix} \frac{5}{12} \\ \frac{1}{12} \\ \frac{1}{12} \end{bmatrix} \|_\infty = \frac{5}{12}$$

$$\begin{aligned} x^{(2)} &= x^{(1)} + D^{-1}(b - Ax^{(1)}) \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{4} \end{bmatrix} \right) \\ &= \begin{bmatrix} -\frac{5}{12} \\ -\frac{1}{8} \\ \frac{1}{4} \end{bmatrix} \end{aligned}$$

$$\|x^{(2)} - x^*\|_\infty = \| \begin{bmatrix} 0 \\ -\frac{1}{24} \\ \frac{1}{12} \end{bmatrix} \|_\infty = \frac{1}{12}$$

d) Write a code for the Gauss-Seidel method, and apply it to find the solution of the system $Ax = b$ with 10^{-5} digits of accuracy. That is $\|x^k - x^*\| \le 10^{-5}$ where x^* is the solution of the system $x^* = A^{-1}b$ and x^k is the kth iterate of Gauss-Seidel.

Solution.

```
function[xGauss, error]=Gauss(A,b,x0,maxiter)
v = diag(A);
D = diag(v);
L = tril(A,-1);
xsol = inv(A)*b;
x = x0;
error = zeros(maxiter+1,1);
error(1) = norm(x0-xsol,inf);
Iter = zeros (maxiter+1,1);
Iter(1) = 1;
for i = 0:maxiter
    x= x + inv(L+D)*(b-A*x);
    Error = norm(x-xsol,inf);
    error(i+1)= Error;
    Iter(i+1)=i+1;
end
xGauss=x;
plot(error,Iter, '*')
end
A=[3,-1,1;0,2,1;-1,1,4]; b=[-1;0;1]; x0=[1;1;0];maxiter=10
[xGauss, error] = Gauss(A,b,x0,maxiter)
```

This gives us results of

```
xGauss =
    -0.416666698527586
   -0.0833333616965369
     0.166666665792238
error =
     0.416666666666667
    0.0416666666666666
    0.0173611111111111
   0.00151909722222224
  0.000693238811728336
  5.90257683898421e-05
  2.64255434724303e-05
  3.56047132765713e-06
```

```r
errorRate<-function(x,y){
  sum=0
  for (i in 1:dim(x)[1]) {
    a<-(x[i]-y[i])^2
    sum=sum+a
  }
  error=sqrt(sum)
  return(error)
}
GaussS=function(A, error,x_0,x,b){
  A<-as.matrix(A)
  C<-matrix(0,nrow=dim(A)[1], ncol=dim(A)[2])
  for (i in 1:dim(A)[1]) {
    C[i,1:i]=A[i,1:i]
  }
  C=solve(C)
  while (errorRate(x,x_0)>=error) {
    x_0=x_0+C%*%(b-A%*%x_0)
  }
  results<-list(estimate=x_0, Error=errorRate(x,x_0))
  return(results)
}
A<-matrix(c(3,-1,1,0,2,1,-1,1,4), ncol=3, byrow = T)
error<-10^(-5)
x_0<-matrix(c(1,1,0), ncol=1)
x<-matrix(c(-5/12,-1/12,2/12), ncol=1)
b<-matrix(c(-1,0,1), ncol=1)
GaussS(A, error,x_0,x,b)
```

```
## $estimate
##              [,1]
## [1,] -0.41667023
## [2,] -0.08333176
## [3,]  0.16666538
##
## $Error
## [1] 4.099228e-06
```