MATH 503: Mathematical Statistics
Dr. Kimberly F. Sellers, Instructor
Homework 8

1. Let $X_1, \ldots, X_n$ denote a random sample from a Poisson distribution with parameter θ, $0 < \theta < \infty$. Let $Y = \sum_{i=1}^n X_i$ and let $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If we restrict our considerations to decision functions of the form $\delta(y) = b + y/n$, where b does not depend on y, show that $R(\theta, \delta) = b^2 + \theta/n$. What decision function of this form yields a uniformly smaller risk than every other decision function of this form? With this solution, say δ, and $0 < \theta < \infty$, determine $\max_\theta R(\theta, \delta)$ if it exists.

2. Let $X_1, \ldots, X_n$ denote a random sample from a $\mathrm{N}(\mu, \theta)$ distribution, $0 < \theta < \infty$, where μ is unknown. Let $Y = \sum_{i=1}^n (X_i - \bar{X})^2/n$ and let $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If we consider decision functions of the form $\delta(y) = by$, where b does not depend on y, show that $R(\theta, \delta) = \frac{\theta^2}{n^2}[(n^2-1)b^2 - 2n(n-1)b + n^2]$. Show that $b = \frac{n}{n+1}$ yields a minimum risk decision function of this form. Note that $\frac{nY}{n+1}$ is not an unbiased estimator of θ. With $\delta(y) = \frac{ny}{n+1}$ and $0 < \theta < \infty$, determine $\max_\theta R(\theta, \delta)$ if it exists.

3. Let $X_1, \ldots, X_n$ denote a random sample from a $\mathrm{N}(\theta, \sigma^2)$ distribution, where $-\infty < \theta < \infty$ and σ^2 is a given positive number. Let $Y = \bar{X}$ denote the mean of the random sample. Take the loss function to be $L[\theta, \delta(y)] = |\theta - \delta(y)|$. If θ is an observed value of the random variable Θ, that is, $\mathrm{N}(\mu, \tau^2)$, where $\tau^2 > 0$ and μ are known numbers, find the Bayes' solution $\delta(y)$ for a point estimate θ.

4. Let $X_1, \ldots, X_n$ be Poisson(λ), and let λ have a gamma(α, β) distribution, the conjugate family for the Poisson.
   (a) Find the posterior distribution of λ.
   (b) Calculate the posterior mean and variance.

5. Let Y_n be the nth order statistic of a random sample of size n form a distribution with pdf $f(x \mid \theta) = \frac{1}{\theta}, 0 < x < \theta$, zero elsewhere. Take the loss function to be $L[\theta, \delta(y)] = [\theta - \delta(y_n)]^2$. Let θ be an observed value of the random variable Θ, which as pdf $\pi(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}, \alpha < \theta < \infty$, zero elsewhere, with $\alpha > 0$, $\beta > 0$. Find the Bayes' solution $\delta(y_n)$ for a point estimate of θ.