MATH 502, HOMEWORK ASSIGNMENTS

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ABSTRACT. HWK is assigned by the dates of the class. It is assigned typically on the day on or before the Tue of each week. It is due by 11:59pm of the following Friday. I will try to grade it by the following Monday.

Each problem is worth 2 points. Get 1 point if there is a minor mistake, 0 for major mistake.

1. Hwk for week 1

Hwk 1.1 Write the following system of eqs in matrix form.

$$\begin{cases} x_1 - 3x_2 + 2x_3 = 1, \\ 3x_1 - 2x_2 + 4_3 = 2, \\ 4x_1 + 2x_2 - 2x_3 = 3. \end{cases} \tag{1.1}$$

Hwk 1.2 Write the matrix equation

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 2 \\ -2 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}. \tag{1.2}$$

back to a system of individual eqs.

Hwk 1.3: Hwk problem 1.3 on the Page 7 of Chapter 1 under the Modules link of Canvas.

Hwk 1.4. Problems of 1, 3, 5 of Homework 2.1 and 2.2 on page 5 Chapter 2 under the Modules link of Canvas.

Date: April 16, 2022.

Hwk 1.5. Based on the geometric meaning of the ODE, roughly sketch solutions of

$$y' = \frac{x^2 - 1}{y^2 + 1}$$

over the interval $0 \leq x \leq 2$ with the following initial point.

a) $y(0) = 1$.

b) $y(0) = -1$.

Use at 4 least steps (in x) to reach $x = 2$.

2. Hwk for week 3

1. Find the A^{-1} for

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \end{bmatrix}.$$

Then use it to solve

$$AX = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & -1 \\ 2 & -1 \end{bmatrix}.$$

Can we say that if A^{-1} exist, then the equation $AX = B$ has one and only one solution? Explain.

2. A and B are square matrices of the same size. Assume all inverse matrices in following exist. Show the following:

a) $A(I + A)^{-1} = (I + A^{-1})^{-1}$.

b) $(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B$

c) $(I + AB)^{-1}A = A(I + BA)^{-1}$.

3. A diagonal matrix is a matrix whose off-diagonal elements are $0's$, eg

$$D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33}. \end{bmatrix}$$

It has a simpler notation $D = diag(a_{11}, a_{22}, a_{33})$. Show that $A^n = diag(a_{11}^n, a_{22}^n, a_{33}^n)$ for any integer $n \geq 1$.

The above result can be stated as "to get D^n for a diagonal matrix D, I only have to n power the diagonal element". Is the same true for non-diagonal matrices?

Generalize the above result to $p(D)$ where $p(x) = c_n x^n + c_{n-1}x^{n-1} + ... + c_0$ is a polynomial and c's are constants.

4. Is it true that the product of two upper triangular matrices is also upper triangular?

5. An **elementary matrix** is the result of doing one row operation on I.

a) Make examples of elementary matrices of size 4×4 generated by 3 kinds of row operations, resulting in 3 elementary matrices: E_1, E_2, E_3

b) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \multicolumn{4}{l}{\dots\dots\dots\dots} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Compute EA and AE for the E_1, E_2, E_3 obtained in a). What can you say about the effect of multiplying E to A?

6. A permutation matrix if the result of doing row interchanging on I many times.

a) Make an example of a 4×4 permutation matrix, P.

b) Observe

$$P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

What is the effect of multiplying P to $\mathbf{x}$?

c) Show that $P^{-1} = P^T$.

7. A circulant matrix is a matrix of the form

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & a_2 & \dots & a_{n-3} & a_{n-2} \\ \multicolumn{5}{l}{\dots\dots\dots\dots\dots\dots\dots} \\ \multicolumn{5}{l}{\dots\dots\dots\dots\dots\dots\dots} \\ \multicolumn{5}{l}{\dots\dots\dots\dots\dots\dots\dots} \\ a_2 & a_3 & a_4 & a_5 & \dots & a_n & a_1 \end{bmatrix}$$

Each row is the previous row cycled forward by one step. The n-by-n permutation matrix C_n is

$$C_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \multicolumn{5}{l}{\dots\dots\dots\dots\dots} \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Show that

$$A = \sum_{k=0}^{n-1} a_{k+1} C_n^k.$$

8. A lower Heisenberg matrix is a bi-diagonal matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 & 0 \\ 0 & a_{32} & a_{33} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & a_{n-1,n} & a_{nn} \end{bmatrix}$$

Show that as long as $a_{jj} \neq 0$, $j = 1, 2, ..., n$, then the rank of this matrix is n.

9. Find a set of independent rows of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & -4 \\ 0 & 1 & -1 & -2 \end{bmatrix}.$$

What is A's rank?

3. Week3, Part B

B3.1 Find the general solution of the following ODEs

a) $xy' + 3y = x^4$.

b) $y' - \frac{2x}{x^2+1} y = 1$.

c) $y' + 3y = 4x$.

B3.2 Refer to Figure 5.1 below. Find curves whose subtangents are always the constant k.

B3.3 Refer to Figure 5.1. Find those curves with the feature that the segment between the curve and the y-axis is bisected by the x-axis.

B3.4. I push my bike forward on a horizontal road with the constant force of 50 newton. The road friction force is 10 newton. The air resistance force is $5v$. The total mass of me and the bike is 100kg. Find my velocity at time t (seconds). What is my maximum velocity if I start from 0 speed?

B3.5 A lake has 20,000 m^3 of water and 100kg salt in it at time $t = 0$. Water flowing into the lake carries 0.01kg salt per m^3. In each hour, 1,000 m^3 of water

(a) $x - y/y'$ is the x intercept of the tangent line.
(b) $y - xy'$ is the y intercept of the tangent line.
(c) $x + yy'$ is the x intercept of the normal line.
(d) $y + x/y'$ is the y intercept of the normal line.
(e) $|y/y'|$ is the length AC of the projection on the x axis, of the segment of the tangent AP. The length AC is called the **subtangent.**
(f) $|yy'|$ is the length CB of the projection on the x axis of the segment of the normal BP. The length CB is called the **subnormal.**
(g) The length of the tangent segment $AP = \left| y\sqrt{\frac{1}{(y')^2}+1} \right|$.
(h) The length of the tangent segment $DP = |x\sqrt{1+(y')^2}|$.
(i) The length of the normal segment $PB = |y\sqrt{1+(y')^2}|$.
(j) The length of the normal segment $PE = \left| x\sqrt{1+\frac{1}{(y')^2}} \right|$.

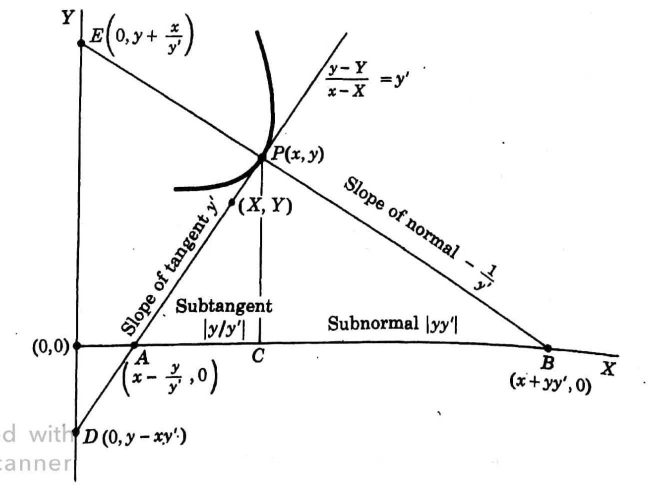

FIGURE 3.1. Subtangent etc. of a curve $y = y(x)$. Fig 13.5 of ODE by Tenenbaum&Pollard

flows into the lake, and $900m^3$ water flows out. Assume that the water in the lake are well mixed. Find the amount of salt in the lake at time $t > 0$.

4. Hwk for week 4

4.1. Part A. .

Summary: When doing +, -, * operations on partitioned matrix, just treat blocks as if they were numbers. The result is the same as what you get if these matrices are not partitioned.

A4.1 Given

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

a) Calculate AB the regular way.

b) Provide two ways to partition A, B. The first allows computation of AB while the other does not.

c) Use the first way in b) to partition to compute AB. Verify that the final result is the same as in a).

A4.2. Assume matrices in the following are with the right size to allow multiplications. Compute

a) $A\begin{bmatrix} B & C & D \end{bmatrix}$.

b) Is $\begin{bmatrix} B & C \end{bmatrix} A$ partitioned right to do the multiplication?

c) $\begin{bmatrix} B \\ C \end{bmatrix} A$

A4.3: Problems 1, 5a) 7, 10, 11, 14 , 20 of reading material PartitionedMatrix(from Leon's book) under Files/Week4:...

A4.4. A data matrix X's $j-$th column records observations the j-th variable x_j. The $i-$ row is for the i-th observation of these x_j's.

a) Make an example of such data matrix X for 3 variables and 4 observations.

b) Find a way to get the average of each x_j by multiplying X by a matrix or vector, on the right or on the left of X.

4.2. **Part B ODE.** .

Problem 3, 6 of Exercises 2.4 in reading material ODE_1_ExistenceUniqueness.pdf, under Files/Week4:...

Read the reading material ODE_2_... upto page 74, before "Bifurcation".

Do Problems 1, 4, 7 of Exercises 2.7 in reading material ODE_2_... under Files/Week4:...

5. HWWK for week 5

A5.1 Compute

$$\det \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ -2 & 2 & -3 & 3 \end{bmatrix}$$

CHAPTER TEST A True or False

For each statement that follows, answer *true* if the statement is always true and *false* otherwise. In the case of a true statement, explain or prove your answer. In the case of a false statement, give an example to show that the statement is not always true. Assume that all the given matrices are $n \times n$.

1. $\det(AB) = \det(BA)$
2. $\det(A - B) = \det(A) - \det(B)$
3. $\det(cA) = c\det(A)$
4. $\det(AB^T) = \det(A^T B)$
5. $\det(A - B) = 0$ implies $A = B$.
6. $\det(A^k) = \det(A)^k$
7. A triangular matrix is nonsingular if and only if its diagonal entries are all nonzero.
8. If $\mathbf{x}$ and $\mathbf{y}$ are two distincts vectors in $\mathbb{R}^n$ and $A\mathbf{x} = A\mathbf{y}$, then $\det(A) = 0$.
9. If A and B are row equivalent matrices, then their determinants are equal.
10. If $A \neq O$, but $A^k = O$ (where O denotes the zero matrix) for some positive integer k, then A must be singular.

CHAPTER TEST B

1. Let A and B be 3×3 matrices with $\det(A) = 4$ and $\det(B) = 6$, and let E be an elementary matrix of type I. Determine the value of each of the following:

   (a) $\det(\frac{1}{2}A)$ **(b)** $\det(B^{-1}A^T)$ **(c)** $\det(EA^2)$

2. Let
$$A = \begin{Bmatrix} x & 2 & 2 \\ 2 & x & -3 \\ -3 & -3 & x \end{Bmatrix}$$
   (a) Compute the value of $\det(A)$. (Your answer should be a function of x.)

   (b) For what values of x will the matrix be singular? Explain.

3. Let
$$A = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 11 \\ 3 & 8 & 14 & 20 \\ 4 & 11 & 20 & 30 \end{Bmatrix}$$
   (a) Compute the LU factorization of A.

   (b) Use the LU factorization to determine the value of $\det(A)$.

4. If A is a nonsingular $n \times n$ matrix, show that AA^T is nonsingular and $\det(AA^T) > 0$.
5. Let A be an $n \times n$ matrix. Show that if $B = S^{-1}AS$ for some nonsingular matrix S, then $\det(B) = \det(A)$.
6. Let A and B be $n \times n$ matrices and let $C = AB$. Use determinants to show that if either A or B is singular, then C must be singular.
7. Let A be an $n \times n$ matrix and let λ be a scalar. Show that
$$\det(A - \lambda I) = 0$$
   if and only if
$$A\mathbf{x} = \lambda\mathbf{x} \quad \text{for some } \mathbf{x} \neq \mathbf{0}$$
8. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$, $n > 1$. Show that if $A = \mathbf{x}\mathbf{y}^T$, then $\det(A) = 0$.
9. Let $\mathbf{x}$ be a nonzero vector in $\mathbb{R}^n$ and let A be an $n \times n$ matrix with the property that $A\mathbf{x} = \mathbf{0}$. Show that $\det(A) = 0$.
10. Let A be a matrix with integer entries. If $|\det(A)| = 1$, then what can you conclude about the nature of the entries of A^{-1}? Explain.

FIGURE 5.1. Ch 2 practice of Leon's linear algebra book

A5.2. Prove that
$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{bmatrix} = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

A5.3: Refer to Fig 5.1. Chapter Test A: 1 - 8. Chapter test B: 2,4,5,6,7,9.

5.1. **hwk 5: Part B ODE.** .

B5.1. Do a bifurcation analysis for $y'(x) = x - x^3 - h$ where h is a parameter.

3. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

(a) The set of all 2×2 diagonal matrices

(b) The set of all 2×2 triangular matrices

(c) The set of all 2×2 lower triangular matrices

(d) The set of all 2×2 matrices A such that $a_{12} = 1$

(e) The set of all 2×2 matrices B such that $b_{11} = 0$

(f) The set of all symmetric 2×2 matrices

(g) The set of all singular 2×2 matrices

6. Determine whether the following are subspaces of $C[-1, 1]$:

(a) The set of functions f in $C[-1, 1]$ such that $f(-1) = f(1)$

(b) The set of odd functions in $C[-1, 1]$

(c) The set of continuous nondecreasing functions on $[-1, 1]$

(d) The set of functions f in $C[-1, 1]$ such that $f(-1) = 0$ and $f(1) = 0$

(e) The set of functions f in $C[-1, 1]$ such that $f(-1) = 0$ or $f(1) = 0$

FIGURE 6.1. From Leon's linear algebra book

CHAPTER TEST A **True or False**

Answer each of the statements that follows as true *or* false. *In each case, explain or prove your answer.*

1. If S is a subspace of a vector space V, then S is a vector space.

2. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^4$.

3. It is possible to find a pair of two-dimensional subspaces S and T of $\mathbb{R}^3$ such that $S \cap T = \{\mathbf{0}\}$.

4. If S and T are subspaces of a vector space V, then $S \cup T$ is a subspace of V.

5. If S and T are subspaces of a vector space V, then $S \cap T$ is a subspace of V.

6. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent, then they span $\mathbb{R}^n$.

7. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ span a vector space V, then they are linearly independent.

8. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are vectors in a vector space V and

$$\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) = \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1})$$

then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly dependent.

FIGURE 6.2. From Leon's linear algebra book

B5.2. I plan to live on the Mars by raising and harvesting fish in a fish tank. The fish population at time t (year) is denoted as $P(t)$ (kg) which grows by logistic growth law. I have to harvest 100 kg per year to survive. The ODE for $P(t)$ is

$$P' = P(1 - P/N) - 100,$$

where the constant N is the tank's carrying capacity. The larger the tank, the larger the N. To determine the tank size, I have to find N. Choose N so that I can survive on the Mars.

6. HWK FOR WEEK 6

A6.1. Problem 3 and 6 shown in Figure 6.1.

A6.2. Are vectors

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

independent?

A6.3 Problems 1-8 in Figure 6.2

A6.4. Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ be independent vectors in $\mathbb{R}^n$,

3. Consider the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

(a) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for $\mathbb{R}^2$.

(b) Why must $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be linearly dependent?

(c) What is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

4. Given the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix}$$

what is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

5. Let

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$$

(a) Show that $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are linearly dependent.

(b) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

(c) What is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

(d) Give a geometric description of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.

6. In Exercise 2 of Section 3.2, some of the sets formed subspaces of $\mathbb{R}^3$. In each of these cases, find a basis for the subspace and determine its dimension.

7. Find a basis for the subspace S of $\mathbb{R}^4$ consisting of all vectors of the form $(a+b, a-b+2c, b, c)^T$, where a, b, and c are all real numbers. What is the dimension of S?

to satisfy in order for $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ to form a basis for $\mathbb{R}^3$?

(c) Find a third vector $\mathbf{x}_3$ that will extend the set $\{\mathbf{x}_1, \mathbf{x}_2\}$ to a basis for $\mathbb{R}^3$.

9. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be linearly independent vectors in $\mathbb{R}^3$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^2$.

(a) Describe geometrically Span$(\mathbf{a}_1, \mathbf{a}_2)$.

(b) If $A = (\mathbf{a}_1, \mathbf{a}_2)$ and $\mathbf{b} = A\mathbf{x}$, then what is the dimension of Span$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{b})$? Explain.

10. The vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix},$$

$$\mathbf{x}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}, \quad \mathbf{x}_5 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

span $\mathbb{R}^3$. Pare down the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ to form a basis for $\mathbb{R}^3$.

11. Let S be the subspace of P_3 consisting of all polynomials of the form $ax^2 + bx + 2a + 3b$. Find a basis for S.

12. In Exercise 3 of Section 3.2, some of the sets formed subspaces of $\mathbb{R}^{2\times 2}$. In each of these cases, find a basis for the subspace and determine its dimension.

13. In $C[-\pi, \pi]$, find the dimension of the subspace spanned by $1, \cos 2x, \cos^2 x$.

14. In each of the following, find the dimension of the subspace of P_3 spanned by the given vectors:

(a) $x, x-1, x^2+1$

FIGURE 7.1. From Leon's linear algebra book

a) Let A be a nonsingular $n \times n$ matrix. Show that $A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_k$ are linearly independent

b) If A is not an invertible $n \times n$ matrix, then must $A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_k$ be linearly dependent?

Read ODE_3_HomogeneousLinearODE.pdf in the canvas, Files, Week6Materials folder and do Problems 1, 5, 9 on page 94.

7. HWK FOR WEEK 7

A7.1 Problems 3,7,11 in Figure 12.1

A7.2 (8 points) Vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is called the standard basis of $\mathbb{R}^2$.

a) Show that vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ also form a basis of $\mathbb{R}^2$.

b) Given any vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, denote its coordinate under the basis $\mathbf{u}_1, \mathbf{u}_2$ as y_1, y_2. Find the relation between $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

The answer should be in $\mathbf{x} = U\mathbf{y}$ form, where the matrix U is called the transition matrix from basis $\mathbf{u}_1, \mathbf{u}_2$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2$.

c) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ also form a basis of $\mathbb{R}^2$. Find the relation of coordinate of a vector under basis $\mathbf{u}_1, \mathbf{u}_2$ and that under the basis $\mathbf{v}_1, \mathbf{v}_2$.

d) Given a system of equations $A\mathbf{x} = \mathbf{b}$, where $x's$ are coordinates under $\mathbf{e}_1, \mathbf{e}_2$, what is this system of equations under the basis $\mathbf{u}_1, \mathbf{u}_2$?

B7.1: A mass m is hanging at the lower end of a spring. The upper end of the spring is fixed on the ceiling. The air resistance force is proportional to the velocity. Find the ODE for the position of the mass at the time t.

Note that you are free to set up the coordinate to describe the position.

8. HWK for week 8

A8.1. For any given

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n,$$

their scalar product is defined as $\mathbf{x}^T\mathbf{y} = x_1y_1 + x_2y_2 + ...x_ny_n$.

a) Show that the scalar product is an inner product.

b) Write out the Cauchy-Schwarz inequality for this inner product in $\sum$ form.

A8.2 Consider the vectors $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \in \mathbb{R}^3$.

a) Find the angle between $\mathbf{x}, \mathbf{y}$.

b)

A8.3 Let $\mathbf{p}$ denote the projection (vector) of $\mathbf{x}$ onto $\mathbf{y}$. Show that $\mathbf{x} - \mathbf{p} \perp \mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

A8.4. Pick n positive numbers $w_1, w_2, ...w_n$. For any given

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n,$$

define

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{w}} := w_1 x_1 y_1 + w_2 x_2 y_2 + ... + w_n x_n y_n.$$

a) Show that $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{w}}$ is an inner product.

b) Write out the Cauchy-Schwarz inequality for this inner product in $\sum$ form.

A8.5. Pick a positive function $w(x) \in C[0,1]$. For any given $f, g \in C[0,1]$, define

$$\langle f, g \rangle_w := \int_0^1 f(x)g(x)w(x)dx.$$

a) Show that $\langle f, g \rangle_w$ is an inner product.

b) Write out the Cauchy-Schwarz inequality for this inner product in integral form.

c) Now take $w(x) \equiv 1$. Show that the functions $\sin(2n\pi x)$, $n = 1, 2, 3, ...$ are orthogonal (ie perpendicular) to each other. Note that $2\sin A \sin B = \cos(A-B) - \cos(A+B)$.

A8.6 a) Show that if $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n$ are orthogonal to each other, then they are linearly independent.

b) Show that in a n-dimensional inner product space, any n vectors that are orthogonal to each other form a basis for the space.

A8.7. a) $\mathbf{v} = (1, 2)$. Find the vector pointing to the same direction as $\mathbf{v}$ and with length 1.

b) Generalize a) to any vector $\mathbf{v}$. This procedure is called unitize or normalize a vector.

A8.8. A matrix A's columns are orthonormal (ie orthogonal and unit) vectors. Compute $A^T A$.

B8.1 (4pts) Solve the following ODEs

a) $x'' + 2x' - 3x = t^2$

b) $x'' + 4x = \cos t$

9. HWK for week 10

A10.1 We know that a matrix Q is orthogonal if and only if $Q^T Q = I$. A permutation matrix is a matrix formed from the identity matrix by reordering its columns. Show that a permutation matrix is an orthogonal matrix.

A10.2 Prove that the transpose of an orthogonal matrix is also an orthogonal matrix.

A10.3. If Q is an $n \times n$ orthogonal matrix and $\mathbf{x}$ and by are nonzero vectors in $\mathbb{R}^n$, then how does the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$ compare with the angle between $\mathbf{x}$ and $\mathbf{y}$? Prove your answer.

A10.4. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ and let $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that H is both orthogonal and symmetric and hence is its own inverse.

A10.5. Let Q be an orthogonal matrix. Show that $|\det(Q)| = 1$.

A10.6. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be an orthonormal set of vectors in an inner product space V, and let $\mathbf{x} = \mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3$ and $\mathbf{y} = 4\mathbf{u}_1 + 5\mathbf{u}_2 + 6\mathbf{u}_3$

(a) Determine the value of $\langle \mathbf{x}, \mathbf{y} \rangle$.

(b) Determine the value of $||\mathbf{x}||$.

A10.7. Let V be an inner product space. For any $\mathbf{v} \in V$, define $||\mathbf{v}|| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. Show that the $||\mathbf{v}||$ defined above is a norm.

B10.1 (4pts) Find the general solution of

a) $x'' + 4x = \cos(2t)$.

b) $x'' + 5' + 6x = e^{-2t} + t$.

B10.2 Write $x''' + tx'' + e^t x' + 2x = \ln t.$ as an equivalent system of 1st order ODEs.

B10.3. Write the system of ODEs

$$\begin{cases} x_1' = x_1 + 3x_2 \\ x_2' = 2x_1 - x_2 + t^2 \end{cases} \tag{9.1}$$

as an equivalent single higher order ODE.

10. Hwk week 11

A11.1 Given the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 1 \\ -2 & 2 & -2 \end{bmatrix}$.

a) Find A's eigenvalues and eigenvectors.

b) Factor A as VDV^{-1}.

c) Use b) to compute A^7 and $A^7 + 3A^4 + 5A^2 + 7I$.

A11.2. a) Show that an invertible matrix A's eigenvalues are not 0.

b) Let λ be an eigenvalue of an invertible matrix A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

A11.3. A, B are square matrices. Show that if $\lambda \neq 0$ is an eigenvalue of AB, then the λ is also an eigenvalue of BA.

B11.1 Find the general solution of

$$\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \mathbf{x}.$$

11. ..

12. HWK 12

A12.0. Continue the problem A11.1. Find $\cos(A)$. e^A.

A12.1 Among the matrices in Figure 12.1, which are Hermitian?

A12.2. Assume A is Hermitian and $\mathbf{x} \in \mathbb{C}^n$. show that $\mathbf{x}^H A\mathbf{x}$ is a real number.

A12.3 Let A and B be Hermitian matrices. Answer true or false for each of the statements that follow. In each case, explain or prove your answer.
(a) The eigenvalues of AB are all real.
(b) The eigenvalues of ABA are all real.

A12.4. Let A be a diagonalizable matrix whose eigenvalues are all either 1 or -1. Show that $A^{-1} = A$.

A12.5. Let A be a $n \times n$ matrix with real entries and let $\lambda_1 = a + bi$ (where a and b are real and $b \neq 0$) be an eigenvalue of A. Let $\mathbf{z}_1 = \mathbf{x} + i\mathbf{y}$ (where x and y both have real entries) be an eigenvector belonging to λ_1 and let $\mathbf{z}_2 = \mathbf{x} - i\mathbf{y}$.
(a) Explain why $\mathbf{z}_1$ and $\mathbf{z}_2$ must be linearly independent.
(b) Show that $\mathbf{y} \neq \mathbf{0}$ and that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

B12.1 (6 pts) Solve the ODE system $\mathbf{x}' = A\mathbf{x}$, where A is
a) $A = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}$
b) $A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$
c) $A = \begin{bmatrix} -4 & 2 \\ -2 & 0 \end{bmatrix}$.

13. Hwk 13

A13.1 a) What is diagonalization of a matrix A?
b) What is the advantage of diagonalizing a matrix?
c) When can you diagonalizing A?

(a) $\begin{bmatrix} 1-i & 2 \\ 2 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2-i \\ 2+i & -1 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{bmatrix}$

(e) $\begin{bmatrix} 0 & i & 1 \\ i & 0 & -2+i \\ -1 & 2+i & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{bmatrix}$

FIGURE 12.1. From Leon's linear algebra book

d) Is a Hermitian matrix always diagonalizable? Why do we factorize as $A = UDU^H$ with unitary matrix U, instead of $A = VDV^H$ as in a)?

e) Is a symmetric real matrix A always diagonalizable, using orthogonal matrix U?

f) What about the diagonalization for general square matrices ?

A13.2 Diagonalize A using a unitary matrix. Note that you need enough many othonormal eigenvectors to form U.

a) $A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$.

b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

A13.3. We know that if a $n \times n$ matrix A has n independent eigenvectors, then $A = V\Lambda V^{-1}$, where V is the matrix with columns being these eigenvectors.

Now prove the converse is also true, that is if $A = V\Lambda V^{-1}$, then the columns of V are n independent eigenvectors of the $n \times n$ matrix A.

A13.4. Show that an eigenvalue, λ, of an unitary matrix must satisfy $|\lambda| = 1$.

A13.5. Let U be an unitary matrix. Show that $|\det(U)| = 1$.

A13.6. Let A be a 2×2 matrix with Schur decomposition UTU^H. Let t_{ij} denotes the ij-element of T. Suppose that $t_{12} \neq 0$. Show that

(a) the eigenvalues of A are $lambda = t_{11}, t_{22}$.

(b) the first column of U, $\mathbf{u}_1$ is an eigenvector of A belonging to $\lambda = t_{11}$.

(c) $\mathbf{u}_2$ is not an eigenvector of A belonging to $\lambda = t_{22}$.

B13.1 (4pts) Roughly plot the phase portrait for $\mathbf{x}' = A\mathbf{x}$ for the following A's. Point out the type of the equilibrium point $\mathbf{x} = \mathbf{0}$.

a) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

b) $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$.

c) $A = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$.

d) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

14. HWK for week 14

A14.1 Show that A and A^T have the same nonzero singular values. How are their singular value decompositions related?

A14.2. Find the singular value decomposition of each of the following matrices:

a) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{bmatrix}$$

has singular value decomposition

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

FIGURE 14.1. For problem A14.3

b) $\begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

A14.3 (2×6 pts) (SVD and solubility of $A\mathbf{x} = \mathbf{b}$)

The matrix A and its $A = U\Sigma V^T$ is given in Figure 14.1

a) Represent A by an outer product expansion $A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_n \mathbf{u}_n \mathbf{v}_n^T$. By the way, $n =$? Are these $\mathbf{u}$'s ($\mathbf{v}$'s) orthonormal?

Note: do b) - e) below using a) only, and without using its numbers as much as possible, so that you can generalize later.

b) Check whether $\mathbf{v}_3$ is a solution of $A\mathbf{x} = \mathbf{0}$.

c) Check whether $\sigma_j^{-1} \mathbf{v}_j$, $j = 1, 2$, is a solution of $A\mathbf{x} = \mathbf{u}_j$.

d) Show that for $A\mathbf{x} = \mathbf{b} \in \mathbb{R}^3$ has a solution if and only if $\mathbf{b}$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

e) Show that all solutions of $A\mathbf{x} = \mathbf{0}$ are linear combinations of $\mathbf{v}_3$.

f) Generalize statements b) - e) for a general matrix A of size $m \times n$.

A14.4 (SVD and the least square solution of over determined $A\mathbf{x} = \mathbf{b}$) The system of eq $A\mathbf{x} = \mathbf{b}$ is call over determined when it has more equations than unknowns. Typically one finds its least square solution by solving its normal eq. $A^T A\mathbf{x} = A^T\mathbf{b}$. This happens when you do linear regressions.

In this problem, we use A's SVD, $A = U\Sigma V^T$, to solve $A^T A\mathbf{x} = A^T\mathbf{b}$. Show that the solution is of the form $\mathbf{x} = V?U^T\mathbf{b}$ and please find the "?".

B14.1. (1×6 pts) Plot the phase portrait of $\mathbf{x}' = A\mathbf{x}$ for the following A's.

a) $A = \begin{bmatrix} 3 & -3 \\ 4 & -3 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} -3 & -3 \\ 4 & 1 \end{bmatrix}$

d) $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

e) $A = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}$

f) $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

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