MATH 503: Mathematical Statistics
Dr. Kimberly F. Sellers, Instructor

Homework 10 Solutions

1. **Observations** (x_i, Y_i)**,** $i = 1, \ldots, n$**, are collected according to the model** $Y_i = \alpha + \beta x_i + \epsilon_i$**, where** $E(\epsilon_i) = 0$**, Var**$(\epsilon_i) = \sigma^2$**, and Cov**$(\epsilon_i, \epsilon_j) = 0$ **if** $i \neq j$**. find the best linear unbiased estimator of** α**.**

To be the BLUE of α, it must satisfy

$$\begin{aligned} E(\sum_{i=1}^n d_i Y_i) = \sum_{i=1}^n d_i E(Y_i) = \sum_{i=1}^n d_i(\alpha + \beta x_i) &\doteq \alpha \\ \alpha\left(\sum_{i=1}^n d_i\right) + \beta\left(\sum_{i=1}^n d_i x_i\right) &= \alpha. \end{aligned} \tag{1}$$

Equation (1) holds iff $\sum_{i=1}^n d_i = 1$ and $\sum_{i=1}^n d_i x_i = 0$, thus we need to satisfy these constraints. Further, to be a best estimator, we need $\text{Var}(\sum_{i=1}^n d_i Y_i) = \sum_{i=1}^n d_i^2 \sigma^2 = \sigma^2 \sum_{i=1}^n d_i^2$ (i.e. $\sum_{i=1}^n d_i^2$) to be minimized.

In accordance with Lemma 11.2.7, let $k = n$, $a_i = d_i x_i$, $v_i = 1/x_i$, $c_i = x_i^2$. This implies that

$$\begin{aligned} \max_{\boldsymbol{a}:\sum_{i=1}^n a_i = 0} \frac{\left(\sum_{i=1}^n a_i v_i\right)^2}{\sum_{i=1}^n a_i^2/c_i} &= \max_{\boldsymbol{dx}:\sum_{i=1}^n d_i x_i = 0} \frac{\left(\sum_{i=1}^n d_i x_i (1/x_i)\right)^2}{\sum_{i=1}^n d_i^2 x_i^2/x_i^2} \\ &= \max_{\boldsymbol{dx}:\sum_{i=1}^n d_i x_i = 0} \frac{\left(\sum_{i=1}^n d_i\right)^2}{\sum_{i=1}^n d_i^2} \\ &= \max_{\boldsymbol{dx}:\sum_{i=1}^n d_i x_i = 0} \frac{1}{\sum_{i=1}^n d_i^2}, \end{aligned}$$

where the x_is are observed and thus known, so $\max_{\boldsymbol{dx}:\sum_{i=1}^n d_i x_i = 0} \frac{1}{\sum_{i=1}^n d_i^2} = \max_{\boldsymbol{d}:\sum_{i=1}^n d_i x_i = 0} \frac{1}{\sum_{i=1}^n d_i^2} = \min_{\boldsymbol{d}:\sum_{i=1}^n d_i x_i = 0} \sum_{i=1}^n d_i^2$, where the latter equation is what we seek to determine. By Lemma 11.2.7, this result is attained at

$$d_i x_i = k x_i^2 \left(\frac{1}{x_i} - \frac{n\bar{x}}{\sum_{i=1}^n x_i^2}\right) \tag{2}$$

because, for this problem, $\bar{v}_c = \frac{\sum_{i=1}^n c_i v_i}{\sum_{i=1}^n c_i} = \frac{\sum_{i=1}^n x_i^2(1/x_i)}{\sum_{i=1}^n x_i^2} = \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2} = \frac{n\bar{x}}{\sum_{i=1}^n x_i^2}$.

Equation (2) implies that $d_i = k x_i \left(\frac{1}{x_i} - \frac{n\bar{x}}{\sum_{i=1}^n x_i^2}\right)$. Further,

$$1 \doteq \sum_{i=1}^n d_i = \sum_{i=1}^n k x_i \left(\frac{1}{x_i} - \frac{n\bar{x}}{\sum_{i=1}^n x_i^2}\right) = k\left[\sum_{i=1}^n 1 - \sum_{i=1}^n \frac{n x_i \bar{x}}{\sum_{i=1}^n x_i^2}\right] = kn\left(1 - \frac{n\bar{x}^2}{\sum_{i=1}^n x_i^2}\right),$$

thus

$$k = \frac{1}{n\left(1 - \frac{n\bar{x}^2}{\sum_{i=1}^n x_i^2}\right)} = \frac{\sum_{i=1}^n x_i^2}{n\left(\sum_{i=1}^n x_i^2 - n\bar{x}^2\right)}$$

and

$$d_i = \frac{\sum_{i=1}^n x_i^2}{n\left(\sum_{i=1}^n x_i^2 - n\bar{x}^2\right)} \cdot \frac{\sum_{i=1}^n x_i^2 - n\bar{x}x_i}{\sum_{i=1}^n x_i^2} = \frac{\sum_{i=1}^n x_i^2 - n\bar{x}x_i}{n\left(\sum_{i=1}^n x_i^2 - n\bar{x}^2\right)}.$$

Claim:

$$\frac{\sum_{i=1}^n x_i^2 - n\bar{x}x_i}{n\left(\sum_{i=1}^n x_i^2 - n\bar{x}^2\right)} = \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}$$

Proof of claim:

$$\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}} = \frac{S_{xx} - n(x_i - \bar{x})\bar{x}}{nS_{xx}}$$

where

$$nS_{xx} = n\sum_{i=1}^n (x_i - \bar{x})^2 = n\sum_{i=1}^n (x_i^2 - 2x_i\bar{x} + \bar{x}^2) = n\left(\sum_{i=1}^n x_i^2 - 2\bar{x}(n\bar{x}) + n\bar{x}^2\right) = n\left(\sum_{i=1}^n x_i^2 - n\bar{x}^2\right)$$

and

$$\begin{aligned} S_{xx} - n(x_i - \bar{x})\bar{x} &= \sum_{i=1}^n (x_i - \bar{x})^2 - nx_i\bar{x} + n\bar{x}^2 = \sum_{i=1}^n (x_i^2 - 2x_i\bar{x} + \bar{x}^2) - nx_i\bar{x} + n\bar{x}^2 \\ &= \sum_{i=1}^n x_i^2 - nx_i\bar{x}, \end{aligned}$$

thus $d_i = \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}$ and $\hat{\alpha} = \sum_{i=1}^n d_i Y_i = \bar{y} - \hat{\beta}\bar{x}$ (by Problem 3a, HW10) is the BLUE of α.

2. **Consider the residuals $\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n$ defined by $\hat{\epsilon}_i = Y_i - \hat{\alpha} - \hat{\beta}x_i$.**

   (a) **Show that $E(\hat{\epsilon}_i) = 0$.**

   (b) **Verify that $\mathbf{Var}(\hat{\epsilon}_i) = \mathbf{Var}(Y_i) + \mathbf{Var}(\hat{\alpha}) + x_i^2\mathbf{Var}(\hat{\beta}) - 2\mathbf{Cov}(Y_i, \hat{\alpha}) - 2x_i\mathbf{Cov}(Y_i, \hat{\beta}) + 2x_i\mathbf{Cov}(\hat{\alpha}, \hat{\beta})$.**

   (c) **Use Lemma 12.2.1 to show that $\mathbf{Cov}(Y_i, \hat{\alpha}) = \sigma^2\left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}\right)$ and $\mathbf{Cov}(Y_i, \hat{\beta}) = \sigma^2\frac{x_i - \bar{x}}{S_{xx}}$, and use these to verify the equation for $\mathbf{Var}(\hat{\epsilon})$, (12.2.23).**

   (a) $E(\epsilon_i) = E(Y_i - \hat{\alpha} - \hat{\beta}x_i) = E(Y_i) - E(\hat{\alpha}) - x_iE(\hat{\beta}) = (\alpha + \beta x_i) - \alpha - \beta x_i = 0.$

   (b)

$$\begin{aligned} Var(\hat{\epsilon}_i) &= Var(Y_i - \hat{\alpha} - \hat{\beta}x_i) \\ &= Var(Y_i) + Var(\hat{\alpha}) + x_i^2Var(\hat{\beta}) - 2Cov(Y_i, \hat{\alpha}) - 2x_iCov(Y_i, \hat{\beta}) + 2x_iCov(\hat{\alpha}, \hat{\beta}) \end{aligned}$$

(c) From Problem 3 of Homework 10, we can represent $\hat{\alpha} = \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{(x_k-\bar{x})\bar{x}}{S_{xx}}\right) Y_k$, therefore

$$\begin{aligned}
Cov(Y_i, \hat{\alpha}_k) &= Cov\left(Y_i, \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{(x_k-\bar{x})\bar{x}}{S_{xx}}\right) Y_k\right) \\
&= Cov\left(Y_i, \left(\frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{S_{xx}}\right) Y_i\right) + Cov\left(Y_i, \sum_{k\neq i} \left(\frac{1}{n} - \frac{(x_k-\bar{x})\bar{x}}{S_{xx}}\right) Y_k\right) \\
&= Cov\left(Y_i, \left(\frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{S_{xx}}\right) Y_i\right) + \sum_{k\neq i} Cov\left(Y_i, \left(\frac{1}{n} - \frac{(x_k-\bar{x})\bar{x}}{S_{xx}}\right) Y_k\right) \\
&= \left(\frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{S_{xx}}\right) Cov(Y_i, Y_i) = \left(\frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{S_{xx}}\right) \sigma^2
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
Cov(Y_i, \hat{\beta}_j) &= Cov\left(Y_i, \sum_{j=1}^{n} \frac{x_j-\bar{x}}{S_{xx}} Y_j\right) \\
&= Cov\left(Y_i, \frac{x_i-\bar{x}}{S_{xx}} Y_i\right) + Cov\left(Y_i, \sum_{j\neq i} \frac{x_j-\bar{x}}{S_{xx}} Y_j\right) \\
&= Cov\left(Y_i, \frac{x_i-\bar{x}}{S_{xx}} Y_i\right) + \sum_{j\neq i} Cov\left(Y_i, \frac{x_j-\bar{x}}{S_{xx}} Y_j\right) \\
&= \frac{x_i-\bar{x}}{S_{xx}} Cov(Y_i, Y_i) = \frac{x_i-\bar{x}}{S_{xx}} \sigma^2
\end{aligned}$$

therefore

$$\begin{aligned}
Var(\hat{\epsilon}_i) &= Var(Y_i) + Var(\hat{\alpha}) + x_i^2 Var(\hat{\beta}) - 2Cov(Y_i, \hat{\alpha}) - 2x_i Cov(Y_i, \hat{\beta}) + 2x_i Cov(\hat{\alpha}, \hat{\beta}) \\
&= \sigma^2 + \frac{\sigma^2}{nS_{xx}} \sum_{i=1}^{n} x_i^2 + \frac{\sigma^2 x_i^2}{S_{xx}} - 2\left(\frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{S_{xx}}\right)\sigma^2 - 2x_i \frac{(x_i-\bar{x})}{S_{xx}} \sigma^2 + 2x_i \left(\frac{-\sigma^2 \bar{x}}{S_{xx}}\right) \\
&= \sigma^2 \left[1 + \frac{1}{nS_{xx}} \sum_{i=1}^{n} x_i^2 + \frac{x_i^2}{S_{xx}} - 2\left(\frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{S_{xx}}\right) - 2x_i \frac{(x_i-\bar{x})}{S_{xx}} - 2x_i \left(\frac{\bar{x}}{S_{xx}}\right)\right] \\
&= \sigma^2 \left[\frac{n-2}{n} + \frac{1}{S_{xx}} \left(\frac{\sum_{j=1}^{n} x_j^2}{n} + x_i^2 + 2\bar{x}(x_i-\bar{x}) - 2x_i(x_i-\bar{x}) - 2x_i\bar{x}\right)\right] \\
&= \sigma^2 \left[\frac{n-2}{n} + \frac{1}{S_{xx}} \left(\frac{\sum_{j=1}^{n} x_j^2}{n} + x_i^2 - 2x_i\bar{x} - 2(x_i-\bar{x})^2\right)\right]
\end{aligned}$$

3. **Fill in the details about the distribution of $\hat{\alpha}$ left out of the proof of Theorem 12.2.1.**

   (a) **Show that the estimator $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$ can be expressed as $\hat{\alpha} = \sum_{i=1}^{n} c_i Y_i$, where $c_i \frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{S_{xx}}$.**

   (b) **Verify that $E(\hat{\alpha}) = \alpha$ and $Var(\hat{\alpha}) = \sigma^2 \left[\frac{1}{nS_{xx}} \sum_{i=1}^{n} x_i^2\right]$.**

   (c) **Verify that $Cov(\hat{\alpha}, \hat{\beta}) = \frac{-\sigma^2 \bar{x}}{S_{xx}}$.**

(a)

$$\begin{aligned}
\hat{\alpha} &= \bar{y} - \hat{\beta}\bar{x} = \bar{y} - \frac{S_{xy}}{S_{xx}}\bar{x} \\
&= \sum_{i=1}^{n} \frac{y_i}{n} - \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{S_{xx}}\bar{x} \\
&= \sum_{i=1}^{n} \frac{y_i}{n} - \frac{\bar{x}\sum_{i=1}^{n}(x_i - \bar{x})y_i}{S_{xx}} + \frac{\bar{x}\sum_{i=1}^{n}(x_i - \bar{x})\bar{y}}{S_{xx}},
\end{aligned}$$

where $\frac{\sum_{i=1}^{n}(x_i-\bar{x})\bar{y}}{S_{xx}} = \frac{\bar{x}\bar{y}}{S_{xx}}(n\bar{x} - n\bar{x}) = 0$, therefore $\hat{\alpha} = \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{S_{xx}}\right)y_i = \sum_{i=1}^{n} c_i y_i$ where $c_i = \frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{S_{xx}}$.

(b)

$$\begin{aligned}
E(\hat{\alpha}) &= E\left(\sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}\right)y_i\right) \\
&= \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}\right)E(y_i) \\
&= \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}\right)(\alpha + \beta x_i) \\
&= \alpha\sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}\right) + \beta\sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}\right)x_i \\
&= \alpha\left(1 - \frac{\bar{x}}{S_{xx}}(n\bar{x} - n\bar{x})\right) + \beta\left(1 - \frac{S_{xx}}{S_{xx}}\right) = \alpha
\end{aligned}$$

Meanwhile,

$$Var(\hat{\alpha}) = Var\left(\sum_{i=1}^{n} c_i Y_i\right) = \sum_{i=1}^{n} c_i^2 Var(Y_i) = \sigma^2\sum_{i=1}^{n} c_i^2$$

where $c_i = \frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{S_{xx}}$.

$$\begin{aligned}
\sum_{i=1}^{n} c_i^2 &= \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}\right)^2 = \sum_{i=1}^{n}\left(\frac{1}{n^2} - \frac{2(x_i - \bar{x})\bar{x}}{nS_{xx}} + \frac{(x_i - \bar{x})^2\bar{x}^2}{S_{xx}^2}\right) \\
&= \frac{n}{n^2} - \frac{2\bar{x}}{nS_{xx}}\sum_{i=1}^{n}(x_i - \bar{x}) + \frac{\bar{x}^2}{S_{xx}^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 \\
&= \frac{1}{n} + \frac{\bar{x}^2 S_{xx}}{S_{xx}^2} = \frac{S_{xx} + n\bar{x}^2}{nS_{xx}},
\end{aligned}$$

where

$$\begin{aligned}
S_{xx} + n\bar{x}^2 &= \sum_{i=1}^{n}(x_i - \bar{x})^2 + n\bar{x}^2 \\
&= \sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2) + n\bar{x}^2 \\
&= \sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + n\bar{x}^2 + n\bar{x}^2 \\
&= \sum_{i=1}^{n} x_i^2 - 2n\bar{x}^2 + 2n\bar{x}^2 = \sum_{i=1}^{n} x_i^2,
\end{aligned}$$

therefore $Var(\hat{\alpha}) = \sigma^2 \sum_{i=1}^{n} c_i^2 = \sigma^2 \left(\frac{1}{nS_{xx}} \sum_{i=1}^{n} x_i^2\right)$.

(c) $Cov(\hat{\alpha}, \hat{\beta})$ where $\hat{\alpha} = \sum_{i=1}^{n} c_i Y_i$ where $c_i = \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}$. Similarly, $\hat{\beta} = \sum_{j=1}^{n} \frac{(x_j - \bar{x})}{S_{xx}} Y_j$ where $d_j = \frac{(x_j - \bar{x})}{S_{xx}}$, therefore $Cov(\hat{\alpha}, \hat{\beta}) = Cov\left(\sum_{i=1}^{n} c_i Y_i, \sum_{j=1}^{n} d_j Y_j\right) = \sigma^2 \sum_{i=1}^{n} c_i d_i$ by Lemma 12.2.1, where

$$\begin{aligned}
\sum_{i=1}^{n} c_i d_i &= \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}\right)\left(\frac{(x_i - \bar{x})}{S_{xx}}\right) \\
&= \frac{1}{nS_{xx}} \sum_{i=1}^{n}(x_i - \bar{x}) - \frac{\bar{x}}{S_{xx}^2} \sum_{i=1}^{n}(x_i - \bar{x})^2 \\
&= \frac{1}{nS_{xx}}(n\bar{x} - n\bar{x}) - \frac{\bar{x}S_{xx}}{S_{xx}^2} = \frac{-\bar{x}}{S_{xx}},
\end{aligned}$$

thus $Cov(\hat{\alpha}, \hat{\beta}) = \frac{-\sigma^2\bar{x}}{S_{xx}}$.

4. **We obtain observations $Y_1, \ldots, Y_n$ which can be described by the relationship $Y_i = \theta x_i^2 + \epsilon_i$, where $x_1, \ldots, x_n$ are fixed constants and $\epsilon_1, \ldots, \epsilon_n$ are iid $\mathbf{N}(0, \sigma^2)$.**

   (a) Find the least squares estimator of θ.

   (b) Find the MLE of θ.

(a) We know that $Y_i = \theta x_i^2 + \epsilon_i$, thus $\epsilon_i = Y_i - \theta x_i^2$. The least squares estimator minimizes

$$\begin{aligned}
RSS &= \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n}(Y_i - \theta x_i^2)^2 \\
\frac{\partial RSS}{\partial \theta} &= 2\sum_{i=1}^{n}(y_i - \theta x_i^2)(-x_i^2) \doteq 0 \\
&\therefore\ -2\left(\sum_{i=1}^{n} x_i^2 y_i - \theta \sum_{i=1}^{n} x_i^4\right) = 0,
\end{aligned}$$

which implies that $\hat{\theta}_{LSE} = \frac{\sum_{i=1}^{n} x_i^2 y_i}{\sum_{i=1}^{n} x_i^4}$ is the least squares estimator of θ.

(b) $Y_i \sim \mathrm{N}(\theta x_i^2, \sigma^2)$, thus it has the density function $f_{Y_i}(y_i) = (2\pi\sigma^2)^{-1/2} \exp\left(\frac{-1}{2\sigma^2}(y_i - \theta x_i^2)^2\right)$, so

$$
\begin{aligned}
L(\theta; \boldsymbol{x}, y) &= (2\pi\sigma^2)^{-n/2} \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \theta x_i^2)^2 \right) \\
\ln L(\theta; \boldsymbol{x}, y) &= \frac{-n}{2} \ln(2\pi\sigma^2) + \left(\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \theta x_i^2)^2 \right) \\
\frac{\partial \ln L(\theta; \boldsymbol{x}, y)}{\partial \theta} &= \frac{-1}{\sigma^2} \sum_{i=1}^{n} (y_i - \theta x_i^2)(-x_i^2) \doteq 0 \\
\therefore & \sum_{i=1}^{n} x_i^2 y_i - \theta \sum_{i=1}^{n} x_i^4 = 0,
\end{aligned}
$$

thus $\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} x_i^2 y_i}{\sum_{i=1}^{n} x_i^4} = \hat{\theta}_{LSE}$, i.e. the MLE equals the least squares estimator of θ.