Homework1

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Problem 1

Geometrically describe with reasoning the following unit ball in $\mathbb{R}^2$.

$B_\infty^{(1)} = \{x \ \in \ \mathbb{R}^2 \mid \|x\|_\infty \leq 1\}$

Response We recall the definition of the L-∞ norm, which is $\|x\|_\infty$. If $x = (x_1, ..., x_n)$, then $\|x\|_\infty = max\{|x_1|, ..., |x_n|\}$, the largest magnitude of any component of x.

$B_\infty^{(1)}$ is the area defined by all vectors whose L-∞ norm is less than or equal to 1.

$\|x\|_\infty = 1$ for all (x_1, x_2) where either $|x_1| = 1$ and $|x_1| \leq 1$ or $|x_1| \leq 1$ and $|x_1| = 1$. This is therefore the shape of a square.

Problem 2

Prove the following triangle inequality.

$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$

Response We recall the definition of the L-2 norm as the Euclidean distance given by $\|x\|_2 = (|x_1|^2 + ... + |x_n|^2)^{\frac{1}{2}}$

We introduce inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$.

Then $\|x + y\|_2^2 = \|x\|_2^2 + 2\langle x, y \rangle + \|y\|_2^2 \leq \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2 \ \|y\|_2$ by Cauchy-Schwarz.

So we have $\|x + y\|_2^2 \leq (\|x\|_2 + \|y\|_2)^2$ which gives us $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$, the triangle inequality for the L-2 norm.

So we have $\|x + y\| \leq \|x\| + \|y\|$.

Problem 3

Prove that for any matrix $A = (a_{ij})_{m \times n}$, the matrix $A^T A$ is symmetric.

Response We know that for any matrix A, B, then $(AB)^T = B^T A^T$.

Consider $(A^T A)^T = A^T (A^T)^T = A^T A$.

We see $(A^T A)^T = A^T A$, so $A^T A$ is symmetric.