Homework 10

Nathan Bick

1. Observations (x_i, Y_i), $i = 1, \ldots, n$, are collected according to the model $Y_i = \alpha + \beta x_i + \varepsilon_i$, where $E(\varepsilon_i) = 0$, $V(\varepsilon_i) = \sigma^2$ and $Cov(\varepsilon_i, \varepsilon_j) = 0$ if $i \neq j$. Find the best linear unbiased estimator of α.

we know if $\sum_{i=1}^n a_i Y_i$ is unbiased for α then

$$E\left(\sum_{i=1}^n d_i Y_i\right) = \sum_{i=1}^n d_i(\alpha + \beta x_i) = \alpha \Rightarrow \sum_{i=1}^n \alpha_i = 1 \text{ and } \sum_{i=1}^n d_i x_i = 0$$

we want to minimize variance, so we see

$$V\left(\sum_{i=1}^n d_i Y_i\right) = \sigma^2 \sum_{i=1}^n d_i^2,$$ so we want to solve:

$$\min \sum_{i=1}^n d_i^2 \text{ s.t. } \sum_{i=1}^n d_i = 1 \text{ and } \sum_{i=1}^n d_i x_i = 0.$$

Try the following: $\sum_{i=1}^n d_i = 1 \Rightarrow d_i = \frac{1}{n} + k(b_i - \bar{b})$ for constants.

$$\sum_{i=1}^n d_i x_i = 0 \Rightarrow k = \frac{-\bar{x}}{\sum_{i=1}^n (b_i - \bar{b})(x_i - \bar{x})}$$ and

$$d_i = \frac{1}{n} - \frac{\bar{x}(b_i - \bar{b})}{\sum_i (b_i - \bar{b})(x_i - \bar{x})}$$

so $$\sum_{i=1}^n d_i^2 = \sum_{i=1}^n \left(\frac{1}{n} - \frac{\bar{x}(b_i - \bar{b})}{\sum_{i=1}^n (b_i - \bar{b})(x_i - \bar{x})}\right)^2$$

$$= \frac{1}{n} + \frac{\bar{x}^2 \sum_{i=1}^n (b_i - \bar{b})^2}{\left(\sum_{i=1}^n (b_i - \bar{b})(x_i - \bar{x})\right)^2} + 0.$$

we see $$\frac{\sum_{i=1}^n (b_i - \bar{b})^2}{\sum_{i=1}^n (b_i - \bar{b})(x_i - \bar{x})^2} \geq \frac{1}{\sum_{i=1}^n (x_i - \bar{x})^2}$$ by Cauchy Schwarz,

and set $b_i = x_i$ minimizes. so we get

$$d_i = \frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{\sum (x_i - \bar{x})^2} \Rightarrow \sum_{i=1}^n d_i Y_i = \bar{Y} - \hat{\beta}\bar{x}$$

2. Consider the residuals $\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n$ defined by
$\hat{\varepsilon}_i = Y_i - \hat{\alpha} - \hat{\beta} x_i$.

(a) Show that $E(\hat{\varepsilon}_i) = 0$

(b) Verify that $V(\hat{\varepsilon}_i) = V(Y_i) + V(\hat{\alpha}) + x_i^2 V(\hat{\beta}) - 2\,Cov(Y_i, \hat{\alpha})$
$- 2x_i Cov(Y_i, \hat{\beta}) + 2x_i Cov(\hat{\alpha}, \hat{\beta})$.

(c) Use Lemma 12.2.1 to show that $Cov(Y_i, \hat{\alpha}) = \sigma^2\left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}\right)$
and $Cov(Y_i, \hat{\beta}) = \sigma^2\left(\frac{x_i - \bar{x}}{S_{xx}}\right)$, and use these to verify the equation for $V(\hat{\varepsilon})$.

(a) $E(\hat{\varepsilon}_i) = E(Y_i - \hat{\alpha} - \hat{\beta} x_i) = \alpha + \beta x_i - \alpha - \beta x_i = 0$

(b) $V(\hat{\varepsilon}_i) = E(Y_i - \hat{\alpha} - \hat{\beta} x_i)^2$ since $E(\hat{\varepsilon}_i) = 0$.

$$= E\left((Y_i - \alpha - \beta x_i) - (\hat{\alpha} - \alpha) - x_i(\hat{\beta} - \beta)\right)^2$$
$$= V(Y_i) + V(\hat{\alpha}) + x_i^2 V(\hat{\beta}) - 2\,Cov(Y_i, \hat{\alpha})$$
$$- 2x_i Cov(Y_i, \hat{\beta}) + 2x_i Cov(\hat{\alpha}, \hat{\beta}).$$

(c) ~~ε_i~~ $Y_i = \hat{\alpha} + \hat{\beta} x_i + \hat{\varepsilon}_i$

Lemma: $Cov\left(\sum_{i=1}^n c_i Y_i, \sum_{i=1}^n d_i Y_i\right) = \left(\sum_{i=1}^n c_i d_i\right)\sigma^2$

we see $\hat{\alpha} = \hat{\beta} x_i + \hat{\varepsilon} - Y_i$, $\hat{\beta} =$

3. Fill in the details about the distribution of $\hat{\alpha}$ left out of the proof of THM 12.2.1.

(a) Show that the estimator $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{x}$ can be expressed as $\hat{\alpha} = \sum_{i=1}^{n} c_i Y_i$ where $c_i = \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}$.

(b) Verify that $E(\hat{\alpha}) = \alpha$ and $V(\hat{\alpha}) = \sigma^2\left(\frac{1}{nS_{xx}} \sum_{i=1}^{n} x_i^2\right)$.

(c) Verify that $Cov(\hat{\alpha}, \hat{\beta}) = \frac{-\sigma^2 \bar{x}}{S_{xx}}$.

(a) $\alpha = \bar{y} - \hat{\beta}\bar{x} = \sum_1^n \frac{1}{n} Y_i - \frac{\bar{x}\sum_1^n (x_i - \bar{x}) Y_i}{\sum_1^n (x_i - \bar{x})^2} = \sum_1^n \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{\sum (x_i - \bar{x})^2}\right) Y_i$

(b) Note $c = \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{\sum_1^n (x_i - \bar{x})^2}$, $\sum_1^n c_i x_i = 0$, $\sum_1^n c_i = 1$.

so $E(\hat{\alpha}) = E\left(\sum_{i=1}^n c_i Y_i\right) = \sum_{i=1}^n c_i(\alpha + \beta x_i) = \alpha$

$V(\hat{\alpha}) = \sum_{i=1}^n c_i^2 V(Y_i) = \sigma^2 \sum_{i=1}^n c_i^2$.

$\sum_{i=1}^n c_i^2 = \sum_{i=1}^n \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{S_{xx}}\right)^2 = \sum_{i=1}^n \frac{1}{n^2} + \frac{\sum \bar{x}^2 (x_i - \bar{x})^2}{(\sum (x_i - \bar{x})^2)^2}$

$= \frac{1}{n} + \frac{\bar{x}^2}{\sum (x_i - \bar{x})^2} = \frac{\sum x_i^2}{nS_{xx}}$. so we have

$V(\hat{\alpha}) = \sigma^2 \frac{1}{nS_{xx}} \sum_i^n x_i^2$.

(c) $\hat{\beta} = \sum_{i=1}^n d_i Y_i$, s.t. $d_i = \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2}$

$Cov(\hat{\alpha}, \hat{\beta}) = Cov\left(\sum_{i=1}^n c_i Y_i, \sum_{i=1}^n d_i Y_i\right) = \sigma^2 \sum_{i=1}^n c_i d_i$

$= \sigma^2 \sum_{i=1}^n \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{S_{xx}}\right)\left(\frac{(x_i - \bar{x})}{S_{xx}}\right) = \frac{-\sigma^2 \bar{x}}{S_{xx}}$.

4. We obtain observations $Y_1, \ldots, Y_n$ which can be described by the relationship $Y_i = \theta x_i^2 + \varepsilon_i$, where $x_1, \ldots, x_n$ are fixed constants and $\varepsilon_1, \ldots, \varepsilon_n$ are iid $N(0, \sigma^2)$.

(a) Find the least squares estimator of θ.

(b) Find the MLE of θ.

(a) $$\frac{\partial}{\partial \theta} \sum_{i=1}^{n} (Y_i - \theta x_i^2)^2 = 2 \sum_{i=1}^{n} (Y_i - \theta x_i^2) x_i^2) = 0$$

$$\Rightarrow \hat{\theta} = \frac{\sum_{i=1}^{n} Y_i x_i^2}{\sum_{i=1}^{n} x_i^4}.$$

(b) $$f(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(Y_i - \theta x_i^2)^2\right)$$

$$\Rightarrow \mathcal{L}(x) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \theta x_i^2)^2\right)$$

$$\Rightarrow \log \mathcal{L} = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \theta x_i^2)^2$$

$\frac{\partial}{\partial \theta} \log \mathcal{L} = \frac{\partial}{\partial \theta} \sum_{i=1}^{n} (Y_i - \theta x_i^2)^2$. Note that this is exactly the same as what we had in previous part, so MLE is $\hat{\theta} = \frac{\sum Y_i x_i^2}{\sum x_i^4}$ and is the same as the least squares estimator.