MATH 503: Mathematical Statistics
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Homework 7

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\mathrm{N}(\mu_0, \sigma^2 = \theta)$ distribution, where $0 < \theta < \infty$ and μ_0 is known. Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ can be based upon the statistic $W = \sum_{i=1}^{n}(X_i - \mu_0)^2/\theta_0$. Determine the null distribution of W and give, explicitly, the rejection rule for a level α test.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample form a Poisson distribution with mean $\theta > 0$.

   (a) Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is based upon the statistic $Y = \sum_{i=1}^{n} X_i$. Obtain the null distribution of Y.

   (b) For $\theta_0 = 2$ and $n = 5$, find the significance level of the test that rejects H_0 if $Y \leq 4$ or $Y \geq 17$.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Beta distribution with $\alpha = \beta = \theta$ and $\Omega = \{\theta : \theta = 1, 2\}$. Show that the likelihood ratio test statistic Λ for testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ is a function of the statistic $W = \sum_{i=1}^{n} \log X_i + \sum_{i=1}^{n} \log(1 - X_i)$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pmf $p(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, where $0 < \theta < 1$. We wish to test $H_0 : \theta = \frac{1}{3}$ versus $H_1 : \theta \neq \frac{1}{3}$.

   (a) Find Λ and $-2\log\Lambda$.

   (b) Determine the Wald-type test.

   (c) What is Rao's score statistic?

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\Gamma(\alpha, \beta)$-distribution where α is known and $\beta > 0$. Determine the likelihood ratio test for $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\mathrm{N}(0, \sigma^2 = \theta)$ distribution, where $\theta > 0$ unknown. Consider $H_0 : \theta = \theta'$ versus $H_1 : \theta < \theta'$. Show that the set $\{(x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i^2 \leq c\}$ is a uniformly most powerful critical region for testing H_0 versus H_1.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\mathrm{N}(0, \sigma^2 = \theta)$ distribution, where $\theta > 0$ unknown. Consider $H_0 : \theta = \theta'$ versus $H_1 : \theta \neq \theta'$. Show that there is no uniformly most powerful test for testing H_0 versus H_1.

8. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\mathrm{N}(\theta, \sigma^2 = 16)$ distribution. Find the sample size n and a uniformly most powerful test of $H_0 : \theta = 25$ against $H_1 : \theta < 25$ with power function $\gamma(\theta)$ so that approximately $\gamma(25) = 0.10$ and $\gamma(23) = 0.90$.

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1, \text{ zero elsewhere,}$$

where $\theta > 0$. Find a sufficient statistic for θ and show that a uniformly most powerful test of $H_0 : \theta = 6$ against $H_1 : \theta < 6$ is based on this statistic.