MATH 503: Mathematical Statistics
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Homework 6

1. Let X have a pdf of the form $f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1$, zero elsewhere, where $\theta \in \{\theta : \theta = 1, 2\}$. To test the simple hypothesis $H_0 : \theta = 1$ against the alternative simple hypothesis $H_1 : \theta = 2$, use the random sample X_1, X_2 of size $n = 2$ and define the critical region $C = \left\{(x_1, x_2) : \frac{3}{4} \leq x_1 x_2\right\}$. Find the power function of the test.

2. Let us say the life of a tire in miles, say X, is normally distributed with mean θ and standard deviation 5000. Past experience indicates that $\theta = 30,000$. The manufacturer claims that the tires made by a new process have mean $\theta > 30,000$. It is possible that $\theta = 35,000$. Check his claim by testing $H_0 : \theta = 30,000$ against $H_1 : \theta > 30,000$. We shall observe n independent values of X, say $x_1, \ldots, x_n$, and we shall reject H_0 (thus accept H_1) if and only if $x \geq c$. Determine n and c so that the power function $\gamma(\theta)$ of the test has the values $\gamma(30,000) = 0.01$ and $\gamma(35,000) = 0.98$.

3. Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample X_1, X_2, X_3, X_4 of size $n = 4$ from a distribution with pdf $f(x;\theta) = \frac{1}{\theta}, 0 < x < \theta$, zero elsewhere, where $\theta > 0$. The hypothesis $H_0 : \theta = 1$ is rejected and $H_1 : \theta > 1$ is accepted if the observed $Y_4 \geq c$.
   (a) Find the constant c so that the significance level is $\alpha = 0.05$.
   (b) Determine the power function of the test.

4. Assume that the weight of cereal in a "10-ounce box" is $\mathrm{N}(\mu, \sigma^2)$. To test $H_0 : \mu = 10.1$ against $H_1 : \mu > 10.1$, we take a random sample of size $n = 16$ and observe $x = 10.4$ and $s = 0.4$.
   (a) Do we reject or fail to reject H_0 at the 5% significance level?
   (b) What is the approximate p-value of this test?

5. Each of 51 golfers hit three golf blls of brand X and three golf balls of brand Y in a random order. Let X_i and Y_i equal the averages of the distances traveled by the brand X and brand Y golf balls hit by the ith golfer, $i = 1, 2, \ldots, 51$. Let $W_i = X_i - Y_i$, $i = 1, 2, \ldots, 51$ and test $H_0 : \mu_W = 0$ against $H_1 : \mu_W > 0$, where μ_W is the mean of the differences. If $\bar{W} = 2.07$ and $s_W^2 = 84.63$, would H_0 be rejected at the 5% significance level? What is the p-value of this test?

6. Let the random variable X have the pdf $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. Consider the simple hypothesis $H_0 : \theta = \theta' = 2$ and the alternative hypothesis $H_1 : \theta = \theta'' = 4$. Let X_1, X_2 denote a random sample of size 2 from this distribution. Show that the best test of H_0 against H_1 may be carried out by use of the statistic $X_1 + X_2$.

7. If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution having pdf of the form $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, zero elsewhere, show that a best critical region for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$ is $C = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : c \leq \prod_{i=1}^{n} x_i\}$.

8. If $X_1, X_2, \ldots, X_n$ is a random sample from a beta distribution with parameters $\alpha = \beta = \theta > 0$, find a best critical region for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$.