BICK-hw11

November 28, 2022

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1 Problem 1

Find the quadratic polynomial $p_2(x)$ that interpolates the data

x	1	2	4
y	-1	-1	2

(a) using the Lagrange method;

(b) using the method of undetermined coefficients.

(a) The Lagrange method is to use the lagrange polynomials in the form:

$p(x) = \sum_{i=0}^{n} y_i l_i(x)$

where $l_i(x)$ is the i th lagrange polynomial:

$l_i(x) = \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j}$

We create the lagrange polynomials for each data point:

$l_0(x) = \frac{(x-2)(x-4)}{(1-2)(1-4)} = \frac{x^2-6x+8}{3}$

$l_1(x) = \frac{(x-1)(x-4)}{(2-1)(2-4)} = \frac{x^2-5x+4}{-2}$

$l_2(x) = \frac{(x-1)(x-2)}{(4-1)(4-2)} = \frac{x^2-3x+2}{6}$

Then we can create the interpolating polynomial:

$p_2(x) = -1\frac{x^2-6x+8}{3} - 1\frac{x^2-5x+4}{-2} + 2\frac{x^2-3x+2}{6} = \frac{x^2-3x}{2}$

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(b) Now we use the method of undetermined coefficients. This is given by solving the system of equations:

$a_0 + a1x_0 + a_2x_0^2 + ... + a_nx_0^n = y_0$

$a_0 + a1x_1 + a_2x_1^2 + ... + a_nx_1^n = y_1$

...

$a_0 + a1x_n + a_2x_n^2 + ... + a_nx_n^n = y_n$

where x_i is the i th data point and y_i is the i th data point. We can write this as a matrix equation:

$$\begin{pmatrix} 1 & x_0 & x_0^2 & ... & x_0^n \\ 1 & x_1 & x_1^2 & ... & x_1^n \\ ... & ... & ... & ... & ... \\ 1 & x_n & x_n^2 & ... & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ ... \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ ... \\ y_n \end{pmatrix}$$

In our problem, we have $n = 2$ so we can write this as:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

We can solve this system of equations to find the coefficients:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3/2 \\ 1/2 \end{pmatrix}$$

This gives us the interpolating polynomial:

$p_2(x) = 0 - \frac{3}{2}x + \frac{1}{2}x^2$

And we see that this is the same result as we achieved in part a using the lagrange method. This is not surprising because th interpolating polynomial is unique.

2 Problem 2

Add the data point $(x_3, y_3) = (3, 0)$ to the data in Problem 1.

(a). Use Newton's method to find the cubic interpolating polynomial $p_3(x)$ for the resulting data.

(b) Find $p_3(x)$ using the Lagrange method.

Newton's method allows us to add an additional term without recalculating the entire polynomial. We can use the formula:

$a_{n+1} = \frac{y_{n+1} - p_n(x_{n+1})}{(x_{n+1} - x_o)...(x_{n+1} - x_n)}$

where a_n is the n th term of the interpolating polynomial.

We know $y_{n+1} = y_3 = 0$ and $x_{n+1} = x_3 = 3$. We can use the previous polynomial to find $p_n(x_{n+1})$:

$p_2(x_3) = 0 - \frac{3}{2}x_3 + \frac{1}{2}x_3^2 = 0 - \frac{3}{2}3 + \frac{1}{2}9 = 0$

We calculate the denominator as:

$(x_{n+1} - x_o)...(x_{n+1} - x) = (3 - 1)(3 - 2)(3 - 4) = -2$

Putting it together, we get:

$a_{n+1} = \frac{0-0}{-2} = 0$

This means that the cubic interpolating polynomial is the same as the quadratic interpolating polynomial, as the coefficient of x^3 is zero.

To find the p_3 using the lagrange method, we must calculate the lagrange polynomials for each data point, as if from the beginning.

$l_0(x) = \frac{(x-2)(x-4)(x-3)}{(1-2)(1-4)(1-3)} = \frac{-x^3+9x^2-26x+24}{6}$

$l_1(x) = \frac{(x-1)(x-4)(x-3)}{(2-1)(2-4)(2-3)} = \frac{x^3-8x^2+19x-12}{2}$

$l_2(x) = \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} = \frac{x^3-6x^2+11x-6}{6}$

$l_3(x) = \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} = \frac{-x^3+7x^2-14x+8}{2}$

Then we can create the interpolating polynomial:

$p_3(x) = -1\frac{-x^3+9x^2-26x+24}{6} - 1\frac{x^3-8x^2+19x-12}{2} + 2\frac{x^3-6x^2+11x-6}{6} + 0\frac{-x^3+7x^2-14x+8}{2} = \frac{x^2-3x}{2}$

We see that we get the same results as we did when using the newton's method.

3 Problem 3

Find the trigonometric function $T(x) = a_0 + a_1 \sin x + a_2 \sin 2x$ that interpolates the following data

x	0	$\pi/2$	$\pi/3$
y	-1	2	1

We can use the given data to use the method of undetermined coefficients and create the system of equations:

$a_0 + a_1 \sin 0 + a_2 \sin 0 = -1$

$a_0 + a_1 \sin \frac{\pi}{2} + a_2 \sin \frac{2\pi}{2} = 2$

$a_0 + a_1 \sin \frac{\pi}{3} + a_2 \sin \frac{2\pi}{3} = 1$

We can solve this using backward substitution. First we see that $a_0 = -1$. We substitute this value and evaluate the sinusoids in second equation:

$-1 + a_1 \sin \pi/2 + a_2 \sin \pi = 2 \Rightarrow -1 + a_1 + 0 = 2 \Rightarrow a_1 = 3$

Finally, we can solve for a_2 in a similar manner, substiuting the values of a_0 and a_1, and evaluating the equation:

$-1 + 3 \sin \frac{\pi}{3} + a_2 \sin \frac{2\pi}{3} = 1 \Rightarrow$

$-1 + 3\frac{\sqrt{3}}{2} + a_2 \frac{\sqrt{3}}{2} = 1 \Rightarrow$

$2 - 3\frac{\sqrt{3}}{2} = a_2 \frac{\sqrt{3}}{2} \Rightarrow$

$a_2 = \frac{4-3\sqrt{3}}{\sqrt{3}}$

4 Problem 4

Find the quartic interpolating polynomial $p_4(x)$ for $f(x) = e^{3x}$ using $x_0 = -1$, $x_1 = -0.5$, $x_2 = 0$, $x_3 = 0.5$, and $x_4 = 1$. For $x = 0.08$

(a) compute $e_4(x) = f(x) - p_4(x)$;

(b) estimate the error $e_4(x)$ using the bound formula given in the lecture.

First we must find the quartic interpolating polynomial. We create the data points x_i and y_i using the function f:

x	-1	-0.5	0	0.5	1
y	$e^{-3} = 0.0497$	$e^{-3/2} = 0.223$	1	$e^{3/2}$=4.486	e^3=20.085

Now we can use the lagrange method to find the interpolating polynomial:

$p_4(x) = 0.0497 * \frac{(x+0.5)(x-0)(x-0.5)(x-1)}{(-1+0.5)(-1)(-1-0.5)(-1-1)} + 0.223 * \frac{(x+1)(x)(x-0.5)(x-1)}{(-0.5+15)(-0.5)(-0.5-0.5)(-0.5-1)} + 1 * \frac{(x+1)(x+0.5)(x-0.5)(x-1)}{(0+1)(0+0.5)(0-0.5)(0-1)} + 4.4486 * \frac{(x+1)(x+0.5)(x)(x-1)}{(0.5+1)(0+0.5)(0.5)(0.5-1)} + 20.085 * \frac{(x+1)(x+0.5)(x-0)(x-0.5)}{(1+1)(1+0.5)(1-0)(0-0.5)}$

This gives us the following polynomial:

$p_4(x) = 0.0497 * (1.5x^4 - 1.5x^3 - 0.375x^2 + 0.375x) + 0.223 * (-10.875x^4 + 5.4375x^3 + 10.875x^2 - 5.4375x) + 0.25 * (x^4 - 0.3125x^2 + 0.0625) + 4.4486 * (-0.1875x^4 - 0.09375x^3 + 0.1875x^2 + 0.09375x) + 20.085 * (-1.5x^4 - 1.5x^3 + 0.375x^2 + 0.375x)$

Then the error is given by:

$f(x) - p_4(x) = e^{3x} - p_4(x)$

The error of interpolation, as discussed in the lecture, is given by:

$e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} \prod_{i=0}^{n}(x - x_i)$

The specific value of ζ depends on x and is not available explicitly, so this can be bounded, if the derivative of f is bounded. In the case of our functino $f(x) = e^{3x}$, the derivative is $|f^{(n)}(x)| \leq 3^n e^{3x}$, so we can try to use this to bound the error:

$|e_n(x)| \leq \frac{3^{n+1}e^{3x}}{(n+1)!} \prod_{i=0}^{n} |x - x_i|$

However, the limit of this derivative is not bounded, so we cannot use this to bound the error. Within an interval, we could bound this by the max value of the function on the interval, but can do no better.