MATH 503: Mathematical Statistics
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Homework 8 Solutions

1. **Let $X_1, \ldots, X_n$ denote a random sample from a Poisson distribution with parameter θ, $0 < \theta < \infty$. Let $Y = \sum_{i=1}^n X_i$ and let $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If we restrict our considerations to decision functions of the form $\delta(y) = b + y/n$, where b does not depend on y, show that $R(\theta, \delta) = b^2 + \theta/n$. What decision function of this form yields a uniformly smaller risk than every other decision function of this form? With this solution, say δ, and $0 < \theta < \infty$, determine $\max_\theta R(\theta, \delta)$ if it exists.**

   $Y = \sum_{i=1}^n X_i$ is Poisson($n\theta$) distributed, therefore $E(Y) = Var(Y) = n\theta$.
   $L(\theta, \delta(y)) = (\theta - \delta(y))^2 = (\theta - (b + y/n))^2$, therefore

$$
\begin{aligned}
R(\theta,\delta) &= E[L(\theta,\delta(y))] = E[(\theta-\delta(y))^2] = E[(\theta-(b+y/n))^2] \\
&= Var(\theta-(b+y/n)) + E^2[\theta-(b+y/n)] \\
&= Var\left(-\frac{1}{n}y + (\theta-b)\right) + \left[E\left(-\frac{y}{n}+\theta-b\right)\right]^2 \\
&= \frac{1}{n^2}Var(y) + \left[-\frac{1}{n}(E(y)) + \theta - b\right]^2 \\
&= \frac{n\theta}{n^2} + \left(\frac{-n\theta}{n} + \theta - b\right)^2 \\
&= \frac{\theta}{n} + b^2.
\end{aligned}
$$

   Find b so that $R(\theta, \delta)$ is minimized implies to find b so that $\frac{\partial}{\partial b}R(\theta, \delta) = 2b = 0$, therefore, $b = 0$, which implies that $\delta(y) = y/n$. Then, $\max_\theta R(\theta, \delta) = \max_\theta \left(\frac{\theta}{n}\right)$ does not exist.

2. **Let $X_1, \ldots, X_n$ denote a random sample from a N(μ, θ) distribution, $0 < \theta < \infty$, where μ is unknown. Let $Y = \sum_{i=1}^n (X_i - \bar{X})^2/n$ and let $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If we consider decision functions of the form $\delta(y) = by$, where b does not depend on y, show that $R(\theta, \delta) = \frac{\theta^2}{n^2}[(n^2 - 1)b^2 - 2n(n - 1)b + n^2]$. Show that $b = \frac{n}{n+1}$ yields a minimum risk decision function of this form. Note that $\frac{nY}{n+1}$ is not an unbiased estimator of θ. With $\delta(y) = \frac{ny}{n+1}$ and $0 < \theta < \infty$, determine $\max_\theta R(\theta, \delta)$ if it exists.**

   $Y = \sum_{i=1}^n (X_i - \bar{X})^2/n$. Let $\delta(y)$ have the form $\delta(y) = by$ and $L(\theta, \delta(y)) = (\theta - \delta(y))^2$. Then,

$$R(\theta,\delta) = E[(\theta-y)^2] = Var(\theta-by) + E^2(\theta-by) = b^2 Var(Y) + [\theta - bE(Y)]^2,$$

where

$$
\begin{aligned}
Y &= \frac{n-1}{n} S^2 \\
nY &= (n-1)S^2 = \theta\left(\frac{(n-1)S^2}{\theta}\right) = \theta W,
\end{aligned}
$$

where $W \sim \chi^2_{n-1}$, therefore $\frac{n}{\theta}Y \sim \chi^2_{n-1}$, implying that

$$
\begin{aligned}
E\left(\frac{n}{\theta}Y\right) = n-1 \quad &\Rightarrow \quad E(Y) = \frac{(n-1)\theta}{n} \\
Var\left(\frac{n}{\theta}Y\right) = 2(n-1) \quad &\Rightarrow \quad Var(Y) = \frac{2(n-1)\theta^2}{n^2}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow R(\theta,\delta) &= b^2\left(\frac{2(n-1)\theta^2}{n^2}\right) + \left(\theta - \frac{b(n-1)\theta}{n}\right)^2 = \frac{\theta^2}{n^2}[2b^2(n-1) + (b(1-n)+n)^2] \\
&= \frac{\theta^2}{n^2}[2nb^2 - 2b^2 + b^2(1-n)^2 + 2nb(1-n) + n^2] \\
&= \frac{\theta^2}{n^2}[b^2(n^2-1) - 2n(n-1)b + n^2] \\
\frac{\partial}{\partial b}R(\theta,\delta) &= \frac{\theta^2}{n^2}[2(n^2-1)b - 2n(n-1)] = 0, \qquad (1)
\end{aligned}
$$

where Equation (**??**) implies that

$$
\begin{aligned}
2(n^2-1)b &= 2n(n-1) \\
b &= \frac{2n(n-1)}{2(n+1)(n-1)} = \frac{n}{n+1},
\end{aligned}
$$

thus $R(\theta,\delta)$ is minimum at $b = \frac{n}{n+1}$, i.e. $\delta(y) = \frac{ny}{n+1}$. With $\delta(y) = \frac{ny}{n+1}$,

$$
R(\theta,\delta) = \frac{\theta^2}{n^2}\left[\frac{n^2(n+1)(n-1)}{(n+1)^2} - \frac{2n^2(n-1)}{n+1} + n^2\right] = 2\theta^2\left(\frac{1}{n+1}\right),
$$

thus $\max_\theta R(\theta,\delta) = \max_\theta\left(\frac{2\theta^2}{n+1}\right)$ does not exist.

3. **Let $X_1,\ldots,X_n$ denote a random sample from a $\mathbf{N}(\theta,\sigma^2)$ distribution, where $-\infty < \theta < \infty$ and σ^2 is a given positive number. Let $Y = \bar{X}$ denote the mean of the random sample. Take the loss function to be $L[\theta,\delta(y)] = |\theta - \delta(y)|$. If θ is an observed value of the random variable Θ, that is, $\mathbf{N}(\mu,\tau^2)$, where $\tau^2 > 0$ and μ are known numbers, find the Bayes' solution $\delta(y)$ for a point estimate θ.**

   $Y = \bar{X}$, therefore $Y \mid \theta \sim \mathrm{N}\left(\theta, \frac{\sigma^2}{n}\right)$, thus

$$
f(y \mid \theta) = \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}\exp\left(\frac{-1}{2\frac{\sigma^2}{n}}(y-\theta)^2\right).
$$

Meanwhile, $\theta \sim \mathrm{N}(\mu, \tau^2)$, thus

$$h(\theta) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{1}{2\tau^2}(\theta-\mu)^2\right).$$

First, finding the posterior distribution of θ given y:

$$\begin{aligned}
g(\theta \mid y) &\propto f(y \mid \theta) \cdot h(\theta) \\
&\propto \exp\left(\frac{-1}{2\frac{\sigma^2}{n}}(y-\theta)^2\right) \cdot \exp\left(-\frac{1}{2\tau^2}(\theta-\mu)^2\right) = \exp\left[\frac{-1}{2\frac{\sigma^2}{n}}(y-\theta)^2 - \frac{1}{2\tau^2}(\theta-\mu)^2\right] = \exp(*),
\end{aligned}$$

where $(*) = -\frac{1}{2\frac{\sigma^2}{n}}(\theta^2 - 2\theta y + y^2) - \frac{1}{2\tau^2}(\theta^2 - 2\mu\theta + \mu^2)$, therefore

$$\begin{aligned}
\Rightarrow \exp(*) &\propto \exp\left[-\frac{1}{2\frac{\sigma^2}{n}}(\theta^2 - 2y\theta) - \frac{1}{2\tau^2}(\theta^2 - 2\mu\theta)\right] \\
&= \exp\left[\frac{-\left[\left(\tau^2 + \frac{\sigma^2}{n}\right)\theta^2 - 2\left(y\tau^2 + \mu\frac{\sigma^2}{n}\right)\theta\right]}{2\tau^2(\sigma^2/n)}\right] \\
&= \exp\left[\frac{\left(\theta - \frac{y\tau^2 + \mu\sigma^2/n}{\tau^2 + \sigma^2/n}\right)^2}{2\frac{\tau^2(\sigma^2/n)}{\tau^2 + \frac{\sigma^2}{n}}}\right],
\end{aligned}$$

i.e. $\theta \mid y \sim \mathrm{N}\left(\frac{y\tau^2+\mu\sigma^2/n}{\tau^2+\sigma^2/n}, \frac{\tau^2(\sigma^2/n)}{\tau^2+\frac{\sigma^2}{n}}\right)$. The Bayes estimator for $L(\theta, \delta(y)) = |\theta - \delta(y)|$ is the median of the conditional distribution of $\theta \mid y$, where the median equals the mean for a symmetric distribution such as the normal distribution. Thus, $\delta(y) = \frac{y\tau^2+\mu\sigma^2/n}{\tau^2+\sigma^2/n}$.

4. **Let $X_1, \ldots, X_n$ be Poisson(λ), and let λ have a gamma(α, β) distribution, the conjugate family for the Poisson.**

   (a) Find the posterior distribution of λ.

   (b) Calculate the posterior mean and variance.

(a) $X_i \mid \lambda \sim$Poisson(λ), thus

$$f(\boldsymbol{x} \mid \lambda) = \prod_{i=1}^{n} f(x_i \mid \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \propto e^{-n\lambda}\lambda^{\sum_{i=1}^n x_i} = e^{-n\lambda}\lambda^y,$$

where $\lambda \sim$Gamma(α, β) implies that $\pi(\lambda) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}e^{-\lambda/\beta}$ is the prior distribution. Thus, the posterior distribution of $\lambda \mid y$ is

$$\begin{aligned}
\pi(\lambda \mid y) = f(y \mid \lambda) \cdot \pi(\lambda) &\propto e^{-n\lambda}\lambda^y\lambda^{\alpha-1}e^{-\lambda/\beta} \\
&= \lambda^{(\alpha+y)-1}e^{-\left(n+\frac{1}{\beta}\right)\lambda} \\
&= \lambda^{(\alpha+y)-1}\exp\left(-\frac{\lambda}{\beta/(\beta n+1)}\right),
\end{aligned}$$

implying that $\lambda \mid y \sim \mathrm{Gamma}\left(\alpha + y, \frac{\beta}{\beta n+1}\right)$.

(b) Given the posterior distribution as determined in part (a), the poster mean and variance are

$$\begin{aligned}
\mathrm{E}(\lambda \mid y) &= (\alpha + y)\left(\frac{\beta}{\beta n + 1}\right) = \frac{\beta(\alpha + y)}{\beta n + 1} \\
\mathrm{Var}(\lambda \mid y) &= (\alpha + y)\left(\frac{\beta}{\beta n + 1}\right)^2 = \frac{\beta^2(\alpha + y)}{(\beta n + 1)^2}
\end{aligned}$$

5. **Let Y_n be the nth order statistic of a random sample of size n form a distribution with pdf $f(x \mid \theta) = \frac{1}{\theta}, 0 < x < \theta$, zero elsewhere. Take the loss function to be $L[\theta, \delta(y)] = [\theta - \delta(y_n)]^2$. Let θ be an observed value of the random variable Θ, which as pdf $\pi(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}, \alpha < \theta < \infty$, zero elsewhere, with $\alpha > 0$, $\beta > 0$. Find the Bayes' solution $\delta(y_n)$ for a point estimate of θ.**

$$\begin{aligned}
f(x \mid \theta) &= \frac{1}{\theta}, 0 < x < \theta \\
F(x \mid \theta) &= \frac{x}{\theta}, 0 < x < \theta \\
\Rightarrow f_{Y_n}(x) &= n[F(x)]^{n-1} f(x) = n\left(\frac{x}{\theta}\right)^{n-1}\left(\frac{1}{\theta}\right) = \frac{nx^{n-1}}{\theta^n}, 0 < x < \theta.
\end{aligned}$$

Meanwhile, the prior distribution is $\pi(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}, \alpha < \theta < \infty$, so the posterior distribution of $\theta \mid y$ is

$$\pi(\theta \mid y_n) = f(y_n \mid \theta) \cdot \pi(\theta) = \frac{ny^{n-1}}{\theta^n} \cdot \frac{\beta\alpha^\beta}{\theta^{\beta+1}} \propto \frac{1}{\theta^{n+\beta+1}},$$

thus we find that $\theta \mid Y_n$ has a Pareto$(\alpha, n + \beta)$ distribution. Given a squared-error loss funciton, the Bayes solution is the associated mean, $\delta(y_n) = E(\theta \mid Y_n) = \frac{(n+\beta)\alpha}{n+\beta-1}$.