Math 501 - 1/26/2021

Probability

Office hours: Thursday 9-10p Saturday 4-5p

Textbook: Available on Amazon and GU library.

Most contents on discord.

Grading: Weekly Homeworks: 65%.
Final: 35%. → Depends on Quarantine status.
could be in person or changed.

Submit as a single pdf, must be pdf.
Can be submitted to LATEX or hadwritten.
Communication on homework is fine,
questions should go to Discord.
Posted on Wednesday, due the following
Wednesday.

Topics for Today

1. Probability Spaces
2. Conditional Probabilities
3. Independence
4. Set Theory (countability vs. uncountability)

1. Probability Space = $(\Omega, \mathcal{F}, P)$ ← *probability space needs these three things.

Ω ↑ sample space; $\mathcal{F}$ ↑ set of Events; P ↖ Probability Measure

__Sample Space__ Ω: Experiment or trial with different outcomes and Ω = {all possible outcomes} = sample space.

__ex__: die roll $\Rightarrow \Omega = \{1,2,3,4,5,6\}$ ← finite

Discrete

days until vaccination $\Rightarrow \Omega = \{1,2,3,\ldots\}$ ← infinite

__Note__: the ~~kind~~ kind of sample space is critical. elements of sample space are not necessarily numbers, are elements abstractly.

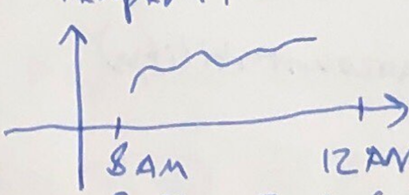
e.g. day of week = {M, T, W, Th, F, Sa, Su}

Two types of Sample Spaces: Discrete or ~~[illegible]~~ Continuous.

__ex__: temperature tomorrow $\Omega = (20°F, 50°F) = (20, 50) \rightarrow$ subset of $\mathbb{R}$ ~~[illegible]~~, $= (-\infty, \infty)$

__Note__: sample spaces can be of "larger" continuous objects like vectors or functions

__ex__: temperatures tomorrow across the whole day



$f: [0,12] \rightarrow (-50, 120)$

$\Rightarrow \Omega = \left\{ \begin{array}{l} \text{all functions} \\ f: [0,12] \rightarrow (-50, 120) \end{array} \right\}$

↑

this is much "bigger"

[illegible]

Set of Events, $\mathcal{F}$

Def: An event is a subset of Ω to which we can assign a probability.

Ex: Die Roll

$\Omega = \{1, 2, 3, 4, 5, 6\} \Rightarrow$ Set of Events $= \mathcal{F} =$

How many possible sets?

$$\text{set of sets} = \mathcal{F} = \big\{ \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}, \dots, \{1,2,3\}, \dots \{1,2,3,4,5,6\} \big\} \Rightarrow 2^6 \text{ sets.}$$

$\mathcal{F}$ is a set of sets

~~Not quite~~ →
~~Next~~ page.

Note: event $A \in \mathcal{F}$ such that $A \subset \Omega$ ↓

$\Rightarrow$ An event A is a subset of Ω. $\mathcal{F}$ is the set of all events, ~~or~~ the set of all subsets of A. ~~part of~~

Def: The power set of Ω is the set of all subsets of Ω, is notated $\mathcal{P}(\Omega)$.

Q: Is $\mathcal{F} = \mathcal{P}(\Omega)$?

Ex: Flip two coins. $\Rightarrow \Omega = \{HH, HT, TH, TT\}$

$$\mathcal{F} = \mathcal{P}(\Omega) = \big\{ \phi, \{HH\}, \{HT\}, \dots \{HH, HT\}, \{HH, TH\}, \dots, \{HH, HT, TH, TT\} \big\}.$$

Q: what do we require of $\mathcal{F}$?
Can we always choose $\mathcal{F} = \mathcal{P}(\Omega)$?

We require of $\mathcal{F}$:

1. $A \in \mathcal{F},\ B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3. $A_1, A_2, \dots, A_n \in \mathcal{F} \Rightarrow A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{F}$

If Ω is <u>finite</u> and <u>discrete</u> then $\mathcal{F} = \mathcal{P}(\Omega)$.

If Ω is <u>continuous</u> then $\mathcal{F} = \mathcal{P}(\Omega)$

$\Rightarrow \mathcal{F} = \mathcal{P}(\Omega)$

Probability Measure P:

A function that maps events to $[0,1]$

$P : \mathcal{F} \to [0,1]$.

Ex: $\Omega = \{1,2,3,4,5,6\}$

$\mathcal{F} = \mathcal{P}(\Omega) = \{\emptyset, \{1\}, \{2\}, \{3\}, \dots, \{1,2,3,4,5,6\}\}$.

we have to assign a probability to all elements of $\mathcal{F}$.

Def: we say an event A occurred if the experiment resulted in an outcome that is in A.

Probability that each event occurred

$$P(\emptyset) =$$
$$P(\{1\}) =$$
$$\vdots$$
$$P(\{1,2,3\}) =$$
$$\vdots$$
$$P(\{1,2,3,4,5,6\}) =$$

For this example we can fill these in, but that is not necessarily true for all $(\Omega, \mathcal{F}, P)$

What rules must $P: \mathcal{F} \to [0,1]$ satisfy?

↳ non trivial to discover, took 200 years to write down.

Axioms of Probability :

1. $P(A) \geq 0$ for all A, $A \in \mathcal{F}$
2. $P(\emptyset) = 0$, $P(\Omega) = 1$
3. $A_1, A_2, A_3, \ldots, A_n \in \mathcal{F}$, each A_i is an event and $A_i \cap A_j = \emptyset$ then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Ex: $A_1 = \{1\}$, $A_2 = \{5,6\}$, $A_3 = \{3\}$

$$P(A_1 \cup A_2 \cup A_3) = P(\{1,3,5,6\})$$
$$= P(\{1\}) + P(\{3\}) + P(\{5,6\}).$$

Recall: Probabilities

$\Omega = \{\text{outcomes}\}$

$\mathcal{F}$ = set of events
event $A \in \mathcal{F}$,
$A \subseteq \Omega$

$P(A)$ = the likelihood/chance of the event A occurring

ex

$\Omega = \{1,2,3,4,5,6\}$

$A = \{1,3,5\}$

$P(A) = 1/2$

What are the rules for all the things above?

Ω can be anything.

$\mathcal{F} \overset{\text{require}}{\Longrightarrow} A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

$A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

ex:

$A = \{1,3,5\}, A^c = \{2,4,6\}$

$A = \{1,2\}, B = \{3\}$

$A \cup B = \{1,2,3\}$

$P \overset{\text{require}}{\Longrightarrow} P(A) \in [0,1]$

$P(\emptyset) = 0, P(\Omega) = 1$

$P(A \cup B) = P(A) + P(B)$

$A \cap B = \emptyset$

Q: How can we build $\mathcal{F}, P$ if Ω is discrete?

Suppose $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$.

Then $\mathcal{F} = \mathcal{P}(\Omega)$

Assign
$$P(\{\omega_1\}) = a_1$$
$$P(\{\omega_2\}) = a_2 \qquad a_i \geq 0$$
$$\vdots \qquad a_1 + a_2 + \cdots + a_n = 1$$
$$P(\{\omega_n\}) = a_n$$

example: $\Omega = \{1,2,3,4,5,6\}$

$P(\{1\}) = 1/6$ or a_1

$P(\{2\}) = 1/6$ or a_2

$\vdots$

$P(\{6\}) = 1/6$ or a_6

Then we can define the pohability of any event $A \in \mathcal{F}$.

e.g. $P(\{\omega_1, \omega_2, \omega_3\}) = P(\{\omega_1\}) + P(\{\omega_2\}) + P(\{\omega_3\})$

$$P(\{2,4,6\}) = P(\{2\}) + P(\{4\}) + P(\{6\})$$
$$= 1/6 + 1/6 + 1/6 = 1/2.$$

Conditional Probability

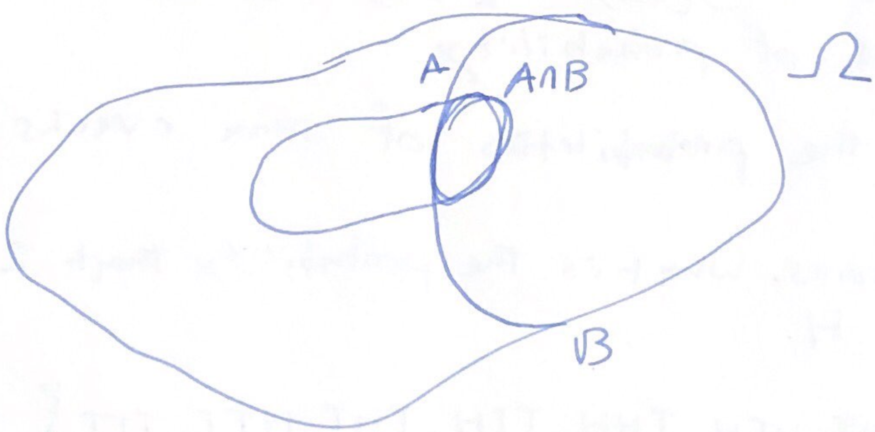
Def: Let A, B be events. The conditional probability of A given B $P(A|B) = \frac{P(A \cap B)}{P(B)}$ where $P(B) \neq 0$.

Intuition: Probability of A assuming an outcome in B occurred.

Ex: $A = \{1\}$, $B = \{1, 3, 5\}$ Rolling a dic.

$$P(\text{roll } 1 \mid \text{roll odd}) = P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(\{1\} \cap \{1,3,5\})}{P(\{1,3,5\})} = \frac{P(\{1\})}{P(\{1,3,5\})} = \frac{1/6}{3/6} = 1/3.$$



In other words, if you assign probabilities to the "atoms" or single outcomes with the above conditions, you can build up the rest of P for all $\mathcal{F}$, for discrete Ω.

For <u>discrete</u> sample space $\Omega = \{w_1, .. w_n\}$

$$\mathcal{F} = \mathcal{P}(\Omega)$$
$$P(\{w_i\}) = a_i \quad \text{st} \quad a_1 + \dots + a_n = 1$$
$$a_i \geq 0.$$

what about <u>continuous</u> sample spaces?

e.g. $\Omega = \mathbb{R}$ or $\Omega = (0, 10)$

$\mathcal{F} = \mathcal{P}(\Omega)$ is big.

How do we define $P(A)$ for $A \in \mathcal{F}$, $A \subseteq \Omega$

<u>Fact</u>: There is no $P : \mathcal{P}(\Omega) \to [0,1]$ that satisfies the axioms of probability.

Let's compute the probabilities of some events.

<u>Ex 1</u>: I flip 3 coins. What is the probability that 2 land on H.

$$\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

$\mathcal{F} = \mathcal{P}(\Omega)$, an event A can be any subset of Ω.

$$P(\{HHH\}) = 1/8, \; P(\{HHT\}) = 1/8, \dots, P(\{TTT\}) = 1/8$$

$$A = \{\text{I flip exactly 2 heads}\} = \{HHT, HTH, THH\} = \{HHT\} \cup \{HTH\} \cup \{THH\}$$

$$\Rightarrow P(A) = P(\{HHT\}) + P(\{HTH\}) + P(\{THH\})$$
$$= 1/8 + 1/8 + 1/8 = 3/8$$