**Homework # 12**
Due 4/30

- Reading: section 8.5. The section discusses characteristic functions which are similar to moment generating functions, but have the advantage that a characteristic function always exists for a r.v. I won't discuss characteristic functions, but do read through the brief discussion in the book.

1. Let $X_1, X_2, X_3, \ldots$ be exponential r.v.'s with rate parameters $\lambda_1, \lambda_2, \lambda_3, \ldots$. Show that $X_1, X_2, X_3, , \ldots$ converge in distribution if and only if $\lambda_1, \lambda_2, \lambda_3, \ldots$ converge. When the $X_i$ converge, what is the distribution to which they converge?

2. A technical issue that I did not mention in class, but which the book states in Definition 8.45, is convergence in distribution for non-continuous r.v. In that case, we have to modify the requirement that $F_n(t) \to F(t)$ for all $t$, with $F_n(t)$ and $F(t)$ being the cdfs of the sequence of r.v. in $n$ and the limit r.v respectively., to the more limited requirement $F_n(t) \to F(t)$ for all $t$ at which $F(t)$ is continuous. As an example, consider the r.v. $X_n$ with $P(X_n = 1/n) = 1$. What r.v. do the $X_n$ converge to in distribution? Discuss the need for the continuity condition in the context of this example.

3. Chapter 8, Problems $1a$ and $1c$

4. Suppose you are given a sequence of samples $\hat{X}_1, \hat{X}_2, \hat{X}_3, \ldots, \hat{X}_n$ drawn from a r.v. $X$ independently. (Recall, a sample of a r.v. is an outcome of the r.v. The $\hat{X}_i$ are therefore numbers.). In this problem we will consider estimating $\mu = E[X]$ by $\hat{\mu}$, where

$$\hat{\mu} = \frac{\hat{X}_1 + \hat{X}_2 + \cdots + \hat{X}_n}{n}. \tag{1}$$

   (a) Explain why $\hat{\mu}$ is a reasonable estimate for $\mu$. Under what situations might it not be a good estimate?
   (b) What is an estimate of $V[X]$ based on LLN?
   (c) Use CLT to estimate the probability that $\hat{\mu}$ differs from $E[X]$ by more than $\frac{1}{2}$.
   (d) Now assume that $X$ is normally distributed. How does this affect your result in (c)?

Note: Much of the theory of statistics involving estimating of means centers on (c) and accounting for the error involved in using CLT for a case of finite $n$.

5. Let $X = (X_1, X_2, \ldots, X_n)$ have joint distribution $\mathcal{N}(0, \Sigma)$. Show that $v \cdot X \sim \mathcal{N}(0, v^T \Sigma v)$ for $v \in \mathbb{R}^n$. (Hint: Start by doing this for diagonal $\Sigma$, in which case the $X_i$ are independent. Then, recalling the fact that $X$ can always be expressed as a rotation by an orthonormal matrix $Q$ of a multivariate normal with independent coordinates, consider rotating $v$ and $X$ simultaneously, which does not alter $v \cdot X$.). This problem demonstrates the important fact that the linear combination of jointly distributed normal r.v.'s is normally distributed.