In earlier lectures, we learned how to solve nonlinear equations, by

- Fixed point method
- Bisection method
- Newton's method
- Secant method, etc.

In this lecture, we learn to solve nonlinear systems such as

$$\begin{cases} x^4 + y^4 - 3 = 0 \\ x^3 - 3xy^2 + 1 = 0 \end{cases}$$

$$\begin{cases} 2x + y = 5 - 2z^2 \\ y^3 + 4z = 0 \\ xy + z = e^z \end{cases}$$

$$\begin{cases} x^T A x + b^T x + c = 0 \\ Bx + d = 0 \end{cases}$$

In contrast to nonlinear equations, in system of nonlinear equations we have more than one variable and more than one equation.

Solution of Nonlinear Systems

In an n dimensional system, we define $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F(x_1, x_2, \dots, x_n) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{bmatrix}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ (scalar-valued), where $i = 1, 2, \dots, n$.

To find $x^* = (x_1^*, \dots, x_n^*)^T \in \mathbb{R}^n$ such that $F(x^*) = 0$, we need to solve

$$\begin{cases} f_1(x_1, x_2, \dots, x_n) = 0 \\ f_2(x_1, x_2, \dots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) = 0 \end{cases}$$

For simplicity, let's focus on a system $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{cases} f_1(x, y) = 0 \\ f_2(x, y) = 0 \end{cases}$$

A solution is given by a pair of numbers,

$$(x^*, y^*)^T \in \mathbb{R}^2$$

such that

$$\begin{cases} f_1(x^*, y^*) = 0 \\ f_2(x^*, y^*) = 0 \end{cases}$$

Fixed Point Method

Suppose that we know that the root $(x^*, y^*)^T$ that we are looking for is such that

$$x^* \in (a, b), \qquad y^* \in (c, d)$$

We reduce the system

$$\begin{cases} f_1(x^*, y^*) = 0 \\ f_2(x^*, y^*) = 0 \end{cases}$$

to an equivalent form

$$\begin{cases} x = x - \alpha f_1(x, y) \\ y = y - \alpha f_2(x, y) \end{cases}$$

where $\alpha \neq 0$.

To find a root

$$(x^*, y^*) \in (a, b) \times (c, d)$$

we take some $(x_0, y_0) \in (a, b) \times (c, d)$ then recursively calculate (x_{k+1}, y_{k+1}) by the formula

$$\begin{cases} x_{k+1} = x_k - \alpha f_1(x_k, y_k) \\ y_{k+1} = y_k - \alpha f_2(x_k, y_k) \end{cases}$$

for $k \geq 0$.
In the above, we assume that $(x_{k+1}, y_{k+1}) \in (a, b) \times (c, d)$

Newton's Method

Recall that for

$$F(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix},$$

its first-order derivative at point $(x,y)^T$ is given by the Jacobian matrix

$$J_F(x,y) = \begin{bmatrix} \partial_x f_1(x,y) & \partial_y f_1(x,y) \\ \partial_x f_2(x,y) & \partial_y f_2(x,y) \end{bmatrix},$$

Assume that we are given $(x_k, y_k)^T$. Similar to one-dimensional case, we use the Taylor's approximation of $F(x,y)$ about $(x_k, y_k)^T$,

$$\begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} \approx \begin{bmatrix} f_1(x_k,y_k) \\ f_2(x_k,y_k) \end{bmatrix} + \begin{bmatrix} \partial_x f_1(x_k,y_k) & \partial_y f_1(x_k,y_k) \\ \partial_x f_2(x_k,y_k) & \partial_y f_2(x_k,y_k) \end{bmatrix} \begin{bmatrix} x - x_k \\ y - y_k \end{bmatrix}$$

Assume that $J_F(x_k, y_k)$ is nonsingular, solve the following system with respect to x and y

$$\begin{bmatrix} f_1(x_k, y_k) \\ f_2(x_k, y_k) \end{bmatrix} + \begin{bmatrix} \partial_x f_1(x_k, y_k) & \partial_y f_1(x_k, y_k) \\ \partial_x f_2(x_k, y_k) & \partial_y f_2(x_k, y_k) \end{bmatrix} \begin{bmatrix} x - x_k \\ y - y_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and denoting the solution by $(x_{k+1}, y_{k+1})^T$ we obtain

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} \partial_x f_1(x_k, y_k) & \partial_y f_1(x_k, y_k) \\ \partial_x f_2(x_k, y_k) & \partial_y f_2(x_k, y_k) \end{bmatrix}^{-1} \begin{bmatrix} f_1(x_k, y_k) \\ f_2(x_k, y_k) \end{bmatrix}$$

Example

Consider the system of nonlinear equations

$$\begin{cases} (x-1)^2 + (y-1)^2 - 1 = 0 \\ x + y - 2 = 0. \end{cases}$$

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Apply Newton's method twice starting with $(x_0, y_0)^T = (0, 2)^T$

- Find $J_F(x, y)$
- Find the inverse of $J_F(x, y)$

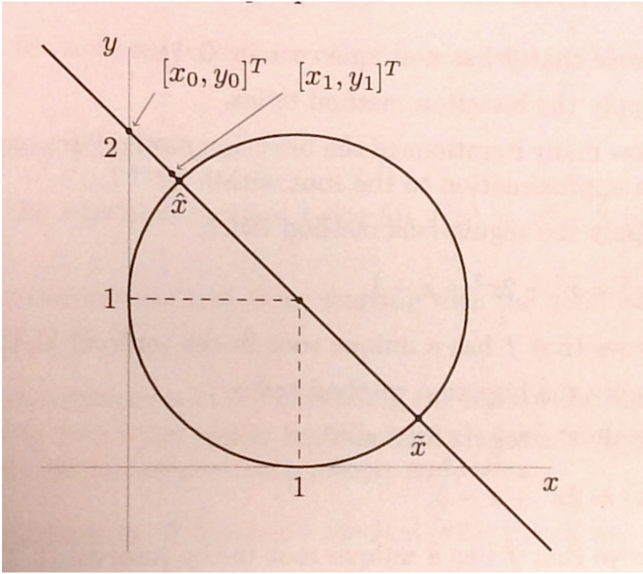
$$J_F(x,y) = \begin{bmatrix} 2(x-1) & 2(y-1) \\ 1 & 1 \end{bmatrix}$$

$$J_F(x,y)^{-1} = \frac{1}{2(x+y)} \begin{bmatrix} 1 & -2(y-1) \\ -1 & 2(x-1) \end{bmatrix}$$

We need to make the following iterates:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \frac{1}{2(x_k - y_k)} \begin{bmatrix} 1 & -2(y_k-1) \\ -1 & 2(x_k-1) \end{bmatrix} \begin{bmatrix} (x_k-1)^2 + (y_k-1)^2 - 1 \\ x_k + y_k - 2 \end{bmatrix}$$

$[x_1, y_1] = [1/4, 7/4]$


y
$[x_0, y_0]^T$
$[x_1, y_1]^T$
2
$\hat{x}$
1
$\tilde{x}$
1
x