LECTURE 3 - 9/20/22

Recall: Linear Systems.

Today: we discuss methods of solving these systems, including direct and others.

Triangular systems: We apply backward/forward substitution for upper/lower triangular matrices.

Gaussian Elimination: $Ax = b \to \bar{A}x = \bar{b}$, $\bar{A}$ is upper triangular.

via elementary row operations, including: (1) change order of rows, (2) scalar multiplication of row (3) replace a row by its sum with a multiple of another row.

Gauss-Jordan: Similar to above, except the goal is to get to I rather than ~~upper~~ upper triangular in the matrix. You get the solution, don't need to do backward substitution.

LU Factorization: Find a lower triangular matrix L and an upper U for A such that $A = LU$.

$$Ax = b \iff L(Ux) = b \iff \underset{\text{forward sub}}{Ly = b} \iff \underset{\text{backward sub to get } x.}{Ux = y}$$

The idea is $\underbrace{(L_{n-1} \cdots L_1)}_{L}A = U$, L is product of multiplication of lower triangular matrices. The inverse of lower triangular is same type.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{pmatrix} \Rightarrow L_1 A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{pmatrix}$$

chosen to zero out elements of first col.

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}. \quad L = (L_2 L_1)^{-1} = L_1^{-1} L_2^{-1} \Rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}.$$

Then check answer $U = L^{-1}A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}$.

General Iterative Method for Solving $Ax=b$

- Start with guess $x^{(0)} \in \mathbb{R}^n$.

Choose matrix C s.t. $\|I_n - CA\| < 1$.

Produce vectors s.t. $x^{(k)} = x^{k-1} + C(b - Ax^{(k-1)})$, $k = 1, 2, \ldots$

Note that $x^{(1)} = x^{(0)} + C(b - Ax^{(0)})$
$x^{(2)} = x^{(1)} + C(b - Ax^{(1)})$ $\ldots$ $x^{(k)}$ ~~series~~ sequence converges to the approximate solution.

This doesn't get actual/exact soln, but it is highly accurate.

Let x^* be solution to $Ax=b$.

$$
\begin{aligned}
x^{(k)} - x^* &= x^{(k-1)} + C(b - Ax^{(k-1)}) - x^* \\
&= x^{(k-1)} + C(Ax^* - Ax^{(k-1)}) - x^* \\
&= (x^{(k-1)} - x^*) - CA(x^{(k-1)} - x^*) \\
&= (I - CA)(x^{(k-1)} - x^*)
\end{aligned}
$$

$\|x^{(k)} - x^*\| = \|I - CA\| \|x^{(k-1)} - x^*\|$ for all k.

$\Rightarrow \|x^{(k)} - x^*\| = \|I - CA\|^j \|x^{(k-j)} - x^*\|$

Let $j = k$.

$\|x^{(k)} - x^*\| = \|I - CA\|^k \|x^{(0)} - x^*\|$

since $\|I - CA\| < 1$ by construction, we ensure convergence.