BICK-hw7

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1 Nathan Bick HW 7

2 Problem 1

Consider the sets

$C = \{(x, y) \mid ||x||_2 \leq y\}$ and $\hat{C} = \{(x, y) \mid ||x||_2^2 \leq y\}$

Determine whether the sets C and $\hat{C}$ are convex or not?

We recall that a set S is convex if, for two elements in S, then the linear combination of these elements is also in S. That is, if $x, y \in S$, then $\lambda x + (1-\lambda)y \in S$.

Let $(x_1, y_1), (x_2, y_2) \in C$, then $||x_1||_2 \leq y_1$ and $||x_2||_2 \leq y_2$.

$\lambda||x_1||_2 \leq \lambda y_1$ and $(1-\lambda)||x_2||_2 \leq (1-\lambda)y_2$. Then $||\lambda x_1 + (1-\lambda)x_2||_2 = ||\lambda x_1||_2 + ||(1-\lambda)x_2||_2 = \lambda||x_1||_2 + (1-\lambda)||x_2||_2$

We then can use the set definition to get the following

$\lambda||x_1||_2 + (1-\lambda)||x_2||_2 \leq \lambda y_1 + (1-\lambda)y_2$

Therefore, $\lambda x + (1-\lambda)y \in C$. This shows that C is convex.

Now we consider the set $\hat{C}$. We can proceed in a very similar way to the set C.

Let $(x_1, y_1), (x_2, y_2) \in \hat{C}$, then $||x_1||_2^2 \leq y_1$ and $||x_2||_2^2 \leq y_2$.

Consider $||\lambda x_1 + (1-\lambda)x_2||_2^2 = \langle \lambda x_1 + (1-\lambda)x_2, \lambda x_1 + (1-\lambda)x_2 \rangle = \langle \lambda x_1, \lambda x_1 \rangle + 2\langle \lambda x_1, (1-\lambda)x_2 \rangle + \langle (1-\lambda)x_2, (1-\lambda)x_2 \rangle$

We then see this is equal to

$\lambda^2||x_1||_2^2 + (1-\lambda)^2||x_2||_2^2 + 2\langle \lambda x_1, (1-\lambda)x_2 \rangle$

$= \lambda^2 y_1 + (1-\lambda)^2 y_2 + 2\langle \lambda x_1, (1-\lambda)x_2 \rangle$

$= \lambda^2 y_1 + (1-\lambda)^2 y_2 + 2\lambda(1-\lambda)\langle x_1, x_2 \rangle$

We use the Cauchy Schwartz inequality to get

$\leq \lambda^2 y_1 + (1-\lambda)^2 y_2 + 2\lambda(1-\lambda)||x_1||_2||x_2||_2$

Following the definition of the set

$\leq \lambda^2 y_1 + (1-\lambda)^2 y_2 + 2\lambda(1-\lambda)\sqrt{y_1}\sqrt{y_2}$

$= \lambda\sqrt{y_1} + (1-\lambda)\sqrt{y_2}$

We see that the set $\hat{C}$ is not convex.

3 Problem 2

Consider the smooth (differentiable) functions $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$. Prove that the function

$f = h \circ g : \mathbb{R}^n \to \mathbb{R}$

where

$f(x) = h(g(x))$ and $dom f = \{x \in dom g | g(x) \in dom h\}$ is convex if one of the following conditions on h and g holds.

(a) If h and g are convex functions, and h is nondeacreasing, or

(b) if h is convex and nonincreasing, and g is concave.

We want to prove that for $x, y \in dom f$, then $(h \circ f)(\lambda x + (1-\lambda)y) \leq \lambda(h \circ g)(x) + (1-\lambda)(h \circ g)$

We consider option (a). know that h and g are convex. Then we see that

$(h \circ g)(\lambda x + (1-\lambda)y) = h(g(\lambda x + (1-\lambda)y))$

By g convex and h nondecreasing, we then get

$\leq h(\lambda f(x) + (1-\lambda)f(y))$

By h convex then

$\leq \lambda h(f(x)) + (1-\lambda)h(g(y)) = \lambda(h \circ g)(x) + (1-\lambda)(h \circ g)(y)$

To consider the option (b), we use the definition of functional conveity that uses the second derivative, which is that a function is convex if the Hessian is positive semidefinite.

If $f = h(g))$ then $f'' = h''(g')^2 + g''h'$. We know h is convex and nonincreasing, and g is concave. Therefore, $g'' < 0$, $h' \leq 0$, and $h'' > 0$. Therefore we see that $f'' > 0$. Therefore, f is convex.