MATH 503: Mathematical Statistics
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Homework 2 Solutions

1. Let X_i, $i = 1, 2, \ldots$, be independent Bernoulli(p) random variables and let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Use the delta method to find the limiting distribution of $g(Y_n) = Y_n(1 - Y_n)$ for $p \neq \frac{1}{2}$.

   Solution: We know that $\sqrt{n}(Y_n - p) \xrightarrow{d} N(0, p(1-p))$. Consider:

$$\begin{array}{ll} g(Y_n) = Y_n(1 - Y_n) = Y_n - Y_n^2 & g(p) = p(1-p) \\ g'(Y_n) = 1 - 2Y_n & g'(p) = 1 - 2p \end{array}$$

   By the delta method, for $p \neq \frac{1}{2}$ (in order to ensure $g'(p) \neq 0$),

$$\sqrt{n}\left(Y_n(1 - Y_n) - p(1-p)\right) \xrightarrow{d} N(0, p(1-p)(1-2p)^2) = N(0, p - 5p^2 + 8p^3 - 4p^4).$$

2. Let $\bar{X}$ be the mean of a random sample from the exponential distribution, Exponential(θ).

   (a) Show that $\bar{X}$ is an unbiased point estimator of θ.

   (b) Using the mgf technique, determine the distribution of $\bar{X}$.

   (c) Use (b) to show that $Y = 2n\bar{X}/\theta$ has a χ^2 distribution with $2n$ degrees of freedom.

   Solution: Consider $\bar{X}$ from a random sample $X_1, \ldots, X_n$ with pdf $f_X(x; \theta) = \frac{1}{\theta}e^{-x/\theta} \Rightarrow E(X) = \theta$ and $M_X(t) = \frac{1}{1-\theta t}$.

   (a) $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n} \theta = \frac{n\theta}{n} = \theta$.

   (b)

$$\begin{aligned} M_{\bar{X}}(t) &= E(e^{\bar{X}t}) = E\left(e^{\frac{t}{n}\sum X_i}\right) = M_{\sum X_i}\left(\frac{t}{n}\right), \text{ where } X_i\text{s are iid} \\ &= M_{\sum X_i}\left(\frac{t}{n}\right) = M_X^n\left(\frac{t}{n}\right) = \left(\frac{1}{1 - \theta\left(\frac{t}{n}\right)}\right)^n = \left(\frac{1}{1 - \frac{\theta t}{n}}\right)^n, \end{aligned}$$

   which is the mgf of a Gamma($n, \frac{\theta}{n}$) distribution, so $\bar{X} \sim \text{Gamma}\left(n, \frac{\theta}{n}\right)$.

   (c) Let $Y = \frac{2n\bar{X}}{\theta}$. Then

$$\begin{aligned} M_Y(t) &= M_{\frac{2n\bar{X}}{\theta}}(t) = E\left(e^{\frac{2n\bar{X}t}{\theta}}\right) = E\left(e^{\bar{X}(2nt/\theta)}\right) \\ &= M_{\bar{X}}\left(\frac{2nt}{\theta}\right) = \left(\frac{1}{1 - \frac{\theta}{n}\left(\frac{2nt}{\theta}\right)}\right)^n = \left(\frac{1}{1-2t}\right)^{2n/2}, \end{aligned}$$

   so $Y \sim \chi^2_{2n}$.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Poisson(θ) distribution, where θ is unknown. Let $Y = \sum_{i=1}^{n} X_i$. Find the distribution of Y and determine c so that cY is an unbiased estimator of θ.

   Solution: By the mgf technique,

$$\begin{aligned} M_Y(t) &= M_X^n(t) \text{ because the } X_i\text{s are iid,} \\ &= \left(e^{\theta(e^t-1)}\right)^n = e^{n\theta(e^t-1)}, \end{aligned}$$

   which is the mgf of the Poisson($n\theta$), so $Y \sim$Poisson($n\theta$).
   Accordingly, $E(cY) = cE(Y) = c(n\theta) \doteq \theta$ to satisfy the unbiasedness requirement, thus $c = \frac{1}{n}$.

4. Let $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics of a random sample of size n $(X_1, X_2, \ldots, X_n)$ from a Weibull distribution of the form $f(x) = cx^b \exp\left\{-\frac{cx^{b+1}}{b+1}\right\}$, $0 < x < \infty$, zero elsewhere. Find the distribution of Y_1.

   Solution: $F_{Y_1}(y) = 1 - P(Y_1 > y) = 1 - P(X_1 > y, ..., X_n > y) = 1 - (1 - F_X(y))^n$, and
   $f_{Y_1}(y) = n(1 - F_X(y))^{n-1} f_X(y)$, where

$$F_X(x) = \int_0^x ct^b e^{\frac{-ct^{b+1}}{b+1}} dt = -e^{\frac{-ct^{b+1}}{b+1}} \mid_0^x = 1 - e^{\frac{-cx^{b+1}}{b+1}},$$

   $\Rightarrow F_{Y_1}(y) = 1 - \left[1 - \left(1 - e^{\frac{-cy^{b+1}}{b+1}}\right)\right] = 1 - e^{\frac{ncy^{b+1}}{b+1}}$ is the cdf, and $f_{Y_1}(y) = ncy^b e^{\frac{-ncy^{b+1}}{b+1}}$ is the pdf of Y_1. Accordingly, $Y_1 \sim$ Weibull$\left(b+1, \frac{b+1}{nc}\right)$.

5. Let X and Y denote independent random variables with respective probability density functions $f(x) = 2x$, $0 < x < 1$, zero elsewhere, and $g(y) = 3y^2$, $0 < y < 1$, zero elsewhere. Let $U = \min(X, Y)$ and $V = \max(X, Y)$. Find the joint pdf of U and V.

   Solution: Consider two cases: (1) $X \leq Y$, and $Y < X$. Case 1 implies that $u = x$ and $v = y$ so "back-solving" produces $x = u$ and $y = v$. Thus, taking the Jacobian, we get $J = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$, and $h(u, v) = f(u)g(v) \mid J \mid = (2u)(3v^2) \cdot 1 = 6uv^2$, $0 < u < v < 1$. Case 2 meanwhile implies that $u = y$ and $v = x$, so (again transforming backwards) we have that $x = v$, $y = u$, and $J = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$, therefore (here) we have that $h(u, v) = f(v)g(u) \mid J \mid = (2v)(3u^2) \cdot 1 = 6u^2v$, $0 < u < v < 1$. Thus, combining the cases, we get $f(u, v) = 6uv^2 + 6u^2v = 6uv(u + v)$, $0 < u < v < 1$.

6. Let $X_1, X_2, \ldots, X_n$ represent a random sample from each of the distributions having the following pdfs or pmfs:

   (a) $f(x; \theta) = \frac{\theta^x e^{-\theta}}{x!}$, $x = 0, 1, 2, \ldots$, $0 \leq \theta < \infty$, zero elsewhere, where $f(0; 0) = 1$
   (b) $f(x; \theta) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$, zero elsewhere
   (c) $f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere
   (d) $f(x; \theta) = e^{-(x-\theta)}$, $\theta <\leq x < \infty$, $-\infty < \theta < \infty$, zero elsewhere.

In each case, find the mle $\hat{\theta}$ of θ.

Solution:

(a)

$$\begin{aligned}
L(\theta;\boldsymbol{x}) &= \frac{\theta^{\sum x_i} e^{-n\theta}}{\prod (x_i!)} \\
\ln L(\theta;\boldsymbol{x}) &= \left(\sum_{i=1}^{n} x_i\right)(\ln\theta) - n\theta - \sum_{i=1}^{n} \ln(x_i!) \\
\frac{\partial \ln L(\theta)}{\partial\theta} &= \frac{\sum_{i=1}^{n} x_i}{\theta} - n = 0 \\
&\Rightarrow \sum_{i=1}^{n} x_i - n\theta = 0 \\
&\Rightarrow \hat{\theta} = \bar{X}.
\end{aligned}$$

(b)

$$\begin{aligned}
L(\theta;\boldsymbol{x}) &= \theta^n \left(\prod x_i\right)^{\theta-1} \\
\ln L(\theta;\boldsymbol{x}) &= n\ln\theta + (\theta-1)\sum \ln x_i \\
\frac{\partial \ln L(\theta)}{\partial\theta} &= \frac{n}{\theta} + \sum \ln x_i = 0 \\
&\Rightarrow n + \theta\sum \ln x_i = 0 \\
&\Rightarrow \hat{\theta} = \frac{-n}{\sum_{i=1}^{n} \ln x_i}.
\end{aligned}$$

(c)

$$\begin{aligned}
L(\theta;\boldsymbol{x}) &= \frac{1}{\theta^n} e^{\frac{-\sum x_i}{\theta}} \\
\ln L(\theta;\boldsymbol{x}) &= -n\ln\theta - \frac{-\sum x_i}{\theta} \\
\frac{\partial \ln L(\theta)}{\partial\theta} &= \frac{-n}{\theta} + \frac{\sum x_i}{\theta^2} = 0 \\
&\Rightarrow -n\theta + \sum x_i = 0 \\
&\Rightarrow \hat{\theta} = \bar{X}.
\end{aligned}$$

(d)

$$L(\theta;\boldsymbol{x}) = e^{n\theta} e^{-\sum x_i} \prod_{i=1}^{n} I_{[\theta,\infty)}(x_i) = e^{n\theta} e^{-\sum x_i} \prod_{i=1}^{n} I_{(-\infty,x_i]}(\theta) = e^{n\theta} e^{-\sum x_i} I_{(-\infty,x_{(1)}]}(\theta),$$

therefore $\hat{\theta} = X_{(1)}$.

7. Suppose $X_1, X_2, \ldots, X_n$ are iid with pdf $f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. Find the MLE of $P(X > k)$, for some $k > 0$ (known).

Solution:

$$P(X > k) = \int_k^\infty \frac{1}{\theta}e^{-x/\theta}dx = -e^{-x/\theta} \mid_k^\infty = -(0 - e^{-k/\theta}) = e^{-k/\theta} = g(\theta),$$

i.e. $P(X > k)$ can be represented as a function (say, $g(\theta)$) of θ. Therefore $g(\hat{\theta})$ is the MLE of $g(\theta)$, where $\hat{\theta}$ is determined from

$$\begin{aligned}
L(\theta; \boldsymbol{x}) &= \frac{1}{\theta^n}e^{-\sum(x_i/\theta)} = \theta^{-n}e^{-\sum x_i/\theta} \\
\ln L(\theta; \boldsymbol{x}) &= -n\ln\theta - \frac{\sum x_i}{\theta} \\
\frac{\partial \ln L(\theta; \boldsymbol{x})}{\partial\theta} &= \frac{-n}{\theta} + \frac{\sum x_i}{\theta^2} = 0 \\
&\Rightarrow -n\theta + \sum x_i = 0 \\
&\Rightarrow \hat{\theta} = \bar{X},
\end{aligned}$$

so $g(\hat{\theta}) = e^{-k/\bar{x}}$ is the MLE for $P(X > k)$.

8. Let $X_1, \ldots, X_n$ be iid with pdf $f(x \mid \theta) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$.

   (a) Find the MLE of θ, and show that its variance converges to 0 as $n \to \infty$.

   (b) Find the method of moments estimator of θ.

   Solution:

   (a) To find the MLE of θ, we find the log-likelihood function and differentiate with respect to θ. Note that we can do so here because the support space does not depend on θ.

$$\begin{aligned}
f(x;\theta) &= \theta x^{\theta-1} \\
L(\theta; \boldsymbol{x}) = \prod_{i=1}^n f(x_i;\theta) &= \theta^n(\prod_{i=1}^n x_i)^{\theta-1} \\
\ln L(\theta; \boldsymbol{x}) &= n\ln\theta + (\theta-1)\sum_{i=1}^n \ln x_i \\
\frac{\partial \ln L(\theta)}{\partial\theta} &= \frac{n}{\theta} + \sum_{i=1}^n \ln x_i = 0 \\
&\Rightarrow \hat{\theta} = \frac{-n}{\sum_{i=1}^n \ln x_i}
\end{aligned}$$

   Meanwhile, to find the variance, let $y_i = -\ln x_i$, $x_i = e^{-y_i}$, $\frac{dx_i}{dy_i} = -e^{-y_i}$, thus

$$f_{Y_i}(y) = f_{X_i}(e^{-y_i}) \mid -e^{-y_i} \mid = \theta e^{-(\theta-1)y}e^{-y} = \theta e^{-\theta y}, \qquad 0 < y < \infty,$$

   thus $Y_i \sim \text{Exponential}(1/\theta) = \text{Gamma}(1, 1/\theta)$.

$\Rightarrow Z = \sum_{i=1}^{n} Y_i$ has a Gamma$(n, 1/\theta)$ distribution whose pdf is

$$f_Z(z) = \frac{1}{\Gamma(n)(1/\theta)^n} z^{n-1} e^{-\theta z}, \qquad 0 < z < \infty.$$

We can use this information to find

$$\begin{aligned} E(\hat{\theta}) &= nE\left(\frac{1}{-\sum \ln x_i}\right) = nE\left(\frac{1}{Z}\right) \\ &= n\int_0^\infty \frac{1}{z} \frac{\theta^n}{\Gamma(n)} z^{n-1} e^{-\theta z} dz = \frac{n\theta^n}{(n-1)!} \frac{(n-2)!}{\theta^{n-1}} \int_0^\infty \frac{\theta^{n-1}}{(n-2)!} z^{n-2} e^{-\theta z} dz = \frac{n\theta}{n-1}. \end{aligned}$$

Meanwhile, $E(\hat{\theta}^2) = E\left(\frac{n}{-\sum \ln x_i}\right) = n^2 E\left(\frac{1}{Z^2}\right)$, where

$$E\left(\frac{1}{Z^2}\right) = \int_0^\infty \frac{1}{z^2} \frac{\theta^n}{\Gamma(n)} z^{n-1} e^{-\theta z} dz = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} \int_0^\infty \frac{\theta^{n-2}}{\Gamma(n-2)} z^{n-3} e^{-\theta z} dz = \frac{\theta^2}{(n-1)(n-2)},$$

thus

$$E(\hat{\theta}^2) = \frac{n^2\theta^2}{(n-1)(n-2)}.$$

Finally,

$$\mathrm{Var}(\hat{\theta}) = E(\hat{\theta}^2) - E^2(\hat{\theta}) = \frac{n^2\theta^2}{(n-1)(n-2)} - \frac{n^2\theta^2}{(n-1)^2} = \frac{n^2\theta^2}{(n-1)^2(n-2)} \to 0.$$

(b) To find the MOM, recall that $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$, therefore $X \sim$Beta$(\theta, 1)$, thus $E(X) = \frac{\theta}{\theta+1}$, which we can estimate as $\bar{X} = \frac{\tilde{\theta}}{\tilde{\theta}+1}$. Back-solving for the MOM estimator $\tilde{\theta}$, we get $\tilde{\theta} = \frac{\bar{X}}{1-\bar{X}}$.