Sensitivity Analysis and Computerized Solutions

Sample Problem from last time

You are assembling fruit baskets from the excess stock at your store. You have 10 bananas, 24 apples, and 16 melons. The baskets come in two configurations.

Mix 1 sells for \$3 and contains 1 banana, 3 apples, and 1 melon.

Mix 2 sells for \$2 and contains 1 banana, 1 apple, and 2 melons.

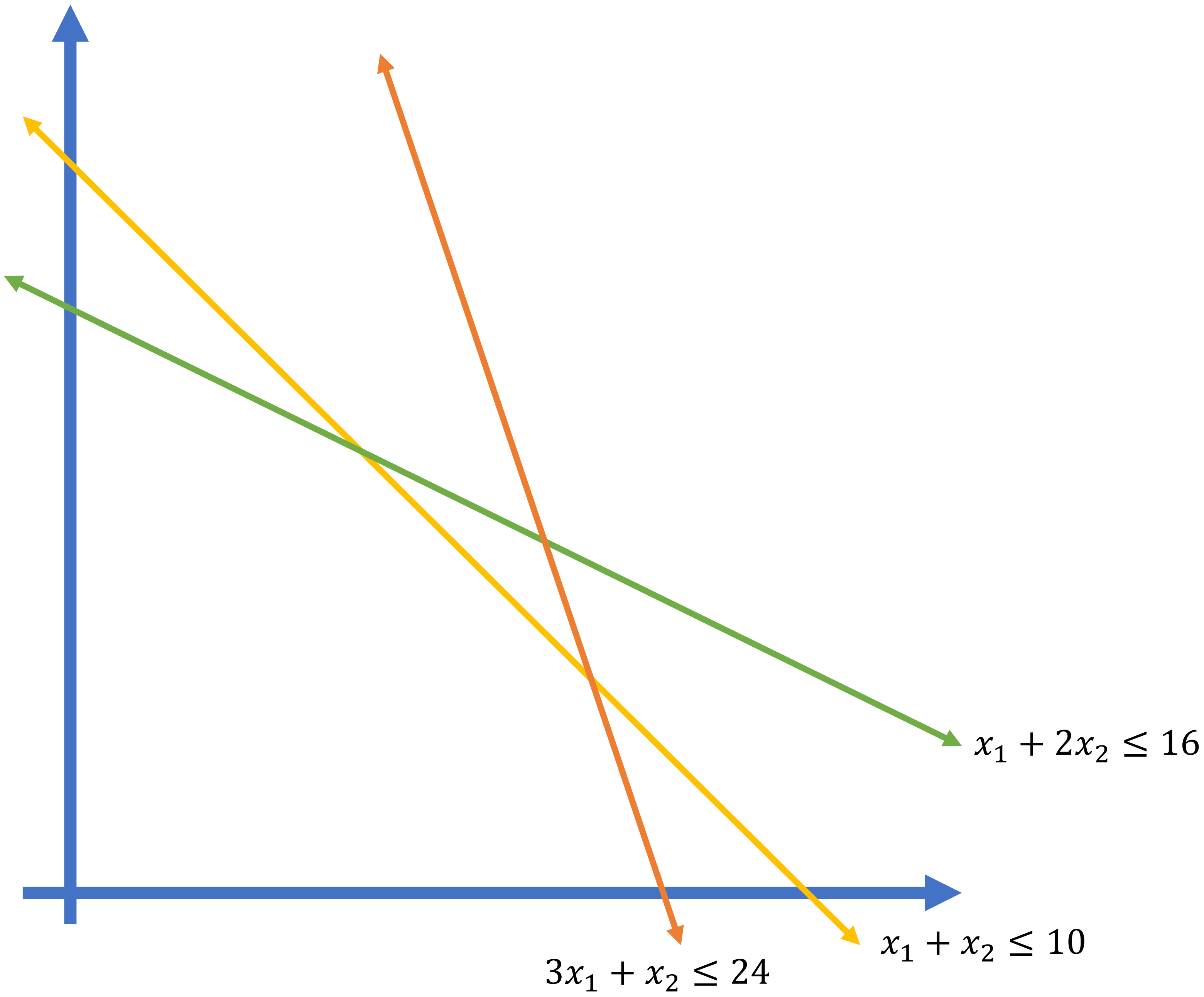
How should you use your remaining fruit to maximize revenue?

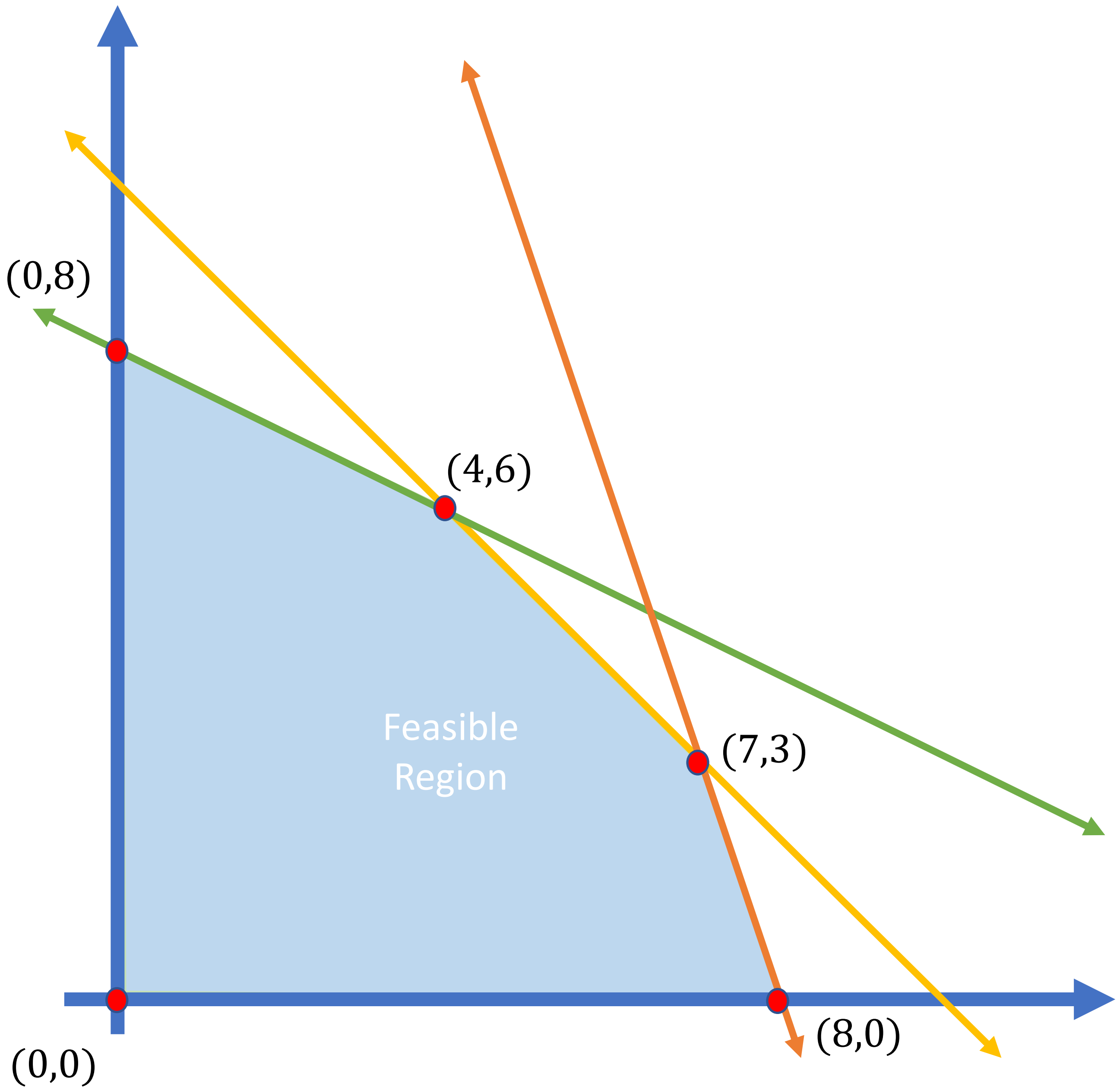
The Simplex Method by Example

Max $3x_1 + 2x_2$

s.t.

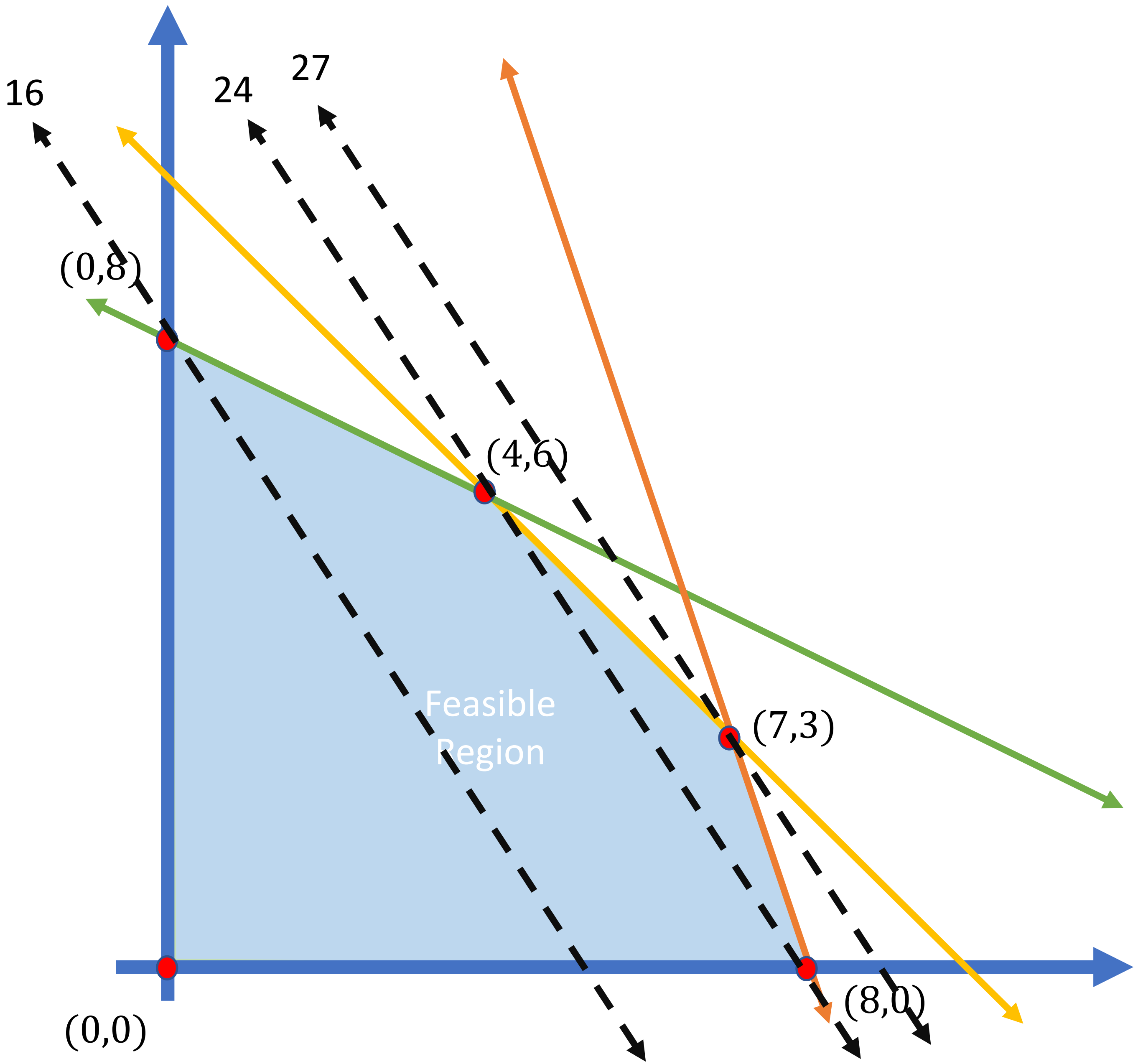
$$
\begin{aligned}
x_1 + x_2 &\leq 10 \\
3x_1 + x_2 &\leq 24 \\
x_1 + 2x_2 &\leq 16 \\
x_1, x_2 &\geq 0
\end{aligned}
$$


$x_1 + 2x_2 \leq 16$
$x_1 + x_2 \leq 10$
$3x_1 + x_2 \leq 24$



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24



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x_1	x_2	Objective
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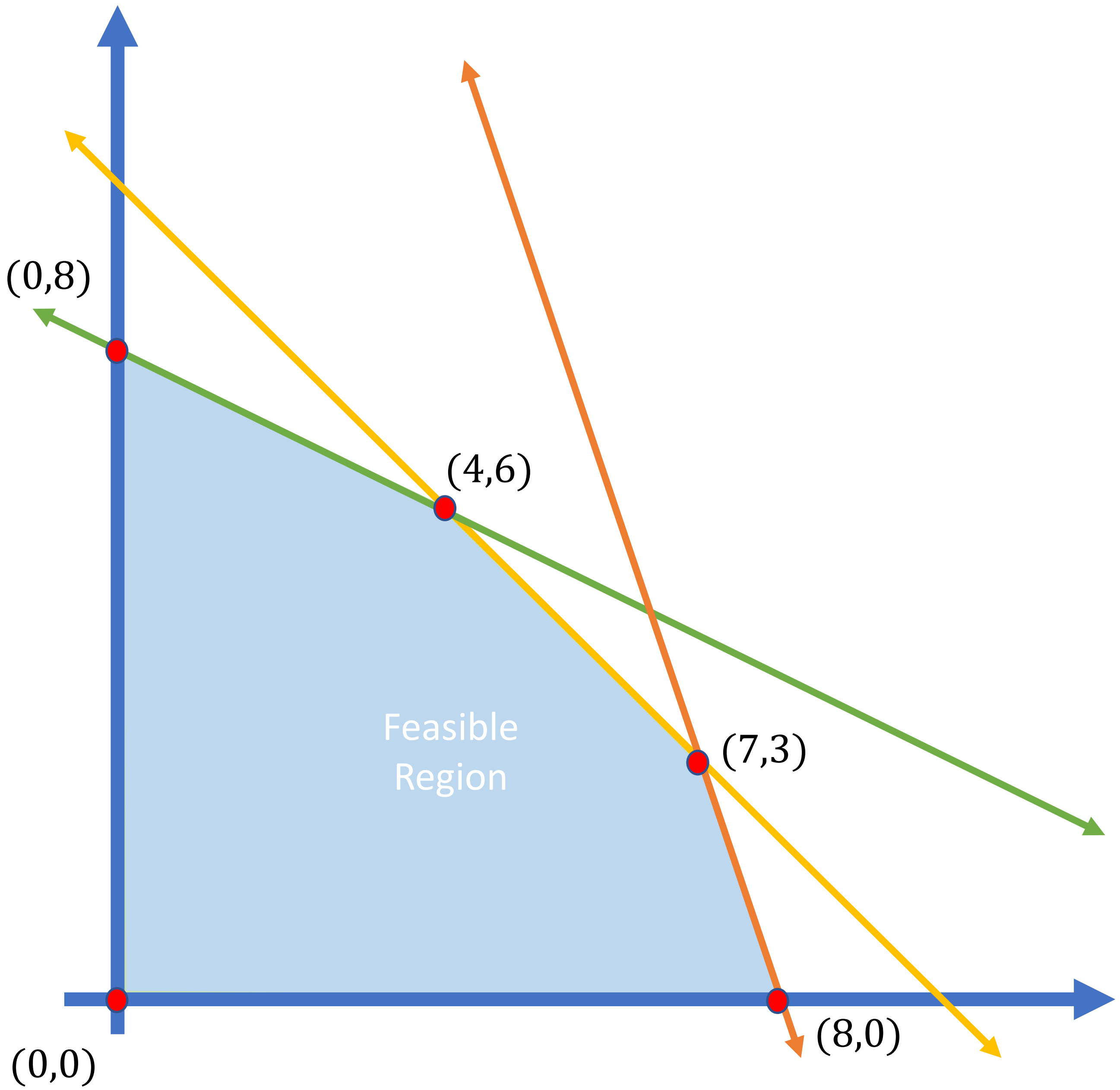


Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

Step 1:

Rewrite the Linear Program in equality form by adding/subtracting slack and surplus variables as needed.
Once in equality form, multiply constraints by -1 as needed to make $b \geq 0$

Max $3x_1 + 2x_2$

s.t.

$$
\begin{aligned}
x_1 \quad + x_2 \quad &\leq 10 \\
3x_1 + x_2 \quad &\leq 24 \\
x_1 \quad + 2x_2 &\leq 16 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Max $3x_1 + 2x_2 = z$

s.t.

$$
\begin{aligned}
x_1 \quad + x_2 \quad + s_1 &= 10 \\
3x_1 + x_2 \quad + s_2 &= 24 \\
x_1 \quad + 2x_2 + s_3 &= 16 \\
x_1, x_2, s_1, s_2, s_3 &\geq 0
\end{aligned}
$$

Initial Simplex Tableau

Max $3x_1 + 2x_2 = z$

s.t.

$$
\begin{aligned}
x_1 \quad + x_2 \quad + s_1 &= 10 \\
3x_1 + x_2 \quad + s_2 &= 24 \\
x_1 \quad + 2x_2 + s_3 &= 16 \\
x_1, x_2, s_1, s_2, s_3 &\geq 0
\end{aligned}
$$

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	-3	-2	0	0	0	0
s_1	1	1	1	0	0	10
s_2	3	1	0	1	0	24
s_3	1	2	0	0	1	16

Initial Simplex Tableau*

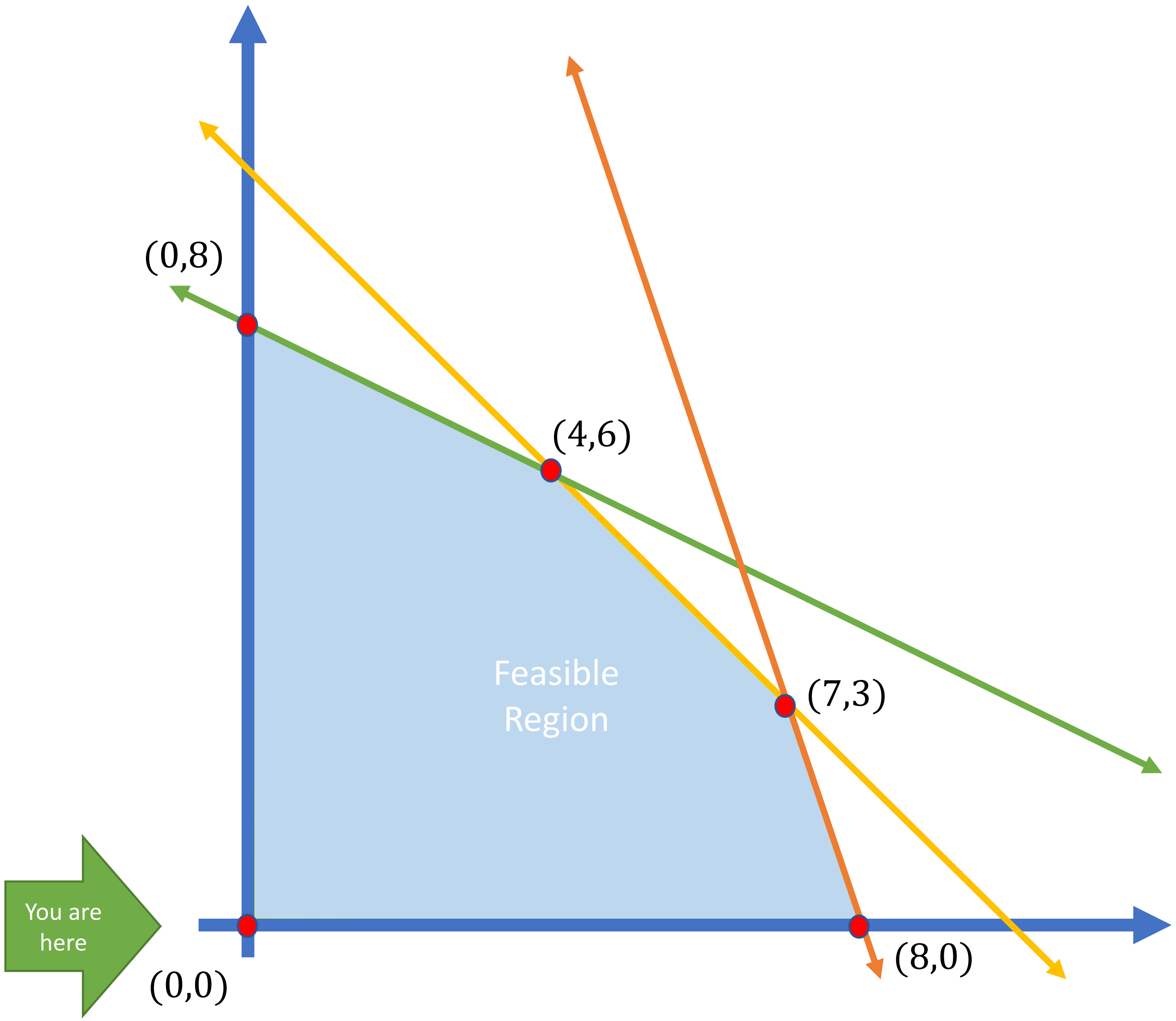
Max $3x_1 + 2x_2 = z$

s.t.

$$\begin{aligned} x_1 \quad + x_2 \quad + s_1 &= 10 \\ 3x_1 + x_2 \quad + s_2 &= 24 \\ x_1 \quad + 2x_2 + s_3 &= 16 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**
z	-3	-2	0	0	0	0
s_1	1	1	1	0	0	10
s_2	3	1	0	1	0	24
s_3	1	2	0	0	1	16

* If you do not have a known basic feasible solution or identity matrix in your tableau, you will need to do the two phase simplex method (which we will do in another example).



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

Choose the pivot column

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**
z	-3	-2	0	0	0	0
s_1	1	1	1	0	0	10
s_2	3	1	0	1	0	24
s_3	1	2	0	0	1	16

- On the objective function line, choose the column with the most negative coefficient to enter the basis
- Next we need to find which line limits the maximum value we can assign to x_1, to do this we use a Ratio Test

Apply the Ratio Test

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**	**Ratio Test**
z	-3	-2	0	0	0	0	
s_1	1	1	1	0	0	10	10/1=10
s_2	3	1	0	1	0	24	24/3=8
s_3	1	2	0	0	1	16	16/1=16

- By taking the ratio or the RHS to the coefficient of the new basis variable, we can see which of the constraints limits the maximum value of the variable entering the basis

Apply the Ratio Test

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**	**Ratio Test**
z	-3	-2	0	0	0	0	
s_1	1	1	1	0	0	10	10/1=10
s_2	3	1	0	1	0	24	24/3=8
s_3	1	2	0	0	1	16	16/1=16

- By taking the ratio or the RHS to the coefficient of the new basis variable, we can see which of the constraints limits the maximum value of the variable entering the basis

Use Elementary Row Operations to eliminate variable from other rows

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**	**Ratio Test**
z	-3	-2	0	0	0	0	
s_1	1	1	1	0	0	10	10/1=10
s_2	3	1	0	1	0	24	24/3=8
s_3	1	2	0	0	1	16	16/1=16

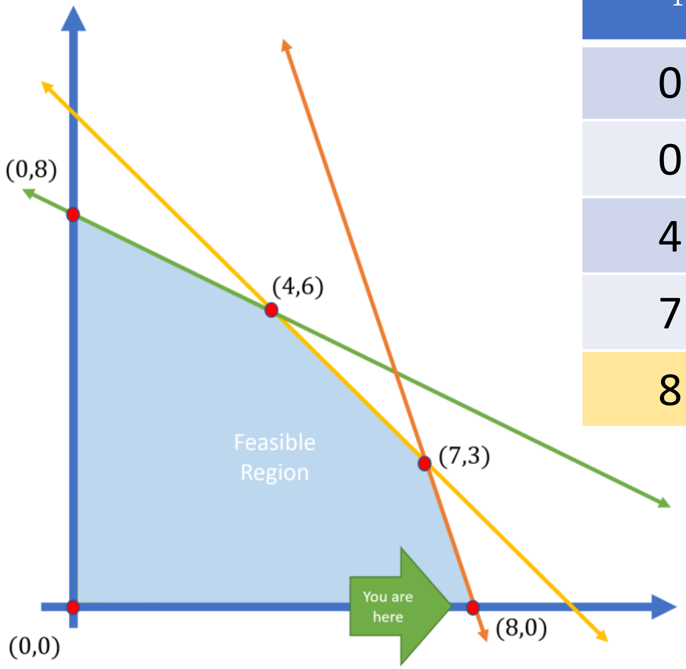
$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**
z	-3	-2	0	0	0	0
s_1	1	1	1	0	0	10
x_1	3/3	1/3	0	1/3	0	24/3
s_3	1	2	0	0	1	16

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**
z	0	-1	0	1	0	24
s_1	0	2/3	1	-1/3	0	2
x_1	1	1/3	0	1/3	0	8
s_3	0	5/3	0	-1/3	1	8

Repeat until no negative numbers in Objective Function

2nd Tableau Form

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	0	-1	0	1	0	24
s_1	0	2/3	1	-1/3	0	2
x_1	1	1/3	0	1/3	0	8
s_3	0	5/3	0	-1/3	1	8



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

Identify Pivot Column and Row

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
z	0	-1	0	1	0	24	
s_1	0	2/3	1	-1/3	0	2	2/(2/3)=3
x_1	1	1/3	0	1/3	0	8	8/(1/3)=24
s_3	0	5/3	0	-1/3	1	8	8/(5/3)=24/5

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	0	0	1.5	0.5	0	27
x_2	0	1	1.5	-0.5	0	3
x_1	1	0	-0.5	0.5	0	7
s_3	0	0	-2.5	0.5	1	3

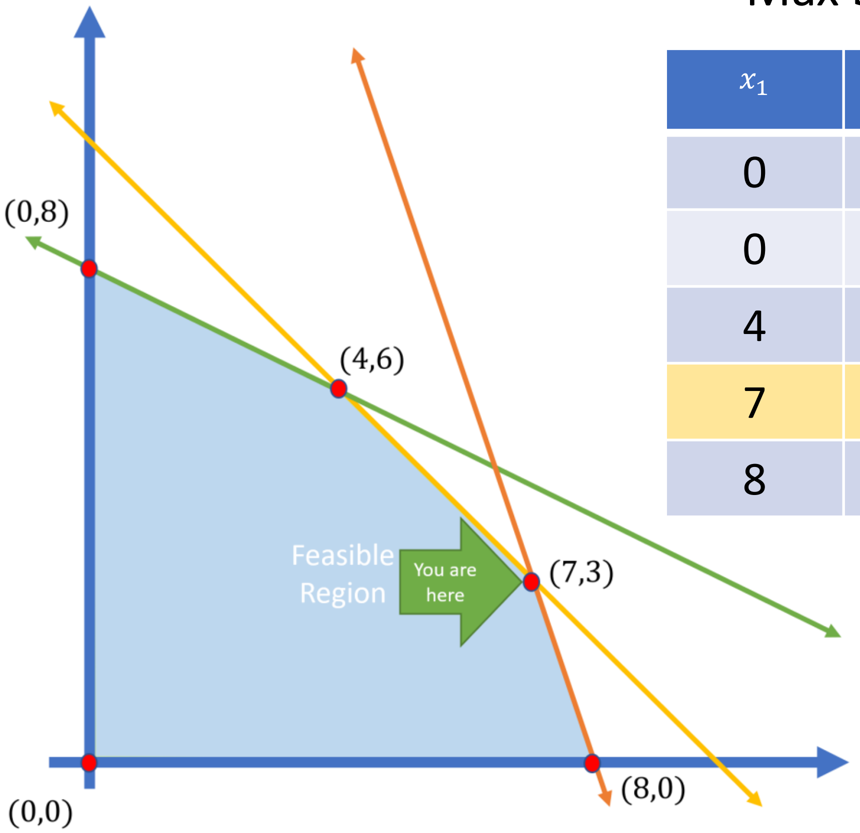
Optimal Solution

Max $3x_1 + 2x_2$
s.t.

$$x_1 + x_2 \leq 10$$
$$3x_1 + x_2 \leq 24$$
$$x_1 + 2x_2 \leq 16$$
$$x_1, x_2 \geq 0$$

3rd Tableau Form

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	0	0	1.5	0.5	0	27
x_2	0	1	1.5	-0.5	0	3
x_1	1	0	-0.5	0.5	0	7
s_3	0	0	-2.5	0.5	1	3



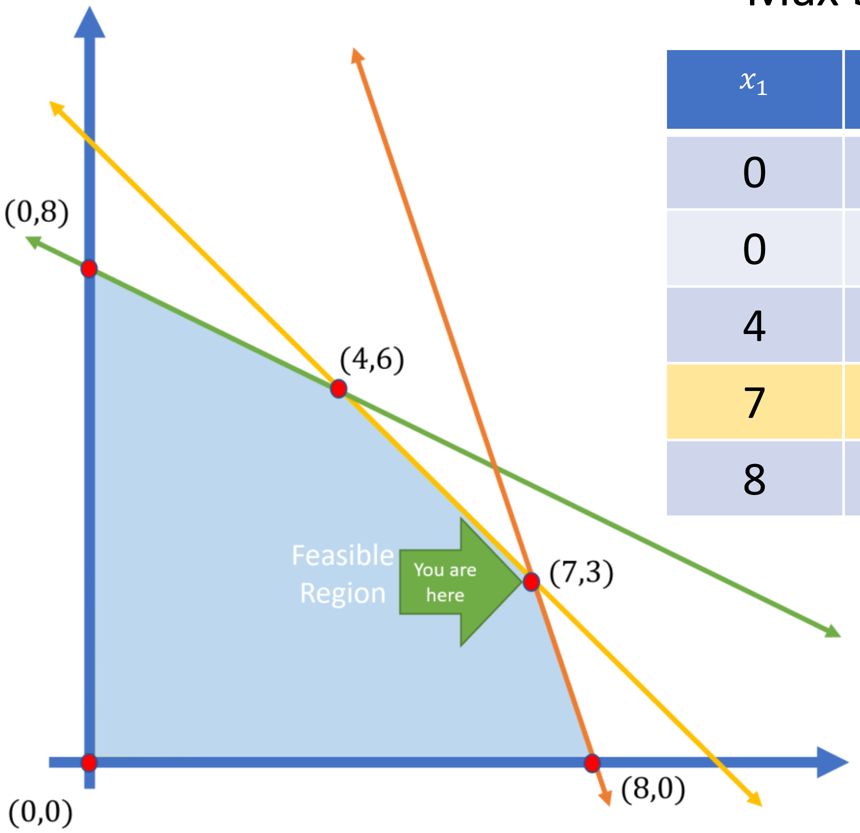
Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

Optimal Solution

3rd Tableau Form

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	0	0	1.5	0.5	0	27
x_2	0	1	1.5	-0.5	0	3
x_1	1	0	-0.5	0.5	0	7
s_3	0	0	-2.5	0.5	1	3



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

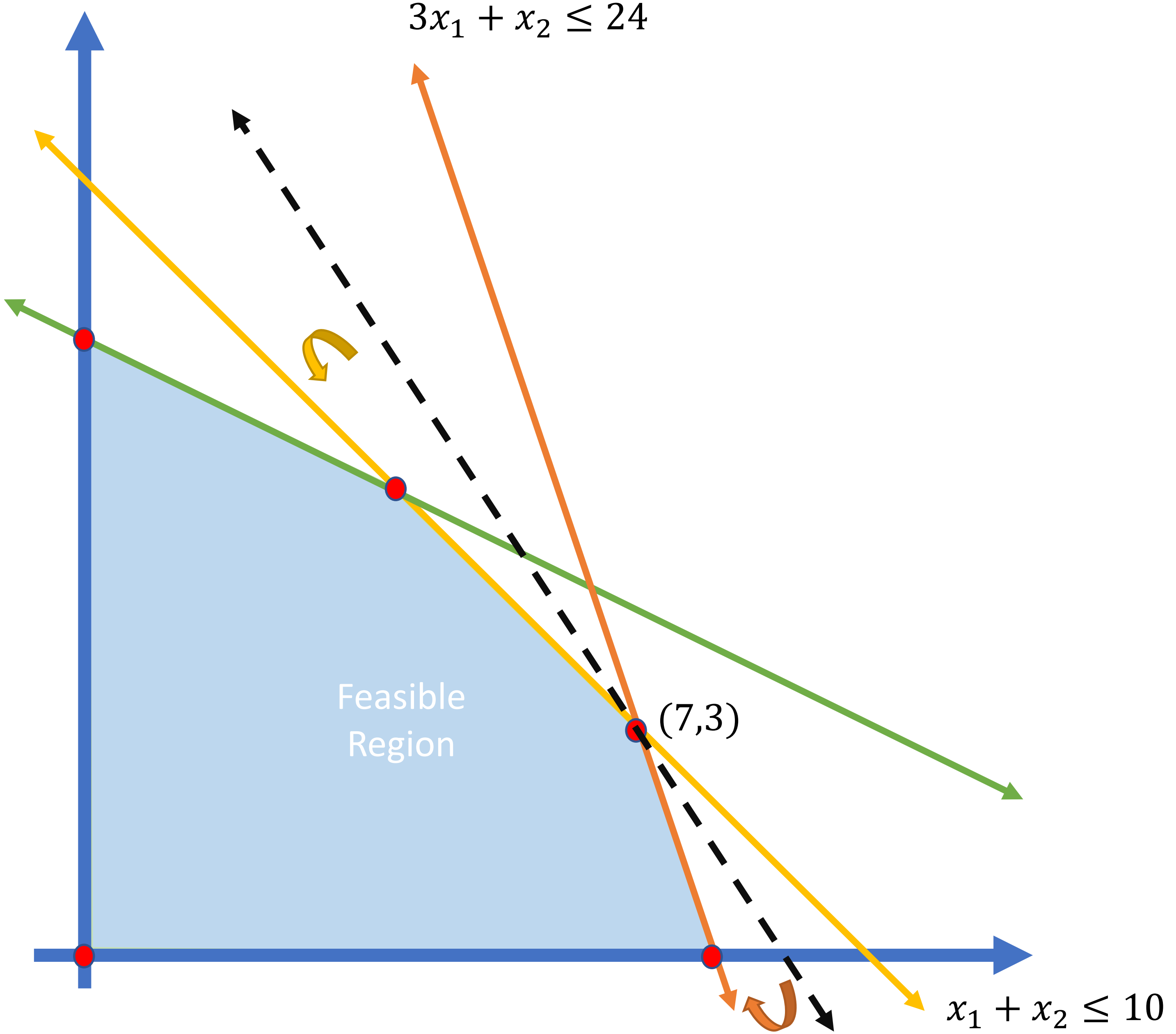
Changing the Problem

- Once we have solved a problem to optimality, we might want to examine how changes to the problem might change our solution.
  - 1. Changes to the objective function
  - 2. Changes to the RHS
  - 3. Adding a constraint
  - 4. Adding a decision variable
  - 5. Changing the constraint slopes

1. Changes to the Objective Function

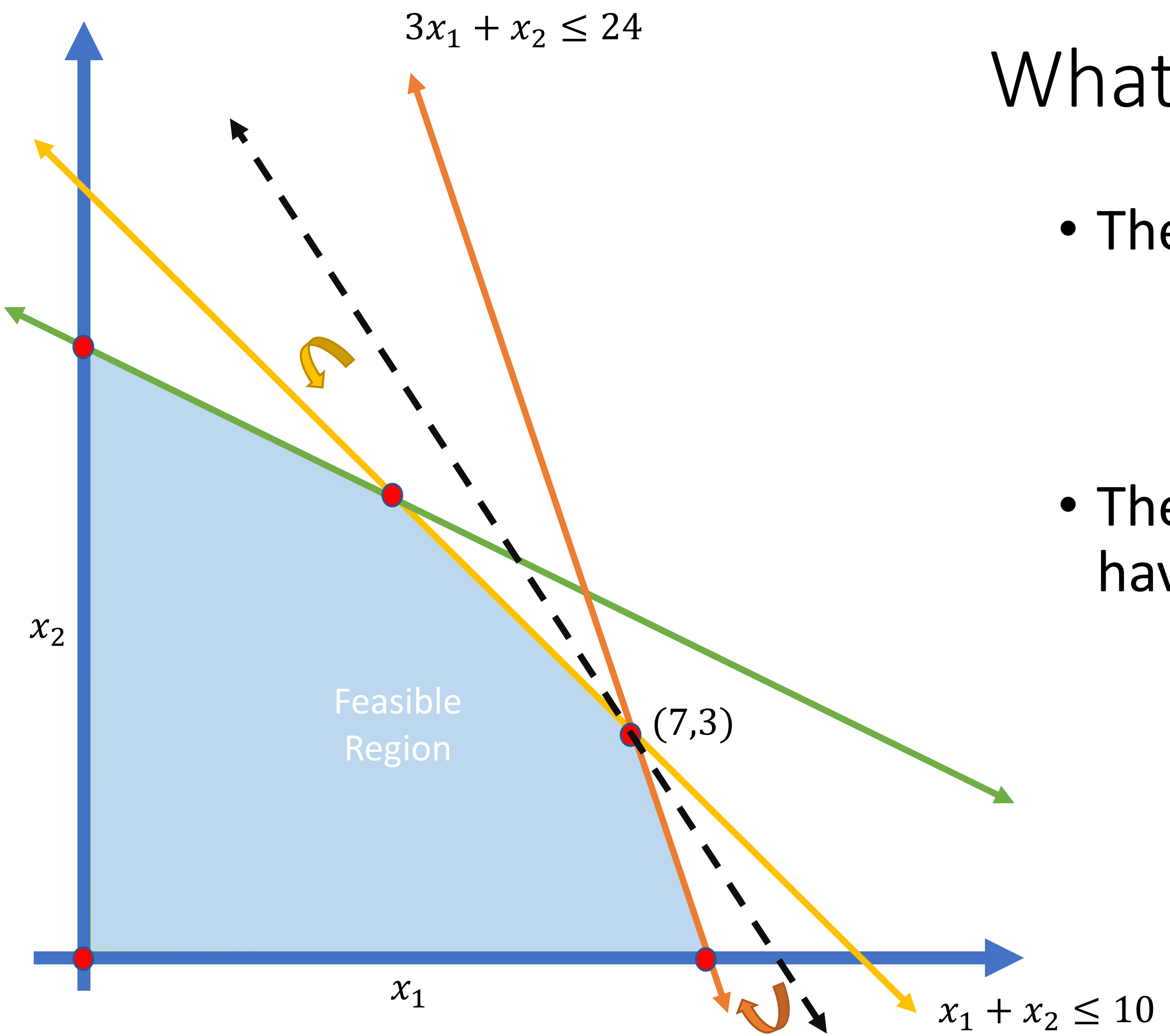
Questions:

- When does changing the price change the solution?
- When does changing the price change the objective function value?
- How big of a change can I make?
- Can I make multiple changes at once?
- When do I need to resolve?



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24



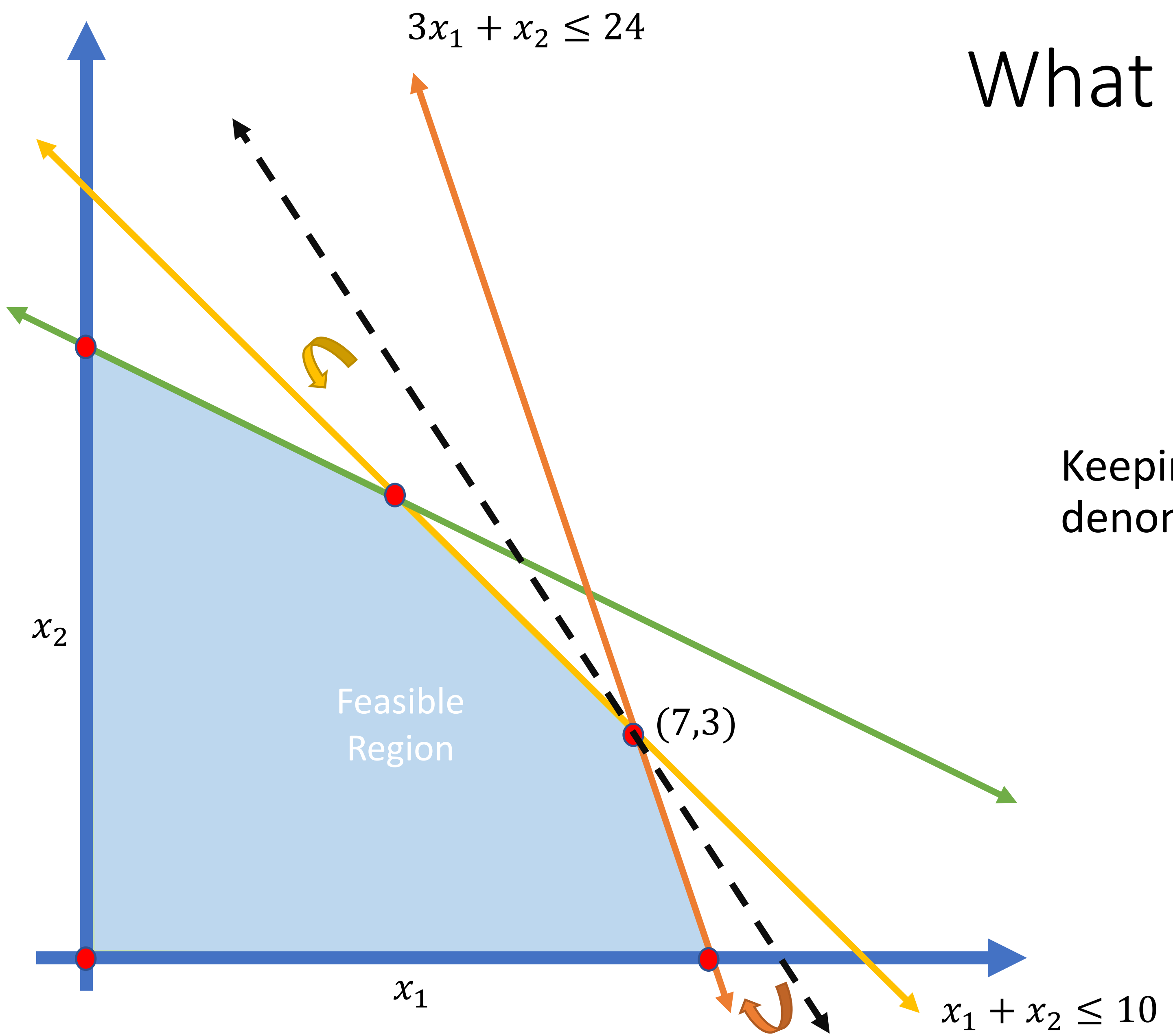
What are the slopes?

- The objective function:

$$c = 3x_1 + 2x_2$$
$$x_2 = -\frac{3}{2}x_1 + c$$

- The active constraint have:

$$x_2 = -x_1 + 10$$
$$x_2 = -3x_1 + 24$$



What are the bounds?

$$-3 \leq -\frac{x_1}{x_2} \leq -1$$

$$-3 \leq -\frac{3}{2} \leq -1$$

Keeping either the numerator or denominator constant:

$$-3 \leq -\frac{6}{2} \leq -1$$

$$-3 \leq -\frac{2}{2} \leq -1$$

$$-3 \leq -\frac{3}{1} \leq -1$$

$$-3 \leq -\frac{3}{3} \leq -1$$

Can I make two changes at once?

$$-3 \leq -\frac{x_1}{x_2} \leq -1$$

	Min	Current	Max
x_1	2	3	6
x_2	1	2	3

$$-3 \leq -\frac{6}{3} \leq -1$$

$$-3 \not\leq -\frac{6}{1} \leq -1$$

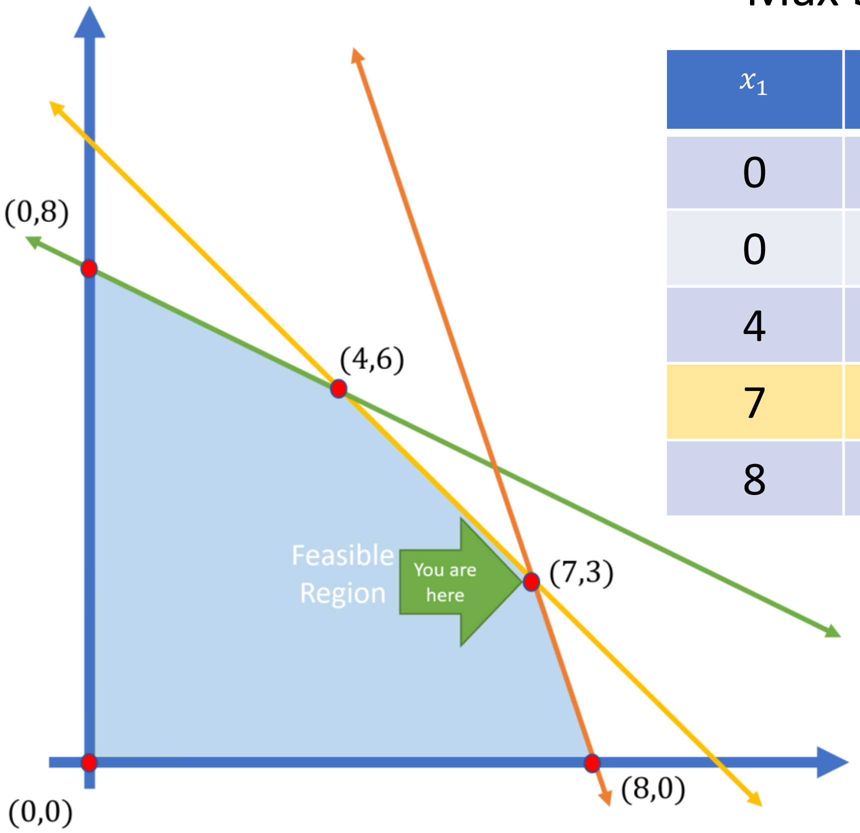
$$-3 \leq -\frac{2}{1} \leq -1$$

$$-3 \leq -\frac{2}{3} \not\leq -1$$

Optimal Solution

3rd Tableau Form

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	$\pm c_1$	$\pm c_2$	1.5	0.5	0	27
x_2	0	1	1.5	-0.5	0	3
x_1	1	0	-0.5	0.5	0	7
s_3	0	0	-2.5	0.5	1	3



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

The 100% Rule

- For all objective function coefficient changes, sum the percentages of the allowable increases and the allowable decreases. If the sum of the percentages is less than or equal to 100%, the optimal solution will not change.

1. Changes to the Objective Function

Questions:

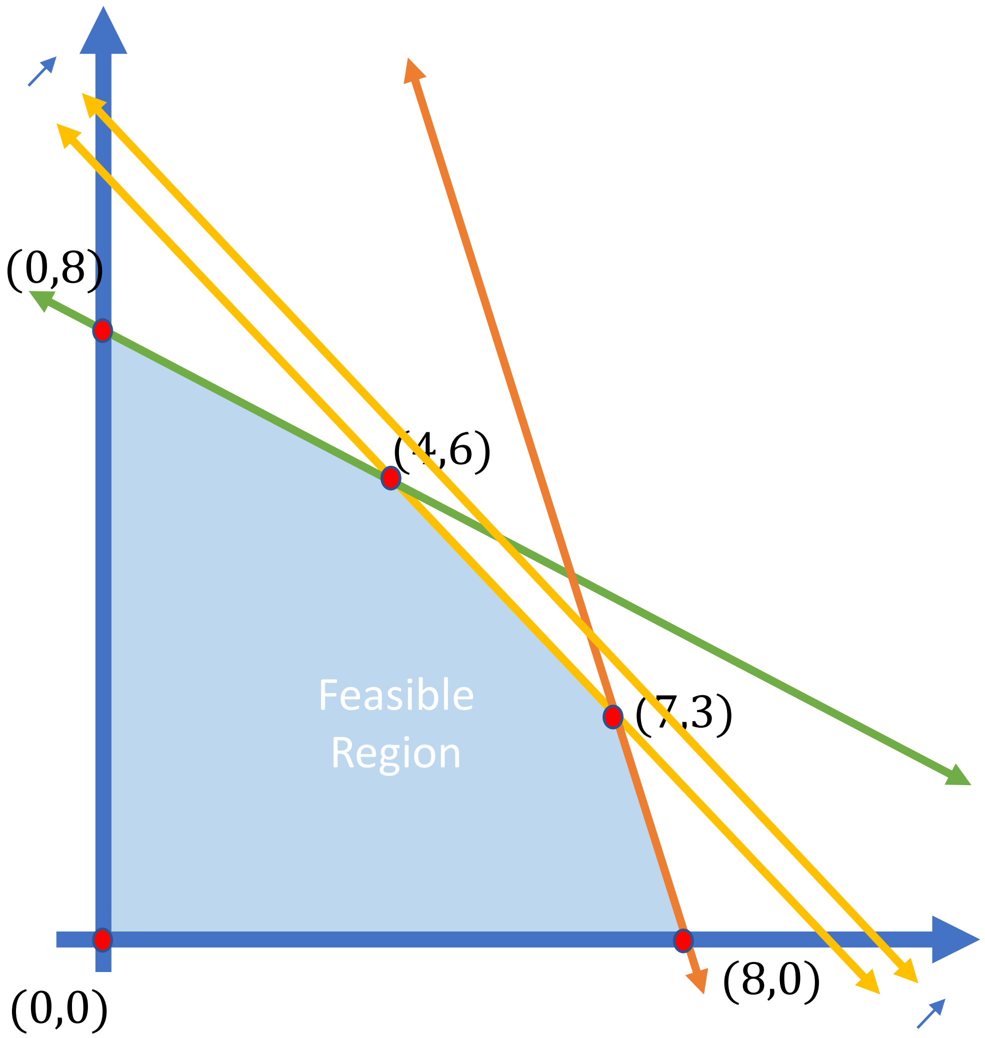
- When does changing the price change the solution?
  - For non-basic variables, not at all so long as the coefficient in z remains positive
  - For basic variables, does not change where the solution is so long as the slope of the cost function does not cross the plane of an active constraint
- When does changing the price change the objective function value?
  - For basic variables proportional to the solution value
- Can I make multiple changes at once?
  - Yes, so long as the changes are <= 100% of the allowable changes for the cost function

2. Changes to the RHS

Questions:

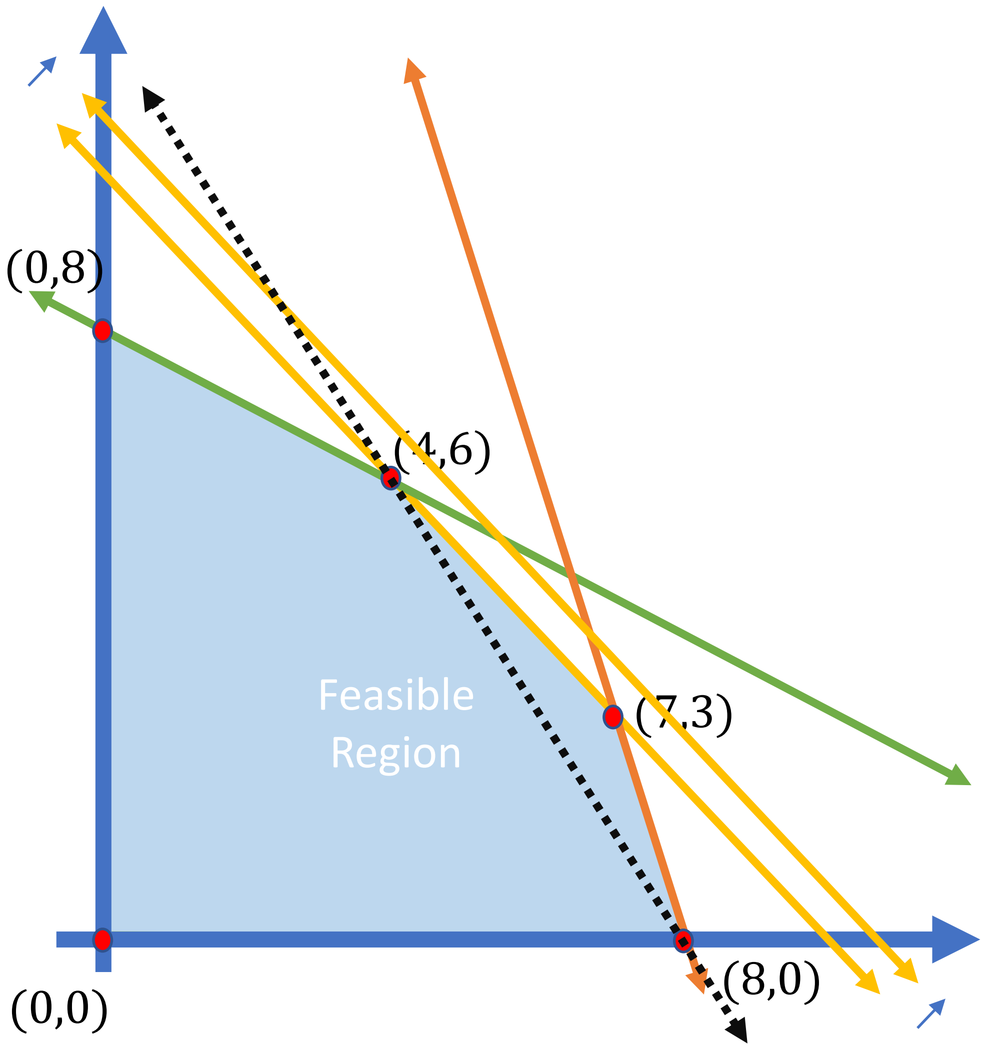
- When does changing the RHS change the solution?
- When does changing the RHS change the objective function value?
- How big of a change can I make?
- Can I make multiple changes at once?
- When do I need to resolve?

How much would an additional banana be worth?



- You are expanding the feasible region, but are you changing where the Optimal Solution is?

How much would an additional banana be worth?



- You are expanding the feasible region, but are you changing where the Optimal Solution is?

The Cost of Constraints

$$
\begin{array}{rl}
-(x_1 \; + x_2 & \leq 10) \\
+(3x_1 + x_2 & \leq 24) \\
\underline{+(x_1 \; + 2x_2} & \underline{\leq 16)} \\
3x_1 + 2x_2 & \leq 30
\end{array}
$$

Objective function

Bound

- We can think of each constraint as having a "cost" or "price" as to how good the objective value can be.
- But could we make a tighter upper bound?

The Cost of Constraints

$$
\begin{array}{rl}
1.5(x_1 + x_2 & \leq 10) \\
+\ .5(3x_1 + x_2 & \leq 24) \\
\hline
3x_1 + 2x_2 & \leq 27
\end{array}
$$

Objective function

Bound

- We can think of each constraint as having a “cost” or “price” as to how good the objective value can be.
- But could we make a tighter upper bound?
- Look the optimal solution is the best upper bound

The Dual Problem

Primal

Max $c^T x$

s.t. $Ax \leq b$

$x \geq 0$

Dual

Min $p^T b$

s.t. $p^T A \geq c$

$p \geq 0$

Forming the Dual

Primal (Max)

Constraint

- $a_i^T x_i \leq \ b_i$
- $a_i^T x_i \geq \ b_i$
- $a_i^T x_i = \ b_i$

Variable

- $x_j \geq 0$
- $x_j \leq 0$
- $x_j \ \ is \quad free$

Dual (Min)

Variable

- $p_i \geq 0$
- $p_i \leq 0$
- $p_i \ \ is \quad free$

Constraint

- $p_j^T a_j \geq \ c_j$
- $p_j^T a_j \leq \ c_j$
- $p_j^T a_j = \ c_j$

Quantifying the Objective Value Change

- Dual Price:
  - “The **improvement** in the value of the optimal solution per unit increase in the RHS of a constraint”
    - A positive *dual price* means an increase for Max problems and a decrease for Min problems
- Shadow/Reduced Price:
  - The actual **change** in the value of the optimal solution per unit increase in the RHS of a constraint
    - A positive *shadow price* means an increase for both Max and Min problems

How much would an additional banana be worth?

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**
z	0	0	1.5	0.5	0	27
x_2	0	1	1.5	-0.5	0	3
x_1	1	0	-0.5	0.5	0	7
s_3	0	0	-2.5	0.5	1	3

- The *shadow* or *dual* price (for maximization) tells you how much the objective function value would increase for 1 additional unit of the associated constraint.
- Only active constraints have positive shadow prices, these can be found from the final state of your simplex tableau

Notice:

Objective Function Change

- Slope of objective function line changed
- Extreme points stayed the same
- Optimal point stayed the same (within certain range)
- Optimal objective value changed

RHS Constraint Change

- Feasible region expands (contracts)
- Extreme points change
- Optimal point will adjust
- Optimal objective function value will change

Additional Reading

- Sensitivity Analysis : http://web.mit.edu/15.053/www/AMP-Chapter-03.pdf
- Duality Theory: http://web.mit.edu/15.053/www/AMP-Chapter-04.pdf