MATH 503 Midterm Exam 2 Solutions

1. **If $X_1, \ldots, X_N$ are iid Binomial $(n, p)$ random variables, find the MVUE of $P(X = n) = p^n$.**

$X_1, \ldots, X_N$ are iid Binomial $(n, p) \Rightarrow$ the pmf

$$
\begin{aligned}
p(X = x) &= \binom{n}{x} p^x (1-p)^{n-x} \\
&= \exp(\ln(\binom{n}{x} \left(\frac{p}{1-p}\right)^x (1-p)^n)) \\
&= \exp(\underbrace{\ln \binom{n}{x}}_{S(x)} + \underbrace{x}_{K(x)} \underbrace{\ln \left(\frac{p}{1-p}\right)}_{P(p)} + \underbrace{n \ln(1-p)}_{Q(p)})
\end{aligned}
$$

has the form of an exponential family, and $Y = \sum_{i=1}^n K(x_i) = \sum_{i=1}^n X_i$ is a (complete) sufficient statistic. Note: an alternative approach is to directly use the Neymann-Fisher Factorization Theorem (NFFT) to show that $Y = \sum_{i=1}^n X_i$ is sufficient for $p$.

Meanwhile, let

$$
W = \begin{cases} 1 & X_1 = n \\ 0 & \text{otherwise.} \end{cases}
$$

By definition, $E(W) = P(X_1 = n) = p^n$, i.e. $W$ is an unbiased estimator of $p^n$. Thus, by the Rao-Blackwell Theorem, and recognizing that $\sum_{i=1}^n X_i \sim Bin(Nn, p)$ and $\sum_{i=2}^n X_i \sim Bin((N-1)n, p)$,

$$
\begin{aligned}
E(W \mid \sum_{i=1}^n X_i = y) &= P(X_1 = n \mid \sum_{i=1}^n X_i = y) \\
&= \frac{P(X_1 = n, \sum_{i=1}^n X_i = y)}{P(\sum_{i=1}^n X_i = y)} = \frac{P(X_1 = n, \sum_{i=2}^n X_i = y - n)}{P(\sum_{i=1}^n X_i = y)} \\
&= \frac{P(X_1 = n) P(\sum_{i=2}^n X_i = y - n)}{P(\sum_{i=1}^n X_i = y)} \\
&= \frac{p^n \binom{(N-1)n}{y-n} p^{y-n} (1-p)^{(N-1)n-(y-n)}}{\binom{Nn}{y} p^y (1-p)^{Nn-y}} = \frac{\binom{(N-1)n}{y-n}}{\binom{Nn}{y}}.
\end{aligned}
$$

2. **The random variables $X_1, \ldots, X_n$ are iid with density**

$$
f_\theta(x) = \exp[-(x - \theta)], \quad x > \theta.
$$

**To test $H_0 : \theta \le 1$ vs. $H_1 : \theta > 1$, a test whose decision rule is to reject $H_0$ if $\min(X_1, \ldots, X_n) > c$ is proposed. Determine $c$ so that this test has size $\alpha$. Determine the associated power function.**

The density $f_\theta(x) = \exp[-(x - \theta)]$ implies that $F(x) = 1 - \exp[-(x - \theta)]$ and $P(X > x) = \exp[-(x - \theta)]$. Thus,

$$
P(X_{(1)} > c) = P(X_1 > c, \ldots, X_n > c) = \prod_{i=1}^n \exp[-(c - \theta)] = \exp(-n(c - \theta)) \quad \doteq \quad \alpha
$$

$$
\begin{aligned}
-n(c-\theta) &= \ln\alpha \\
c-\theta &= -\frac{1}{n}\ln\alpha \\
c &= \theta - \frac{1}{n}\ln\alpha,
\end{aligned}
$$

where, under $H_0$, $\theta \le 1 \Rightarrow c = 1 - \frac{1}{n}\ln\alpha$.

Now, given $c$, the power is determined as

$$
\begin{aligned}
P_\theta(\text{reject } H_0) &= P_\theta(X_{(1)} > 1 - \frac{1}{n}\ln\alpha) \\
&= \exp(-n(1 - \frac{1}{n}\ln\alpha - \theta)) = \exp(-n + \ln\alpha + n\theta) = \alpha\exp(-n(1-\theta)),
\end{aligned}
$$

however, because this is a probability, we need to ensure that it lies between 0 and 1. This constraint only holds for $\theta \le 1 + \frac{1}{n}\ln(1/\alpha)$; otherwise, the power equals 1 for $\theta > 1 + \frac{1}{n}\ln(1/\alpha)$. Hence,

$$
\text{Power} = \begin{cases} \alpha\exp(-n(1-\theta)) & \theta \le 1 + \frac{1}{n}\ln(1/\alpha) \\ 1 & \theta > 1 + \frac{1}{n}\ln(1/\alpha). \end{cases}
$$

Aside: the proof that these constraints must be satisfied....

$$
\begin{aligned}
\alpha\exp(\theta n - n) &\le 1 \\
\exp(\theta n - n) &\le 1/\alpha \\
\theta n - n \le \ln(1/\alpha) \\
\theta n \le n + \ln(1/\alpha) \\
\theta \le 1 + \frac{1}{n}\ln\left(\frac{1}{\alpha}\right)
\end{aligned}
$$

3. **If $S^2$ is the sample variance based on a sample of size $n$ form a normal population, we know that $\frac{((n-1)S^2)}{\sigma^2}$ has a $\chi_(n-1)^2$ distribution. Let the prior distribution for $\sigma^2$ be an inverted gamma distribution IG($\alpha,\beta$), i.e. the pdf is given by**

$$
\pi(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\frac{1}{(\sigma^2)^{\alpha+1}}e^{-1/(\beta\sigma^2)}, \quad 0 < \sigma^2 < \infty,
$$

**where $\alpha$ and $\beta$ are positive constants. What is the posterior distribution of $\sigma^2$?**

Let $X = \frac{((n-1)S^2)}{\sigma^2} \sim \chi^2_{n-1}$, thus it has the pdf

$$
\begin{aligned}
f\left(x = \frac{(n-1)s^2}{\sigma^2}\right) &= \frac{1}{\Gamma\left(\frac{n-1}{2}\right)2^{(n-1)/2}}x^{(n-1)/2-1}e^{-x/2} \\
&= \frac{1}{\Gamma\left(\frac{n-1}{2}\right)2^{(n-1)/2}}\left(\frac{(n-1)s^2}{\sigma^2}\right)^{(n-1)/2-1} e^{-\left(\frac{(n-1)s^2}{\sigma^2}\right)/2}.
\end{aligned}
$$

Meanwhile, the prior distribution for $\sigma^2$ is

$$
\pi(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\frac{1}{(\sigma^2)^{\alpha+1}}e^{-1/(\beta\sigma^2)},
$$

thus the posterior distribution of $\sigma^2$ is

$$\begin{aligned}
&\propto \left(\frac{1}{\sigma^2}\right)^{(n-1)/2-1} e^{-\left(\frac{(n-1)s^2}{\sigma^2}\right)/2} \cdot \frac{1}{(\sigma^2)^{\alpha+1}} e^{-1/(\beta\sigma^2)} \\
&= \left(\frac{1}{\sigma^2}\right)^{(n-1)/2+\alpha} e^{-\left(\frac{(n-1)s^2}{2\sigma^2}\right)-1/(\beta\sigma^2)} \\
&= \left(\frac{1}{\sigma^2}\right)^{(n-1)/2+\alpha} e^{-\left(\frac{(n-1)s^2}{2\sigma^2}\right)-1/(\beta\sigma^2)} \\
&= \left(\frac{1}{\sigma^2}\right)^{(n-1)/2+\alpha} e^{-\left(\frac{(n-1)s^2}{2}+\frac{1}{\beta}\right)^{-1}\frac{1}{\sigma^2}},
\end{aligned}$$

which is the form of an $\mathrm{IG}\left((n-1)/2+\alpha, \left(\frac{(n-1)s^2}{2}+\frac{1}{\beta}\right)^{-1}\right)$ distribution.