Homework # 9

Due 4/9

1. Reading: sections 6.4 and 6.6.

2. This problems covers the Spectral Decomposition Theorem. Let M be a $n \times n$ symmetric matrix.

   (a) Let $q^{(1)}, q^{(2)}, \ldots, q^{(n)}$ be an orthonormal basis of eigenvectors for M and $\lambda_1, \lambda_2, \ldots, \lambda_n$ the corresponding eigenvalues. Let Q be the matrix with columns given by $q^{(1)}, q^{(2)}, \ldots, q^{(n)}$. (A matrix with orthonormal columns is called an orthonormal matrix.) Let D be the diagonal matrix with $D_{ii} = \lambda_i$. Let $e^{(i)}$ be the n-dimensional vector of all zeros, except for a 1 in the ith coordinate. Show $Q^T q^{(i)} = e^{(i)}$, then $DQ^T q^{(i)} = \lambda_i e^{(i)}$, and finally $QDQ^T q^{(i)} = \lambda_i q^{(i)}$. Consequently, show that $Mq^{(i)} = QDQ^T q^{(i)}$ and explain why you can then conclude that $M = QDQ^T$.

   (b) Show that $Q^{-1} = Q^T$.

   (c) Show that $\|Qx\| = \|x\|$ for all $x \in \mathbb{R}^n$ if Q is an $n \times n$ orthonormal matrix. (Hint: Show that $\|Qx\|^2 = x^T Q^T Q x$.)

   (d) In this problem you will show that any 2-d orthonormal matrix involves a rotation and/or a reflection. Let Q be a 2×2 orthonormal matrix with columns $q^{(1)}, q^{(2)}$.

      i. Why can we assume the following for some angle θ?

      $$q^{(1)} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{1}$$

      ii. Show that given the form of $q^{(1)}$ directly above, then $q^{(2)}$ is given by either

      $$q^{(2)} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{2}$$

      or

      $$q^{(2)} = \begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \end{pmatrix} \tag{3}$$

      iii. Compute Qu and Qv where $u = (1, 0)$ and $v = (0, 1)$ and we think of u, v as column vectors. Explain why we can think of Q as a rotation and/or reflection.

3. Let $X \sim \mathcal{N}(0, \Sigma)$ where

$$\Sigma = \begin{pmatrix} 2 & 4 \\ 4 & 10 \end{pmatrix} \tag{4}$$

   (a) Use R to compute the spectral decomposition $\Sigma = QDQ^T$. (You can use Python if you prefer.) Here is the R code. The R function **eigen** returns *values* which are the eigenvalues and *vectors* which are the eigenvectors as column vectors.

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M = matrix(c(2,4,4,10), ncol=2)
eigen(M)
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   (b) Let $Y = Q^T X$. Show that Y is composed of two independent normals and specify their means and variances. (We did this in the lecture. I just want you to go through the argument yourself.

   (c) Draw the level curves of the pdf of Y for the values 1 and 10. Now use the relation $X = QY$ (explain why this relation holds) to draw the level curves of X.

4. Exercie 6.60