Homework 6 Solutions :: MATH 504

Your homework submission must be a single pdf called “LASTNAME-hw5.pdf” with your solutions to all theory problem to receive full credit. All answers must be typed in Latex.

1. (Coding) Apply a fixed point method to find a root of $\cos x = \sin x$ on $[0, \frac{\pi}{2}]$, by converting the equation into a fixed point equation

$$x = g(x) = x + \cos x - \sin x$$

given $x_0 = 0$. Note that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4}$, so $x^* := \frac{\pi}{4} \approx 0.7853982$. Fill out the following table

k	x_k	$g(x_k)$	$e_k = \lvert x_k - x^*\rvert$	e_k/e_{k-1}
1				
10				
20				
30				

Solution. Provided below in both R and Python:

```r
options(digits=10)
x<-c(0)
for(i in 1:25){
  x[i+1]<-x[i]+cos(x[i])-sin(x[i])
}
#Drop initial guess
k<-c(2,6,11,16,21,26)
g<-function(x){
  return(x+cos(x)-sin(x))
}
e<-c()
xk<-c()
gk<-c()
er<-c()
for(i in k){
  xk<-cbind(xk,x[i])
  gk<-cbind(gk,g(x[i]))
  e<-cbind(e, abs(x[i]-pi/4))
  er<-cbind(er,abs(x[i]-pi/4)/abs(x[i-1]-pi/4) )
}
data<-as.data.frame(cbind(t(xk),t(gk),t(e),t(er)))
rownames(data)<-k-1
colnames(data)<-c("x_k","g(x_k)","e_k","e_k/e_k-1")
data
```

```
##              x_k       g(x_k)              e_k    e_k/e_k-1
## 1  1.0000000000 0.6988313211 2.146018366e-01 0.2732395447
## 5  0.7915188509 0.7828629457 6.120687503e-03 0.4141620847
## 10 0.7853235339 0.7854290760 7.462951275e-05 0.4142135547
## 15 0.7853990734 0.7853977865 9.099806927e-07 0.4142135624
## 20 0.7853981523 0.7853981680 1.109567560e-08 0.4142135615
## 25 0.7853981635 0.7853981633 1.352929990e-10 0.4142135910
```

k	x_k	$g(x_k)$	$e_k = \lvert x_k - x^*\rvert$	e_k/e_{k-1}
1	1.0000000000	0.6988313211	0.214601836602551721	0.2732395447
5	0.7915188509	0.7828629457	0.006120687502820354	0.4141620847
10	0.7853235339	0.7854290760	0.000074629512752389	0.4142135547
15	0.7853990734	0.7853977865	0.000000909980692687	0.4142135624
20	0.7853981523	0.7853981680	0.000000011095675601	0.4142135615
25	0.7853981635	0.7853981633	0.000000000135292999	0.4142135910

k	x_k	$g(x_k)$	$e_k = \|x_k - x^*\|$	e_k/e_{k-1}
1	1	0.69883132	0.2146018	0.2732
5	0.79151885	0.78286295	0.00612065	0.4142
10	0.78532353	0.78542908	7.4666115e-05	0.4145
15	0.78539907	0.78539779	8.73378141e-07	0.3910
20	0.78539815	0.78539817	4.76982274e-08	4.859
25	0.78539816	0.78539816	3.64672588e-08	0.9875

```
g=@(x) x + cos(x) - sin(x);

%# Start out iteration loop
x0 = 0;
x1 = g(x0);

iterations = 1;% # iteration counter


while (iterations<24)
    iterations = iterations + 1;
    x0 = x1;
    x1 = g(x0);
end
iterations
x1

format long
g_xk = x1 + cos(x1) - sin(x1)

e_k = abs(x1- 0.7853982)
```

2. Let $f(x) = x^6 - x - 1$.

   a. Use 4 iterations of the Newton's method with $x_0 = 2$ to get an approximate root for this equation.

Solution: First, note that $f(x) \in C^1$, since $f'(x) = 6x^5 - 1$ is continuous in $\mathbb{R}$ (thus satisfying the assumption needed to use Newton's method). Then, according to Newton's method, we would select x_{k+1} according to the following formula:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^6 - x_k - 1}{6x_k^5 - 1}.$$

Using $x_0 = 2$ as a starting point, we would then have that:

$$x_1 = 2 - \frac{64 - 2 - 1}{6 * 32 - 1} = 1.680628.$$

Similarly,

$$x_2 = 1.680628 - \frac{22.533569 - 1.680628 - 1}{79.44694869} \approx 1.43073872.$$

We would repeat these procedure several times until we obtain x_7. Resorting to code, we have the following:

```
f<-function(x){
  return(x^6-x-1)
}
fp<-function(x){
  return(6*x^5-1)
}
newton<-function(x_0, f1, f2){
  return(x_0-f1(x_0)/f2(x_0))
}
X<-c(2)
for(i in 1:7){
  X[i+1]<-newton(X[i], f, fp)
}
X
```

```
## [1] 2.000000 1.680628 1.430739 1.254971 1.161538 1.136353 1.134731 1.134724
```

b. Use 4 iterations of the Secant method with $x_0 = 2$ and $x_1 = 1$ to get an approximate root for this equation.

Solution: According to the secant method, the points will be chosen as follows:

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}.$$

For example,

$$x_2 = 1 - (-1)\frac{1-2}{(-1)-61} \approx 1.01613.$$

Using code, we have the following:

```
Y<-c(2,1)
secant<-function(x0, x1, f1, f2){
  return(x1-f(x1)*(x1-x0)/(f(x1)-f(x0)))
}
for(i in 1:7){
  x0<-X[i]
  x1<-X[i+1]
  Y[i+2]<-secant(x0,x1, f,fp)
}
Y
```

```
## [1] 2.000000 1.000000 1.526535 1.318671 1.190388 1.141280 1.134826 1.134724
## [9] 1.134724
```

3. Consider the equation $e^{100x}(x-2) = 0$. Apply Newton's method several times with $x^0 = 1$. What do you observe?

Solution. It is clear that the root of the equation is $x = 2$. We apply Newton's method and we observe that the solution is decreasing and going farther away from the solution. The reason is the initial value may not be close enough to the solution for the Newton's method to converge.