Final Project :: MATH 504 :: Due Dec 14th, 11:59 pm

Your submission must be a single pdf called "LASTNAME-final.pdf". All answers must be typed in Latex. Insert your m.files at the end and make sure to include your images and graphs. 20% of your final course grade will be based on this project.

1. (2pts) Given a matrix $u \in \mathbb{R}^{m\times n}$, white two codes with signature

   - function v=dxb(u): returns a matrix v having backward difference of u row-wise.
   - function v= dyb(u): returns a matrix v having backward difference of u column-wise

   Note that forward differences (dxf(u) and dyf(u)) are already given in the lecture. Create those functions as well in your workspace, you will need them.

2. (3pts) Upload an image either from your phone or some available images from the matlab. Make it a grayscale image, and if it has more than two channels, just pick one of the channels to work with. Read the image (imread) and add some Gaussian noise (imsnoise) to the image. Use imshow to visualize both original grayscale image (u0) and your noisy image (f).

3. The isotropic total variation image denoising problem is given by

   $$\min_u \ f(u) = TV_{iso}(u) + \frac{\mu}{2}\|u - f\|^2$$

   where $\mu > 0$ and $f \in \mathbb{R}^{m\times n}$ is the noisy image. Note that the gradient of objective function as discussed in the class is given by

   $$Df(u) = -\nabla^- \cdot \Big[\frac{\nabla^+ u}{\sqrt{\|\nabla^+ u\|^2 + \epsilon^2}}\Big] + \mu(u - f)$$

   where $\nabla^+ u = (\nabla_x^+ u, \nabla_y^+ u)$ where $\nabla_x^+ u$ and $\nabla_y^+ u$ are forward differences along the horizontal and vertical directions, respectively. More specifically,

   $$(\nabla u)_{i,j} = (\nabla_x^+ u, \nabla_y^+ u)_{i,j} = (u_{i+1,j} - u_{i,j}\ ,\ u_{i,j+1} - u_{i,j}) \in \mathbb{R}^2$$

   where $u_{ij} \in \mathbb{R}$ refers to the (i, j) component of the image. Moreover,

   $$\|\nabla u_{i,j}^+\|^2 = (\nabla_x^+ u)^2 + (\nabla_y^+ u)^2 = (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2$$

   In the lecture, the functions dxf and dyf correspond with these two operators. Given two matrices P_1 and P_2, then the divergence operator for $P = (P_1, P_2)$ is given by $\nabla \cdot P = \nabla_x P_1 + \nabla_y P_2$ but for the stability of our method we use backward differences (dxb and dyb, obtained in part 1.) for the gradient operators.

a. (5pts) Write a program that exploit the gradient descent method to denoise the noisy image. Use a fixed stepsize and initialize u^0 with zero image (zeros (m,n)).

   The iterate of the gradient descent method with stepsize α applied to this image denoising problem is given by for $k = 0, 1, 2, \ldots$

$$\begin{aligned} u^{(k+1)} &= u^{(k)} - \alpha Df(u^k) \\ &= u^k - \alpha\Big(- \nabla_x^- \Big[\frac{\nabla^+ u^k}{\sqrt{|\nabla^+ u^k|^2 + \epsilon^2}} \Big] - \nabla_y^- \Big[\frac{\nabla_y^+ u^k}{\sqrt{|\nabla^+ u^k|^2 + \epsilon^2}} \Big] + \mu(u^k - f) \Big) \end{aligned}$$

b. (4pts) Note that μ has an important role in the model. If it is too large, the model would forces the image stay close to the given noisy image. If it is too small, the restored image will be blurry. Run the gradient descent method with different values for μ and discuss your observation. What is the best range for μ to get a good denoised image.

c. (2pts) Displace the denoised image with your best parameter μ.

d. (4pts) Graph the image error

$$\frac{1}{mn} \sum_i \sum_j |u_{ij}^k - u0_{ij}|^2$$

   versus iteration number k and show that as the iteration increases the error decreases. $m \times n$ is the size of the image.