LECTURE 4 – 9/19/2022

Introduction to the Simplex Method

HW posted, due in 2 weeks.

Key Ideas from Last time:

- Polytope, convex combination of vectors, convex hull, extreme point x

$\{x \in \mathbb{R}^n \mid Ax \geq b\}$ → $\sum \lambda_i = 1$, $\sum \lambda_i x_i$

convex hull → all convex combinations

extreme point → cannot find 2 vectors v, w s.t. $tv + (1-t)w = x$

- In the typical cases, when we search for the extreme points, we have very good performance

Convex functions and convex sets/Simplex

~~A function is convex if~~

① Vertex $\iff$ extreme point ⟷ nonempty bounded Polyhedral is the convex hull of its extreme points

② local min on convex function $\iff$ global min.

Recall: Two linear transformations are equivalent if they have the same solutions.
Linear transformations preserved under linear ops, scalar mult, adding rows.

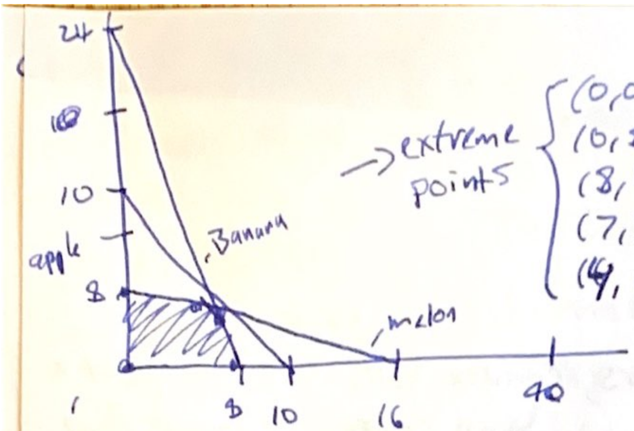
Recall: Equality form vs. inequality form.

Ex: Assembling fruit baskets. 10 Bananas, 24 apples, 16 melons. Two configurations. Mix 1 sells for \$3, contains 1 banana, 3 apple, 1 melon. Mix 2 sells for \$2, contains 1 banana, 2 apple, 2 melon. How should you configure to maximize profit?

$$\max\ 3x_1 + 2x_2$$
$$\text{s.t.}\quad \begin{aligned} x_1 + x_2 &\leq 10 \\ 3x_1 + 2x_2 &\leq 24 \\ 1x_1 + 2x_2 &\leq 16 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$\iff$

$$\max\ 3x_1 + 2x_2 = z$$
$$\text{s.t.}\quad \begin{aligned} x_1 + x_2 + S_1 &= 10 \\ 3x_1 + 2x_2 + S_2 &= 24 \\ x_1 + 2x_2 + S_3 &= 16 \\ x_1, x_2 &\geq 0 \\ S_1, S_2, S_3 &\geq 0. \end{aligned}$$



→ extreme points
- (0,0) – 0
- (0,8) – 16
- (8,0) – 24
- (7,3) – 27 ← optimum.
- (4,6) – 24

To do Simplex Method:

Step 1: Transform to equality form
Rewrite in grid/tabular form

Basis	x_1	x_2	S_1	S_2	S_3	RHS
z	-3	-2	0	0	0	0
S_1	1	1	1	0	0	10
S_2	3	1	0	1		24
S_3	1	2	0	0	1	16

See in slides the operations.

	x_1	x_2	S_1	S_2	S_3	RHS
z	0	0	3/2	1/2	0	27
x_2	0	1	3/2	-1/2	0	3
x_1	1	0	-1/2	1/2	0	7
S_3	0	0	-5/2	1/2	1	3

Sensitivity Analysis:

How much would an additional fruit be worth?

↳ we already have 3 left-over melons, so no value in getting more.

↳ what about banana? Changes the constraint, expands the feasible region. Then quantify the new objective.

Dual Price: ~~increase~~ improvement in optimal solution per unit increase in RHS of constraint. Positive means increase for max problem, decrease for min problem.

Shadow Price: ~~Stead~~ actual change in value of optimal solution per unit increase in the RHS of a constraint.

Cost of Constraints:

Each constraint has a cost/price to how good the objective function can be.

$$-(x_1+x_2 \le 10)$$
$$+(3x_1+x_2 \le 24)$$
$$+(x_1+2x_2 \le 16)$$
$$\overline{3x_1+2x_2 \le 30.}$$

The Dual Problem:

Primal	Dual
Max $c^T x$	Min $p^T b$
s.t. $Ax \le b$	s.t. $p^T A \ge c$
$x \ge 0$	$p \ge 0$

Forming Dual (on Midterm).

Primal (Max)	Dual (Min)
Constraint	Variable.
$a_i^T x_i \le b$	$p_i \ge 0$
$a_i^T x_i \ge b_i$	$p_i \le 0$
$a_i^T x_i = b_i$	p_i free
Variable	constraint
x_i	

look in Slides to complete.

What if I change my prices? changes to the objective function.

This impacts the tangency point.

The 100% Rule

For all objective function coef changes, sum the percentages of the allowable increases/decreases. Must be $\le 100\%$.

Note:
- Obj func change.
  - Slope of objective func line changes
  - extr pts same
  - optimal pt same
  - opt val changed.
- RHS constraints
  - Feasible rgn changed
  - extreme pts change.
  - opt pt adjust.
  - opt val change.

Two Phases Simplex Method

Phase I: 1. Put the problem in equality form. ~~Stla~~ Multiply some of the constraints by -1 so that $b \geq 0$.

2. Introduce artificial variables $y_1, y_2, \ldots, y_m$ for each constraint and apply the simplex method maximizing for $-\sum y_i$.

3. a) If optimal value is not 0, the problem is infeasible.
   b) If 0 and no artificial vars then remove corresponding cols, remainder is feasible basis.
   c) If ith basis var is artificial then
      (i) if all non-artificial entries 0 then drop b/c redundant.
      (ii) if jth entry is non-zero, pivot and have x_j enter basis.

I.2: Max $x_1 + x_2 + x_3$

s.t. $x_1 + x_2 + x_3 = 3$

$-x_1 + 2x_2 + 6x_3 = 2$

$4x_2 + 9x_3 = 5$

$3x_3 + x_4 = 1$

$x_1, \ldots, x_4 \geq 0$

$\rightarrow$ Max $-y_1 + y_2 + y_3 + y_4$

~~add~~ $y_1, \ldots, y_4$.

st.