LECTURE 10 - 3/28/22

Eigenvalues and Eigenvectors

Def: If $A\vec{v} = \lambda \vec{v}$ for some $\vec{v} \neq 0$, ~~$\lambda \in \mathbb{R}$~~ λ a number, then call $\vec{v}$ an eigenvector of A and λ and eigenvalue of A.

THM: If $\vec{v}$ is an eigenvector of A, then so is $c\vec{v}$, c scalar.

Pf: Since $\vec{v}$ is an eigenvector of A, then $A\vec{v} = \lambda\vec{v}$ for some λ. Then $A(c\vec{v}) = \lambda(c\vec{v}) \Leftrightarrow c\vec{v}$ is an eigenvector of A. ~~with eigenvalue c etc.~~

THM: Eigenvectors for different eigenvalues of A are independent.

Pf: Given eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, and $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_k$. We use induction on k.

Consider $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$.

when $k=1$, $\vec{v}_1, \lambda_1$ only eigenvector/value. $\vec{v}_1$ is independent if and only if $\vec{v}$ is not $\vec{0}$. This is true by definition.

when $k=2$. $\vec{v}_1, \vec{v}_2, \lambda_1, \lambda_2$. Consider $c_1\vec{v}_1 + c_2\vec{v}_2 = 0 \Rightarrow$

① $c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \vec{0} \Rightarrow$ ② $c_1\lambda_1 v_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$ system of Eq $\Rightarrow \lambda_2$①$-$② $\Rightarrow$

$\Rightarrow c_1\underbrace{(\lambda_2 - \lambda_1)}_{\substack{\lambda_1 \neq \lambda_2 \\ \neq 0}}\underbrace{\vec{v}_1}_{\substack{\neq 0 \\ \text{by def}}} = \vec{0} \Rightarrow$ therefore $c_1 = 0. \Rightarrow c_2 = 0 \Rightarrow$ So $\vec{v}_1, \vec{v}_2$ independent

The induction step is left for us.

THM: If $A_{n\times n}$ matrix has n independent eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then $A = V\Lambda V^{-1}$

③ $f(A) \Rightarrow$ when $A = V\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}V^{-1}$, $A^k = V\begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}V^{-1}$.

so $p(A) = A^3 + A + I = V\begin{pmatrix} \lambda_1^3 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^3 \end{pmatrix}V^{-1} + V\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} + I$.

$= V\begin{pmatrix} \lambda_1^3 + \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^3 + \lambda_n \end{pmatrix}V^{-1} \Rightarrow$ for any polynomial function $p(x)$

$p(A) = V\begin{pmatrix} p(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & p(\lambda_n) \end{pmatrix}V^{-1}$, ~~[illegible]~~, that is, we only apply to the eigenvalues.

Generalizing further, if $f(x)$ has a Taylor series

e.g. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ (like a poly but only many powers).

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

Then we can use this for dealing with a matrix like before

$$e^A = V\begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n}\end{pmatrix}V^{-1}, \quad \log(1-A) = V\begin{pmatrix} \log(1-\lambda_1) & & 0 \\ & & \\ 0 & & \log(1-\lambda_n)\end{pmatrix}V^{-1}.$$

Def: If $A = V\Lambda V^{-1}$, then $f(A) = V\begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n)\end{pmatrix}V^{-1}$.

In other words, we define all functions like the polynomial case shows (it's a natural logical extension).

Consider now A with complex eigenvalues.

e.g. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $\lambda = \pm i$. Solving $A\vec{v} = \pm i\vec{v}$ gives $\vec{v}$ complex. Therefore, we have to extend from $\mathbb{R}^n$ to $\mathbb{C}^n$.

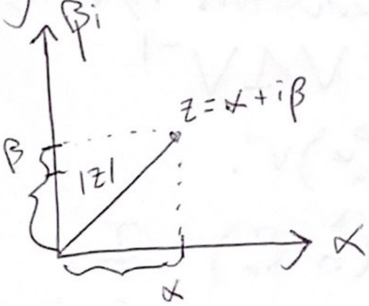
Note $\mathbb{C}^n$ is defined as vectors $\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 i \\ \vdots \\ \alpha_n + \beta_n i \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + i\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$

$= \vec{\alpha} + i\vec{\beta}$.

Quick Review of Complex Number z, Complex Vector $\vec{z}$.

If $z = \alpha + i\beta$, then $\bar{z} = \alpha - i\beta$ is called the conjugate of z.

e.g. $z = 2+3i$, $\bar{z} = 2-3i$, $|z| = \sqrt{\alpha^2 + \beta^2}$. Then $z\bar{z} = |z|^2$



Now consider complex vector $\vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n$

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + i\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \vec{\alpha} + i\vec{\beta}$$

$$\bar{\vec{z}} = \begin{pmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{pmatrix} = \vec{\alpha} - i\vec{\beta}$$

Inner product for complex vectors $\vec{w}, \vec{z}$

Def $\langle \vec{w}, \vec{z} \rangle = \vec{w}^T \vec{z}$. Let's check. No

Recall Inner product requirements.

~~Check~~ ① $\langle \vec{x}, \vec{x} \rangle \geq 0$ and $=0$ iff $x=0$.

If $\vec{z}=i$, then this fails.

Instead we consider $\langle \vec{w}, \vec{z} \rangle = \overline{\vec{w}^T} \vec{z}$ (conjugate)

Check ① $\langle \vec{z}, \vec{z} \rangle = (\bar{z}_1 \cdots \bar{z}_n) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$

Def: $\overline{\vec{z}^T} = \vec{z}^H$ (Notation) complex/transpose/Hermitian conjugate

so the inner product is $\langle \vec{w}, \vec{z} \rangle = \vec{w}^H \vec{z}$.

Check ② $\langle \vec{w}, \vec{z} \rangle \overset{?}{=} \langle \vec{z}, \vec{w} \rangle$ symmetry. This is not true.

$\langle \vec{w}, \vec{z} \rangle = \vec{w}^H \vec{z} = \overline{\vec{w}^T} \vec{z}$

$\langle \vec{z}, \vec{w} \rangle = z^H \vec{w} = \overline{z^T} w$

so ② is modified to $\overline{\langle \vec{w}, \vec{z} \rangle} = \langle \vec{z}, \vec{w} \rangle$

Note $z=\bar{z}$, then z real

$z=-\bar{z}$ then z imag

Example: $A = \begin{pmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{pmatrix}$

$A^H = \overline{A^T} = \begin{pmatrix} 1-2i & 5-6i \\ 3-4i & 7-8i \end{pmatrix}$

$(A_1 A_2)^H = A_2^H A_1^H$

Recall $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

$(A \pm B)^H = A^H \pm B^H$

$(A^{-1})^H = (A^H)^{-1}$

$(cA)^H = \bar{c} A^H$

Hermitian Matrix.

Recall def symmetric matrix, $A = A^T$.

Now define A is Hermitition (conjugate symmetric) if $A = A^H$.

Ex: $A = \begin{pmatrix} 1 & 3+4i \\ 3-4i & 2 \end{pmatrix}$. Then $A^H = \begin{pmatrix} 1 & 3-4i \\ 3+4i & 2 \end{pmatrix}$. We see $A = A^H$. But $A \neq A^T$.

If A is Hermitian, then A's diagonal entries are real.

THM: If A is Hermitian, then A's eigenvalues are real.

(want to show $\lambda = \bar{\lambda}$.

Since λ is eigenvalue of A then $A\vec{v} = \lambda\vec{v}$ for $\vec{v} \neq 0$.

$\Rightarrow \bar{A}\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}} \Rightarrow \vec{v}^H A^H = \bar{\lambda}\vec{v}^H$

$\Rightarrow (\vec{v}^H A^H)(A\vec{v}) = \bar{\lambda}\lambda \vec{v}^H\vec{v} = \bar{\lambda}\cdot\lambda\langle\vec{v},\vec{v}\rangle \neq 0$.

$\Rightarrow v^H A A v = v^H A \lambda v = \lambda v^H A v = \lambda^2 v^H v$

$\Rightarrow \bar{\lambda}\lambda\langle v,v\rangle = \lambda^2\langle v,v\rangle \Rightarrow \bar{\lambda}\lambda = \lambda^2 \Rightarrow \bar{\lambda} = \lambda, 0$

$\Rightarrow \lambda$ is real.

Cor: If A is symmetric and real $(A^T = A^H = A)$ special case of Hermitian.

then A's eigenvalues are real.

THM: ~~If A is Hermitian then A's eigen vectors are orthogonal if they belong to~~

If A is Hermitian, then the eigenvectors of different eigenvalues are orthogonal.

Pf: Known: $A = A^H$, $\lambda_1 \neq \lambda_2$ of A. $\Rightarrow Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$
$v_1 \quad v_2$

Then $v_1^H A_H^H = \bar{\lambda}_1 v_1^H$

$\Rightarrow v_1^H A^H v_2 = \bar{\lambda}_1 v_1^H v_2$ $\Rightarrow = \lambda_1 v_1^H v_2$.

$\Rightarrow v_1 \overset{''}{A} v_2 = \lambda_2 v_1^H v_2$

so we have ~~$\bar{\lambda}_1 = \lambda_1$~~

$\underbrace{(\lambda_2 - \lambda_1)}_{\neq 0}\langle v_1, v_2\rangle = 0$

$\Rightarrow \langle v_1, v_2\rangle = 0$ ◁

so v_1, v_2 orthogonal.

ODE $\vec{x}' = A\vec{x}$, A const $n \times n$ matrix.

THM: If $A_{n\times n}$ has n independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and
(i) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$ is the general solution.

RMK: eigenvectors of different eigenvalues are independent. So, we have $<n$ indep evectors only if some λ repeats.

Ex: $\vec{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\vec{x}$. To find A's λs:

solve: $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-2)^2 = 0 \Rightarrow \lambda = 2, 2.$

e-$\vec{v}$'s for $\lambda = 2$.

solve $(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} 2-2 & 1 \\ 0 & 2-2 \end{pmatrix}\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}$

$\Rightarrow v_2 = 0$, v_1 free. so $v_1 = c\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\Rightarrow x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + ?$ $\Rightarrow$ try te^{2t}.

Try $x_2 = te^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\vec{x}' = A\vec{x} = te^{2t}A\vec{v} = 2te^{2t}\vec{v}$

$= e^{2t} + 2te^{2t}\vec{v}$ $\longrightarrow$ not equal, extra term.

So we try $x_2 = te^{2t}\vec{v} + e^{2t}\vec{w} \longrightarrow$ TBD.

$\vec{x}' = A\vec{x} = te^{2t}A\vec{v} + e^{2t}A\vec{w}$

$= e^{2t}\vec{v} + 2te^{2t}\vec{v} + 2e^{2t}\vec{w}$

$\Rightarrow \vec{v} + 2\vec{w} = A\vec{w} \Leftrightarrow \vec{v} + \lambda\vec{w} = A\vec{w} \Rightarrow \vec{v} = (A - \lambda I)\vec{w}$

$\begin{pmatrix} 2-2 & 1 \\ 0 & 2-2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} w_2 = 1 \\ w_1 = \text{free} = 0 \end{matrix} \Rightarrow \vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

so $\vec{x}_2 = te^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now get general sol'n.

$\vec{x} = c_2 e^{2t}\left[t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

This motivates a second part of the THM above.

THM (ii) when a λ repeats, say λ_1, so that (i) fails, then

Then $\vec{x}_2 = e^{\lambda t}(t\vec{v}+\vec{w})$ where $\vec{w}$ is a solution of $\vec{v}_1 = (A-\lambda I)\vec{w}$

Then we can get generate the solution.

(iii) If $\lambda = \alpha \pm i\beta$ and we want a real general solution.

Ex: Consider $\vec{z}_{\pm} = e^{(\alpha \pm i\beta)t}\vec{v}_{\pm}$ are 2 sols - complex valued

$= (\vec{Re}) \pm i(\vec{Im})$. Then, $\vec{Re}$ and $\vec{Im}$ are two real solutions.

$Re = \frac{1}{2}(z_+ + z_-) \Rightarrow$ Linear combination of 2 solutions of $\vec{x}' = A\vec{x}$.

$\Rightarrow$ it is a solution

$Im = \frac{1}{2i}(z_+ - z_-) \Rightarrow$ also a solution.

So for (iii): we just write $z_{\pm} = e^{(\alpha \pm i\beta)t}\vec{v}_{\pm}$

$= Re + i\,Im$ then Re & Im are two real solutions.

Result: $e^{(\alpha+i\beta)t}(\vec{r}+i\vec{s}) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{r}+i\vec{s})$

$$= e^{\alpha t}\left[(\vec{r}\cos\beta t - \vec{s}\sin\beta t) + i(\vec{r}\sin\beta t + \vec{s}\cos\beta t)\right]$$

Giving us two solutions.

$\vec{x}_1 = e^{\alpha t}[\vec{r}\cos\beta t - \vec{s}\sin\beta t]$

$\vec{x}_2 = e^{\alpha t}[\vec{r}\sin\beta t - \vec{s}\cos\beta t]$.

Ex: Solve $\vec{x}' = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\vec{x}$. Find λ.

$\begin{vmatrix} 1-\lambda & 3 \\ -3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 3^2 = 0 \Rightarrow \lambda = 1 \pm 3i$

Then we must find the eigenvectors.

$(A-\lambda I)\vec{v}=\vec{0}$:

$$\lambda = 1+3i \Rightarrow \begin{pmatrix} 1-3i-1 & 3 \\ -3 & 1-1-3i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -3i v_1 + 3v_2 = 0 \\ -3v_1 - 3i v_2 = 0 \end{cases} \Rightarrow \begin{matrix} v_2 = i v_1 \\ v_1 \text{ free} \end{matrix}$$

$\Rightarrow \vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$, a complex vector.

$$\lambda = \underset{\alpha}{\underset{\uparrow}{1}} \pm \underset{\beta}{\underset{\uparrow}{3}}i, \quad \vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \underset{r}{\underset{\uparrow}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}} + i\underset{s}{\underset{\uparrow}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}}$$

Now we have the information we need to fill out the ~~equation~~ formulas that we have for x_1, x_2. $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$.

$$\vec{x} = c_1 e^t \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 3t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin 3t\right] + c_2 e^t\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 3t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos 3t\right].$$