Homework 9

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A11.1 Given the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 1 \\ -2 & 2 & -2 \end{pmatrix}$

(a) Find A's eigenvalues and eigenvectors.

(b) Factor A as VDV^{-1}.

(c) use b to compute A^7 and $A^7 + 3A^4 + 5A^2 + 7I$.

(a) To find the eigenvalues of A, we solve $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & -1-\lambda & 1 \\ -2 & 2 & -2-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -\lambda & 1 \\ -\lambda & -2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-\lambda)\begin{vmatrix} 1 & 1 \\ 1 & -2-\lambda \end{vmatrix} = (3-\lambda)(-\lambda)\begin{vmatrix} 1 & 1 \\ 0 & -3-\lambda \end{vmatrix} = (3-\lambda)(-\lambda)(-3-\lambda)$$

$$= (3-\lambda)(\lambda)(3+\lambda) = 0 \Rightarrow \lambda = 3, -3, 0.$$

once we have the eigenvalues we can find the eigenvectors by solving $(A - \lambda I)\vec{v} = 0$ for each λ.

$$\lambda_1 = 3: \begin{pmatrix} 3-3 & 0 & 0 \\ 1 & -1-3 & 1 \\ -2 & 2 & -2-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & -4 & 1 \\ -2 & 2 & -5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} -3 \\ -1/2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -3: \begin{pmatrix} 3+3 & 0 & 0 \\ 1 & -1+3 & 1 \\ -2 & 2 & -2+3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 0: \begin{pmatrix} 3+0 & 0 & 0 \\ 0 & -1+0 & 0 \\ 0 & 0 & -2+0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(b). To factor, we have all the information

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 1 \\ -2 & 2 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}.$$

(c) Once we have a diagonalization, we have a simple way to apply functions such as polynomials to A.

In general $f(A) = V \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} V^{-1}$.

$$A^7 \Rightarrow \begin{pmatrix} -3 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3^7 & 0 & 0 \\ 0 & -3^7 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

$A^7 + 3A^4 + 5A^2 + 7I$

$$\begin{pmatrix} -3 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \cdot 3^7 + 5 \cdot 3^2 + 7 & 0 & 0 \\ 0 & 3(-3)^7 + 5(3)^2 + 7 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}.$$

A11.2 (a) Show that an invertible matrix A's eigenvalues are not 0.

(b) Let λ be an eigenvalue of an invertible matrix A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

(a) If A is an invertible matrix and λ is an eigenvalue of A, then $A\vec{v} = \lambda\vec{v}$ for some $\vec{v} \neq 0$. If $\lambda = 0$ then $A\vec{v} = 0$. since A^{-1} exists, then $\vec{v} = \vec{0}$. This is a contradiction, so it must be that $\lambda \neq 0$.

(b) we have $A\vec{v} = \lambda\vec{v}$. $\iff$ $A^{-1}A\vec{v} = \lambda A^{-1}\vec{v}$
$\iff$ $\vec{v} = \lambda A^{-1}\vec{v} \iff \frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$. This shows us that $\frac{1}{\lambda}$ is the eigenvalue for A^{-1}! Note that $\vec{v} \neq 0$,

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B A11.3 A, B are square matrices. Show that if $\lambda \neq 0$ is an eigenvalue of AB, then the λ is also an eigenvalue of BA.

If λ is an eigenvalue of AB, then $AB\vec{v} = \lambda\vec{v}$ for $\vec{v} \neq 0$.

Consider $\vec{w} = B\vec{v}$.

Then $BA\vec{w} = BAB\vec{v} = B(AB\vec{v}) = B(\lambda\vec{v}) = \lambda B\vec{v}$
$= \lambda\vec{w}$.

we therefore see that λ is also an eigenvalue for BA.

B11.1 Find the general solution of $x' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} x$

we check if $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ is diagonalizable.

Find eigenvalues of A by solving $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = 0 \Rightarrow (\lambda-4)(\lambda-1) \Rightarrow \lambda = 4, \lambda = 1.$$

Now find eigenvectors by solving $(A - \lambda I)\vec{v} = 0$ for each λ.

$$\lambda = 4: \begin{pmatrix} 3-4 & 1 \\ 2 & 2-4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow v_1 - v_2 = 0 \Rightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 1: \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \vec{v}_2 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}.$$

Therefore we diagonalize $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}^{-1}$.

Then the solution is given by

$$x = c_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}.$$