Homework 14 Nathan Bick

B16.1.2. Consider the competing species model

$$\begin{cases} x' = x(2-2x) - 0.5xy \\ y' = y(1-\frac{1}{2}y) - Pxy \end{cases}$$

with parameter P. We are interested in the effect of the parameter P, describing how fast x species consumes y, on the solution behavior

(a) Compute the Jacobian matrix J for this system.

(b) Find all the equilibrium solutions. Let (x_0, y_0) denote the equilibrium where the two species coexist.

(c) For the equilibrium (x_0, y_0) found in part (b), plot the curve $(tr(x_0,y_0), det(x_0,y_0))$ in a trace-determinant plane for values of the parameter P from 0 to 2. Describe any bifurcations that occur.

(a) system is $\begin{cases} x' = 2x - 2x^2 - \frac{1}{2}xy = f_1 \\ y' = y - \frac{1}{2}y^2 - Pxy = f_2 \end{cases}$

$$J = \begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{pmatrix} = \begin{pmatrix} 2 - 4x - \frac{1}{2}y & -\frac{1}{2}x \\ -Py & 1 - y - Px \end{pmatrix}$$

(b) Equilibria are found via $x' = y' = 0$.

$$2x - 2x^2 - \frac{1}{2}xy = 0 \iff x\left[2 - 2x - \frac{1}{2}y\right] = 0$$

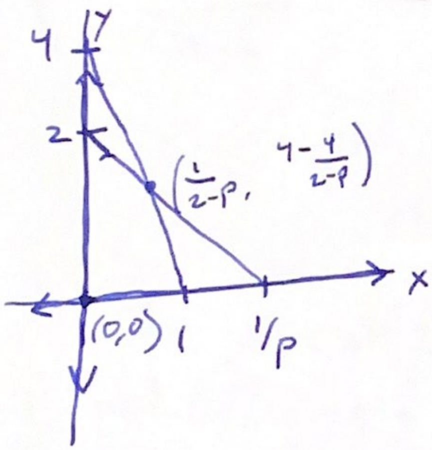
$\Rightarrow x = 0$, then $2 - \frac{1}{2}y = 0 \Leftrightarrow y = 2$.

or $\left[2 - 2x - \frac{1}{2}y\right] = 0 \Rightarrow y = 4 - 4x$

$$y - \frac{1}{2}y^2 - Pxy = 0 \iff y\left[1 - \frac{1}{2}y - Px\right] = 0$$

$\Rightarrow y = 0$, then $1 - Px = 0 \Rightarrow x = \frac{1}{P}$.

or $1 - \frac{1}{2}y - Px = 0 \Rightarrow y = 2 - 2Px$.



The equilibrium point where both species coexist, which is equivalent that $x \neq 0$ and $y \neq 0$ is at the point $\left(\frac{1}{2-P}, 4 - \frac{4}{2-P}\right)$

(c) we must calculate the value of J at (x_0, y_0).

$$\left.\begin{pmatrix} 2-4x-\frac{1}{2}y & -\frac{1}{2}x \\ -py & 1-y-px \end{pmatrix}\right|_{\left(\frac{1}{2-p},\, 4-\frac{4}{2-p}\right)} =$$

$$\begin{pmatrix} 2-4\left(\frac{1}{2-p}\right)-\frac{1}{2}\left(4-\frac{4}{2-p}\right) & -\frac{1}{2}\left(\frac{1}{2-p}\right) \\ -p\left(4-\frac{4}{2-p}\right) & 1-\left(4-\frac{4}{2-p}-p\left(\frac{1}{2-p}\right)\right) \end{pmatrix} =$$

$$\begin{pmatrix} \frac{2}{p-2} & \frac{1}{2(p-2)} \\ \cancel{-4p(1-p)} \; \frac{-4(p-1)p}{p-2} & \frac{2-4p}{p-2} \end{pmatrix} = J.$$

we want to consider $tr(J)$ and $det(J)$.

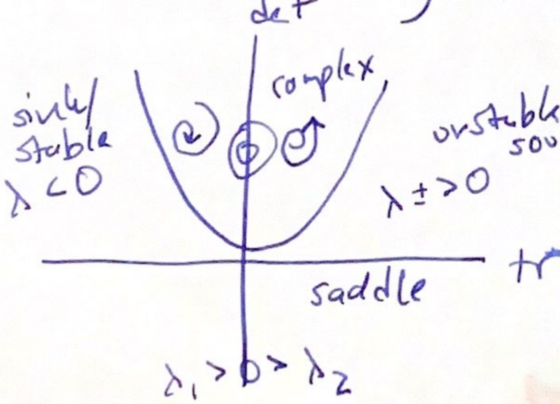
$$tr(J) = \frac{2}{p-2} + \frac{2-4p}{p-2} = \frac{4-4p}{p-2}.$$

$$det(J) = \frac{(2-4p)2}{(p-2)^2} + \frac{4(p-1)p}{2(p-2)^2}$$

$$= \frac{4[(2-4p)+(p-1)p]}{2(p-2)^2} = \frac{p^2-9p+4}{(p-2)^2}$$

The sign of λ in $det(J-\lambda I)$ depends on $tr(J)$, $det(J)$. using

$$\lambda_\pm = \frac{1}{2}\left[tr \pm \sqrt{tr^2 - 4det}\right] = 0$$



we see that

$$tr^2 = \frac{16p^2-32p+16}{(p-2)^2} = \frac{16(p^2-2p+1)}{(p-2)^2}$$

so $tr^2 > 4det$. so not complex.

If $p>1$, then $\lambda > 0$, $p<1$, $\lambda < 0$. so $p=1$ is coexisting and bifurcation point.