MATH 503: Mathematical Statistics
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Homework 1 Solutions

1. If C_1 and C_2 are independent events, show that C_1' and C_2 are also independent.

   Solution: C_1 and C_2 are independent events $\Rightarrow P(C_1 \mid C_2) = P(C_1)$. To show that C_1' and C_2 are also independent, we need to show that $P(C_1' \mid C_2) = P(C_1')$.

$$\begin{aligned} P(C_1' \mid C_2) &= 1 - P(C_1 \mid C_2) \\ &= 1 - P(C_1) \text{ because } C_1 \perp C_2 \\ &= P(C_1'). \end{aligned}$$

2. Find the constant c so that

$$p(x) = c\left(\frac{2}{3}\right)^x, \qquad x = 1, 2, 3, \ldots, \qquad \text{zero elsewhere}$$

   is a pmf.

   Solution: Determining the value for c relies on satisfying the constraint, $\sum p(x) = 1$.

$$\sum_{x=1}^{\infty} c\left(\frac{2}{3}\right)^x = \sum_{x=0}^{\infty} c\left(\frac{2}{3}\right)^x \cdot \frac{2}{3} = \frac{2}{3}c \cdot \frac{1}{1-(2/3)} = 2c = 1.$$

   therefore $c = \frac{1}{2}$.

3. Determine the value of c that makes

$$f(x) = c\sin(x), \qquad 0 < x < \frac{\pi}{2}$$

   a pdf.

   Solution: Determining the value for c relies on satisfying the constraint, $\int f(x)dx = 1$.

$$\int_0^{\pi/2} c\sin(x)dx = c(-\cos x \mid_0^{\pi/2}) = -c(\cos\frac{\pi}{2} - \cos(0)) = -c(0-1) = c = 1$$

   therefore $c = 1$.

4. Let X have a pmf $p(x) = \frac{1}{3}$, $x = 1, 2, 3$, zero elsewhere. Find the pmf of $Y = 2X + 1$.

   Solution:

$$P(Y = 2X+1) = P\left(X = \frac{Y-1}{2}\right) = \frac{1}{3}, \qquad y = 3, 5, 7.$$

5. Let X have a pdf $f(x) = \frac{x^2}{9}$, $0 < x < 3$, zero elsewhere. Find the pdf of $Y = X^3$.

Solution: $y = x^3$ so $x = y^{1/3}$, and $dx = \frac{1}{3}y^{-2/3}$ where $0 < y < 27$, so the pdf of Y is

$$g(y) = f(y^{1/3}) \cdot \left(\frac{1}{3y^{2/3}}\right) = \frac{(y^{1/3})^2}{9} \cdot \left(\frac{1}{3y^{2/3}}\right) = \frac{1}{27}, \qquad 0 < y < 27.$$

6. Let $f(x) = 2x, 0 < x < 1$, zero elsewhere, be the pdf of X.
   (a) Compute $E(1/X)$.
   (b) Find the cdf and the pdf of $Y = 1/X$.
   (c) Compute $E(Y)$ and compare the result with the answer obtained in Part (a).

   Solution:

   (a) $E\left(\frac{1}{X}\right) = \int_0^1 \frac{1}{x} \cdot 2x dx = 2$.
   (b) $y = \frac{1}{x} \Rightarrow x = \frac{1}{y} = y^{-1} \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2}$, so

   $$f(y) = f\left(\frac{1}{y}\right) \mid \frac{dx}{dy} \mid = \left(\frac{2}{y}\right)\left(\frac{1}{y^2}\right) = \frac{2}{y^3}, \qquad 1 < y < \infty,$$

   and

   $$F(y) = \int_1^y \frac{2}{t^3} dt = -t^{-2} \mid_1^y = -\frac{1}{y^2} + 1, \text{ i.e. } F(y) = \begin{cases} 0 & y \le 1 \\ 1 - \frac{1}{y^2} & y > 1 \end{cases}$$

   (c) $E(Y) = \int_1^\infty y \cdot \frac{2}{y^3} dy = \int_1^\infty 2y^{-2} dy = \frac{-2}{y} \mid_1^\infty = 0 - (-2) = 2$, which agrees with (a).

7. Let X be a random variable with a pdf $f(x)$ and mgf $M(t)$. Suppose f is symmetric about 0, i.e. $f(-x) = f(x)$. Show that $M(-t) = M(t)$.

Solution:

$$M(-t) = E(e^{-Xt}) = \int_{-\infty}^{\infty} e^{-xt} f(x) dx = \int_{-\infty}^{\infty} e^{-xt} f(-x) dx, \tag{1}$$

by the symmetry of f. Performing substitution (letting $y = -x$, $dy = -dx$), we get that Equation (1) equals

$$\int_{\infty}^{-\infty} e^{yt} f(y)(-dy) = -\left(-\int_{-\infty}^{\infty} e^{yt} f(y) dy\right) = \int_{-\infty}^{\infty} e^{yt} f(y) dy = M(t),$$

so $M(-t) = M(t)$.

8. Let X_1 and X_2 be two independent random variables. Suppose that X_1 and $Y = X_1 + X_2$ have Poisson distributions with means μ_1 and $\mu > \mu_1$, respectively. Find the distribution of X_2.

Solution: By definition, X_1 has mgf $M_{X_1}(t) = e^{\mu_1(e^t - 1)}$, and $Y = X_1 + X_2$ has mgf $M_Y(t) = e^{\mu(e^t - 1)}$. By further definition, however, $M_Y(t) = M_{X_1 + X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$ by independence of X_1 and X_2.

$$\Rightarrow M_{X_2}(t) = \frac{M_Y(t)}{M_{X_1}(t)} = \frac{e^{\mu(e^t - 1)}}{e^{\mu_1(e^t - 1)}} = e^{(\mu - \mu_1)(e^t - 1)},$$

which is the mgf of a Poisson$(\mu - \mu_1)$ random variable. Thus $X_2 \sim$ Poisson$(\mu - \mu_1)$.

9. Suppose X is a random variable with the pdf $f(x)$ which is symmetric about 0, i.e. $f(-x) = f(x)$. Show that $F(-x) = 1 - F(x)$, for all x in the support of X.

Solution: $1 - F(x) = 1 - \int_{-\infty}^{x} f(t)dt = \int_{x}^{\infty} f(t)dt$. Let $y = -t$, so $t = -y$, $dy = -dt$, $dt = -dy$. By substitution,

$$\begin{aligned} 1 - F(x) &= \int_{-x}^{-\infty} f(-y)(-dy) \\ &= \int_{-\infty}^{-x} f(-y)dy \\ &= \int_{-\infty}^{x} f(y)dy \text{ because } f \text{ is symmetric} \\ &= F(-x). \end{aligned}$$

10. Let X_n have a gamma distribution with parameter $\alpha = n$ and β, where β is not a function of n. Let $Y_n = X_n/n$. Find the limiting distribution of Y_n.

Solution: $X_n \sim \text{Gamma}(n, \beta) \Rightarrow M_{X_n}(t) = \left(\frac{1}{1-\beta t}\right)^n$. Let $Y_n = \frac{X_n}{n}$. By definition,

$$M_{Y_n}(t) = E(e^{Y_n t}) = E\left(e^{\frac{X_n}{n}t}\right) = E\left(e^{X_n \frac{t}{n}}\right) = M_{X_n}\left(\frac{t}{n}\right) = \left(\frac{1}{1 - \frac{\beta t}{n}}\right)^n = \left[\left(1 - \frac{\beta t}{n}\right)^n\right]^{-1}.$$

One can rephrase the problem as "Find the distribution of some random variable Y where $Y_n \xrightarrow{d} Y$." Using the mgf technique, we are then interested to find the form of some mgf $M_Y(t)$ so that $\lim_{n\to\infty} M_{Y_n}(t) = M_Y(t)$.

$$\lim_{n\to\infty} M_{Y_n}(t) = \lim_{n\to\infty} \left(\frac{1}{\left(1 - \frac{\beta t}{n}\right)}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1 - \frac{\beta t}{n}\right)^n} = \frac{1}{\lim_{n\to\infty} \left(1 - \frac{\beta t}{n}\right)^n} = \frac{1}{e^{-\beta t}} = e^{\beta t} \doteq M_Y(t)$$

because $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for some x. This resulting mgf is the mgf for $P(Y = \beta) = 1$, i.e. the degenerate distribution at β, i.e.

$$\begin{cases} Y = \beta & \text{with probability } 1 \\ Y \neq \beta & \text{with probability } 0 \end{cases} \quad \text{whose cdf is } F_Y(y) = \begin{cases} 0, & y < \beta \\ 1, & y \geq \beta. \end{cases}$$

Alternative solution: If you instead used the alternate formulation of a Gamma(α, β) mgf $\left(\text{namely } M(t) = \left(\frac{1}{1-t/\beta}\right)^\alpha\right)$ for some parameters α, β, then the solution proceeds as follows:

$X_n \sim \text{Gamma}(n, \beta) \Rightarrow M_{X_n}(t) = \left(\frac{1}{1-t/\beta}\right)^n$. Let $Y_n = \frac{X_n}{n}$. By definition,

$$M_{Y_n}(t) = E(e^{Y_n t}) = M_{X_n}\left(\frac{t}{n}\right) = \left(\frac{1}{1 - \frac{t}{n\beta}}\right)^n = \left[\left(1 - \frac{t}{n\beta}\right)^n\right]^{-1}.$$

We are interested to find $\lim_{n\to\infty} M_{Y_n}(t)$.

$$\lim_{n\to\infty} M_{Y_n}(t) = \lim_{n\to\infty} \left(\frac{1}{\left(1 - \frac{t}{n\beta}\right)}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1 - \frac{t}{n\beta}\right)^n} = \frac{1}{\lim_{n\to\infty} \left(1 - \frac{t}{n\beta}\right)^n} = \frac{1}{e^{-t/\beta}} = e^{t/\beta} \doteq M_Y(t)$$

because $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for some x. This resulting mgf is the mgf for $P(Y = 1/\beta) = 1$, i.e. the degenerate distribution at $1/\beta$, i.e.

$$\begin{cases} Y = 1/\beta & \text{with probability } 1 \\ Y \neq 1/\beta & \text{with probability } 0 \end{cases} \quad \text{whose cdf is } F_Y(y) = \begin{cases} 0, & y < 1/\beta \\ 1, & y \geq 1/\beta. \end{cases}$$

For future reference (so there is no confusion), all distributions are to be defined as in the Casella and Berger text.

11. The Pareto distribution, with parameters α and β, has pdf

$$f(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}, \qquad \alpha < x < \infty, \qquad \alpha > 0, \qquad \beta > 0.$$

(a) Verify that $f(x)$ is a pdf.

(b) Derive the mean of this distribution.

Solution:

(a) To show that $f(x)$ is a pdf, we must show that $f(x)$ is non-negative and integrates to 1.

i. Because $x > \alpha > 0$, clearly $f(x)$ is non-negative for all x.

ii.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_{\alpha}^{\infty} \frac{\beta\alpha^\beta}{x^{\beta+1}}dx = \beta\alpha^\beta \int_{\alpha}^{\infty} x^{-\beta-1}dx \\ &= \beta\alpha^\beta \left(\frac{-1}{\beta}x^{-\beta}\right) \Big|_{\alpha}^{\infty} = \frac{-\alpha^\beta}{x^\beta} \Big|_{\alpha}^{\infty} = -(0-1) = 1. \end{aligned}$$

(b)

$$E(X) = \int_{\alpha}^{\infty} x \cdot \frac{\beta\alpha^\beta}{x^{\beta+1}}dx = \beta\alpha^\beta \int_{\alpha}^{\infty} x^{-\beta}dx = \beta\alpha^\beta \left(\frac{1}{1-\beta}x^{1-\beta}\right) \Big|_{\alpha}^{\infty}. \tag{2}$$

Note that this integration holds only if $\beta > 1$; the result is undefined for $\beta = 1$, and is infinite if $\beta < 1$. Assuming $\beta > 1$, Equation (2) becomes

$$\frac{-\beta\alpha^\beta}{\beta-1}\left(\frac{1}{x^{\beta-1}}\right) \Big|_{\alpha}^{\infty} = \frac{-\beta\alpha^\beta}{\beta-1}\left(0 - \frac{1}{\alpha^{\beta-1}}\right) = \frac{\beta\alpha^\beta}{(\beta-1)\alpha^{\beta-1}} = \frac{\beta\alpha}{\beta-1} \text{ for } \beta > 1.$$