Homework 1
Due Monday October 3rd 6:30 pm

Please label all graphs clearly. Homework can be handwritten and scanned or photographed. Please submit to Canvas as a pdf (preferred), docx, or jpg. Make sure the image has sufficient resolution to make out all writing.

1. For the following Linear Program:

$$\text{Max } 20X + 25Y$$

$$
\begin{aligned}
X + \ Y &\le 10 \\
2X + 3Y &\le 27 \\
2X + \ Y &\le 16 \\
X, Y &\ge 0
\end{aligned}
$$

   a. Graph the feasible region.
   b. Identify the extreme points of the feasible region.
   c. Find the optimal solution.

2. For the following Linear Program:

$$\text{Max } 80X + 65Y$$

$$
\begin{aligned}
X + \ Y &\le 12 \\
2X + \ Y &\le 20 \\
-3X + 4Y &\le 20 \\
X, Y &\ge 0
\end{aligned}
$$

   a. Graph the feasible region.
   b. Identify the extreme points of the feasible region.
   c. Find the optimal solution.
   d. Write the problem in equality form.
   e. What are the values of the slack variables at the optimal solution?

3. You oversee hiring for a major teaching hospital. The hospital has recently expanded the number of beds available in the Intensive Care Unit (ICU) from 12 to 20. The hospital currently assigns 3 teaching doctors and 8 residents to the ICU, for each of three shifts per day. Due to their experience teaching doctors can oversee up to 2 patients per shift while residents are only able to treat one. Experienced doctors are paid $280,000 annually compared to only $90,000 for residents. Because it is a teaching hospital a ratio of no more than three residents per teaching doctor must be maintained. The hospital does not wish to fire any of its current employees and is only looking to hire new ones.

    a.) What are your decision variables? Constraints? Graph the feasible region.
    b.) How many new doctors and residents should the hospital hire to minimize the total annual salary of the ICU? What is the salary budget needed for the ICU?
    c.) Are any constraints redundant? If so which and why.
    d.) Is the problem infeasible? Explain.
    e.) Is the problem unbounded? If so, what real world constraint should be added?

4. You are managing the account of an investor who wishes to invest their money in two stock funds. Stock fund X invests in 60% small cap and 40% large cap stocks, with a 2% (of principal) management fee and an average return of 10% of principal per year. Stock fund Y invests in 30% small cap and 70% large cap stocks, with a .5% (of principal) management fee and an average return of 7% of principal per year. The investor wants at least 50% of their investments in large cap stocks, with a total management fee of less than 1.5%.

    a.) What are your decision variables? Constraints?
    b.) How should you invest the money in order to maximize the net return on principal per year?
    c.) Write the problem in standard form and equality form.
    d.) What are the slack/surplus values for each constraint at the optimal value? Explain what each means in the context of the problem.

5. Prove that a local minimum of a convex function on a convex set is globally optimal on that set. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $S \subset \mathbb{R}$ be a convex set. Let x^* be an element of S. Suppose that x^* is a local minimum for f, that is, there exists $\varepsilon \geq 0$ such that $f(x^*) \leq f(x)$ for all $x \in S$ for which $\|x - x^*\| \leq \varepsilon$. Prove that x^* is globally optimal, that is $f(x^*) \leq f(x)$ for all $x \in S$.