MATH 504

LECTURE 1 - 8/30/2022

Homeworks must be typed using LaTeX, with other things causing point deduction.

Vectors and Matrices - Review

Linear (Vector) space.

Def: Let a set V be given, let addition & multiplication be defined such that if $x, y \in V$, $\alpha \in \mathbb{R}$, then $x+y \in V$, $\alpha x \in V$. V is linear space if it has additional properties

(commutativity, associativity, distributive, compatability, identity wrt add, unit, inverse
$x+y=y+x$, $(x+y)+z=x+(y+z)$, $\alpha(x+y)=\alpha x+\alpha y$, $(\alpha\beta)x=\alpha(\beta x)$

The elements of V are called vectors.

Def: A real n dimensional vector is ordered set of reals $\{x_1, \dots, x_n\}$ written $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ (a column vector). Recall can have row vector.

Def: $\mathbb{R}^n$ is set that contains vectors of dim n, called n-dim coordinate space.

Def: Addition of vectors is done entry-wise $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}$

Def: Given vectors $v^{(1)}, v^{(2)}, \dots, v^{(k)}$ in V, $c_1, \dots, c_k \in \mathbb{R}$, $c_1 v^{(1)} + \dots c_k v^{(k)}$ is linear combination.

Def: The set of all lin combs of $v^{(1)}, \dots, v^{(k)}$ is called the span, denoted $\text{span}(v^{(1)}, \dots, v^{(k)})$.

Ex: $\text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right) = \mathbb{R}^2$ plane in $\mathbb{R}^3$.

Def: Linear independent vectors $v^{(1)}, \dots, v^{(k)}$ if

$$c_1 v^{(1)} + \cdots + c_k v^{(k)} = \vec{0} \Rightarrow c_1 = c_2 = \cdots = c_k = 0$$

Otherwise, linear dependent.

Def: A set of vectors $v_1, \dots, v_k$ form a basis for V iff

(1) $v_1, \dots, v_k$ linearly independent

(2) $\text{span}(v_1, \dots, v_k) = V \leftarrow$ span V.

THM: If $V = \text{span}(v_1, \dots, v_k)$, any $m > k$ vectors in V are linearly dependent.

Any two bases $v_1, \dots, v_k$ and $u_1, \dots, u_m$ of V must contain $m = k = \dim(V)$ number of vectors, called dimension of V.

Def: A function $\|\cdot\| : V \to \mathbb{R}^+$ is called a norm in V if for any $\vec{x}, \vec{y} \in V$ and any scalar α the following hold

- $\|\vec{x}\| \geq 0$ with equality iff $\vec{x} = 0$.
- $\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|$
- $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ $\leftarrow$ called Triangle Inequality

The norm assigns a positive number to each non-zero vector, is zero exactly when zero vector.

Def: A linear space with a norm is called a normed linear space (not always the case).

Def: A p-norm is defined by

$$\|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p \right)^{1/p}$$

This gives us ℓ_2 norm: $\|x\|_2 = (|x_1|^2 + \dots + |x_n|^2)^{1/2}$

ℓ_1 norm: $\|x\|_1 = |x_1| + \cdots + |x_n|$

ℓ_∞ norm "sup norm": $\|x\|_\infty = \max\{|x_1|, \cdots |x_n|\}$.

Lemma: Minkowski's Inequality: triangle ineq holds for p-norm

Def: ε ball for ℓp norms

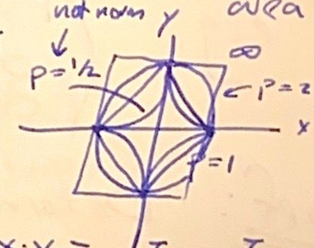
$$\varepsilon \ge 0, \ B_{\ell_p}(\varepsilon) = B_p(\varepsilon) = \{a \in \mathbb{R}^n \mid \|a\|_p \le \varepsilon\}$$

Ex: unit ball in $\mathbb{R}^2$

$$B_2(1) = \{a \in \mathbb{R}^2 \mid \|a\|_2 \le 1\} = \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, \sqrt{a_1^2 + a_2^2} \le 1 \right\}$$

This is the unit circle in $\mathbb{R}^2$ (including the interior area).

Lemma: General equivalance of ℓp norms



Def: (1) The inner product of two vectors

algebraic: $x, y \in \mathbb{R}^2$ is defined by $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i = x^T y = y^T x = \langle y, x\rangle \in \mathbb{R}$

(2) If θ is the angle between two vectors

geometric: $x, y \in \mathbb{R}^n$, then the inner product is given by

$$\langle x, y\rangle = \|x\|_2 \|y\|_2 \cos\theta$$

Note: $\langle x, x\rangle = \|x\|_2^2$ ← connection with ℓ_2 norm.

$\langle x, y\rangle = 0 \Leftrightarrow x \perp y$ ← orthogonality.

Def: Cauchy-Schwarz Inequality.

Given $x, y \in \mathbb{R}^n$, $|\langle x, y\rangle| \le \|x\|_2 \|y\|_2$

Can be generalized to other norms →

Def: $|\langle x, y\rangle| \le \|x\|_p \|x\|_q$ if $\frac{1}{p} + \frac{1}{q} \le 1$.

Lemma: Triangle Inequality: $\|x + y\|_2 \le \|x\|_2 + \|y\|_2$

Matrices

Def: A real matrix given $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ $a_{ij} \in \mathbb{R}$.

A has m rows, n cols.

Def: Identity matrix $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ ← 1's on diagonal and 0 elsewhere.

$IA = AI = A$.

Def: Triangular Matrices.

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 1 & & 0 \\ 3 & 5 & 0 \\ -1 & -2 & 6 \end{pmatrix}$. A matrix is diagonal if it is both upper and triangular.

Def: Determinants.

$$\det(A) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{vmatrix}$$ for $n=2$ $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

Def: Transpose: rows become columns (or reverse).

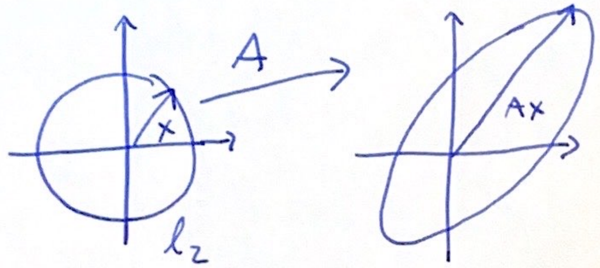
- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A+B)^T = A^T + B^T$
- $A = A^T \iff A$ symmetric

Def: Matrix norm $\leftarrow$ analogous to vector norm.

Matrix norm induced by vector norms.

$$\|A\| = \max \{ \|Ax\| \mid x \in \mathbb{R}^n, \|x\| = 1 \} = \max_{\|x\|=1} \|Ax\|.$$

$$\Leftrightarrow \|A\| = \max \left\{ \frac{\|Ax\|}{\|x\|} \mid x \in \mathbb{R}^n, x \neq 0 \right\}.$$



$\leftarrow$ Find the vector with largest norm after transformation by A.

$\Rightarrow \|A\| =$ maximum magnitude of a unit ball after transformation by A.

Matrix p-norms.

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| \text{ (max of abs column sums).}$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)} \text{ (max singular value of A)}$$

$$\|A\|_\infty = \max_{1 \le j \le m} \sum_{j=1}^{n} |a_{ij}| \text{ (max of abs row sums).}$$

Frobenius norm: $\|A\|_F = \sqrt{\text{trace}(A^T A)} = \sqrt{\text{trace}(A A^T)}$

Eigenvectors and Eigenvalues

Let A be $n \times n$ square matrix.

x is an eigenvector and λ is an eigenvalue of A if

$$Ax = \lambda x.$$

Eigenvectors are those whose direction is preserved under action of A, but length may change.

Finding eigen-pairs is left for review.