MATH 503: Mathematical Statistics

Lecture 4: Properties of Point Estimators II

Reading: Sections 6.1-6.2, 7.3

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Today's Topics

- Recap: Sufficient statistics
- Uniform minimum variance unbiased estimators (UMVUEs)
  - Rao-Blackwell Theorem
  - Completeness
  - Lehmann-Scheffé Theorem
  - Uniqueness
- Exponential families
- Comments connecting Rao-Blackwell and Lehmann-Scheffé

Sufficiency

Let $X_1, \ldots, X_n$ denote a random sample of size n from a distribution that has pdf/pmf $f(x;\theta), \theta \in \Omega$. Let $Y_1 = u_1(X_1, \ldots, X_n)$ be a statistic whose pdf/pmf is $f_{Y_1}(y_1;\theta)$. Then Y_1 is a sufficient statistic for θ iff.

$$\frac{f(x_1;\theta)f(x_2;\theta)\cdots f(x_n;\theta)}{f_{Y_1}[u_1(x_1, \ldots, x_n);\theta]} = H(x_1, \ldots, x_n),$$

where $H(x_1, \ldots, x_n)$ does not depend on $\theta \in \Omega$.

Neyman-Fisher Factorization Thm

Let $X_1, \ldots, X_n$ denote a random sample from a distribution that has pdf/pmf $f(x;\theta), \theta \in \Omega$. The statistic $Y_1 = u_1(X_1, \ldots, X_n)$ is a sufficient statistic for θ iff. we can find two nonnegative functions, k_1 and k_2, such that

$$f(x_1;\theta)f(x_2;\theta)\cdots f(x_n;\theta) = k_1[u_1(x_1, \ldots, x_n);\theta] \cdot k_2(x_1, \ldots, x_n)$$

where $k_2(x_1, \ldots, x_n)$ does not depend on θ.

Uniform Minimum Variance Unbiased Estimators (UMVUEs)

- For a given positive integer n, $Y = u(X_1, \dots, X_n)$ is a <u>uniform minimum variance unbiased estimator</u> (UMVUE) of the parameter θ
  - if Y is unbiased, and
  - if the variance of Y is less than or equal to the variance of every other unbiased estimator of θ.

Rao-Blackwell Theorem

(Hogg, McKean, & Craig)

C.R. Rao

David Blackwell

Let $X_1, \dots, X_n$, n a fixed positive integer, denote a random sample from a distribution that has pdf/pmf $f(x;\theta), \theta \in \Omega$. Let $Y_1 = u_1(X_1, \dots, X_n)$ be a sufficient statistic for θ, and let $Y_2 = u_2(X_1, \dots, X_n)$, not a function of Y_1 alone, be an unbiased estimator of θ. Then $E(Y_2 \mid y_1) = \varphi(y_1)$ defines a statistic $\varphi(Y_1)$. This statistic $\varphi(y_1)$ is a function of the sufficient statistic for θ; it is an unbiased estimator of θ; and its variance is less than that of Y_2.

Rao-Blackwell Theorem

(Casella & Berger)

C.R. Rao

David Blackwell

Let W be any unbiased estimator of $\tau(\theta)$, and let T be a sufficient statistic of θ. Define $\phi(T) = E(W|T)$. Then $E_\theta \phi(T) = \tau(\theta)$ and $\text{Var}_\theta \phi(T) \leq \text{Var}_\theta W$ for all θ, that is, $\phi(T)$ is a uniformly better unbiased estimator of $\tau(\theta)$.

Notes re. Rao-Blackwell Thm.

- If we know a sufficient statistic for the parameter exists, the MVUE will be a function of the sufficient statistic.
- This does not mean that we first need to find an unbiased statistic!
- Focus on functions of sufficient statistics

Theorem

- Let $X_1, \dots, X_n$ denote a random sample from a distribution that has pdf/pmf $f(x;\theta), \theta \in \Omega$. If a sufficient statistic $Y_1 = u_1(X_1, \dots, X_n)$ for θ exists and if a MLE $\hat{\theta}$ of θ, also exists uniquely, then $\hat{\theta}$ is a function of $Y_1 = u_1(X_1, \dots, X_n)$.
- **The point:** MLEs are functions of sufficient statistics.

Example

Let $X_1, \dots, X_n$ denote a random sample from a distribution that has pdf $f(x;\theta) = \theta e^{-\theta x},\ 0 < x < \infty$.

1. Find a sufficient statistic for θ.
2. Find the MLE of θ.
3. Determine a MVUE of θ.

SEE ATTACHED

Example

$X_1, \ldots, X_n \sim f(x;\theta) = \theta e^{-\theta x}, \quad 0 < x < \infty \qquad (\text{Note}: X_1, \ldots, X_n \sim Exp(1/\theta))$

① $\prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n e^{-\theta \sum_1^n x_i} = \underbrace{\theta^n e^{-\theta \sum_1^n x_i}}_{k_1(\sum x_i;\theta)} \cdot \underbrace{1}_{k_2(x)}$

$\therefore$ by NFFT, $Y = \sum_1^n X_i$ is sufficient for θ.

② $\mathcal{L}(\theta; \underline{x}) = \prod_{i=1}^{n} f(x_i;\theta) = \theta^n e^{-\theta \sum_1^n x_i}$

$\ln \mathcal{L}(\theta; \underline{x}) = n \ln \theta - \theta \sum_1^n x_i$

$\frac{\partial \ln \mathcal{L}}{\partial \theta} = \frac{n}{\theta} - \sum_1^n x_i = 0 \quad \therefore \quad \frac{n}{\theta} = \sum x_i$

$$\Longrightarrow \boxed{\hat{\theta} = \frac{n}{\sum X_i} = \frac{1}{\overline{X}}}$$

③ $X_i \sim Exp(1/\theta) = Gamma(1, 1/\theta)$ iid $\Rightarrow \sum_1^n X_i \sim Gamma(n, 1/\theta)$

Pf) $M_{\sum X_i}(t) = E(e^{t\sum X_i}) = E(e^{t(X_1 + \cdots + X_n)}) = E(e^{tX_1 + \cdots + tX_n})$

$= E(e^{tX_1}) \cdots E(e^{tX_n})$ because X_i's indpt

$= E^n(e^{tX})$ because X_i's iid

$= [M_X(t)]^n = \left(\frac{1}{1 - t/\theta}\right)^n$ which is the mgf of $Gamma(n, 1/\theta)$ r.v. $\therefore \sum X_i \sim Gamma(n, 1/\theta)$ ∎

$E\left(\frac{n}{\sum X_i}\right) = n E\left(\frac{1}{\sum X_i}\right) = n \int_0^\infty \frac{1}{y} \cdot \frac{1}{\Gamma(n)(1/\theta)^n} y^{n-1} e^{-y/(1/\theta)} dy$

$= \frac{n \Gamma(n-1)(1/\theta)^{n-1}}{\Gamma(n)(1/\theta)^n} \cancel{\int_0^\infty \frac{1}{\Gamma(n-1)(1/\theta)^{n-1}} y^{(n-1)-1} e^{-y/(1/\theta)} dy}^{\,1}$

$= \frac{n \Gamma(n-1)\theta}{(n-1)\Gamma(n-1)} = \frac{n\theta}{n-1}$

Example

(3) cont.

$\therefore \mathbb{E}(\hat{\theta}) = \mathbb{E}\left(\frac{n}{\sum X_i}\right) = \frac{n\theta}{n-1}$

$\Rightarrow \frac{n-1}{n}\mathbb{E}(\hat{\theta}) = \frac{\cancel{n-1}}{\cancel{n}}\left(\frac{\cancel{n}\theta}{\cancel{n-1}}\right) = \theta$

where $\frac{n-1}{n}\mathbb{E}(\hat{\theta}) = \mathbb{E}\left(\frac{n-1}{n}\hat{\theta}\right) = \mathbb{E}\left(\frac{n-1}{\cancel{n}} \cdot \frac{\cancel{n}}{\sum X_i}\right) = \mathbb{E}\left(\frac{n-1}{\sum X_i}\right)$

$\therefore \frac{n-1}{\sum X_i}$ is unbiased estimator of θ $\therefore$ by Rao-Blackwell Thm.,

$\frac{n-1}{\sum X_i}$ is MVUE of θ.

Completeness

Let the random variable Z have a pdf/pmf that is one member of the family $\{h(z;\theta): \theta \in \Omega\}$. If the condition $E[u(Z)] = 0$, for every $\theta \in \Omega$, requires that $u(z)$ be zero except on a set of points that has probability zero for each $h(z;\theta): \theta \in \Omega$, then the family $\{h(z;\theta): \theta \in \Omega\}$ is called a complete family of pdfs/pmfs.

Note: One-to-one functions of complete sufficient statistics are themselves complete sufficient.

Example 1

Let $X_1, \ldots, X_n$ ~Poisson(θ) iid.

1. Determine a sufficient statistic for θ.
2. What is the pdf associated with this statistic?
3. Show that this statistic is complete.

SEE ATTACHED

Example 1

$X_1, \ldots, X_n \sim \text{Poisson}(\theta)$ iid $\quad f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}; \quad x = 0,1,2,\ldots$

① $$\prod_{i=1}^{n} f(x_i;\theta) = \frac{e^{-n\theta}\theta^{\Sigma x_i}}{\prod_1^n x_i!} = \underbrace{e^{-n\theta}\theta^{\Sigma x_i}}_{k_1(\Sigma x_i;\theta)} \cdot \underbrace{\frac{1}{\prod_1^n x_i!}}_{k_2(\underline{x})}$$

$\therefore$ by NFFT, $Y = \Sigma X_i$ is sufficient for θ.

② $$M_Y(t) = \mathbb{E}(e^{Yt}) = \mathbb{E}(e^{t\Sigma X_i}) = [\mathbb{E}(e^{tX})]^n = [M_X(t)]^n$$

↑ because X_is iid

$$= (e^{\theta(e^t-1)})^n = e^{n\theta(e^t-1)}$$, which is the mgf of Poisson$(n\theta)$ rv.

$\Rightarrow Y \sim \text{Poisson}(n\theta)$

③ $$\mathbb{E}(g(Y)) = \sum_{y=0}^{\infty} g(y)\cdot\frac{e^{-n\theta}(n\theta)^y}{y!} = \cancel{e^{-n\theta}}\sum_{y=0}^{\infty} g(y)\frac{(n\theta)^y}{y!} \doteq 0$$

where $\sum_{y=0}^{\infty} g(y)\frac{(n\theta)^y}{y!} = g(0) + g(1)(n\theta) + g(2)\frac{(n\theta)^2}{2} + g(3)\frac{(n\theta)^3}{6} + \ldots = 0$ where $n, \theta > 0$

$$\mathbb{E}(g(Y)) = 0 \ \forall\theta \iff \begin{cases} g(0) = 0 \\ g(1)(n\theta) = 0 \Rightarrow g(1) = 0 \\ g(2)\frac{(n\theta)^2}{2} = 0 \Rightarrow g(2) = 0 \\ \vdots \end{cases} \Rightarrow g(k) = 0 \ \forall k = 0,1,2,\ldots$$

$\Rightarrow Y$ is complete sufficient for θ.

Example 2 (C&B, Ex. 6.2.23)

Let $X_1, \dots, X_n \sim$Uniform$(0,\theta)$ iid, $\theta > 0$. Show $X_{(n)}$ is complete sufficient for θ.

$$\prod_1^n f(x_i) = \prod_1^n \frac{1}{\theta} I_{(0,\theta)}(x_i) = \frac{1}{\theta^n} I_{(0,\theta)}(x_{(n)}) = \underbrace{\frac{1}{\theta^n} I_{(0,\theta)}(x_{(n)})}_{k_1(x_{(n)};\theta)} \cdot \underbrace{1}_{k_2(\underline{x})}$$

$\therefore$ by NFFT, $Y = X_{(n)}$ is sufficient for θ.

$$f_Y(y) = nF^{n-1}(y)f(y) = n\left(\frac{y}{\theta}\right)^{n-1}\left(\frac{1}{\theta}\right) = \frac{ny^{n-1}}{\theta^n}, \quad 0<y<\theta$$

$$E(g(Y)) = \int_0^\theta g(y)\cdot \frac{ny^{n-1}}{\theta^n}\,dy \doteq 0 \quad \forall \theta>0$$

$$= \frac{n}{\theta^n}\int_0^\theta g(y)y^{n-1}\,dy = 0$$

Differentiating both sides wrt $\theta \Rightarrow g(\theta)\theta^{n-1} = 0 \ \forall\theta>0$

$\Leftrightarrow g(\theta) = 0 \ \forall\theta$

$\therefore X_{(n)}$ is complete sufficient.

Example 3 (C&B, Ex. 6.2.22)

Let $T \sim$Binomial(n,p), $0 < p < 1$. Show T is complete.

$$E(g(T)) = \sum_{t=0}^n g(t)\binom{n}{t}p^t(1-p)^{n-t} = (1-p)^n\sum_{t=0}^n g(t)\binom{n}{t}\left(\frac{p}{1-p}\right)^t \doteq 0 \quad \forall\, 0<p<1$$

$$= g(0) + g(1)\left[n\underbrace{\left(\frac{p}{1-p}\right)}_{>0}\right] + g(2)\left[\frac{n(n-1)}{2}\underbrace{\left(\frac{p}{1-p}\right)^2}_{>0}\right] + \dots + g(n)\underbrace{\left(\frac{p}{1-p}\right)^n}_{>0} = 0 \quad \forall\, 0<p<1$$

This can only hold for all p, $0<p<1$ iff.

$$\begin{cases} g(0) = 0 \\ ng(1) = 0 \\ \frac{n(n-1)}{2}g(2) = 0 \\ \vdots \\ g(n) = 0 \end{cases} \Leftrightarrow g(0) = g(1) = g(2) = \dots = g(n) = 0$$

$\therefore T$ is complete sufficient for p.

Lehmann-Scheffé Theorem

Let $X_1, \dots, X_n$, n a fixed positive integer, denote a random sample from a distribution that has pdf/pmf $f(x;\theta), \theta \in \Omega$, let $Y_1 = u_1(X_1, \dots, X_n)$ be a sufficient statistic for θ, and let the family $\{f_{Y_1}(y_1;\theta): \theta \in \Omega\}$ be complete. If there is a function of Y_1 that is an unbiased estimator of θ, then this function of Y_1 is the unique UMVUE of θ.

Uniqueness

- In most instances, if there is one function $\varphi(Y_1)$ that is unbiased, then it is the only unbiased estimator based on the sufficient statistic Y_1
- Lehmann-Scheffe $\Rightarrow$ unbiased estimators based on complete sufficient statistics are unique.

How to Determine UMVUEs?

- Expected value of complete sufficient statistic
- Conditional expectation of unbiased estimate given sufficient statistic

Example 4

Let a random sample of size n be taken from a distribution of the discrete type with pmf $f(x;\theta) = \frac{1}{\theta}$, $x = 1,2,\ldots,\theta$, where θ is an unknown positive integer.

1. Show that the largest observation, say $Y = X_{(n)}$, of the sample is a complete sufficient statistic for θ.
2. Prove that $[Y^{n+1} - (Y-1)^{n+1}]/[Y^n - (Y-1)^n]$ is the unique UMVUE of θ.

SEE ATTACHED

Example 4

① $f(x;\theta) = \frac{1}{\theta}$, $x = 1, 2, \dots, \theta$

$$\prod_1^n f(x_i;\theta) = \prod_1^n \frac{1}{\theta} I_{\{1,2,\dots,\theta\}}(x_i) = \underbrace{\frac{1}{\theta^n} I_{\{1,2,\dots,\theta\}}(x_{(n)})}_{k_1(x_{(n)};\theta)} \cdot \underbrace{1}_{k_2(\underline{x})}$$

$\Rightarrow Y = X_{(n)}$ is sufficient for θ.

$$f_{X_{(n)}}(y) = \mathbb{P}(X_{(n)} = y) = \mathbb{P}(X_{(n)} \le y) - \mathbb{P}(X_{(n)} \le y-1)$$ because discrete pmf

$$= F^n(y) - F^n(y-1) = \left(\frac{y}{\theta}\right)^n - \left(\frac{y-1}{\theta}\right)^n = \frac{1}{\theta^n}\left[y^n - (y-1)^n\right]$$

$$\mathbb{E}(g(Y)) = \sum_{y=1}^{\theta} g(y) \cdot \frac{1}{\theta^n}\left[y^n - (y-1)^n\right] = \cancel{\frac{1}{\theta^n}} \sum_{y=1}^{\theta} g(y)\left[y^n - (y-1)^n\right] \doteq 0$$

$$= g(1)(1-0) + g(2)(2^n - 1) + g(3)(3^n - 2^n) + \dots +$$
$$+ g(\theta-1)\left((\theta-1)^n - (\theta-2)^n\right) + g(\theta)\left(\theta^n - (\theta-1)^n\right) \doteq 0$$

$$= [g(1) - g(2)] + 2^n[g(2) - g(3)] + \dots + (\theta-1)^n[g(\theta-1) - g(\theta)] + \theta^n g(\theta) = 0$$

Because this relationship must hold $\forall \theta > 0$, this implies

$g(\theta) = 0 \Rightarrow g(\theta-1) - g(\theta) = 0$

$\Rightarrow \quad \dots \quad = 0$

$\Rightarrow g(2) - g(3) = 0$

$\Rightarrow g(1) - g(2) = 0$

$\therefore g(i) = 0 \ \forall i \Rightarrow X_{(n)}$ complete sufficient for θ.

Example 4 (cont.)

② $$\mathbb{E}\left(\frac{Y^{n+1}-(Y-1)^{n+1}}{Y^{n}-(Y-1)^{n}}\right) = \sum_{y=1}^{\theta} \frac{y^{n+1}-(y-1)^{n+1}}{\cancel{y^{n}-(y-1)^{n}}} \cdot \frac{1}{\theta^{n}}\cancel{\left(y^{n}-(y-1)^{n}\right)}$$

$$= \frac{1}{\theta^{n}} \sum_{y=1}^{\theta} \left[y^{n+1}-(y-1)^{n+1}\right]$$

$$= \frac{1}{\theta^{n}}\left[(\cancel{1}-0)+(\cancel{2^{n+1}}-\cancel{1})+(3^{n+1}-\cancel{2^{n+1}})+\cancel{\cdots}+(\theta^{n+1}-(\cancel{\theta-1})^{n+1})\right]$$

$$= \frac{\theta^{n+1}}{\theta^{n}} = \theta$$

$\therefore \dfrac{Y^{n+1}-(Y-1)^{n+1}}{Y^{n}-(Y-1)^{n}}$ is unbiased for θ and Y is complete sufficient

for θ $\Rightarrow$ by Lehmann-Scheffe, $\dfrac{Y^{n+1}-(Y-1)^{n+1}}{Y^{n}-(Y-1)^{n}}$ is UMVUE of θ.

Exponential Family/Class

A pdf of the form

$$f(x;\theta) = \exp[p(\theta)K(x) + S(x) + q(\theta)], x \in S^*$$

is said to be a member of the regular exponential class of probability density or mass functions if

1. S^*, the support of X, does not depend on θ
2. $p(\theta)$ is a nontrivial continuous function of $\theta \in \Omega$
3. Finally,
   - If X is a continuous rv then each of $K'(x) \not\equiv 0$ and $S(x)$ is a continuous function of $x \in S^*$
   - If X is a discrete rv then $K(x)$ is a nontrivial function of $x \in S^*$

Example 5

Show that the Normal($0, \sigma^2 = \theta$) distribution is a member of the regular exponential class.

$$f(x) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta}(x-0)^2} = (2\pi\theta)^{-\frac{1}{2}} \exp\left(\frac{-1}{2\theta} x^2\right), \underbrace{-\infty < x < \infty}_{\text{support does not depend on } \theta}$$

$$= \exp\Big[\underbrace{-\tfrac{1}{2}\ln(2\pi\theta)}_{q(\theta)} \underbrace{- \tfrac{1}{2\theta}}_{p(\theta)} \underbrace{x^2}_{K(x)} + \underbrace{0}_{S(x)}\Big]$$

i.e. $S(x) = 0$

$p(\theta) = -\frac{1}{2\theta}$ nontrivial

$K(x) = x^2$ (i.e. $K'(x) \neq 0$)

$q(\theta) = -\frac{1}{2}\ln(2\pi\theta)$

Example 6

Is the Uniform$(0, \theta)$ distribution a member of the regular exponential class?

$$f(x) = \frac{1}{\theta}, \quad \underbrace{0 < x < \theta}_{\text{Support for } x \text{ depends on } \theta}$$

NO! Unif $(0, \theta)$ is not a member

What about for a random sample?

$$\prod_1^n f(x_i) = \prod_1^n \exp\left[p(\theta)K(x_i) + S(x_i) + q(\theta)\right]$$

$$= \exp\left[p(\theta)\sum_{i=1}^n K(x_i) + \sum_{i=1}^n S(x_i) + nq(\theta)\right]$$

$$= \underbrace{\exp\left[p(\theta)\sum_{i=1}^n K(x_i) + nq(\theta)\right]}_{k_1\left(\sum_{i=1}^n K(x_i); \theta\right)} \underbrace{\exp\left[\sum_{i=1}^n S(x_i)\right]}_{k_2(\underline{x})}$$

Result: $Y_1 = \sum_{i=1}^n K(x_i)$ is a sufficient statistic for θ.

Theorem

Let $X_1, \dots, X_n$, denote a random sample from a distribution that represents a regular case of the exponential class, with pdf/pmf given by

$$f(x;\theta) = \exp[p(\theta)K(x) + S(x) + q(\theta)], x \in S^*$$

Consider the statistic $Y_1 = \sum_{i=1}^{n} K(x_i)$. Then,

1. The pdf/pmf of Y_1 has the form,
$$f_{Y_1}(y_1;\theta) = R(y_1)\exp[p(\theta)y_1 + nq(\theta)]$$
for $y_1 \in S^*_{Y_1}$ and some function $R(y_1)$. Neither $S^*_{Y_1}$ nor $R(y_1)$ depend on θ.
2. $E(Y_1) = -nq'(\theta)/p'(\theta)$
3. $\text{Var}(Y_1) = n[1/p'(\theta)]^3\{p''(\theta)q'(\theta) - q''(\theta)p'(\theta)\}$

Example 7

1. Consider $X \sim \text{Poisson}(\theta)$. Show that it is a member of the regular exponential class.
2. For a random sample, $X_1, \dots, X_n \sim \text{Poisson}(\theta)$, determine the sufficient statistic, Y_1.
3. Use the above theorem to verify $E(Y_1)$ and $V(Y_1)$.

SEE ATTACHED

Example 7

① $X \sim \text{Poisson}(\theta) \therefore f(x;\theta) = \dfrac{e^{-\theta}\theta^{x}}{x!}$, $\underbrace{x = 0, 1, 2, \ldots\ldots}_{\text{support space doesn't depend on } \theta}$

$$f(x;\theta) = \exp\left[\ln\left(\frac{e^{-\theta}\theta^{x}}{x!}\right)\right]$$

$$= \exp\left[\underbrace{-\theta}_{q(\theta)} + \underbrace{x}_{K(x)}\underbrace{\ln\theta}_{p(\theta)} \underbrace{- \ln(x!)}_{S(x)}\right]$$

$\therefore$ it is a member of the regular exponential class.

② $Y_1 = \sum_{i=1}^{n} K(x_i) = \sum_{i=1}^{n} X_i$ is sufficient statistic of θ.

③ $p(\theta) = \ln\theta$ $\qquad q(\theta) = -\theta$

$p'(\theta) = 1/\theta = \theta^{-1}$ $\qquad q'(\theta) = -1$

$p''(\theta) = -\theta^{-2} = \dfrac{-1}{\theta^2}$ $\qquad q''(\theta) = 0$

$$\therefore E(Y_1) = \frac{-nq'(\theta)}{p'(\theta)} = \frac{-n(-1)}{1/\theta} = n\theta \checkmark$$

$$V(Y_1) = n\left(\frac{1}{p'(\theta)}\right)^3 \left(p''(\theta)q'(\theta) - q''(\theta)p'(\theta)\right)$$

$$= n\left(\frac{1}{1/\theta}\right)^3 \left(\left(\frac{-1}{\theta^2}\right)(-1) - \cancel{(0)(1/\theta)}\right)$$

$$= n\theta^{\cancel{3}}\left(\cancel{\frac{1}{\theta^2}}\right)$$

$$= n\theta \checkmark$$

Theorem

Let $f(x;\theta), \gamma < \theta < \delta$, be a pdf/pmf of a rv X whose distribution is a regular case of the exponential class. Then if $X_1, X_2, \dots, X_n$ (where n is a fixed positive integer) is a random sample from the distribution of X, the statistic $Y_1 = \sum_{i=1}^{n} K(X_i)$ is a sufficient statistic for θ and the family $\{f_{Y_1}(y_1;\theta): \gamma < \theta < \delta\}$ of pdfs of Y_1 is complete. That is, Y_1 is a complete sufficient statistic for θ.

Implication: After determining the sufficient statistic, $Y_1 = \sum_{i=1}^{n} K(X_i)$, we form a function, $\varphi(Y_1)$, so that $\mathrm{E}(\varphi(Y_1)) = \theta$ implies $\varphi(Y_1)$ is unique and UMVUE of θ.

Example 8

Consider $X_1, \dots, X_n \sim$Normal(θ, σ^2) iid, σ known. Show that $Y_1 = \sum_{i=1}^{n} X_i$ is complete sufficient. Determine the unique UMVUE of θ.

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-1}{2\sigma^2}(x-\theta)^2} = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[\frac{-1}{2\sigma^2}(x-\theta)^2\right]$$

$$= \exp\left[\frac{-1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x^2 - 2\theta x + \theta^2)\right]$$

$$= \exp\Big[\underbrace{\frac{-1}{2}\ln(2\pi\sigma^2) - \frac{\theta^2}{2\sigma^2}}_{q(\theta)} \underbrace{- \frac{x^2}{2\sigma^2}}_{S(x)} + \underbrace{\frac{\theta}{\sigma^2}}_{p(\theta)} \underbrace{x}_{K(x)}\Big]$$ is exponential family

$\therefore Y_1 = \sum_1^n K(x_i) = \sum_1^n x_i$ is complete sufficient for θ

$$E(Y_1) = E\left(\sum_1^n X_i\right) = \sum_1^n E(X_i) = \sum_1^n \theta = n\theta$$

$\Rightarrow \frac{Y_1}{n} = \frac{\sum X_i}{n} = \bar{X}$ is unbiased for θ

$\Rightarrow \bar{X}$ is unique UMVUE by Lehmann–Scheffé Thm.

Example 9

Let $X_1, \ldots, X_n \sim$ Bernoulli(θ) iid, $0 < \theta < 1$. Find the UMVUE of θ.

$f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$ — support does not depend on θ

$= \exp[x \ln\theta + (1-x)\ln(1-\theta)]$

$= \exp\left[\underbrace{x}_{K(x)} \underbrace{\ln\left(\frac{\theta}{1-\theta}\right)}_{p(\theta)} + \underbrace{\ln(1-\theta)}_{q(\theta)} + \underbrace{0}_{S(x)}\right]$ is an exponential family

$\therefore Y = \sum_1^n K(X_i) = \sum_1^n X_i$ is complete sufficient for θ

$E(Y) = E\left(\sum_1^n X_i\right) = \sum_1^n E(X_i) = \sum_1^n \theta = n\theta$

$\therefore \frac{Y}{n} = \frac{\sum X_i}{n} = \bar{X}$ unbiased for θ $\therefore \bar{X}$ UMVUE by Lehmann-Scheffé Thm.

SEE ATTACHED for alternate solution to Example 9.

Example 10

Let a random sample of size n, i.e. $X_1, \ldots, X_n$, be taken from a distribution that has the pdf $f(x;\theta) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right) I_{(0,\infty)}(x)$. Find the MLE and the UMVUE of $P(X_1 \leq 2)$.

SEE ATTACHED

Example 9 by Rao-Blackwellization

$X_1, \ldots, X_n \sim \text{Bernoulli}(\theta)$ iid, $0<\theta<1 \Rightarrow f(x) = \theta^x (1-\theta)^{1-x}; x=0,1$

$$\prod_{i=1}^{n} f(x_i) = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i} = \underbrace{\left(\frac{\theta}{1-\theta}\right)^{\Sigma x_i}(1-\theta)^n}_{k_1(\sum_1^n x_i;\, \theta)} \cdot \underbrace{1}_{k_2(\underline{x})}$$

$\therefore$ by NFFT, $Y = \sum_1^n X_i$ sufficient for θ.

Consider $X_1 = \begin{cases} 1 & \text{w.p. } \theta \\ 0 & \text{w.p. } 1-\theta \end{cases}$. By definition, $\mathbb{E}(X_1) = 1 \cdot \mathbb{P}(X_1=1) + 0 \cdot \mathbb{P}(X_1=0)$

$$= 1(\theta) + \cancel{0(1-\theta)}$$

$$= \theta$$

$\therefore X_1$ unbiased for θ. Thus, by the Rao-Blackwell Thm., consider

$$\mathbb{E}\left(X_1 \,\middle|\, \sum_1^n X_i = y\right) = \mathbb{P}\left(X_1 = 1 \,\middle|\, \sum_1^n X_i = y\right) = \frac{\mathbb{P}(X_1=1, \sum_1^n X_i = y)}{\mathbb{P}(\sum_1^n X_i = y)}$$

$$= \frac{\mathbb{P}(X_1=1, \sum_2^n X_i = y-1)}{\mathbb{P}(\sum_1^n X_i = y)} = \frac{\mathbb{P}(X_1=1)\mathbb{P}(\sum_2^n X_i = y-1)}{\mathbb{P}(\sum_1^n X_i = y)} \text{ by independence}$$

where $\sum_2^n X_i \sim \text{Bin}(n-1, \theta)$ and $\sum_1^n X_i \sim \text{Bin}(n, \theta)$

$$= \frac{\cancel{(\theta)}\binom{n-1}{y-1}\cancel{\theta^{y-1}}\cancel{(1-\theta)^{(n-1)-(y-1)}}}{\binom{n}{y}\cancel{\theta^{y}}\cancel{(1-\theta)^{n-y}}} = \frac{\binom{n-1}{y-1}}{\binom{n}{y}} = \frac{\cancel{(n-1)!}\ y\cancel{!}\ \cancel{(n-y)!}}{\cancel{(y-1)!}\cancel{[(n-1)-(y-1)]!}\ n\cancel{!}}$$

$$= \frac{y}{n} = \frac{\sum_1^n X_i}{n} = \bar{X}$$

$\therefore$ by the Rao-Blackwell Thm., $\bar{X}$ is UMVUE of θ.

Example 10

$$\prod_1^n f(x_i;\theta) = \prod_1^n \frac{1}{\theta} e^{-x_i/\theta} = \frac{1}{\theta^n} e^{-\Sigma x_i/\theta} = \underbrace{\frac{1}{\theta^n} e^{\frac{-\Sigma x_i}{\theta}}}_{k_1(\Sigma x_i;\theta)} \cdot \underbrace{1}_{k_2(\underline{x})}$$

$\therefore$ by NFFT, $Y = \Sigma X_i$ sufficient for θ.

Alternatively, $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} = \exp\left[\ln\left(\frac{1}{\theta} e^{-x/\theta}\right)\right]$

$$= \exp[\underbrace{-\ln\theta}_{q(\theta)} \underbrace{-\tfrac{1}{\theta}}_{p(\theta)} \underbrace{x}_{K(x)} + \underbrace{0}_{S(x)}]$$ is an exponential family

$\therefore Y = \sum_1^n K(X_i) = \sum_1^n X_i$ complete sufficient for θ.

Let $Z = \begin{cases} 1 & X_1 \le 2 \\ 0 & \text{otherwise} \end{cases}$. Then $\mathbb{E}(Z) = 1 \cdot \mathbb{P}(X_1 \le 2) + \cancel{0 \cdot \mathbb{P}(X_1 > 2)} = \mathbb{P}(X_1 \le 2)$

$\Rightarrow Z$ unbiased for $\mathbb{P}(X_1 \le 2)$.

By Rao-Blackwell Thm., $\mathbb{E}(Z|Y=y) = \mathbb{P}(X_1 \le 2 | Y = \sum_1^n X_i = y)$ where

$$f(x_1|y) = f(x_1 | \textstyle\sum_1^n X_i = y) = \frac{f(x_1, \sum_1^n x_i = y)}{f(\sum_1^n x_i = y)} = \frac{f(x_1, \sum_2^n x_i = y - x_1)}{f(\sum_1^n x_i = y)}$$

$$= \frac{f(x_1) f_{n-1}(y-x_1)}{f_n(y)}$$ by independence, where $\sum_2^n X_i \sim \text{Gamma}(n-1,\theta)$, $\sum_1^n X_i \sim \text{Gamma}(n,\theta)$

$$= \frac{\left(\frac{\cancel{1}}{\cancel{\theta}} \cancel{e^{-x_1/\theta}}\right)\left(\frac{1}{\Gamma(n-1)\cancel{\theta^{n-1}}} (y-x_1)^{(n-1)-1} \cancel{e^{-\frac{(y-x_1)}{\theta}}}\right)}{\frac{1}{\Gamma(n)\cancel{\theta^n}} y^{n-1} \cancel{e^{-y/\theta}}} = \frac{\Gamma(n)}{\Gamma(n-1)} \cdot \frac{(y-x_1)^{n-2}}{y^{n-1}}, \quad 0<x<y$$

$$= \frac{(n-1)\cancel{\Gamma(n-1)}}{\cancel{\Gamma(n-1)}} \cdot \frac{(y-x_1)^{n-2}}{y^{n-1}} = \frac{(n-1)(y-x_1)^{n-2}}{y^{n-1}}, \quad 0<x<y$$

Example 10 (cont.)

$$\mathbb{P}(X_1 \le 2 \mid y) = \int_0^2 f(x_1 \mid y)\,dx_1 = \int_0^2 \frac{(n-1)(y-x_1)^{n-2}}{y^{n-1}}\,dx_1$$

$$= \frac{1}{y^{n-1}} \int_0^2 (n-1)(y-x)^{n-2}\,dx_1$$

$$= \frac{1}{y^{n-1}} \left(-(y-x)^{n-1} \Big|_0^2 \right)$$

$$= \frac{-1}{y^{n-1}} \left[(y-2)^{n-1} - y^{n-1} \right]$$

$$= 1 - \left(\frac{y-2}{y} \right)^{n-1}$$

$\Rightarrow 1 - \left(\frac{\sum_1^n X_i - 2}{\sum_1^n X_i} \right)^{n-1}$ is UMVUE of $\mathbb{P}(X_1 \le 2)$ by Rao-Blackwell Thm.