MATH 503: Mathematical Statistics
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Homework 5

1. **Let $X_1, X_2, \ldots, X_n$ be a random sample from each of the following distributions involving the parameter θ. In each case, find the MLE of θ and show that it is a sufficient statistic for θ and hence a minimal sufficient statistic.**
   (a) **Binomial(1,θ), where $0 < \theta < 1$.**
   (b) **Poisson with mean $\theta > 0$.**
   (c) **Gamma with $\alpha = 3$ and $\beta = \theta > 0$.**
   (d) **N$(\theta, 1)$ where $-\infty < \theta < \infty$.**
   (e) **N$(0, \theta)$ where $0 < \theta < \infty$.**

2. **Let $Y_1 < Y_2 < Y_3 < Y_4$ denote the order statistics of a random sample of size $n = 4$ from a distribution having pdf $f(x;\theta) = \frac{1}{\theta}, 0 < x < \theta$, zero elsewhere, where $0 < \theta < \infty$. Argue that the complete sufficient statistic, Y_4 for θ, is independent of each of the statistics $\frac{Y_1}{Y_4}$ and $\frac{Y_1+Y_2}{Y_3+Y_4}$.**

3. **Let $Y_1 < \cdots < Y_n$ be the order statistics of a random sample from a N(θ, σ^2), $-\infty < \theta < \infty$, distribution. Show that the distribution of $Z = Y_n - \bar{X}$ does not depend on θ. Thus $\bar{Y} = \sum_{i=1}^n Y_i/n$, a complete sufficient statistic for θ, is independent of Z.**

4. **Let $X_1, X_2, \ldots, X_n$ be iid with the distribution N(θ, σ^2), $-\infty < \theta < \infty$. Prove that a necessary and sufficient condition that the statistics $Z = \sum_{i=1}^n a_i X_i$ and $Y = \sum_{i=1}^n X_i$, a complete sufficient statistic for θ, are independent is that $\sum_{i=1}^n a_i = 0$.**

5. **Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with pdf $f(x;\theta) = \frac{1}{2}\theta^3 x^2 e^{-\theta x}$, $0 < x < \infty$, zero elsewhere, where $0 < \theta < \infty$.**
   (a) **Find the MLE of θ. Is it unbiased? Hint: find the pdf of $Y = \sum_{i=1}^n X_i$ and then compute $E(\hat{\theta})$.**
   (b) **Argue that Y is a complete sufficient statistic for θ.**
   (c) **Find the UMVUE of θ.**
   (d) **Show that $\frac{X_1}{Y}$ and Y are independent.**
   (e) **What is the distribution of $\frac{X_1}{Y}$?**

6. **If $X_1, \ldots, X_N$ are iid Binomial(n, p) random variables, find the UMVUE of $\theta = p^n = P(X_1 = n)$.**