LECTURE 3 - 9/12/22

Course will focus on Python implementation rather than R.

Recall: Structure of OR problem: objective function, constraints

e.g. $\min c'x$

s.t. $Ax \le b \iff Ax = b$ ← can go back/forth if use slack/etc.

$x \ge 0$

- Feasible regions
- Solve LP graphically
- Standard form
- Equivalence
- Redundant Constraints
- Unboundedness/Infeasibility

Today: Turning problems into LP.

Farmer and the Salesperson

Alice works for a company selling feed blend Y to farmer Bob with 100 cattle. Diet minimum 100 units calcium, 1,500 units protein, 20,000 calories.

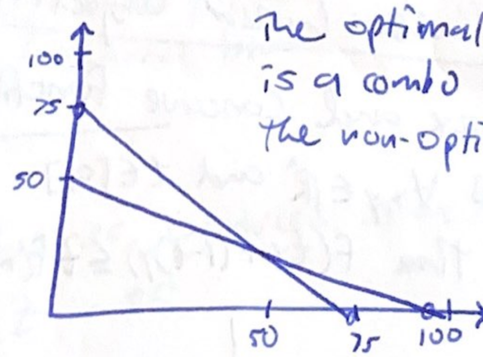
Currently use formula X costs \$0.015/oz providing 1 unit calcium, 400 calories, 20 units protein. Y costs \$0.02/oz. Should Bob buy Y rather than X.

Each cow gets 100 units of X ⇒ 100 calcium, 2000 protein, 40,000 cal.
\$150 cost for ranch per day.

Y isn't better alone, how about in combination?

LP: $\min 0.015x + 0.02y$

s.t.
calcium: $x + 2y \ge 100$
protein: $20x + 20y \ge 1500$
cals: $400x + 250y \ge 20000$
non-negativity: $x, y \ge 0$



The optimal sol'n is a combo with the non-optimal product

Linear Regression

Typically done using ordinary least squares wherein we minimize $\|y-\beta X\|^2$ → easy to solve, has good characteristics, but there's susceptibility to errors due to 2nd order term.

$\uparrow$ L_2 norm

Another option is Least Absolute Deviation, minimizing $|\varepsilon|$.

$|y-\beta X|$ ← L_1 Norm.

How can we represent this: ① $\min \sum |y_i - \beta x_i|$

$\quad$ s.t. $\beta \in (-\infty, \infty)$

Note that in this we have x not as decision variable but as observed data. We want to linearize this:

$$\begin{aligned} \min\ & \sum z_i \\ \text{s.t. }\ & y_i - \beta x_i \le z_i \\ & \beta x_i - y_i \le z_i \end{aligned}$$

Another formulation is ②: $\min \sum |y_i - \beta x_i|$ s.t. $\beta \in (-\infty, \infty)$ } Need to linearize again.

$$\Rightarrow \begin{aligned} \min\ & \sum e_i^+ + e_i^- \\ \text{s.t. }\ & y_i - \beta x_i = e_i^+ - e_i^- \\ & \beta \in (-\infty, \infty) \\ & e_i^\pm \ge 0 \end{aligned}$$

Piecewise Linear Objectives

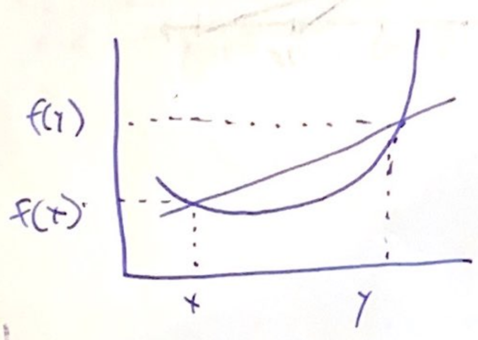
Convex and Concave functions:

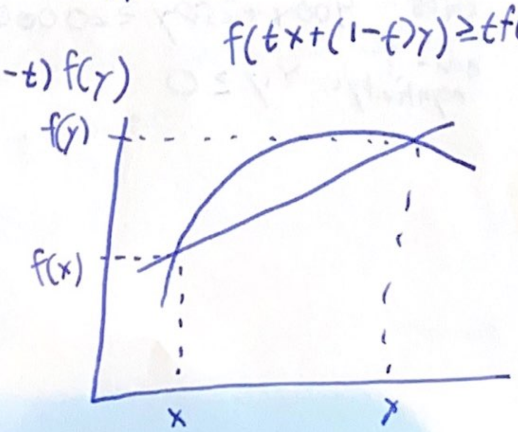
↳ $\forall x, y \in \mathbb{R}^n$ and $t \in [0,1]$

Then $f(tx + (1-t)y) \le t f(x) + (1-t) f(y)$

→ $\forall x, y \in \mathbb{R}^n$, $t \in [0,1]$, then

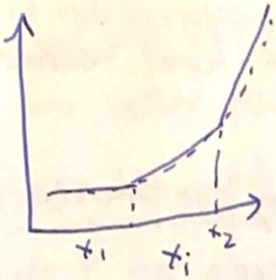
$f(tx + (1-t)y) \ge t f(x) + (1-t) f(y)$





Extreme Points

Can we try to use piecewise-linear approximations for convex cost functions?



→ can we expand the LP?

Given $\min c'x$
s.t. $Ax \le b, x \ge 0$.

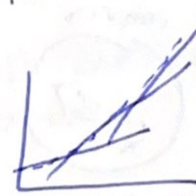
$$x_i \to x_{i,1} + x_{i,2} + x_{i,3}$$
$$c_i x_i \to c_{i,1} x_{i,1} + c_{i,2} x_{i,2} + c_{i,3} x_{i,3}$$
$$\text{s.t. } c_{i,1} < c_{i,2} < c_{i,3}$$
$$0 \le x_{i,1} \le x_1$$
$$0 \le x_{i,2} \le x_2 - x_1$$
$$0 \le x_{i,3}$$

} Break up into decision variables. We need monotonicity for this trick.

Alternatively, can use min max method for convex.

Let $f_1, f_2, \dots, f_m$ be linear functions. Then $\max(f_1, \dots, f_m)$ is convex.

Then consider $\min z$ s.t. $f_1(x) \le z$, $f_2(x) \le z$, etc.



We can also consider the max min method for concave.

Notice: minmax $\neq$ maxmin, in general.

Note: convex for min
concave for max } In general, best practice.

Case study:

Cancer is 2nd leading cause of death, 1/4 deaths.
In 2019, 1.76 million diagnosed. 60% cancer patients treated with radiation.
Radiation is most common form of external beam radiation Therapy.
X-Ray photon Proton is getting more common.

Formulation Problem: Galaxy Industries produces space rays R and zappers Z. R profit is \$8, Z is \$5.

$$\max\ 8r+5z$$
$$\text{s.t.}\quad r-z \le 350$$
$$r, z \ge 0$$

} Note we don't have a bound on r, z. So there isn't a solution yet. We observe the unboundedness due to likely ~~constraint~~ missing such a total addressable market.

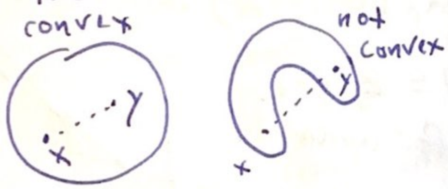
<u>Geometry</u>: The set $\{x \in \mathbb{R}^n \mid a'x = b\}$ is a <u>hyperplane</u>.

The set $\{x \in \mathbb{R}^n \mid a'x \le b\}$ is a <u>half space</u>.

<u>Convex Set</u>: S is convex if $\forall x, y \in S$, $t \in [0,1]$, then $(tx + (1-t)y) \in S$

Line, planes, hyperplanes, half spaces are convex sets. → If not empty.

The intersection of convex sets ~~of.~~ is convex. → Recall from Real Analysis / Topology.



<u>Polyhedron/Polytope</u>: $\{x \in \mathbb{R}^n \mid Ax \ge b\}$, A $m \times n$, $b \in \mathbb{R}^m$.

A 3D (or more) convex region with linear/faceted sides.

polygon / Polyhedron / Polytope
2D / 3D / 4+D.

<u>Convex Hull</u>: Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be non-negative scalars such that $\sum \lambda_i = 1$, then $\sum \lambda_i x_i$ is said to be a <u>convex combination</u> of the vectors $x_1, \dots, x_n$.

A <u>convex hull</u> is the set of all convex combinations of $x_1, \dots, x_n$.

<u>Binding Constraint</u>: If a vector x^* satisfies $a_i' x^* = b_i$ for some i in M_1, M_2, M_3, the constraint is binding.

Slack vars $= 0$.

So at faces of polyhedron.

Define polyhedron.

$$a_i' x^* \ge b_i,\ i \in M_1$$
$$\le b_i\ \ i \in M_2$$
$$= b_i\ \ i \in M_3$$

Extreme Points

Let P be a polyhedron. $x \in P$ is extreme point of P if we cannot find $y, z \in P$, $y, z \neq x$ such that $ty + (1-t)z = x$, $t \in [0,1]$.
x is an extreme point if it is not a convex combination of any two other points.

Basic Feasible Solutions.

Let P be a polyhedron defined by linear inequality and equality constraints and let $x^* \in \mathbb{R}^n$.
The vector x^* is a basic solution if:

- All equality constraints are active
- n linear independent of the above.

If x^* is a basic solution that satisfies all of the constraints we say it is a basic feasible solution.
If there is at least one basic feasible solution, either there is an optimal solution or the problem is unbounded.

Let P be a nonempty polyhedron. x^* in P. Then.

$$\left.\begin{array}{l} x^* \text{ vertex} \\ x^* \text{ extreme point} \\ x^* \text{ basic feasible solution.} \end{array}\right\} \text{these are equivalent.}$$

Degeneracy: A basic solution $x \in \mathbb{R}^n$ is degenerate if more than n of the constraints are active at x.

Important Thms:

THM: A nonempty and bounded polyhedron is the convex hull of its extreme points.

THM: A linear programming problem min $c'x$ over polyhedron P ($Ax \geq b$). If P has at least on extreme point, then the optimal cost is either $-\infty$ or there exists an extreme point that is optimal.

Next Time we will investigate the simplex method.
HW to be posted Wednesday, due ~10 day later Sat.
↳ Readings.