MATH 503: Mathematical Statistics

Lecture 10: Linear Regression

Reading: C&B Sections 11.3,12.1-12.2.4

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Today's Topics

- What's the point?
- Method of least squares
- Best linear unbiased estimators (BLUEs)
- Simple regression model assumptions
- Point estimation
- Sampling distributions
- Inference and testing

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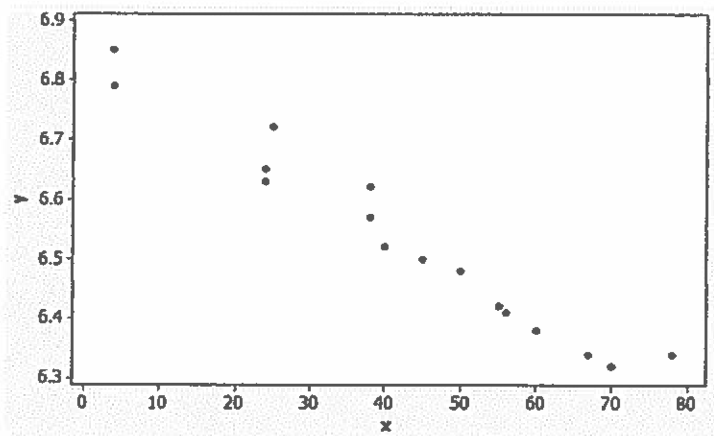
What's the point?

Given the values (x,y), we want to see if there is a relationship between X and Y.



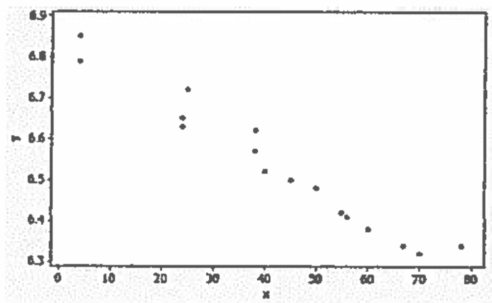
What's the point? (cont.)

- Simple (linear) regression refers to regression with one predictor variable
- "Linear" regression ⇒ linear in the parameters

- Which of the following are linear models?
  - ✓ $Y_i = \alpha + \beta x_i + \epsilon_i$
  - ✓ $\log(Y_i) = \alpha + \beta x_i^2 + \epsilon_i$
  - X $Y_i = \alpha + \beta^2 x_i + \epsilon_i$

    not linear in β

What's the point?



- For simple regression, we want to find a line $\hat{Y}_i = \hat{\alpha} + \hat{\beta}x_i$ that best describes the relationship displayed in the scatterplot.
- We may think of the value $\hat{Y}_i = \hat{\alpha} + \hat{\beta}x_i$ as predicting Y_i, and then define the ith residual as $r_i = Y_i - \hat{Y}_i = Y_i - (\hat{\alpha} + \hat{\beta}x_i)$. To judge the quality of the fit of the line, examine the r_i's.

Notation

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$S_{xx} = \sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$S_{yy} = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

$$S_{xy} = \sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})$$

Method of Least Squares

The method of least squares chooses the line that has the smallest residual sum of squares, $RSS = \sum_{i=1}^{n} r_i^2$

$$RSS = \sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} (Y_i - (\alpha + \beta x_i))^2 = \sum_{i=1}^{n} (Y_i - \alpha - \beta x_i)^2$$

$$\frac{\partial RSS}{\partial \alpha} = \cancel{2} \sum_{i=1}^{n} (Y_i - \alpha - \beta x_i)\cancel{(-1)} = 0$$

$$\sum_{i=1}^{n} Y_i - n\alpha - \beta \sum_{i=1}^{n} x_i = 0 \longrightarrow \boxed{\hat{\alpha} = \frac{\sum Y_i - \beta \sum x_i}{n} = \bar{Y} - \hat{\beta}\bar{x}}$$

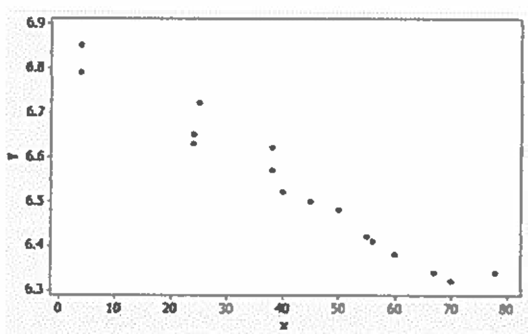
$$\frac{\partial RSS}{\partial \beta} = \cancel{2} \sum_{i=1}^{n} (Y_i - \alpha - \beta x_i)(\cancel{-}x_i) = 0$$

$$\sum x_i Y_i - \alpha \sum x_i - \beta \sum x_i^2 = 0$$

$$\sum x_i Y_i - \frac{1}{n}\left(\sum Y_i - \beta \sum x_i\right)\sum x_i - \beta \sum x_i^2 = 0$$

$$\hat{\beta} = \frac{\sum x_i Y_i - \frac{(\sum x_i)(\sum Y_i)}{n}}{\sum x_i^2 - \frac{(\sum x_i)^2}{n}} = \frac{S_{xy}}{S_{xx}}$$ (see scrap for details)

Why is the least squares approach reasonable?



- Least squares is only one way to fit lines, and it has good and bad properties
  - Good: easily computable and have some nice mathematical properties
  - Bad: heavily influenced by outliers

Show $S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n} x_i y_i - \frac{(\sum_1^n x_i)(\sum_1^n y_i)}{n}$

Pf | $S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$

$$= \sum_{i=1}^{n}(x_i y_i - \bar{x} y_i - x_i \bar{y} + \bar{x}\bar{y})$$

$$= \sum_{i=1}^{n} x_i y_i - \bar{x}\sum_{i=1}^{n} y_i - \bar{y}\sum_{i=1}^{n} x_i + n\bar{x}\bar{y}$$

$$= \sum_1^n x_i y_i - \left(\frac{\sum_1^n x_i}{n}\right)\sum_1^n y_i - \cancel{\left(\frac{\sum_1^n y_i}{n}\right)\sum_1^n x_i} \cancel{+ n\left(\frac{\sum x_i}{n}\right)\left(\frac{\sum y_i}{n}\right)}$$

$$= \sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n} \quad \checkmark$$

Show $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n}$

Pf | $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)$

$$= \sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_1^n x_i + n\bar{x}^2$$

$$= \sum_{i=1}^{n} x_i^2 - 2\bar{x}(n\bar{x}) + n\bar{x}^2$$

$$= \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$$

$$= \sum_{i=1}^{n} x_i^2 - \cancel{n}\left(\frac{\sum x_i}{\cancel{n}}\right)\left(\frac{\sum x_i}{n}\right)$$

$$= \sum_{i=1}^{n} x_i^2 - \frac{(\sum_1^n x_i)^2}{n} \quad \checkmark$$

Another Reasonable Approach

- Use horizontal distances instead of vertical distances
- The resulting line would be

$$x^* = a^* + b^*y$$

where $b^* = \frac{S_{xy}}{S_{yy}}$ and $a^* = \bar{x} - b^*\bar{y}$

- Re-expressing the line as a function of y on x implies

$$\hat{y} = \frac{-a^*}{b^*} + \frac{1}{b^*}x$$

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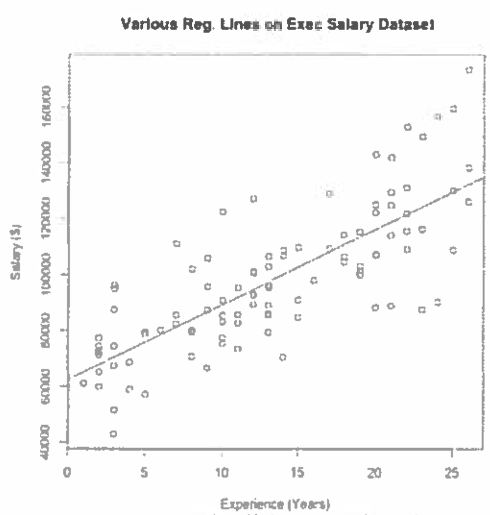
What's the difference?

- If the two lines were the same, then the slopes would be equal, i.e. $b/(1/b^*) = 1$
- In actuality,

$$b/(1/b^*) = bb^* = \frac{(S_{xy})^2}{S_{xx}S_{yy}} \le 1$$

- Problem when there is no distinction between predictor and response variables



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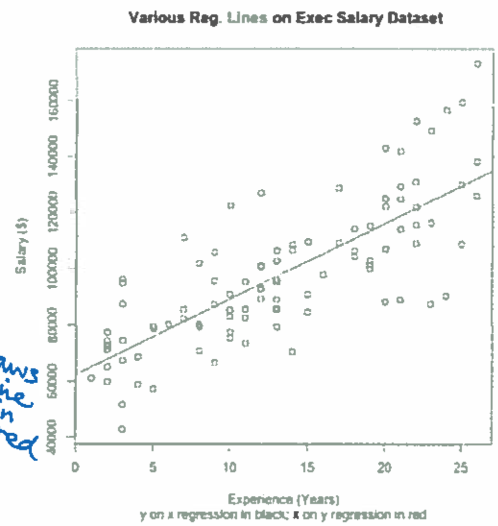
R code (if you were wondering)

In RStudio, can use "Import Dataset"; "txt"

```
> exec <- read.table("MATH 651/data/
   ExecSalary.txt")
> View(exec)
> plot(exec$Experience,
   exec$Salary,xlab="Experience
   (Years)", ylab="Salary ($)")
> exec.lm1 <- lm(Salary~Experience,
   data=exec)
> exec.lm2 <- lm(Experience~Salary,
   data=exec)
> abline(exec.lm1)
> abline(-exec.lm2$coefficients[1]/
   exec.lm2$coefficients[2],
   1/exec.lm2$coefficients[2],col=2)
> title(main="Various Reg. Lines on
   Exec Salary Dataset", sub="y on x
   regression in black; x on y
   regression in red")
```

draws line in red



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Best Linear Unbiased Estimators (BLUEs)

- Setup:
  - Assume x_i's known & fixed
  - y_i's observed values from uncorrelated rv's Y_i's
  - Consider model $Y_i = \alpha + \beta x_i + \epsilon_i$, where ϵ_i's uncorrelated rv's with E(ϵ_i)=0 and Var(ϵ_i)=σ^2 unknown
  - Goal: determine estimates for α, β
- Restrict choice of estimators to class of linear estimators (i.e. of the form $\sum_{i=1}^{n} d_i Y_i$ where d_i's known & fixed)

  "Unbiased" is self-explanatory

  "Best" refers to estimator with smallest variance

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Example

What specifications must be in place to satisfy a BLUE of β?

Estimator has the form $\sum_{i=1}^{n} d_i Y_i$ where

"Unbiased" $\Rightarrow \mathbb{E}\left(\sum_{i=1}^{n} d_i Y_i\right) = \sum_{i=1}^{n} d_i \mathbb{E}(Y)$

$= \sum_{i=1}^{n} d_i (\alpha + \beta x_i)$

$= \alpha \sum_{i=1}^{n} d_i + \beta \sum_{i=1}^{n} d_i x_i \stackrel{\cdot}{=} \beta$

ie. $\sum_{i=1}^{n} d_i = 0$ and $\sum_{i=1}^{n} d_i x_i = 1$

"Best" $\Rightarrow Var\left(\sum_{i=1}^{n} d_i Y_i\right)$ minimized, where

$Var\left(\sum_{i=1}^{n} d_i Y_i\right) = \sum_{i=1}^{n} d_i^2 Var(Y_i) = \sigma^2 \sum_{i=1}^{n} d_i^2$

$\therefore$ we want to minimize $\sum_{i=1}^{n} d_i^2$

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Result
(Casella & Berger, Lemma 11.2.7)

Let $(v_1, \dots, v_k)$ be constants and let $(c_1, \dots, c_k)$ be positive constants. Then, for

$A = \{\boldsymbol{a} = (a_1, \dots, a_k) : \sum_{i=1}^{k} a_i = 0\}$,

$$\max_{\boldsymbol{a} \in A} \left\{ \frac{\left(\sum_{i=1}^{k} a_i v_i\right)^2}{\sum_{i=1}^{k} a_i^2 / c_i} \right\} = \sum_{i=1}^{k} c_i (v_i - \bar{v}_c)^2,$$

where $\bar{v}_c = \frac{\sum_{i=1}^{k} c_i v_i}{\sum_{i=1}^{k} c_i}$. The maximum is attained at any $\boldsymbol{a}$ of the form $a_i = K c_i (v_i - \bar{v}_c)$ where K is a nonzero constant.

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What is the BLUE of β?

Using Lemma 11.2.7 ($k = n, v_i = x_i, c_i = 1, a_i = d_i$), d_i's maximize

$$\frac{(\sum_{i=1}^n d_i x_i)^2}{\sum_{i=1}^n d_i^2} = \frac{1}{\sum_{i=1}^n d_i^2} \Leftrightarrow \text{minimize} \sum_{i=1}^n d_i^2$$

Among all d_i's that satisfy $\underbrace{\sum_{i=1}^n d_i = 0}_{\text{by the BLUE constraint}}$, assuming d_i has the form

$$d_i = Kc_i(v_i - \bar{v}_c) = K(x_i - \bar{x}), \qquad i = 1, \dots, n$$

Thus, because $d_i = K(x_i - \bar{x})$

$$\underbrace{1 \doteq}_{\text{by the constraint}} \sum_{i=1}^n d_i x_i = \sum_{i=1}^n K(x_i - \bar{x})x_i = KS_{xx} \therefore K = \frac{1}{S_{xx}}$$

$$\Rightarrow d_i = K(x_i - \bar{x}) = \frac{x_i - \bar{x}}{S_{xx}} \text{ and } \sum d_i Y_i = \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{S_{xx}}\right) Y_i = \frac{S_{xy}}{S_{xx}}$$

Note:
$$S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x}) = \sum_{i=1}^n (x_i - \bar{x})x_i - \bar{x}\underbrace{\sum_{i=1}^n (x_i - \bar{x})}_{0} = \sum_{i=1}^n (x_i - \bar{x})x_i$$

BLUE Results

- $b = \frac{S_{xy}}{S_{xx}}$ is the BLUE of β.
- $\text{Var}(b) = \sigma^2 \sum_{i=1}^n d_i^2 = \frac{\sigma^2}{S_{xx}} = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2}$
- Similar analysis used to determine BLUE for α $\quad \mathbb{E}\left(\sum_{i=1}^n d_i Y_i\right) = \alpha \sum_{i=1}^n d_i + \beta \sum_{i=1}^n d_i x_i \doteq \alpha \Rightarrow \sum_{i=1}^n d_i = 1$ and $\sum_{i=1}^n d_i x_i = 0$
- Constants $d_1, \dots, d_n$ must satisfy

$$\sum_{i=1}^n d_i = 1 \quad \text{and} \quad \sum_{i=1}^n d_i x_i = 0$$

Model & Distribution Assumptions

- Conditional normal model:
  1. x_is known and fixed; y_is observed from Y_is
  2. $Y_i = \alpha + \beta x_i + \epsilon_i;\ i = 1, \dots, n$ holds (linearity of the model)
  3. $\epsilon_i \sim \mathrm{N}(0, \sigma^2)$ iid

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Model & Distribution Assumptions (cont.)

- Bivariate normal model:
  1. x_is can be observed from X_is; y_is observed from Y_is
  2. $(X_i, Y_i) \sim$ BivariateNormal$\left(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho\right)$
  3. $E(Y \mid x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_x}(x - \mu_x)$

     $$= \left(\mu_Y - \rho \frac{\sigma_Y}{\sigma_x}\mu_x\right) + \left(\rho \frac{\sigma_Y}{\sigma_x}\right)x$$
  4. $\mathrm{Var}(Y \mid x) = \sigma_Y^2(1 - \rho^2)$

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Point Estimation

- Inference based on point estimators, intervals, tests same for both models
- Determine MLEs for α, β, σ^2 under conditional normal model:

$$Y_i \sim \mathrm{N}(\alpha + \beta x_i, \sigma^2), \qquad i = 1, \dots, n,$$

i.e.

$$Y_i = \alpha + \beta x_i + \epsilon_i, \qquad i = 1, \dots, n$$

where $\epsilon_i \sim \mathrm{N}(0, \sigma^2)$

SEE SCRAP

Point Estimation (cont.)

- $\hat{\alpha}, \hat{\beta}$ BLUEs for $\alpha, \beta \Rightarrow$ both are unbiased
- $\widehat{\sigma^2} = \frac{1}{n} RSS$ biased for σ^2 because

$$E\left(\widehat{\sigma^2}\right) = \frac{n-2}{n} \sigma^2$$

- What is an unbiased estimator for σ^2?

$$\sigma^2 = \mathbb{E}\left(\frac{n}{n-2} \widehat{\sigma^2}\right) = \mathbb{E}\left(\frac{\cancel{n}}{n-2} \cdot \frac{RSS}{\cancel{n}}\right) = \mathbb{E}\left(\frac{RSS}{n-2}\right)$$

$\therefore \frac{RSS}{n-2}$ is unbiased estimator for σ^2

$Y_i \sim N(\alpha + \beta x_i, \sigma^2)$ where α, β, σ^2 unknown

$$f(y_i ; \underbrace{\alpha+\beta x_i}_{\mu_i}, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(y_i - \alpha - \beta x_i)^2}$$

$$\mathcal{L}(\alpha, \beta, \sigma^2 ; y) = (2\pi\sigma^2)^{-n/2} \exp\left[\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2\right]$$

$$\log \mathcal{L}(\alpha, \beta, \sigma^2 ; y) = \frac{-n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$$

$$\frac{\partial \log \mathcal{L}(\alpha, \beta, \sigma^2)}{\partial \alpha} = \cancel{\frac{-1}{\sigma^2}} \sum (y_i - \alpha - \beta x_i)\cancel{(-1)} = 0$$

$$\sum y_i - n\alpha - \beta \sum x_i = 0 \quad \therefore \boxed{\hat{\alpha} = \frac{\sum y_i - \beta \sum x_i}{n} = \bar{Y} - \beta \bar{X}}$$

$$\frac{\partial \log \mathcal{L}(\alpha, \beta, \sigma^2)}{\partial \beta} = \frac{-1}{\sigma^2} \sum (y_i - \alpha - \beta x_i)(-x_i) = 0$$

$$\sum x_i y_i - \alpha \sum x_i - \beta \sum x_i^2 = 0$$

$$\sum x_i y_i - \left(\frac{\sum y_i}{n} - \beta \frac{\sum x_i}{n}\right) \sum x_i - \beta \sum x_i^2 = 0$$

$$\sum x_i y_i - \frac{\sum x_i \sum y_i}{n} - \beta \left(\sum x_i^2 - \frac{(\sum x_i)^2}{n}\right) = 0$$

$$\boxed{\therefore \hat{\beta} = \frac{\sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}}{\sum x_i^2 - \frac{(\sum x_i)^2}{n}} = \frac{S_{xy}}{S_{xx}}}$$

$$\frac{\partial \log \mathcal{L}(\alpha, \beta, \sigma^2)}{\partial \sigma^2} = \frac{-n}{2}\left(\frac{\cancel{2\pi}}{\cancel{2\pi}\sigma^2}\right) + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2 = 0$$

$$2\sigma^4 \left(\frac{-n}{2\sigma^2} + \frac{\sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2}{2\sigma^4}\right) = 0\,(2\sigma^4)$$

$$-n\sigma^2 + \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2 = 0$$

$$\boxed{\therefore \hat{\sigma^2} = \frac{\sum_{i=1}^{n} (y_i - \hat{\alpha} - \hat{\beta} x_i)^2}{n} = \frac{RSS}{n}}$$

Summarizing the extent to which the line fits the data: s

- Error standard deviation, σ, represents average size of the error
- σ tells how far off, on average, we expect line to be in predicting a value y at any given x_i
- Estimated by $s = \sqrt{s^2}$ where

$$s^2 = \frac{1}{n-2}\sum_{i=1}^{n}\left(y_i - (\hat{\alpha} + \hat{\beta}x_i)\right)^2 = \frac{RSS}{n-2}$$

  called the "residual mean squared error"
- Thought of as the standard deviation of the residuals
- Provides summary of the average deviation of Y_i values from the corresponding values predicted by the line
- Has the same units as Y

Sampling Distributions Theorem

Under conditional normal regression model, sampling distributions of $\hat{\alpha}, \hat{\beta}$, and S^2 are

$$\hat{\alpha} \sim \mathrm{N}\left(\alpha, \frac{\sigma^2}{nS_{xx}}\sum_{i=1}^{n} x_i^2\right)$$

$$\hat{\beta} \sim \mathrm{N}\left(\beta, \frac{\sigma^2}{S_{xx}}\right)$$

with $\mathrm{Cov}(\hat{\alpha}, \hat{\beta}) = \frac{-\sigma^2\bar{x}}{S_{xx}}$. Further, $(\hat{\alpha}, \hat{\beta})$ and S^2 are independent and $\frac{(n-2)S^2}{\sigma^2} \sim \chi^2_{n-2}$.

Inference Results

$$\frac{\hat{\alpha} - \alpha}{S\sqrt{\left(\sum_{i=1}^{n} x_i^2\right)/(nS_{xx})}} \sim t_{n-2}$$

and

$$\frac{\hat{\beta} - \beta}{S/\sqrt{S_{xx}}} \sim t_{n-2}$$

This serves as the basis for determining CIs, decision rules for hypothesis tests!

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Confidence Intervals for Slope

- To compute the $100(1-\alpha)\%$ CI, use

$$\hat{\beta} \pm z_{\alpha/2} \cdot \frac{S}{\sqrt{S_{xx}}}$$

- For small samples, substitute $t_{\alpha/2,n-2}$ for $z_{\alpha/2}$. Thus, we use

$$\hat{\beta} \pm t_{\alpha/2,n-2} \cdot \frac{S}{\sqrt{S_{xx}}}$$

as $100(1-\alpha)\%$ CI for β.

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Model Utility Test (t-test)

- Understanding the association (increasing or decreasing tendency) between two variables can be essential in analyses
  - Assume that y is approximately linear in x
  - Consider the possibility that the slope of the line is zero, i.e. $H_0: \beta = 0$ vs. $H_1: \beta \neq 0$

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Testing Approaches

- There are three approaches to solve this hypothesis test:
  - Find $100(1-\alpha)\%$ CI for β: $\hat{\beta} \pm t_{\alpha/2,n-2} \cdot \frac{S}{\sqrt{S_{xx}}}$
  - Using p-value or rejection region method associated with t-statistic,
    $$t = \frac{\hat{\beta}}{S/\sqrt{S_{xx}}}$$
    and t-distribution with $n-2$ degrees of freedom
  - Use ANOVA with $F_{1,n-2}(\alpha)$: $\left(\frac{\hat{\beta}}{S/\sqrt{S_{xx}}}\right)^2 = \frac{\hat{\beta}^2}{S^2/S_{xx}} > F_{1,n-2}(\alpha)$

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Simple Regression ANOVA Table

Source	df	Sum of Squares	Mean square	F statistic
Regression (slope)	1	$\text{SS(Reg)} = S_{xy}^2/S_{xx}$	$\text{MS(Reg)} = S_{xy}^2/S_{xx}$	$\text{F} = \frac{\text{MS(Reg)}}{\text{MSE}}$
Residual	$n-2$	$\text{SSE} = \sum_{i=1}^n \hat{\epsilon}^2$	$\text{MSE} = \frac{\text{SSE}}{n-2}$	
Total	$n-1$	$\text{SST} = \sum_{i=1}^n (y_i - \bar{y})^2$		

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Summarizing the extent to which the line fits the data: R^2

- R^2 interpreted as fraction of variability in Y attributable to the regression (i.e. proportion of variability in Y explained by X);

$$R^2 = 1 - \frac{\text{SSE}}{\text{SST}} = \frac{\text{SSReg}}{\text{SST}} = \frac{\sum_{i=1}^n (\hat{y}_i - \bar{y})^2}{\sum_{i=1}^n (y_i - \bar{y})^2} = \frac{S_{xy}^2}{S_{xx}S_{yy}}$$

where SSE = "sum of squares due to error" = s^2, and SST = "total sum of squares" = $\sum_{i=1}^n (y_i - \bar{y})^2$

- $\frac{\text{SSE}}{\text{SST}}$ is proportion of variability in Y attributable to error
- Interpreted as "proportion of variability of Y explained by X"

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Coefficient of Determination (cont.)

- $0 \leq R^2 \leq 1$
- R^2 is dimensionless (no reference units)
- No universal rule as to what constitutes a "large R^2"

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Example (and R code)

The prevalence of respiratory symptoms was recorded for 9 groups of subjects exposed to differing levels of dust in their work environment. Dust exposure was measured as particules/ft^3/year scaled by 10^6. The direct outcome variable is "relative risk", the ratio of symptom prevalence at a given exposure level to symptom prevalence in the absence of workplace dust.

```
> dust <-
  data.frame(exposure=c(75,100,150,350,600,900,1300,1650,2250),
  RR=c(1.10,1.05,0.97,1.9,1.83,2.45,3.70,3.52,4.16))
> summary(lm(RR ~ exposure,data=dust))
```

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R Output

```
Call:
lm(formula = RR ~ exposure, data = dust)

Residuals:
    Min       1Q   Median      3Q      Max
-0.34055 -0.13997 -0.05667  0.02818  0.66226

Coefficients:
              Estimate   Std. Error  t value  Pr(>|t|)
(Intercept)  1.0359939  0.1688447   6.136   0.000474 ***
exposure     0.0015398  0.0001541   9.993   2.15e-05 ***
—
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3363 on 7 degrees of freedom
Multiple R-Squared: 0.9345,     Adjusted R-squared: 0.9251
F-statistic: 99.85 on 1 and 7 DF,  p-value: 2.150e-05
```

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SAS Code

```
data symptoms;
 input exposure RR;
 cards;
75 1.10
100 1.05
150 0.97
350 1.9
600 1.83
900 2.45
1300 3.70
1650 3.52
2250 4.16
;
proc print data=symptoms; run;

proc glm data=symptoms;
 model RR=exposure;
 ;
 run;
```

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SAS Output

The GLM Procedure
Dependent Variable: RR

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	1	11.29121174	11.29121174	99.85	<.0001
Error	7	0.79154382	0.11307769		
Corrected Total	8	12.08275556			

R-Square	Coeff Var	Root MSE	RR Mean
0.934490	14.63459	0.336270	2.297778

Source	DF	Type I SS	Mean Square	F Value	Pr > F
exposure	1	11.29121174	11.29121174	99.85	<.0001

Source	DF	Type III SS	Mean Square	F Value	Pr > F
exposure	1	11.29121174	11.29121174	99.85	<.0001

Parameter	Estimate	Standard Error	t Value	Pr > |t|
Intercept	1.035993934	0.16884466	6.14	0.0005
exposure	0.001539804	0.00015409	9.99	<.0001

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