Lecture 1: Vectors and Matrices

Matrices have many applications in diverse fields of science, commerce and social science. Matrices are used in

- Computer graphics
- Optics
- Cryptography
- Economics
- Robotics and animation
- Wireless communication and signal processing

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Lecture 1

1. Part 1: Vectors
   - Linear vector space
   - Vectors and their basic properties
   - Vector norms
   - Inner products
   - Cauchy-Schwarz inequality
   - Triangle inequality, reverse triangle inequality
2. Part 2: Matrices

Linear (Vector) Sapce

Definition

Let a set V be given and let the operations of additions $(+)$ and scalar multiplication (.) be defined on V such that for any $x, y \in V$ and any scalar α we have $x + y \in V$ and $\alpha x \in V$ (In other word, V be closed under $+$ and .). The set V is a linear space if for any $x, y, z \in V$ and any scalars α, β the following are satisfied

- $x + y = y + x$ (commutativity)
- $(x + y) + z = x + (y + z)$ (associativity)
- $\alpha(x + y) = \alpha x + \alpha y$ (distributive)
- $(\alpha + \beta)x = \alpha x + \beta x$ (distributive)
- $(\alpha\beta)x = \alpha(\beta x)$ (compatibility)
- $\exists 0 \in V$ s.t. $x + 0 = x$ (additive Identity)
- $\exists -x \in V$ s.t. $x + (-x) = 0$ (inverse element of addition)
- $1.x = x$ (multiplicative Identity)

The elements of V called vectors.

Definition

A real n dimensional vector is an *ordered set* of n real numbers $\{x_1, x_2, \dots, x_n\}$ and it us usually written in the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{(column vector)}$$

or

$$x = [x_1, x_2, \dots, x_n] \qquad \text{(row vector)}$$

We will also write

$$x = [x_i]_{i=1}^n.$$

The numbers $x_1, x_2, \dots, x_n$ are called the components of x.

For the sake of clarity, by a vector we will mean a column vector. For a vector, the corresponding row vector will be denoted by x^T.

$\mathbb{R}^n$

Definition

The set $\mathbb{R}^n$ consists of all n-tuples of real numbers, for any real number n, It is called the "$n-$dimensional real coordinate space".

Example.

$$[1, 2, 0, 0]^T \in \mathbb{R}^4$$

$$[10] = 10 \in \mathbb{R}^1 = \mathbb{R}$$

Sum of Vectors; Scalar Multiplication

Definition

Given vectors

$$x = [x_i]_{i=1}^n, \quad y = [y_i]_{i=1}^n.$$

their sum is defined by

$$x + y = [x_i + y_i]_{i=1}^n.$$

For a vector x and the sacalar α, the sacalar multiplication αx is defined by

$$\alpha x = [\alpha x_i]_{i=1}^n$$

Example. $x = [1, 2, 3]^T, y = [4, 5, 6]^T$ and $\alpha = 2$ we have

$$x + y = [1 + 4, 2 + 5, 3 + 6]^T = [5, 7, 9]^T;$$

and

$$\alpha x = [2 \cdot 1, 2 \cdot 2, 2 \cdot 3]^T = [2, 4, 6]^T$$

Definition

Given vectors $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ in V, a sum in the form

$$c_1 v^{(1)} + c_2 v^{(2)} + \cdots + c_k v^{(k)}$$

where $c_1, c_2, \ldots, c_k$ are scalars, is called a <u>linear combination</u> of $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$.

Definition

The set of <u>ALL linear combinations</u> of $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ is called the span of $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ and denoted by

$$\mathrm{span}(\mathrm{v}^{(1)}, \mathrm{v}^{(2)}, \ldots, \mathrm{v}^{(\mathrm{k})}).$$

Example. What's the span of $v^{(1)} = [1, 2, 3]^T$ and $v^{(2)} = [2, 4, 6]^T$?

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Definition

Vectors $v^{(1)}, v^{(2)}, \dots, v^{(k)} \in V$ are called linearly independent if

$$c_1 v^{(1)} + c_2 v^{(2)} + \cdots + c_k v^{(k)} = 0$$

implies that

$$c_1 = \cdots = c_k = 0.$$

Otherwise, the vectors are linearly dependent.

Example. Show that the vectors

$$v^{(1)} = [1, 2]^T \quad v^{(2)} = [1, 3]^T$$

are linearly independent, and the vectors

$$v^{(1)} = [0, 2]^T \quad v^{(2)} = [0, 3]^T$$

are linearly dependent.

Definition

The vectors $v^{(1)}, v^{(2)}, \ldots, v^{(k)} \in V$ form a basis of the space V if and only if

- $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ are linearly independent
- $\mathrm{span}(\mathrm{v}^{(1)}, \mathrm{v}^{(2)}, \ldots, \mathrm{v}^{(\mathrm{k})}) = \mathrm{V}$

Example. The vectors $v^{(1)} = [1, 0]^T$ and $v^{(2)} = [0, 1]^T$ form a basis for $\mathbb{R}^2$.

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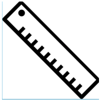
Example. The vectors $v^{(1)} = [1, 0]^T$ and $v^{(2)} = [0, 1]^T$ form a basis for $\mathbb{R}^2$.

Theorem

The following statements are valid

- *If $V = \mathrm{span}(v^{(1)}, v^{(2)}, \ldots, v^{(k)})$, then any set of $m > k$ vectors in V is linearly dependent.*
- *Any two bases $(v^{(1)}, v^{(2)}, \ldots, v^{(k)})$ and $(u^{(1)}, u^{(2)}, \ldots, u^{(m)})$ of V contain equal number of vectors, $m = k$. This number is called the dimension of V.*

Vector Norms (General Definition)



Definition

A function $\|\cdot\| : V \to \mathbb{R}^+ := [0, +\infty)$ is called a norm in V if for any $x, y \in V$ and any scalar α the following hold

- $\|x\| \geq 0$ with equality if and only if $x = 0$
- $\|\alpha x\| = |\alpha|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

- Assigns a positive number to each non-zero vector
- Is only zero if the vector is an all-zero vector

Definition

A vector space equipped with a norm $\|x\|$, denoted by $(V, \|\cdot\|)$ is called a <u>normed linear space</u>.

ℓ_p Norms, $p \in \mathbb{R}_+$

Definition (ℓ_p norm)

A p-norm of a vector $x = [x_i]_{i=1}^n \in \mathbb{R}^n$ is defined by

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

for any real $p \geq 1$.

- ℓ_2 norm: $p = 2, \|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$
- ℓ_1 norm: $p = 1, \|x\|_1 = \sum_{i=1}^n |x_i|$
- ℓ_∞ norm: $p = \infty, \|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$

Lemma (Minkowski's inequality)

$$1 \leq p \leq \infty, \quad a, b \in V, \quad \|\mathrm{a} + \mathrm{b}\|_p \leq \|\mathrm{a}\|_p + \|\mathrm{b}\|_p$$

Find $\ell_2, \ell_1, \ell_\infty$ for

$$x = [1, 0, -2]^T$$

ϵ balls with different ℓ_p norms

Definition (ℓ_p ball)

$$\epsilon \geq 0, \quad B_{\ell_p}(\epsilon) = B_p(\epsilon) = \{a \in \mathbb{R}^n \mid \|a\|_p \leq \epsilon\}$$

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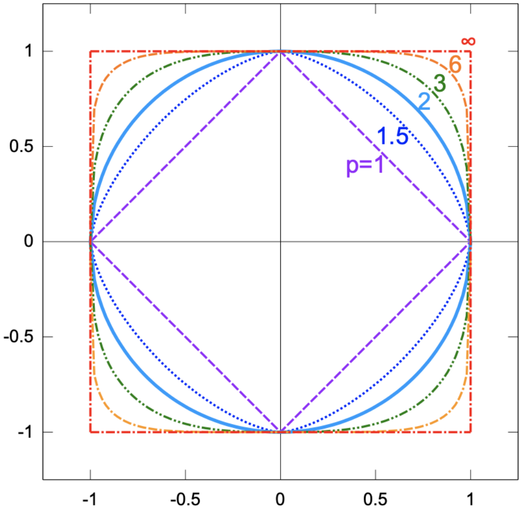
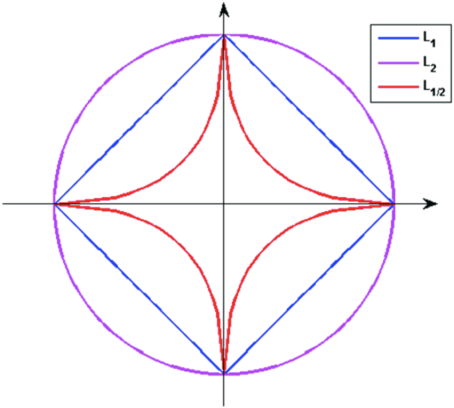
Example. Describe unit balls ($\epsilon = 1$)for different norms $\ell_1, \ell_\infty, \ell_2$ in $\mathbb{R}^2$.

Lemma (General equivalence of ℓ_p norms)

$$1 \leq p < q, \quad \|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q$$



ℓ_p with $p < 1$



Inner Product or Dot Product

Definition

(1)The inner product of two vectors $x, y \in \mathbb{R}^n$ is defined by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i = x^T y = y^T x = \langle y, x\rangle \in \mathbb{R} \qquad \text{algebraic representation}$$

(2) If θ be the angle between two vectors $x, y \in \mathbb{R}^n$, then the inner product of x and y is defined by

$$\langle x, y\rangle = \|x\|_2 \cdot \|y\|_2 \cos\theta \qquad \text{geometric representation}$$

$$\langle [1, 0, -1]^T, [2, 1, 3]^T\rangle = (1)(2) + (0)(1) + (-1)(3) = 2 + 0 - 3 = -1$$

Notes on Inner Product

- (Connection with ℓ_2 norm)

$$\langle x, x \rangle = \sum_{i=1}^{n} x_i^2 = \|x\|_2^2$$

- (Orthogonality) If $\langle x, y \rangle = 0$, then x, y are orthogonal to each other.
- (Different expression for equations)

$$ax^2 + bx + c = \langle [a, b, c], [x^2, x, 1] \rangle$$

- $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ called the Euclidean n-space.

Cauchy-Schwarz inequality

Lemma (Cauchy-Schwarz inequality)

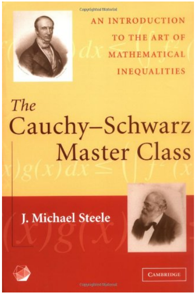
Given $x, y \in \mathbb{R}^n$,

$$|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2$$

or, equivalently

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \sqrt{\sum_{i=1}^{n} |x_i|^2} \sqrt{\sum_{i=1}^{n} |y_i|^2}$$

- Probably the most important inequality out there!
- There is a book solely devoted to this inequality.
- Is used to prove the triangle inequality (up coming)



Generalization of the Cauchy-Schwarz inequality

Definition (Hölder's inequality)

$$|\langle x, y\rangle| \leq \|x\|_p \|y\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1,\ p, q \in [1, \infty]$$

or, equivalently

$$\Big|\sum_{i=1}^{n} x_i y_i\Big| \leq \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} |y_i|^q\Big)^{\frac{1}{q}}$$

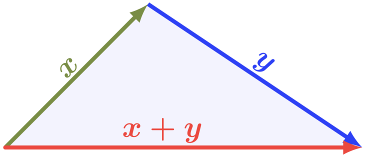
- Is used to prove the Minkowski inequality, generalization of the triangle inequality

Triangle inequality

Lemma (Triangle inequality)

Given $x, y \in \mathbb{R}^n$,

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2.$$



- When does this inequality hold with equality?

Matrics

- Matrices, and their basic properties
- Matrix norms
- Eigenvalues and eigenvectors
- Eigenvalue decomposition
- Spectral Theorem
- Matrix decomposition

Matrices

Definition

A <u>real matrix</u> is a rectangular array of real numbers composed of rows and columns. We write

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

for a matrix of m rows and n columns.

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for a matrix of m rows and n columns.

- If $m = 1$, then A becomes a row vector
- If $n = 1$, then A becomes a column vector
- Given $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$,
  $$A = B \quad \Longleftrightarrow \quad a_{ij} = b_{ij}, \quad \forall i, j$$
  $$A + B = [a_{ij} + b_{ij}]_{m\times n}.$$

Identity Matrix

Definition

(Identity Matrix)

$$I_n = [\delta_{ij}]_{n \times n}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $IA = AI = A$ (We often denote $I := I_n$)

Triangular Matrices

Definition

A matrix $A = [a_{ij}]_{n \times n}$ is called

- (Upper triangular) if $a_{ij} = 0$ for all $i > j$
- (Lower triangular) if $a_{ij} = 0$ for all $i < j$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ -1 & -2 & 6 \end{bmatrix}$$

A is both upper triangular and lower triangular iff A is diagonal ($a_{ij} = 0$ for all $i \neq j$)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Determinants of square matrices

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

- If $n = 1$, $det(A) = a_{11}$
- If $n = 2$, $det(A) = a_{11}a_{22} - a_{12}a_{21}$
- If $n \geq 2$, choose the row i

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} \; a_{ij} \; det(A_{ij}),$$

where A_{ij} is the matrix obtained from A by removing its ith and jth column.

Example. Find the determinant of

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Matrix Transpose

Definition

(Matrix Transpose)

$$A = [a_{ij}]_{m \times n} \quad A^T = [a_{ji}]_{n \times m}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A+B)^T = A^T + B^T$
- If $A = A^T$, then A is called a **Symmetic Matrix**

Matrix norms

The function

$$\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}^{+}$$

is a norm if for given $\alpha \in \mathbb{R}$, $A, B \in \mathbb{R}^{m\times n}$ it holds

- $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = [0]_{m,n}$
- $\|\alpha A\| = |\alpha|\|A\|$
- $\|A + B\| \leq \|A\| + \|B\|$

When $m = n$, submultiplicative property holds

Lemma

$$\|AB\| \leq \|A\|\|B\|$$

Matrix Norms induced by Vector Norms

Definition

$$\|A\| = \max\{\|Ax\| : x \in \mathbb{R}^n \quad \|x\| = 1\} = \max_{\|x\|=1} \quad \|Ax\|$$

or equivalently

$$\|A\| = \max\{\frac{\|Ax\|}{\|x\|} : x \in \mathbb{R}^n \quad x \neq 0\}$$

$\|A\|$ = maximum magnitude of a unit ball after transformation by A In our course, we assume that the norm on Ax and the vector x are the same, and equal to the norm imposed on A. Although, they can be also different.

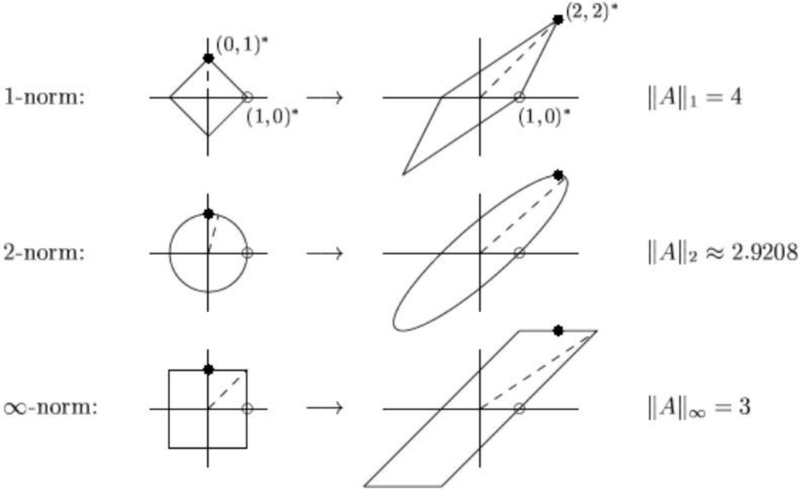
Definition

(p norms)

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_p$$

Geometric Interpretation

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$



Matrix p-norms

- $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$ (maximum of the absolute column sums)
- $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$ (the maximum singular value of A)
- $\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$ (maximum of the absolute row sums)

Double check for the previous example.

- Frobenius norm:
  - Definition: $\|A\|_F = \sqrt{\sum_{i,j} a_{i,j}^2}$
  - Alternative definition: $\|A\|_F = \sqrt{\mathrm{trace}(A^T A)} = \sqrt{\mathrm{trace}(AA^T)}$
  - trace of a matrix is obtained by summing all the diagonal elements

Find $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$, and $\|A\|_F$

$$A = \begin{bmatrix} 1 & -6 \\ 3 & 2 \end{bmatrix}$$

Find $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$, and $\|A\|_F$

$$A = \left[\begin{array}{cc} 1 & -6 \\ 3 & 2 \end{array} \right]$$

$$\|A\|_1 = \max\{1+3, 6+2\} = 8$$

$$\|A\|_2 = \sqrt{\lambda_{\max} \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix}} = \sqrt{40} = 2\sqrt{10}$$

$$\|A\|_\infty = \max\{1+6, 3+2\} = 7$$

$$\|A\|_F = \sqrt{1+36+9+4} = \sqrt{50}$$

Eigenvectors and eigenvalues

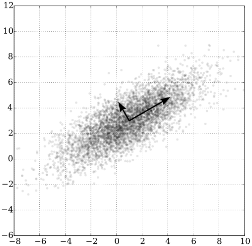
- Let $A \in \mathbb{R}^{n \times n}$ square matrix
- x is an eigenvector and λ is an eigenvalue of A if

$$A\mathsf{x} = \lambda \mathsf{x}$$

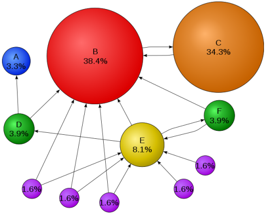
- **Intuition:** eigenvectors are vectors in $\mathbb{R}^n$ whose direction is preserved under action of A; however, length may change.

Applications

Dimensionality Reduction/PCA. The principal components correspond the the largest eigenvalues of $A^T A$, and this yields the least squared projection onto a smaller dimensional hyperplane, and the eigenvectors become the axes of the hyperplane. Dimensionality reduction is extremely useful in machine learning and data analysis as it allows one to understand where most of the variation in the data comes from.



The Google Page Rank algorithm. The largest eigenvector of the graph of the internet is how the pages are ranked.



The eigenvalues of a matrix A are the roots of the characteristic polynomial

$$p(\lambda) = det(A - \lambda I_n)$$

where I_n is an identity matrix.
Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 5-\lambda \end{bmatrix} = (1-\lambda)(5-\lambda) = 0 \qquad \lambda_1 = 1, \lambda_2 = 5$$

To find the eigenvector: find v such that $(A - \lambda I)v = 0$

$$\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, $v_2^{(1)} = 0$, choose $v^{(1)} = [1, 0]^T$. Similarly, $v^{(2)} = [0, 1]^T$.

Properties

- The **determinant** of any matrix is equal to the **product of its eigenvalues**
- The **trace** of any matrix is equal to the **sum of its eigenvalues**
- The eigenvalues of a nonsingular matrix A are nonzero, and the eigenvalues if A^{-1} are reciprocals of the eigenvalues of A.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \quad \lambda_1 = \lambda_2 = 1, \lambda_3 = 5$$

$$\det(A) = 1 \cdot 1 \cdot 5 = 5 \quad \mathit{trace}(A) = 1 + 1 + 5 = 7$$

Hilbert Matrix

$$H_{i,j} = \frac{1}{i+j-1}$$

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}.$$

What are the properties of Hilbert matrix?

Hilbert Matrix

$$H_{i,j} = \frac{1}{i+j-1}$$

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}.$$

What are the properties of Hilbert matrix?

- Symmetric
- positive definite (all its eigenvalues are positive)
- totally positive (every submatrix has positive determinant).
- Its condition number grows rapidly with n; indeed for the 2-norm the asymptotic growth rate is $\mathrm{e}^{3.5n}$.
- H = hilb(n), or j = 1:n; H = 1./(j'+j-1), n: size of matrix

Lehmer Matrix

$$A_{i,j} = \frac{\min(i,j)}{\max(i,j)} = \begin{cases} i/j & j \geq i \\ j/i & i > j \end{cases}$$

$$A_2 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}; \qquad A_2^{-1} = \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & \mathbf{4/3} \end{pmatrix};$$

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$$A_4 = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1 & 2/3 & 1/2 \\ 1/3 & 2/3 & 1 & 3/4 \\ 1/4 & 1/2 & 3/4 & 1 \end{pmatrix}; \quad A_4^{-1} = \begin{pmatrix} 4/3 & -2/3 & & \\ -2/3 & 32/15 & -6/5 & \\ & -6/5 & 108/35 & -12/7 \\ & & -12/7 & \mathbf{16/7} \end{pmatrix}.$$

Lehmer Matrix

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What are the properties of Lehmer matrix?

- $A_{n\times n}$ is a submatrix of $A_{m\times m}$ for $m \geq n$.
- The values of elements diminish toward zero away from the diagonal, where all elements have value 1.
- The inverse of a Lehmer matrix is a tridiagonal matrix, where the superdiagonal and subdiagonal have strictly negative entries.
- A rather peculiar property of their inverses is that $A^{-1}_{n\times n}$ is nearly a submatrix of $A^{-1}_{m\times m}$ except for the (n, n) element of $A^{-1}_{n\times n}$.
- A = gallery('lehmer',n)