MATH 503: Mathematical Statistics
Dr. Kimberly F. Sellers, Instructor

Homework 9 Problems

1. Let $X_{1j}, X_{2j}, \ldots, X_{a_j j}$ represent independent random samples of sizes a_j from a normal distribution with means μ_j and variances σ^2, $j = 1, 2, \ldots, b$. Show that

$$\sum_{j=1}^{b}\sum_{i=1}^{a_j}(X_{ij} - \bar{X}_{\cdot\cdot})^2 = \sum_{j=1}^{b}\sum_{i=1}^{a_j}(X_{ij} - \bar{X}_{\cdot j})^2 + \sum_{j=1}^{b} a_j(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot})^2,$$

   or $Q' = Q_3' + Q_4'$. Here,

$$\bar{X}_{\cdot\cdot} = \frac{\sum_{j=1}^{b}\sum_{i=1}^{a_j} X_{ij}}{\sum_{j=1}^{b} a_j} \text{ and } \bar{X}_{\cdot j} = \frac{\sum_{i=1}^{a} X_{ij}}{a_j}.$$

   If $\mu_1 = \mu_2 = \ldots = \mu_b$, show that Q'/σ^2 and Q_3'/σ^2 have chi-square distributions. Note that Q_3' and Q_4' are independent, and hence Q_4'/σ^2 also has a chi-square distribution.

2. Solve the following using R or SAS: The following are observations associated with independent random samples from three normal distributions having equal variances and respective means μ_1, μ_2, μ_3.

I	II	III
0.5	2.1	3.0
1.3	3.3	5.1
-1.0	0.0	1.9
1.8	2.3	2.4
	2.5	4.2
		4.1

   Compute the F-statistic that is used to test $H_0 : \mu_1 = \mu_2 = \mu_3$.

3. Solve the following by hand, and by using either R or SAS: Let μ_1, μ_2, μ_3 be, respectively, the means of three normal distributions with a common but unknown variance σ^2. In order to test, at the $\alpha = 0.05$ significance level, the hypothesis $H_0 : \mu_1 = \mu_2 = \mu_3$ against all possible alternative hypotheses, we take an independent random sample of size 4 from each of these distributions. Determine whether we reject or fail to reject H_0 if the observed values from these three distributions are, respectively:

X1	5	9	6	8
X2	11	13	10	12
X3	10	6	9	9

4. Let $X_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$ denote independently normally distributed random variables, where $\epsilon_{ij} \sim \mathrm{N}(0, \sigma^2)$; $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, b$. Show that the maximum likelihood estimator of α_i, β_j, and μ are $\hat{\alpha}_i = \bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot}$, $\hat{\beta}_j = \bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}$, and $\hat{\mu} = \bar{X}_{\cdot\cdot}$, respectively.

5. Solve the following using either R or SAS: Given the following observations in a two-way classification with $a = 3$, $b = 4$, and $c = 2$, compute the F-statistics used to test that all interactions are equal to zero ($\gamma_{ij} = 0$), all column means are equal ($\beta_j = 0$), and all row means are equal ($\alpha_i = 0$), respectively.

		Column			
		1	2	3	4
Row	1	3.1	4.2	2.7	4.9
		2.9	4.9	3.2	4.5
	2	2.7	2.9	1.8	3.0
		2.9	2.3	2.4	3.7
	3	4.0	4.6	3.0	3.9
		4.4	5.0	2.5	4.2