MATH 503: Mathematical Statistics

Lecture 7: Hypothesis Testing II

Reading: C&B Chp. 8, HMC Sec. 6.3

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Today's Topics

- Maximum likelihood tests
  - Likelihood Ratio Test
  - Wald Test
  - (Rao's) Score Test
- Uniformly most powerful (UMP) tests
  - Monotone Likelihood Ratio

Likelihood Ratio Test (LRT)

- Let $X_1, \ldots, X_n$ be iid with pdf $f(x;\theta)$
- Consider $H_0\!: \theta \in \Theta_0$ vs. $H_1\!: \theta \in \Theta_0'$
- Let $\hat{\theta}$ denote the MLE of θ
- Consider the ratio of two likelihoods, namely

$$\Lambda = \frac{\sup\limits_{\Theta_0} L(\theta \mid \boldsymbol{x})}{\sup\limits_{\Theta} L(\theta \mid \boldsymbol{x})}$$

- Question: what are the bounds for Λ?

Likelihood Ratio Test (cont.)

- By definition,

  Λ close to 1 if H_0 true

  Λ "small" if H_1 true

- For specified significance level α, decision rule says to reject H_0 in favor of H_1 if $\Lambda \leq c$, where c chosen st. $\alpha = P_{\theta_0}[\Lambda \leq c]$

Steps to Performing LRT

- Identify hypotheses
- Determine likelihood function, $L(\theta; \boldsymbol{x})$
- Find associated MLEs $\hat{\theta} \in \Theta$, and $\hat{\theta}_0 \in \Theta_0$
- Determine likelihood ratio Λ (simplify as necessary)
- Determine appropriate decision rule based on $\Lambda \leq c$

Example 1

Let $X_1, \ldots, X_n$ be iid Exponential(θ). Determine the appropriate LRT for testing $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$.

Example 2

- Let $X_1, \dots, X_n$ be iid with pdf
  $$f(x \mid \theta) = e^{-(x-\theta)}, \quad x \geq \theta, \quad -\infty < \theta < \infty$$
- Consider $H_0\colon \theta \leq \theta_0$ vs. $H_1\colon \theta > \theta_0$. Determine the appropriate LRT.

Example 3

Let $X_1, \dots, X_n$ be iid Normal(θ, σ^2), $-\infty < \theta < \infty$ unknown, $\sigma^2 > 0$ known. Consider $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$. Determine the appropriate LRT.

Theorem

- Assume that the appropriate regularity conditions hold:
  - Pdfs are distinct, i.e. $\theta \neq \theta' \Rightarrow f(x_i;\theta) \neq f(x_i;\theta')$
  - Pdfs have common support for all θ
  - The point θ_0 is an interior point in Ω
  - Pdf is twice differentiable as a function of θ
  - Integral $\int f(x;\theta)\mathrm{d}x$ can be differentiated twice under the integral sign as a function of θ
  - The pdf $f(x;\theta)$ is three times differentiable as a function of θ. Further, for all $\theta \in \Omega$, there exists a constant c and function $M(x)$ s.t. $\left|\frac{\partial^3}{\partial\theta^3}\log f(x;\theta)\right| \leq M(x)$, with $E_{\theta_0}[M(x)] < \infty$, for all $\theta_0 - c < \theta < \theta_0 + c$ and all x in the support of X.

Theorem (cont.)

- Under H_0, $-2\log\Lambda \to \chi_1^2$ in distribution.
- Decision rule: Reject H_0 in favor of H_1 if
$$\chi_L^2 = -2\log\Lambda \geq \chi_1^2(\alpha)$$

Wald Test

- The Wald Test statistic: $\chi_W^2 = \left[\sqrt{nI(\hat{\theta})}\left(\hat{\theta} - \theta_0\right)\right]^2$
- Taylor expansion implies

$$-2\log\Lambda = 2\left(l\left(\hat{\theta}\right) - l(\theta_0)\right) = \left[\sqrt{nI(\theta_0)}\left(\hat{\theta} - \theta_0\right)\right]^2 + \underbrace{R_n^*}_{\xrightarrow{p} 0}$$

where $I\left(\hat{\theta}\right) \xrightarrow{p} I(\theta_0)$ $\therefore$ $\chi_W^2 = \left[\sqrt{nI(\hat{\theta})}\left(\hat{\theta} - \theta_0\right)\right]^2 \xrightarrow{d} \chi_1^2$

- Decision rule: Reject H_0 in favor of H_1 if $\chi_W^2 \geq \chi_1^2(\alpha)$
- Under H_0, $\chi_W^2 - \chi_L^2 \xrightarrow{p} 0$

Rao's Score Test

- $\chi_R^2 = \left(\frac{l'(\theta_0)}{\sqrt{nI(\theta_0)}}\right)^2$, where scores are $S(\theta) = \left(\frac{\partial \log f(X_1;\theta)}{\partial\theta}, \ldots, \frac{\partial \log f(X_n;\theta)}{\partial\theta}\right)'$ and $l'(\theta_0) = \sum_{i=1}^n \frac{\partial \log f(X_i;\theta)}{\partial\theta}$

- Decision rule: Reject H_0 in favor of H_1 if

$$\chi_R^2 \geq \chi_1^2(\alpha)$$

Example 4

- Let $X_1, \dots, X_n$ be a random sample from Poisson(θ), $\theta > 0$. Test $H_0{:}\,\theta = 2$ vs. $H_1{:}\,\theta \neq 2$ using (a) $-2\log\Lambda$, (b) a Wald-test statistic, (c) Rao's score statistic.

Uniformly Most Powerful Tests

- The critical region C is a uniformly most powerful (UMP) critical region of size α for testing the simple hypothesis H_0 against an alternative composite hypothesis H_1 if the set C is a best critical region of size α for testing H_0 against each simple hypothesis in H_1.

- A test defined by this critical region C is called a uniformly most power (UMP) test, with significance level α, for testing the simple hypothesis H_0 against the alternative composite hypothesis H_1.

Notes re. UMP Tests

- UMP tests don’t always exist
- When they do exist, Neymann-Pearson can help determine them.
- UMP tests are based on sufficient statistics

How do we determine the best critical region?

Neymann-Pearson Thm.: Let $X_1, \ldots, X_n$ (n, *a* positive fixed integer) denote a random sample from a distribution that has pdf/pmf $f(x;\theta)$. Then the likelihood of $X_1, \ldots, X_n$ is

$$L(\theta;\boldsymbol{x}) = \prod_{i=1}^{n} f(x_i;\theta).$$

Let θ' and θ'' be distinct fixed values of θ s.t. $\Omega = \{\theta: \theta = \theta', \theta''\}$, and let k be a positive number.

Neymann-Pearson Thm. (cont.)

Let C be a subset of the sample space s.t.

a) $\frac{L(\theta';\boldsymbol{x})}{L(\theta'';\boldsymbol{x})} \leq k$, for each point $\boldsymbol{x} \in C$

b) $\frac{L(\theta';\boldsymbol{x})}{L(\theta'';\boldsymbol{x})} \geq k$, for each point $\boldsymbol{x} \in C^c$

c) $\alpha = P_{H_0}[\boldsymbol{X} \in C]$

Then C is a best critical region of size α for testing the simple hypothesis $H_0\colon \theta = \theta'$ vs. $H_1\colon \theta = \theta''$.

Example 5

Let $X_1, \ldots, X_n$ be a random sample from a distribution that is N$(0, \theta)$, where the variance θ is an unknown positive number. Show that there exists a UMP test with significance level α for testing $H_0: \theta = \theta'$ vs. $H_1: \theta > \theta'$.

Example 6

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution N$(\theta, 1)$, where θ is unknown. Does a UMP test of $H_0: \theta = \theta'$ vs. $H_1: \theta \neq \theta'$ exist?

Example 7

Let $X_1, \ldots, X_{10}$ be a random sample of size 10 from a Poisson(θ) distribution. Find a best critical region for testing $H_0: \theta = 0.1$ vs. $H_1: \theta = 0.5$. Is this region uniformly most powerful for $H_0: \theta = 0.1$ vs. $H_1: \theta > 0.1$?

For One-sided Hypotheses…

- Consider $H_0: \theta \leq \theta'$ vs. $H_1: \theta > \theta'$
  [or, analogously, $H_0: \theta \geq \theta'$ vs. $H_1: \theta < \theta'$]

- For some families of pdfs and hypotheses, we can obtain general forms of UMP tests

- Introduce the monotone likelihood ratio….

Monotone Likelihood Ratio

[In CB] A family of pdfs or pmfs $\{g(t \mid \theta): \theta \in \Theta\}$ for a univariate random variable T with real-valued parameter θ has a monotone likelihood ratio (MLR) if, for every $\theta_2 > \theta_1$, $g(t \mid \theta_2)/g(t \mid \theta_1)$ is a monotone (nonincreasing or nondecreasing) function of t on $\{t: g(t \mid \theta_1) > 0 \text{ or } g(t \mid \theta_2) > 0\}$

[In HMC] The likelihood $L(\theta; \boldsymbol{x})$ has monotone likelihood ratio (MLR) in the statistic $y = u(\boldsymbol{x})$, if for $\theta_1 < \theta_2$, the ratio $L(\theta_1; \boldsymbol{x})/L(\theta_2; \boldsymbol{x})$ is a monotone function of $y = u(\boldsymbol{x})$.

Example 8

Let $X_1, \dots, X_n$ be a random sample from a Bernoulli distribution with parameter $p = \theta$, where $0 < \theta < 1$. Show that this distribution satisfies the MLR property.

Karlin-Rubin Theorem

Consider testing $H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$. Suppose that T is a sufficient statistic for θ and the family of pdfs or pmfs $\{g(t \mid \theta): \theta \in \Theta\}$ of T has a nondecreasing MLR*. Then for any t_0, the test that rejects H_0 if and only if $T > t_0$ is a UMP level α test, where $\alpha = P_{\theta_0}(T > t_0)$.

* as defined in CB.

Example 9

Let $X_1, \ldots, X_n$ be a random sample whose distribution can be represented as an exponential family. Show that this distribution satisfies the MLR property, if $p(\theta)$ is monotone.

Unbiased Tests

- A test is <u>unbiased</u> if its power never falls below the significance level

- Examples:
  - MP test of simple H_0 vs. simple H_1
  - One-sided tests based on MLR pdfs