Homework 1 :: MATH 504 :: Solution

1. Geometrically describe with reasoning the following unit ball in $\mathbb{R}^2$

$$B_\infty(1) = \{x \in \mathbb{R}^2 : \quad \|x\|_\infty \leq 1\}.$$

$\|x\|_\infty = \max\{|x_1|, |x_2|\} \leq 1$. Consider two cases.

- Case 1) $|x_1| = 1$ (i.e. $x_1 = 1$ or $x_1 = -1$) and $|x_2| \leq 1$.
- Case 2) $|x_2| = 1$ (i.e. $x_2 = 1$ or $x_2 = -1$) and $|x_2| \leq 1$.

The result will be a square of width 2 centered at the origin

2. Prove the following triangle inequality

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2, \qquad \forall x, y \in \mathbb{R}^n.$$

When does the equality hold?

Solution.

$$\begin{aligned}
\|x + y\|_2^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|_2\|y\|_2 + \|y\|^2 \\
&= (\|x\|_2 + \|y\|_2)^2
\end{aligned}$$

Take square roots to get

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$$

Equality holds when $x = y$.

3. Prove that for any matrix $A = [a_{i,j}]_{m\times n}$, the matrix $A^\mathsf{T} A$ is symmetric.

Solution.

$$(A^\mathsf{T} A)^\mathsf{T} = A^\mathsf{T}(A^\mathsf{T})^\mathsf{T} = A^\mathsf{T} A.$$