BICK-hw11

November 27, 2022

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1 Problem 1

Find the quadratic polynomial $p_2(x)$ that interpolates the data

x	1	2	4
y	-1	-1	2

(a) using the Lagrange method;

(b) using the method of undetermined coefficients.

(a) The Lagrange method is to use the lagrange polynomials in the form:

$p(x) = \sum_{i=0}^{n} y_i l_i(x)$

where $l_i(x)$ is the i th lagrange polynomial:

$l_i(x) = \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j}$

We create the lagrange polynomials for each data point:

$l_0(x) = \frac{(x-2)(x-4)}{(1-2)(1-4)} = \frac{x^2-6x+8}{3}$

$l_1(x) = \frac{(x-1)(x-4)}{(2-1)(2-4)} = \frac{x^2-5x+4}{-2}$

$l_2(x) = \frac{(x-1)(x-2)}{(4-1)(4-2)} = \frac{x^2-3x+2}{6}$

Then we can create the interpolating polynomial:

$p_2(x) = -1\frac{x^2-6x+8}{3} - 1\frac{x^2-5x+4}{-2} + 2\frac{x^2-3x+2}{6} = \frac{x^2-3x}{2}$

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(b) Now we use the method of undetermined coefficients. This is given by solving the system of equations:

$a_0 + a1x_0 + a_2x_0^2 + ... + a_nx_0^n = y_0$

$a_0 + a1x_1 + a_2x_1^2 + ... + a_nx_1^n = y_1$

...

$a_0 + a1x_n + a_2x_n^2 + ... + a_nx_n^n = y_n$

where x_i is the i th data point and y_i is the i th data point. We can write this as a matrix equation:

$$\begin{pmatrix} 1 & x_0 & x_0^2 & ... & x_0^n \\ 1 & x_1 & x_1^2 & ... & x_1^n \\ ... & ... & ... & ... & ... \\ 1 & x_n & x_n^2 & ... & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ ... \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ ... \\ y_n \end{pmatrix}$$

In our problem, we have $n = 2$ so we can write this as:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

We can solve this system of equations to find the coefficients:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3/2 \\ 1/2 \end{pmatrix}$$

This gives us the interpolating polynomial:

$p_2(x) = 0 - \frac{3}{2}x + \frac{1}{2}x^2$

And we see that this is the same result as we achieved in part a using the lagrange method. This is not surprising because th interpolating polynomial is unique.

2 Problem 2

Add the data point $(x_3, y_3) = (3, 0)$ to the data in Problem 1.

(a). Use Newton's method to find the cubic interpolating polynomial $p_3(x)$ for the resulting data.

(b) Find $p_3(x)$ using the Lagrange method.

Newton's method allows us to add an additional term without recalculating the entire polynomial. We can use the formula:

$a_{n+1} = \frac{y_{n+1} - p_n(x_{n+1})}{(x_{n+1} - x_o)...(x_{n+1} - x_n)}$

where a_n is the n th term of the interpolating polynomial.

We know $y_{n+1} = y_3 = 0$ and $x_{n+1} = x_3 = 3$. We can use the previous polynomial to find $p_n(x_{n+1})$:

$p_2(x_3) = 0 - \frac{3}{2}x_3 + \frac{1}{2}x_3^2 = 0 - \frac{3}{2}3 + \frac{1}{2}9 = 0$

We calculate the denominator as:

$(x_{n+1} - x_o)...(x_{n+1} - x) = (3 - 1)(3 - 2)(3 - 4) = -2$

Putting it together, we get:

$a_{n+1} = \frac{0-0}{-2} = 0$

This means that the cubic interpolating polynomial is the same as the quadratic interpolating polynomial, as the coefficient of x^3 is zero.

To find the p_3 using the lagrange method, we must calculate the lagrange polynomials for each data point, as if from the beginning.

$l_0(x) = \frac{(x-2)(x-4)(x-3)}{(1-2)(1-4)(1-3)} = \frac{-x^3+9x^2-26x+24}{6}$

$l_1(x) = \frac{(x-1)(x-4)(x-3)}{(2-1)(2-4)(2-3)} = \frac{x^3-8x^2+19x-12}{2}$

$l_2(x) = \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} = \frac{x^3-6x^2+11x-6}{6}$

$l_3(x) = \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} = \frac{-x^3+7x^2-14x+8}{2}$

Then we can create the interpolating polynomial:

$p_3(x) = -1\frac{-x^3+9x^2-26x+24}{6} - 1\frac{x^3-8x^2+19x-12}{2} + 2\frac{x^3-6x^2+11x-6}{6} + 0\frac{-x^3+7x^2-14x+8}{2} = \frac{x^2-3x}{2}$

We see that we get the same results as we did when using the newton's method.

3 Problem 3

Find the trigonometric function $T(x) = a_0 + a_1 \sin x + a_2 \sin 2x$ that interpolates the following data

x	0	$\pi/2$	$\pi/3$
y	-1	2	1

We can use the given data to use the method of undetermined coefficients and create the system of equations:

$a_0 + a_1 \sin 0 + a_2 \sin 0 = -1$

$a_0 + a_1 \sin \frac{\pi}{2} + a_2 \sin \frac{2\pi}{2} = 2$

$a_0 + a_1 \sin \frac{\pi}{3} + a_2 \sin \frac{2\pi}{3} = 1$

We can solve this using backward substitution. First we see that $a_0 = -1$. We substitute this value and evaluate the sinusoids in second equation:

$-1 + a_1 \sin \pi/2 + a_2 \sin \pi = 2 \Rightarrow -1 + a_1 + 0 = 2 \Rightarrow a_1 = 3$

Finally, we can solve for a_2 in a similar manner, substiuting the values of a_0 and a_1, and evaluating the equation:

$-1 + 3 \sin \frac{\pi}{3} + a_2 \sin \frac{2\pi}{3} = 1 \Rightarrow$

$-1 + 3\frac{\sqrt{3}}{2} + a_2 \frac{\sqrt{3}}{2} = 1 \Rightarrow$

$2 - 3\frac{\sqrt{3}}{2} = a_2 \frac{\sqrt{3}}{2} \Rightarrow$

$a_2 = \frac{4-3\sqrt{3}}{\sqrt{3}}$

4 Problem 4

Find the quartic interpolating polynomial $p_4(x)$ for $f(x) = e^{3x}$ using $x_0 = -1$, $x_1 = -0.5$, $x_2 = 0$, $x_3 = 0.5$, and $x_4 = 1$. For $x = 0.08$

(a) compute $e_4(x) = f(x) - p_4(x)$;

(b) estimate the error $e_4(x)$ using the bound formula given in the lecture.

First we must find the quartic interpolating polynomial. We create the data points x_i and y_i using the function f:

x	-1	-0.5	0	0.5	1
y	$e^{-3} = 0.0497$	$e^{-3/2} = 0.223$	1	$e^{3/2}$=4.486	e^{3}=20.085

Now we can use the lagrange method to find the interpolating polynomial:

$p_4(x) = 0.0497 * \frac{(x+0.5)(x-0)(x-0.5)(x-1)}{(-1+0.5)(-1)(-1-0.5)(-1-1)} + 0.223 * \frac{(x+1)(x)(x-0.5)(x-1)}{(-0.5+15)(-0.5)(-0.5-0.5)(-0.5-1)} + 1 * \frac{(x+1)(x+0.5)(x-0.5)(x-1)}{(0+1)(0+0.5)(0-0.5)(0-1)} + 4.4486 * \frac{(x+1)(x+0.5)(x)(x-1)}{(0.5+1)(0+0.5)(0.5)(0.5-1)} + 20.085 * \frac{(x+1)(x+0.5)(x-0)(x-0.5)}{(1+1)(1+0.5)(1-0)(0-0.5)}$