Chapter 2

Ordinary Differential Equations

Differential equations are in the background of most of the critical mathematical modelling. In this chapter, we will introduce the basic theory for ordinary differential equations.

2.1 Terminology

A *differential equation of the first order* is an equation of the form

$$y' = f(x, y) \tag{2.1}$$

with a given function $f(x, y)$. Here $y = y(x)$ is a function of x and y' stands for $\mathrm{d}y/\mathrm{d}x$. A function $y(x)$ is called a solution of this equation if for all x,

$$y'(x) = f(x, y(x)). \tag{2.2}$$

Example 2.1. *A simple example is*

$$y' = y.$$

Since one knows that the exponential function equals its own derivative, $(e^x)'$ $= e^x$*, then*

$$y(x) = e^x$$

is one solution of $y' = y$*. Note that* $y = Ce^x$ *is also a solution for any constant* C*.*

It was observed very early by Newton, [1] Leibniz and Euler that the solution usually contains a free parameter, so that it is only uniquely determined when an *initial value*

$$y(x_0) = y_0 \tag{2.3}$$

is prescribed.

A *differential equation of the second order* for y is of the form

$$y'' = f(x, y, y'). \tag{2.4}$$

Here, the solution usually contains *two* parameters and is only uniquely determined by *two* initial values

$$y(x_0) = y_0, \quad y'(x_0) = y'_0. \tag{2.5}$$

2.2 First Order Equations

2.2.1 Separable Variables

Let us consider a population whose initial value is y_0, with constant per-capita birth rate given as β, and constant per-capita death rate given as α. Our aim is to find the population size at any time t. The first step is to determine an equation for the population. We assume that the population can only change due to births or deaths, neglecting here any immigration or emigration. Also, we assume that this change in population at any time is proportional to the size of the population at that time. This yields the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \beta y - \alpha y, \quad y(t_0) = y_0. \tag{2.6}$$

[1] Many mathematicians have studied differential equations and contributed to the field, but we should remark that Newton can be considered as the creators of the field. A simple example is Newton's second law of motion - the relationship between the displacement x and the time t of the object under the force F, which leads to the differential equation

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = F(x(t)),$$

for the motion of a particle of constant mass m. In general, F depends on the position $x(t)$ of the particle at time t, and so the unknown function $x(t)$ appears on both sides of the differential equation, as is indicated in the notation $F(x(t))$.

Equation (2.6) is a particular case of a separable equation. Here Eq.(2.1) is the particular case

$$y' = f(x)g(y). \tag{2.7}$$

Dividing by $g(y)$ and integrating we obtain the *general solution*[2]

$$\int \frac{dy}{g(y)} = \int f(x)dx + C.$$

Example 2.2 (General Solution)**.** *Solve $y' = x$.*

Solution: We write $\mathrm{d}y/\mathrm{d}x = x$, and multiply by $\mathrm{d}x$ so $\mathrm{d}y = x\mathrm{d}x$. Take the integral in both sides of the equation,

$$y = \int x\mathrm{d}x = x^2/2 + C.$$

The formula for y is known as the general solution.

Example 2.3 (Boundary Conditions)**.** *Solve $y' = x$, with $y(0) = 1$.*

Solution: From example 2.2 we have the general solution

$$y(x) = x^2/2 + C.$$

The constant of integration C will be determined by the initial condition. Using, $y(0) = 1$ implies $C = 1$ and then

$$y(x) = x^2/2 + 1.$$

Example 2.4. *Solve $y' = y^{-5}$, with $y(0) = 1$.*

Solution: We write $\mathrm{d}y/\mathrm{d}x = y^{-5}$, and multiply by $y^5\mathrm{d}x$ so $y^5\mathrm{d}y = \mathrm{d}x$. Take the integral in both sides of the equation, $y^6/6 = x + C$. The general solution is given by

$$y = (6x + 6C)^{1/6}.$$

For instance, $y(0) = 1$ implies $C = 1/6$ and then we have the particular solution

$$y(x) = (6x + 1)^{1/6}.$$

[2]Leibniz 1691, in a letter to Huygens

Example 2.5. *Solve* $z' \cos(z) = 1$, *with* $z(0) = \pi/4$.

Solution: This equation is equivalent to $\cos(z)\mathrm{d}z = \mathrm{d}t$, or $\sin(z) = t + C$. The general solution is given by

$$z = \sin^{-1}(t + C).$$

Setting $t = 0$ gives $\pi/4 = \sin^{-1}(C)$, or $C = \sin(\pi/4) = \sqrt{2}/2$, so the solution is

$$z(t) = \sin^{-1}(t + \sqrt{2}/2).$$

Example 2.6. *Solve* $y' = y^{-5} \cos(x)$, *with* $y(0) = -1$.

Solution: Write as $y^5 dy = \cos(x)\mathrm{d}x$ or $y^6/6 = \sin(x) + C$. The general solution is given by

$$y = (6\sin(x) + 6C)^{1/6}.$$

Given $y(0) = -1$ then $1/6 = 0 + C$, $C = 1/6$,

$$y(x) = (6\sin(x) + 1)^{1/6}.$$

Example 2.7. *Solve* $x' = x^2/t^2$, *with* $x(1) = 1$.

Solution: This separates into $x^{-2}dx = t^{-2}dt$ and integrates to

$$-x - 1 = -t - 1 + C,$$

where the initial condition gives $-1 = -1 + C$ or $C = 0$ and so

$$x(t) = t.$$

Note that $t = 0$ could not be used to place an initial condition because $f(x, t) = x^2/t^2$ is undefined for $t = 0$.

Example 2.8. *Solve* $x' = x^{-1}e^t$, *with* $x(0) = 1$.

Solution: This separates into $xdx = e^t dt$ and integrates to $x^2 = e^t + C$, so we have the general solution

$$x(t) = \sqrt{e^t + C}.$$

The initial condition gives $1 = \sqrt{1 + C}$ or $C = 0$ and so

$$x(t) = \sqrt{e^t} = e^{t/2}.$$

A special example of the linear equations Eq.(2.7) is

$$y' = h(x)y,$$

which possesses the solution

$$y(x) = CR(x), \quad R(x) = \exp\{H(x)\}, \tag{2.8}$$

where H is the antiderivative of h, i.e., $H' = h$. For the solution of the population model in Eq.(2.6) we set $h(t) = r$, so $H(r) = rt$ then the general solution is given by

$$y(t) = CR(t) = Ce^{rt}.$$

Homework 2.1. *Find the general solution to the following problems*

1. $x' = t^{-2}$
2. $y' = y^{-2}$
3. $y' = y^{-1}t^2$
4. $x' = \cos(x)$
5. $y' = -1/\sin(y)$

Homework 2.2. *Find the solution of the following initial value problems*

1. $y' = t^{-2}$, $y(1) = 0$
2. $x' = -1/\sin(x)$, $x(0) = \pi/2$
3. $y' = y^{-1}\cos(t)$, $y(\pi) = 1$
4. $x' = x^{-2}e^t$, $x(0) = 1$
5. $z' = \sin(t)e^z$, $z(\pi) = 0$

Homework 2.3. *For the population growth model we include γ which is the rate of deaths by crowding we obtain the logistic equation*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx - \frac{r}{K}x^2.$$

Find the general solution to the differential equation of this model.

2.2.2 Inhomogeneous linear equation

For the first order equations we have the inhomogeneous case

$$y' = h(x)y + g(x). \tag{2.9}$$

We define as H the antiderivative of h, or $H' = h$. The multiplication by $e^{-H(x)}$ results in

$$e^{-H(x)}y' - h(x)e^{-H(x)}y = e^{-H(x)}g(x),$$

or

$$\left(e^{-H(x)}y\right)' = e^{-H(x)}g(x).$$

Using the notation in (2.9), the general solution in the form

$$y(x) = R(x)\left\{\int_{x_0}^{x} \frac{g(s)}{R(s)}ds + C\right\}. \tag{2.10}$$

Example 2.9. *Solve $x' - 7x = 1$, with $x(0) = 0$.*

Solution: Here $h(t) = -7$, so $H(t) = -7t$. Multiply by e^{-7t} to get $(xe^{-7t})' = e^{-7t}$ or $xe^{-7t} = (-1/7)e^{-7t} + C$, the general solution is

$$x(t) = (-1/7) + Ce^{7t}.$$

If $x(0) = 0$ then $C = 1/7$ and

$$x(t) = (e^{7t} - 1)/7.$$

Example 2.10. *Solve $x' + 4tx = 5t$, with $x(0) = 1/4$*

Solution: Here $h(t) = 4t$, $H(t) = 2t^2$ and the integrating factor is $\exp(2t^2)$. Multiplying by the integrating factor, $(x' + 4tx)e^{2t^2} = \left(xe^{2t^2}\right)' = 5te^{2t^2}$. The integral of this equation is

$$xe^{2t^2} = \frac{5}{4}\int 4te^{2t^2}\mathrm{d}t = \frac{5}{4}e^{2t^2} + C, \quad \text{and so } x(t) = \frac{5}{4} + Ce^{-2t^2} \text{ for all } t.$$

An initial condition would be needed to evaluate to constant C. If, say, $x(0) = 1/4$ then $C = -1$.

Don't be fooled by the case where $h(t) = 0$, or $x'(t) = g(t)$. This is trivial and the solution is $x(t) = G(t)$ where G is any indefinite integral of g.

Example 2.11 (Terminal Velocity Example)**.** *Suppose a sky diver with terminal velocity* $200 ft/sec$ *steps out of a hovering helicopter and drops straight down subject to* $v'(t) = \alpha(v_T - v(t))$.

Here $v_T = 200$ and $\alpha v_T = g = 32 ft/sec^2$ so the equation simplifies to

$$v' = (32/200)(200 - v)$$

or

$$v' + 0.16v = 32. \tag{2.11}$$

This is a first order linear equation with $h(t) = 0.16$ and $g(t) = 32$. The integrating factor is $e^{0.16t}$ and the equation is equivalent to $(ve^{0.16t})' = 32e^{0.16t}$. Integrating, $ve^{0.16t} = (32/0.16)e^{0.16t} + C$, or

$$v = 200 + Ce^{-0.16t}.$$

Since $v(0) = 0$ then $C = -200$ and

$$v(t) = 200(1 - e^{-0.16t})$$

for all t.

Example 2.12 (continuation to terminal velocity)**.** *Now suppose the parachute can only open when the velocity reaches* $150 ft/sec$. *How long will that take?*

Solution: To find out, set $v(t) = 150$ in the above equation and solve for t. In fact, $(1 - e^{-0.16t}) = 3/4$ or $e^{-0.16t} = 1/4$. Taking logarithms, $-0.16t = \ln(1/4) = -\ln(4)$ so $t = \ln(4)/0.16 = 8.66 sec$.

Homework 2.4. *Find the solution of the following initial value problems*

1. $x' + 2x = 1$, $x(0) = 1$
2. $y' + 2y = e^{-t}$, $y(0) = 1$
3. $z' - 2tz = t$, $z(0) = 1$
4. $x' - t^3x = e^{\frac{1}{4}t^4}$, $x(0) = 1$

Homework 2.5. *Consider the falling object equation Eq.(**??**). Assume that the terminal velocity of a sky diver is* $180 ft/sec$ *and* $v(0) = 0$. *The gravity constant is* $32 ft/sec^2$. *Find the velocity* $v(t)$ *for all* t. *Say the goal is to pull the rip cord when* $v = 100 ft/sec$. *Find the time* t *when* $v = 100 ft/sec$.

2.2.3 Total differential equations

Notice that we can encounter *non-linear* differential equation of the type

$$P(x,y) + Q(x,y)y' = 0, \tag{2.12}$$

where both P and Q are functions that depend on both x and y. It is found to be immediately that Eq.(2.12) is solvable if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{2.13}$$

One can then find by integration a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = P, \quad \frac{\partial F}{\partial y} = Q.$$

Therefore (2.12) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x, y(x)) = 0,$$

which the solutions expressed by

$$F(x, y(x)) = C.$$

For the case when (2.12) is not satisfied, Clairaut and Euler investigated the multiplication of (2.12) by a suitable factor $M(x,y)$, which sometimes allows the equation

$$MP + MQy' = 0$$

to satisfy (2.13).

2.3 Euler's Method

The simplest way to get an insight into the solution of a first-order differential equation is to draw the slope field. This is accomplish by using Eq.(2.1) for a grid of point (x_i, y_i) over an axis $x - y$. In a simplified manner, we use the relation

$$\frac{\Delta y_i}{\Delta x_i} \approx f(x_i, y_i).$$

Figure 2.1: Slope field plots from a) example 2.2 and b) example 2.3.

We show in Fig.2.1 the slope field that results example 2.3-2.4.

MATLAB programming note 2.1.

We can use the function "drawslope" to visualize the general solution of a first-order differential equation. We use $y' = f(x, y) = x$ in example 2.2

```
>> f1 = inline('x','x','y')
f1 =
     Inline function:
     f1(x,y) = x
>> slopefield(f1)
>> slopefield(f1,[0 2],[0 2])
>> xlabel('x')
>> ylabel('y')
```

And define the initial solution $y(0) = 1$ as in example 2.3 that results in $y(x) = x^2/2 + 1$.

```
>> y = inline('(x.^2)/2 + 1','x');
>> hold on
>> t = linspace(0,2,256);
>> plot(x,y(x),'r')
>> xlim([0 2])
>> ylim([0 2])
```

The resultant plots are shown in figure 2.1

The most natural computational solution procedure (other than the slope field) is choosing an initial point (x_0, y_0) and following the direction specified there. For example, after moving a short distance:

- Re-evaluate the slope at the new point (x_1, y_1),
- Move farther according to the new slope
- Repeat the process.

There will be some errors associated with the progress, but we will address these issues later.

Given the initial value problem for the first order equation

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x \in [a, b], \tag{2.14}$$

define a grid of $n + 1$ points

$$a = x_0 < x_1 < ... < x_n = b$$

for the independent variable x with equal step size h. From the formal definition of the first derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = \lim_{h \to 0} \frac{y(x + h) - y(x)}{h}$$

we have the following useful first order (or two point) approximations

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x}(x) &= \frac{y(x + h) - y(x)}{h} - \frac{h}{2}y''(c), \quad c \in [t, t + h], \quad \text{(Forward difference)} \\
\frac{\mathrm{d}y}{\mathrm{d}x}(x) &= \frac{y(x) - y(x - h)}{h} + \frac{h}{2}y''(c), \quad c \in [t - h, t], \quad \text{(Backward difference)}
\end{aligned}$$

or the second order (or three point) approximation

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = \frac{y(x + h) - y(x - h)}{2h} + \frac{h^2}{6}y'''(c), \quad c \in [t - h, t + h], \quad \text{(Centered difference)}$$

The accuracy of the previous formulas depends on the size of h. So we can expect that when h is small, these formulas approximate the derivative more accurately. So by making h small, we increase the number n from our original grid. We assume the relation

$$h = (b - a)/n.$$

By using the two-point forward difference in Eq.(2.14) we have the relation

$$y(x + h) - y(x) \approx hf(x, y(x)).$$

If we denote as $y_j = y(x_j)$ then we can formally define the Euler method

Definition 2.1 (Euler's method)**.** *The solution y to the initial value problem Eq.(2.14) can be found by the sequence*

$$y_{j+1} = y_j + hf(x_j, y_j).$$

for $j = 0, ..., n-1$.

Let's use the following results from previous sections

Example 2.13. *Use the Euler's method to solve $y' = x$, with $y(0) = 1$, $x \in [0, 1]$. The exact solution is*

$$y(x) = x^2/2 + 1.$$

Solution: We will use the program `ODEsolver_euler` to compute the solution using Euler's method. First we define $f(x, y)$

```
>> f1 = inline('x','x','y');
```

then obtain the solution for $n = 11$

```
>> [y,x] = ODEsolver_euler(f1,1,0,1,11);
```

the first entry is the function definition, the second input is $y_0 = 1$, the third and four $a = 0$,$b = 1$ and finally the fifth argument is $n = 11$. Defining the exact function $y(x)$ we made the comparison

```
>> ye = x.^2/2+1;
>> [x y ye abs(y-ye)]
ans =

         0    1.0000    1.0000         0
    0.1000    1.0000    1.0050    0.0050
    0.2000    1.0100    1.0200    0.0100
    0.3000    1.0300    1.0450    0.0150
    0.4000    1.0600    1.0800    0.0200
    0.5000    1.1000    1.1250    0.0250
    0.6000    1.1500    1.1800    0.0300
    0.7000    1.2100    1.2450    0.0350
    0.8000    1.2800    1.3200    0.0400
    0.9000    1.3600    1.4050    0.0450
    1.0000    1.4500    1.5000    0.0500
```

Here, notice that the first column is x, the second is the solution from Euler's approach, the third is the exact solution, and the fourth is the absolute value of the difference (or error). As we can expect, the error increases as x increases. An alternate way to measure the error is given by the use of the $\|\cdot\|_\infty$

```
>> norm(y-ye,inf)
ans =
    0.0500
```

Notice that this error coincides with the difference of the numerical and exact solution at $x = b$. As we mentioned before, the error increases at each step, so it is not surprising that the most significant difference is the end-point of the interval.

Example 2.14. *Use the Euler's method to solve* $y' = y^{-5}$*, with* $y(0) = 1$*,* $x \in [01]$*. The exact solution is*

$$y(x) = (6x + 1)^{1/6}.$$

Solution: We proceed as the previous example for $n = 11$

```
>> f2 = inline('1./(y.^5)','x','y');
>> [y,x] = ODEsolver_euler(f2,1,0,1,11);
>> ye = (6*x+1).^(1/6);
>> norm(y-ye,inf)
ans =
    0.0221
```

Since $h = 1/n = 1/10 = 0.1$, we argue from the derivative approximation that the result will be improve by reducing h or increasing the number of points t. We try the solution for $h = 0.01$ and $h = 0.001$

```
>> [y,x] = ODEsolver_euler(f2,1,0,1,101);
>> ye = (6*x+1).^(1/6);
>> norm(y-ye,inf)
ans =
    0.0019

>> [y,x] = ODEsolver_euler(f2,1,0,1,1001);
```

```
>> ye = (6*x+1).^(1/6);
>> norm(y-ye,inf)
ans =
   1.8421e-04
```

The error for this method can be defined in different ways that depend on the particular application. The error at each step of the numerical method is given by

$$e_j = |y_j - y(x_j)|, \quad 0 \le j \le n,$$

so the "local error" is defined as

$$E_{\text{loc}} = \max_j e_j,$$

and the global error will be defined as

$$E_{\text{glob}} = \sqrt{\sum_{j=1}^{n} e_j^2}.$$

If we denote as $\mathbf{y}$ (the vector of n entries y_j solutions to Euler's method) and as $\mathbf{y}_e$ (the vector of n entries $y(x_j)$ solution the initial value problem), we have the definitions. The *local error* is given by

$$E_{\text{loc}} = \|\mathbf{x} - \mathbf{x}_e\|_\infty,$$

and the *global error* is given by

$$E_{\text{glob}} = \|\mathbf{x} - \mathbf{x}_e\|_2.$$

In many instances it is useful to define the *relative error* as

$$RE = \frac{\|\mathbf{x} - \mathbf{x}_e\|_2}{\|\mathbf{x}_e\|_2}.$$

As a common procedure of numerical solutions, we will require certain precision to the resultant solution. A solution with *p digits of precision* is

$$E_{\text{loc}} < \frac{1}{2} \times 10^{-p}.$$

Example 2.15. *Use the Euler's method to solve*

$$y' + 4xy = 5x,$$

with $x(0) = 1/4$, $x \in [0, 1]$. *The solution should be within* 4 *digits of precision.*

Solution: The exact solution is given by

$$y(x) = \frac{5}{4} - e^{-2x^2}.$$

Now we use the MATLAB commands

```
>> f3 = inline('5*x - 4*x*y','x','y');
>> [y,x] = ODEsolver_euler(f3,1/4,0,1,11);
>> ye = 5/4 - exp(-2*x.^2);
>> norm(y-ye,inf)
ans =
    0.0511

>> [y,x] = ODEsolver_euler(f3,1/4,0,1,101);
>> ye = 5/4 - exp(-2*x.^2);
>> norm(y-ye,inf)
ans =
    0.0046

>> [y,x] = ODEsolver_euler(f3,1/4,0,1,1001);
>> ye = 5/4 - exp(-2*x.^2);
>> norm(y-ye,inf)
ans =
   4.6052e-04

>> [y,x] = ODEsolver_euler(f3,1/4,0,1,10001);
>> ye = 5/4 - exp(-2*x.^2);
>> norm(y-ye,inf)

ans =

   4.6009e-05
```

Notice that we needed $n > 10000$ steps to obtain the 4 digits precision!

The following result can be found in many numerical analysis textbooks

Theorem 2.1 (Euler's Method Convergence)**.** *Given a Lipschitz function f on its second variable on $[a, b] \times [\alpha, \beta]$, Euler's method for the solution of initial value problem Eq.(2.14) returns the following estimate*

$$e_j \leq \frac{M}{2L} h \left(e^{L(t_j - a)} - 1 \right). \tag{2.15}$$

Here M be an upper bound for $|y''|$ on $[a, b]$.

This results explains the outcome in example 2.15. The exponent of h in the error bound Eq.(2.15) is known as the order of convergence. Euler's method is an order 1 method, which means that will slowly converge to the exact solution.

Homework 2.6. *Use Euler's method to approximate the solution of the following initial value problems for $h = 0.1$*

1. $y' = t^{-2}$, $y(1) = 0$, $t \in [1, 3]$,
2. $x' = -1/\sin(x)$, $x(0) = \pi/2$, $t \in [0, 1]$,
3. $y' = y^{-1}\cos(t)$, $y(\pi) = 1$, $t \in [\pi, 2\pi]$,
4. $x' = x^{-2}e^t$, $x(0) = 1$, $t \in [0, 1]$,
5. $z' = \sin(t)e^z$, $z(\pi) = 0$, $t \in [\pi, 2\pi]$,

Homework 2.7. *Use Euler's method to approximate the solution of the following initial value problems for $t \in [0, 1]$*

1. $x' + 2x = 1$, $x(0) = 1$,
2. $y' + 2y = e^{-t}$, $y(0) = 1$,
3. $z' - 2tz = t$, $z(0) = 1$,
4. $x' - t^3x = e^{\frac{1}{4}t^4}$, $x(0) = 1$,

Report the n required to obtain a precision of 2 and 4 digits.

2.4 Second Order Equations

The general second order linear equation is of the form (or can be brought to the form)

$$y'' + p(x)y' + q(x)y = g(x). \tag{2.16}$$

The term linear refers to the fact that y and its derivatives appear as linear terms, such as $y'' + y = 0$. An example of a nonlinear second order ODE is $y'' + yy' + y^2 = 0$. The equation is called *homogeneous* if $g(x) = 0$ and non-homogeneous otherwise.

The y' term can usually be removed by a change of variable. For arbitrary coefficients $p(x)$, $q(x)$ and $g(x)$ the all-purpose equation cannot generally be solved in closed form. Besides linear equations with constant coefficients, whose solutions for the second order case were already known to Newton, several tricks of reduction are possible[3].

For the second order equations the initial conditions take the form that $y(x_0)$ and $y'(x_0)$ are specified. The standard initial value problem for a second order linear equation is

$$y'' + p(x)y' + q(x)y = g(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1,$$

with $p(x)$, $q(x)$ and $g(x)$ known and y_0 and y_1 specified. The symbols t, s, etc are also used to denote the independent variable. Various other characters are used to denote the dependent variable as well.

[3]We make the substitution

$$y = e^{P(x)},$$

where $P' = p$. The derivatives of this function contain only derivatives of p of lower order

$$y' = pe^{P(x)}, \quad y'' = \left[p^2 + p'\right] e^{P(x)}$$

so that the insertion of this into the differential equation Eq.(2.16), after division by y, lead to a *lower order* equation

$$p^2 + p' = a(x)p + b(x)$$

which, however, is nonlinear (Riccati (1723), Euler (1728)).

2.4.1 Homogeneous, constant coefficients

Consider the general form

$$y'' + py' + qy = 0, \tag{2.17}$$

where p and q are constants. In general, solutions are available assuming the form $y = e^{rx}$. By plugging this solution into Eq.(2.15) we obtain

$$e^{rx}\left(r^2 + pr + q\right) = 0,$$

where the *characteristic equation* is defined as $r^2 + pr + q = 0$. Notice that if r solves the characteristic equation then $y = e^{rx}$ is a solution to Eq.(2.15). There are typically two roots r_1 and r_2, and it can be proved that the general solution is as follows (meaning that all solutions are expressible in this form),

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \tag{2.18}$$

If it should happen that $r_1 = r_2$ then the second solution defaults to xe^{rx}, where r is the single root.

Example 2.16. *Solve* $y'' - 3y' + 2y = 0$.

Solution: Here $r^2 - 3r + 2 = (r-1)(r-2)$, so Eq.(2.17) becomes

$$y = c_1 e^{x} + c_2 e^{2x}.$$

Example 2.17. *Solve* $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = -1$.

Solution: Since $y = c_1 e^{x} + c_2 e^{2x}$, then $1 = y(0) = c_1 + c_2$ and $-1 = y'(0) = c_1 + 2c_2$. Solving for the c's gives $c_1 = 3$, $c_2 = -2$,

$$y = 3e^{x} - 2e^{2x}.$$

Example 2.18. *Solve* $y'' - 2y' + y = 0$.

Solution: Here $r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$ is the only root. The general solution is

$$y = c_1 e^{x} + c_2 x e^{x}.$$

The constants can be found using initial conditions as in the previous example.

That really is the whole story for equation Eq.(2.17) when the roots are real. This is simple stuff.

The roots of $r^2 + br + c = 0$ can be complex, say

$$r_1 = \alpha + i\beta, \quad \text{and} \quad r_2 = \alpha - i\beta.$$

The complex solution corresponding to the "+" sign is $e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$. It turns out (can be shown) that independent real solutions are $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$, and the general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x). \tag{2.19}$$

Example 2.19. *Solve* $y'' + 6y' + 10y = 0$.

Solution: The characteristic equation is $r^2 + 6r + 10 = 0$, with root $-3 + i$ and its conjugate $-3 - i$. The general solution is

$$y = e^{-3x}(c_1\cos(x) + c_2\sin(x)).$$

If initial conditions $y(0) = 1$ and $y'(0) = 1$ are given, then $1 = y(0) = c_1$, so

$$y = e^{-3x}\cos(x) + c_2 e^{-3x}\sin(x).$$

Thus

$$y' = -3e^{-3x}\cos(x) - e^{-3x}\sin(x) + c_2 e^{-3x}(\cos(x) - 3\sin(x)),$$

and $1 = y'(0) = -3 + c_2$, so $c_2 = 4$ and finally

$$y = e^{-3x}(\cos(x) + 4\sin(x)).$$

An even simpler suite of examples is the family of equations $y'' + \beta^2 y = 0$, where there is no exponential because the roots are purely imaginary. The general solution here is $y = c_1\cos(\beta x) + c_2\sin(\beta x)$. This is a key example.

Example 2.20. *Solve* $y'' + 4y = 0$, *with* $y(0) = 1$, $y'(0) = 0$.

Solution: The roots are $2i$ and $-2i$ so the general real solution is

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$

Setting $x = 0$ gives $1 = y(0) = c_1$, while $0 = y'(0) = 2c_2$, and thus

$$y = \cos(2x).$$

Homework 2.8. *Find the general solution of the following homogeneous problems*

1. $y'' + 4y' + 4y = 0$
2. $y'' + 2y' - 15y = 0$
3. $y'' + 4y = 0$
4. $y'' + 2y' + 2y = 0$
5. $y'' + y' + y = 0,$

Homework 2.9. *Find the solution of the following initial value problems*

1. $y'' - y' - 2y = 0,\ y(0) = y'(0) = 1$
2. $y'' - 4y' - 21y = 0,\ y(0) = y'(0) = 1$
3. $y'' + 16y = 0,\ y(0) = y'(0) = 0$
4. $y'' + 2y' + 5y = 0,\ y(0) = 1,\ y'(0) = 0$
5. $y'' + y' + y = 0,\ y(0) = 1,\ y'(0) = 0$

Project 2.1. *This goes back to the constant coefficient equation Eq.(2.17), $y'' + py' + qy = 0$, and the solution Eq.(2.17) but where the roots are equal, $r_1 = r_2$. For equal roots Eq.(2.17) gives only one independent solution. To find a second independent solution first suppose that the roots are different, $r_1 \neq r_2$, and show that the solution $y(t)$ with $y(0) = 0$ and $y'(0) = 1$ is given by*

$$y(x) = \frac{e^{r_2 x} - e^{r_1 x}}{r_2 - r_1}$$

Now use L'Hospital's Rule to show that the limit of $y(x)$ as $r_2 \to r_1$ is $y(x) = xe^{r_1 x}$. This is a justification that, in the case of equal roots $r_1 = r_2 = r$, the second independent solution is xe^{rx}.

Project 2.2. *Consider the equation $y'' + py' + qy = 0$, with constant p and q. For a certain constant α define $z = ye^{-\alpha x}$. Show that the original differential equation reduces to a differential equation in the variable z with no z' term for the right choice of α. What is that value of α in terms of p and q?*

Project 2.3. *ODEs in the form $x^2y'' + pxy' + qy = 0$, where p and q are constants, are called Cauchy-Euler or equi-dimensional equations. Show that there are solutions of the form $y = x^s$, for real or complex numbers s, and provide a means of finding the appropriate s values. Illustrate with the equation $x^2y'' + 4xy' + 2y = 0$.*

2.4.2 Nonhomogeneous Equations

The generic constant coefficient equation with "forcing term" $g(x)$ has the form

$$y'' + py' + qy = g(x). \tag{2.20}$$

If one can find a single and specific solution $y_p(x)$ to Eq.(2.20), which will be referred to as a *particular solution*, then the general solution to Eq.(2.20) turns out to be

$$y = y_h + y_p, \tag{2.21}$$

where y_h is the general solution of the associated homogeneous equation Eq.(2.17). Then solving Eq.(2.20) amounts to two things:

1. Solving the associated Eq.(2.17) in general (as has been covered)

2. Finding a single particular solution y_p to Eq.(2.20).

Example 2.21 (Trivial example)**.** *For $y'' - 3y' + 2y = 1$, $y_p = 1/2$ by inspection.*

Solution: The general solution is therefore

$$y = c_1e^x + c_2e^{2x} + (1/2).$$

Suppose initial values $y(0) = 1$, $y'(0) = 0$ were prescribed. Then $y' = c_1e^x + 2c_2e^{2x}$ and plugging in $x = 0$ and using the initial conditions gives

$c_1 + c_2 + (1/2) = 1$, $c_1 + 2c_2 = 0$, or $c_1 = 1$, $c_2 = -1/2$ and consequently the unique solution is

$$y = e^x - (1/2)e^{2x} + (1/2).$$

Notice that initial conditions are applied to the whole solution $y = y_h + y_p$, not just y_h.

When the function $g(x)$ consists of polynomials, exponentials, sines and cosines, then a particular solution can be found using the method of undetermined coefficients. This is a fairly involved procedure in its most general form. However, in this course the full generality will not be needed. These notes will take up only two cases.

Case 1. The forcing term $g(x)$ has the form $g(x) = Ke^{mx}$.

Form of y_p: In this case

$$y_p = Ax^s e^{mx}, \quad A = \text{constant},$$

where s is the smallest among 0, 1 or 2 such that $x^s e^{mx}$ is not a solution of the homogeneous equation Eq.(2.17).

Example 2.22. *Find the general solution to* $y'' - 3y' + 2y = 10e^{3x}$.

Solution: The characteristic equation is $r^2 - 3r + 2 = (r - 1)(r - 2) = 0$ so the roots are $r = 1$ and $r = 2$. The homogeneous solution is thus

$$y_h = c_1 e^x + c_2 e^{2x}.$$

In this problem $m = 3$ and so $y_p = Ax^s e^{3x}$; s can be taken to be $s = 0$ because $Ax^0 e^{3x} = Ae^{3x}$ is not a solution of the homogeneous equation. Consequently

$$y_p = Ae^{3x}.$$

Differentiating, $y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x}$ and plugging in gives $[9A - 3(3A) + 2A]e^{3x} = 10e^{3x}$. Canceling the exponential term and solving for A results in $A = 5$, so the general solution is

$$y = c_1 e^x + c_2 e^{2x} + 5e^{3x}.$$

The constants c_1 and c_2 would be found using initial conditions.

Example 2.23. *Find the solution to* $y'' - 3y' + 2y = 10e^{3x}$*, with initial conditions* $y(0) = y'(0) = 0$.

Solution: From the previous example we get that the general solution is

$$y = c_1 e^x + c_2 e^{2x} + 5e^{3x}.$$

If $y(0) = y'(0) = 0$ then $c_1 + c_2 + 5 = 0$ and $c_1 + 2c_2 + 15 = 0$ yield $c_1 = 5$ and $c_2 = 10$ for

$$y = 5e^x + 10e^{2x} + 5e^{3x}.$$

Notice that the homogeneous initial conditions $y(0) = y'(0) = 0$ will result in the trivial solution $y(x) = 0$ unless the nonhomogeneous term $g(x)$ is not identically 0.

Example 2.24 (Modified Example)**.** *Find the solution to* $y'' - 3y' + 2y = 10e^{2x}$ *with initial conditions* $y(0) = 0$*,* $y'(0) = 0$.

Solution: The homogeneous solution is still

$$y_h = c_1 e^x + c_2 e^{2x}.$$

This time $m = 2$, the particular solution takes the form

$$y_p = Ax^s e^{2x}$$

but s can no longer be 0 because e^{2x} is a solution of the homogeneous equation. However, xe^{2x} is not a solution of the homogeneous equation and thus $s = 1$. Remember that s is the smallest among 0, 1 or 2 such that $x^s e^{2x}$ is not a solution of the homogeneous equation; in particular, $s = 2$ would be incorrect. Now differentiate $y_p = Axe^{2x}$ to obtain $y_p' = Ae^{2x} + 2Axe^{2x}$, $y_p'' = 4Ae^{2x} + 4Axe^{2x}$, plug into the original equation to get

$$(4Ae^{2x} + 4Axe^{2x}) - 3(Ae^{2x} + 2Axe^{2x}) + 2(Axe^{2x}) = 10e^{2x}.$$

Drop the exponentials, note that all the terms containing x as a factor cancel and see that $A = 10$. The general solution is then

$$y = c_1 e^x + c_2 e^{2x} + 10xe^{2x}.$$

(It's a coincidence that both input and output constants are 10.) To solve the initial value problem $y'' - 3y' + 2y = 10e^{2x}$, $y(0) = 0$, $y'(0) = 0$, note that $y' = c_1e^x + 2c_2e^2x + 10e^{2x} + 20xe^{2x}$, and setting $x = 0$ yields $c_1 + c_2 = 0$ and $c_1 + 2c_2 + 10 = 0$. Solving together gives $c_1 = 10$, $c_2 = -10$ and thus

$$y = 10e^x - 10e^{2x} + 10xe^{2x}.$$

What kind of problem can have $s = 2$? There's only one way and that is when the m in $g(x) = Ke^{mx}$ is a double root of the characteristic equation. An example is $y'' - 4y' + 4y = e^{2x}$, where the characteristic equation is $r^2 - 4r + 4 = (r - 2)2$ and there is a double root at $r = 2$. The particular solution is $y_p = Ax^2e^{2x}$. The constant is found by plugging in as before, so there is really not much new here.

Case 2. The forcing term $g(x)$ is of the form $g(x) = e^{mx}(K\cos(\gamma x) + M\sin(\gamma x))$.
This time the particular solution has the form $y_p = x^se^{mx}(A\cos(\beta x) + B\sin(\beta x))$, where again s is the smallest among 0, 1 or 2 such that no term in y_p is a solution of the homogeneous equation.

Example 2.25. *Solve* $y'' - 2y' + y = 5\cos(x)$, $y(0) = y'(0) = 0$.

The characteristic equation is $r^2 - 2r + 1 = 0$, or $r = 1$ as a double root. The homogeneous solution is then

$$y_h = c_1e^x + c_2xe^x.$$

To match up the terms in Case 2, $m = 0$, $K = 5$, $\gamma = 1$ and $M = 0$. Since the homogeneous solution has no sine or cosine solutions, then $s = 0$ for this example, that is,

$$y_p(x) = A\cos(x) + B\sin(x).$$

One solves for A and B by substitution, namely $y_p' = -A\sin(x) + B\cos(x)$, $yp'' = -A\cos(x) - B\sin(x)$. Plugging into the nonhomogeneous equation and collecting terms, the coefficient of $\cos(x)$ is $-A - 2(B) + A = 5$, while the $\sin(x)$ coefficient is $-B - 2(-A) + B = 0$, as $g(x)$ has no sine term. Solving yields $B = -5/2$ and $A = 0$. Therefore $y = c_1 e^x + c_2 x e^x - (5/2)\sin(x)$. The derivative is

$$y' = c_1 e^x + c_2 x e^x + c_2 e^x - (5/2)\cos(x),$$

so setting $x = 0$ gives $c_1 = 0$ and $c_1 + c_2 - (5/2) = 0$, or $c_2 = 5/2$. Finally,

$$y = (5/2) x e^x - (5/2)\sin(x).$$

If one adds exponential terms to the right hand side of the previous example, then the computations are more tedious but the ideas are exactly the same.

Example 2.26. *How about $y'' + 4y = 3\sin(2x)$, $y(0) = y'(0) = 0$? This is a key example.*

This time

$$y_p = x^s(A\cos(2x) + B\sin(2x)),$$

but s cannot be 0 because $\cos(2x)$ and $\sin(2x)$ are solutions of the homogeneous equation. Since $x\cos(2x)$ and $x\sin(2x)$ are not solutions, then $s = 1$. The homogeneous solution is

$$y_h = c_1\cos(2x) + c_2\sin(2x),$$

and y_p takes the form

$$y_p = x(A\cos(2x) + B\sin(2x)).$$

The constants A and B are found as usual from substituting y_p into the original equation and equating sine and cosine coefficients. Doing the calculations, $y_p' = (A\cos(2x) + B\sin(2x)) + x(-2A\sin(2x) + 2B\cos(2x))$ and $y_p'' = 2(-2A\sin(2x) + 2B\cos(2x)) + x(-4A\cos(2x) - 4B\sin(2x))$. Substitute these into $y'' + 4y = 3\sin(2x)$ and equate the sine and cosine terms to get $-4A = 3$, $4B = 0$, $A = -3/4$, $B = 0$ and finally $y_p = -(3/4)x\cos(2x)$. The general solution is $y = c_1\cos(2x) + c_2\sin(2x) - (3/4)x\cos(2x)$. Taking the derivative,

$$y' = -2c_1\sin(2x) + 2c_2\cos(2x) - (3/4)\cos(2x) + (3/2)x\sin(2x),$$

and setting $x = 0$ in both y and y' gives $c_1 = 0$ and $2c_2 - (3/4) = 0$ so $c_2 = 3/8$. The unique solution is then

$$y = (3/8)\sin(2x) - (3/4)x\cos(2x).$$

In this problem $\beta = 2$ represents a natural frequency of $y'' + 4y = 0$ this is discussed further in later sections. When the right hand side involves a term oscillating with the natural frequency one says that the equation exhibits resonance.

Homework 2.10. *Find the solution of the following initial value problems*

1. $y'' + 2y' - 15y = e^{3x}$, $y(0) = y'(0) = 1$
2. $y'' - 4y = e^{x}$, $y(0) = y'(0) = 0$
3. $y'' + 4y = e^{2x}$, $y(0) = y'(0) = 0$
4. $y'' + 2y' + 2y = e^{-x}$, $y(0) = y'(0) = 0$

Homework 2.11. *Find the solution of the following initial value problems*

1. $y'' + 4y = 5\sin(x)$, $y(0) = y'(0) = 0$
2. $y'' + 4y = 5\sin(2x)$, $y(0) = y'(0) = 0$.
3. $y'' + 4y' + 4y = \cos(x)$, $y(0) = y'(0) = 0$
4. $y'' + 2y' - 15y = \sin(x)$, $y(0) = y'(0) = 0$

Homework 2.12. *Find the solution to the following initial value problem*

1. $y'' + 2y' + 2y = e^{-x}\sin(x)$, $y(0) = y'(0) = 0$

Project 2.4. *Show that the solution of the initial value problem* $y'' + \alpha y = \cos(\beta t)$, $y(0) = y'(0) = 0$ *is*

$$y_\beta(t) = \frac{\cos(\beta t) - \cos(\alpha t)}{\alpha^2 - \beta^2}, \quad \text{for } \alpha \neq \beta.$$

Use L'Hospital's Rule to show that the limit as $\beta \to \alpha$ of $y_\beta(t)$ is $t\sin(at)/(2\alpha)$. Verify that this is the solution with $\beta = \alpha$.

2.5 Existence Theory

2.5.1 Transforming into a System of Equations

There are many reasons to transform the way in which a given differential equation with a real independent variable x and real or complex dependent variable $y(x)$ is written. The new form is that of a vector equation. To briefly describe the vector setting, the idea is to replace Eq.(2.1) by a vector equation of the form $\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}(x) = \mathbf{f}(x, \mathbf{y}(x))$, where

$$\mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} f_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) \\ \vdots \\ f_n(x, \mathbf{y}) \end{pmatrix} \tag{2.22}$$

However, there are infinitely many solutions to Eq.(2.22), and to make the problem well posed one attaches an "initial condition" in the form

$$\mathbf{y}(x_0) = \mathbf{y}_0, \quad (x_0 \text{ and } \mathbf{y}_0 \text{ constants }) \tag{2.23}$$

and then Eq.(2.22) - Eq.(2.23) together comprise an initial value problem.

Here the dimension n can be any positive integer. The case $n = 1$ is Eq.(2.1). The case $n = 2$ contains second order equations. As covered in section 2.4, the equation $y'' + p(x)y' + q(x)y = f(x)$ is equivalent to

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q(x) & -p(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) \end{pmatrix} \tag{2.24}$$

where one defines $y_1 = y$ and $y_2 = y'$. Note that $y_1' = y_2$ and

$$x_2' = y'' = -q(x)y - p(x)y' + f(x) = -q(x)x_1 - p(x)x_2 + f(x).$$

In other words, Eq.(2.24) is equivalent to $y'' + p(x)y' + q(x)y = f(x)$. Any theory supporting Eqs.(2.22)–(2.23) then pertains to general second order ODEs. The identification of $y_1 = y$ and $y_2 = y'$ is called the "standard embedding." It basically embeds the scalar (1-dimension) variable $y(x)$ as a vector in 2-D space.

Example 2.27. *Re-express the second order ODE $y'' + 5y' - 6y = 0$ as a system.*

Solution: We use the substitution $y_1 = y$ and $y_2 = y'$ then

$$y_1' = y_2, \quad y_2' + 5y_2 - 6y_1 = 0.$$

This can be written in matrix form as

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ 6y_1 - 5y_2 \end{pmatrix}$$

and this becomes

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ 6y_1 - 5y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6 & -5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{f}(\mathbf{y})$$

Example 2.28. *Same question for $y'' - ty = \cos(2t)$, $y(0) = 1$, $y'(0) = 0$.*

Solution: This is an initial value problem whose system equivalent is obtained by using the same thinking. Let $\mathbf{y}(t) = (y_1, y_2)^T = (y(t), y'(t))^T$ and use the ODE to see that

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{y}) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(2t) \end{pmatrix}$$

What about the initial values? They are just $\mathbf{y}(0) = (1, 0)^T$.

Example 2.29. *The embedding applies to nonlinear equations as well. Take the equation $y'' - y' - y^3 = 0$*

Solution: Using the same idea

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_1^3 + y_2 \end{pmatrix} \text{ and } \mathbf{f}\left(t, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} y_2 \\ y_1^3 + y_2 \end{pmatrix}$$

The first use here of the embedding will be to solve second order ODEs numerically. Almost all codes for solving ODEs numerically are written in vector language, as will now be explained. For this reason, second order ODEs to be solved numerically are first written in vector form.

Example 2.30. *A single second order ODE of the type $y'' + by' + cy = 0$ is equivalent to a system of type Eq.(2.7).*

Solution: One lets $y_1 = y$ and $y_2 = y'$ and then notes that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (y_2' = y'' = -qy - py' = -qy_1 - py_2)$$

Homework 2.13. *Transform the following second order differential equations into first order linear systems*

1. $y'' + 4y' + 4y = 0$
2. $y'' + 2y' - 15y = 0$
3. $y'' + 4y = 0$
4. $y'' + 2y' + 2y = 0$
5. $y'' + y' + y = 0,$

2.5.2 Existence Theory

Going back to the initial value problem (IVP) Eqs.(2.22)–(2.23), namely

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{f}(t, \mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2.25}$$

there is a fundamental existence and uniqueness theorem which can be stated as follows.

Theorem 2.2 (Existence and Uniqueness Theorem)**.** *Consider the IVP Eq.(2.25) where $\mathbf{f}(t, \mathbf{x})$ is piecewise continuous in a neighborhood of the point $(t_0, \mathbf{x}_0)$. Then Eq.(2.25) has a unique solution valid in some neighborhood of t_0.*

The hypothesis of piecewise continuity can be considerably weakened. The neighborhood of existence $(t_0 - \epsilon, t_0 + \delta)$ can be predicted only in special cases. If the equation is linear, then the interval is at least as large as the largest interval around t_0 for which the coefficients are piecewise continuous.

Example 2.31. *For instance, the simple 1-D problem , $x' = -1/(t-1)$, $x(0) = -1$, has solution $x(t) = 1/(t-1)$, which has a singularity at the same place as the coefficient $-1/(t-1)$. This is typical of linear equations. But the nonlinear IVP $x' = x^2$, $x(0) = x_0$, has solution $x(t) = x_0/(1 - x_0 t)$ which blows up at $t = 1/x_0$ despite the fact that the right hand side of $x' = x^2$ is continuous everywhere.*

Theorem 2.3 (Linear Existence and Uniqueness Theorem)**.** *If* $\mathbf{f}(t, \mathbf{y})) = [\mathbf{A}](t)\mathbf{y}$ *and* $[\mathbf{A}](t)$ *is piecewise continuous on an interval* $[a, b]$*, then there is a unique solution to Eq.(2.25) on all of* $[a, b]$*.*

Assume that a system can be express in the form

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}(t) = [\mathbf{A}]\,\mathbf{y}(t) \tag{2.26}$$

where $[\mathbf{A}]$ is a matrix with constant coefficients.

Suppose that a eigen-vector $\mathbf{x}_\lambda \neq 0$ and eigen-value λ can be found such that

$$[\mathbf{A}]\,\mathbf{x}_\lambda = \lambda \mathbf{x}_\lambda \tag{2.27}$$

and consider the function $\mathbf{y}(t) = e^{\lambda t}\mathbf{x}_\lambda$. Differentiation shows that $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t) = \lambda e^{\lambda t}\mathbf{x}_\lambda$ and, on the other hand,

$$[\mathbf{A}]\,\mathbf{x}(t) = e^{\lambda t}\,[\mathbf{A}]\,\mathbf{x}_\lambda = \lambda e^{\lambda t}\mathbf{x}_\lambda = \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}(t)$$

In other words, $\mathbf{y}(t) = e^{\lambda t}\mathbf{x}_\lambda$ is a solution to Eq.(2.26).

Example 2.32. *Consider*

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{y}(t), \quad \mathbf{y}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Note that the pairs $\lambda = 3$, $\mathbf{x}_\lambda = (1, 2)^T$ and $\lambda = -1$, $\mathbf{x}_\lambda = (1, -2)^T$ are eigendata. The two solutions are $\mathbf{y}_1 = e^{3t}(1, 2)^T$ and $\mathbf{y}_2 = e^{-t}(1, -2)^T$. Now postulate that the unique solution is of the form

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

for certain constants c_1 and c_2. Set $t = 0$ to get the resulting linear equations

$$\begin{cases} c_1 + c_2 &= 3 \\ 2c_1 - 2c_2 &= 5 \end{cases}$$

whose solution is $c_1 = 11/4$, $c_2 = 1/4$. Here $\mathbf{f}(t, \mathbf{y}) = [\mathbf{A}](t)\mathbf{y}$ where $[\mathbf{A}](t) = [\mathbf{A}]$ is constant and thus continuous for all t. The unique solution, defined for all t, is thus

$$\mathbf{y}(t) = e^{3t} \begin{pmatrix} 11/4 \\ 11/2 \end{pmatrix} + e^{-t} \begin{pmatrix} 1/4 \\ -1/2 \end{pmatrix}$$

Example 2.33. *Consider the system* $\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}(t) = [\mathbf{A}]\,\boldsymbol{y}(t)$*, where*

$$[\mathbf{A}] = \begin{pmatrix} -1/2 & 0 \\ 1/2 & -1 \end{pmatrix}$$

The eigenvalues are obtained from

$$\begin{vmatrix} \lambda + 1/2 & 0 \\ -1/2 & \lambda + 1 \end{vmatrix} = (\lambda + 1/2)(\lambda + 1) = 0.$$

One can check that e-vectors corresponding to $-1/2$ and -1 are, respectively,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Homework 2.14. *Find the unique solution to the IVP* $\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}(t) = [\mathbf{A}]\,\mathbf{y}(t)$ *with*

$$[\mathbf{A}] = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Homework 2.15. *Find the unique solution to* $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t) = [\mathbf{A}]\,\mathbf{x}(t)$ *where*

$$[\mathbf{A}] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -3 & 0 \\ -6 & 0 & 5 \end{pmatrix} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$