MATH 503: Mathematical Statistics
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Homework 10

1. Observations (x_i, Y_i), $i = 1, \ldots, n$, are collected according to the model $Y_i = \alpha + \beta x_i + \epsilon_i$, where $E(\epsilon_i) = 0$, $\text{Var}(\epsilon_i) = \sigma^2$, and $\text{Cov}(\epsilon_i, \epsilon_j) = 0$ if $i \neq j$. find the best linear unbiased estimator of α.

2. Consider the residuals $\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n$ defined by $\hat{\epsilon}_i = Y_i - \hat{\alpha} - \hat{\beta} x_i$.
   (a) Show that $E(\hat{\epsilon}_i) = 0$.
   (b) Verify that $\text{Var}(\hat{\epsilon}_i) = \text{Var}(Y_i) + \text{Var}(\hat{\alpha}) + x_i^2 \text{Var}(\hat{\beta}) - 2\text{Cov}(Y_i, \hat{\alpha}) - 2x_i \text{Cov}(Y_i, \hat{\beta}) + 2x_i \text{Cov}(\hat{\alpha}, \hat{\beta})$.
   (c) Use Lemma 12.2.1 to show that $\text{Cov}(Y_i, \hat{\alpha}) = \sigma^2 \left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}} \right)$ and $\text{Cov}(Y_i, \hat{\beta}) = \sigma^2 \frac{x_i - \bar{x}}{S_{xx}}$, and use these to verify the equation for $\text{Var}(\hat{\epsilon})$, (12.2.23).

3. Fill in the details about the distribution of $\hat{\alpha}$ left out of the proof of Theorem 12.2.1.
   (a) Show that the estimator $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$ can be expressed as $\hat{\alpha} = \sum_{i=1}^n c_i Y_i$, where $c_i \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}$.
   (b) Verify that $E(\hat{\alpha}) = \alpha$ and $Var(\hat{\alpha}) = \sigma^2 \left[\frac{1}{nS_{xx}} \sum_{i=1}^n x_i^2 \right]$.
   (c) Verify that $Cov(\hat{\alpha}, \hat{\beta}) = \frac{-\sigma^2 \bar{x}}{S_{xx}}$.

4. We obtain observations $Y_1, \ldots, Y_n$ which can be described by the relationship $Y_i = \theta x_i^2 + \epsilon_i$, where $x_1, \ldots, x_n$ are fixed constants and $\epsilon_1, \ldots, \epsilon_n$ are iid $\text{N}(0, \sigma^2)$.
   (a) Find the least squares estimator of θ.
   (b) Find the MLE of θ.