LECTURE 3 - 2/9/2021

Topics for today:

1. set theory (finite, countable, uncountable)
2. discrete r.v.
3. expectations/expected values.

we want to be able to think about the "sizes" of sample spaces.

A finite (discrete) sample space $\Omega = \{w_1, w_2, \dots, w_n\}$.

A countably infinite (discrete) sample space $\Omega = \{w_1, w_2, \dots\}$.

An uncountable sample space $\Omega = \mathbb{R}$, $[0,1]$

Def: A set Ω is finite if there exists $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ such that $f(x) : [1, 2, \dots, n] \longrightarrow \Omega$, where f is one-to-one and onto.

Def: A set Ω is infinitely countable if there exists
$$f(x) : \mathbb{Z}^+ \to \Omega$$

Theorem: $\mathbb{R}$, $[0,1]$ are uncountable (not countable).

How do these facts about sets help us in probability?

finite sample space
$$\Omega = \{w_1, w_2, \dots, w_n\}.$$
$$\mathcal{F} = \mathcal{P}(\Omega)$$
$$P(w_i) = a_i$$
$$a_1 + a_2 + \dots + a_n = 1$$
$$a_i \ge 0$$
Define $P(A) = \sum_{w_i \in A} P(w_i) = \sum_{w_i \in A} a_i$

We created probability measures as in this structure.

In a countable infinite space it is similar:

$$\Omega = \{w_1, w_2, w_3, \dots\}$$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

$$P(w_i) = a_i$$

$$\sum_{i=1}^{\infty} a_i = 1, \ a_i \geq 0$$

$$P(A) = \sum_{w_i \in A} P(w_i) = \sum_{w_i \in A} a_i$$

we can still construct probability measures in this way for countably infinite sample spaces

Note: this just requires infinite sums, limits, etc.

we will deal with uncountable sample space later, but note:

If $\Omega = \mathbb{R}$

$P(w_i) = a_i$ for all $w_i \in \Omega$

$a_1 + a_2 + \dots = 1$ ← this is not possible for uncountable to do this sum in this ordered way.

$\sum_{w \in \Omega} a_w = 1$ ← Also not possible because we can't add incrementally.

Likewise, all components of our method of building probability measures collapses.

⇒ we cannot add up uncountably many summands

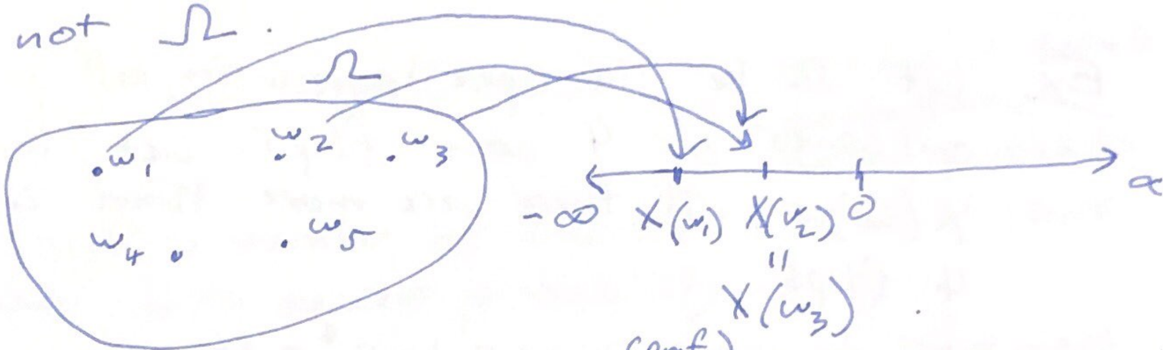
↳ we will revisit this.

Random Variables

Def: A discrete random ~~var~~ variable is a function $X : \Omega \to \mathbb{R}$ where Ω is a sample space of $(\Omega, \mathcal{F}, P)$ and the range or image of X ($\mathrm{Im}(X)$) takes on only countably many values.

Last week's definition stated that Ω be finite. However, what needs to be countable or finite is $\mathrm{Im}(X)$, not Ω.

Recall:



Def: The probability mass function (pmf) or distribution of X is a function $p(z) : \mathbb{R} \to [0,1]$ defined by

$$p(z) = P(X = z) = P(\{\omega \mid X(\omega) = z\}).$$

Bernoulli Random Variable: $X = 0$ or $X = 1$ (a coin flip)

Ex $\quad \left.\begin{array}{l} \Omega = \{0, 1\} \\ P(0) = q,\ P(1) = 1 - q \\ (\Omega, \mathcal{P}(\Omega), P) \end{array}\right]$ ← no X here yet, just a probability space.

$X(0) = 0,\ X(1) = 1.$

$\Rightarrow \quad P(X = 0) = P(\{\omega \mid X(\omega) = 0\}) = q$

$P(X = 1) = 1 - q$

$\Rightarrow X = \begin{cases} 0 & \text{prob } q \\ 1 & \text{prob } 1-q \end{cases}$] this is pmf.

Note that there is only <u>one</u> parameter to the Bernoulli Random Variable q

$1-q$ is known as the success probability, the probability that you get a 1.

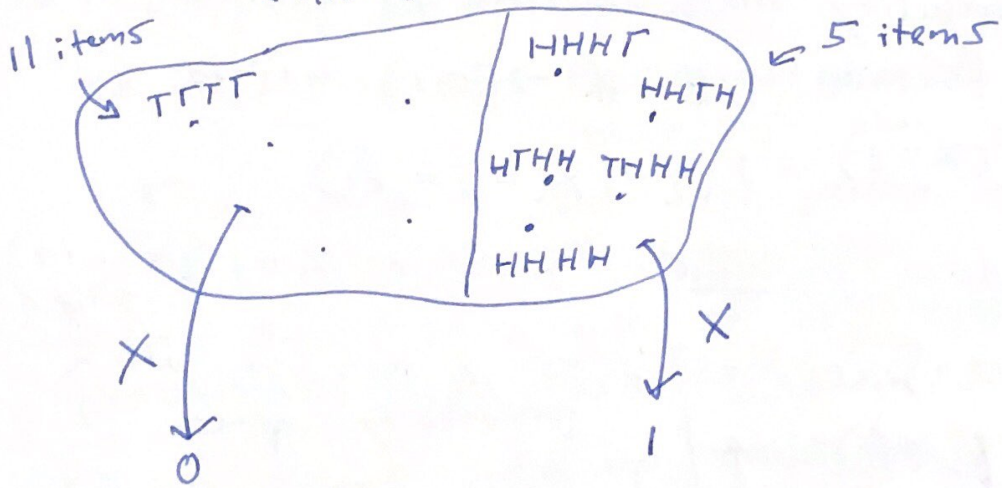
Ex $\Omega = \{H, T\}$

$$P(H) = q$$
$$P(T) = 1-q$$

$$X(H) = 0 \qquad X = \begin{cases} 0 & \text{prob } q \\ 1 & \text{prob } 1-q \end{cases} \leftarrow \text{success probability.}$$
$$X(T) = 1$$

Ex Let Ω be the space representing all possible outcomes of 4 coin flips. Let for $w \in \Omega$, $X(w) = 1$ if there are more than 2 H in the 4 flips.

$$\Omega = \left\{ \begin{matrix} HHHH, \\ HHHT, \end{matrix} \ldots \right\}, \quad |\Omega| = 2^4 = 16.$$



$$P(X=1) = \frac{5}{16}, \quad P(X=0) = \frac{11}{16}.$$

so X is a Bernoulli r.v. with success probability $5/16$.

Ex $X = \begin{cases} 0 & \text{prob } 1-q \\ 1 & \text{prob } q \end{cases}$ → where is Ω? Doesn't matter.

Note; as in this example, we often don't get an Ω, but there is always an Ω "behind the scenes"!

↳ Theorem 2.7 in the Book: we can always construct an $(\Omega, \mathcal{F}, P)$ to match X.

Binomial r.v. Binomial(n,p), n = number of trials
p = ~~probability~~ success probability.

Binomial(n,p) = number of successful trials if there are n trials with success probability p.

$$X = \begin{cases} 0 & \text{prob } \binom{n}{0} p^0 (1-p)^n \\ 1 & \binom{n}{1} p^1 (1-p)^{n-1} \\ 2 & \vdots \\ \vdots & \vdots \\ n & \end{cases}$$

note that these must be in $[0,1]$ and the sum is 1.

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \dots, n.$$

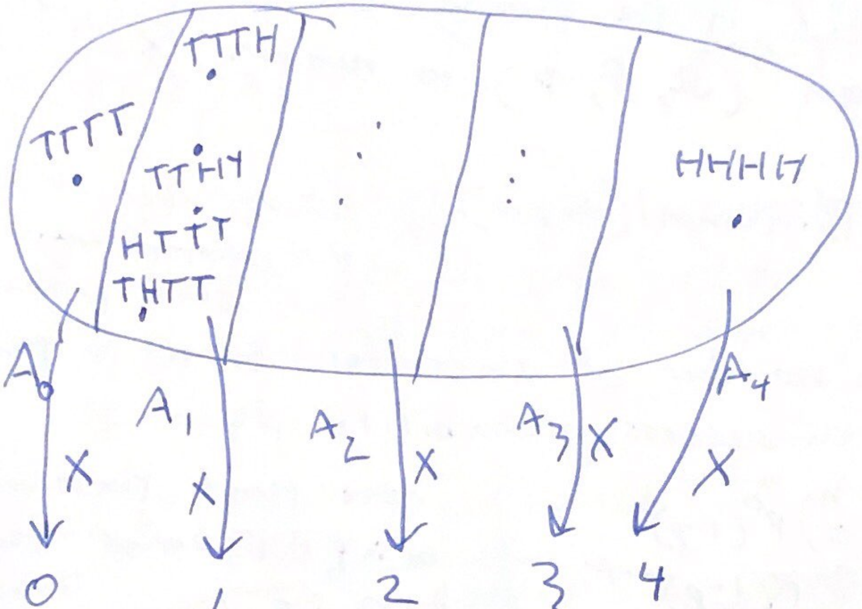
Therefore it must be true that $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$ and this can be proven ~~and is called the Binomial theor..~~

$$(p + (1-p))^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$ ← the Binomial theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$ ← in more general terms

~~Ex~~ <u>Ex</u> Let Ω be the space representing all possible outcomes of 4 coin flips. $X(\omega) = \#$ of H in the outcome ω. (same Ω from above).

$\Omega = \{HHHH, THHH, \dots\}$, $|\Omega| = 16$.



the number of ways of getting k H out of n tosses, which is $\binom{n}{k}$.

$$P(X=k) = \left(\tfrac{1}{2}\right)^4 |A_k| = \binom{4}{k}\left(\tfrac{1}{2}\right)^4$$

~~In general~~ $\Rightarrow$ $P(X=k) = \binom{4}{k}\left(\tfrac{1}{2}\right)^k\left(\tfrac{1}{2}\right)^{4-k}$

So X is binomial$(4, \tfrac{1}{2})$.

Geometric r.v.

Geometric(p) ~ the number of trials (flips of coin) needed before a success is achived, with probability of success in each trial of p.

Note: this is a countably infinite.

$$X = \begin{cases} 1 & \text{prob } p \\ 2 & \text{prob } p(1-p) \\ 3 & \text{prob } p(1-p)^2 \\ 4 & \\ \vdots & \vdots \\ \cancel{a} & \end{cases}$$

$\rightarrow$ ~~prob~~ $P(X=k) = p(1-p)^{k-1}$ for $k = 1, 2, \ldots$

$$\sum_{k=1}^{\infty} P(X=k) = \sum_{k=1}^{\infty} p(1-p)^{k-1} = p \sum_{k=1}^{\infty} (1-p)^{k-1} = p\left(\frac{1}{p}\right) = 1.$$

quick proof: $\cancel{T = 1 + a + a^2}$

$$\sum_{j=0}^{\infty} a^j = a^0 + a^1 + a^2 + a^3 + \cdots$$

$$T = 1 + a + a^2 + a^3 + \cdots$$

$$- \quad aT = a + a^2 + a^3 + a^4 + \cdots$$

$$T - aT = 1$$

$$\Rightarrow T(1-a) = 1 \Rightarrow T = \frac{1}{1-a}.$$

~~Ex: Let Ω be the space of all possible infinitely many coin flips.~~

~~$\Omega = \{ HHH\cdots, THH\cdots$~~

Ex: I flip a coin repeatedly until it lands H. Let X be the number of flips. The coin lands H with probability p.

$$\Omega = \{H, TH, TTH, TTTH, \dots\}.$$

$$P(X=k) = (1-p)^{k-1}p \Rightarrow X \text{ is geometric with success probability } p$$

or X is geometric(p).

Notation: $X \sim \text{Geometric}(p)$. ← This means X has this pmf or dist.

Poisson r.v.: X can have values $0, 1, 2, 3, \dots$

$$P(X=k) = \frac{e^{-\mu}\mu^k}{k!} \quad \text{for } k = 0, 1, 2, 3, \dots$$

μ is the mean.

If $X \sim \text{Poisson}(\mu)$

$Y \sim \text{Binomial}(n, p)$

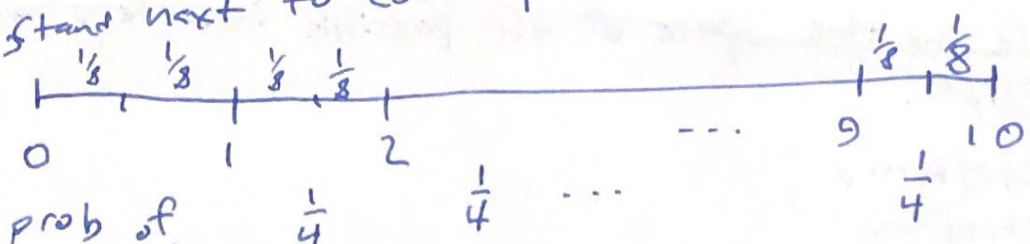
If we choose $p = \mu/n$, then

$$\lim_{n\to\infty} \text{Binomial}(n, \tfrac{\mu}{n}) \sim \text{Poisson}(\mu).$$

Intuitively, the more trials with lower probability.

Stand next to covid positive for ~~hours~~ minutes.

Ex:



Now split into half minutes.

prob of infection $\frac{1}{4}$

$\text{Binomial}(10, \frac{1}{4})$ in 1 minute increments

If we look at 2x increments $\text{Binomial}(20, \frac{1}{8})$, and we could do $Binomial(40, \frac{1}{16})$ in $\frac{1}{4}$ min increments

Note: $10 \cdot \frac{1}{4} = 2.5$, $20 \cdot \frac{1}{8} = 2.5$, $40 \cdot \frac{1}{16} = 2.5$.

It looks like $2.5 = \mu$. ~~So~~ So $p = \mu/n$.

This is like $\lim_{n \to \infty} Binomial(n, \frac{\mu}{n}) \sim Poisson(\mu)$

The proof is in the book and will be in HW.

Examples of things that follow Poisson:

1) # infection events (as above).
2) # highway deaths in a year
3) # gamma-rays given off by radioactive substance.

<u>Summary</u>: Remember these random variables:

Bernoulli(p) $\rightarrow$ like coin flip

Binomial(n,p) $\rightarrow$ like # of H in n coin flips

Geometric(p) $\rightarrow$ like # of flips before a H.

Poisson(μ) $\rightarrow$ like # of infections at a constant rate.

<u>Recall</u>: week 1: Prob space $(\Omega, \mathcal{F}, P)$, $P(A)$ = prob of an event A.

week 2/3: X r.v. why?

<u>Q</u>: on <u>average</u>, what happens in etc?
requires numbers / implies number.

<u>Q</u>: what is the probability to an <u>extreme</u> outcome? (p-value)

$\Rightarrow$ Random Variables help us quantify our prob. spaces.

Expected Value: (mean).

Def: The expected value of r.v. X is given by

$$E[X] = \sum_{z \in Im(X)} z P(X = z)$$

Ex: Bernoulli(p). $X = \begin{cases} 0 & \text{prob } 1-p \\ 1 & \text{prob } p. \end{cases}$

$$E[X] = 0 \cdot P(X=0) + 1 \cdot P(X=1)$$
$$= 0 + p$$

$$\Rightarrow E[X] = p.$$

Ex: $X = \begin{cases} 7 & \text{prob } 1/4 \\ 23 & \text{prob } 3/4 \end{cases}$

$$\Rightarrow E[X] = 7 \cdot P(X=7) + 23 \cdot (P(X=23))$$
$$= 7\left(\frac{1}{4}\right) + 23 \cdot \left(\frac{3}{4}\right)$$

Ex: Binomial(n, p)

$X \sim \text{Binomial}(100, \frac{1}{2})$

$$E[X] = \sum_{k=0}^{100} kP(X=k) = \sum_{k=0}^{100} k\binom{100}{k}\left(\frac{1}{2}\right)^{100}$$
$$= \left(\frac{1}{2}\right)^{100} \sum_{k=0} k\binom{100}{k} = ?$$

Note: $X \sim \text{Binomial}(n, p)$
$\Rightarrow E[X] = np$

Functions of r.v. (nothing to do with expected value, but touch on this is needed).

Given X, we might want to think about functions of X such as x^2, x^5, $2x^2+x+3$, e^{-5x}, etc.

function

$$g(x): \mathbb{R} \to \mathbb{R}$$
$$g(x) = x^2$$
$$g(x) = x^5$$
$$g(x) = 2x^2 + x + 3$$
$$g(x) = e^{5x}$$

these are not random, "pre-calc".

Then define $Y \sim g(x)$, e.g. $Y \sim x^2$, $\boxed{Y \sim 2x^2 + x + 3}$

$$X = \begin{cases} 0 & \text{prob } 1-p \\ 1 & \text{prob } p \end{cases} \Rightarrow Y = \begin{cases} 3 & \text{prob } 1-p \\ 6 & \text{prob } p \end{cases}$$

This "manual" translation may be too computationally cumbersome, especially for infinite cases.

(Back to Expected Value)

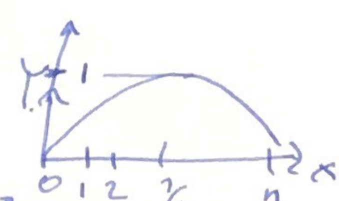
$$E[Y] = 3 \cdot P(Y=3) + 6\,P(Y=6)$$
$$E[g(x)] = g(0) \cdot P(X=0) + g(1) \cdot P(X=1)$$

when you want to get expected value of a function.

Ex: $X \sim \text{Binomial}(n,p) \Rightarrow X$ takes vals $0, \dots, n$

↳ This range "blows up", how to control?

$$Y \sim 4\frac{X}{n}\left(1 - \frac{X}{n}\right) \Rightarrow \begin{array}{l} X = 0,\ Y = 0 \\ X = n,\ Y = 0 \\ X = \frac{n}{2},\ Y = 1 \end{array}$$



so Y is always between $[0, 1]$.

$$\left.\begin{aligned} Y &\sim 4\frac{X}{n}\left(1-\frac{X}{n}\right) \\ g(X) &= \frac{4X}{n}\left(1-\frac{X}{n}\right) \end{aligned}\right\} \Rightarrow Y \sim g(X) \rightarrow \text{not 1-1.}$$

$\Rightarrow E[Y] =$ what values does Y take on?
what is its pmf?

But we know X well...

$$E[g(X)] = \sum_{k=0}^{n} g(k) P(X=k) = E[Y]$$

Proof of this is in the Book.

$$= \sum_{k=0}^{n} 4\frac{k}{n}\left(1-\frac{k}{n}\right)\binom{n}{k} p^k (1-p)^{n-k}$$

we did this by the fact that we know dist of X but not of Y.