MATH 503: Mathematical Statistics
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Homework 2

1. Let X_i, $i = 1, 2, \ldots$, be independent Bernoulli(p) random variables and let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Use the delta method to find the limiting distribution of $g(Y_n) = Y_n(1 - Y_n)$ for $p \neq \frac{1}{2}$.

2. Let $\bar{X}$ be the mean of a random sample from the exponential distribution, Exponential(θ).
   (a) Show that $\bar{X}$ is an unbiased point estimator of θ.
   (b) Using the mgf technique, determine the distribution of $\bar{X}$.
   (c) Use (b) to show that $Y = 2n\bar{X}/\theta$ has a χ^2 distribution with $2n$ degrees of freedom.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Poisson(θ) distribution, where θ is unknown. Let $Y = \sum_{i=1}^{n} X_i$. Find the distribution of Y and determine c so that cY is an unbiased estimator of θ.

4. Let $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics of a random sample of size n $(X_1, X_2, \ldots, X_n)$ from a Weibull distribution of the form $f(x) = cx^b \exp\left\{-\frac{cx^{b+1}}{b+1}\right\}$, $0 < x < \infty$, zero elsewhere. Find the distribution of Y_1.

5. Let X and Y denote independent random variables with respective probability density functions $f(x) = 2x$, $0 < x < 1$, zero elsewhere, and $g(y) = 3y^2$, $0 < y < 1$, zero elsewhere. Let $U = \min(X, Y)$ and $V = \max(X, Y)$. Find the joint pdf of U and V.

6. Let $X_1, X_2, \ldots, X_n$ represent a random sample from each of the distributions having the following pdfs or pmfs:
   (a) $f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}$, $x = 0, 1, 2, \ldots$, $0 \leq \theta < \infty$, zero elsewhere, where $f(0;0) = 1$
   (b) $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$, zero elsewhere
   (c) $f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere
   (d) $f(x;\theta) = e^{-(x-\theta)}$, $\theta <\leq x < \infty$, $-\infty < \theta < \infty$, zero elsewhere.

   In each case, find the mle $\hat{\theta}$ of θ.

7. Suppose $X_1, X_2, \ldots, X_n$ are iid with pdf $f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. Find the MLE of $P(X > k)$, for some $k > 0$ (known).

8. Let $X_1, \ldots, X_n$ be iid with pdf $f(x \mid \theta) = \theta x^{\theta - 1}$, $0 < x < 1$, $0 < \theta < \infty$.

   (a) Find the MLE of θ, and show that its variance converges to 0 as $n \to \infty$.

   (b) Find the method of moments estimator of θ.