test

October 27, 2022

1 Problem 1

Consider the sets

$C = \{(x, y) \mid ||x||_2 \leq y\}$ and $\hat{C} = \{(x, y) \mid ||x||_2^2 \leq y\}$

Determine whether the sets C and $\hat{C}$ are convex or not?

We recall that a set S is convex if, for two elements in S, then the linear combination of these elements is also in S. That is, if $x, y \in S$, then $\lambda x + (1 - \lambda)y \in S$.

Let $(x_1, y_1), (x_2, y_2) \in C$, then $||x_1||_2 \leq y_1$ and $||x_2||_2 \leq y_2$.

We see $\lambda||x_1||_2 \leq \lambda y_1$

and $(1 - \lambda)||x_2||_2 \leq (1 - \lambda)y_2$.

Then $||\lambda x_1 + (1 - \lambda)x_2||_2$

$= ||\lambda x_1||_2 + ||(1 - \lambda)x_2||_2$

$= \lambda||x_1||_2 + (1 - \lambda)||x_2||_2$

We then can use the set definition to get the following

$\lambda||x_1||_2 + (1 - \lambda)||x_2||_2 \leq \lambda y_1 + (1 - \lambda)y_2$

Therefore, $\lambda x + (1 - \lambda)y \in C$. This shows that C is convex.