MATH 502, HOMEWORK ASSIGNMENTS

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ABSTRACT. Final Exam for Math 502

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Attention: This exam is for you to do independently. Communication to any other person, except the teacher, about this exam is disallowed. You can consult your notes, previous homework and textbooks. Each problem is 10 points.

Please note the closing time at the canvas link. Give yourself enough time before the closing time to do this exam.

——————————- **Exam problems:**——————-

A1. (20pts) A is a square matrix. Show the following statements.

a) If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.

b) An eigenvector $\mathbf{v}$ of A is also an eigenvector of A^k.

c) (A major reason for the success of eigenvalues) If $A = UDU^{-1}$, then $A^k = UD^kU^{-1}$.

d) (D's form) If $A = UDU^{-1}$ and D is diagonal, then U's columns must be independent eigenvectors of A.

A2. (5pts) Prove or disprove the statement "If σ is a singular value of A, then σ^k is a singular value of A^k." Compare this with the problem A1 (a).

A3. (5pts) Assume A's non-zero singular values are $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r$. Show that the non-zero singular values of AA^TA are $\sigma_1^3, \sigma_2^3, ..., \sigma_r^3$.

A4. (10pts) Find the SVD for the matrix $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

Date: May 10, 2022.

A5. (10pts) (Optimization and eigenvalues) Recall that we use the Lagrange multiplier method in Cal III to find the point $\mathbf{x}$ that maximizes $f(\mathbf{x})$ under the constraint $g(\mathbf{x}) = 0$. The process can be lengthy.

Now, we solve the same problem for qudratic forms using linear algebra. That is to find the maximum point $\mathbf{x}$ and value of $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ while requiring $\mathbf{x}^T\mathbf{x} = 1$. Here A is a real symmetric matrix. Do the following:

a) Can the $n \times n$ matrix A be diagonalized as UDU^T? What are the matrices U, D made of?

b) Are U's column vectors $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n$ independent? Can I write any vector, $\mathbf{x}$, of size n as

$$\mathbf{x} = c_1\mathbf{u}_1, +c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n?$$

Explain.

c) If the answer for c) is "I can", then compute $\mathbf{x}^T\mathbf{x}$ and $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

d) Denote the largest eigenvalue of A as λ_1. Show that maximum point of $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ under the constraint $\mathbf{x}^T\mathbf{x} = 1$ is $\mathbf{x} =$ an eigenvector for λ_1. And the maximum value of $f(\mathbf{x})$ under the constraint is λ_1.

A6. (10 pts) Assume A is a positive definite matrix of size $n \times n$. Show that $\langle \mathbf{x}, \mathbf{y} \rangle$ defined as $\mathbf{x}^T A\mathbf{y}$ is an inner product. State the corresponding Cauchy-Schwarz inequality.

B1. (10pts) Solve the ODE system $\mathbf{x}' = A\mathbf{x}$ for the following matrices.

a) $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$.

b) $A = \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$.

B2. (10 pts) Roughly (qualitatively) graph the phase portrait of $\mathbf{x}' = A\mathbf{x}$.

a) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

c) $A = \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}$.

d) $A = \begin{bmatrix} 3 & 3 \\ -6 & -3 \end{bmatrix}$.

B3. (10pts) Consider the ODE system

$$\begin{cases} x' = (x-1)(x+y-2), \\ y' = y(4x+0.5y-4). \end{cases} \tag{1.1}$$

a) Find its equilibrium points.

b) Identify the type of each equilibrium point.

c) Roughly graph the phase portrait of the ODE system.

B4. (10 pts) The amount of heat needed to heat up my house by $1C^o$ is 20 KJ. My air heater produces 5 KJ of heat per minute. At time $t = 0$, the room temperature inside is $10C^o$.

a) Assuming there is no loss of heat from my house to the outside of the house. Find the room temperature T at time t.

b) However, houses lose heat to the cold air outside. The speed of heat loss is proportional to the temperature difference $(T - T_{out})$. Assume the proportion constant is 0.2 and the temperature outside is a constant $5C^o$. Find the room temperature T at time t.

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