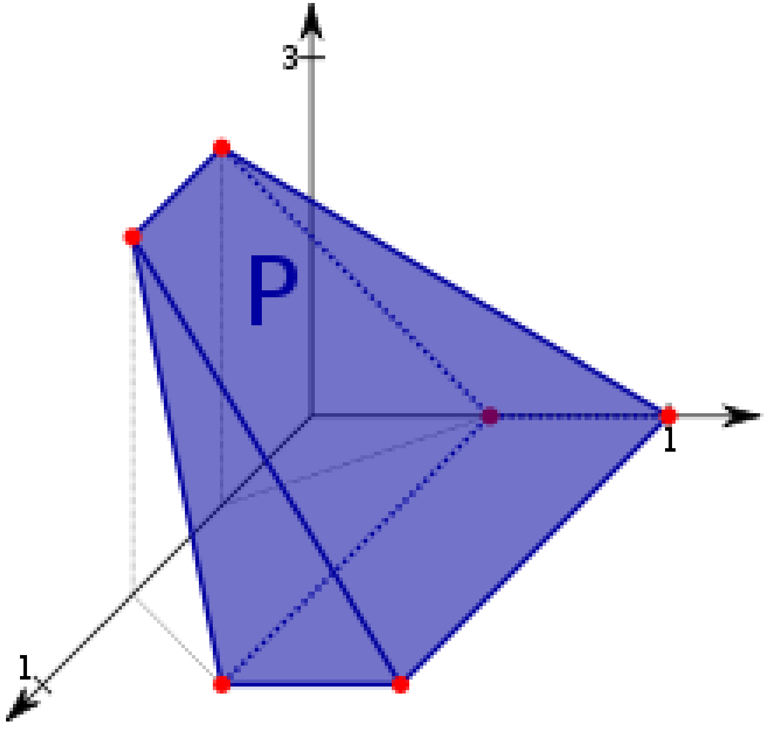
Introduction to the Simplex Method

Stuart Price

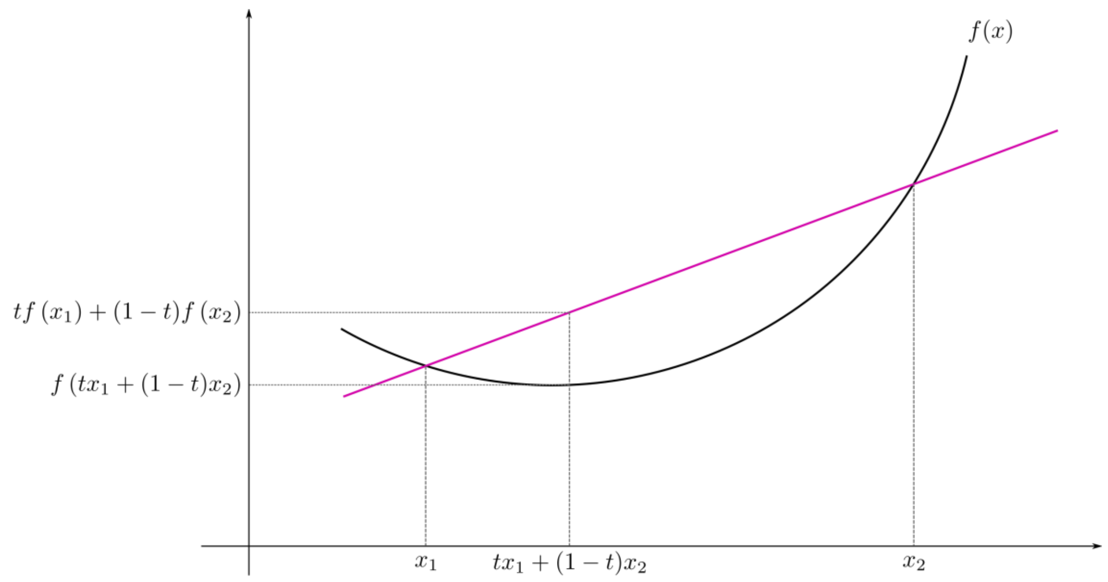
September 19th, 2022

Key Ideas from Last Time



- We are defining a polyhedron to be a set described in the form $\{x \in \mathbb{R}^n | Ax \geq b\}$ where A is an m x n matrix and $b \in \mathbb{R}^m$
- Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be non-negative scalars such that $\sum \lambda_i = 1$, then $\sum \lambda_i \, x_i$ is said to be a **convex combination** of vectors $x_1, x_2, \ldots, x_k$
- A **convex hull** of the vectors $x_1, x_2, \ldots, x_k$ is the set of all convex combinations of those vectors
- Let P be a polyhedron. A vector $x \in P$ is an **extreme point** of P if we cannot find two vectors $y, z \in P$, both different from x, and a scalar $t \epsilon [0,1]$ such that $\mathrm{x} = (ty + (1-t)z)$

Convex Functions and Convex Sets



- A function is **convex** if $\forall x, y \in \mathbb{R}^n$ $and\ t\epsilon[0,1]$ then $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$

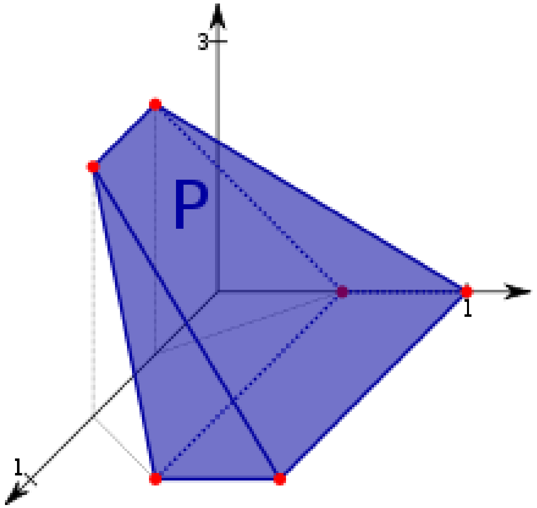
- A local minimum, x^* , of a convex function, f , on a convex set, S , is also the global minimum for the function, f , on the set, S.

Two Key Ideas Needed for Simplex Method

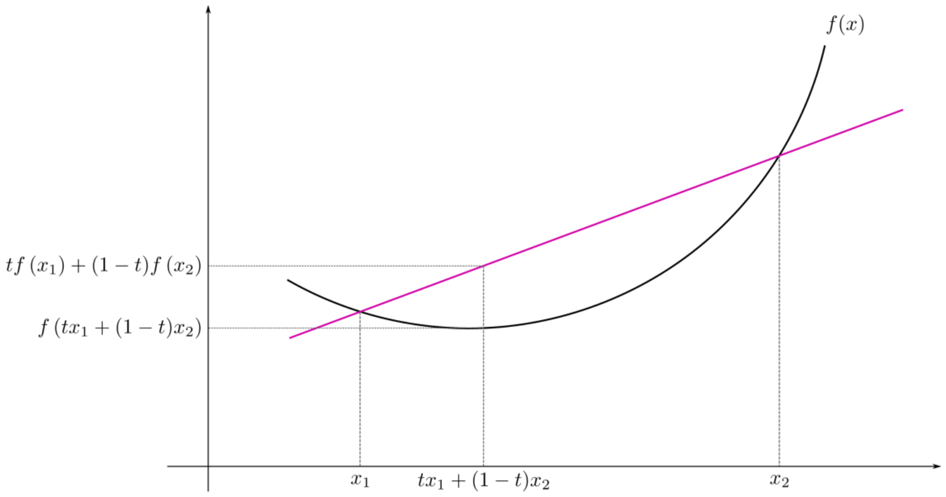
1: Let P be a nonempty polyhedron and let x^* be in P. Then the following are equivalent:

- x^* is a vertex
- x^* is an extreme point

A non-empty and bounded polyhedral is the convex hull of its extreme points.



2: A local minimum, x^* , of a convex function, f , on a convex set, S , is also the global minimum for the function, f , on the set, S



Equivalent Linear Programs and Row Operations

- Recall that two systems of equations are equivalent if the set of solutions to one is equal to the solutions to the other i.e.
$$\{x: Ax = b\} = \{x: \bar{A}x = \bar{b}\}$$
- The solution to a linear program is preserved under row operations that include:
  - Multiplying a row by a non-zero constant
  - Adding a scalar multiple of another row

Equality Form

max $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

s.t.

$$\mathbf{Ax=b}$$

$$\mathbf{x \geq 0}$$

- **x** is a vector (n) of the decision variables
  - **x** = $(x_1, x_2, \dots, x_n)$
- **c** is a vector (n) of the coefficients of the objective function
  - $\boldsymbol{c}$ = $(c_1, c_2, \dots, c_n)$
- **A** is a $m \times n$ matrix of the constraint coefficients
- **b** is a vector (m)of the right-hand side of the constraint equations
  - $\boldsymbol{b}$ = $(b_1, b_2, \dots, b_m)$

Sample Problem

You are assembling fruit baskets from the excess stock at your store. You have 10 bananas, 24 apples, and 16 melons. The baskets come in two configurations.

Mix 1 sells for \$3 and contains 1 banana, 3 apples, and 1 melon.

Mix 2 sells for \$2 and contains 1 banana, 1 apple, and 2 melons.

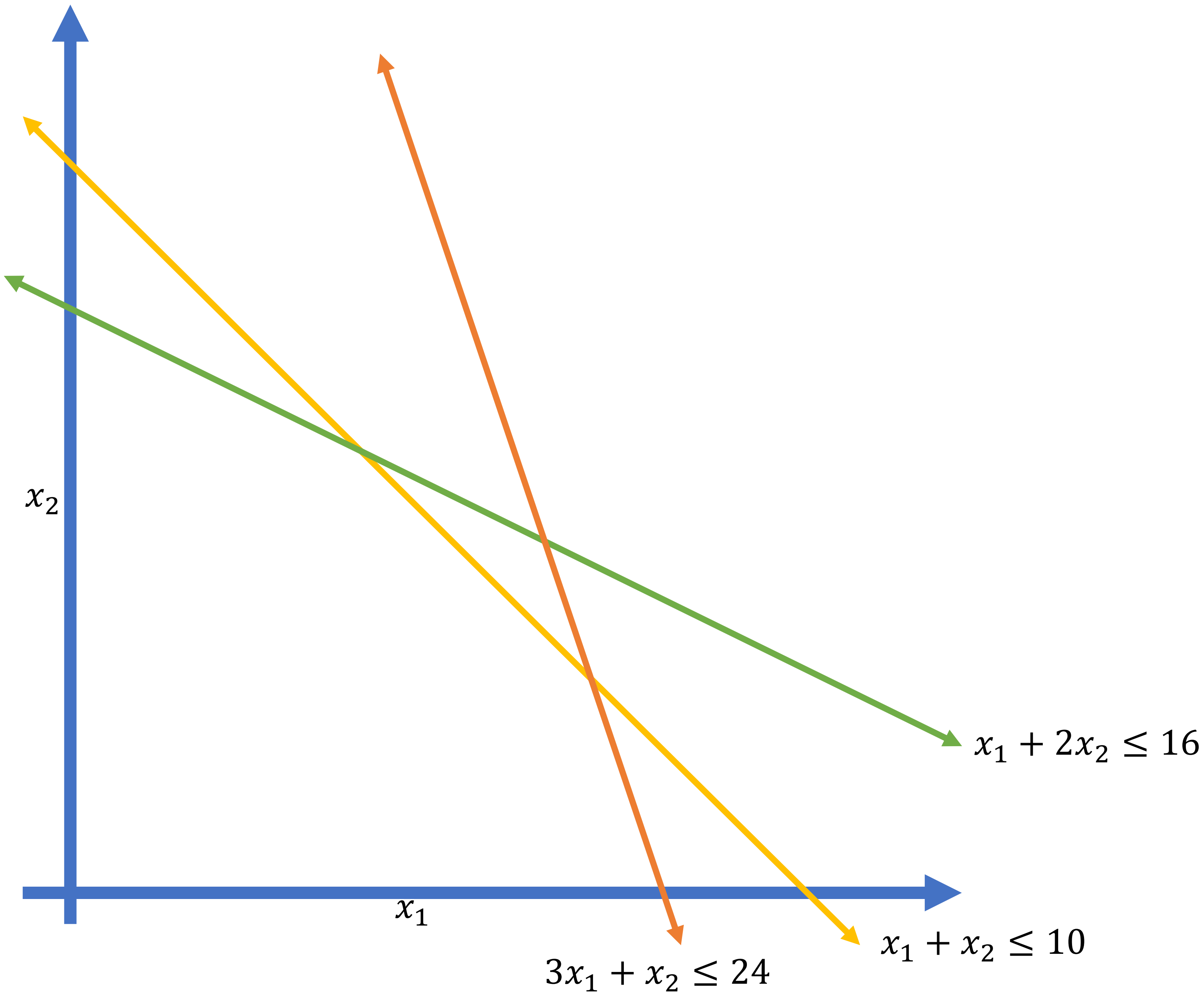
How should you use your remaining fruit to maximize revenue?

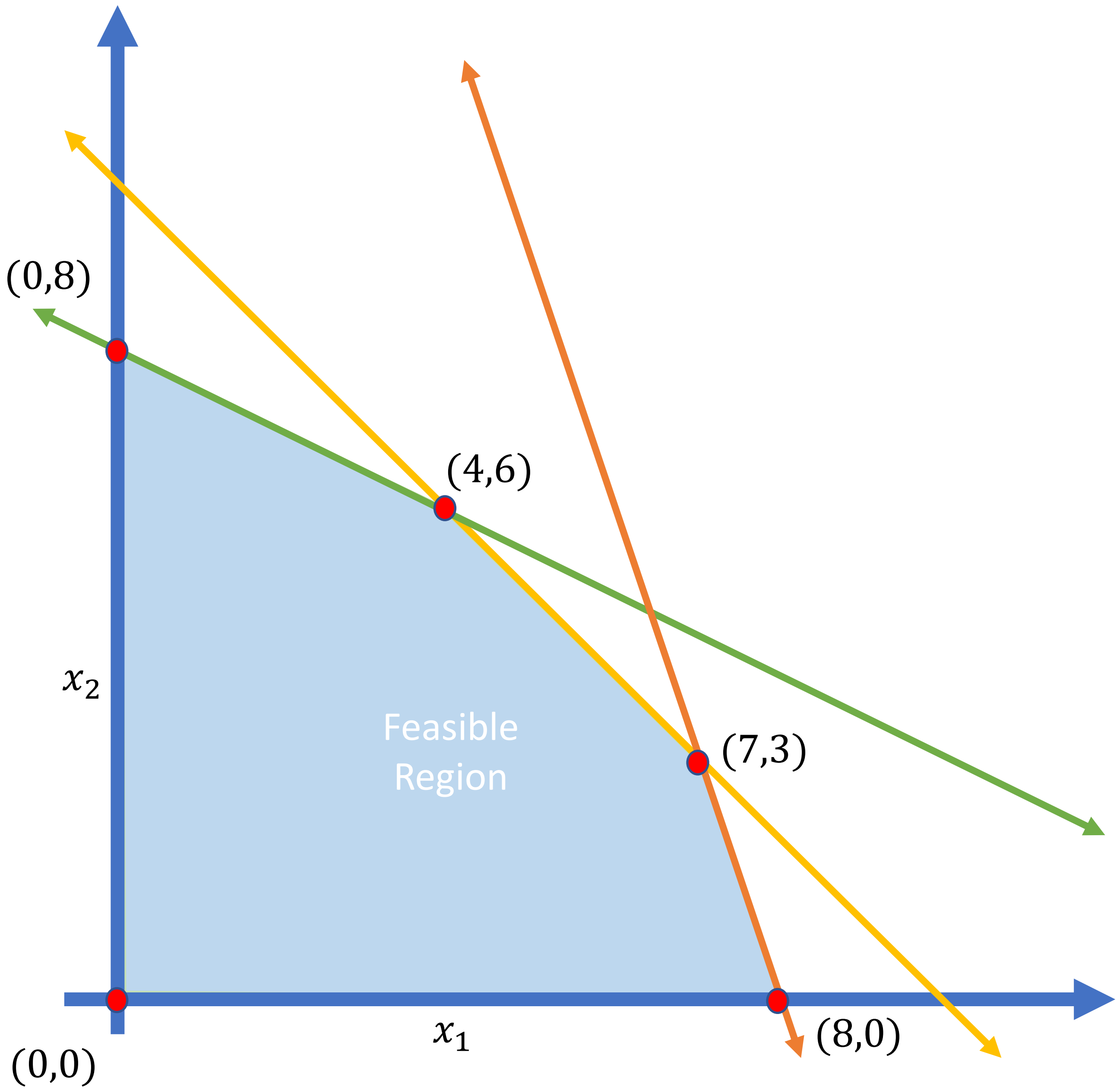
The Simplex Method by Example

Max $3x_1 + 2x_2$

s.t.

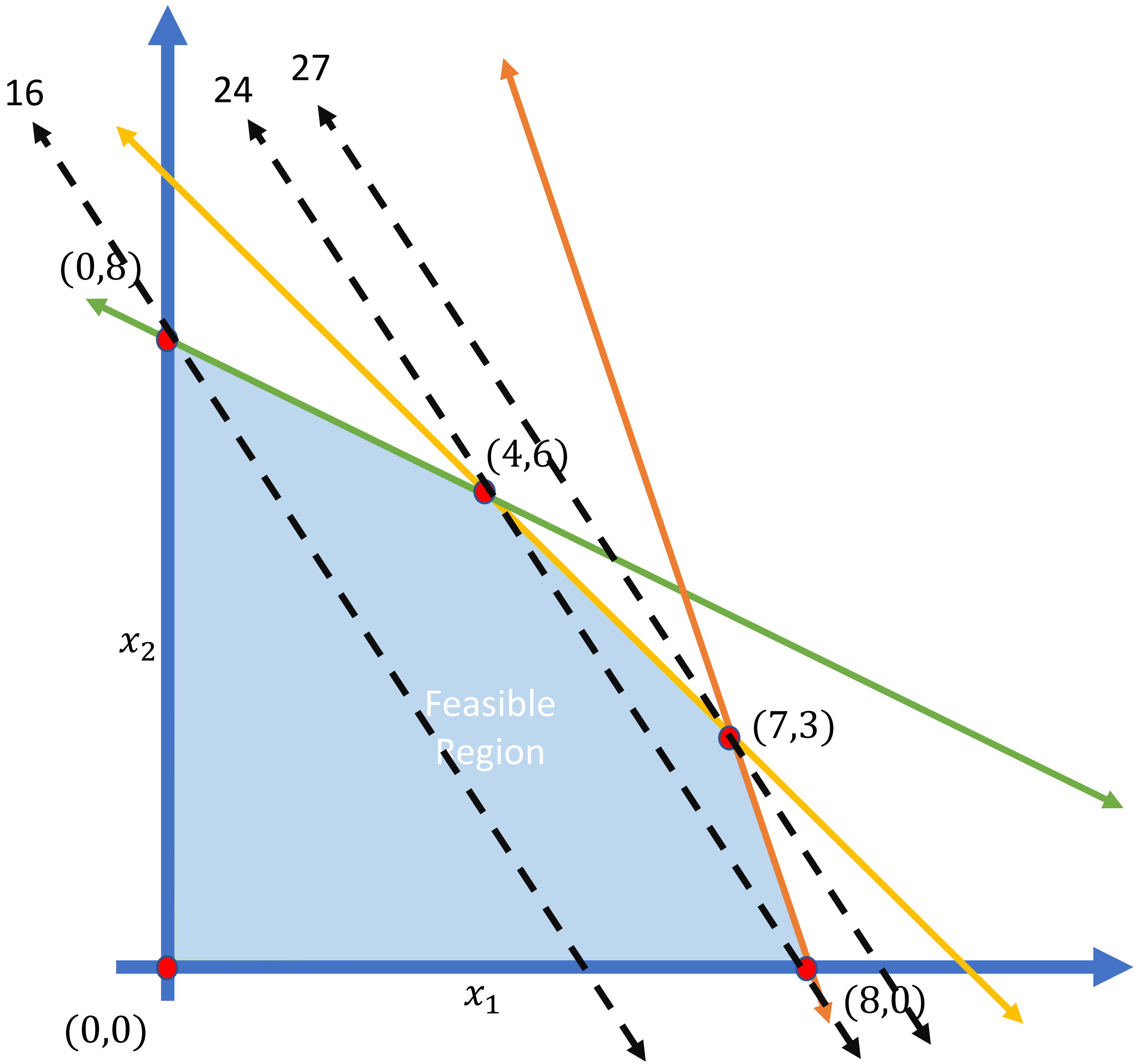
$$\begin{aligned} x_1 \quad + x_2 \quad &\leq 10 \\ 3x_1 + x_2 \quad &\leq 24 \\ x_1 \quad + 2x_2 &\leq 16 \\ x_1, x_2 &\geq 0 \end{aligned}$$


x_2
x_1
$x_1 + 2x_2 \leq 16$
$x_1 + x_2 \leq 10$
$3x_1 + x_2 \leq 24$



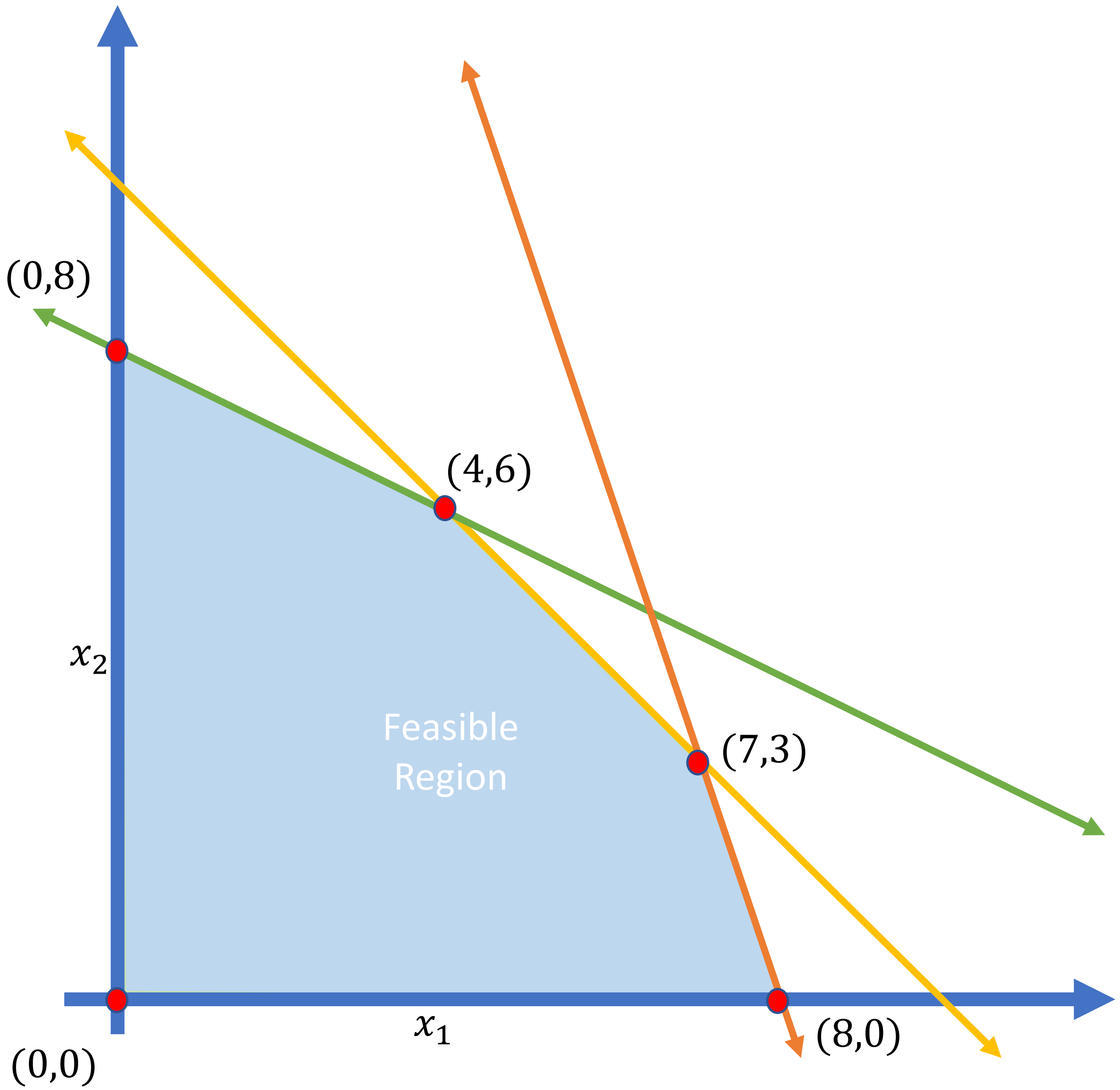
Max $3x_1 + 2x_2$

x_1	x_2	**Objective**
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

Step 1:

Rewrite the Linear Program in equality form by adding/subtracting slack and surplus variables as needed.
Once in equality form, multiply constraints by -1 as needed to make $b \geq 0$

Max $3x_1 + 2x_2$

s.t.

$$
\begin{aligned}
x_1 \quad + x_2 \quad &\leq 10 \\
3x_1 + x_2 \quad &\leq 24 \\
x_1 \quad + 2x_2 &\leq 16 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Max $3x_1 + 2x_2 = z$

s.t.

$$
\begin{aligned}
x_1 \quad + x_2 \quad + s_1 &= 10 \\
3x_1 + x_2 \quad + s_2 &= 24 \\
x_1 \quad + 2x_2 + s_3 &= 16 \\
x_1, x_2, s_1, s_2, s_3 &\geq 0
\end{aligned}
$$

Initial Simplex Tableau

Max $3x_1 + 2x_2 = z$

s.t.

$$
\begin{aligned}
x_1 \quad + x_2 \quad + s_1 &= 10 \\
3x_1 + x_2 \quad + s_2 &= 24 \\
x_1 \quad + 2x_2 + s_3 &= 16 \\
x_1, x_2, s_1, s_2, s_3 &\geq 0
\end{aligned}
$$

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	-3	-2	0	0	0	0
s_1	1	1	1	0	0	10
s_2	3	1	0	1	0	24
s_3	1	2	0	0	1	16

Initial Simplex Tableau*

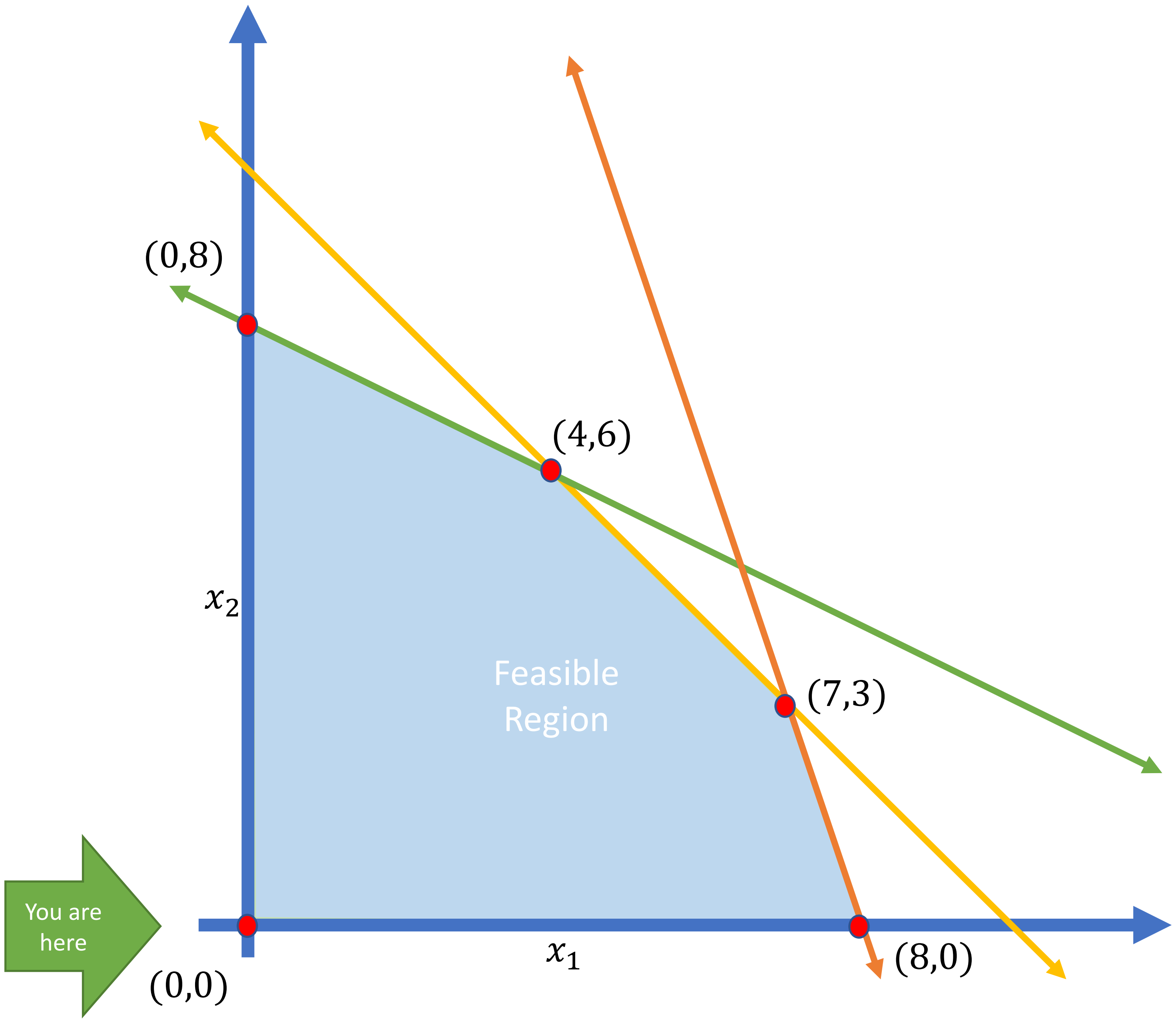
Max $3x_1 + 2x_2 = z$

s.t.

$$x_1 \quad + x_2 \quad + s_1 = 10$$
$$3x_1 + x_2 \quad + s_2 = 24$$
$$x_1 \quad + 2x_2 + s_3 = 16$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**
z	-3	-2	0	0	0	0
s_1	1	1	1	0	0	10
s_2	3	1	0	1	0	24
s_3	1	2	0	0	1	16

* If you do not have a known basic feasible solution or identity matrix in your tableau, you will need to do the two phase simplex method (which we will do in another example).



Max $3x_1 + 2x_2$

x_1	x_2	**Objective**
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

Choose the pivot column

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	-3	-2	0	0	0	0
s_1	1	1	1	0	0	10
s_2	3	1	0	1	0	24
s_3	1	2	0	0	1	16

- On the objective function line, choose the column with the most negative coefficient to enter the basis
- Next, we need to find which line limits the maximum value we can assign to x_1, to do this we use a Ratio Test

Apply the Ratio Test

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**	**Ratio Test**
z	-3	-2	0	0	0	0	
s_1	1	1	1	0	0	10	10/1=10
s_2	3	1	0	1	0	24	24/3=8
s_3	1	2	0	0	1	16	16/1=16

- By taking the ratio or the RHS to the coefficient of the new basis variable, we can see which of the constraints limits the maximum value of the variable entering the basis

Apply the Ratio Test

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**	**Ratio Test**
z	-3	-2	0	0	0	0	
s_1	1	1	1	0	0	10	10/1=10
s_2	3	1	0	1	0	24	24/3=8
s_3	1	2	0	0	1	16	16/1=16

- By taking the ratio or the RHS to the coefficient of the new basis variable, we can see which of the constraints limits the maximum value of the variable entering the basis

Use Elementary Row Operations to eliminate variable from other rows

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS	Ratio Test
z	-3	-2	0	0	0	0	
s_1	1	1	1	0	0	10	10/1=10
s_2	3	1	0	1	0	24	24/3=8
s_3	1	2	0	0	1	16	16/1=16

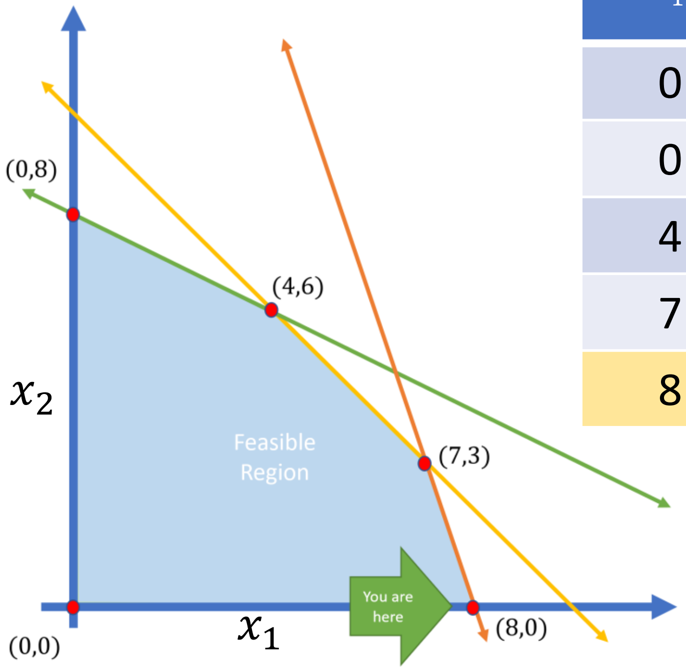
$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	-3	-2	0	0	0	0
s_1	1	1	1	0	0	10
x_1	3/3	1/3	0	1/3	0	24/3
s_3	1	2	0	0	1	16

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	0	-1	0	1	0	24
s_1	0	2/3	1	-1/3	0	2
x_1	1	1/3	0	1/3	0	8
s_3	0	5/3	0	-1/3	1	8

Repeat until no negative numbers in Objective Function

2nd Tableau Form

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	0	-1	0	1	0	24
s_1	0	2/3	1	-1/3	0	2
x_1	1	1/3	0	1/3	0	8
s_3	0	5/3	0	-1/3	1	8



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

Identify Pivot Column and Row

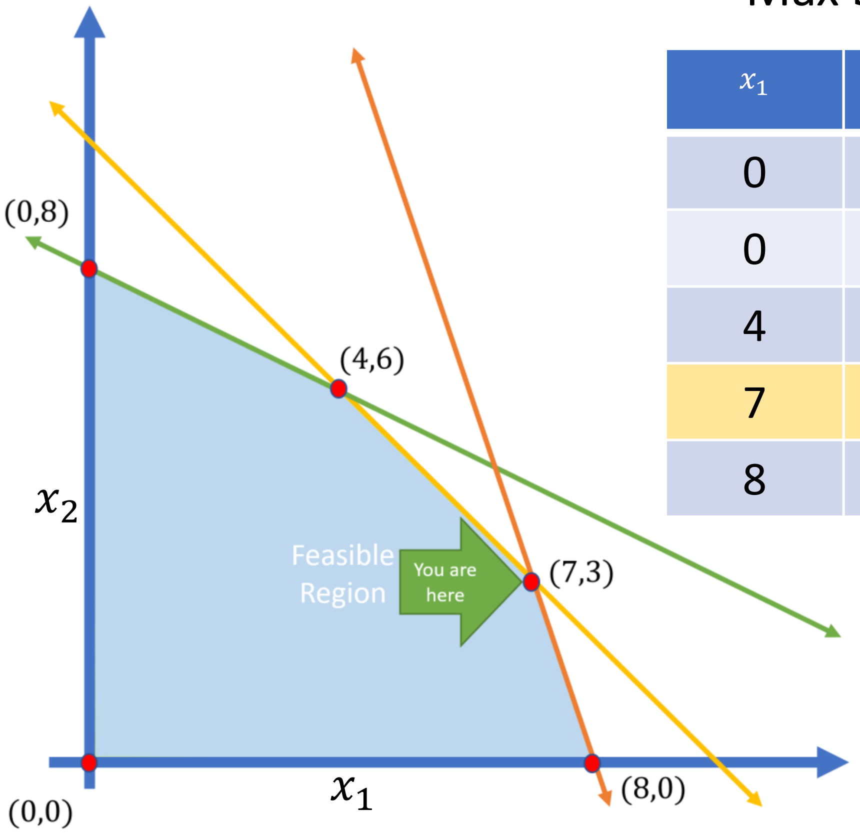
$Basis$	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
z	0	-1	0	1	0	24	
s_1	0	2/3	1	-1/3	0	2	2/(2/3)=3
x_1	1	1/3	0	1/3	0	8	8/(1/3)=24
s_3	0	5/3	0	-1/3	1	8	8/(5/3)=24/5

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	0	0	1.5	0.5	0	27
x_2	0	1	1.5	-0.5	0	3
x_1	1	0	-0.5	0.5	0	7
s_3	0	0	-2.5	0.5	1	3

Optimal Solution

3rd Tableau Form

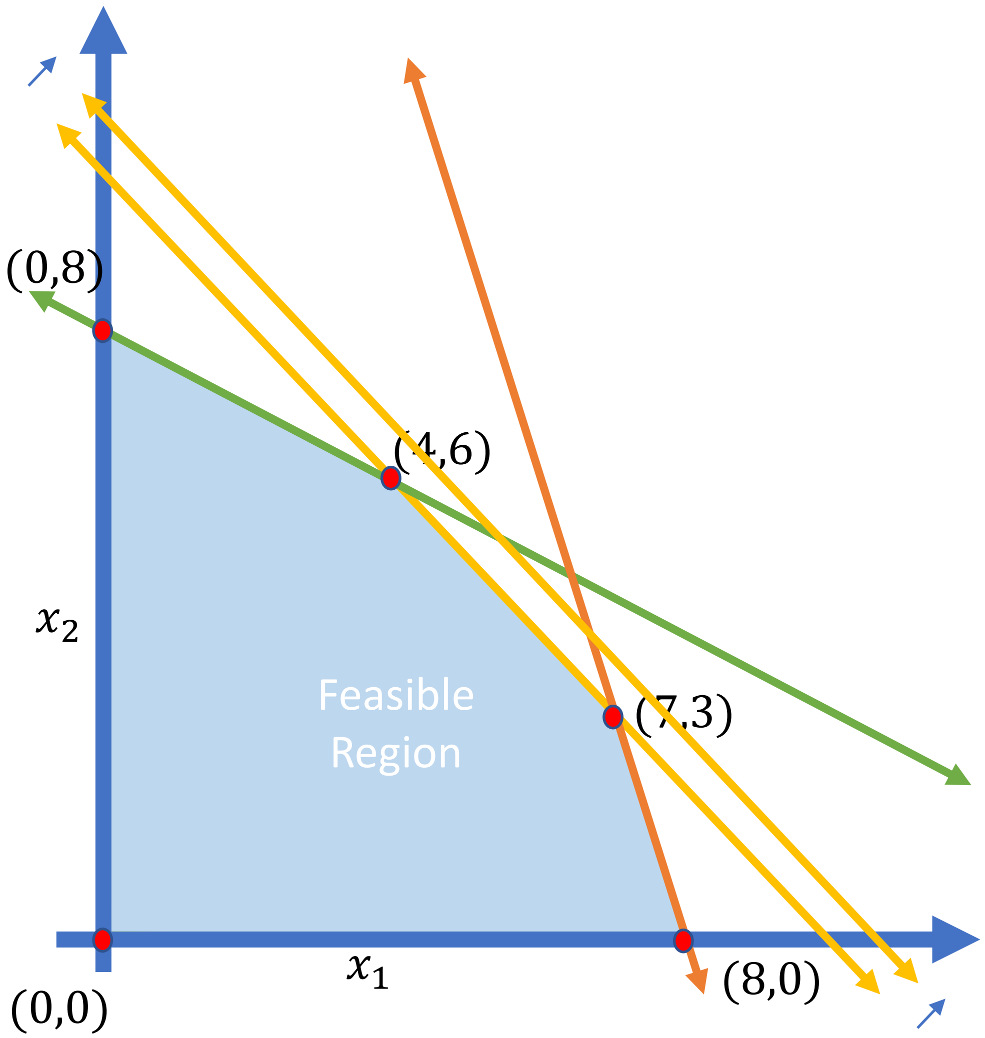
$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**
z	0	0	1.5	0.5	0	27
x_2	0	1	1.5	-0.5	0	3
x_1	1	0	-0.5	0.5	0	7
s_3	0	0	-2.5	0.5	1	3



Max $3x_1 + 2x_2$

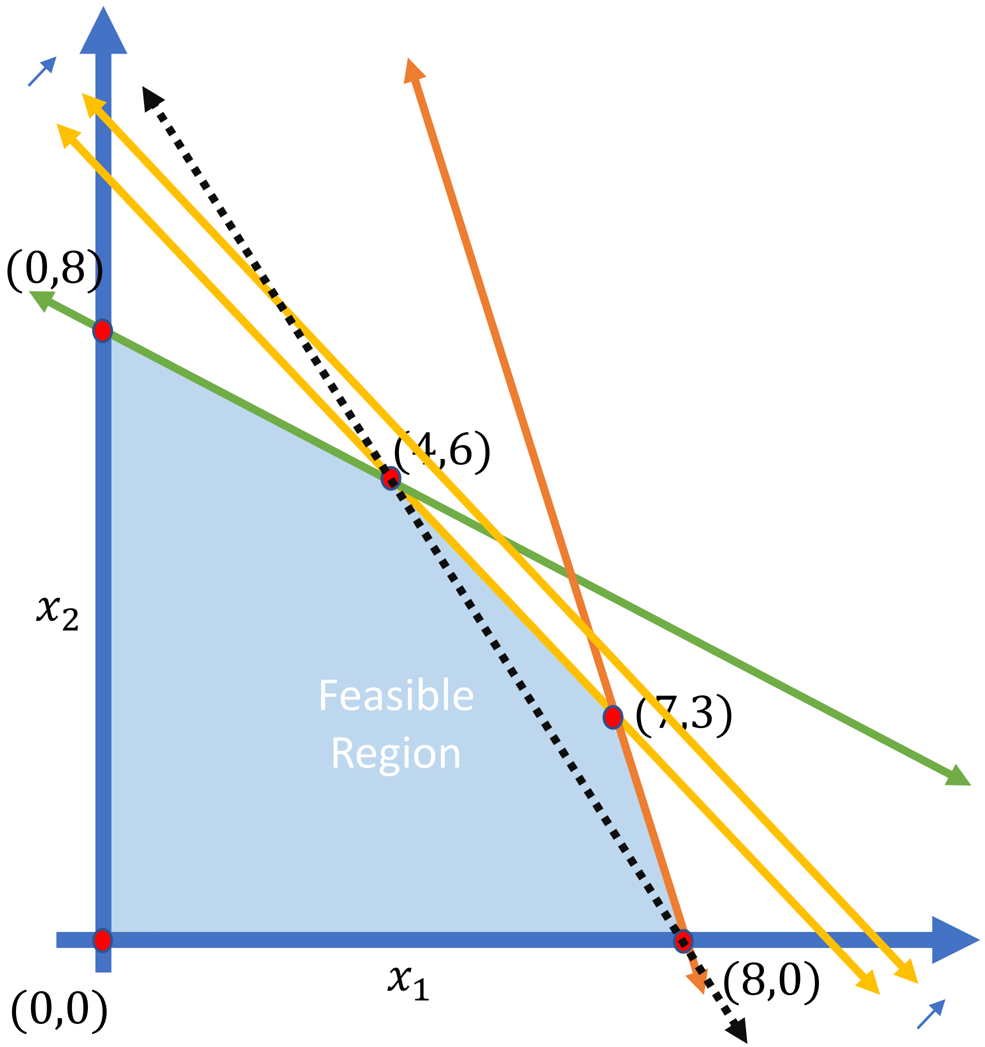
x_1	x_2	**Objective**
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

How much would an additional banana be worth?



- You are expanding the feasible region, but are you changing where the Optimal Solution is?

How much would an additional banana be worth?



- You are expanding the feasible region, but are you changing where the Optimal Solution is?

Quantifying the Objective Value Change

- Dual Price:
  - "The improvement in the value of the optimal solution per unit increase in the RHS of a constraint"
    - A positive *dual price* means an increase for Max problems and a decrease for Min problems
- Shadow Price:
  - The actual change in the value of the optimal solution per unit increase in the RHS of a constraint
    - A positive *shadow price* means an increase for both Max and Min problems

How much would an additional banana be worth?

$Basis$	x_1	x_2	s_1	s_2	s_3	**RHS**
z	0	0	1.5	0.5	0	27
x_2	0	1	1.5	-0.5	0	3
x_1	1	0	-0.5	0.5	0	7
s_3	0	0	-2.5	0.5	1	3

- The *shadow* or *dual* price (for maximization) tells you how much the objective function value would increase for 1 additional unit of the associated constraint.
- Only active constraints have positive shadow prices, these can be found from the final state of your simplex tableau

The Cost of Constraints

$$
\begin{array}{rl}
-(x_1 + x_2 & \leq 10) \\
+(3x_1 + x_2 & \leq 24) \\
+(x_1 + 2x_2 & \leq 16) \\
\hline
3x_1 + 2x_2 & \leq 30
\end{array}
$$

Objective function

Bound

- We can think of each constraint as having a "cost" or "price" as to how good the objective value can be.
- But could we make a tighter upper bound?

The Dual Problem

Primal

Max $c^T x$

s.t. $Ax \leq b$

$x \geq 0$

Dual

Min $p^T b$

s.t. $p^T A \geq c$

$p \geq 0$

Forming the Dual

Primal (Max)

Constraint

- $a_i^T x_i \leq \ b_i$
- $a_i^T x_i \geq \ b_i$
- $a_i^T x_i = \ b_i$

Variable

- $x_j \geq 0$
- $x_j \leq 0$
- $x_j \ \ is \quad free$

Dual (Min)

Variable

- $p_i \geq 0$
- $p_i \leq 0$
- $p_i \ \ is \quad free$

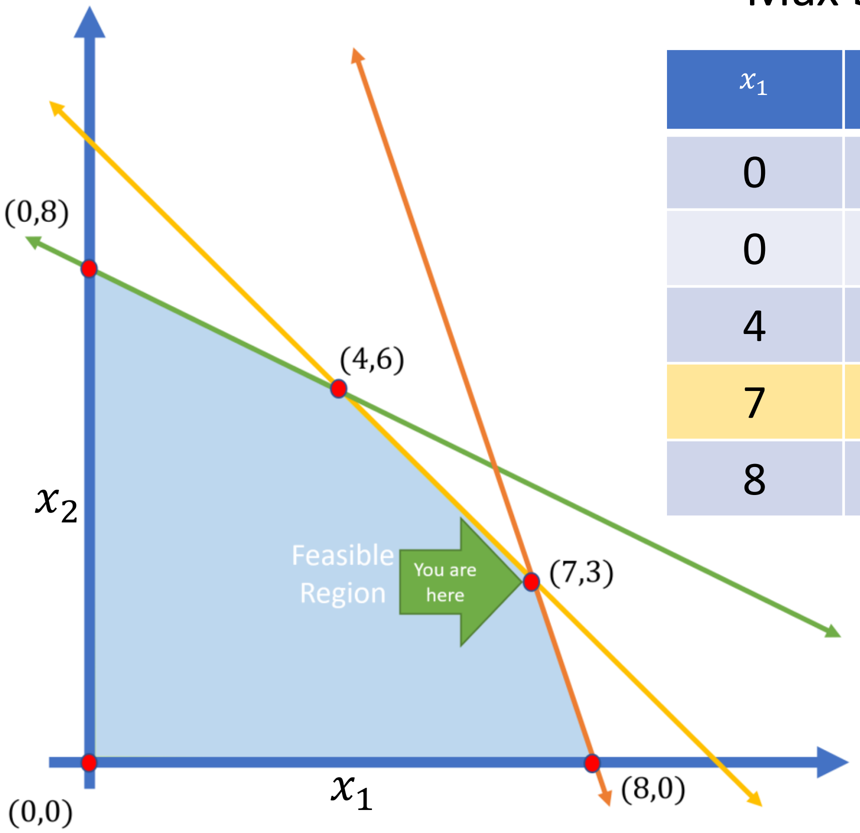
Constraint

- $p_j^T a_j \geq \ c_j$
- $p_j^T a_j \leq \ c_j$
- $p_j^T a_j = \ c_j$

Optimal Solution

3rd Tableau Form

$Basis$	x_1	x_2	s_1	s_2	s_3	RHS
z	0	0	1.5	0.5	0	27
x_2	0	1	1.5	-0.5	0	3
x_1	1	0	-0.5	0.5	0	7
s_3	0	0	-2.5	0.5	1	3



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

The Cost of Constraints

$$
\begin{array}{rl}
1.5(x_1 + x_2 & \leq 10) \\
+\ .5(3x_1 + x_2 & \leq 24) \\
\hline
3x_1 + 2x_2 & \leq 27
\end{array}
$$

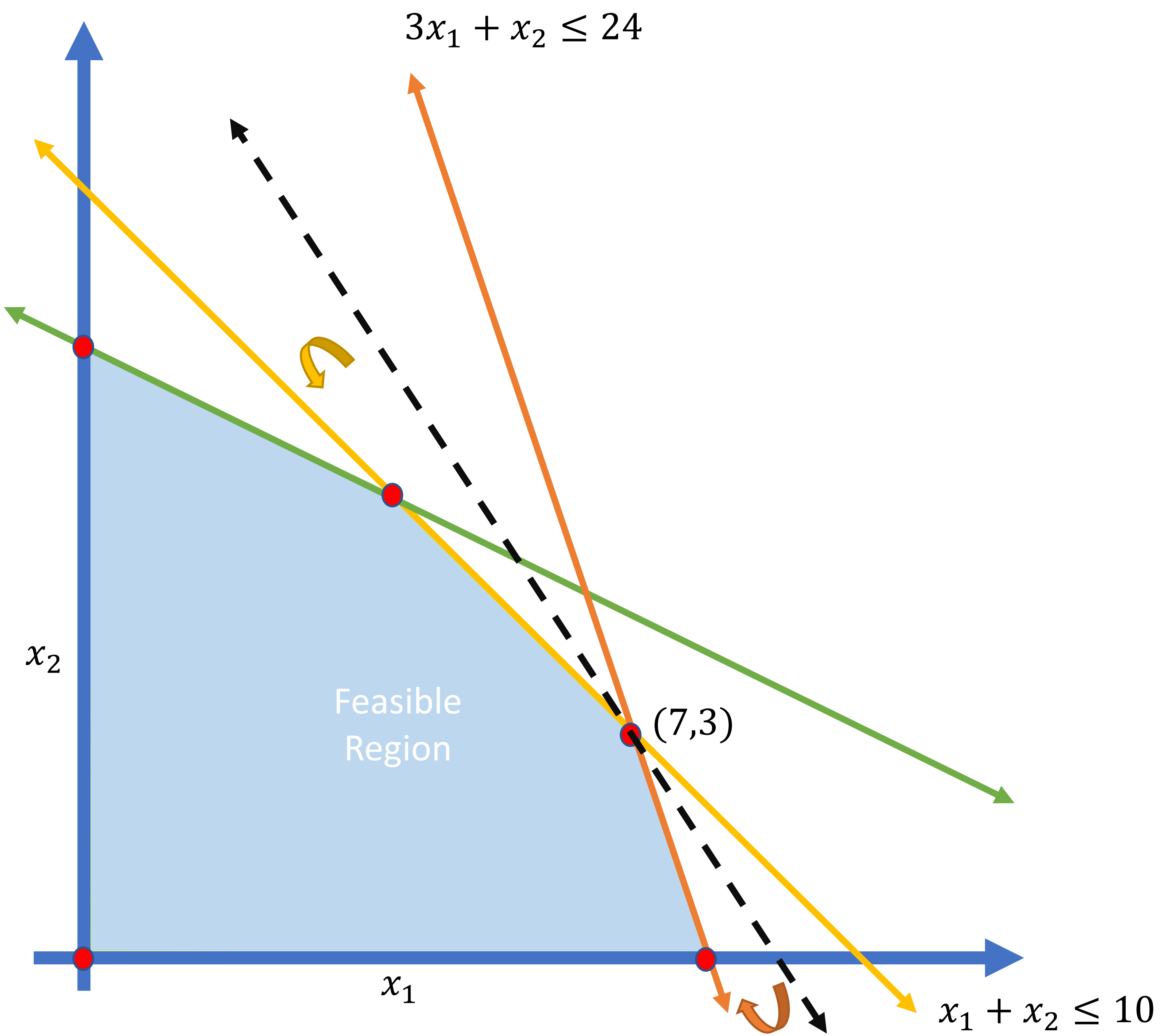
Objective function

Bound

- We can think of each constraint as having a “cost” or “price” as to how good the objective value can be.
- But could we make a tighter upper bound?
- Look the optimal solution is the best upper bound

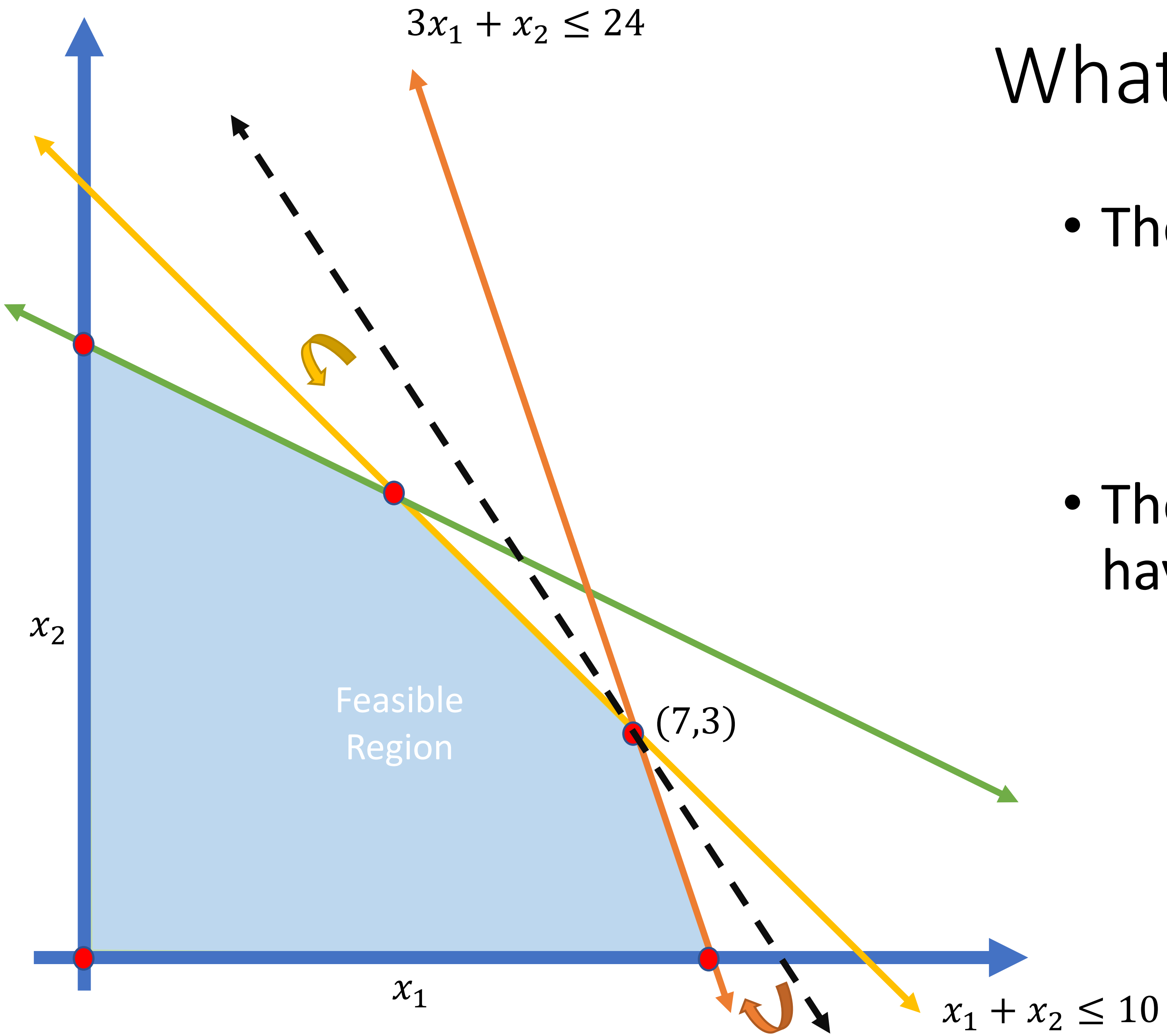
What if I want to change my prices?

- If I want to raise the price of Mix 2 would I need to make a different amount of each to maximize my revenue?
- What if Mix 1 wasn’t selling and I had to discount it?
- Can I change both prices at once?
- When do I need to resolve?



Max $3x_1 + 2x_2$

x_1	x_2	Objective
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24



What are the slopes?

- The objective function:

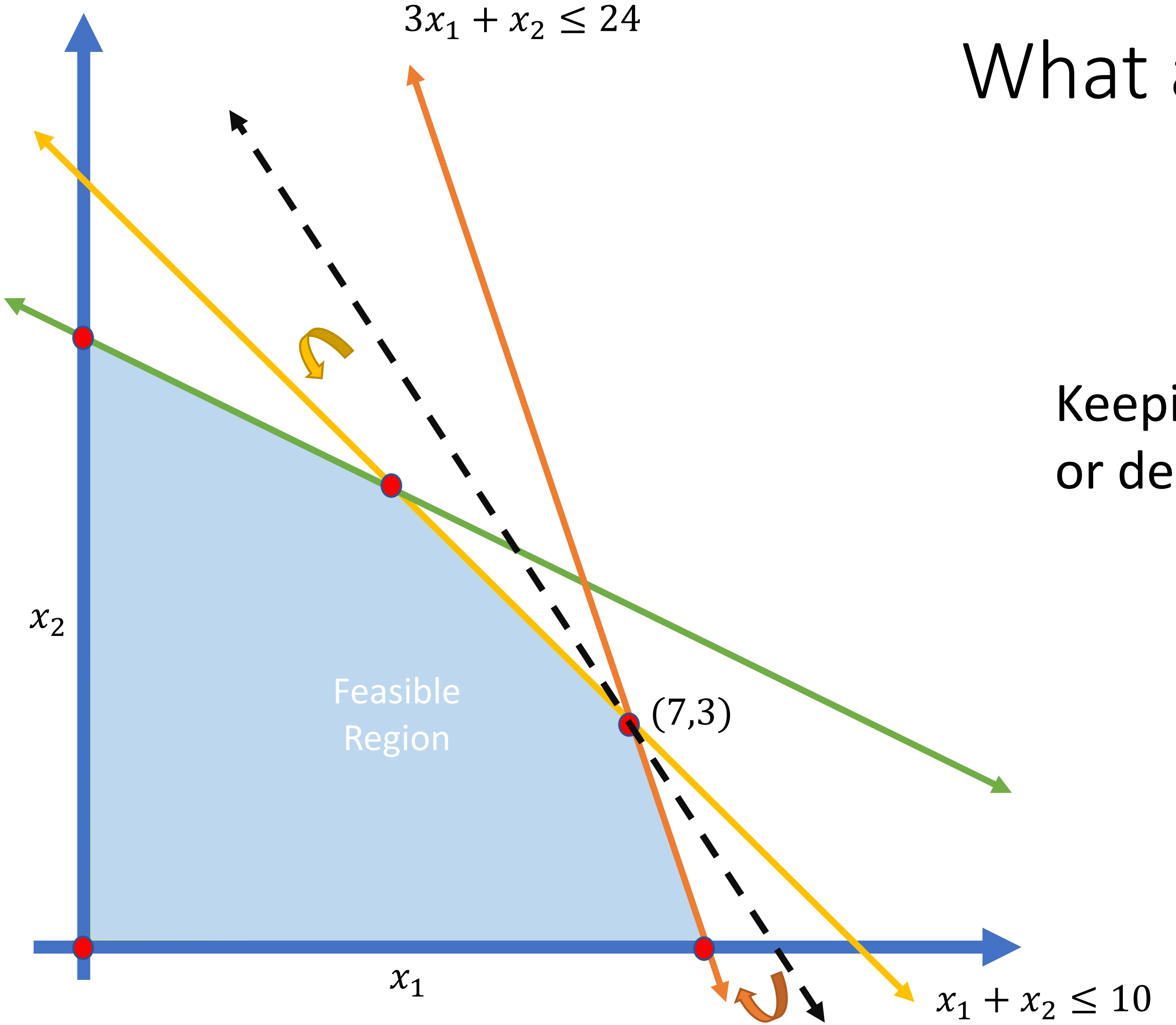
$$c = 3x_1 + 2x_2$$

$$x_2 = -\frac{3}{2}x_1 + c$$

- The active constraint have:

$$x_2 = -x_1 + 10$$

$$x_2 = -3x_1 + 24$$



What are the bounds?

$$-3 \leq -\frac{3}{2} \leq -1$$

Keeping either the numerator or denominator constant:

$$-3 \leq -\frac{6}{2} \leq -1$$

$$-3 \leq -\frac{2}{2} \leq -1$$

$$-3 \leq -\frac{3}{1} \leq -1$$

$$-3 \leq -\frac{3}{3} \leq -1$$

Can I make two changes at once?

	Min	Current	Max
x_1	2	3	6
x_2	1	2	3

$$-3 \leq -\frac{6}{3} \leq -1$$

$$-3 \not\leq -\frac{6}{1} \leq -1$$

$$-3 \leq -\frac{2}{1} \leq -1$$

$$-3 \leq -\frac{2}{3} \not\leq -1$$

The 100% Rule

- For all objective function coefficient changes, sum the percentages of the allowable increases and the allowable decreases. If the sum of the percentages is less than or equal to 100%, the optimal solution will not change.

Notice:

Objective Function Change

- Slope of objective function line changed
- Extreme points stayed the same
- Optimal point stayed the same (within certain range)
- Optimal objective value changed

RHS Constraint Change

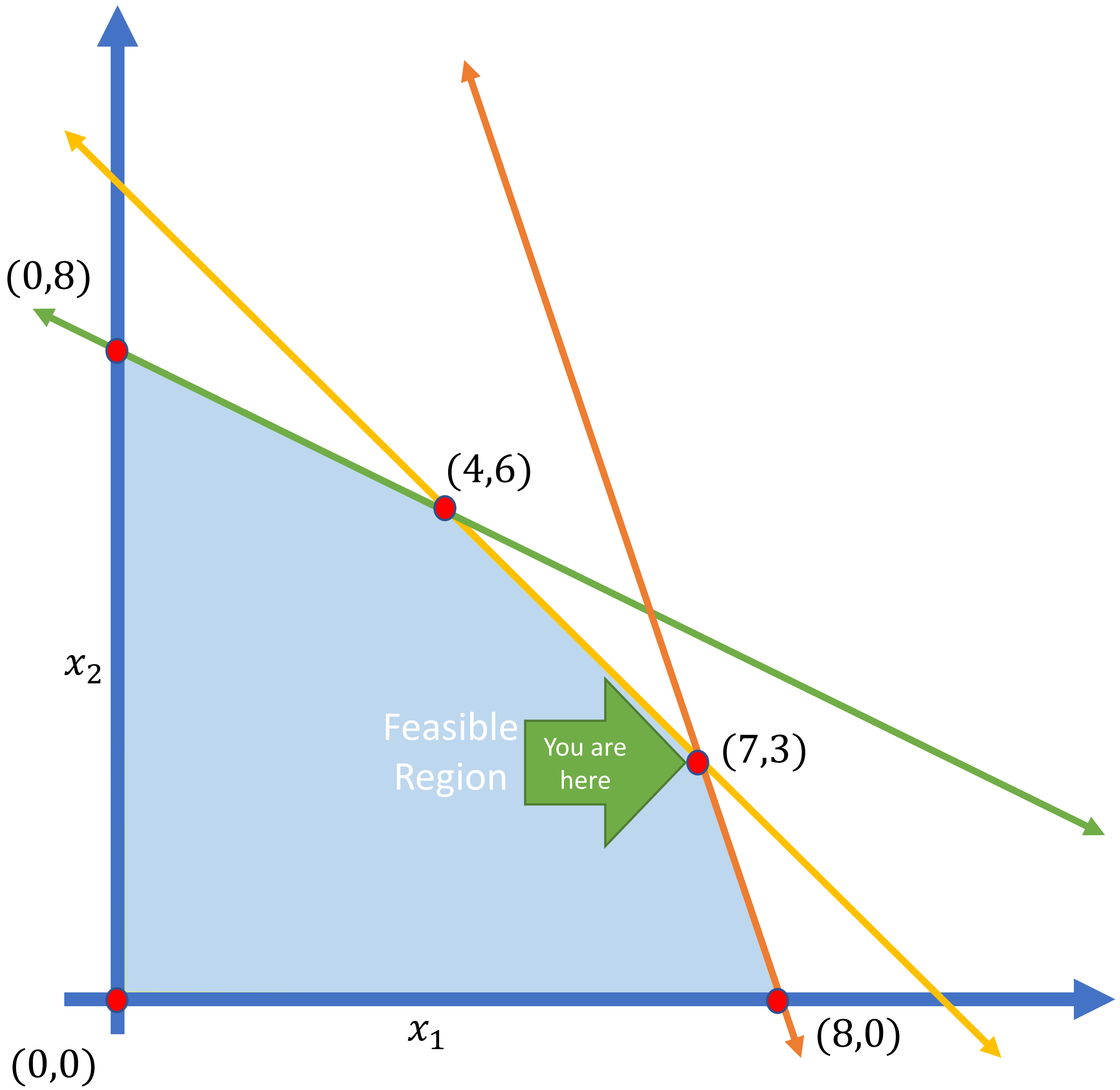
- Feasible region expands (contracts)
- Extreme points change
- Optimal point will adjust
- Optimal objective function value will change

The Simplex Method by Example

Max $3x_1 + 2x_2$

s.t.

$$\begin{aligned} x_1 \quad + x_2 \quad &\leq 10 \\ 3x_1 + x_2 \quad &\leq 24 \\ x_1 \quad + 2x_2 &\leq 16 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Max $3x_1 + 2x_2$

x_1	x_2	**Objective**
0	0	0
0	8	16
4	6	24
7	3	27
8	0	24

Phase I: Finding a Basic Feasible Solution

1. Put the problem in equality form. Multiply some of the constraints by -1, so that $b \geq 0$
2. Introduce artificial variables $y_1, y_2, \dots, y_m$ for each constraint and apply the simplex method maximizing for $-\sum y_i$
3.
    a.) If the optimal value is not zero then the problem is infeasible

    b.) If the optimal value is zero and no artificial variables are in the final basis then remove the corresponding columns and what remains is a feasible basis

    c.) If the ith basic variable is an artificial one then,

    i.) if all non-artificial entries are 0 the constraint is redundant and drop

    ii.) if the jth entry is non-zero pivot and have x_j enter the basis

Phase II: Solve the Original Problem

4. Use the final basis and tableau obtained from Phase I
5. Compute the reduced costs of all variables for the initial basis, using the cost coefficient of the original problem
6. Apply the simplex method to solve

Example Problem

Max $x_1 + x_2 + x_3$

s.t.

$$\begin{aligned} x_1 \quad + 2x_2 + 3x_3 &= 3 \\ x_1 \quad - 2x_2 - 6x_3 &= -2 \\ 4x_2 + 9x_3 &= 5 \\ 3x_3 &\leq 1 \end{aligned}$$

$$x_1, x_2, x_3 \geq 0$$

Phase I: Step 1

Put the problem in equality form. Multiply some of the constraints by -1, so that $b \geq 0$

Max $x_1 + x_2 + x_3$

s.t.

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 3 \\
x_1 - 2x_2 - 6x_3 &= -2 \\
4x_2 + 9x_3 &= 5 \\
3x_3 &\leq 1 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Max $x_1 + x_2 + x_3$

s.t.

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 3 \\
-x_1 + 2x_2 + 6x_3 &= 2 \\
4x_2 + 9x_3 &= 5 \\
3x_3 + x_4 &= 1 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

Phase I: Step 2

Introduce artificial variables $y_1, y_2, \dots, y_m$ for each constraint and apply the simplex method maximizing for $-\sum y_i$

Max $x_1 + x_2 + x_3$

s.t.

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 3 \\
-x_1 + 2x_2 + 6x_3 &= 2 \\
4x_2 + 9x_3 &= 5 \\
3x_3 + x_4 &= 1 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

Max $-y_1 + y_2 + y_3 + y_4$

s.t.

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 + y_1 &= 3 \\
-x_1 + 2x_2 + 6x_3 + y_2 &= 2 \\
4x_2 + 9x_3 + y_3 &= 5 \\
3x_3 + x_4 + y_4 &= 1 \\
x_1, x_2, x_3, x_4 &\geq 0 \\
y_1, y_2, y_3, y_4 &\geq 0
\end{aligned}
$$

Phase I: Step 2

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS
z_{Init}	0	0	0	0	1	1	1	1	?
y_1	1	2	3	0	1	0	0	0	3
y_2	-1	2	6	0	0	1	0	0	2
y_3	0	4	9	0	0	0	1	0	5
y_4	0	0	3	1	0	0	0	1	1

Once you have put the problem in a Simplex Tableau, start by subtracting the basis vectors from the objective line

Phase I: Step 2

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS	Ratio
z_{Init}	0	-8	-21	-1	0	0	0	0	-11	
y_1	1	2	3	0	1	0	0	0	3	3/3
y_2	-1	2	6	0	0	1	0	0	2	2/6
y_3	0	4	9	0	0	0	1	0	5	5/9
y_4	0	0	3	1	0	0	0	1	1	1/3

Identify the most negative coefficient in the objective function, then apply the ratio test

Phase I: Step 2

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS	Ratio
z_{Init}	0	-8	-21	-1	0	0	0	0	-11	
y_1	1	2	3	0	1	0	0	0	3	3/3
y_2	-1	2	6	0	0	1	0	0	2	2/6
y_3	0	4	9	0	0	0	1	0	5	5/9
y_4	0	0	3	1	0	0	0	1	1	1/3

Note that two rows share the same ratio, I can choose either row to leave the basis.

Phase I: Step 2

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS	Ratio
z_{Init}	0	-8	-21	-1	0	0	0	0	-11	
y_1	1	2	3	0	1	0	0	0	3	3/3
y_2	-1	2	6	0	0	1	0	0	2	2/6
y_3	0	4	9	0	0	0	1	0	5	5/9
y_4	0	0	3	1	0	0	0	1	1	1/3

Add pivot column variable to the basis and remove from other rows.

Phase I: Step 2

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS
z_{Init}	-7/2	-1	0	-1	0	7/2	0	0	-4
y_1	3/2	1	0	0	1	-1/2	0	0	2
x_3	-1/6	1/3	1	0	0	1/6	0	0	1/3
y_3	3/2	1	0	0	0	-3/2	1	0	2
y_4	1/2	-1	0	1	0	-1/2	0	1	0

Add pivot column variable to the basis and remove from other rows.

Phase I: Step 2

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS	Ratio
z_{Init}	-7/2	-1	0	-1	0	7/2	0	0	-4	
y_1	3/2	1	0	0	1	-1/2	0	0	2	4/3
x_3	-1/6	1/3	1	0	0	1/6	0	0	1/3	
y_3	3/2	1	0	0	0	-3/2	1	0	2	4/3
y_4	1/2	-1	0	1	0	-1/2	0	1	0	0

Note: you only need to apply the ratio test to rows with positive coefficients of the pivot column. Since the RHS is zero this corresponds to a degenerative point.

Phase I: Step 2

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS
z_{Init}	0	-8	0	6	0	0	0	7	-4
y_1	0	4	0	-3	1	1	0	-3	2
x_3	0	0	1	1/3	0	0	0	1/3	1/3
y_3	0	4	0	-3	0	0	1	-3	2
x_1	1	-2	0	2	0	-1	0	2	0

Note: Degenerative pivots don’t improve the objective function value

Phase I: Step 2

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS	Ratio
z_{Init}	0	-8	0	6	0	0	0	7	-4	
y_1	0	4	0	-3	1	1	0	-3	2	½
x_3	0	0	1	1/3	0	0	0	1/3	1/3	
y_3	0	4	0	-3	0	0	1	-3	2	½
x_1	1	-2	0	2	0	-1	0	2	0	

Phase I: Step 3

No longer needed

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS
z_{Init}	0	0	0	0	2	2	0	1	0
x_2	0	1	0	-3/4	1/4	1/4	0	-3/4	1/2
x_3	0	0	1	1/3	0	0	0	1/3	1/3
y_3	0	0	0	0	-1	-1	1	0	0
x_1	1	0	0	1/2	1/2	-1/2	0	1/2	1

redundant

The Objective Function Value is 0 and 1 artificial variable remains in the basis, but all coefficients for non-artificial variables are 0 meaning the constraint is redundant and can be removed from the problem. The columns for the artificial constraints can also now be removed.

Side Note on Redundant Constraint:

Max $x_1 + x_2 + x_3$

s.t.

$$
\begin{aligned}
x_1 \quad + 2x_2 + 3x_3 \qquad\qquad &= 3 \\
-x_1 + 2x_2 + 6x_3 \qquad\qquad &= 2 \\
4x_2 + 9x_3 \qquad\qquad &= 5 \\
3x_3 + x_4 &= 1
\end{aligned}
$$

$$x_1, x_2, x_3, x_4 \geq 0$$

- Notice that the third constraint is simply the addition of the first two constraints!
- Therefore one of the three is redundant and does not change the feasible region of the problem

Phase 2: Step 4

Use the final basis and tableau obtained from Phase I

Basis	x_1	x_2	x_3	x_4	RHS
z					
x_2	0	1	0	-3/4	1/2
x_3	0	0	1	1/3	1/3
x_1	1	0	0	1/2	1

Phase 2: Step 5

Compute the reduced costs of all variables for the initial basis, using the cost coefficient of the original problem

Basis	x_1	x_2	x_3	x_4	RHS
z	-1	-1	-1	0	?
x_2	0	1	0	-3/4	1/2
x_3	0	0	1	1/3	1/3
x_1	1	0	0	1/2	1

Basis	x_1	x_2	x_3	x_4	RHS
z	0	0	0	1/12	11/6
x_2	0	1	0	-3/4	1/2
x_3	0	0	1	1/3	1/3
x_1	1	0	0	1/2	1

Phase II: Step 6

Basis	x_1	x_2	x_3	x_4	RHS
z	0	0	0	1/12	11/6
x_2	0	1	0	-3/4	1/2
x_3	0	0	1	1/3	1/3
x_1	1	0	0	1/2	1

Since all coefficients in the objective function are non-negative, we are already at our optimal answer.

Is there another way we can prove it is optimal?

Max $x_1 + x_2 + x_3$

s.t.

$$x_1 \quad + 2x_2 + 3x_3 \; = 3$$

$$x_1 \quad -2x_2 - 6x_3 \; = -2$$

$$4x_2 + 9x_3 = 5$$

$$3x_3 \leq 1$$

$$x_1, x_2, x_3 \geq 0$$

$$\frac{3}{4}(x_1 \quad + 2x_2 + 3x_3 \; = 3)$$

$$+\frac{1}{4}(\,x_1 \quad -2x_2 - 6x_3 \; = -2)$$

$$+\frac{1}{12}(3x_3 \; \leq 1) \rightarrow$$

$$x_1 + x_2 + x_3 \leq \frac{11}{6}$$