Sensitivity/Stability of a System

Recall: So far, in the previous lectures we talked about how to solve linear systems of equations. For square systems, direct methods and Iterative methods. For underdetermined systems, we used least squares methods. This lecture, we aim to learn about the system's stability. Consider the following pair of linear system

$$\begin{cases} x_1 - x_2 = 1 \\ x_1 - 1.001x_2 = 0 \end{cases} \qquad [x_1^*, x_2^*] = [1001, 1000]$$

$$\begin{cases} x_1 - x_2 = 1 \\ x_1 - 0.999x_2 = 0 \end{cases} \qquad [x_1^*, x_2^*] = [-999, -1000]$$

The system is sensitive, since **small variation in the input values** resulted a **significant change in the solution**. Note that computer is always having the roundoff error. You think you put b in the computer, but because of round off error you actually solve $b + \Delta b$, where Δb is the round off error.

Is there anyway to measure the system's sensitivity?

Relative Error/Change

Recall:

$$\text{Relative Error} = \frac{|\text{measured} - \text{real}|}{|\text{real}|}$$

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be an smooth function of x. To quantify how sensitive $f(x)$ is to changes in x we use

$$\text{Relative Error of } f = \left| \frac{f(x + \Delta x) - f(x)}{f(x)} \right|$$

Similarly, we need to consider relative error for x,

$$\text{Relative Error of } x = \frac{|x + \Delta x - x|}{|x|} = \frac{|\Delta x|}{|x|}.$$

Condition Number

Definition

The **condition number** is define by the worst possible relative changes of f divided by the relative changes to x

$$\kappa(f) = \text{largest}\left\{\frac{\text{relative change of } f}{\text{relative change in } x}\right\} = \max_{x,\Delta x}\left|\frac{\frac{f(x+\Delta x)-f(x)}{f(x)}}{\frac{\Delta x}{x}}\right|$$

Note: we don't care about the specific point hence we consider the larger among all x and Δx.

The condition number of the linear system $Ax = b$ denoted by $\kappa(A)$, obtained by setting $f(x) = Ax$.

Definition

If $\kappa(A)$ is large, it is said that A is **badly conditioned** or **ill-condition**. Moreover, the system $Ax = b$ is said to be unstable.

Definition

A large condition number also means that the matrix is close to being **singular**.

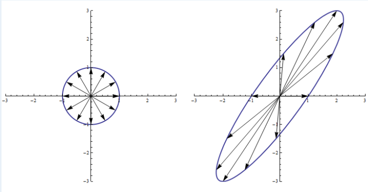
How can we find the condition number of A, $\kappa(A)$?

Definition

Recall. Norm of a matrix

$$\|A\| = \max_{z \in \mathbb{R}^n} \frac{\|Az\|}{\|z\|} = \max_{z \in \mathbb{R}^n, \|z\|=1} \|Az\|$$

Let $z \in \mathbb{R}^2$ with $\|z\| = 1$, then Az maps the unit circle to some ellipse, or some new surface. The norm of a matrix is the size of the farthest point in the new mapping:



Theorem

If A is a $n \times n$ symmetric matrix, then

$$\|A\|_2 = \max_{z \in \mathbb{R}^n, \|z\|_2 = 1} \|Az\|_2 = |\lambda_1|$$

where λ_1 is the dominant eigenvalue.

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Proof. Since A is $n \times n$ and symmetric, by the Spectral Decomposition Theorem, there is an orthonormal eigenvalue basis $q^{(1)}, \dots, q^{(n)}$ corresponding to $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

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Hence for any vector $z \in \mathbb{R}^n$ we have

$$z = \sum_{i=1}^{n} w_i q^{(i)}$$

and furthur

$$Az = \sum_{i=1}^{n} w_i A q^{(i)} = \sum_{i=1}^{n} w_i \lambda_i q^{(i)}$$

By the ℓ_2 norm and the inner product rule we have

$$\begin{aligned}\|Az\|_2^2 = (Az)^T(Az) &= \left(\sum_{i=1}^{n} w_i\lambda_i q^{(i)}\right)^T \left(\sum_{j=1}^{n} w_j\lambda_j q^{(j)}\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \lambda_i \lambda_j {q^{(i)}}^T q^{(j)} = \sum_{i=1}^{n} w_i^2 {\lambda_i}^2\end{aligned}$$

We assume $\|z\|_2 = 1$, then $\sum_{i=1}^{n} w_i^2 = \|z\|^2 = 1$. Now the problem

$$\max_{z\in\mathbb{R}^n, \|z\|_2=1} \|Az\|_2^2$$

is equivalently written as follows

$$\max_{w_i} \sum {w_i}^2{\lambda_i}^2 \qquad s.t. \quad \sum w_i^2 = 1$$

Then the optimal solution is $w^* = (1, 0, 0, \ldots, 0)^T$, that is $w_1^* = 1$, and $w_i^* = 0, \;\; 2 \le i \le n$. Hence

$$\max_{\|z\|_2=1} \|Az\|_2^2 = \lambda_1^2 \qquad \to \qquad \|A\|_2 = \max_{\|z\|_2=1} \|Az\|_2 = \sqrt{\lambda_1^2} = |\lambda_1|$$

Theorem

If A is a $n \times m$ matrix (not a symmetric matrix), then

$$\|A\|_2 = \max_{x \in \mathbb{R}^m, \|x\|_2 = 1} \|Ax\|_2 = \sqrt{\alpha_1}$$

where α_1 is the dominant eigenvalue of $A^T A$.

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Proof. $\|Ax\|_2^2 = (Ax)^T(Ax) = x^T A^T Ax$. The matrix $A^T A$ is a symmetric matrix and positive definite. By the Spectral Decomposition Theorem we have

$$A^T A = \Phi^T \Lambda \Phi$$

where

$$\Phi = [\phi^{(1)}, \ldots, \phi^{(m)}] \quad \Lambda = [\alpha_1, \ldots, \alpha_m]$$

where

$$\phi^{(i)}{}^T \phi^{(j)} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$$

Now

$$x^TA^TAx = x^T\Phi^T\Lambda\Phi x = (\Phi x)^T\Lambda(\Phi x) = y^T\Lambda y = \sum_{i=1}^{n}\alpha_i y_i^2$$

where $y = \Phi x$.
Let $\|x\|_2 = 1$, then $\|y\|_2(= \|x\|_2) = 1$. Now the optimization problem

$$\max_{x\in\mathbb{R}^m, \|x\|_2=1} \|Ax\|_2^2$$

is equivalent to

$$\max_{y\in\mathbb{R}^m} \sum {y_i}^2\alpha_i \qquad s.t. \quad \sum y_i^2 = 1$$

By the fact that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$. The optimal value for y is $y^* = (1, 0, 0, \ldots, 0)$.
Therefore,

$$\|A\|_2^2 = \max_{x\in\mathbb{R}^m, \|x\|_2=1} \|Ax\|_2^2 = \max_{y\in\mathbb{R}^m, \|y\|_2=1} \sum {y_i}^2\alpha_i = \alpha_1$$

Thus $\|A\|_2 = \sqrt{\alpha_1}$.

Condition Number of Matrix A

Theorem

Let A be an $n \times n$ symmetric matrix, then the condition number is given by

$$\kappa(A) = \frac{|\lambda_1|}{|\lambda_n|}$$

where λ_1 and λ_n are the largest and smallest eigenvalues of A, respectively. When κ is so large, the matrix A is so sensitive to changes in x.

Theorem

Let A be an $n \times m$ matrix, then the condition number is given by

$$\kappa(A) = \frac{\sqrt{\sigma_1}}{\sqrt{\sigma_n}},$$

where σ_1 and σ_n are the largest and smallest eigenvalues of $A^T A$, respectively. When κ is so large, the matrix A is so sensitive to changes in x.

Proof. Let $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ (square matrix of size n) and define $f(x) = Ax$.

$$\frac{\text{relative change of } f}{\text{relative change in } x} = \frac{\frac{\|f(x+\Delta x)-f(x)\|}{\|f(x)\|}}{\frac{\|\Delta x\|}{\|x\|}} = \frac{\frac{\|A(x+\Delta x)-Ax\|}{\|Ax\|}}{\frac{\|\Delta x\|}{\|x\|}} = \frac{\frac{\|A\Delta x\|}{\|Ax\|}}{\frac{\|\Delta x\|}{\|x\|}}$$

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We need to maximize to find the worst possible case

$$\begin{aligned}
\max_{\Delta x, x} \frac{\frac{\|A\Delta x\|}{\|Ax\|}}{\frac{\|\Delta x\|}{\|x\|}} &= \max_{\Delta x, x} \frac{\|A\Delta x\|}{\|\Delta x\|} \frac{\|x\|}{\|Ax\|} \\
&= \max_{\Delta x} \frac{\|A\Delta x\|}{\|\Delta x\|} \max_{x} \frac{\|x\|}{\|Ax\|} \\
&= \max_{\Delta x} \frac{\|A\Delta x\|}{\|\Delta x\|} \max_{x} \frac{1}{\|Ax\|/\|x\|} \\
&= \max_{\Delta x} \frac{\|A\Delta x\|}{\|\Delta x\|} \frac{1}{\min_x \|Ax\|/\|x\|} = \frac{|\lambda_1|}{|\lambda_n|}
\end{aligned}$$

Q. How can we show $\min_x \|Ax\|/\|x\| = |\lambda_n|$?

Sensitivity of System $Ax = b$ wrt the Change in b

Let's altering the rhs of the linear system, by replacing b with $b + \delta b$, and think δb is the error in b. This results the error δx in the solution, so we have

$$A(x + \delta x) = b + \delta b$$

Note that

$$\begin{aligned} A\,x &= b \\ A\,\delta x &= \delta b \end{aligned}$$

Fact: $|\lambda_{\min}|\|x\|_2 \leq \|Ax\|_2 \leq |\lambda_{\max}|\|x\|_2$ we have

$$\|b\|_2 = \|Ax\|_2 \leq |\lambda_{\max}|\|x\|_2$$

$$|\lambda_{\min}|\ \|\delta x\| \leq \|A\delta x\|_2 = \|\delta b\|_2. \quad \rightarrow \quad \frac{1}{\|\delta b\|} \leq \frac{1}{|\lambda_{\min}|}\frac{1}{\|\delta x\|_2}$$

By the last two inequality we obtain

$$\frac{\|b\|_2}{\|\delta b\|_2} \leq \frac{|\lambda_{\mathsf{max}}|}{|\lambda_{\mathsf{min}}|} \frac{\|x\|}{\|\delta x\|}$$

Rearrange and use the fact that $\kappa(A) = |\lambda_{\mathsf{max}}|/|\lambda_{\mathsf{min}}|$ we get

$$\frac{\|\delta x\|}{\|x\|_2} \leq \kappa(A)\ \frac{\|\delta b\|_2}{\|b\|_2}$$

- $\frac{\|\delta b\|}{\|b\|}$: the relative change in the right hand side
- $\frac{\|\delta x\|}{\|x\|}$: the relative change in the solution
- If $\kappa(A)$ is too large, small changes in the right hand side can cause a large change in the solution, hence the error in the solution would be too large.

Example. Let's consider the following matrix

$$A = \begin{bmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{bmatrix}$$

Note that here A is not symmetric, eigenvalues of $A^T A$ are

$$\sigma_1 = 162.2999 \qquad \sigma_2 = 0.0001$$

hence $\kappa(A) = \sqrt{\sigma_1/\sigma_2} = 1273.96$ which is pretty big. Let consider two different right hand side vectors:

$$b^{(1)} = \begin{bmatrix} 4.1 \\ 9.7 \end{bmatrix} \quad b^{(2)} = \begin{bmatrix} 4.11 \\ 9.7 \end{bmatrix}$$

- The solution of $Ax = b^{(1)}$ is $x^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- The solution of $Ax = b^{(2)}$ is $x^{(2)} = \begin{bmatrix} 0.34 \\ 0.97 \end{bmatrix}$
- $\text{Relative change} = \frac{\|x^{(1)} - x^{(2)}\|}{\|x^{(1)}\|} = 1.1732.$