Homework 2 :: MATH 504 :: Solution

Your homework submission must be a single pdf called "LASTNAME-hw1.pdf" with your solutions to all theory problem to receive full credit. All answers must be typed in Latex. Submission should be done on Canvas.

1. Consider the quadratic function

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 5x_3^2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3 + x_1 - x_2.$$

(a) Choose a matrix A and vector b so that with $x = (x_1, x_2, x_3)$, $f(x) = x^T Ax + b \cdot x$.
(b) Choose another matrix B, such that $A \neq B$ and $B = B^T$ so that $f(x) = x^T Bx + b^T x$.
(c) Determine the gradient vector for f.

Solution.
(a)

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

(c)

$$\nabla f(x) = 2Bx + b = \begin{bmatrix} 2x_1 + 2x_2 - 2x_3 + 1 \\ 2x_1 + 2x_2 + 4x_3 - 1 \\ -2x_1 + 4x_2 + 10x_3 \end{bmatrix} = \begin{bmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \partial_{x_3} f \end{bmatrix}$$

Notation: $\partial_{x_i} f = \partial f / \partial x_i$.

2. Use the Spectral Decomposition Theorem and determine the eigenvalue decomposition of the matrix A, given by

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Solution.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

3. Find the linear and quadratic approximation of the following function

$$f(x) = \exp(x_1^2 + x_2^2 + x_3^2)$$

at $\bar{x} = (0, 0, 0)^\mathsf{T}$

Solution.

$$f(\bar{x}) = 1$$

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 e^{x_1^2+x_2^2+x_3^2} \\ 2x_2 e^{x_1^2+x_2^2+x_3^2} \\ 2x_3 e^{x_1^2+x_2^2+x_3^2} \end{bmatrix} \quad \nabla f(\bar{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\nabla^2 f(x_1, x_2, x_3) = \begin{bmatrix} (2+4x_1^2)e^{x_1^2+x_2^2+x_3^2} & 4x_1x_2 e^{x_1^2+x_2^2+x_3^2} & 4x_1x_3 e^{x_1^2+x_2^2+x_3^2} \\ 4x_2x_1 e^{x_1^2+x_2^2+x_3^2} & (2+4x_2^2)e^{x_1^2+x_2^2+x_3^2} & 4x_2x_3 e^{x_1^2+x_2^2+x_3^2} \\ 4x_3x_1 e^{x_1^2+x_2^2+x_3^2} & 4x_3x_2 e^{x_1^2+x_2^2+x_3^2} & (2+4x_3^2)e^{x_1^2+x_2^2+x_3^2} \end{bmatrix}$$

$$\nabla^2 f(\bar{x}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} f(x_1, x_2, x_3) &\approx f(\bar{x}) + \nabla f(\bar{x})^\mathsf{T}(x-\bar{x}) + \frac{1}{2}(x-\bar{x})^\mathsf{T}\nabla^2 f(\bar{x})(x-\bar{x}) \\ &= 1 + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - 0 \\ x_2 - 0 \\ x_3 - 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 - 0 \\ x_2 - 0 \\ x_3 - 0 \end{bmatrix}^\mathsf{T} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 - 0 \\ x_2 - 0 \\ x_3 - 0 \end{bmatrix} \\ &= x_1^2 + x_2^2 + x_3^2 + 1 \end{aligned}$$

4. Determine whether the following quadratic function has a min, max, or saddle point. Explain why?

$$f(x_1, x_2) = 2x_1^2 - x_2^2 - x_1x_2 + 5x_2 - 1$$

Solution.

$$f(x) = (x_1, x_2) \underbrace{\begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (0, 5) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 1.$$

The eigenvalues of A are 2 and -1. Since one is positive and one is negative, f has a saddle point.