Formulations and Geometry of Linear Programs

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September 12th, 2022

Last Time

- Feasible Region
- Solving LPs Graphically
- Standard Form
- Equivalence of Linear Programs
- Redundant Constraints
- Unboundedness
- Infeasible

Outline

- Additional Formulation Example
- Absolute Value
- Piecewise Linear Convex
- Min Max
- Geometry Review
- Basic Feasible Solutions

The Farmer and The Salesperson

by Ingrid Kallick for the program cover of the 1996 annual meeting of the American Astronomical Association

Alice works for a company that produces cow feed. She is currently trying to sell a new blend (Y Feed Mix) to Bob, a successful farmer with 100 dairy cattle. Bob's feeds a rigid diet to his cows. Each cow receives a daily minimum of; 100 units of calcium, 1,500 units of protein, and 20,000 calories.

Right now, Bob uses a product called Formula X that costs \$0.015/ounce and provides 1 unit of calcium, 400 calories, and 20 units of protein.

The new product contains 2 units of calcium, 250 calories, and 20 units of protein at a cost of \$0.02/ounce. Should Bob buy any of the new product? How will it change his monthly budget?

How much does Bob spend now?

- Feeds each cow 100 ounces of Formula X
  - Calcium: 100*1=100 units
  - Protein: 100*20=2000 units
  - Calories 100*400=40,000 units
- Total cost=
  - 100*100*0.015=$150.00

What does Alice notice?

- Bob’s cows receive
  - the exact amount of calcium needed
  - An excess of 500 units of protein per cow
  - An excess of 20,000 calories per cow (double)
- What about Y Feed Mix ?
  - 2 units of calcium per ounce
  - 250 calories per ounce
  - 20 units of protein per ounce
  - Cost=$0.020

Would Bob want to only use Alice’s new feed?

- Would need to feed each cow only 80 ounces
  - Calcium: 80*2=160
  - Protein: 20*80=1,600
  - Calories: 250*80=20,000
- Total Cost:
  - 100*80*0.02=$160.00

- NO! It is more expensive

The Decision Variables

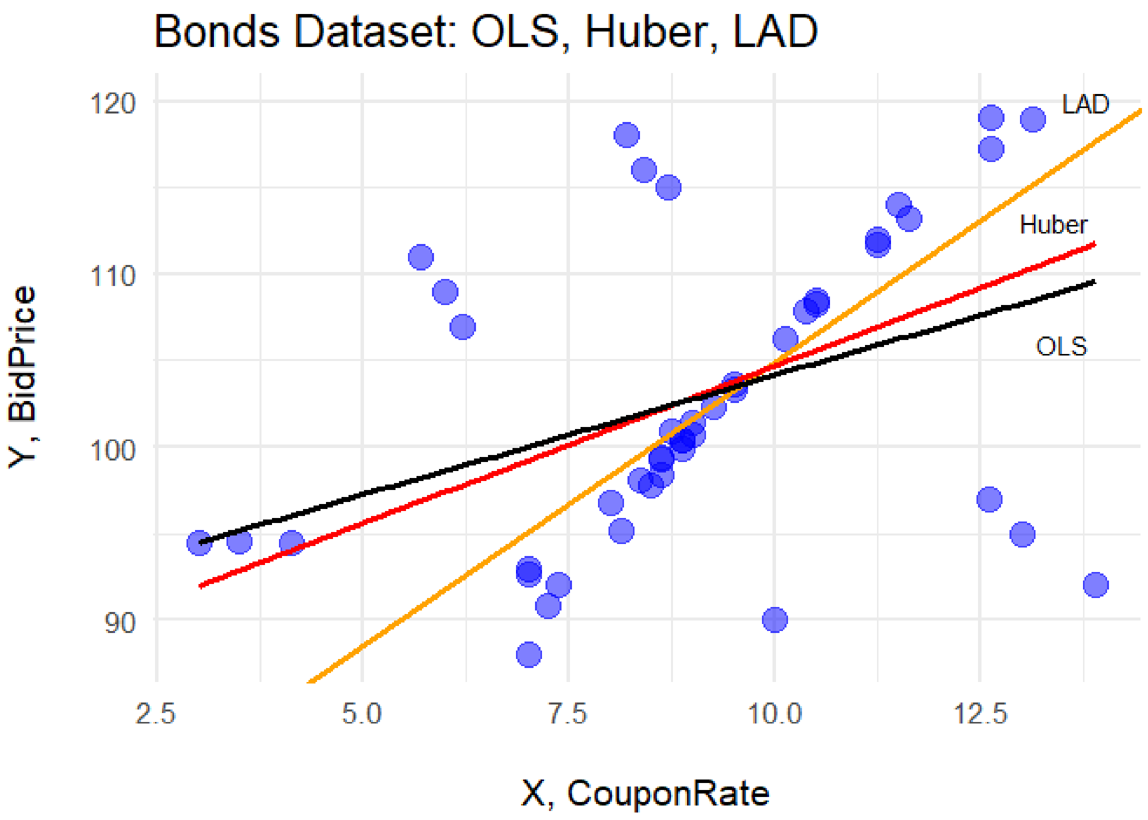
- X = ounces of Formula X to be used per cow
- Y = ounces of Y Feed Mix to be used per cow

- We will use the decision variables above, however, there are other acceptable formulations:
  - % of each feed used per ounce
  - Ounces of feed to be used in total

LP Formulation

- Objective Function
  - Minimize $0.015X + 0.02Y$
- Constraints
  - Calcium : $1X + 2Y \geq 100$
  - Protein : $20X + 20Y \geq 1500$
  - Calories : $400X + 250Y \geq 20{,}000$
  - Non-negativity : $X, Y \geq 0$

Absolute Value



rpubs.com/MauricioClaudio/885472

- Traditional regression typically refers to Ordinary Least Squares regression that seeks to minimize
  - $\|\boldsymbol{y} - \boldsymbol{X}\beta\|^2$
- Least Absolute Deviations is an alternative form of regression that instead of the L_2 norm uses the L_1 norm
  - $|\boldsymbol{y} - \boldsymbol{X}\beta|$

Formulation 1:

- Min $\sum|y_i - \beta x_i|$
- s.t. $\beta \in (-\infty, \infty)$

- Note that x in this case does not represent the decision variable, instead the data that was observed

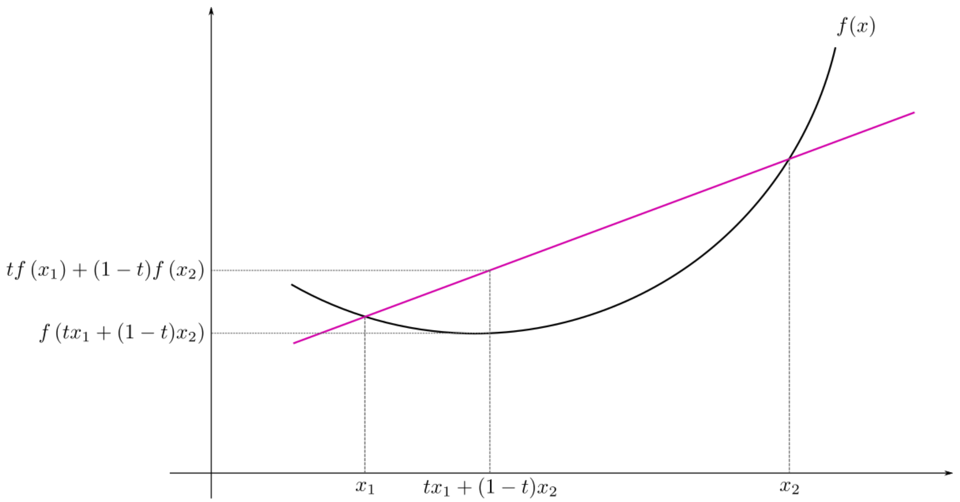
- Min $\sum z_i$
- s.t. $y_i - \beta x_i \leq z_i$
  $\beta x_i - y_i \leq z_i$
- $\beta \in (-\infty, \infty)$
- Note that this trick does NOT work for Maximization or Minimization of an Absolute Value with a negative cost coefficient

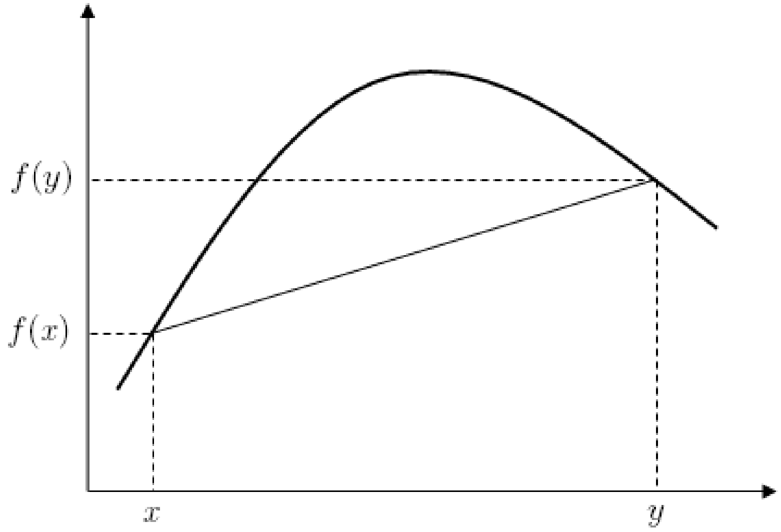
Formulation 2:

- Min $\sum|y_i - \beta x_i|$
- s.t. $\beta \in (-\infty, \infty)$
- Note that x in this case does not represent the decision variable, instead the data that was observed

- Min $\sum e_i^+ + e_i^-$
- s.t. $y_i - \beta x_i = e_i^+ - e_i^-$
- $\beta \in (-\infty, \infty)$
- $e_i^\pm \geq 0$
- Note that this trick does NOT work for Maximization or Minimization of an Absolute Value with a negative cost coefficient

Piecewise Linear Objectives

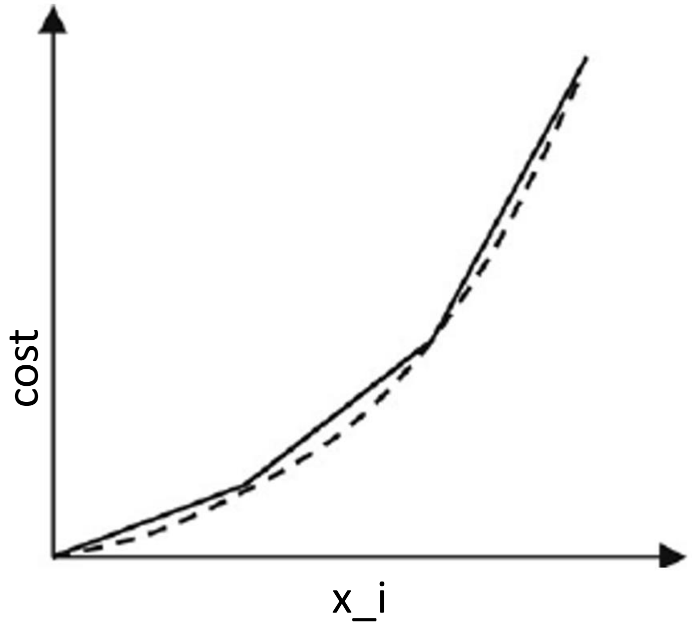




- A function is **convex** if $\forall x, y \in \mathbb{R}^n$ $and\ t\epsilon[0,1]$ then $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$
- A function is **concave** if $\forall x, y \in \mathbb{R}^n$ $and\ t\epsilon[0,1]$ then $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$

Can we use Convex Piecewise-linear Cost Functions?

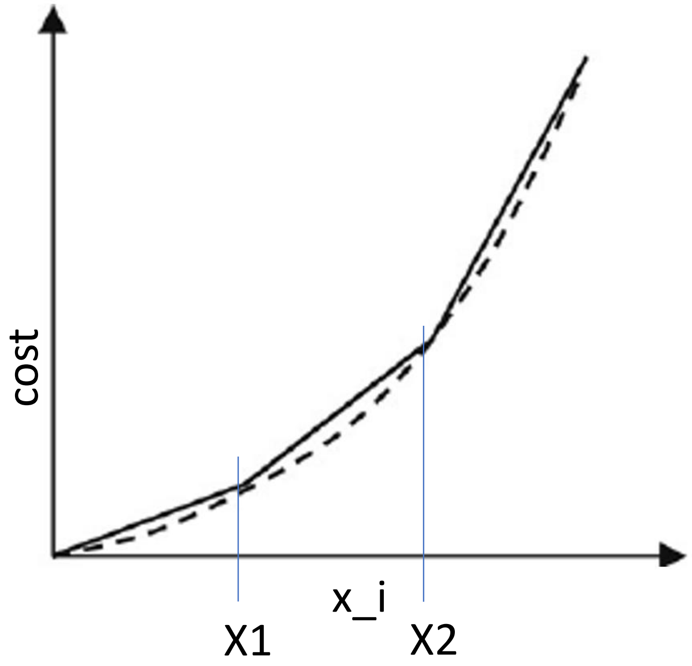
Suppose we want:



- Given:
  - Minimize: c'x
  - s.t.: Ax≤b
  - x≥0
- Can we transform our linear program to accommodate the new cost function?

Can we use Convex Piecewise-linear Cost Functions?

Suppose we want:



- Given:
  - Minimize: c'x
  - s.t.: Ax≤b
  - x≥0
- $x_i \rightarrow x_{i,1} + x_{i,2} + x_{i,3}$
- $c_i x_i \rightarrow c_{i,1} x_{i,1} + c_{i,2} x_{i,2} + c_{i,2} x_{i,3}$
- s.t. $c_{i,1} < c_{i,2} < c_{i,3}$
- $0 \leq x_{i,1} \leq X1$
- $0 \leq x_{i,2} \leq X2 - X1$
- $0 \leq x_{i,3}$

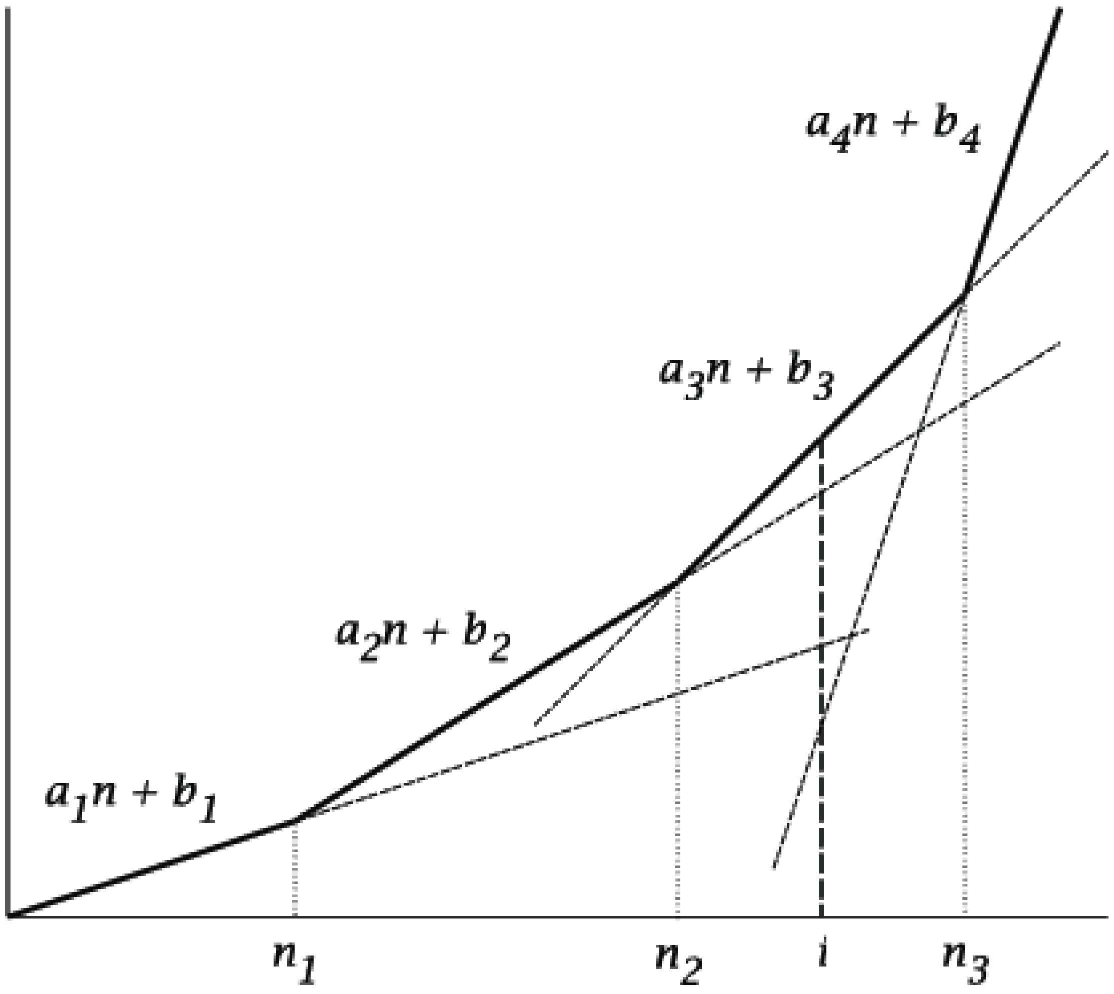
Alternatively: Min Max

- Let $f_1, f_2, \dots, f_m$ be linear functions of the decision variables, then max($f_1, f_2, \dots, f_m$) is convex. Then we can change min max into:
- min z

  s.t. f_1(x) $\leq z$

  f_2(x) $\leq z$

  …



What about Max Min?

- max z

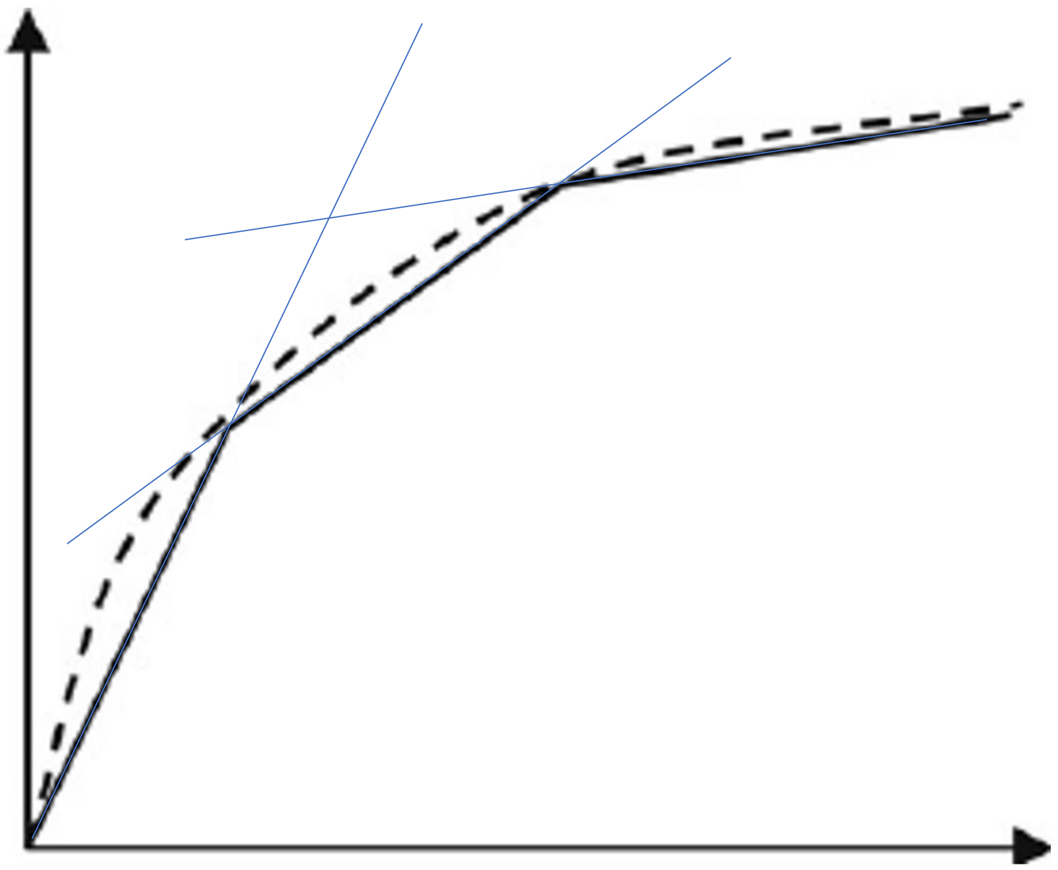
  s.t. f_1(x) $\geq z$

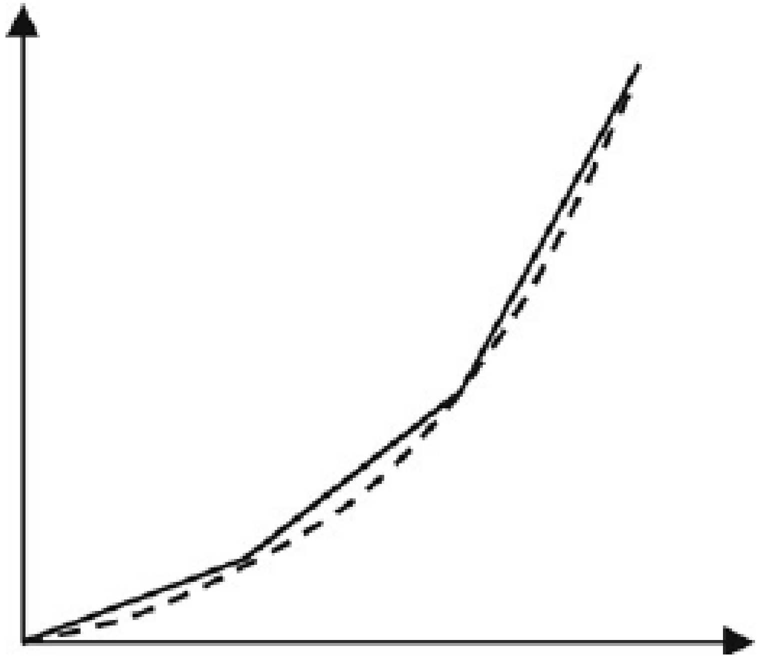
  f_2(x) $\geq z$

  …

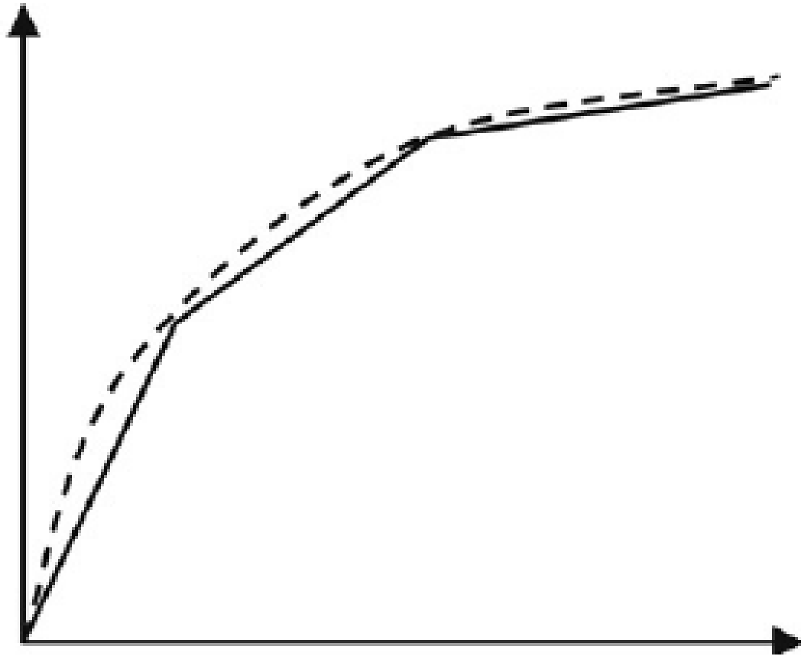
- But remember in general:

$$\min_x \max_{i=1,2,3} f_i(x) \neq \max_{i=1,2,3} \min_x f_i(x)$$





- Convex functions work for minimization



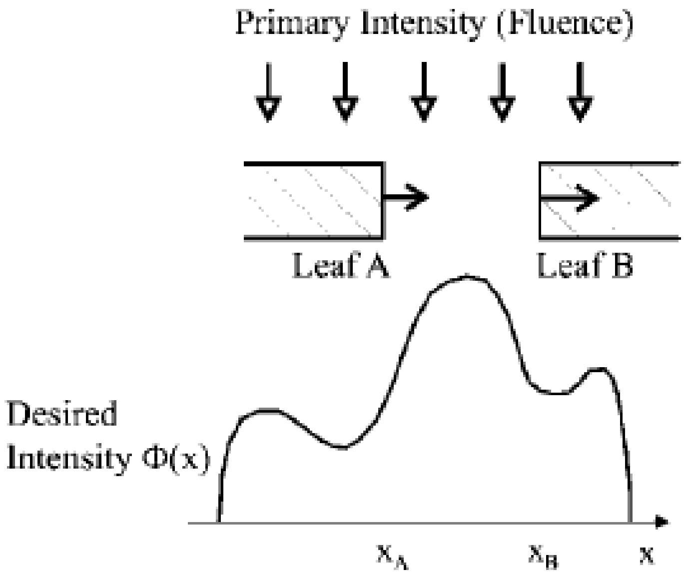
- Concave functions work for maximization

Radiation Therapy Problem

- Cancer is the second leading cause of death in the United States, with nearly 1 of every 4 deaths attributable to cancer
- In 2019, an estimated 1.76 million Americans will be diagnosed with some form of cancer
- Approximately 60% of all U.S. patients with cancer are treated with radiation therapy, most of them with external beam radiation therapy
- Intensity-Modulated Radiation Therapy (IMRT) is the most common form of external beam radiation therapy

- IMRT uses a gantry arm to allow radiation to be delivered from multiple angles

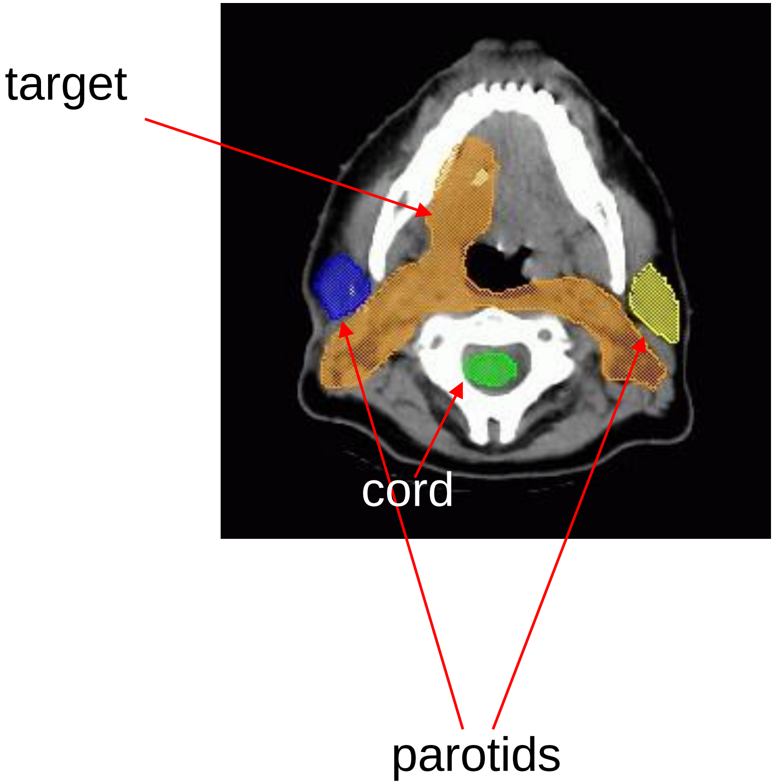
- A multi-leaf collimator is adjusted for each beam angle to shape the radiation



- IMRT can adjust the amount of radiation received at each pixel offering far greater control than previous treatments such as 3D conformal mapping radiation therapy (3DCRT)
- The multi-leaf collimator is dynamically adjusted controlling the intensity to each pixel

- Images from Webb 2003 ©2003 by The British Institute of Radiology

Identify Tumor and OAR



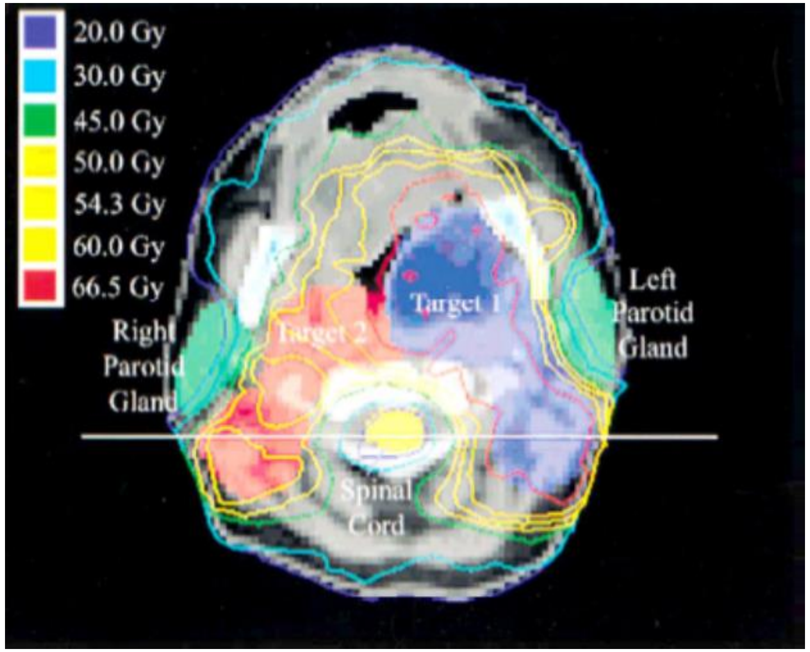
- A number of sensitive structures including the parotids (salivary glands), spinal cord, and lymph nodes
- Different tissues have different tolerance to radiation
- The tumor and immediate adjacent tissue form the planning treatment volume (PTV) for which target radiation levels are set

Penalty Score

$$S = \sum_{i \in OAR \cup PTV} \beta_i \max(A_i - d_i, 0) + \sum_{i \in PTV} \beta_i \max(d_i - A_i, 0)$$

- β_i penalty weight
- d_i desired dose level, in Greys (Gy)
- A_i actual dose level
- OAR set of organs at risk
- PTV set of tissue in planned treatment volume

Penalty Score



Constraint	Desired level	Weight
Less than 66% of the left parotid receiving	26 Gy	3
Less than 33% of the left parotid receiving	32 Gy	3
Less than 66% of the right parotid receiving	26 Gy	3
Less than 33% of the right parotid receiving	32 Gy	3
Less than 90% of the oral mucosa receiving	30 Gy	8
Less than 30% of the oral mucosa receiving	40 Gy	8
Maximum spinal cord dose	45 Gy	15
Maximum brain stem dose	54 Gy	15
More than 95% of the low-risk PTV receiving	54 Gy	6
Less than 5% of the low-risk PTV receiving	59.4 Gy	6
More than 95% of the high-risk PTV receiving	59.4 Gy	6
Less than 5% of the high-risk PTV receiving	70 Gy	6

Formulation Problem: Galaxy Industries

Galaxy Industries is a branch of a toy company that produces children's toys. They are currently considering two new products: (1) space ray water gun and (2) zapper water gun. Galaxy Industries is still in the beginning phases of new product introduction and has only conducted limited market research so far. They have determined that a reasonable profit to be made on space ray guns is $8 and $5 for zapper guns. In addition, it has been decided that space ray sales will not exceed zapper sales by more than 350 units per week. Galaxy Industries would like to know how many of each type of gun to produce per week to maximize profit.

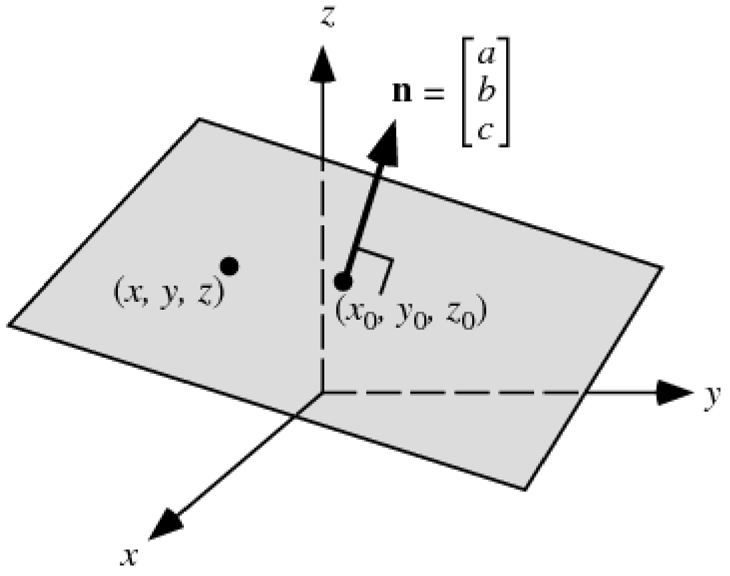
Key Components of the Linear Program

- Decision Variables
  - x_1= number of space ray water guns to sell per week
  - x_2= number of zapper water guns to sell per week
- Objective Function
  - Maximize Profit=$8x_1+5x_2$
- Constraints
  - Market Research: $x_1-x_2<=350$
  - Non-negativity

Some Geometry

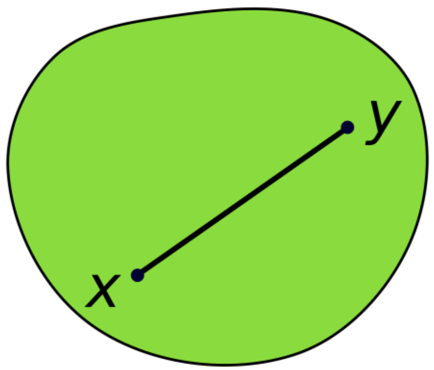
Let a be a nonzero vector in $\mathbb{R}^n$ and let b be a scalar.

- The set $\{x \in \mathbb{R}^n | a'x = b\}$ is called a **hyperplane**
- The set $\{x \in \mathbb{R}^n | a'x \leq b\}$ is called a **halfspace**

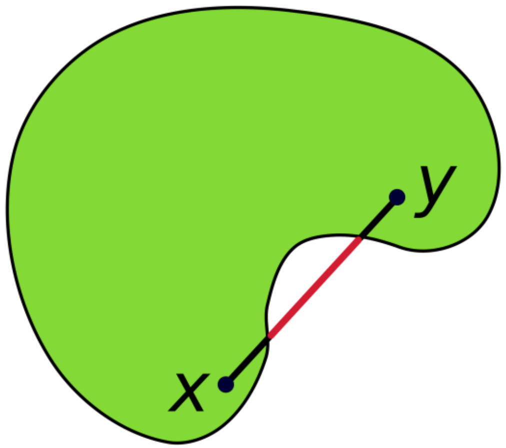


Convex Set

Convex



Non- Convex



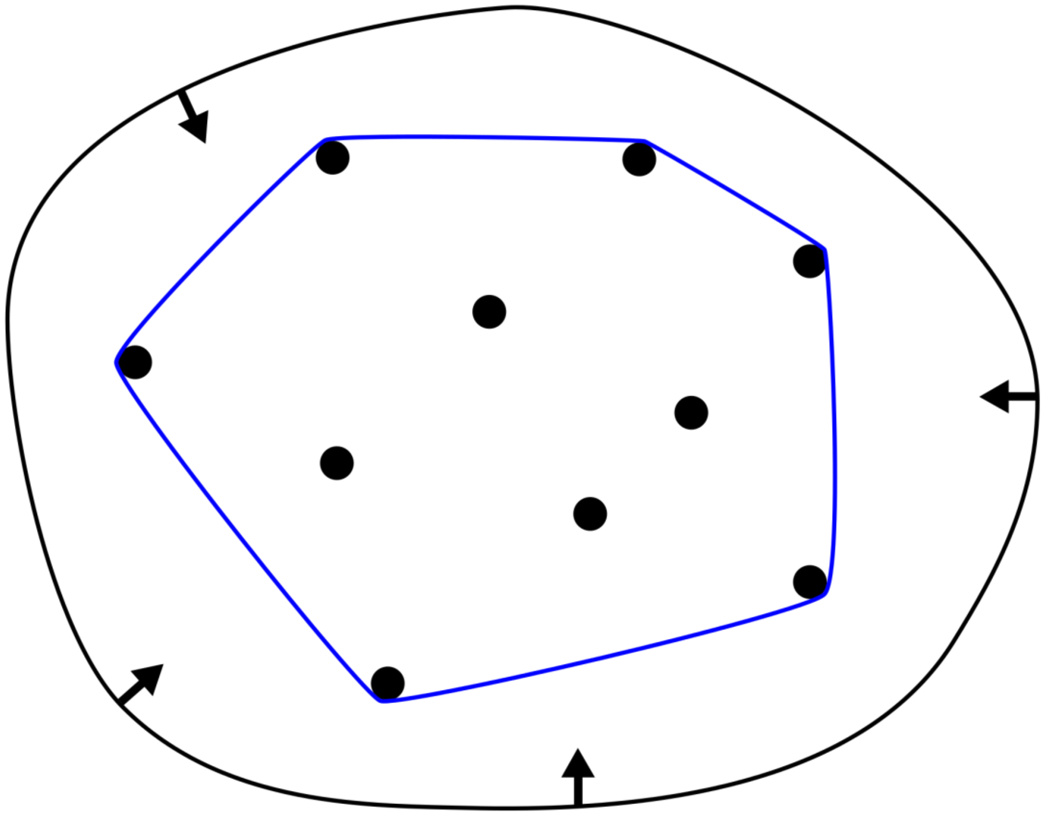
- A set, S, is **convex** if $\forall x, y \in S$ and $t\epsilon[0,1]$ then

$$(tx + (1-t)y) \in S$$

- Note that a **line segment** is a convex set, as is a **hyperplane** and **halfspace**

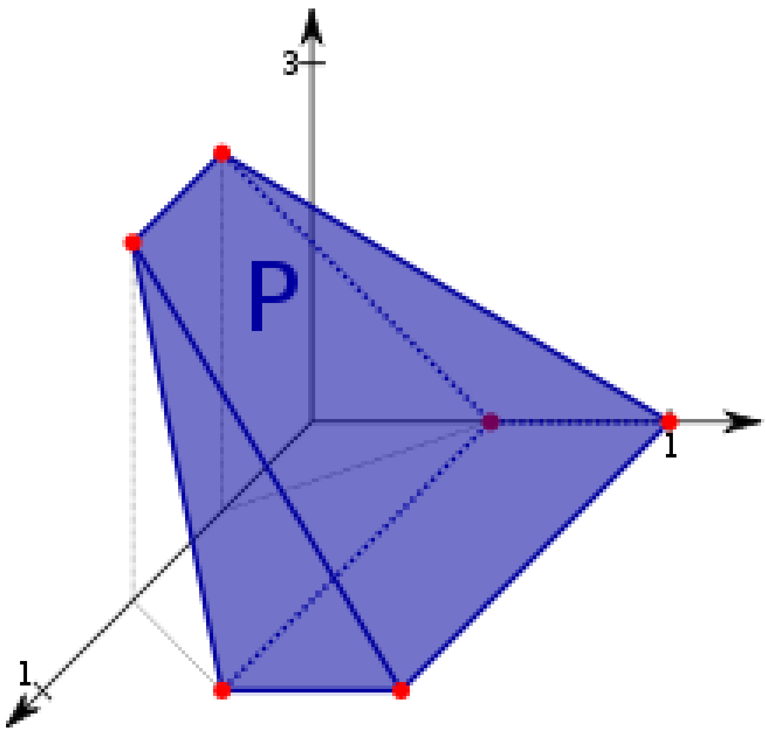
- The **intersection** of a collection of convex sets is convex

Convex Hull



- Let $\lambda_{1,}\lambda_{2,\dots,}\,\lambda_k$ be non-negative scalars such that $\sum \lambda_i = 1$, then $\sum \lambda_i\, x_i$ is said to be a **convex combination** of vectors $x_{1,}x_{2,\dots,}\,x_k$
- A **convex hull** of the vectors $x_{1,}x_{2,\dots,}\,x_k$ is the set of all convex combinations of those vectors

Polyhedron/Polytope



- We are defining a polyhedron to be a set described in the form $\{x \in \mathbb{R}^n | Ax \geq b\}$ where A is an m x n matrix and $b \in \mathbb{R}^m$

Binding Constraints

Let us define a polyhedron such that:

- $a'_i x^* \geq b_i \text{ , } i \in M_1$
- $a'_i x^* \leq b_i \text{ , } i \in M_2$
- $a'_i x^* = b_i, i \in M_3$

- If a vector x^* satisfies $a'_i x^* = b_i$ for some i in M_1, M_2, or M_3, then we say the corresponding constraint is **binding** or **active**.
- If a constraint is binding, than any corresponding slack or surplus variables used put it in equality form are 0.

Extreme points

- Let P be a polyhedron. A vector $x \in P$ is an **extreme point** of P if we cannot find two vectors $y, z \in P$, both different from x, and a scalar $t \epsilon [0,1]$ such that $\mathrm{x} = (ty + (1-t)z)$

- i.e. x is an extreme point if it is not a convex combination of any two other points in the convex set

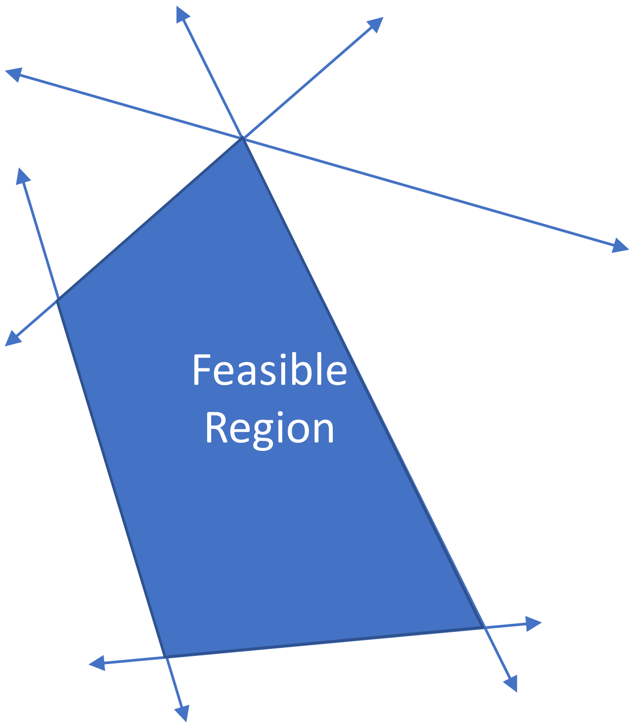
Basic Feasible Solutions

- Let P be a polyhedron defined by linear inequality and equality constraints and let $x^* \in \mathbb{R}^n$.
- The vector x^* is a basic solution if:
  - All equality constraints are active.
  - Out of the constraints that are active at x^* , there are n of them that are linearly independent.
- If x^* is a basic solution that satisfies all of the constraints, we say that it is a **basic feasible solution**.

Basic Feasible Solution

- Let P be a nonempty polyhedron and let x^* be in P. Then the following are equivalent:
  - x^* is a vertex
  - x^* is an extreme point
  - x^* is a basic feasible solution

Degeneracy



- A basic solution $x \in \mathbb{R}^n$ is said to be **degenerate** if more than n of the constraints are active at x

Important Theorems

Theorem: *A nonempty and bounded polyhedron is the convex hull of its extreme points*

Theorem: *A linear programming problem of minimizing $c'x$ over a polyhedron P ($Ax \geq b$). If P has at least one extreme point, then the optimal cost is either $-\infty$ or there exists an extreme point which is optimal*

Next Time

- The Simplex Method!