Mathematical Formulation and Graphical Solutions for LP

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Outline

- Mathematical Programming
- Linear Programs
- Solving LPs Graphically
- Standard Form and Equivalence of Linear Programs
- Redundant Constraints, Unboundedness, …

Mathematical Programming

- *Optimize*

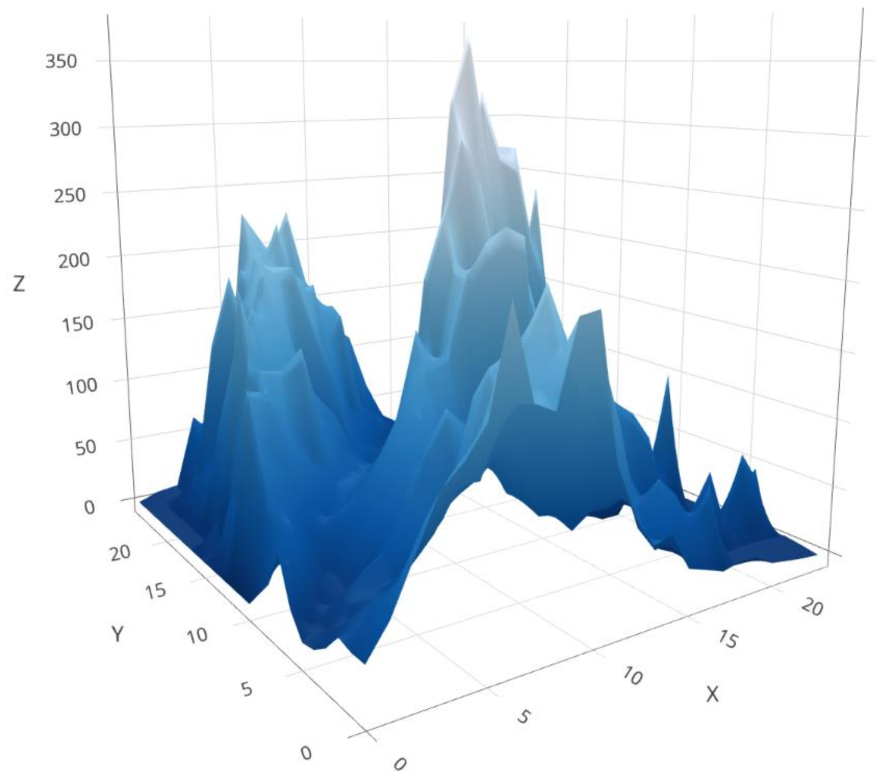
$$f(x_1, x_2, \dots, x_n)$$

- *Subject to*

$$\left.\begin{matrix} g_1(x_1, x_2, \dots, x_n) \\ \dots \\ g_m(x_1, x_2, \dots, x_n) \end{matrix}\right\} \begin{matrix} \leq \\ = \\ \geq \end{matrix} \left\{\begin{matrix} b_1 \\ \dots \\ b_2 \end{matrix}\right.$$

Key Components

- Decision Variables $(x_1, x_2, \ldots, x_n)$
  - The controllable input to the model
- Objective Function (or cost function)
  - A mathematical representation of the goal (the quantity to be minimized or maximized)
- Constraints
  - An equation or inequality that defines the feasible solution set



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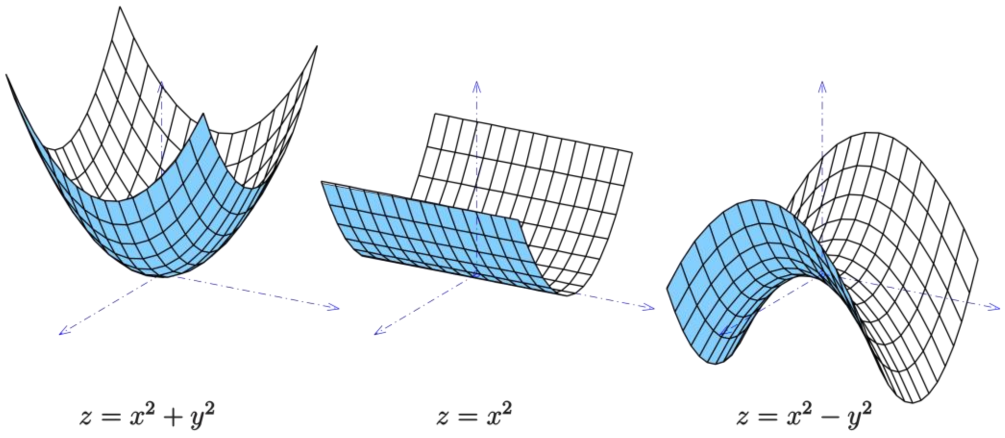
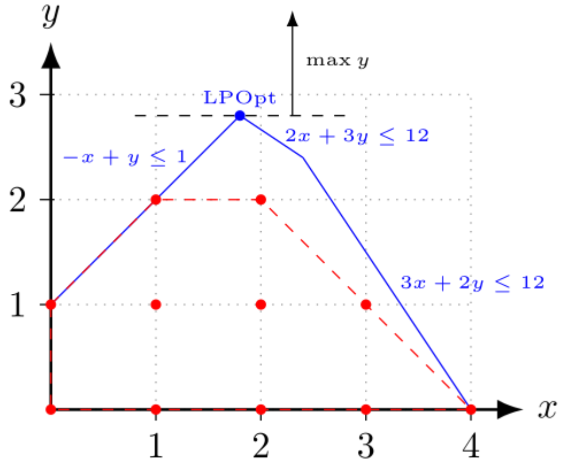
Some Types of Mathematical Programming

- Convex Programming- with a convex (min)/ concave (max) objective function and convex constraint set with continuous variables
- Quadratic Programming - a quadratic objective function and linear constraints
- Linear Programming (LP) – linear objective function
- Mixed-Integer Programming (MIP)- Mix of integer and continuous variables
- Integer Programming (IP) – Integer variables





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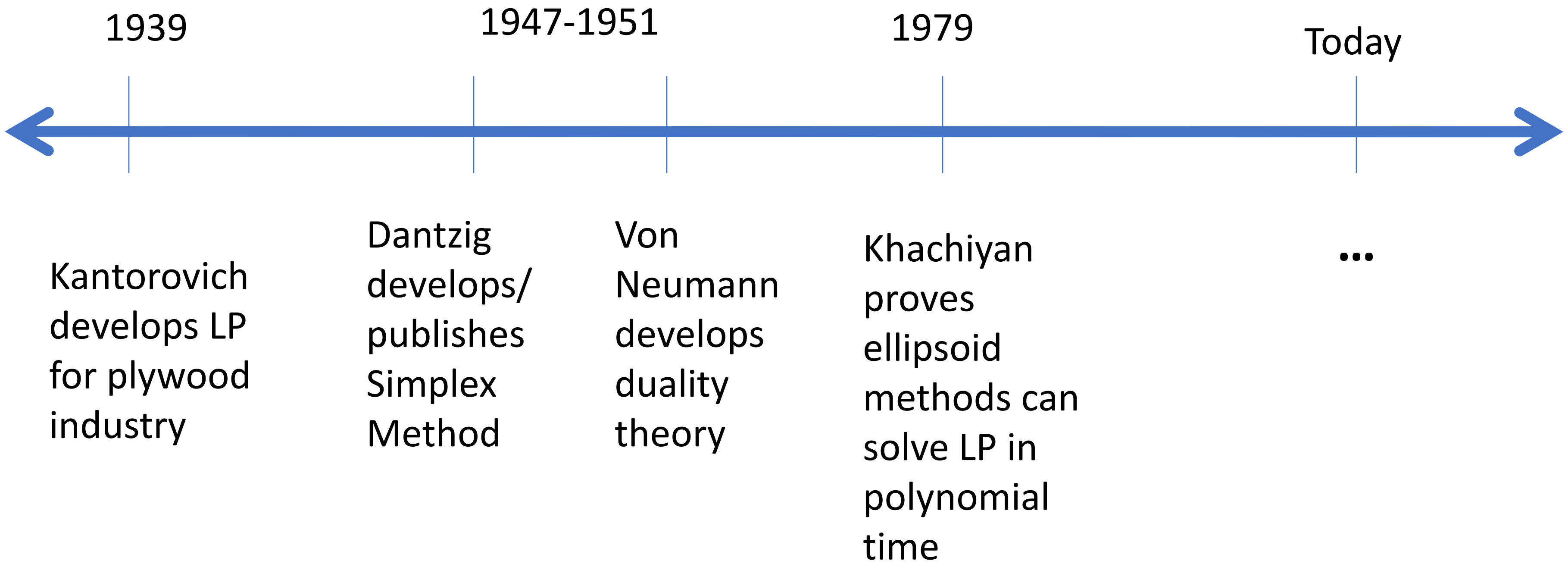
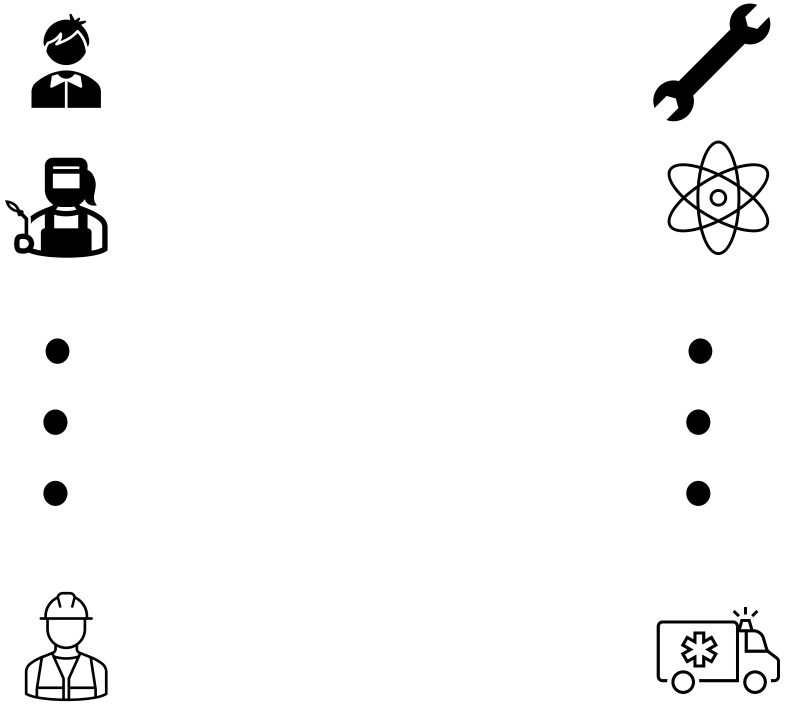
A Brief History of Linear Programming



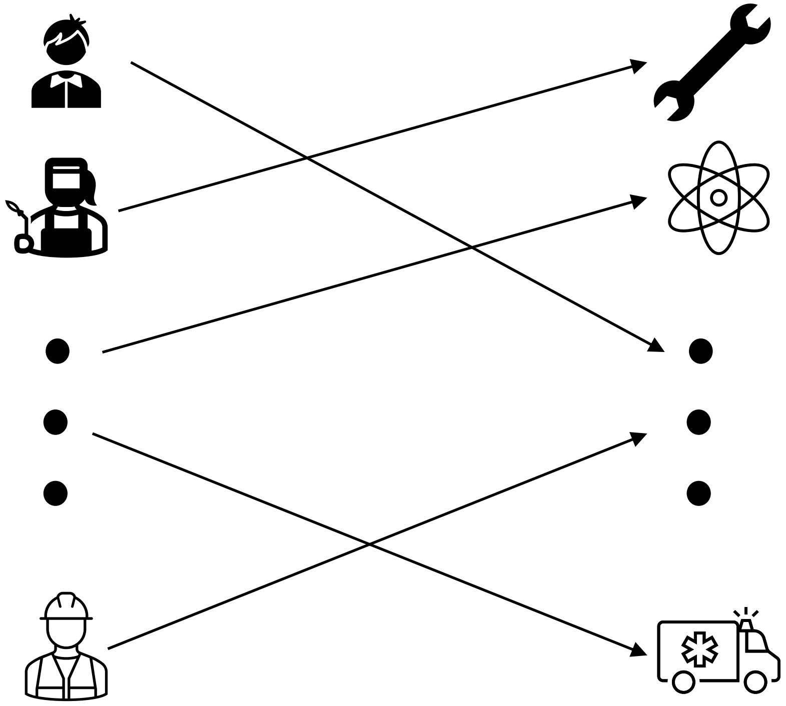
The Power of Linear Programming:

- George Danztig’s famous example
  - Assigning 70 people to 70 jobs in an optimal manner
- Compare 2 approaches:
  - Checking every option
  - Using the Simplex Method



Checking Every Option:

- Let's say it takes .0000001s for a computer to evaluate the "score" of one option (that's 10,000,000 evaluations per second!)
  - There are 70! = 1.2×10^{100} options
- How long will that take?
  - 3.33×10^{89} hours
  - 1.38×10^{88} days
  - 3.805×10^{85} years
- An LP can solve the problem in seconds!



Assumptions of Linear Programming:

Linear Constraints and Objectives

- Proportionality
  - All contributions of variables to the objective function and constraints must be proportional
- Additivity
  - Value of the objective function and use of total resources can be found by *summing* across decision variables
  - No cross products

Assumptions of Linear Programming:

- Divisibility
  - Each variable must be **continuous**
  - Decision variables can take any value that satisfies the constraints
- Certainty
  - All data provided is deterministic
  - No probabilistic ranges
- Non-negativity (often listed but more of a formalism)
  - Variables are not allowed to take on negative values in formal statements

An Example:

You have recently inherited a large sum of money and you want to use this money to create an investment portfolio. You have two investment options:

- A bond fund, return of 6%
- A stock fund, return of 10%

You want to invest at least 30% in the bond fund and want to obtain at least a 7.5% return

Decision Variables:

- x_1=% of the inheritance that you invest in the bond fund
- x_2=% of the inheritance that you invest in the bond fund

Objective Function:

- Want to maximize possible return

- Mathematically:
  - Maximize: $0.06*x_1+0.10*x_2$

Constraints

- The % invested in the bond fund has to be at least 30%
  - $X_1 \geq 0.30$
- The overall return has to be at least 7.5%
  - $0.06*x_1+0.10*x_2 \geq 0.075$

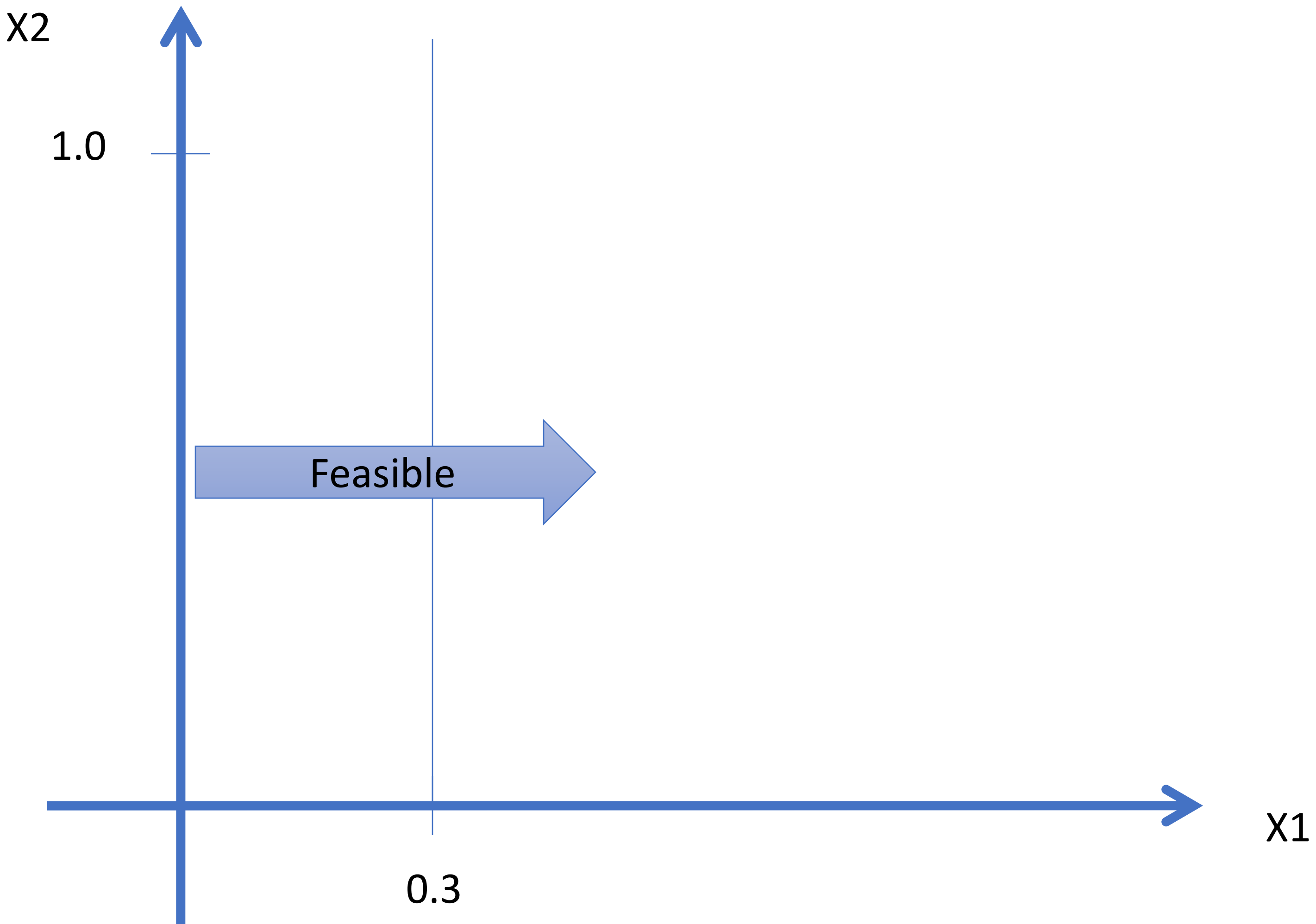
Constraints

- The % invested in the bond fund has to be at least 30%
  - $X_1 \geq 0.30$
- The overall return has to be at least 7.5%
  - $0.06*x_1+0.10*x_2 \geq 0.075$
- The sum of the variables must be less than or equal to 1
  - $x_1+x_2 \leq 1$
  - $X_2 \geq 0$

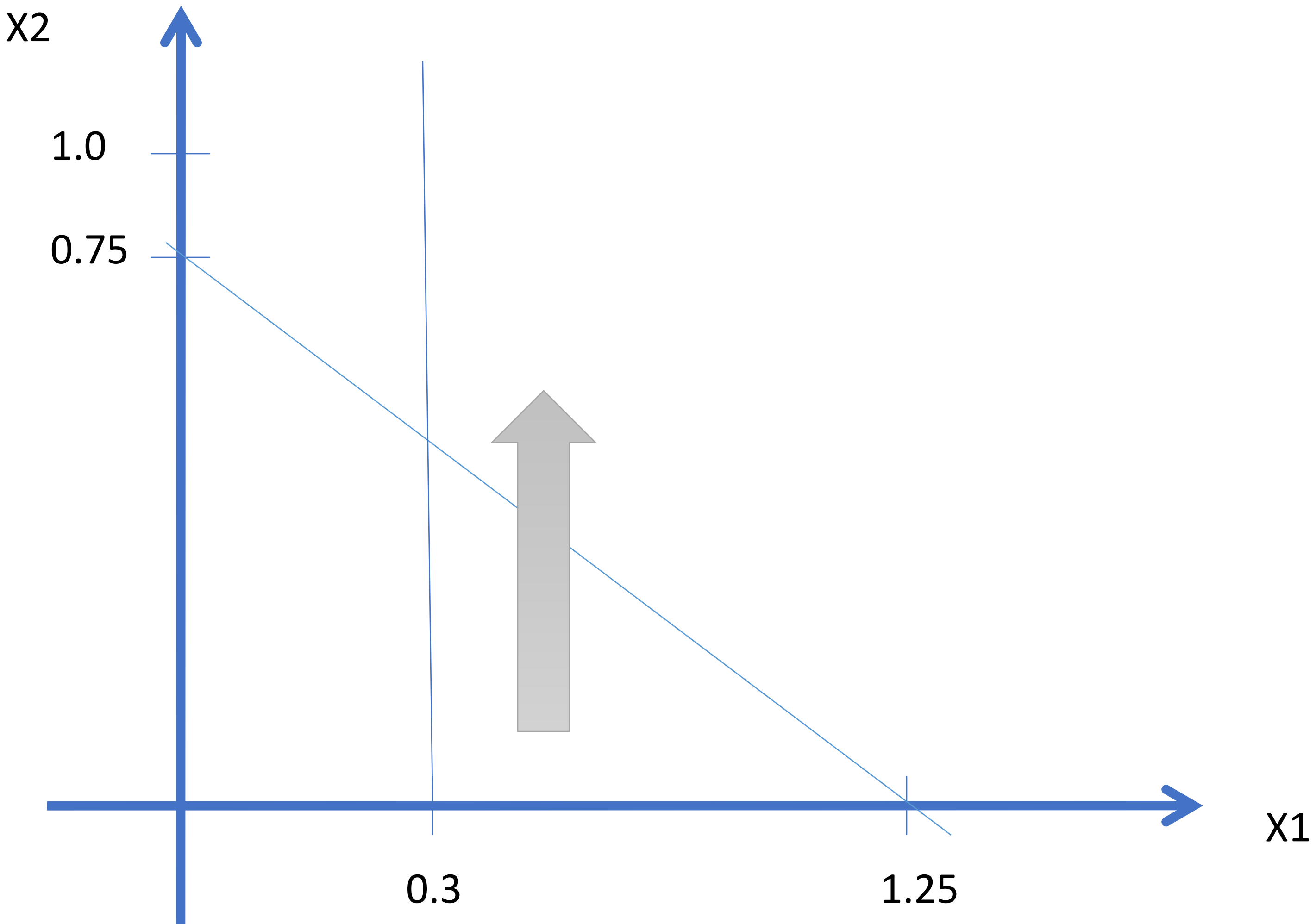
Solving Graphically

- Step 1:
  - Finding the feasible region

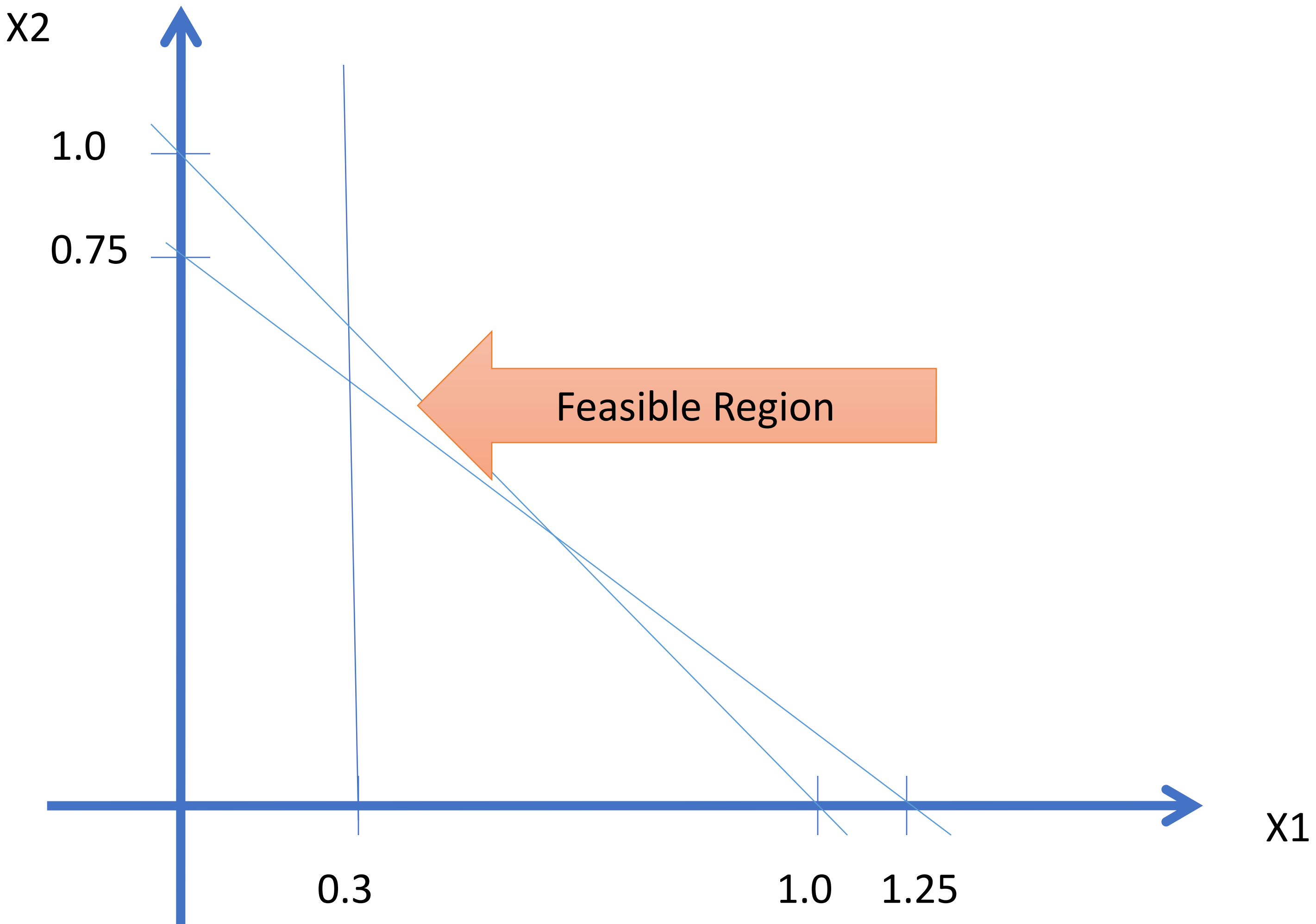
Graphical Representation



Graphical Representation



Graphical Representation



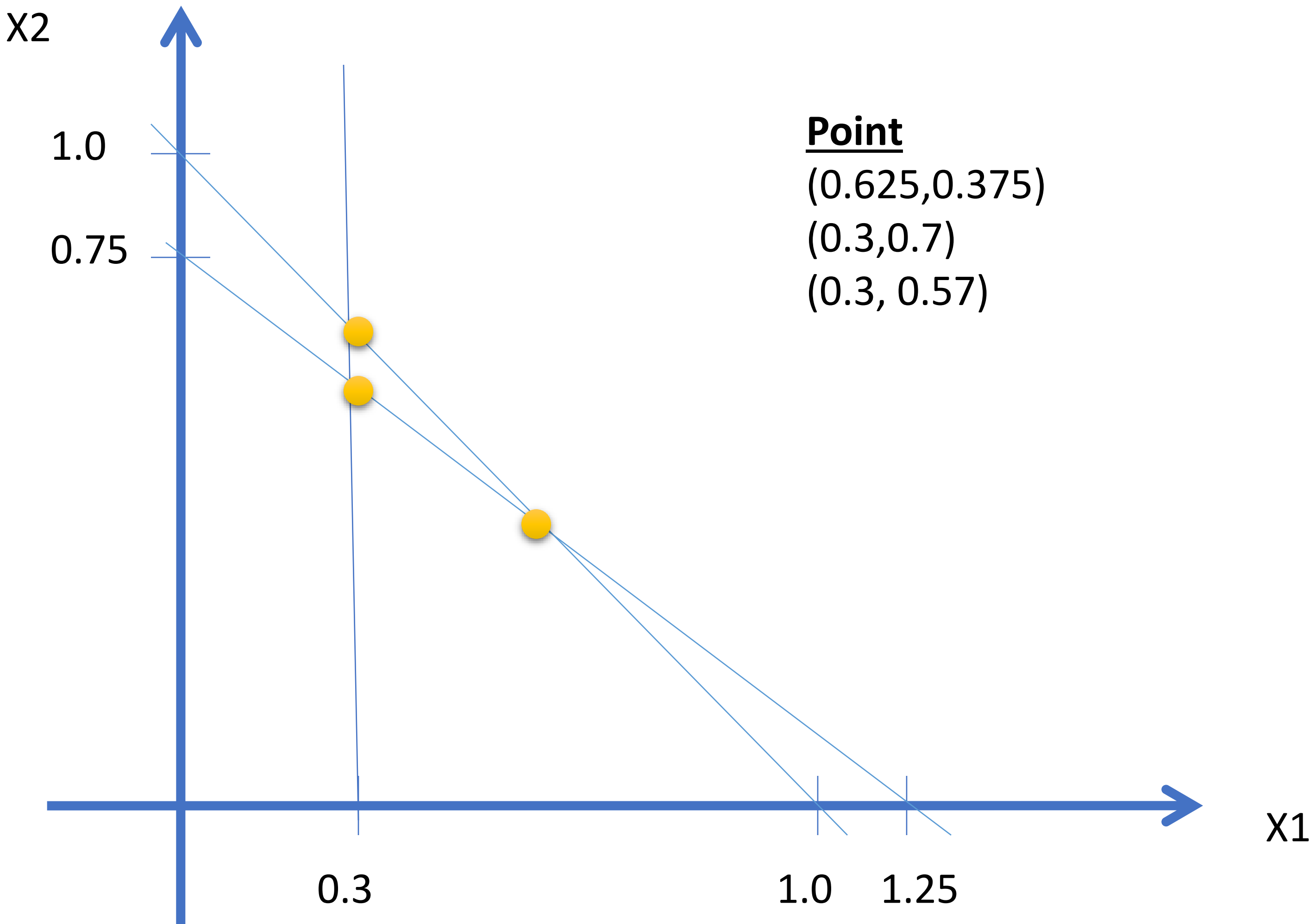
Corner Point (Extreme Point)

- Points that mark the boundaries of the feasible region
  - In a two –variable model, these are intersections of two constraints

Solving Graphically

- Step 1:
  - Finding the feasible region
- Step 2:
  - Identifying the corner solutions

Graphical Representation



Point

(0.625,0.375)

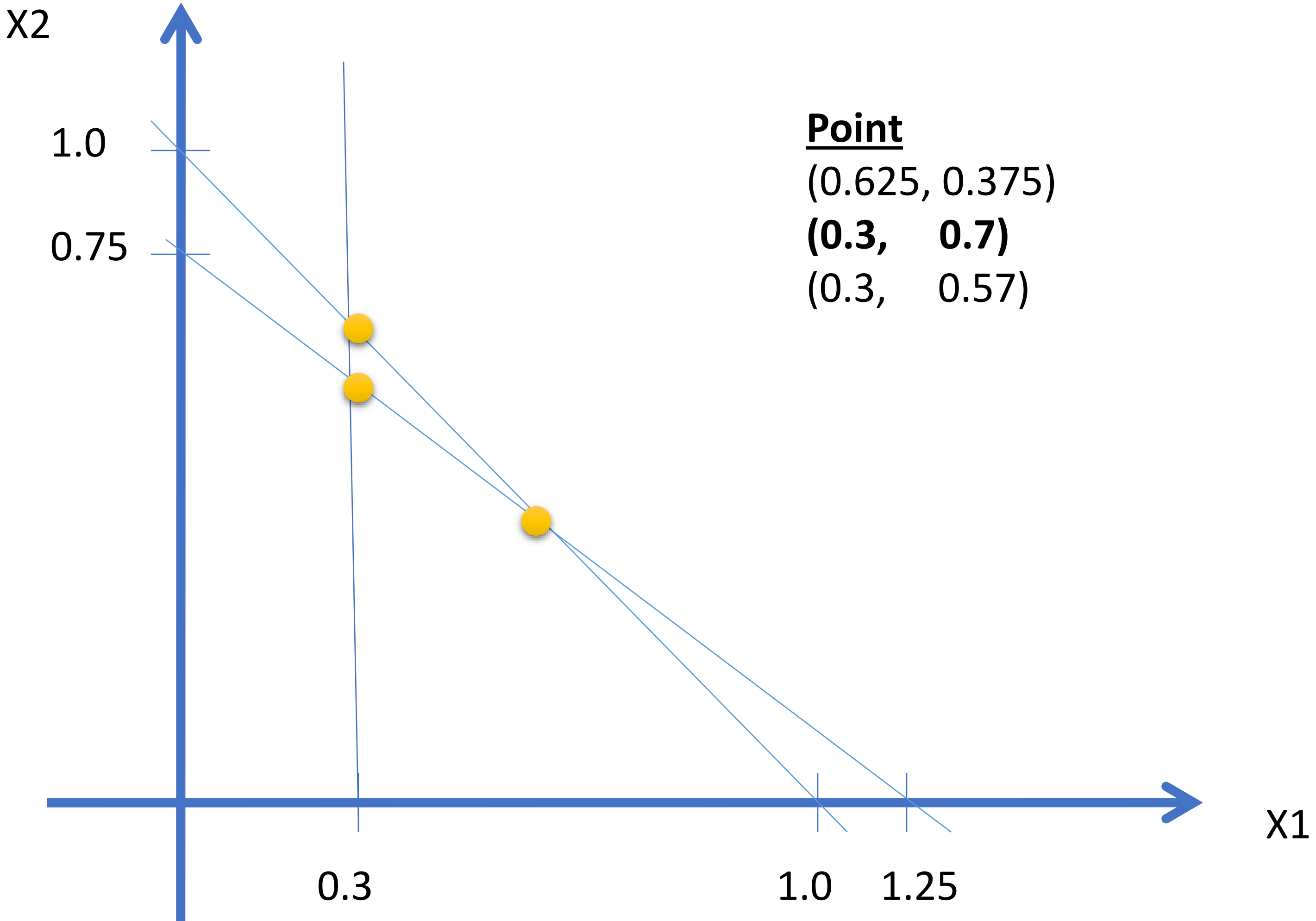
(0.3,0.7)

(0.3, 0.57)

Solving Graphically

- Step 1:
  - Finding the feasible region
- Step 2:
  - Identifying the corner solutions
- Step 3:
  - Evaluate each corner solution at the profit function

Graphical Representation



Point	Return
(0.625, 0.375)	0.075
(0.3, 0.7)	**0.088**
(0.3, 0.57)	0.075

Exercise

- You are in charge of planning production for an upholstery company. The company currently upholsters two different types of products. You are trying to determine how many of each to produce per day. A couch sells for $300, takes 1 hour to make, and uses 30 square feet of fabric. A recliner sells for $200, takes 2 hours to make, and uses 10 square feet of fabric. You have up to an 8 hour workday and 90 square feet of fabric.
- **Your goal is to plan production for the day to maximize the value of the products manufactured.**

General/Standard/Canonical/Equality Form

?

Ax=b
Ax≤b

?

Standard
Canonical
Equality
General

- Frustratingly/ironically different sources will define “Standard Form” differently, ultimately it doesn’t matter as transformation between the forms is what is important

- For purposes of homework/test questions it will be explicit whether equalities or inequalities are requested

General Form

max c'x

s.t.

Ax≤b

x≥0

- **x** is a vector (*n*) of the decision variables
  - **x** = $(x_1, x_2, \dots, x_n)$
- **c** is a vector (*n*) of the coefficients of the objective function
  - $\boldsymbol{c}$ = $(c_1, c_2, \dots, c_n)$
- **A** is a $m \times n$ matrix of the constrain coefficients
- **b** is a vector (*m*)of the right-hand side of the constraint equations
  - $\boldsymbol{b}$ = $(b_1, b_2, \dots, b_m)$

Standard Form

max c'x

s.t.

Ax=b

x≥0

- **x** is a vector (n) of the decision variables
  - **x** = $(x_1, x_2, \ldots, x_n)$
- **c** is a vector (n) of the coefficients of the objective function
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  - $\boldsymbol{b}$ = $(b_1, b_2, \ldots, b_m)$

Tricks for translating between forms

- Switching inequalities:
- Equality to inequalities:
- Unbounded to bounded:

- a ≤ b → -a ≥ -b
- a = b → a ≤ b
  a ≥ b
- $x \, \epsilon (-\infty, \infty) \rightarrow x = \ x^+ - x^-$
  $x^+, x^- \geq 0$

Transforming Constraints to Equalities

Slack Variables

- In less than or equal to constraints (≤)
- An Example:
  - $5x_1+8x_2 \leq 30$
  - If $x_1=1$ and $x_2=2$ we have that the value of the constraint is:
  - $5(1)+8(2)=21$
  - To make the constraint equal, we need to add 9 units to make up the "slack"
- **$5x_1+8x_2+s_1=30$**

Surplus Variables

- In greater than or equal to constraints (≥)
- An Example:
  - $10x_1+2x_2 \geq 14$
  - If $x_1=1$ and $x_2=3$ we have that the value of the constraint is:
  - $10(1)+2(3)=16$
  - To make the constraint equal, we need to take away 2 units of "surplus"
- **$10x_1+2x_2-s_2=14$**

Transforming the Objective Function

Original Form

- Max:
  - x_1+x_2
- s.t.
  - $x_1 \leq 100$
  - $x_2 \leq 80$
  - $2x_1+4x_2 \leq 400$
  - $x_1, x_2 \geq 0$
- Optimal solution:
  - **$x_1 = 100$**
  - **$x_2 = 50$**
  - **O.F. value=150**

Equality Form

- Max:
  - $x_1+x_2 +0*s_1+0*s_2+0*s_3$
- s.t.
  - $x_1+s_1 = 100$
  - $x_2+s_2 = 80$
  - $2x_1+4x_2 +s_3 = 400$
  - $x_1, x_2, s_1, s_2, s_3, \geq 0$
- Optimal solution:
  - **$x_1 = 100$**
  - **$x_2 = 50$**
  - $s_1 = 0$
  - $s_2 = 30$
  - $s_3 = 0$
  - **O.F. value=150**

What affect do slack/surplus variables have on an LP?

- Increases the number of variables
- Changes the outward appearance of the constraints
- Will NOT change the feasible region, objective function value or the optimal solution!
  - The problems are equivalent

Equivalence:

- We have seen:
  - Equivalence by adding variables
  - Equivalence by altering constraints
- What about:
  - Altering the original objective function coefficients?
  - Deleting/adding constraints?

Multiplying Objective Function by a Constant

- This is ok if we multiply *everything* by a (positive) constant
  - Do not change the *relative* importance of a variable
  - Do not change the slope of the objective function line

Multiplying by a constant

Original Form

- Max:
  - x_1+x_2
- s.t.
  - $x_1 \leq 100$
  - $x_2 \leq 80$
  - $2x_1+4x_2 \leq 400$
  - $x_1, x_2 \geq 0$
- Optimal solution:
  - **x_1 =100**
  - **x_2=50**
  - **O.F. value=150**

This is ok

- Max:
  - $5x_1+5x_2$
- Optimal solution:
  - **x_1 =100**
  - **x_2=50**
  - **O.F. value=750**

Multiplying by a constant

Original Form

- Max:
  - x_1+x_2
- s.t.
  - $x_1 \leq 100$
  - $x_2 \leq 80$
  - $2x_1+4x_2 \leq 400$
  - $x_1, x_2 \geq 0$
- Optimal solution:
  - **x_1 =100**
  - **x_2=50**
  - **O.F. value=150**

This is not ok

- Max:
  - x_1+5x_2
- Optimal solution:
  - **x_1 =40**
  - **x_2=80**
  - **O.F. value=440**

Adding a constant to the Objective Function

- If a number is not associated with any variables, it does not affect the overall optimization
  - Think of these constants as sunk costs

Adding a constant

Original Form

- Max:
  - x_1+x_2
- s.t.
  - $x_1 \leq 100$
  - $x_2 \leq 80$
  - $2x_1+4x_2 \leq 400$
  - $x_1, x_2 \geq 0$
- Optimal solution:
  - **x_1 =100**
  - **x_2=50**
  - **O.F. value=150**

This is ok

- Max:
  - $x_1+x_2+10{,}000$
- Optimal solution:
  - **x_1 =100**
  - **x_2=50**
  - **O.F. value=10,750**

Redundant Constraints

- Constraints whose existence does not affect the feasible region
- Two types:
  - A constraint that is a constant multiple of another constraint
  - A constraint whose restrictions do not affect the boundaries of the feasible region
- These constraints can be removed from the problem and the linear program will remain equivalent

Two Types of Redundant Constraints

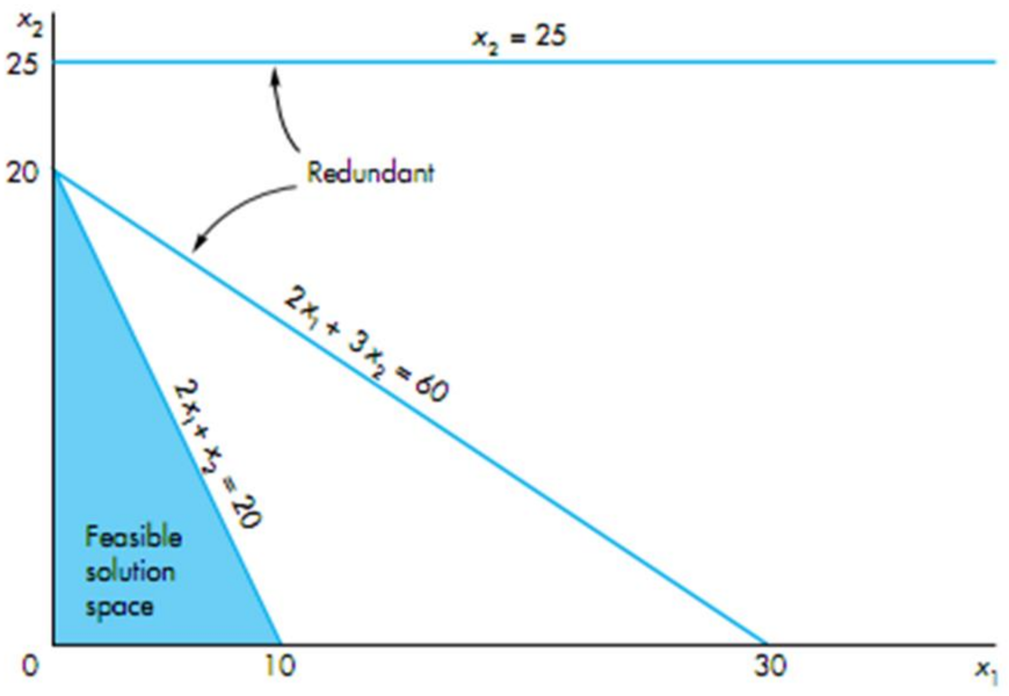
Type 1

Redundant Constraints:

- $2x_1+4x_2 \leq 400$
- $8x_1+16x_2 \leq 1600$

- These two constraints reduce to the same inequality.
- Neither constraint provides additional information to the problem
- We only need one

Redundant Constraint Type 2



- Set of constraints:
  - $x_2 \leq 25$
  - $2x_1 + 3x_2 \leq 60$
  - $2x_1 + x_2 \leq 20$
- Do each of these constraints affect the feasible region…

Infeasibility

- The situation when the linear program has no solution
- This occurs when:
  - The feasible region is empty (e.g. there is no solution that satisfies all constraints)
- A computer solver will identify that a problem is infeasible, but it is up to the programmer to find out why [we will discuss this more when we talk about sensitivity analysis]

Infeasibility (Example)

- Min
  - $2x_1+4x_2$
- s.t.
  - $2x_1+4x_2 \leq 10$
  - $x_2 \geq 5$
  - $x_1, x_2 \geq 0$
- Optimal solution?

Unboundedness (Of the feasible region)

- When the feasible region extends infinitely
  - The region is not bounded in at least one direction
- This is ok when the objective function value does not improve in the unbounded direction.

Unboundedness (Of the optimal solution)

- The situation when the feasible region allows the objective function value to be:
  - Infinitely large in a maximization problem
  - Infinitely small in a minimization problem

Solution Unboundedness (Example)

- Max
  - $3x_1+6x_2$
- s.t.
  - $x_1+x_2 \geq 10$
  - $x_1 \leq 5$
  - $x_2 \geq 4$
  - $x_1, x_2 \geq 0$

- Optimal Solution?

Alternative Optimal Solutions

- The case when more than one optimal solution exists
- This means:
  - There are multiple variable valuations that will produce the same (optimal) objective function value

Alternative Optimal Solutions (Example)

- Max
  - $x_1+1/2x_2$
- s.t.
  - $2x_1+x_2\leq 4$
  - $x_1+2x_2\leq 3$
  - $x_1, x_2\geq 0$

- Points to graph:
- Constraint 1
  - (0,4) and (2,0)
- Constraint 2
  - (0,1.5) and (3,0)
- Intersection Point
  - (5/3, 2/3)

Implications of Special Cases in the real world

- Infeasibility
  - An indication that the current restrictions present in your problem currently make it impossible to satisfy all of the company's needs
  - You may need to compromise on something or make more resources available

- Unboundedness (of the solution)
  - You may be forgetting a restriction

- Redundant Constraints
  - One or more of your concerns does not need to be considered and you may be able to focus on fewer restrictions

A Trail Mix Problem:

- You work for a snack food company that wants to branch out into the Trail Mix market. After significant market research, it has been determined that your company will be producing three different kinds of trail mix targeted at the three distinct market segments identified. Your job is to determine the optimal amount (in packages) of each type of Trail Mix to produce. You are given the following information:

Data for the Trail Mix Problem

- Three types of Trail Mix
  - Type 1 (mostly nuts)
  - Type 2 (heavy on chocolate)
  - Type 3 (an even mix)
- Each bag weighs 1lb
- You have 5 ingredients:
  - Peanuts (200lbs)
  - Almonds (175lbs)
  - M&Ms (210 lbs)
  - Chocolate covered raisins (150lbs)
  - Pretzels (190lbs)
- Profits Per Bag:
  - Type 1: $0.25
  - Type 2: $0.35
  - Type 3: $0.32
- Market Restrictions:
  - Sell no more than 180 bags of Type 1, 150 bags of Type 2, and 175 bags of Type 3

- Type 1 Trail Mix
  - 35% or 0.35lbs Peanuts
  - 35% or 0.35lbs Almonds
  - 10% or 0.10lbs M&Ms
  - 10% or 0.10lbs Raisins
  - 10% or 0.10lbs Pretzels
- Type 2 Trail Mix
  - 12.5% or 0.125lbs Peanuts
  - 12.5% or 0.125lbs Almonds
  - 33% or 0.33lbs M&Ms
  - 27% or 0.27lbs Raisins
  - 15% or 0.15lbs Pretzels
- Type 3 Trail Mix
  - 20% or 0.20lbs Peanuts
  - 20% or 0.20lbs Almonds
  - 20% or 0.20lbs M&Ms
  - 20% or 0.20lbs Raisins
  - 20% or 0.20lbs Pretzels

Suggested Reading

Linear Programming Formulation

- *Linear Programming: A Concise Introduction,* T. Ferguson, pages 1-9, https://www.math.ucla.edu/~tom/LP.pdf
- *An Introduction to Linear Programming,* S. Miller, pages 1-7, https://web.williams.edu/Mathematics/sjmiller/public_html/BrownClasses/54/handouts/LinearProgramming.pdf
- *Introduction to Linear Optimization,* D. Bertsimas and J. Tsitsiklis, Chapter 1

Linear Algebra Review [*review as needed*]

- *Linear Algebra Review and Reference,* Z. Kolter and C. Do, http://cs229.stanford.edu/section/cs229-linalg.pdf