MATH 502, HOMEWORK ASSIGNMENTS

HAITAO FAN

ABSTRACT. HWK is assigned by the dates of the class. It is assigned typically on the day on or before the Tue of each week. It is due by 11:59pm of the following Friday. I will try to grade it by the following Monday.

Each problem is worth 2 points. Get 1 point if there is a minor mistake, 0 for major mistake.

1. Hwk for week 1

Hwk 1.1 Write the following system of eqs in matrix form.

$$\begin{cases} x_1 - 3x_2 + 2x_3 = 1, \\ 3x_1 - 2x_2 + 4_3 = 2, \\ 4x_1 + 2x_2 - 2x_3 = 3. \end{cases} \tag{1.1}$$

Hwk 1.2 Write the matrix equation

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 2 \\ -2 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}. \tag{1.2}$$

back to a system of individual eqs.

Hwk 1.3: Hwk problem 1.3 on the Page 7 of Chapter 1 under the Modules link of Canvas.

Hwk 1.4. Problems of 1, 3, 5 of Homework 2.1 and 2.2 on page 5 Chapter 2 under the Modules link of Canvas.

Date: February 15, 2022.

Hwk 1.5. Based on the geometric meaning of the ODE, roughly sketch solutions of

$$y' = \frac{x^2 - 1}{y^2 + 1}$$

over the interval $0 \le x \le 2$ with the following initial point.

a) $y(0) = 1$.

b) $y(0) = -1$.

Use at 4 least steps (in x) to reach $x = 2$.

2. Hwk for week 3

1. Find the A^{-1} for

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \end{bmatrix}.$$

Then use it to solve

$$AX = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & -1 \\ 2 & -1 \end{bmatrix}.$$

Can we say that if A^{-1} exist, then the equation $AX = B$ has one and only one solution? Explain.

2. A and B are square matrices of the same size. Assume all inverse matrices in following exist. Show the following:

a) $A(I + A)^{-1} = (I + A^{-1})^{-1}$.

b) $(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B$

c) $(I + AB)^{-1}A = A(I + BA)^{-1}$.

3. A diagonal matrix is a matrix whose off-diagonal elements are $0's$, eg

$$D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33}. \end{bmatrix}$$

It has a simpler notation $D = diag(a_{11}, a_{22}, a_{33})$. Show that $A^n = diag(a_{11}^n, a_{22}^n, a_{33}^n)$ for any integer $n \ge 1$.

The above result can be stated as "to get D^n for a diagonal matrix D, I only have to n power the diagonal element". Is the same true for non-diagonal matrices?

Generalize the above result to $p(D)$ where $p(x) = c_n x^n + c_{n-1}x^{n-1} + ... + c_0$ is a polynomial and c's are constants.

4. Is it true that the product of two upper triangular matrices is also upper triangular?

5. An **elementary matrix** is the result of doing one row operation on I.

a) Make examples of elementary matrices of size 4×4 generated by 3 kinds of row operations, resulting in 3 elementary matrices: E_1, E_2, E_3

b) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \multicolumn{4}{l}{\dotfill} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Compute EA and AE for the E_1, E_2, E_3 obtained in a). What can you say about the effect of multiplying E to A?

6. A permutation matrix if the result of doing row interchanging on I many times.

a) Make an example of a 4×4 permutation matrix, P.

b) Observe

$$P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

What is the effect of multiplying P to $\mathbf{x}$?

c) Show that $P^{-1} = P^T$.

7. A circulant matrix is a matrix of the form

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & a_2 & \dots & a_{n-3} & a_{n-2} \\ \dots\dots & & & & & & \\ \dots\dots & & & & & & \\ \dots\dots & & & & & & \\ a_2 & a_3 & a_4 & a_5 & \dots & a_n & a_1 \end{bmatrix}$$

Each row is the previous row cycled forward by one step. The n-by-n permutation matrix C_n is

$$C_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots\dots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Show that

$$A = \sum_{k=0}^{n-1} a_{k+1} C_n^k.$$

8. A lower Heisenberg matrix is a bi-diagonal matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 & 0 \\ 0 & a_{32} & a_{33} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & a_{n-1,n} & a_{nn} \end{bmatrix}$$

Show that as long as $a_{jj} \neq 0$, $j = 1, 2, ..., n$, then the rank of this matrix is n.

9. Find a set of independent rows of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & -4 \\ 0 & 1 & -1 & -2 \end{bmatrix}.$$

What is A's rank?

3. Week3, Part B

B3.1 Find the general solution of the following ODEs

a) $xy' + 3y = x^4$.

b) $y' - \frac{2x}{x^2+1} y = 1$.

c) $y' + 3y = 4x$.

B3.2 Refer to Figure 5.1 below. Find curves whose subtangents are always the constant k.

B3.3 Refer to Figure 5.1. Find those curves with the feature that the segment between the curve and the y-axis is bisected by the x-axis.

B3.4. I push my bike forward on a horizontal road with the constant force of 50 newton. The road friction force is 10 newton. The air resistance force is $5v$. The total mass of me and the bike is 100kg. Find my velocity at time t (seconds). What is my maximum velocity if I start from 0 speed?

B3.5 A lake has 20,000 m^3 of water and 100kg salt in it at time $t = 0$. Water flowing into the lake carries 0.01kg salt per m^3. In each hour, 1,000 m^3 of water

(a) $x - y/y'$ is the x intercept of the tangent line.
(b) $y - xy'$ is the y intercept of the tangent line.
(c) $x + yy'$ is the x intercept of the normal line.
(d) $y + x/y'$ is the y intercept of the normal line.
(e) $|y/y'|$ is the length AC of the projection on the x axis, of the segment of the tangent AP. The length AC is called the **subtangent.**
(f) $|yy'|$ is the length CB of the projection on the x axis of the segment of the normal BP. The length CB is called the **subnormal.**
(g) The length of the tangent segment $AP = \left| y\sqrt{\frac{1}{(y')^2} + 1} \right|$.
(h) The length of the tangent segment $DP = |x\sqrt{1+(y')^2}|$.
(i) The length of the normal segment $PB = |y\sqrt{1+(y')^2}|$.
(j) The length of the normal segment $PE = \left| x\sqrt{1 + \frac{1}{(y')^2}} \right|$.

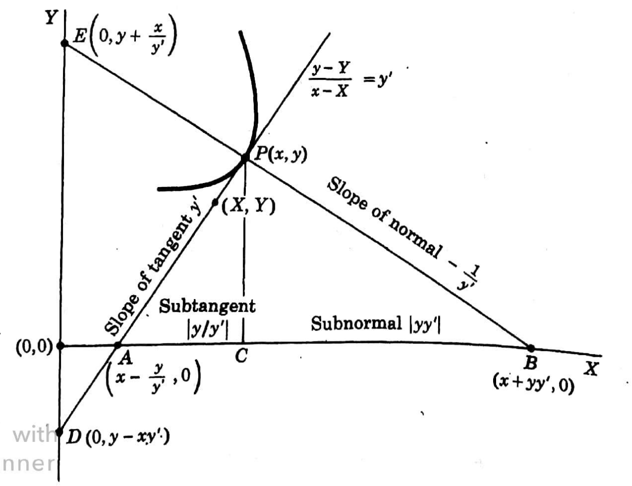

FIGURE 3.1. Subtangent etc. of a curve $y = y(x)$. Fig 13.5 of ODE by Tenenbaum&Pollard

flows into the lake, and $900m^3$ water flows out. Assume that the water in the lake are well mixed. Find the amount of salt in the lake at time $t > 0$.

4. Hwk for week 4

4.1. Part A. .

Summary: When doing +, -, * operations on partitioned matrix, just treat blocks as if they were numbers. The result is the same as what you get if these matrices are not partitioned.

A4.1 Given

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

a) Calculate AB the regular way.

b) Provide two ways to partition A, B. The first allows computation of AB while the other does not.

c) Use the first way in b) to partition to compute AB. Verify that the final result is the same as in a).

A4.2. Assume matrices in the following are with the right size to allow multiplications. Compute

a) $A\begin{bmatrix} B & C & D \end{bmatrix}$.

b) Is $\begin{bmatrix} B & C \end{bmatrix} A$ partitioned right to do the multiplication?

c) $\begin{bmatrix} B \\ C \end{bmatrix} A$

A4.3: Problems 1, 5a) 7, 10, 11, 14 , 20 of reading material PartitionedMatrix(from Leon's book) under Files/Week4:...

A4.4. A data matrix X's $j-$th column records observations the j-th variable x_j. The $i-$ row is for the i-th observation of these x_j's.

a) Make an example of such data matrix X for 3 variables and 4 observations.

b) Find a way to get the average of each x_j by multiplying X by a matrix or vector, on the right or on the left of X.

4.2. **Part B ODE.** .

Problem 3, 6 of Exercises 2.4 in reading material ODE_1_ExistenceUniqueness.pdf, under Files/Week4:...

Read the reading material ODE_2_... upto page 74, before "Bifurcation".

Do Problems 1, 4, 7 of Exercises 2.7 in reading material ODE_2_... under Files/Week4:...

5. HWWK for week 5

A5.1 Compute

$$\det \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ -2 & 2 & -3 & 3 \end{bmatrix}$$

CHAPTER TEST A **True or False**

For each statement that follows, answer *true* if the statement is always true and *false* otherwise. In the case of a true statement, explain or prove your answer. In the case of a false statement, give an example to show that the statement is not always true. Assume that all the given matrices are $n \times n$.

1. $\det(AB) = \det(BA)$
2. $\det(A - B) = \det(A) - \det(B)$
3. $\det(cA) = c\det(A)$
4. $\det(AB^T) = \det(A^T B)$
5. $\det(A - B) = 0$ implies $A = B$.
6. $\det(A^k) = \det(A)^k$
7. A triangular matrix is nonsingular if and only if its diagonal entries are all nonzero.
8. If $\mathbf{x}$ and $\mathbf{y}$ are two distincts vectors in $\mathbb{R}^n$ and $A\mathbf{x} = A\mathbf{y}$, then $\det(A) = 0$.
9. If A and B are row equivalent matrices, then their determinants are equal.
10. If $A \neq O$, but $A^k = O$ (where O denotes the zero matrix) for some positive integer k, then A must be singular.

CHAPTER TEST B

1. Let A and B be 3×3 matrices with $\det(A) = 4$ and $\det(B) = 6$, and let E be an elementary matrix of type I. Determine the value of each of the following:
   (a) $\det(\frac{1}{2}A)$ **(b)** $\det(B^{-1}A^T)$ **(c)** $\det(EA^2)$
2. Let
$$A = \begin{pmatrix} x & 2 & 2 \\ 2 & x & -3 \\ -3 & -3 & x \end{pmatrix}$$
   (a) Compute the value of $\det(A)$. (Your answer should be a function of x.)
   (b) For what values of x will the matrix be singular? Explain.
3. Let
$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 11 \\ 3 & 8 & 14 & 20 \\ 4 & 11 & 20 & 30 \end{pmatrix}$$
   (a) Compute the LU factorization of A.
   (b) Use the LU factorization to determine the value of $\det(A)$.
4. If A is a nonsingular $n \times n$ matrix, show that AA^T is nonsingular and $\det(AA^T) > 0$.
5. Let A be an $n \times n$ matrix. Show that if $B = S^{-1}AS$ for some nonsingular matrix S, then $\det(B) = \det(A)$.
6. Let A and B be $n \times n$ matrices and let $C = AB$. Use determinants to show that if either A or B is singular, then C must be singular.
7. Let A be an $n \times n$ matrix and let λ be a scalar. Show that
$$\det(A - \lambda I) = 0$$
   if and only if
$$A\mathbf{x} = \lambda\mathbf{x} \quad \text{for some } \mathbf{x} \neq \mathbf{0}$$
8. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$, $n > 1$. Show that if $A = \mathbf{x}\mathbf{y}^T$, then $\det(A) = 0$.
9. Let $\mathbf{x}$ be a nonzero vector in $\mathbb{R}^n$ and let A be an $n \times n$ matrix with the property that $A\mathbf{x} = \mathbf{0}$. Show that $\det(A) = 0$.
10. Let A be a matrix with integer entries. If $|\det(A)| = 1$, then what can you conclude about the nature of the entries of A^{-1}? Explain.

FIGURE 5.1. Ch 2 practice of Leon's linear algebra book

A5.2. Prove that
$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{bmatrix} = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

A5.3: Refer to Fig 5.1. Chapter Test A: 1 - 8. Chapter test B: 2,4,5,6,7,9.

5.1. hwk 5: Part B ODE. .

B5.1. Do a bifurcation analysis for $y'(x) = x - x^3 - h$ where h is a parameter.

3. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

(a) The set of all 2×2 diagonal matrices

(b) The set of all 2×2 triangular matrices

(c) The set of all 2×2 lower triangular matrices

(d) The set of all 2×2 matrices A such that $a_{12} = 1$

(e) The set of all 2×2 matrices B such that $b_{11} = 0$

(f) The set of all symmetric 2×2 matrices

(g) The set of all singular 2×2 matrices

6. Determine whether the following are subspaces of $C[-1, 1]$:

(a) The set of functions f in $C[-1, 1]$ such that $f(-1) = f(1)$

(b) The set of odd functions in $C[-1, 1]$

(c) The set of continuous nondecreasing functions on $[-1, 1]$

(d) The set of functions f in $C[-1, 1]$ such that $f(-1) = 0$ and $f(1) = 0$

(e) The set of functions f in $C[-1, 1]$ such that $f(-1) = 0$ or $f(1) = 0$

FIGURE 6.1. From Leon's linear algebra book

CHAPTER TEST A **True or False**

Answer each of the statements that follows as true *or* false. *In each case, explain or prove your answer.*

1. If S is a subspace of a vector space V, then S is a vector space.

2. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^4$.

3. It is possible to find a pair of two-dimensional subspaces S and T of $\mathbb{R}^3$ such that $S \cap T = \{\mathbf{0}\}$.

4. If S and T are subspaces of a vector space V, then $S \cup T$ is a subspace of V.

5. If S and T are subspaces of a vector space V, then $S \cap T$ is a subspace of V.

6. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent, then they span $\mathbb{R}^n$.

7. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ span a vector space V, then they are linearly independent.

8. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are vectors in a vector space V and

$$\operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) = \operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1})$$

then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly dependent.

FIGURE 6.2. From Leon's linear algebra book

B5.2. I plan to live on the Mars by raising and harvesting fish in a fish tank. The fish population at time t (year) is denoted as $P(t)$ (kg) which grows by logistic growth law. I have to harvest 100 kg per year to survive. The ODE for $P(t)$ is

$$P' = P(1 - P/N) - 100,$$

where the constant N is the tank's carrying capacity. The larger the tank, the larger the N. To determine the tank size, I have to find N. Choose N so that I can survive on the Mars.

6. HWK FOR WEEK 6

A6.1. Problem 3 and 6 shown in Figure 6.1.

A6.2. Are vectors

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

independent?

A6.3 Problems 1-8 in Figure 6.2

A6.4. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be independent vectors in $\mathbb{R}^n$,

a) Let A be a nonsingular $n \times n$ matrix. Show that $A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_k$ are linearly independent

b) If A is not an invertible $n \times n$ matrix, then must $A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_k$ be linearly dependent?

Read ODE_3_HomogeneousLinearODE.pdf in the canvas, Files, Week6Materials folder and do Problems 1, 5, 9 on page 94.

Department of Mathematics, Georgetown University, Washington, DC 20057, USA

Email address: `fanh@georgetown.edu`