MATH 503: Mathematical Statistics
Dr. Kimberly F. Sellers, Instructor
Homework 3

1. Given $f(x;\theta) = \frac{1}{\theta}$, $0 < x < \theta$, zero elsewhere, with $\theta > 0$, formally compute the reciprocal of $nE\left\{\left[\frac{\partial \log f(X;\theta)}{\partial\theta}\right]^2\right\}$. Compare this with the variance of $(n+1)Y_n/n$, where Y_n is the largest observation of a random sample of size n from this distribution. Comment.

2. Let X be $N(0,\theta)$, $0 < \theta < \infty$, where $\text{Var}(X) = \theta$.
   (a) Find the Fisher information $I(\theta)$.
   (b) If $X_1, X_2, \ldots, X_n$ is a random sample from this distribution, show that the MLE of θ is an efficient estimator of θ.

3. Let $\bar{X}$ be the mean of a random sample of size n from a $N(\theta, \sigma^2)$ distribution, $-\infty < \theta < \infty$, $\sigma^2 > 0$. Assume that σ^2 is known. Show that $\bar{X}^2 - \frac{\sigma^2}{n}$ is an unbiased estimator of θ^2 and find its efficiency.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a geometric distribution that has pmf $f(x;\theta) = (1-\theta)^x\theta$, $x = 0, 1, 2, \ldots$, $0 < \theta < 1$, zero elsewhere. Show that $\sum_{i=1}^n X_i$ is a sufficient statistic for θ.

5. Show that the sum of the observations of a random sample of size n from a gamma distribution has pdf $f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, is a sufficient statistic for θ.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a beta distribution with parameters $\alpha = \theta$ and $\beta = 2$. Show that the product $X_1X_2\ldots X_n$ is a sufficient statistic for θ.

7. Show that the product of the sample observations is a sufficient statistic for $\theta > 0$ if the random sample is taken from a gamma distribution with parameters $\alpha = \theta$ and $\beta = 6$.

8. What is the sufficient statistic for θ if the sample arises from a beta distribution in which $\alpha = \beta = \theta > 0$?