Homework 1

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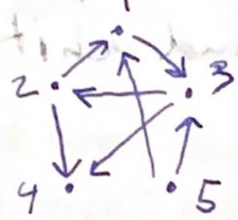
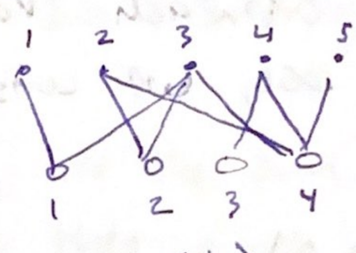
1. Consider the following two networks:



(a)



(b)

Network (a) is directed. Network (b) is bipartite undirected. Write down:

(i) the adjacency matrix of (a).

(ii) the incidence matrix of (b).

(iii) the projection matrix for the projection of (b) onto its black vertices.

(i) An adjacency matrix indicates whether there is an edge between nodes i and j as ~~dash~~ indicated using

$$A_{ij} = \begin{cases} 1 & \text{if there exists an edge between node } i \text{ and } j \\ 0 & \text{o/w.} \end{cases}$$

In the case of an undirected graph.

Moreover, if the graph is directed,

$$A_{ij} = \begin{cases} 1 & \text{if there exists an edge from } j \text{ to } i. \\ 0 & \text{o/w.} \end{cases}$$

In the case of (a) we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

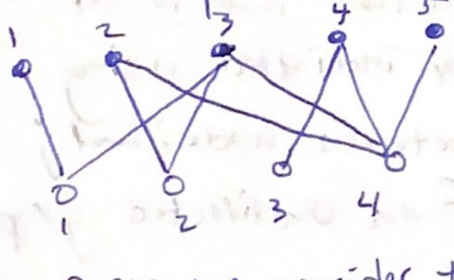
Note that we indicate the in-ward edges. The diagonal entries are defined to be 0.

(ii) An incidence matrix indicates whether or not there is a pair ~~of a given edge and node~~. elements, one each from two classes of objects. In this case, we consider the black and white nodes.

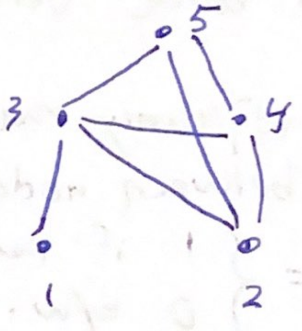
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

(iii) A projection matrix indicates the mutual group membership of the items in one type from a bipartite network (defined by two items ~~each~~ both connected to the same item of the other type).

our bipartite network is



$\Rightarrow$



once we consider the projection network, we can use the formula $P = B^T B$

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

which yields us $\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$

2. Consider a bipartite network, with its two types of vertices, and suppose there are n_1 vertices of type 1 and n_2 vertices of type 2. Show that the mean degrees c_1 and c_2 of the two types are related by $c_2 = \frac{n_1}{n_2} c_1$.

we know that if s_1 and s_2 are the sum of degrees of type 1 and type 2, respectively, then $s_1 = s_2 = L$.

The definitions of c_1, c_2 are given by

$$c_1 = \frac{s_1}{n_1}, \quad c_2 = \frac{s_2}{n_2}.$$

Therefore $n_1 c_1 = n_2 c_2 \iff c_2 = \frac{n_1}{n_2} c_1$.

3. Consider a connected planar network with n vertices and m edges. Let f be the number of faces of the network, i.e., areas bound by edges. The "outside" of the network is also considered a face.

(i) Write down the values of n, m, and f for a network with a single vertex and no edges.

(ii) How do n, m, and f change when we add a single vertex to the network along with a single edge attaching it to another vertex.

(iii) How do n, m, and f change when we add a single edge between two extant vertices (or a self-edge attached to just on vertex), in such a way as to maintain the planarity of the network?

(iv) Hence by induction prove a general relation between n, m, and f for all connected planar networks.

(v) Now suppose that our network is simple (i.e. it contains no multi-edges or self-edges). Show that the mean degree c of a simple, connected, planar network is strictly less than 6.

(i) A single vertex with no edges: • Face

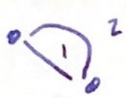
$n=1,\ m=0,\ f=1.$

(ii)  Face This network: $n=2,\ m=1,\ f=1.$

(iii) 

Face This network: $n=2,\ m=2,\ f=2.$

(iv) First, by exploration of two case, we see the pattern of the relationship continue.



$\rightarrow$ 

$\Rightarrow n=2,\ m=3,\ f=3$

$\searrow$ 

$\Rightarrow n=3,\ m=3,\ f=2$

we get the following observation: $n-m+f=2$ ~~edges~~ on the number of ~~faces~~.

So we can prove by induction. We know that $n-m+f=2$ for the base case of a single vertex with no edges in (i).

Now if we assume this is true for the case of ~~...~~, ~~...~~. So graph with k edges. $n=k$,

Consider a graph with $k+1$ edges. The graph either does or does not have a cycle. If it doesn't have a cycle, then it is a tree. ~~...~~ with ~~...~~ $k+2$ nodes. so we see $(k+2)-(k+1)+f=2 \Rightarrow f=1$, which is correct.

If there is a cycle, then the cycle creates 2 faces. If we remove an edge in the cycle this reduces the count of faces by 1. So the smaller graph has one fewer face. We know the formula holds for this smaller graph.

Then $n-(m+1)+(f+1)=2$ is true. So proven.

(v) Now we consider a simple, planar network and its mean degree c.

Since the planar network is simple, there are no multiedges, so every face is bounded by 3 edges. so $\sum \text{degree}(f) \geq 3f$ for the graph.

Each edge bounds 2 faces, so $\sum \text{degree}(f) \leq 2m$.

Together we find $3f \leq 2m$. Recalling the formula from previous part, we combine with above:

$$\left.\begin{array}{l} 3f \leq 2m \\ 3f = 3m - 3n - 6 \end{array}\right\} \Rightarrow 2m \geq 3m - 3n - 6 \Rightarrow m \leq 3n - 6.$$

If $\underset{\text{mean degree}}{c} \geq 6$ then $\dfrac{\sum \text{degree}(n)}{n} \geq 6 \Rightarrow 2m \geq 6n \Rightarrow m \geq 3n$.

This contradicts $m \leq 3n - 6$. So it must be the case that $c < 6$.

4. One possible definition of the trophic level x_i of a species in a directed food web is given in section 5.3.1 as the mean of the trophic levels of the species' prey plus one.

(i) Show that x_i is the ith element of the vector

$$x_i = 1 + \frac{1}{k_i^{in}} \sum_j A_{ij} x_j$$

(ii) The expression doesn't work for species with no prey because the vector element diverges, and are given trophic level of 1. Suggest a modification of the calculation that will correctly assign trophic levels to these species, and hence to all species. This show that x, and D is defined

$$x = (D - A)^{-1} D \cdot \mathbb{1},$$

where D is diagonal matrix, A is symmetric adjacency matrix, and $\mathbb{1} = (1, 1, 1, \ldots)$.

(i) First we interpret the meaning of the expression.

A_{ij} is the incidence matrix. In this case of the food web, which is directed, its entry is nonzero if j eats i.

x_j is the trophic level of j.

k_i^{in} is the sum of edges in to i.

This means the sum of trophic levels of species j that eat i, divided by count of links to i.

So this is the average trophic level of ~~predators~~ prey plus 1.

(ii) In following the definition in part (i), we may use $D = \begin{pmatrix} k_1^{in} & & 0 \\ & \ddots & \\ 0 & & k_n^{in} \end{pmatrix}$