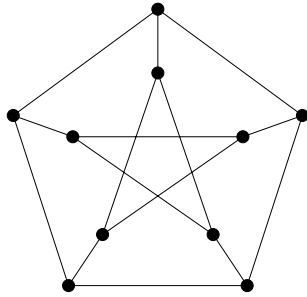# MATH517 Midterm Exam (Total 40 points)

1. **[8 Points]** Using Kuratowski's theorem, prove that this network is not planar:

2 In Section 5.3.1, we gave one possible definition of the trophic level x_i of a species in a (directed) food web as the mean of the trophic levels of the species' prey, plus one.

a) **[4 Points]** Show that x_i, when defined in this way, satisfies

$$x_i = 1 + \frac{1}{k_i^{\text{in}}} \sum_j A_{ij} x_j.$$

b) **[4 Points]** This expression does not work for autotrophs—species with no prey—because the corresponding vector element is undefined. Such species are usually given a trophic level of one. Suggest a modification of the calculation that will correctly assign trophic levels to these species, and hence to all species. Thus, show that x_i can be calculated as the ith element of a vector

$$\mathbf{x} = (\mathbf{D} - \mathbf{A})^{-1} \mathbf{D} \cdot \mathbf{1},$$

and specify how the matrix $\mathbf{D}$ is defined.

3 As we saw in Section 7.1.3, the Katz centrality in vector form satisfies the equation $\mathbf{x} = \alpha \mathbf{A} \mathbf{x} + \mathbf{1}$ (which is Eq. (7.6) with the conventional choice $\beta = 1$).

a) **[3 Points]** Show that the Katz centrality can also be written in series form as $\mathbf{x} = \mathbf{1} + \alpha \mathbf{A} \mathbf{1} + \alpha^2 \mathbf{A}^2 \mathbf{1} + \dots$

b) **[3 Points]** Hence, argue that in the limit where α is small but non-zero, the Katz centrality is essentially equivalent to degree centrality.

c) **[3 Points]** Conversely, in the limit $\alpha \to 1/\kappa_1$, where κ_1 is the largest (most positive) eigen-value of the adjacency matrix, argue that $\mathbf{x}$ becomes proportional to the leading eigenvector, which is simply the eigenvector centrality.

Thus, the Katz centrality can be thought of as a one-parameter family of centralities, parametrized by $\alpha \in [0, 1/\kappa_1]$, which includes the degree centrality and the eigenvector centrality at the two limits of the range and interpolates between them everywhere in between.

4. [8 Points] Suppose a directed network takes the form of a tree with all edges pointing inward towards a central node:
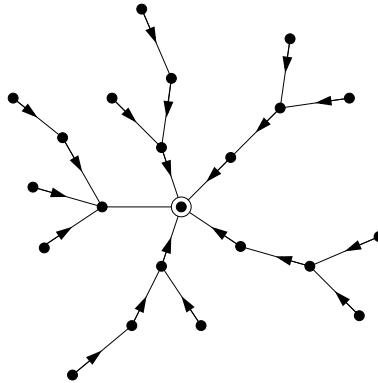
What is the PageRank centrality of the central node in terms of the single parameter α appearing in the definition of PageRank and the distances d_i from each node i to the central node?

5 [7 Points] Consider the following studies:

A 2016 article on the website FiveThirtyEight.com described the results of a study by Eitan Hersh and Yair Ghitza of political alignment among couples in the United States. Hersh and Ghitza estimated the fractions of opposite-sex partners with each combination of major-party alignment (which in the US means Democratic, Independent, or Republican) to be as follows:

		Women			
		Democrat	Independent	Republican	Total
Men	Democrat	0.25	0.04	0.03	0.32
	Independent	0.06	0.15	0.05	0.26
	Republican	0.06	0.05	0.30	0.41
	Total	0.37	0.24	0.38	

Assuming these results to be representative of the network of relationships, calculate the modularity of the network with respect to political persuasion.