Uniqueness Spectra of Graphs

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Abstract

In this study, we examine the **Uniqueness Spectrum** of graphs, which refers to the set of integers that indicate the number of edges that can be assigned distinct colors in a proper edge coloring. We focus on the conditions required for a graph to achieve **full spectrum**, defined as the ability to realize all possible uniqueness values. Our results establish specific criteria for attaining full spectrum and highlight key structural properties of graphs that impact the uniqueness of edge colorings.

Key Definitions

To study the uniqueness spectrum of a graph, we begin with the following key definitions.

A **graph** $G = (V, E)$ consists of a set V of vertices and a set $E \subseteq \binom{V}{2}$ of edges. The **degree** of a vertex $v \in V$, denoted $d(v)$, is the number of edges incident to v. The **maximum degree** of G, denoted $\Delta(G)$, is defined by $\Delta(G) = \max_{v \in V} d(v)$.

The **maximum matching** of G, denoted $\alpha'(G)$, is the largest set of edges in G such that no two edges in the set share a common vertex.

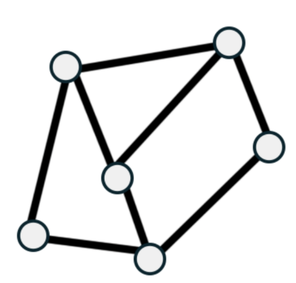

Figure 1. Graph with $\alpha'(G) = 3$ and $\Delta(G) = 3$

The **chromatic index** of G, denoted $\chi'(G)$, is the minimum number of colors required to assign to E so that adjacent edges have distinct colors (a **proper edge coloring**). According to **Vizing's Theorem** [2], for any graph G,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

A proper edge coloring of G is k**-unique** if exactly k edges have unique colors (i.e., each of these colors appears on only one edge in G).

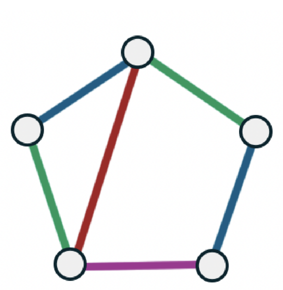

Figure 2. 2-unique Graph

The **uniqueness spectrum** of G, denoted $Spec(G)$, is the set of all integers k for which G has a k-unique coloring.

A graph G is **full spectrum** if $Spec(G) = \{0, 1, \ldots, \|G\| - 2, \|G\|\}$, where $\|G\| = |E|$, meaning G can achieve all possible uniqueness values except $\|G\| - 1$.

Characterizing Full Spectrum Graphs

To determine when a graph achieves full spectrum, we establish the following conditions.

The **net graph** N_6 serves as an exception in the main theorem, as it does not achieve full spectrum (see Figure 8).

The **main theorem** states that a graph G with at least three edges is **full spectrum** if and only if $2\chi'(G) \leq \|G\|$, $\alpha'(G) \geq 3$, and G is not N_6.

Supporting Lemmas

The following lemmas explore conditions for a graph to achieve full spectrum, focusing on $2\chi'(G) < \|G\|$ and $2\chi'(G) > \|G\|$. The case $2\chi'(G) = \|G\|$ is challenging, limiting the possibility of 1-unique colorings.

Lemma 1. For any graph G, no $\|G\| - 1$-unique coloring exists.

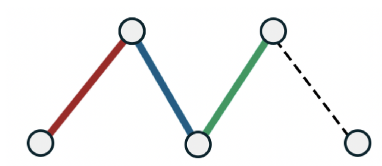

Figure 3. $\|G\| - 1$-unique coloring attempt

Lemma 2. For color classes C_i and C_j in a coloring of G, if $|C_i| = 1$ and $|C_j| \geq 3$, some edge in C_j can be recolored to C_i.

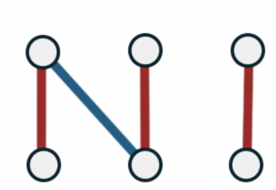

Figure 4. Example of a recolorable edge from C_j

Lemma 3. If $2\chi'(G) > \|G\|$, then G is not full spectrum.

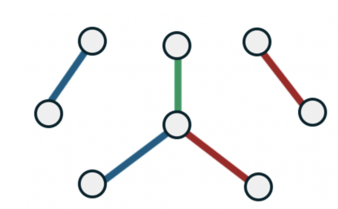

Figure 5. Graph with $2\chi'(G) > \|G\|$: No 0-uniqueness

Chain Lemma. For any graph G with a k-unique coloring, if $k \leq \|G\| - 4$, then G also has a $(k + 2)$-unique coloring.

Corollary 1. If G is both 0-unique and 1-unique, then G is full spectrum.

Corollary 2. Every graph G with at least three edges and $2\chi'(G) < \|G\|$ has is full spectrum.

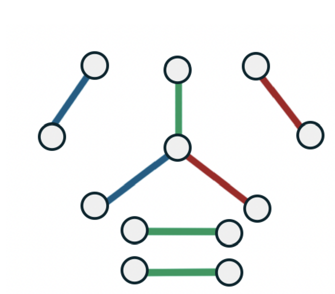

Figure 6. Graph with $2\chi'(G) < \|G\|$

Examples

The following are examples of graphs that are full spectrum, meaning they satisfy all conditions from the main theorem.

Full Spectrum Example. The path graph P_5 can achieve both a 0-unique and a 1-unique coloring, demonstrating that it meets the criteria for full spectrum.

The following images illustrate these colorings side by side:

Figure 7. Left. 0-unique coloring, Right. 1-unique coloring

Other Examples. Cycle graphs or additional paths meeting the theorem's criteria also exhibit full spectrum when colored similarly.

Non-Full Spectrum Example. The **net graph** N_6 below does not meet full spectrum conditions. Its unique structure makes a 1-unique coloring impossible.

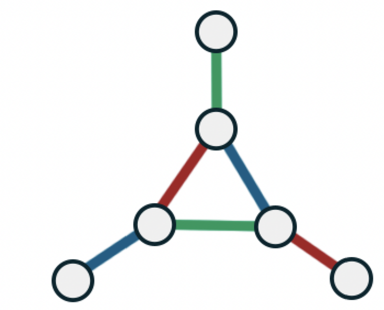

Figure 8. 0-Unique Coloring of N_6

Future Work

Future research will explore the uniqueness spectrum of r-uniform hypergraphs. An r-uniform hypergraph $H = (V, E)$ consists of a vertex set V and edge set $E \subseteq \binom{V}{r}$, where each edge contains exactly r vertices. This investigation will focus on how the conditions for achieving full spectrum may generalize within this specific context.

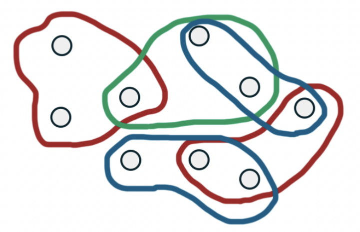

Figure 9. 1-unique 3-uniform hypergraph

References

[1] Yunus Bidav and Neal Bushaw. Uniqueness spectra of graphs. *Unpublished Manuscript*, 2024.

[2] V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz*, (3):25–30, 1964.

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