Uniqueness Spectra of Graphs

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Joint Work With

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Brief Background

- Graph coloring is something that is widely studied.
- There's many famous results such as the Four Color Theorem, Brooks' Theorem, Vizing's Theorem, Ramsey Theory, and many others.
- This is a very active field of research both in applied and pure mathematics.

Basic definitions

Definition

A graph $G = (V, E)$ is an ordered pair where V is a set containing elements called vertices and E is a set of paired vertices which are called edges. For edge $e = \{v_1, v_2\}$ we call v_1 and v_2 the endpoints.

Definition

Two edges are incident if they share an endpoint.

Definition

For a given graph G we denote the number of vertices as $|G|$ and the number of edges as $\|G\|$.

Graph

- We can draw a graph by representing the vertices as circles and the edges as lines which connect them.

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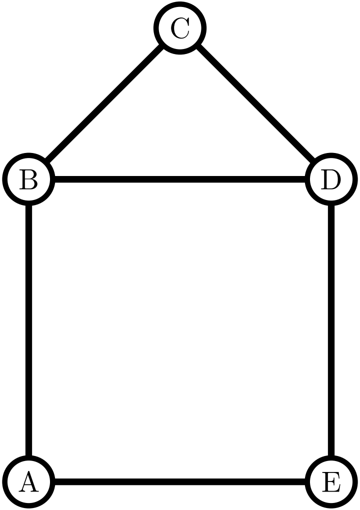


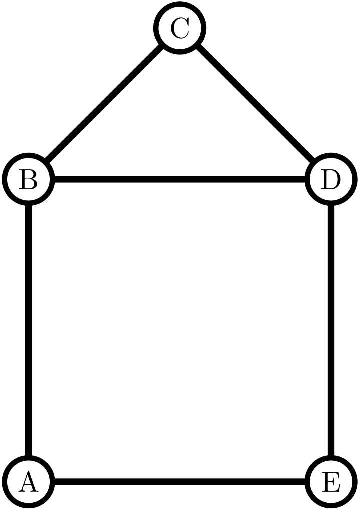
Coloring Definitions

Definition

A k-**edge-coloring** of a graph $G = (V, E)$ is a function $c : E \to [k]$. We call c **proper** if no pair of incident edges is colored the same. The **chromatic index of** G, written $\chi'(G)$, is the minimum $k \in \mathbb{N}$ for which a proper k-edge-coloring of G exists.

- We can think of edge coloring as assigning a color to an edge.
- We call this coloring proper if whenever two edges share an endpoint they are colored differently.
- We can then think of the chromatic index as the minimum number of colors you need to properly edge color a graph.

Let's Edge Color Something!



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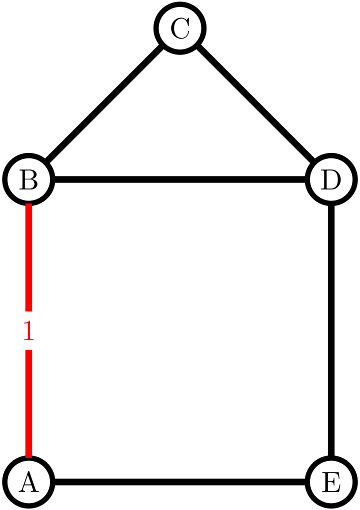

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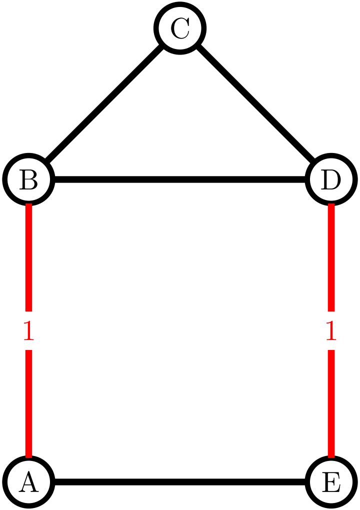

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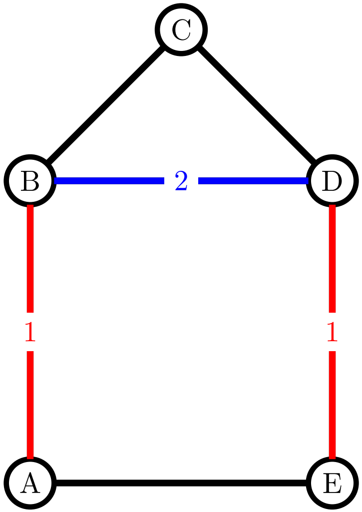

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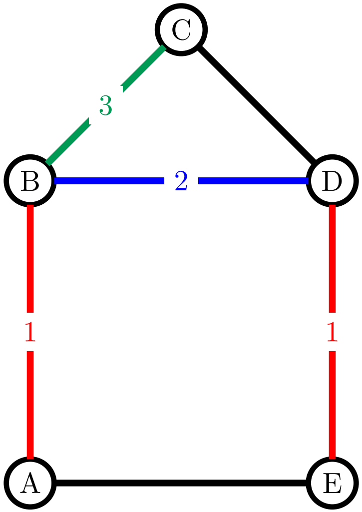

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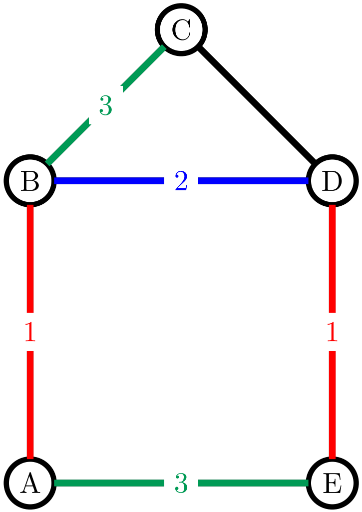

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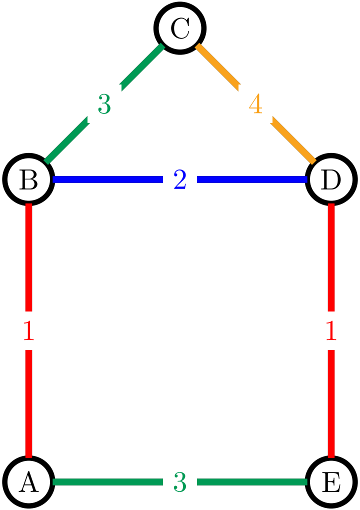

K-Uniqueness

Definition

Given a proper k-edge-coloring c of a graph G, we say that c is k**-unique** if there are exactly k edges of G whose colors are not repeated elsewhere in the coloring c.

- A coloring is k-unique when exactly k edges have colors which appear nowhere else.

More Coloring!

More Coloring!

This is 0-unique

More Coloring!

More Coloring!

This is 1-unique

More Coloring!

$\text{Spec} = \{0, 1\}$

More Coloring!

$\text{Spec} = \{0, 1, 3\}$

More Coloring!

Spec $= \{0, 1, 3, 5\}$

More Coloring!

$\text{Spec} = \{0, 1, 2, 3, 5\}$

Important observations

Observation

If we recolor an edge from a color used exactly twice into a new color, our uniqueness goes up by two.

Observation

If we recolor an edge from a color used three (or more) times into a new color, our uniqueness goes up by one.

Uniqueness Spectrum

Definition (Bushaw-Bednar 2022)

The **uniqueness spectrum** of a graph G is the set of natural numbers k for which a k-unique coloring of G exists. We denote this set by $\mathrm{Spec(G)}$.

Definition

We say that a graph G is **full spectrum** whenever $\mathrm{Spec(G)} = \{0, 1, \ldots, \|\mathrm{G}\| - 2, \|\mathrm{G}\|\}$.

- Why is $\|G\| - 1$ missing?
- What kinds of graphs are full spectrum? What kinds are not?

Guaranteed Values

Lemma

For any graph G, no $\|G\| - 1$-unique coloring can exist.

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Guaranteed Values

Lemma

For any graph G, there always exists a $\|G\|$-unique coloring

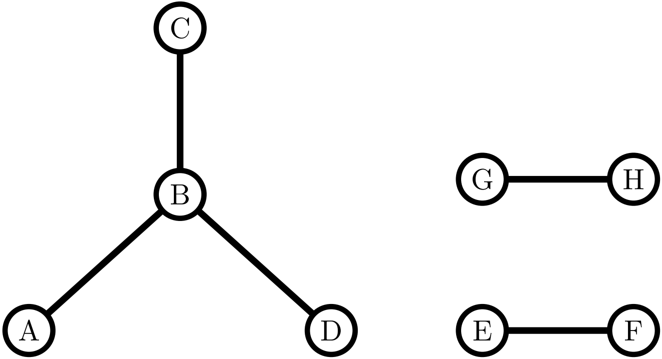
- Just color every edge something different!

Graphs Which Aren't Full Spectrum

Lemma

If G is a graph with $2\chi'(G) > \|G\|$, then G is not full spectrum. Specifically, G no 0-unique coloring.

Graphs Which Aren't Full Spectrum



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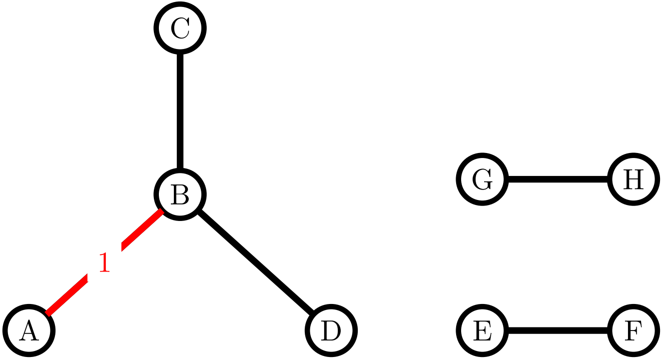

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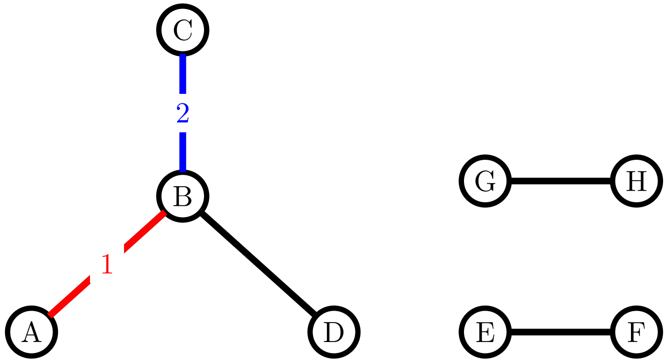

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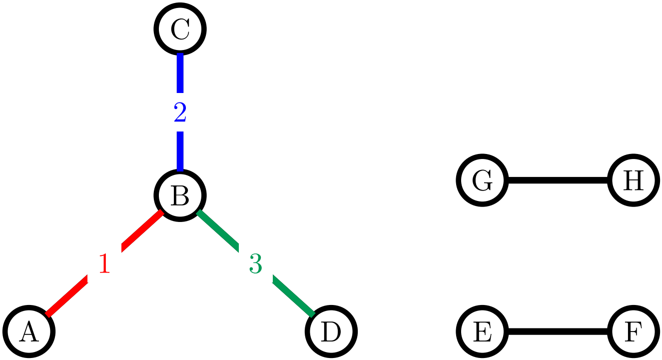

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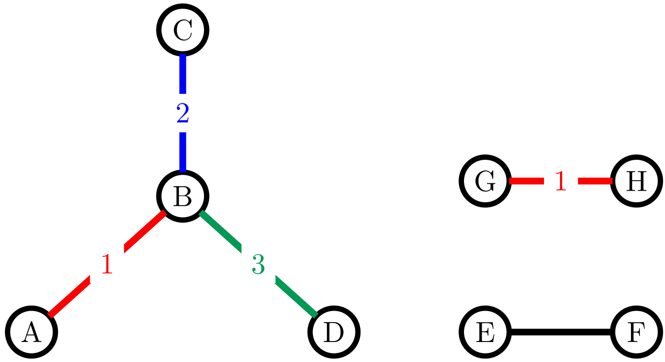

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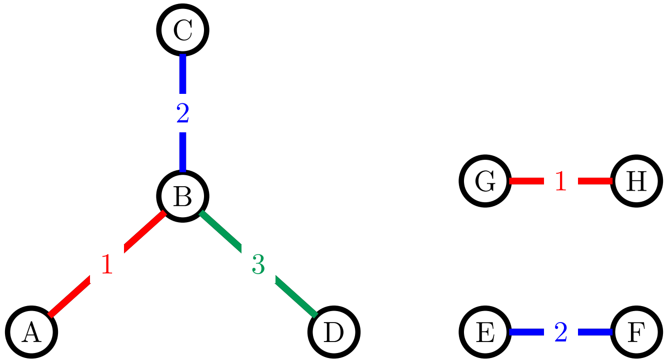

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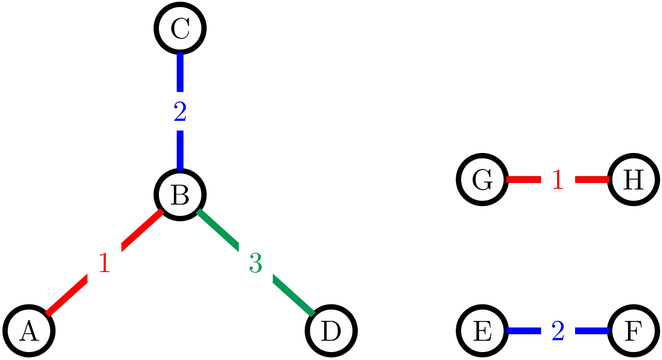

Not enough edges!

Chain Theorem

Theorem (Chain Theorem)

Given graph G with a k-unique coloring, if $k \leq \|G\| - 4$, then G also has a $(k+2)$-unique coloring

- There's an important observation.
- If a graph G has a 0-unique and 1-unique coloring it must have $2, 3, 4, \ldots, \|G\| - 3, \|G\| - 2$ in the spectrum.
- But we know that $\|G\| - 1$ is never in the spectrum and that $\|G\|$ is always in the spectrum!

Corollary

If graph G is 0-unique and 1-unique, then it is full spectrum.

Graphs Which are Full Spectrum

Lemma

If G is a graph such that $2\chi'(G) < \|G\|$, then it is 0-unique.

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If G is a graph such that $2\chi'(G) < \|G\|$, then it is full spectrum.

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- We have that a graph is full spectrum if $2\chi'(G) < \|G\|$.

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- We have that a graph is not full spectrum if $2\chi'(G) > \|G\|$.
- What about if $2\chi'(G) = \|G\|$?

The Middle Case

Definition

A matching is a set of disjoint edges.

- Just like a color!

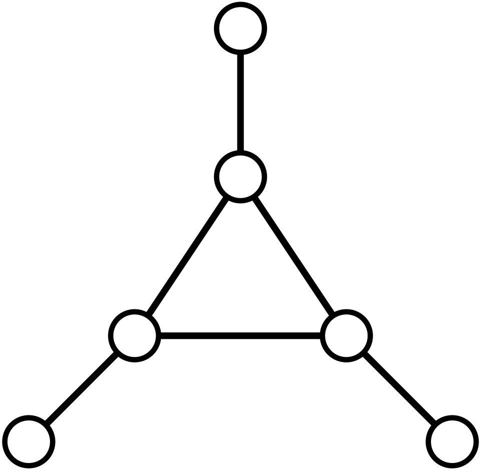
Definition

The matching number of a graph G, denoted as $\alpha'(G)$, is the size of the largest matching in G.

Lemma

If G has $2\chi'(G) = \|G\|$ and $\alpha'(G) \geq 3$ and G is not N_6, then G is full spectrum.

The Net Graph — N_6



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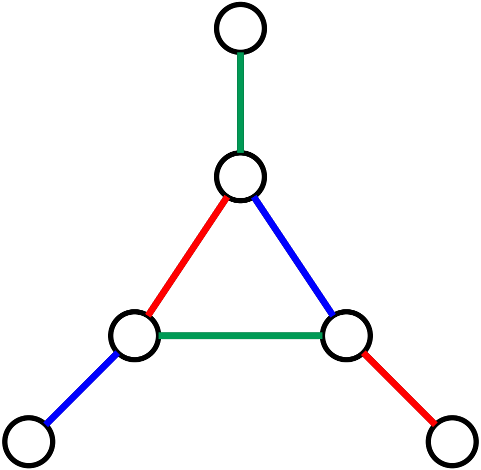

Final Result

Theorem

A graph G is full spectrum if and only if we have $2\chi'(G) \leq \|G\|$, $\alpha'(G) \geq 3$, and G is not N_6.

Common Graphs which are full spectrum

Question

What kinds of graphs have full spectrum?

- Paths with 5 or more edges.
- Cycles with 6 or more edges.
- Any that satisfies the final result!

Looking Forward

- Hypergraphs!

Conclusion

Thank you everyone for listening!
Any questions?