PGE 337 Lecture 2: Probability

Lecture outline . . .

- Probability Definitions
- Venn Diagrams
- Frequentist Concepts
- Bayesian Concepts

Introduction

General Concepts

Statistics

Probability

Univariate

Bivariate

Time Series Analysis

Spatial Analysis

Machine Learning

Uncertainty Analysis

Note: some slides were modified from Dr. Zoya Heidari's and Dr. Larry Lake's PGE 337 Course

Comments

- We have group office hours 3:30 – 5:00 pm we can work through Anaconda, R and GSLIB software installation and example problems.
- I'm open to suggestions. My goal is to help you all learn. Let me know if you have ideas on improving the class anytime.

Statistics Moment By…

- 2 mins.

Probability and Statistics
What should you learn from this lecture?

- **Fundamentals of Statistics and Probability**
  - **Fundamentals of Probability**
    - » **Basic Definitions and Rules**
    - » **Venn Diagram**
    - » **Conditional Probability**
    - » **Probability tree**
    - » **Bayes' Theorem**
    - » **Applications of Probability in Decision Making**

Probability Helps in Making Decisions

For example:

- What is the probability that a well is a success? – *drill the well*
- What is the probability that a valve has a crack? – *replace the valve*
- What is the probability that a seismic survey finds a reservoir? – *acquire the seismic*
- What is the probability that a reservoir seal will fail? – *inject the CO_2*

Most of our decisions involve uncertainty:

- By quantifying probability we can make better decisions.

Probability Definitions
What is Probability?

Measure of the likelihood that an event will occur. For random experiments and well-defined settings (such as coin tosses):

$$\text{Prob}(A) = \text{P}(A) = \lim_{n \to \infty} \left(\frac{n(A)}{n(\Omega)} \right)$$

frequentist approach to probability is the limit of relative frequency over a large number of trails.

where:

n(A) = number of times event A occurred
n(Ω) = number of trails

Example: Possibility of drilling a dry hole for the next well, encountering sandstone at a location ($\mathbf{u}_\alpha$), exceeding a rock porosity of 15% at a location ($\mathbf{u}_\alpha$).

Probability Concepts
Venn Diagrams

Venn Diagrams are a tool to communicate probability

Experiments (Sampling) (J): Establishment of conditions that produce an outcome.

Simple Event (x): A single outcome of an experiment.

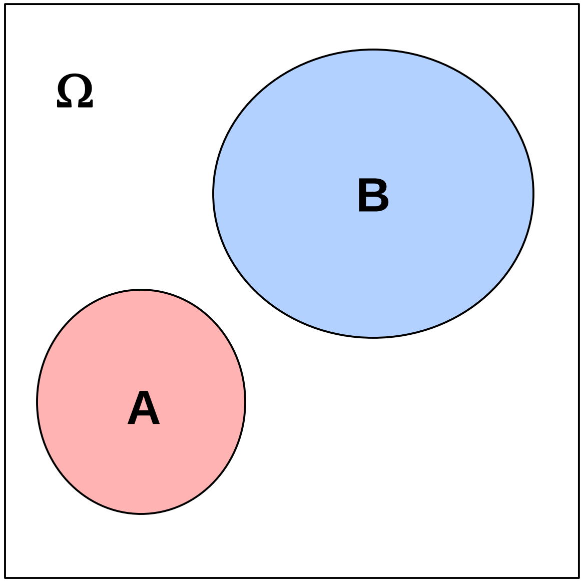
Event (A, B, …): Collection of simple events.

Occurrence of A: A has occurred if the outcome of experiment (sampling) belongs to it.

Sample Space (Ω): Collection of all possible events.

What do we learn from a Venn diagram?

- size of regions = probability of occurrence
- overlap = probability of joint occurrence
- excellent tool to visualize marginal, joint and conditional probability.



Venn Diagram – illustration of events and relations to each other.

Probability Definitions
Venn Diagram Example

Experiments (Sampling) (J):

- Facies determined from a set of well cores (N=3,000 measures at 1 foot increments)

Sample Space (Ω):

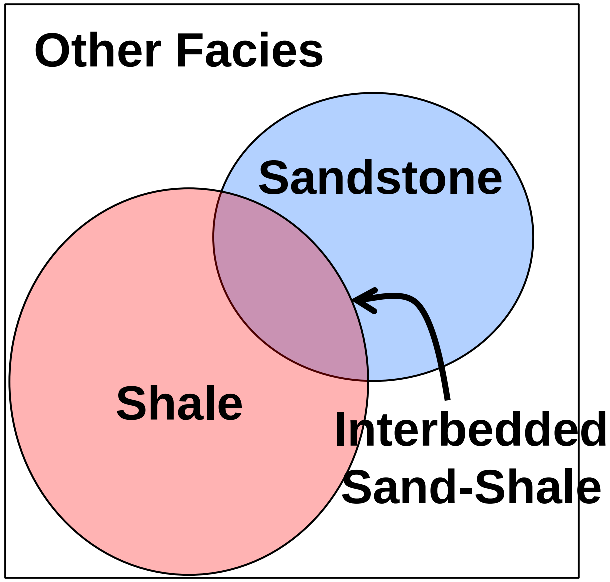
- Facies for the N=3,000 core measures

Event (A, B, …):

- Facies = {Sandstone, Interbedded Sandstone and Shale, Shale and other facies}

Venn Diagram Tells Us About Probability:

- Prob{Other Facies} > Prob{Shale} > Prob{Sandstone} > Prob{Interbedded} = Prob{Shale and Sandstone}
- Prob{Sandstone and Shale given Sandstone } < Prob{Sandstone}



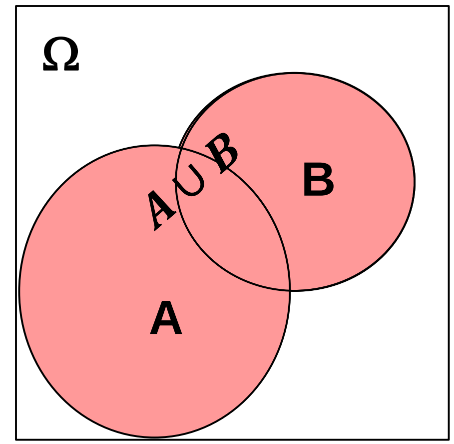
Venn Diagram – illustration of events and relations to each other.

Probability Definitions
Probability Operators

Union of Events:

- All outcomes in the sample space that belong to either event A or B

$$A \cup B = \{x : x \in A \; or \; x \in B\}$$

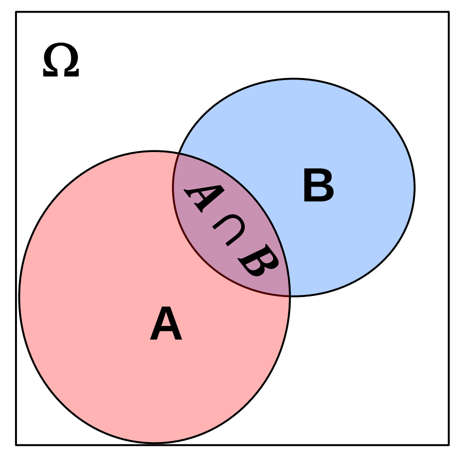


Venn Diagram – illustrating union.

Intersection of Events:

- All outcomes in the sample space that belong to both events A and B

$$A \cap B = \{x : x \in A \; and \; x \in B\}$$

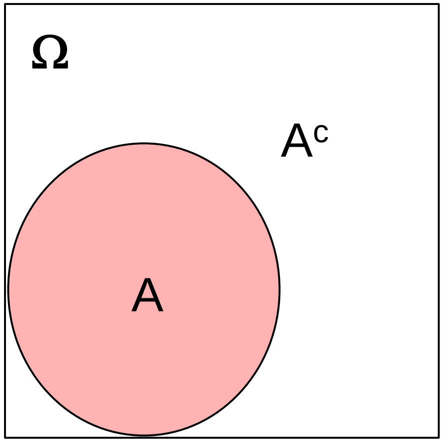


Venn Diagram – illustrating intersection.

Probability Definitions
Probability Operators

Complementary Events: A^c

- All outcomes in the sample space that do not belong to A

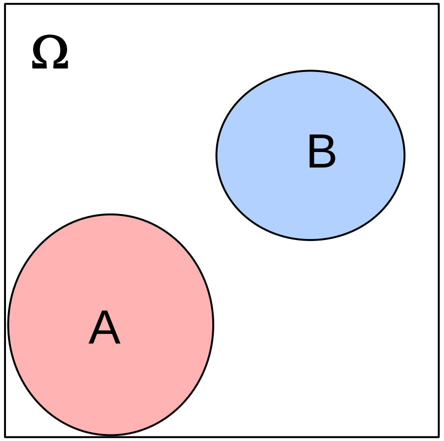


Venn Diagram – illustrating complementary events.

Mutually Exclusive Events:

- The events that do not intersect or do not have any common outcomes

$$A \cap B = \emptyset \rightarrow \text{Null Set}$$



Venn Diagram – illustrating mutually exclusive.

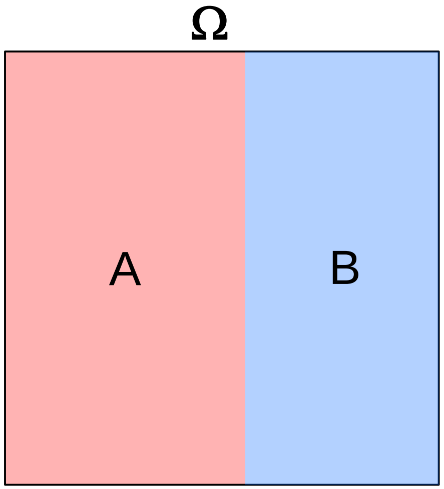
Probability Definitions
Probability Operators

Exhaustive, Mutually Exclusive Sequence of Events:

- The sequence of events whose union is equal to the sample space

$$A_1 \cup A_2 \cup \ \ldots \cup A_n = \Omega$$

- For example, all the samples are either A or B



Venn Diagram – illustrating exhaustive events.

Probability Definitions
Now We Refine Probability

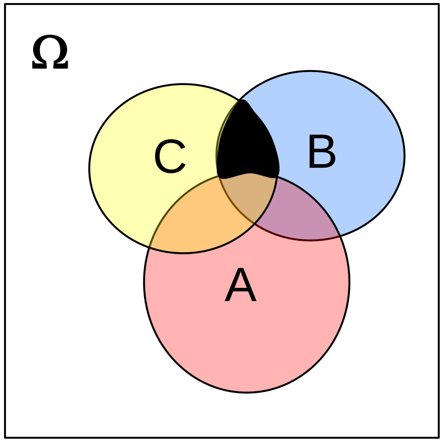
$$\text{Prob}(A) = \text{P}(A) = \lim_{n\to\infty}\left(\frac{Area(A)}{Area(\Omega)}\right)$$

where:

Area(A) = area of A / total area = P(A)
Area(Ω) = total area / total area = probability of any possible outcome = P(Ω) = 1.0

Example: Possibility of drilling a dry hole for the next well (A^C), encountering sandstone at a location ($\mathbf{u}_\alpha$)(B), exceeding a rock porosity of 15% at a location ($\mathbf{u}_\alpha$)(C).

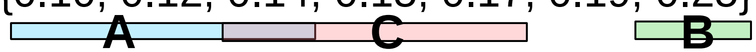
Prob($\boldsymbol{A^C} \cap \boldsymbol{B} \cap \boldsymbol{C}$) = Area($\boldsymbol{A^C} \cap \boldsymbol{B} \cap \boldsymbol{C}$) / Area(Ω)



Probability Definitions
Test Your Knowledge

Example: A petrophysicist has measured porosity of 7 core samples from a carbonate formation in the laboratory as follows:

Sample Space: {0.10, 0.12, 0.14, 0.15, 0.17, 0.19, 0.25}



We would like to investigate the following events:

- Event A: Porosity values of less than 0.15, {0.10, 0.12, 0.14}
- Event B: Porosity values of greater than 0.20, {0.25}
- Event C: Porosity values of between 0.14 and 0.17 {0.14, 0.15, 0.17}

Union of Events:

$A \cup B =$ $\qquad$ $B \cup C =$ $\qquad$ $A \cup C =$

Intersection of Events:

$A \cap B =$ $\qquad$ $B \cap C =$ $\qquad$ $A \cap C =$

Complementary Events:

$A^c =$ $\qquad$ $B^c =$ $\qquad$ $C^c =$

Mutually Exclusive Events:

$A \cap B =$ $\qquad$ $B \cap C =$

All Events:

$A \cup B \cup C =$

Probability Definitions
Test Your Knowledge

Example: A petrophysicist has measured porosity of 7 core samples from a carbonate formation in the laboratory as follows:

Sample Space: {0.10, 0.12, 0.14, 0.15, 0.17, 0.19, 0.25}

We would like to investigate the following events:

- Event A: Porosity values of less than 0.15, {0.10, 0.12, 0.14}
- Event B: Porosity values of greater than 0.20, {0.25}
- Event C: Porosity values of between 0.14 and 0.17 {0.14, 0.15, 0.17}

Union of Events:

$A \cup B = \{0.10, 0.12, 0.14, 0.25\}$ $\quad B \cup C = \{0.14, 0.15, 0.17, 0.25\}$ $\quad A \cup C = \{0.10, 0.12, 0.14, 0.15, 0.17\}$

Intersection of Events:

$A \cap B = \phi$ $\quad A \cap C = \{0.14\}$ $\quad B \cap C = \phi$

Complementary Events:

$A^c = \{0.15, 0.17, 0.19, 0.25\}$ $\quad B^c = \{0.10, 0.12, 0.14, 0.15, 0.17, 0.19\}$ $\quad C^c = \{0.10, 0.12, 0.19, 0.25\}$

Mutually Exclusive Events:

$A \cap B = \phi$ $\quad B \cap C = \phi$

All Events:

$A \cup B \cup C = \{0.10, 0.12, 0.14, 0.15, 0.17, 0.25\}$

Probability Definitions
Test Your Knowledge, Frequentist

Example: A petrophysicist has measured porosity of 7 core samples from a carbonate formation in the laboratory as follows:

Sample Space: {0.10, 0.12, 0.14, 0.15, 0.17, 0.19, 0.25}

We would like to investigate the following events:

- Event A: Porosity values of less than 0.15, {0.10, 0.12, 0.14} $P(A)$ = 3/7
- Event B: Porosity values of greater than 0.20, {0.25} $P(\text{B})$ = 1/7
- Event C: Porosity values of between 0.14 and 0.17 {0.14, 0.15, 0.17} $P(C)$ = 3/7

Union of Events:

$A \cup B = \{0.10, 0.12, 0.14, 0.25\}$ $\quad P(A \cup B) = 4/7$

$B \cup C = \{0.14, 0.15, 0.17, 0.25\}$ $\quad P(B \cup C) = 4/7$

$A \cup C = \{0.10, 0.12, 0.14, 0.15, 0.17\}$ $\quad P(A \cup C) = 5/7$

Intersection of Events:

$A \cap B = \phi$, $\text{P}(A \cap B) = 0$ $\qquad B \cap C = \{0.14\}$, $\text{P}(B \cap C) = 1/7$ $\qquad A \cap C = \phi$, $P(A \cap C) = 0$

Complementary Events:

$A^c = \{0.15, 0.17, 0.19, 0.25\}$ P = 4/7 $\quad B^c = \{0.10, 0.12, 0.14, 0.15, 0.17, 0.19\}$ P = 6/7 $\quad C^c = \{0.10, 0.12, 0.19, 0.25\}$ P = 4/7

Mutually Exclusive Events:

$A \cap B = \phi$ $\text{P}(A \cap B) = 0$ $\qquad B \cap C = \phi$ $\qquad P(B \cap C) = 0$

Exhaustive Sequence of Events:

$A \cup B \cup \text{C} = \{0.10, 0.12, 0.14, 0.15, 0.17, 0.19, 0.25\} = \Omega$, $\text{P}(A \cup B \cup \text{C}) = 1.0$

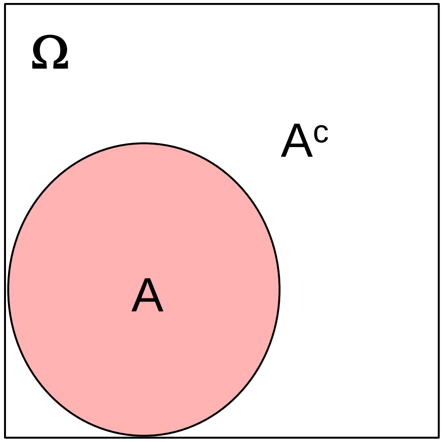
Probability Definitions
Probability Concepts

Non-negativity, Normalization:

- Fundamental probability constraints
  - Bounded $0 \leq P(A) \leq 1$
  - Closure $P(\Omega) = 1$
  - Null Sets $P(\phi) = 0$

Complimentary Events:

- Closure $P(A^c) + P(A) = 1$



Venn Diagram – illustrating complementary events.

Probability Definitions
Probability by Venn Diagram

The Addition Rule:

$$P(A \cup B) = \mathrm{P(A) + P(B) - P}(A \cap B)$$

Must account for the intersection!

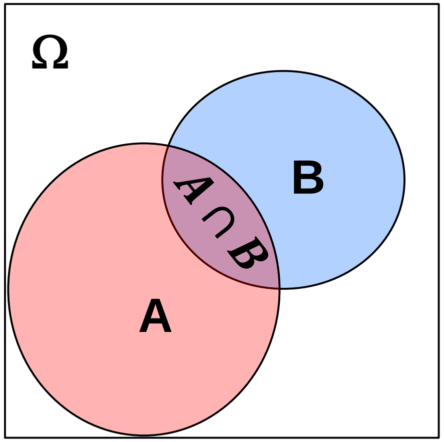
If mutually exclusive events:

$$A_i \cap A_j = \emptyset, \forall\, i \neq j$$

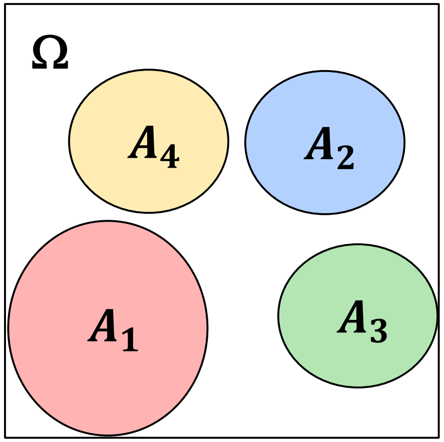
then,

$$\mathrm{P}\left(\bigcup_{i=1}^{k} \mathrm{A}_i\right) = \sum_{i=1}^{k} \mathrm{P}(\mathrm{A}_i)$$

no intersections to account for.



Venn Diagram – illustrating intersection.



Venn Diagram – illustrating intersection.

Probability Definitions Addition Rule Example

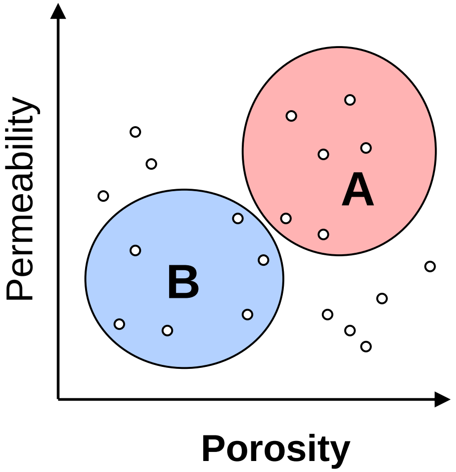
Calculate the following probabilities for event A and B:
Note Event A: Sandstone and Event B: Shale

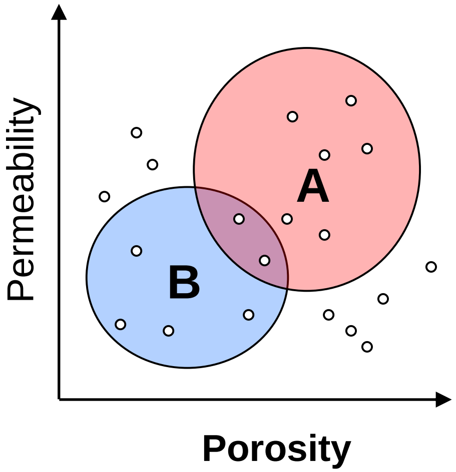
$$P(A) =$$

$$P(B) =$$

$$P(A \cap B) =$$

$$P(A \cup B) =$$





$$P(A) =$$

$$P(B) =$$

$$P(A \cap B) =$$

$$P(A \cup B) =$$

Probability Definitions Addition Rule Example

Calculate the following probabilities for event A and B:

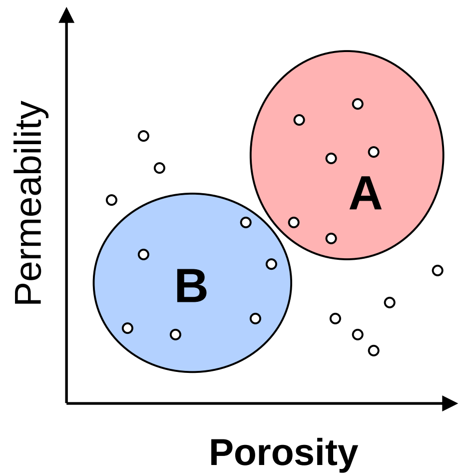
Note Event A: Sandstone and Event B: Shale

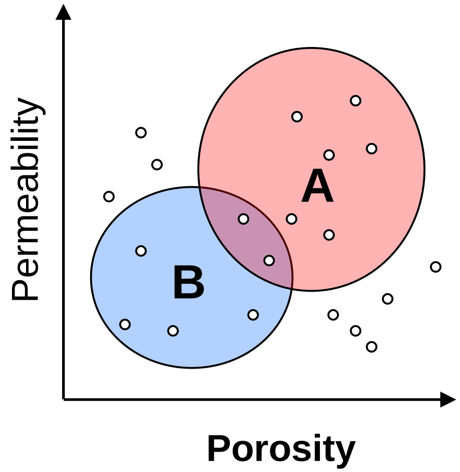
$$\mathrm{P}(A) = {}^{6}/_{20} = 30\%$$

$$\mathrm{P}(B) = {}^{6}/_{20} = 30\%$$

$$\mathrm{P}(A \cap B) = {}^{0}/_{20} = 30\%$$

$$\mathrm{P}(A \cup B) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P}(A \cap B)$$
$$= 30\% + 30\% - 0\% = 60\%$$





$$\mathrm{P}(A) = {}^{8}/_{20} = 40\%$$

$$\mathrm{P}(B) = {}^{6}/_{20} = 30\%$$

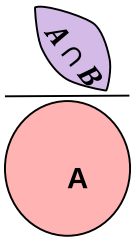
$$\mathrm{P}(A \cap B) = {}^{2}/_{20} = 10\%$$

$$\mathrm{P}(A \cup B) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P}(A \cap B)$$
$$= 40\% + 30\% - 10\% = 60\%$$

Probability Definitions
Conditional Probability

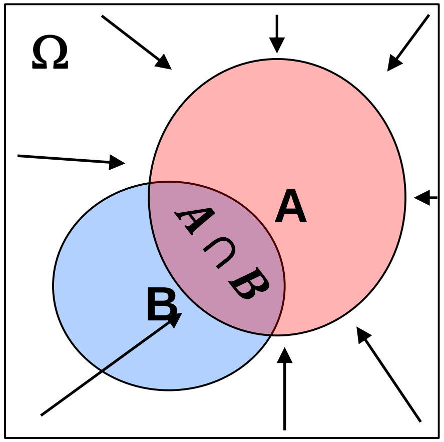
Probability of B given A occurred? $P(B \mid A)$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$



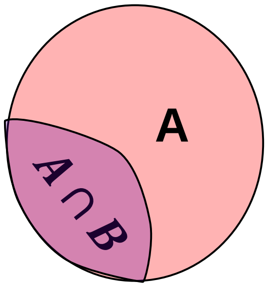
$P(A \cap B)$

$P(A)$



Conceptually we shrink space of possible outcomes.

A occurred so we shrink our space to only event A.



Probability Definitions
Conditional, Marginal and Joint Probability

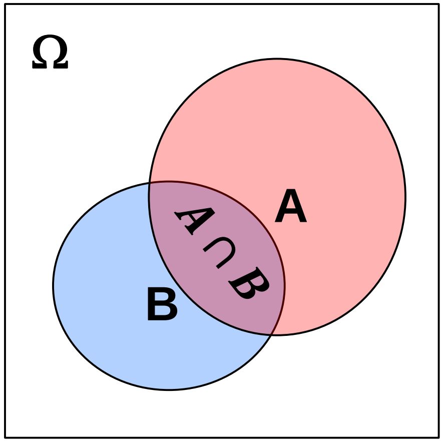
Probability of B given A occurred? $P(B \mid A)$

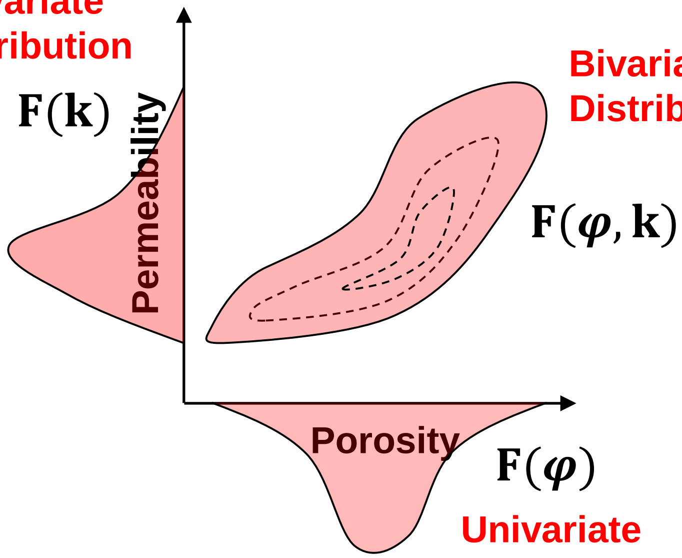
Conditional Probability

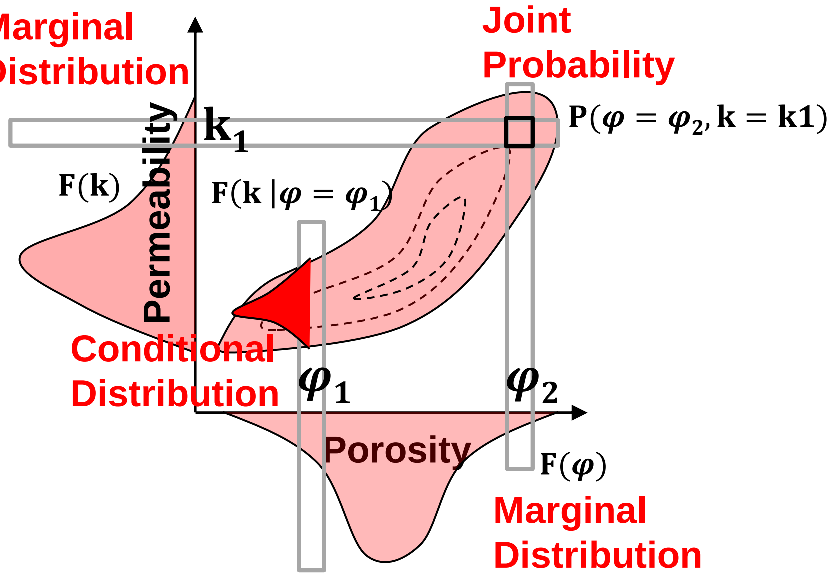
Joint Probability

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \text{ and } B)}{P(A)}$$

Marginal Probability







Probability Definitions
Conditional, Marginal and Joint Probability

Marginal Probability: Probability of an event, irrespective of any other event

$$P(X), P(Y)$$

Conditional Probability: Probability of an event, given another event is already true.

$$P(X\ given\ Y), P(Y\ given\ X)$$

$$P(X \mid Y), P(Y|X)$$

Joint Probability: Probability of multiple events occurring together.

$$P(X\ and\ Y), P(Y\ and\ X)$$

$$P(X\ \cap\ Y), P(Y \cap X)$$

$$P(X, Y), P(Y, X)$$

Probability Definitions
Conditional Probability

General Form for Conditional Probability?

$$P(C \mid B, A) = \frac{P(A \cap B \cap C)}{P(A \cap B)}$$

Recall:

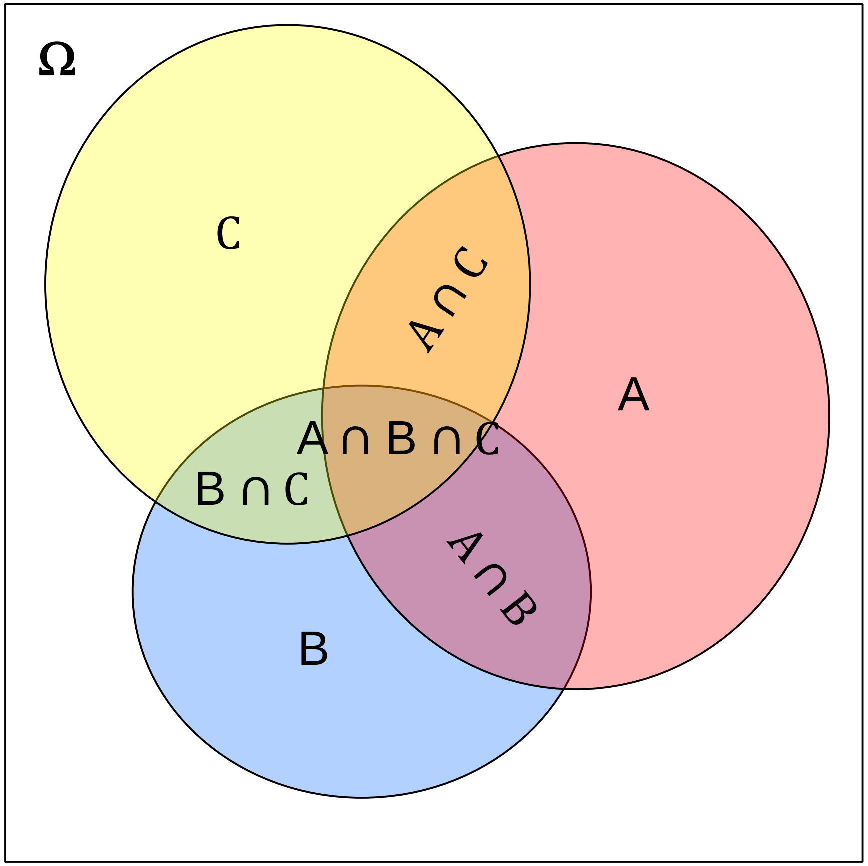
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Substitute:

$$P(C \mid B, A) = \frac{P(A \cap B \cap C)}{P(B|A)P(A)}$$

Reorganize:

$$P(A \cap B \cap C) = P(C \mid B, A)P(B|A)P(A)$$



General Form, Recursion of Conditionals

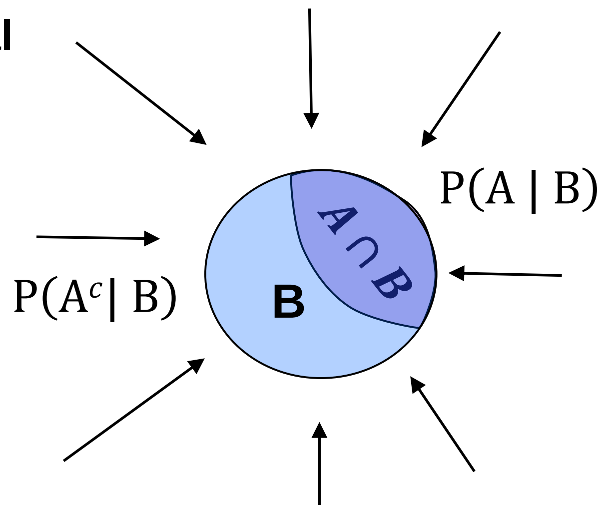
$$P(A_1 \cap \cdots \cap A_n) = P(A_n|A_{n-1},\ldots,A_1)P(A_{n-1}|A_{n-2},\ldots,A_1)\ \ldots\ P(A_1)$$

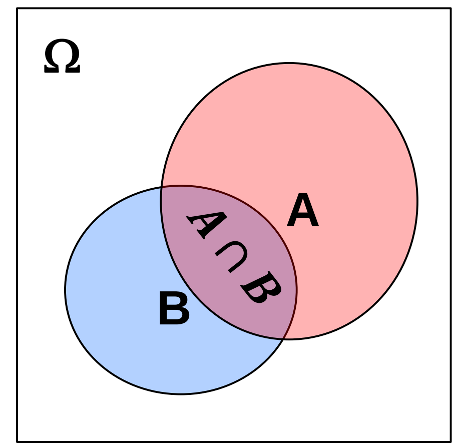
Probability Definitions
Conditional Probability

Other Relations with Conditional Probability

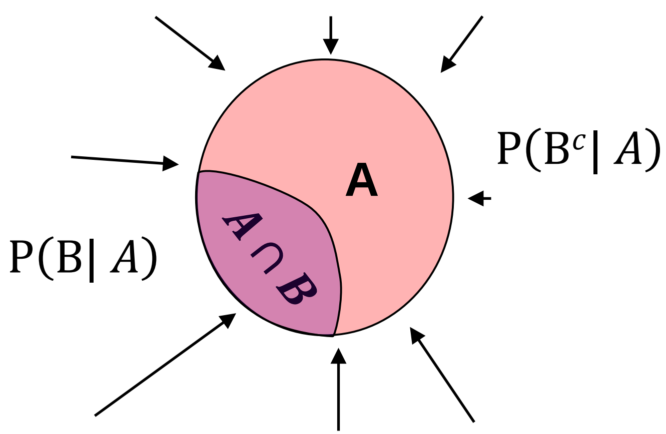
- Closure with conditional probabilities:

$$P(A \mid B) + P(A^c \mid B) = 1$$





$$P(B \mid A) + P(B^c \mid A) = 1$$



Probability Definitions
Conditional Probability Examples

Recall:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

For Case 1 calculate:

$P(A \mid B) =$

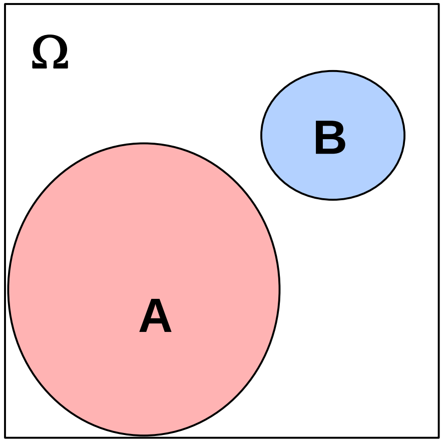
$P(B \mid A) =$

For Case 2 calculate:

$P(A \mid B) =$

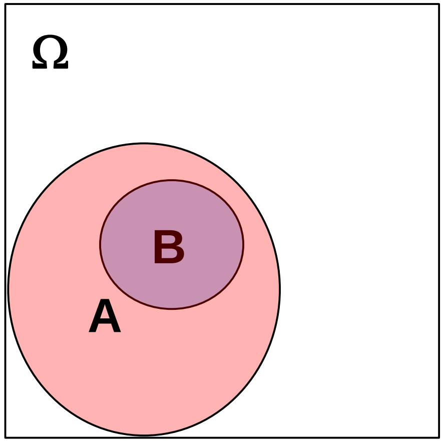
$P(B \mid A) =$

Case 1:



Venn Diagram – case 1.

Case 2:



Venn Diagram – case 2.

Probability Definitions
Conditional Probability Examples

Recall:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

For Case 1 calculate:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\emptyset}{P(B)} = \emptyset$$

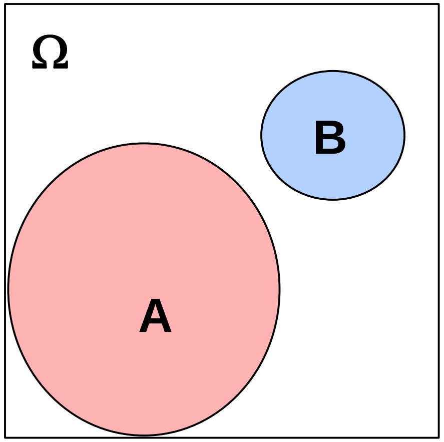
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\emptyset}{P(A)} = \emptyset$$

For Case 2 calculate:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)} = 1, \text{since } P(A \cap B) = P(B)$$

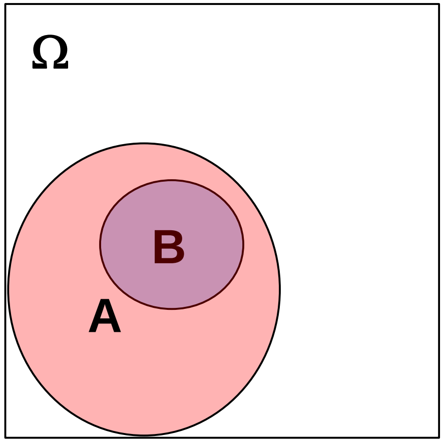
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)}, \text{since } P(A \cap B) = P(B)$$

Case 1:



Venn Diagram – case 1.

Case 2:



Venn Diagram – case 2.

Probability Definitions
Conditional Probability Examples

Question: Calculate the following probabilities for events A and B:

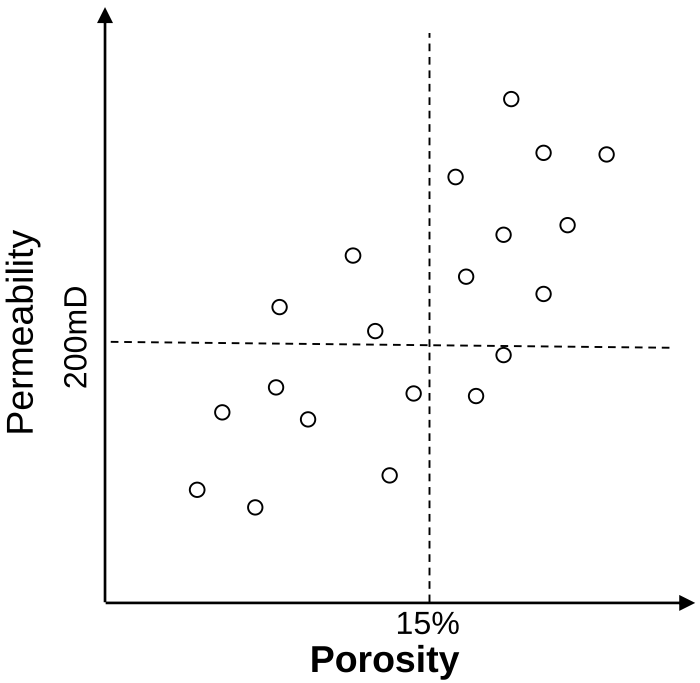
Event A: Porosity $> 15\%$
Event B: Permeability > 200 mD

For Case 1 calculate:

$P(A \mid B) =$

$P(B \mid A) =$

Bonus Question: How much information does event B tell you about event A and visa versa?



Probability Definitions Conditional Probability Examples

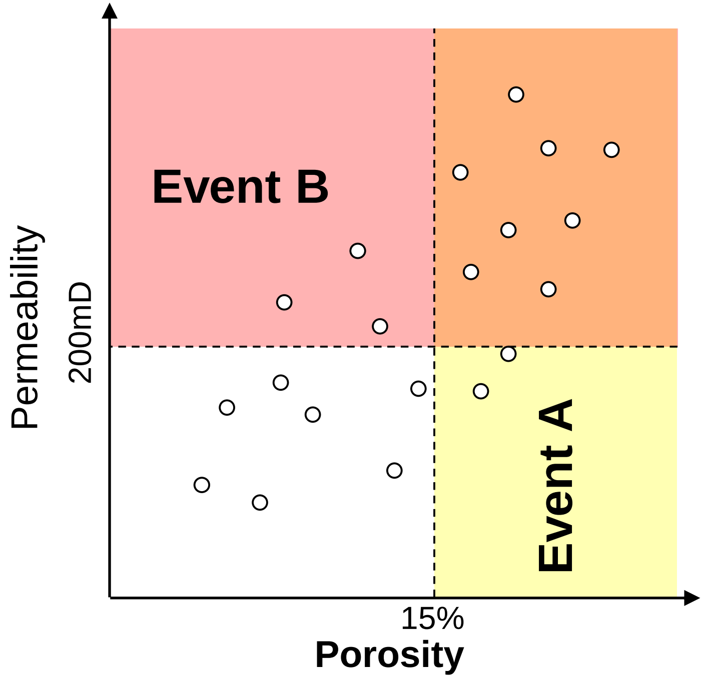
Question: Calculate the following probabilities for events A and B:

Event A: Porosity > 15%
Event B: Permeability > 200 mD

For Case 1 calculate:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{8/20}{11/20} = 8/11$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{8/20}{10/20} = 8/10$$



Bonus Question: How much information does B tell you about A and visa versa?
$P(A) = 10/20$, $P(A|B) = 8/11$ Probability from 50% → 73%
$P(B) = 11/20$, $P(B|A) = 8/10$ Probability from 55% → 80%
We cannot work with A and B independently, they provide information about each other.

Probability Definitions
Conditional, Marginal and Joint Probability

Joint Distribution:

$$f_{XY}(x, y)$$

Marginal Distribution:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y)dy$$

Conditional Distribution:

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

Table of Frequencies

Porosity (%) \ Fraction Shale (%)	10%	30%	50%	70%	90%
25%	1	1	0	0	0
20%	2	3	2	0	0
15%	1	2	2	1	0
10%	0	0	2	3	2
5%	0	0	1	1	1

Probability Definitions
Conditional, Marginal and Joint Probability

Joint Distribution:

$$f_{XY}(x,y)$$

Marginal Distribution:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)dy$$

Conditional Distribution:

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Table of Joint Probabilities

Porosity (%) \ Fraction Shale (%)	10%	30%	50%	70%	90%
25%	4%	4%	0	0	0
20%	8%	12%	8%	0	0
15%	4%	8%	8%	4%	0
10%	0	0	8%	12%	8%
5%	0	0	4%	4%	4%

Probability Definitions
Conditional, Marginal and Joint Probability

Given these joint probabilities calculate the:

Marginal Distributions:

Vsh	**10%**	**30%**	**50%**	**70%**	**90%**
$f_{Vsh}(v_{sh})$					

Porosity	**5%**	**10%**	**15%**	**20%**	**25%**
$f_{\varphi}(\varphi) =$					

Conditional Distribution:

Vsh	**10%**	**30%**	**50%**	**70%**	**90%**
$f_{Vsh\|\varphi}(v_{sh}\| \varphi = 15\%) =$					

Table of Joint Probabilities

Porosity (%) \ **Fraction Shale (%)**	**10%**	**30%**	**50%**	**70%**	**90%**
25%	4%	4%	0	0	0
20%	8%	12%	8%	0	0
15%	4%	8%	8%	4%	0
10%	0	0	8%	12%	8%
5%	0	0	4%	4%	4%

Probability Definitions

Conditional, Marginal and Joint Probability

Given these joint probabilities calculate the:

Marginal Distributions:

Vsh	**10%**	**30%**	**50%**	**70%**	**90%**
$f_{Vsh}(v_{sh})$	16%	24%	28%	20%	12%

Porosity	**5%**	**10%**	**15%**	**20%**	**25%**
$f_{\varphi}(\varphi) =$	12%	28%	24%	28%	8%

Conditional Distribution:

Vsh	**10%**	**30%**	**50%**	**70%**	**90%**
	20%	40%	40%	0	0

$$f_{Vsh|\varphi}(v_{sh}|\,\varphi = 15\%) = f_{Vsh,\varphi}(v_{sh}, \varphi = 15\%)/f_{\varphi}(\varphi = 15\%)$$

Table of Joint Probabilities

Porosity (%) \ Fraction Shale (%)	10%	30%	50%	70%	90%
25%	4%	4%	0	0	0
20%	8%	12%	8%	0	0
15%	4%	8%	8%	0	0
10%	0	4%	8%	12%	8%
5%	0	0	4%	4%	4%

Probability Definitions
Multiplication Rule

The Multiplication Rule:

$$\mathsf{P}(A \cap B) = \mathrm{P}(\mathrm{B}|\mathrm{A})\ \mathrm{P}(A)$$

If events A and B are independent**:**

$\mathrm{P}(B|A) = \mathrm{P}(\mathrm{B})$

Knowing something about A does nothing to help predict B. Then by substitution:

$$\mathsf{P}(A \cap B) = \mathrm{P}(\mathrm{B})\ \mathrm{P}(A)$$

The general form given independence for all cases, $i = 1, \ldots, k$:

$$\mathsf{P}\left(\cap_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{k}} \mathrm{A_i}\right) = \prod_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{k}} \mathrm{P}(\mathrm{A_i})$$

Probability Definitions Multiplication Rule Example

Given there is independence between fluid type and porosity:

Event A = Oil
Event B = Porosity > 10%

Given: $\mathrm{P(A)} = 30\%$ and $\mathrm{P(B)} = 50\%$

What is the P($A \cap B$)?

Given there is independence between fluid type, porosity and saturation:

Event A = Oil
Event B = Porosity > 10%
Event C = S_{oil} > 40%

Given: $\mathrm{P(A)} = 30\%, \mathrm{P(B)} = 50\%, \mathrm{P(C)} = 25\%$

What is the P($A \cap B \cap C$)?

Probability Definitions
Multiplication Rule Example

Given there is independence between fluid type and porosity:

Event A = Oil
Event B = Porosity > 10%

Given: $\mathrm{P(A)} = 30\%$ and $\mathrm{P(B)} = 50\%$

What is the P($A \cap B$)? 30% x 50% = 15%

Given there is independence between fluid type, porosity and saturation:

Event A = Oil
Event B = Porosity > 10%
Event C = S_{oil} > 40%

Given: $\mathrm{P(A)} = 30\%$, $\mathrm{P(B)} = 50\%$, $\mathrm{P(C)} = 10\%$

What is the P($A \cap B \cap C$)? 30% x 50% x 10% = 1.5%

Probability Definitions
Evaluating Independence

Events A and B are independent if and only if:

$$\mathrm{P}(A \cap B) = \mathrm{P}(\mathrm{B})\mathrm{P}(A)$$

or

$$\mathrm{P}(\, A|B\,) = \mathrm{P}(A) \quad \text{and} \quad \mathrm{P}(\, B|A\,) = \mathrm{P}(\boldsymbol{B})$$

General Form:

Events $A_1, A_2, \ldots, A_n$ are independent if:

$$\mathrm{P}\left(\cap_{\boldsymbol{i=1}}^{\boldsymbol{k}} \mathrm{A_i}\right) = \prod_{\boldsymbol{i=1}}^{\boldsymbol{k}} \mathrm{P}(\mathrm{A_i})$$

Probability Definitions
Evaluating Independence Example

Example: Facies F1, F2 and F3 in 10 wells:

Position	Well 1	Well 2	Well 3	Well 4	Well 5	Well 6	Well 7	Well 8	Well 9	Well 10
Top	F3	F2	F2	F1	F1	F1	F2	F2	F1	F1
Middle	F1	F1	F1	F1	F2	F2	F1	F2	F2	F2
Bottom	F2	F2	F2	F3	F3	F3	F3	F3	F3	F2

Event A_1 = F1 is middle facies
Event A_2 = F3 is bottom facies

$\mathrm{P}(\boldsymbol{A_1} \cap \boldsymbol{A_2}) = \mathrm{P}(\boldsymbol{A_1})\mathrm{P}(\boldsymbol{A_2})$ or

$\mathrm{P}(\boldsymbol{A_1}|\boldsymbol{A_2}) = \mathrm{P}(\boldsymbol{A_1})$ and $\mathrm{P}(\boldsymbol{A_2}|\boldsymbol{A_1}) = \mathrm{P}(\boldsymbol{A_2})$

Question: are events A1 and A2 independent?

Probability Definitions
Evaluating Independence Example

Example: Facies F1, F2 and F3 in 5 wells:

Position	Well 1	Well 2	Well 3	Well 4	Well 5	Well 6	Well 7	Well 8	Well 9	Well 10
Top	F3	F2	F2	F1	F1	F1	F2	F2	F1	F1
Middle	F1	F1	F1	F1	F2	F2	F1	F2	F2	F2
Bottom	F2	F2	F2	F3	F3	F3	F3	F3	F3	F2

Event A_1 = middle facies if F1
Event A_2 = bottom facies is F3

$\mathrm{P}(\boldsymbol{A_1} \cap \boldsymbol{A_2}) = \mathrm{P}(\boldsymbol{A_1})\mathrm{P}(\boldsymbol{A_2})$ or

$\mathrm{P}(\boldsymbol{A_1}|\boldsymbol{A_2}) = \mathrm{P}(\boldsymbol{A_1})$ and $\mathrm{P}(\boldsymbol{A_2}|\boldsymbol{A_1}) = \mathrm{P}(\boldsymbol{A_2})$

Question: are events A1 and A2 independent?

$$\mathrm{P}(\boldsymbol{A_1}) = {}^{5}/_{10} = 50\%, \mathrm{P}(\boldsymbol{A_2}) = {}^{6}/_{10} = 60\%, \mathrm{P}(\boldsymbol{A_1} \cap \boldsymbol{A_2}) = {}^{2}/_{10} = 20\%$$

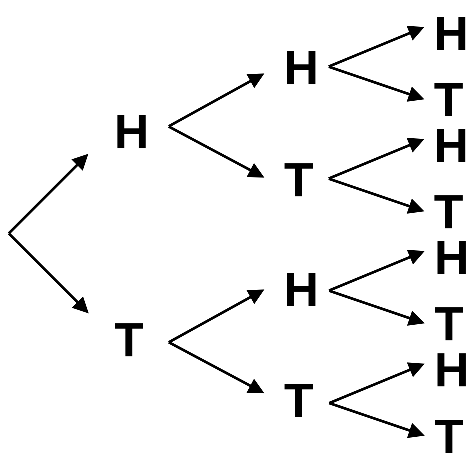
$$\mathrm{P}(\boldsymbol{A_1})\mathrm{P}(\boldsymbol{A_2}) = \mathbf{50\% \cdot 60\% = 30\%} \neq \mathrm{P}(\boldsymbol{A_1} \cap \boldsymbol{A_2}) = {}^{2}/_{10} = 20\% \quad \text{Not independent.}$$

Probability Definitions
Probability Tree

Coin Flip:

Events $A_1,\ldots,A_3$ are 3 coin flips:

A_1 A_2 A_3



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

$P(A_1, A_2, A_3) = P(A_1)\ P(A_2)\ P(A_3)$

$0.5 \times 0.5 \times 0.5 = 0.125$

…

…

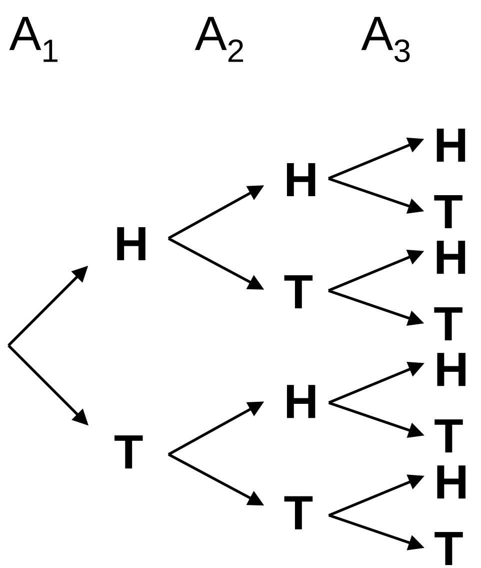
$0.5 \times 0.5 \times 0.5 = 0.125$

Sum of all outcomes $= 1$

All these outcomes are equiprobable.

Probability Definitions
Probability Tree

Coin Flip:
What is the probability of only one tails?



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

$$\mathbf{P}(A_1, A_2, A_3) = \mathbf{P}(A_1)\,\mathbf{P}(A_2)\,\mathbf{P}(A_3)$$

0.5 x 0.5 x 0.5 = 0.125

…

…

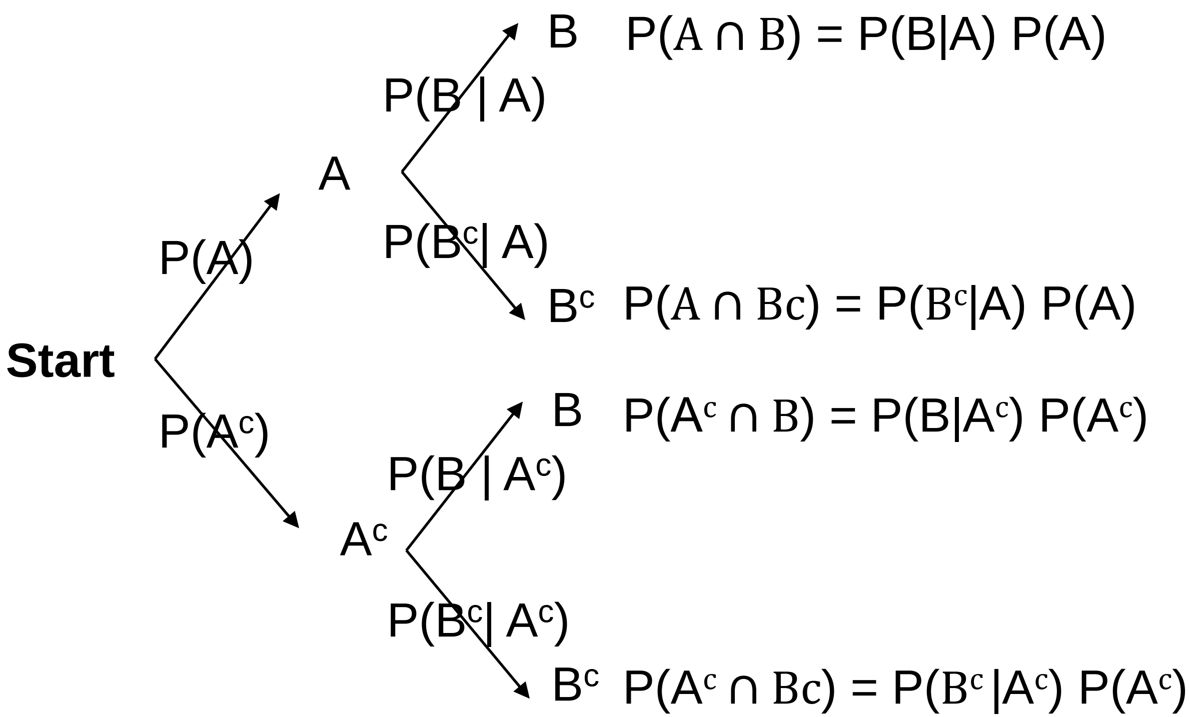
0.5 x 0.5 x 0.5 = 0.125

0.125 + 0.125 + 0.125 = 0.375

Sum of probability of :
the specified outcomes

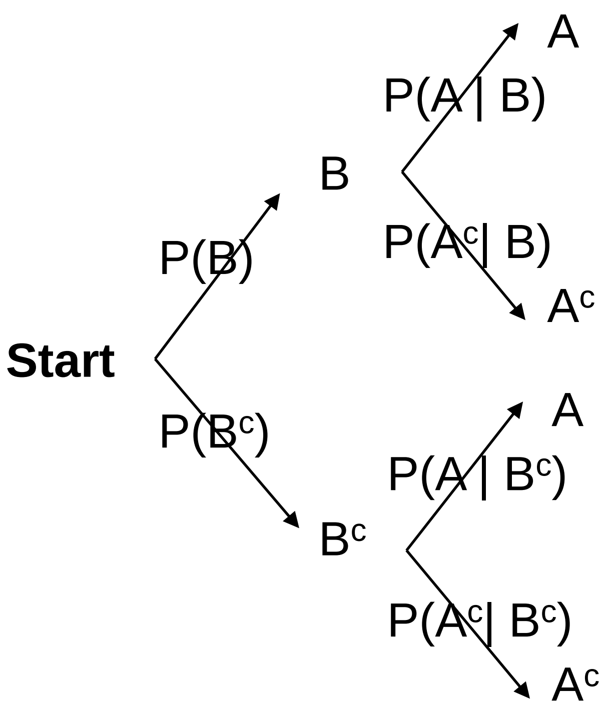
Probability Definitions
Probability Tree

General Form of a Probability Tree



Probability Definitions
Probability Tree

General Form of a Probability Tree



$$P(B \cap A) = P(A|B)\,P(B)$$

$$P(B \cap A^c) = P(A^c|B)\,P(B)$$

$$P(B^c \cap A) = P(A|B^c)\,P(B^c)$$

$$P(B^c \cap A^c) = P(A^c|B^c)\,P(B^c)$$

Probability Definitions
Bayesian Statistics

Product Rule:

$$P(B \cap A) = P(A|B)\, P(B)$$

$$P(A \cap B) = P(B|A)\, P(A)$$

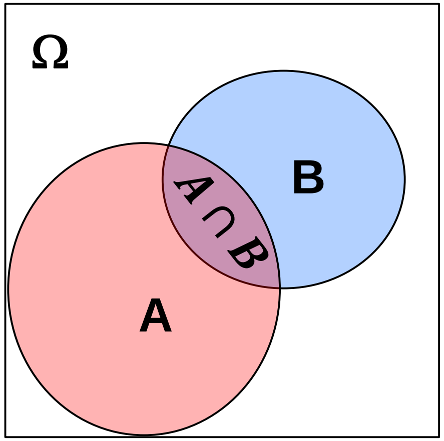
It follows that:

$$P(B \cap A) = P(A \cap B)$$

Therefore we combine two product rules, substitute:

$$P(A|B)\, P(B) = P(B|A)\, P(A)$$

We get Bayes' Theorem!



Venn Diagram – illustrating intersection.

Probability Definitions
Bayesian Theorem

Bayes' Theorem:

Make a easy adjustment and we get the popular form.

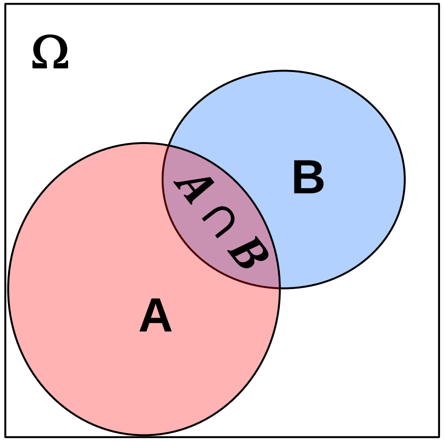
$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Observations:

1. We are able to get P(A | B) from P(B | A) as you will see this often comes in handy.
2. Each term is known as:

$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Evidence}}$$

3. Prior should have no information from likelihood.
4. Evidence term is usually just a standardization to ensure closure.



Venn Diagram – illustrating intersection.

Note: we got to Bayes' Theorem by fundamental frequentist approaches.

Probability Definitions
Bayesian Theorem Example

Bayes Theorem:

A common example of Bayes' Theorem for modeling the subsurface.

Model Updating with a New Data Source:

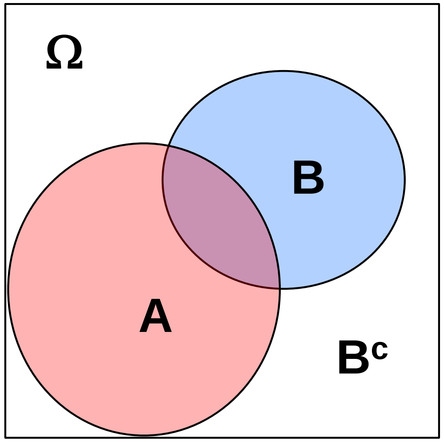
$$P(\text{Model} \mid \text{New Data}) = \frac{\overbrace{P(\text{New Data} \mid \text{Model})}^{\textbf{Likelihood}}\ \overbrace{P(\text{Model})}^{\textbf{Prior}}}{\underbrace{P(\text{New Data})}_{\textbf{Evidence}}}$$

Probability Definitions
Bayesian Theorem Example

Bayes Theorem:

Alternative form, symmetry:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} \qquad P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}$$



Venn Diagram – illustrating intersection.

Alternative form to calculate evidence term:

Given: $P(A) = \underbrace{P(A|B)\,P(B)}_{P(A \text{ and } B)} + \underbrace{P(A|B^c)\,P(B^c)}_{P(A \text{ and } B^c)}$

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B^c)\,P(B^c)}$$

Probability and Statistics
Bayesian Methods

Bayesian approaches allow you to solve statistical problems that are not possible otherwise. Consider:

Event A	Event B
You have a disease	You test positive for the disease
There is fault compartmentalization	Geologist says there's a fault
Low permeability of a sample	The laboratory measure is low
A valve will fail	X-ray test is positive
You drill a dry well	Seismic AVO response looks poor

In all of these cases you need to calculate:

$$P\left(\begin{matrix}\text{Something is}\\\text{Happening}\end{matrix} \;\middle|\; \begin{matrix}\text{Looks like}\\\text{its happening}\end{matrix}\right) = \frac{P\left(\begin{matrix}\text{Looks like}\\\text{its happening}\end{matrix} \;\middle|\; \begin{matrix}\text{Something is}\\\text{Happening}\end{matrix}\right) P\left(\begin{matrix}\text{Something is}\\\text{Happening}\end{matrix}\right)}{P\left(\begin{matrix}\text{Looks like}\\\text{its happening}\end{matrix}\right)}$$

Probability and Statistics
Bayesian Methods

Bayesian approaches allow you to solve statistical problems that are not possible otherwise. Consider:

Correct Detection Rate x Occurrence Rate

$$P\left(\begin{matrix}\text{Something is}\\ \text{Happening}\end{matrix} \,\middle|\, \begin{matrix}\text{Looks like}\\ \text{its happening}\end{matrix}\right) = \frac{P\left(\begin{matrix}\text{Looks like}\\ \text{its happening}\end{matrix} \,\middle|\, \begin{matrix}\text{Something is}\\ \text{Happening}\end{matrix}\right) P\left(\begin{matrix}\text{Something is}\\ \text{Happening}\end{matrix}\right)}{P\left(\begin{matrix}\text{Looks like}\\ \text{its happening}\end{matrix}\right)}$$

All Detection Rate (included false positives)

Often these terms are much easier to collect:

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B^c)\,P(B^c)}$$

Let's try this out next.

Probability and Statistics Bayesian Methods

Example: Prior information at a site suggests a channel feature exists at a given location with probability of 60%. We decide to further investigate this information using seismic data.

Seismic survey can show a feature
- is present with 90% probability if it really is present
- is not present with a probability 70% if it really is not

$P(A) = 0.6$
$P(B|A) = 0.9$
$P(B^c|A^c) = 0.7$

***A*=The feature is present**
***B* =Seismic shows the feature**
***A^c* =The feature not present**
***B^c* =Seismic does not show the feature**

Will seismic information be useful?

Probability and Statistics Bayesian Methods

Example: Prior information at a site suggests a channel feature exists at a given location with probability of 60%. We decide to further investigate this information using seismic data.

Seismic survey can show a feature
- is present with 90% probability if it really is present
- is not present with a probability 70% if it really is not

***A*=The feature is present**
***B* =Seismic shows the feature**
***A^c^* =The feature not present**
***B^c^* =Seismic does not show the feature**

$P(A) = 0.6$

$P(B|A) = 0.9$

$P(B^c|A^c) = 0.7$

$P(B|A^c) = 1 - P(B^c|A^c) = 0.3$

$P(A^c) = 1- P(A) = 0.4$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)} = \frac{\overbrace{(0.9)(0.6)}^{\text{True Positive}}}{\underbrace{(0.9)(0.6)}_{\text{True Positive}} + \underbrace{(0.3)(0.4)}_{\text{False Positive}}} = 82\%$$

Probability and Statistics
Bayesian Methods

Example: One in every thousand BOPs has a serious crack. X-ray analysis has a 99% chance of detecting the crack correctly. If the BOP does not have a crack, there is a 2% chance that the X-ray detects a crack. The rate of BOP cracks is 0.1%. A BOP has been X-rayed and the result is positive. What is the chance that the BOP actually does have a crack?

Solution:

***A* = BOP has cracks** **P(A|B) = ?**

***B* = BOP tests positive**

A^c = BOP does not have cracks

B^c = BOP did not test positive

P(A) = 0.001 – crack rate

P(B|A) = 0.99 – true positive

P(B|A^c) = 0.02 – false positive

Probability and Statistics
Bayesian Methods

Example: One in every thousand BOPs has a serious crack. X-ray analysis has a 99% chance of detecting the crack correctly. If the BOP does not have a crack, there is a 2% chance that the X-ray detects a crack. The rate of BOP cracks is 0.1%. A BOP has been X-rayed and the result is positive. What is the chance that the BOP actually does have a crack?

Solution:

***A* = BOP has cracks** **$P(A|B)$ = ?**

***B* = BOP tests positive**

A^c = BOP does not have cracks

B^c = BOP did not test positive

$P(A) = 0.001$ – crack rate

$P(A^c) = 0.999$ – not cracked rate

$P(B|A) = 0.99$ – true positive

$P(B|A^c) = 0.02$ – false positive

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)} = \frac{\overbrace{(0.99)(0.001)}^{\text{True Positive}}}{\underbrace{(0.99)(0.001)}_{\text{True Positive}} + \underbrace{(0.02)(0.999)}_{\text{False Positive}}} = 4.7\%$$

Probability of a crack in the BOP given a positive crack test is only 4.7%! Why?
Cracks are very unlikely + high false positive rate (2%)!

Probability and Statistics
Bayesian Theorem General Form

General Form:

$$P(A_i \mid B) = \frac{P(B|A_i)\,P(A_i)}{P(B)}$$

if non-overlapping

$$A_i \bigcap A_j = \emptyset, i \neq j$$

and exhaustive

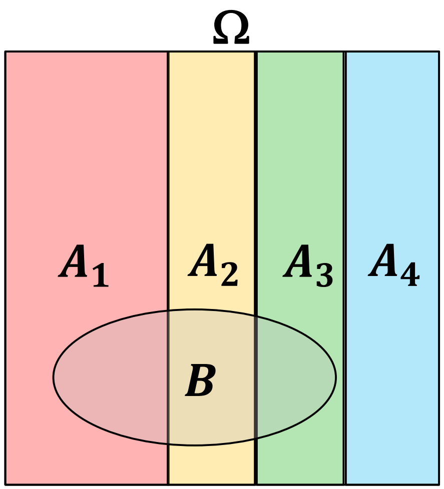
$$\bigcup_{i=1}^{K} A_i = \Omega$$

then:

$$P(B) = \sum_{k=1}^{K} P(B|\,A_i)\,P(A_i) = \sum_{k=1}^{K} P(B|\,A_i)\,P(A_i)$$

we substitute:

$$P(A_i \mid B) = \frac{P(B|A_i)\,P(A_i)}{\sum_{k=1}^{K} P(B|\,A_i)\,P(A_i)} = \frac{P(B|A_i)\,P(A_i)}{\sum_{k=1}^{K} P(B, A_i)}$$



Venn Diagram – illustrating exhaustive, mutually exclusive series.

Probability and Statistics
What should you learn from this lecture?

Introduction

General Concepts

Statistics

Probability

Univariate

Bivariate

Time Series Analysis

Spatial Analysis

Machine Learning

Uncertainty Analysis

Lecture outline . . .

- Probability Definitions
- Venn Diagrams
- Frequentist Concepts
- Bayesian Concepts