Moments of Inertia

Using torsional pendulums to calculate moments of inertia

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Objectives

- Broadly, we want to estimate moments of inertia for a vast assortment of objects using a torsional pendulum.

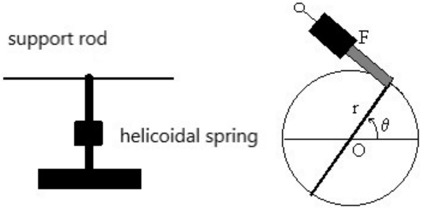


Figure: Torsional Pendulum

Objectives

- More specifically we want to:
  1. Use a rod with two masses of variable distance to estimate the torsional coefficient of the pendulum.
  2. Use that estimation to determine the mass and moment of inertia of a disk, a very effective way to check the validity of our
  3. Use the torsional coefficient to directly calculate the moment of inertia for a variety of different objects (A Solid Cylinder, a Hollow Cylinder, a Solid Wooden Disk and a Solid Sphere)

What is a moment of inertia and how do we calculate it?

- It encodes information about the mass (distribution) and the geometry and is used to describe rotations.
- For our purposes, we are not referring to the Inertia Tensor, but only to one of its components along a principal axis.

Definitions

Operational:

$$\boldsymbol{\Gamma}_{\text{net}} = I\ddot{\phi}\boldsymbol{k}$$

Theoretical:

$$I = \int_{\Omega} r^2 dm$$

Using the theoretical definition

- While it is beyond the point of this presentation to do the explicit integrals and calculations, the results (that can be found in any introductory physics textbook) will still be important to give us something to compare our experimental results to:

	Solid Cylinder	Hollow Cylinder	Disk	Solid Sphere
I_{cm}	$\frac{1}{2}MR^2$	MR^2	$\frac{1}{2}MR^2$	$\frac{2}{5}MR^2$

Using the operational definition

The net torque given by our torsional pendulum will be:

$$-D\phi\boldsymbol{k} = \boldsymbol{\Gamma}_{\mathrm{net}}$$

Applying our definition, we get that:

$$\ddot{\phi} + \frac{D}{I}\phi = 0$$

Which is just a SHO of period:

$$T^2 = \frac{4\pi^2}{D} I$$

That is, we can calculate moment of inertia using the period of the oscillations of our pendulum if we know the torsional coefficient, D

Problem: We don't know the torsional coefficient D

Our strategy to circumvent this problem here will be to exploit a very well-known property of moments of inertia, known commonly as Steiner's Parallel Axis Theorem.

$$I' = I_{\text{cm}} + mR^2$$

Expressions for our moments of inertia

For each of our experimental setups, a different expression will make sense. For the first one, we will use:

$$T^2 = \frac{4\pi^2}{D}\left(I_r + 2I_c + 2m_c R^2\right)$$

For the second, it makes more sense to use:

$$T^2 = \frac{4\pi^2}{D}\left(I_{\mathrm{cm}} + mr^2\right)$$

Finally, for last chunk of this experiment, we will use the values we calculated for D in the first sections to directly apply the formula:

$$T^2 = \frac{4\pi^2}{D} I$$

Basic Experimental Outline

1. Estimating the torsional coefficient

   We use one rod and two cylindrical masses that we can move and measure its period, we measure the period for different positions of the masses. With a linear regression, we calculate the torsional coefficient.

2. The punctured disk

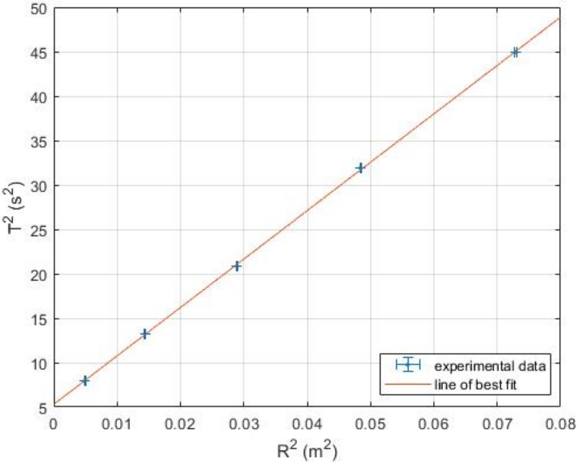
   We get a punctured disk and measure the period of its oscillations for different displacements from the center of mass. With a linear regression, we can calculate its mass and moment of inertia (corresponding to the different parameters of our regression).

3. The rest of the objects

   We measure their period of oscillation. With a direct application of the basic formula, we calculate the torsional coefficient.

Results: Torsional Coefficient

Figure: Square period of oscillation as a function of the square of the distance of the masses and the center of mass.



Results: Torsional Coefficient

$$a_1 = (544 \pm 3) \quad s^2 m^{-2}$$
$$b_1 = (5.30 \pm 0.14) \quad s^2$$

Now, if we compare them with our original equations,

$$a = \frac{4\pi^2 2m_c}{D}$$

That is,

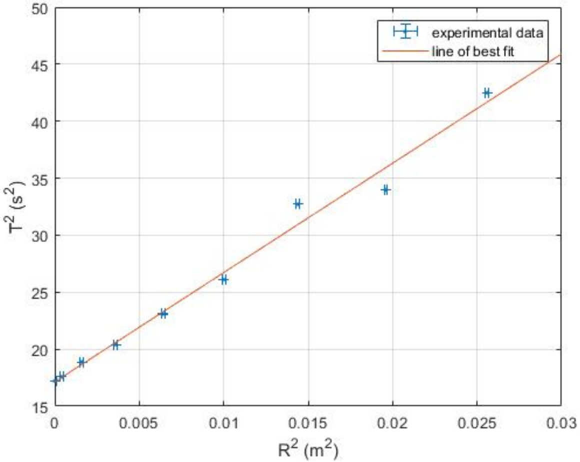
$$D = \frac{4\pi^2 2m_c}{a} = 3.45 \pm 0.04 \cdot 10^{-2} \quad \text{N m}$$

It is simple to turn this into a 95% confidence interval, since it has $\nu = 6 - 2 = 4$ DOF, which we can easily look for in a table of the coverage factor. We get that:

$$D_{95\%} = 0.0345 \pm 0.0012 \quad \text{N m}$$

Results: Disk

Figure: Square period of oscillation as a function of the square of the distance from the axis of rotation to the center of mass.



Results: Disk

$$a_2 = (959 \pm 41) \quad s^2 m^{-2}$$
$$b_2 = (17 \pm 0.52) \quad s^2$$

By direct comparison, we get:

$$a = \frac{4\pi^2 m_d}{D}$$
$$b = \frac{4\pi^2 I_{\mathrm{cm}}}{D}$$

That is;

$$m_d = \frac{aD}{4\pi^2} = 0.84 \pm 0.05 \quad \mathrm{kg}$$

$$I_{\mathrm{cm}} = \frac{bD}{4\pi^2} = 0.0141 \pm 0.0007 \quad \mathrm{kg\ m^2}$$

Analysis: Disk

To check if our result is valid, we can compare those results to the measured mass and theoretical moment of inertia:

Table: Comparison between the theoretical and experimental masses and moments of inertia for the disk.

	Theoretical	Experimental
m (kg)	0.69	0.84 ± 0.05
I_{cm} (kg m^2)	0.0139	0.0141 ± 0.0007

Results and Analysis: Various Objects

With this one, it makes the most sense to display and compare the results at the same time:

Table: The experimental and theoretical moments of inertia for all the objects.

	Solid Cylinder	Hollow Cylinder	Wooden disk	Sphere
I_{exp} (kg m^2)	$(6.3 \pm 0.4) \cdot 10^{-4}$	$(1.1 \pm 0.5) \cdot 10^{-3}$	$(2.2 \pm 0.1) \cdot 10^{-3}$	$(2.17 \pm 0.08) \cdot 10^{-3}$
I_{theo} (kg m^2)	$6.8 \cdot 10^{-4}$	$0.80 \cdot 10^{-3}$	$1.8 \cdot 10^{-3}$	$2.14 \cdot 10^{-4}$

Final Discussion

- While the results themselves were not groundbreaking (or even useful), they were close enough to the geometrical one to draw some heuristics about measurements in general:
  - Having more two unknown variables in a relation does not always mean you have to do more measurements. In this, we were able to decouple the equations by use of Steiner's Theorem.
  - Having a variable you want to measure as (part of) a parameter in linear regression yields more precise results than having it directly related to a measured variable.
  - It is not always obvious when this is possible. In particular, this problem required some creativity with regards to Steiner's Theorem make it so.

Bibliography

Herbert Goldstein, Charles P. Poole, and John L. Safko. *Classical mechanics*. Pearson, 2014

Hugh D Young. *Sears and Zemansky's university physics*. 12th ed. Pearson Addison-Wesley, 2008