A VARIATIONAL QUANTUM ALGORITHM FOR THE BLACK-SCHOLES PDE WITH STOCHASTIC VOLATILITY

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1. Abstract

We assess the use of Variational Quantum Imaginary Time Evolution(VarQITE) in pricing the European Option in the Black-Scholes model with stochastic volatility. We give the full description of the decomposition of the PDE's infinitesimal generator to quantum gates in the polynomial order, make it implementable on Noisy Intermediate-Scale Quantum (NISQ) devices. This work highlights the potential of variational quantum algorithms for pricing options with more complex modelling.

2. The Hull-White model

For an European Option with volatility $\sqrt{y_t}$, and an underlying stock with price x_t, we have two stochastic processes:

$$dx_t = \mu x_t dt + \sqrt{y_t} x_t dW_1$$
$$dy_t = a y_t dt + b y_t dW_2$$

(ρ is correlation between dW_1 and dW_2) Let $u(x, y, t)$ be the value of the option on the underlying stock at time t. Then, it satisfies[HW87]

$$\frac{\partial u}{\partial t} + \left(rx\frac{\partial}{\partial x} + \frac{1}{2}x^2 y \frac{\partial^2}{\partial x^2} + b\rho x y^{\frac{3}{2}} \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} b^2 y^2 \frac{\partial^2}{\partial y^2} + ay\frac{\partial}{\partial y} - r \right) u = 0$$

We can express this in a Schrodinger-type equation. Let $a = b = r = 1$ and $\rho = 1/2$, we then apply $\tau = T - t$ to evolve backwards, and a wick rotation $\epsilon = -i\tau$ to get:

$$-i\frac{\partial u}{\partial \epsilon} = \widetilde{\mathfrak{G}} u \quad \equiv \quad -i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

$$\hat{H} = \widetilde{\mathfrak{G}} = \left(x\frac{\partial}{\partial x} + \frac{1}{2}x^2 y\frac{\partial^2}{\partial x^2} + \frac{1}{2}xy^{\frac{3}{2}}\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} - \mathbb{I} \right)$$

We use a variational method to evolve a trial wavefunction $|\phi(\boldsymbol{\theta}(t))\rangle$ under $\hat{H} = \sum_i \lambda_i h_i$, parameterised by $\boldsymbol{\theta}$ that approximates $|\psi\rangle$.

3. Variational Imaginary Time Evolution

By McLachlan's variational principle[FJO21], $\boldsymbol{\theta}$ satisfies the following ODEs:

$$A(t)\boldsymbol{\theta}(\mathbf{t})' = C(t)$$

With the matrix $A(t)$, vector $C(t)$:

$$A_{i,j}(t) = \Re\left(\frac{\partial \langle \phi(\boldsymbol{\theta}(t))|}{\partial \theta_i} \frac{\partial |\phi(\boldsymbol{\theta}(t))\rangle}{\partial \theta_j} \right)$$
$$C_i(t) = \Re\left(-\sum_i \lambda_i \frac{\partial \langle \phi(\boldsymbol{\theta}(t))|}{\partial \theta_i} h_i |\phi(\boldsymbol{\theta}(t))\rangle \right)$$

Where $A_{i,j}(t)$ and $C_i(t)$ are evaluated on a quantum circuit. We can then evolve the parameter $\boldsymbol{\theta}$ as $\boldsymbol{\theta}(t+\delta t) \approx \boldsymbol{\theta}(t) + A^{-1}(t)C(t)\delta t$ to update $|\phi(\boldsymbol{\theta}(t))\rangle$ at each time increment.

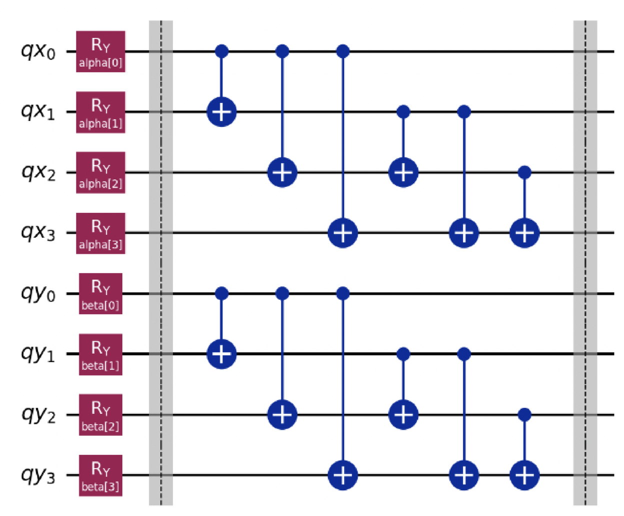

Figure 1: one block of ansatz in $|\phi(\boldsymbol{\theta}(t))\rangle$

4. Generator decomposition

To decompose $\hat{H}$ into $\sum_i \lambda_i h_i$, sum of unitaries, consider a 16×16 grid of qubit states, achieved by 8 qubits, with states $\{|0\rangle, |1\rangle, \cdots |2^8 - 1\rangle\}$. We embed u_{ij} into this grid, where u_{ji} corresponds to the probability amplitude of the state $|i, j\rangle$. We discretize $\hat{H} = \widetilde{\mathfrak{G}}$ as follows:

$$\widetilde{\mathfrak{G}} = \sum_{i_1=0}^{2^4-1}\sum_{i_2=0}^{2^4-1}\sum_{j_1=0}^{2^4-1}\sum_{j_2=0}^{2^4-1}[\widetilde{\mathfrak{G}}]_{(i_1,i_2)\times(j_1,j_2)}|i_1,i_2\rangle\langle j_1,j_2|$$

By the finite difference method, we can approximate the derivative operators[Kub+21], where $\widetilde{\mathfrak{G}} =$

$$\begin{aligned}
&\sum_{j_1=0}^{2^4-1}\sum_{j_2=0}^{2^4-1}\left(\frac{x^4y^2}{\Delta x^2} + \frac{x^2y^3+y^4}{\Delta y^2} - 1\right)|j_1,j_2\rangle\langle j_1,j_2| + \sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2}\left(\frac{1}{2}\frac{x^4y^2}{\Delta x^2} - \frac{1}{2}\frac{x}{\Delta x}\right)|j_1+1,j_2\rangle\langle j_1,j_2| \\
&+ \sum_{j_1=1}^{2^4-1}\sum_{j_2=1}^{2^4-1}\left(\frac{1}{2}\frac{x^4y^2}{\Delta x^2} + \frac{1}{2}\frac{x}{\Delta x}\right)|j_1-1,j_2\rangle\langle j_1,j_2| + \sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2}\left(\frac{1}{2}\frac{x^2y^3+y^4}{\Delta y^2} - \frac{1}{2}\frac{y}{\Delta y}\right)|j_1,j_2+1\rangle\langle j_1,j_2| \\
&+ \sum_{j_1=1}^{2^4-1}\sum_{j_2=1}^{2^4-1}\left(\frac{1}{2}\frac{x^2y^3+y^4}{\Delta y^2} + \frac{1}{2}\frac{y}{\Delta y}\right)|j_1,j_2-1\rangle\langle j_1,j_2| + \sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2}\left(\frac{1}{4}\frac{x^3y^{5/2}}{\Delta x\Delta y}\right)|j_1\pm 1,j_2\pm 1\rangle\langle j_1,j_2| \\
&\approx -\mathbb{I}^{\otimes 8} + \frac{1}{2}[V_-(4) - V_+(4)]D(4)\otimes\mathbb{I}^{\otimes 4} + \mathbb{I}^{\otimes 4}\otimes\frac{1}{2}[V_-(4) - V_+(4)]D(4)
\end{aligned}$$

Where $V_+(n)$, $V_-(n)$, $D(n)$ are composed of the following unitary operators:

$$V_+(n) = \sum_{i=0}^{2^n-2}|i+1\rangle\langle i| = \text{CycInc}(n)\frac{1}{2}(C^{n-1}Z + I^{\otimes n}) \quad V_-(n) = \sum_{i=1}^{2^n-1}|i-1\rangle\langle i| = \frac{1}{2}(C^{n-1}Z + I^{\otimes n})\text{CycDec}(n)$$

$$D(n) = \sum_{i=0}^{2^n-1} i|i\rangle\langle i| = \frac{2^n-1}{2}I^{\otimes n} - \sum_{i=1}^{n}2^{n-i-1}Z_i$$

$C^{n-1}Z$ is a (n-1)-qubit controlled Pauli Z gate, I is the identity gate, and Z_i is the Pauli Z gate acting on the i-th qubit.

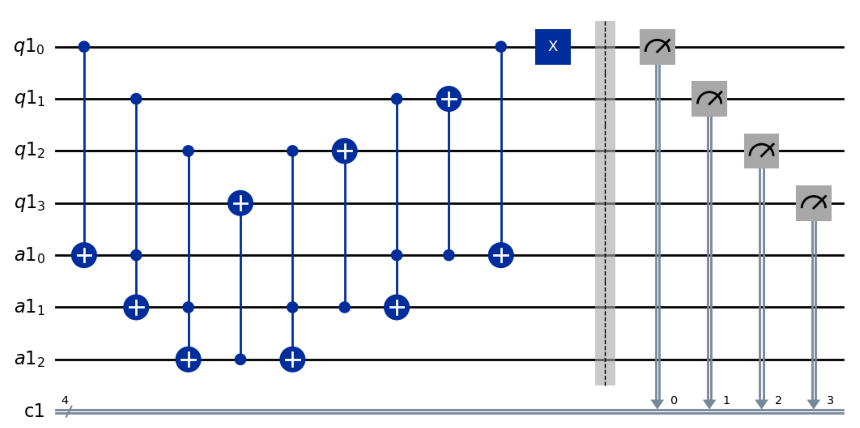

Figure 2: CycInc(4)

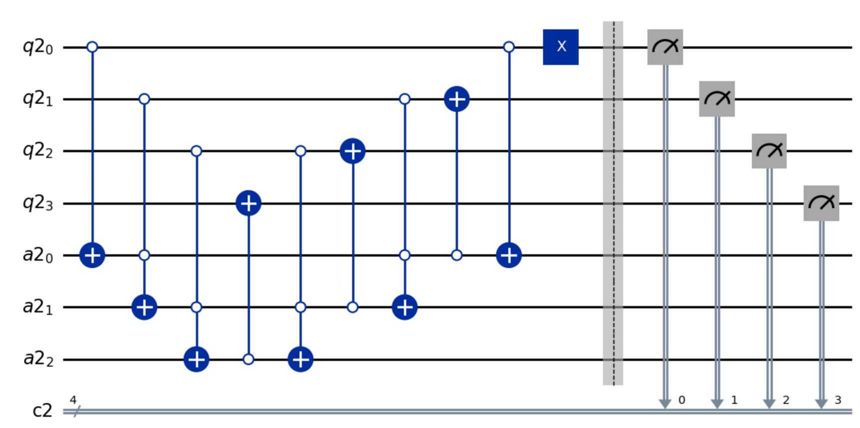

Figure 3: CycDec(4)

5. Numerical Simulation

We set the strike price $K = 1/2$, and $0 \le y \le 1$, $0 \le x \le 1$, $T = 0.1$ second, and $\delta t = 0.001$ second. From a visual comparison, the simulation doesn't match existing results in literature that well[FJO21] mostly due to our approximation of the generator up to second order. In future investigations, we can use higher-order approximation and more qubits for a more accurate, higher resolution result.

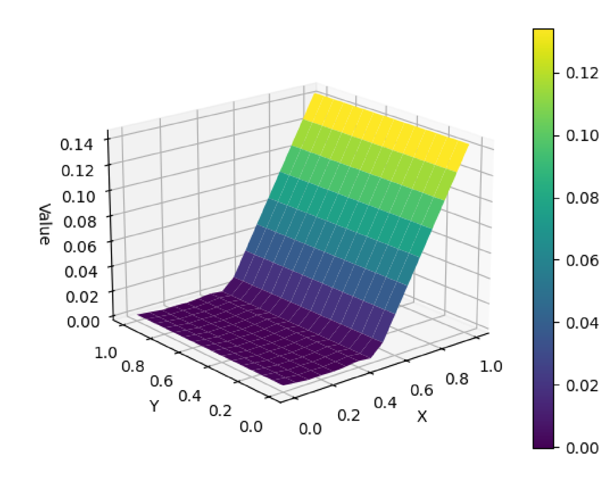

Figure 4: $u(x, y, 0)$

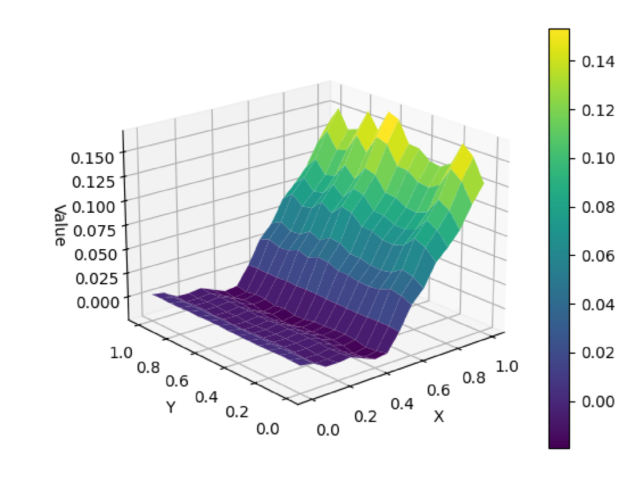

Figure 5: $u(x, y, 0.05)$

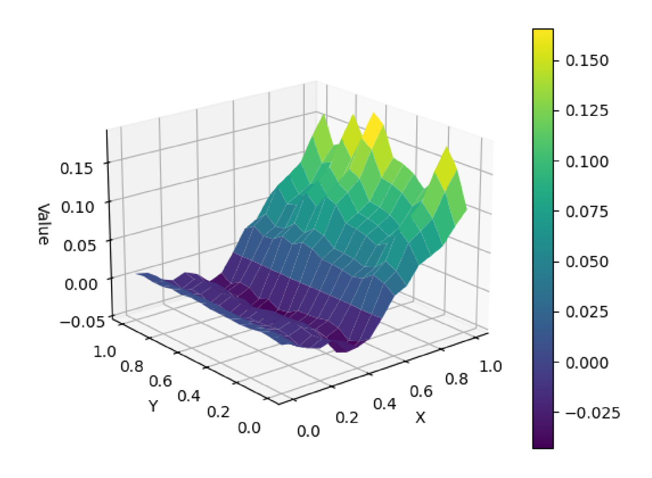

Figure 6: $u(x, y, 0.1)$

References

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[FJO21] Filipe Fontanela, Antoine Jacquier, and Mugad Oumgari. *A Quantum algorithm for linear PDEs arising in Finance.* 2021.

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