Reading project on solitons, instantons and twistors

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These are notes taken while reading the book: Solitons, Instantons, and Twistors by Maciej Dunajski. The note will mainly contain derivations that I find important, illuminating examples, background knowledge for which I have gaps, some questions/insights that I make while reading, and answers to some exercises.

Contents

1 Action and the Lagrangian (Prerequisite)

For an n particle system, we want to study its dynamics, i.e. how the particles evolve and interact. To build a motivation for constructing a language to describe the evolution of such a system, consider first the scenario of a single particle.

TODO: Add some motivation

That the Lagrangian depends only on q and q', but not higher derivatives, is related to the fact that one expects the dynamics of a system to be determined uniquely once we specify q and q' at some initial time $t = t_1$.

Theorem 1.1 (Principle of stationary action (Hamilton's principle))**.** *For some fixed time interval* $[t_1, t_2]$*, Let* $\mathbf{q}(t) \in \mathbb{R}^n$ *be a parameterized path in the generalized coordinates. Then, the system will evolve from* $\mathbf{q}(t_1)$ *to* $\mathbf{q}(t_2)$ *such that the following quantity, know as action:*

$$\mathcal{S}[\mathbf{q}(t)] = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \mathbf{q}'(t), t)\, dt$$

is extremised.

Following Theorem 1.1, we extremise the action by finding a critical path function $\mathbf{q}(t)$ such that

$$\lim_{\delta\mathbf{q}(t)\to 0} \frac{\delta\mathcal{S}}{\delta\mathbf{q}(t)} = \lim_{\delta\mathbf{q}(t)\to 0} \frac{\mathcal{S}[\mathbf{q}(t)] - \mathcal{S}[\mathbf{q}(t) + \delta\mathbf{q}(t)]}{\delta\mathbf{q}(t)} = 0$$

where $\delta\mathbf{q}(t)$ is a small variation in $\mathbf{q}(t)$, with the boundary condition $\delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = 0$. We then arrive at the following theorem, which reformulates the critical condition in terms of a system of PDEs:

Theorem 1.2 (Euler-Lagrange equation)**.** *The critical path* $\mathbf{q}(t)$ *which extremises* $S[\mathbf{q}(t)]$ *satisfies*

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{q}'}\right) - \frac{\partial\mathcal{L}}{\partial\mathbf{q}} = 0$$

Proof. By considering first order variations, where $\delta\mathbf{q}(t) = \epsilon\mu(t)$, and considering first order Taylor expansion of $\mathcal{L}$ around $\mathbf{q}(t)$, we have

$$\begin{aligned}
\mathcal{S}[\mathbf{q}(t) + \delta\mathbf{q}(t)] &= \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t) + \epsilon\eta(t), \mathbf{q}'(t) + \epsilon\eta'(t), t)\, dt \\
&= \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \mathbf{q}'(t), t) + \epsilon\eta\frac{\partial\mathcal{L}}{\partial\mathbf{q}} + \epsilon\eta'\frac{\partial\mathcal{L}}{\partial\mathbf{q}'}\, dt + \mathcal{O}(\epsilon^2) \\
&= \mathcal{S}[\mathbf{q}(t)] + \epsilon\int_{t_1}^{t_2} \eta\frac{\partial\mathcal{L}}{\partial\mathbf{q}} + \eta'\frac{\partial\mathcal{L}}{\partial\mathbf{q}'}\, dt + \mathcal{O}(\epsilon^2) \\
&= \mathcal{S}[\mathbf{q}(t)] + \epsilon\left(\int_{t_1}^{t_2} \eta\frac{\partial\mathcal{L}}{\partial\mathbf{q}}\, dt + \eta\frac{\partial\mathcal{L}}{\partial\mathbf{q}'}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{q}'(t)}\, dt\right) + \mathcal{O}(\epsilon^2) \\
&= \mathcal{S}[\mathbf{q}(t)] + \epsilon\int_{t_1}^{t_2} \eta\left(\frac{\partial\mathcal{L}}{\partial\mathbf{q}} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{q}'(t)}\right) dt + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Next, manipulating this expression, we get

$$\lim_{\delta\mathbf{q}(t)\to 0}\frac{\delta\mathcal{S}}{\delta\mathbf{q}(t)} = \lim_{\delta\mathbf{q}(t)\to 0}\frac{1}{\eta}\int_{t_1}^{t_2}\eta\left(\frac{\partial\mathcal{L}}{\partial\mathbf{q}} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{q}'(t)}\right)dt = 0$$
$$\iff \frac{\partial\mathcal{L}}{\partial\mathbf{q}} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{q}'(t)} = 0$$

by the fundamental theorem of calculus of variation. □

Remark. *This proof treats $\mathbf{q}'$ and $\mathbf{q}$ as independent variables.*

Exercise 1. *On the change of generalized coordinates, B7.1 Classical Mechanics: Sheet 1, Section B, Question 3.*

Solution 1. *Consider a coordinate transformation:*

$$\mathbf{q}\mapsto\mathbf{q}(\tilde{\mathbf{q}},t)$$

which results in a transformation of the Lagrangian:

$$\mathcal{L}(\mathbf{q},\mathbf{q}',t)\mapsto\mathcal{L}(\mathbf{q}(\tilde{\mathbf{q}},t),\mathbf{q}'(\tilde{\mathbf{q}},\tilde{\mathbf{q}}',t),t)$$

We name the transformed Lagrangian $\mathcal{L}^*(\tilde{\mathbf{q}},\tilde{\mathbf{q}}',t)$ to simplify the notation. By the chain rule, we first establish that:

$$q_a' = \frac{\partial q_a}{\partial\tilde{\mathbf{q}}}\frac{\partial\tilde{\mathbf{q}}}{\partial t} + \frac{\partial q_a}{\partial t} = \sum_{b=1}^{n}\frac{\partial q_a}{\partial\tilde{q}_b}\tilde{q}_b{}' + \frac{\partial q_a}{\partial t}$$

Next, we consider the Euler-Lagrange equation of $\mathcal{L}^*(\tilde{\mathbf{q}},\tilde{\mathbf{q}}',t)$:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}^*}{\partial\tilde{q}_a{}'}\right) - \frac{\partial\mathcal{L}^*}{\partial\tilde{q}_a} &= \frac{d}{dt}\left(\sum_{b=1}^{n}\frac{\partial\mathcal{L}}{\partial q_b'}\frac{\partial q_b'}{\partial\tilde{q}_a{}'}\right) - \sum_{b=1}^{n}\frac{\partial\mathcal{L}}{\partial q_b}\frac{\partial q_b}{\partial\tilde{q}_a} + \frac{\partial\mathcal{L}}{\partial q_b'}\frac{\partial q_b'}{\partial\tilde{q}_a}\\
&= \sum_{b=1}^{n}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial q_b'}\right)\frac{\partial q_b'}{\partial\tilde{q}_a{}'} + \frac{d}{dt}\left(\frac{\partial q_b'}{\partial\tilde{q}_a{}'}\right)\frac{\partial\mathcal{L}}{\partial q_b'} - \frac{\partial\mathcal{L}}{\partial q_b}\frac{\partial q_b}{\partial\tilde{q}_a} - \frac{\partial\mathcal{L}}{\partial q_b'}\frac{\partial q_b'}{\partial\tilde{q}_a}\\
&= \sum_{b=1}^{n}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial q_b'}\right)\frac{\partial q_b'}{\partial\tilde{q}_a{}'} - \frac{\partial\mathcal{L}}{\partial q_b}\frac{\partial q_b}{\partial\tilde{q}_a} + \frac{\partial\mathcal{L}}{\partial q_b'}\left(\frac{d}{dt}\left(\frac{\partial q_b'}{\partial\tilde{q}_a{}'}\right) - \frac{\partial q_b'}{\partial\tilde{q}_a}\right)
\end{aligned}$$

From what we previously established on q_a', we interchange a and b in the expression, and take partial derivative with respect to $\tilde{q}_a{}'$ to get $\dfrac{\partial q_b'}{\partial\tilde{q}_a{}'} = \dfrac{\partial q_b}{\partial\tilde{q}_a}$. Substitute back, we get

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}^*}{\partial\tilde{q}_a{}'}\right) - \frac{\partial\mathcal{L}^*}{\partial\tilde{q}_a} &= \sum_{b=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial q_b'}\right) - \frac{\partial\mathcal{L}}{\partial q_b}\right]\frac{\partial q_b}{\partial\tilde{q}_a} + \frac{\partial\mathcal{L}}{\partial q_b'}\left(\frac{\partial}{\partial\tilde{q}_a}\left(\frac{dq_b}{dt}\right) - \frac{\partial q_b'}{\partial\tilde{q}_a}\right)\\
&= \sum_{b=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial q_b'}\right) - \frac{\partial\mathcal{L}}{\partial q_b}\right]\frac{\partial q_b}{\partial\tilde{q}_a}
\end{aligned}$$

as required. since for a (non-singular) coordinate transformation the Jacobian matrix $\dfrac{\partial q_b}{\partial\tilde{q}_a}$ is invertible, we must have

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}^*}{\partial\tilde{q}_a{}'}\right) - \frac{\partial\mathcal{L}^*}{\partial\tilde{q}_a} = 0 \iff \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial q_b'}\right) - \frac{\partial\mathcal{L}}{\partial q_b} = 0$$

Definition 1.1 (Conserved quantity). *A conserved quantity is a function $F(\mathbf{q}, \mathbf{q}', t)$ that is constant along solutions to the Euler-Lagrange equations, where*

$$0 = \frac{d}{dt} F(\mathbf{q}(t), \mathbf{q}'(t), t)$$

Definition 1.2 (Generator of infinitesimal deformation). *A generator of infinitesimal deformation is a function $\rho = \rho(\mathbf{q}, \mathbf{q}', t)$ which "generates" first order variation to an arbitrary path $\mathbf{q}(t)$, where*

$$\mathbf{q}^*(t) = \mathbf{q}(t) + \delta\mathbf{q}(t)$$

$$\delta\mathbf{q}(t) = \epsilon\boldsymbol{\rho}(\mathbf{q}, \mathbf{q}', t) = \epsilon\boldsymbol{\rho}(t)$$

Remark (Extension to vector fields). *A more general perspective is to treat $\boldsymbol{\rho}$ as a vector field X acting on the paths. Let $\mathcal{Q}$ denote the configuration space in which the system evolves. If we can describe a particle in $\mathbb{R}^3$, for an n particle system, we have $\mathcal{Q} \subseteq \mathbb{R}^{3n}$. The tangent bundle $T\mathcal{Q}$ is then the union of all tangent spaces $T_q\mathcal{Q}$ where $\mathbf{q} \in \mathcal{Q}$:*

$$T\mathcal{Q} = \bigcup_{\mathbf{q}\in\mathcal{Q}} T_q\mathcal{Q}$$

One can show that the Tangent bundle is indeed a differentiable structure. Then, the generator of infinitesimal deformation

$$X : \ \mathcal{Q} \times \mathbb{R} \to T\mathcal{Q}$$

where

$$X(q,t) = \sum_i y_i(q) \left(\frac{\partial}{\partial q_i}\right)_q + \frac{\partial}{\partial t}$$

is a smooth map that assigns a tangent vector to each point on the configuration space (plus time dimension) that indicates the direction in which a path should be deformed (If ρ is a also function of $\mathbf{q}'$, we map from $T\mathcal{Q} \times \mathbb{R}$ instead). Thus, the infinitesimal deformation of a path can be equivalently expressed as

$$\left.\frac{d\mathbf{q}(t)}{d\epsilon}\right|_{\epsilon=0} = X\Big|_{\mathbf{q}(t),t}$$

Remark (Exponentiation of operator). *We can uncover more intuition by looking at why $\boldsymbol{\rho}(q,t) = X(q,t)$ is named the "generator" of infinitesimal deformation.*

Indeed, it is intimately connected to the infinitesimal generator of continuous symmetry, and the idea of flow under a vector field. Suppose there exist some arbitrary operator $A(\epsilon)$ acting on $\mathbf{q}(t)$, where $A(\epsilon) : \mathcal{Q} \times \mathbb{R} \to \mathcal{Q}$ is a one-parameter family of diffeomorphisms:

$$A(\epsilon) : \ (\mathbf{q}(t), \epsilon) \mapsto A(\epsilon)\mathbf{q}(t)$$

After we take the first order Taylor expansion of $A(\epsilon)$ at $\epsilon = 0$, we get

$$A(\epsilon) = 1 + \epsilon A'(0) + O(\epsilon^2)$$

when ϵ is infinitesimal (where $\epsilon \to 0$), and $A'(0)$ the infinitesimal generator, in definition. This gives us

$$\begin{aligned} \mathbf{q}^*(t) &= (1 + \epsilon A'(0))\mathbf{q}(t) \\ &= \mathbf{q}(t) + \epsilon A'(0)\mathbf{q}(t) \end{aligned}$$

Comparing with Definition 1.2 , we identify $\boldsymbol{\rho}(\mathbf{q}, t) = X\Big|_{\mathbf{q}(t),t} = A'(0)\mathbf{q}(t)$.

Connecting back to the vector field formulation in our previous remark, and in line with the notation of diffeomorphisms, we write $\mathbf{q}^*(t) = \phi_\epsilon(\mathbf{q}(t))$ to get

$$\begin{cases} \dfrac{d}{d\epsilon}\phi_\epsilon(\mathbf{q}(t)) = A'(0)\phi_\epsilon(\mathbf{q}(t)) \\ \quad \phi_0(\mathbf{q}(t)) = \mathbf{q}(t) \end{cases}$$

Solving for $\mathbf{q}(t)$, we have

$$\begin{aligned} &\int \frac{1}{\phi_\epsilon(\mathbf{q}(t))} d\phi_\epsilon(\mathbf{q}(t)) = \int A'(0) d\epsilon \\ &\Leftrightarrow \ln(\phi_\epsilon(\mathbf{q}(t))) = A'(0)\epsilon + C \\ &\Leftrightarrow \phi_\epsilon(\mathbf{q}(t)) = e^{\epsilon A'(0)}\mathbf{q}(t) \end{aligned}$$

Where the exponential $e^{\epsilon A'(0)}$ represents the flow generated by the vector field, infinitesimally deforming $\mathbf{q}(t)$. The $A'(0)$ does not depend explicitly on time t, i.e. time-independent, which indicates a global symmetry, where the same transformation is applied at every moment.

Remark (Lie Groups and Their Action on Paths)**.** *We can further extend this idea into the realms of group theory, which is robust in describing symmetries.*

A Lie group G is a group that is also a smooth manifold. Continuous symmetries like rotation or translation is modelled by how G acts on the configuration space via

$$\mathbf{q}(t) \mapsto \mathbf{q}^*(t) = g \cdot \mathbf{q}(t), \quad g \in G.$$

The diffeomorphism we introduced earlier is now a one-parameter subgroup $\{g(\epsilon)\}_{\epsilon \in \mathbb{R}}$ satisfing $g(0) = e$ (the identity) and

$$g(\epsilon) = e + \epsilon\, A'(0) + O(\epsilon^2).$$

The Lie algebra $\mathfrak{g}$ of G is the tangent space at the identity e and its elements are the infinitesimal generators *of the group action. The exponential map* $\exp : \mathfrak{g} \to G$ *sends an element* $X \in \mathfrak{g}$ *to a group element via*

$$\exp(\epsilon X) = 1 + \epsilon X + \frac{\epsilon^2}{2!} X^2 + \cdots .$$

Definition 1.3 (Infinitesimal symmetry)**.** *ρ generates an infinitesimal symmetry of $\mathcal{L}$ if there exists a function $f(\mathbf{q}(t), \mathbf{q}'(t), t)$ such that for all paths $\mathbf{q}(t)$, we have*

$$\frac{\partial}{\partial \epsilon} \mathcal{L}(\mathbf{q}^*(t), \mathbf{q}^{*\prime}(t), t) \bigg|_{\epsilon=0} = \frac{d}{dt} f(\mathbf{q}(t), \mathbf{q}'(t), t)$$

In the language of variational calculus, the left hand side is equivalent to $\frac{\delta \mathcal{L}}{\epsilon}$, where in other words, this expression says the first order variation in the Lagrangian is a total time derivative.

To see this, note that by definition,

$$\begin{aligned} \delta \mathcal{L} &= \mathcal{L}(\mathbf{q}(t) + \delta \mathbf{q}(t), (\mathbf{q}(t) + \delta \mathbf{q}(t))', t) - \mathcal{L}((\mathbf{q}(t), \mathbf{q}'(t), t)) \\ &= \mathcal{L}(\mathbf{q}(t) + \epsilon \rho(t), (\mathbf{q}(t) + \epsilon \rho(t))', t) - \mathcal{L}((\mathbf{q}(t), \mathbf{q}'(t), t)) \end{aligned}$$

Thus we have

$$\begin{aligned} \lim_{\delta \mathbf{q}(t) \to 0} \frac{\delta \mathcal{L}}{\epsilon} &= \rho(t) \lim_{\delta \mathbf{q}(t) \to 0} \frac{\mathcal{L}(\mathbf{q}(t) + \epsilon \rho(t), (\mathbf{q}(t) + \epsilon \rho(t))', t) - \mathcal{L}((\mathbf{q}(t), \mathbf{q}'(t), t))}{\epsilon \rho(t)} \\ &= \frac{\partial \delta \mathbf{q}(t)}{\partial \epsilon} \cdot \frac{\partial \mathcal{L}}{\partial \delta \mathbf{q}(t)} \bigg|_{\delta \mathbf{q}(t)=0} = \frac{\partial \mathcal{L}}{\partial \epsilon} \bigg|_{\epsilon=0} \end{aligned}$$

Remark. *An alternative formulation of infinitesimal/continuous symmetry is via one-parameter family of maps, where we consider*

$$\mathbf{q}(t) \mapsto \mathbf{Q}(s, t)$$

such that $\mathbf{Q}(0, t) = \mathbf{q}(t)$, and $\mathbf{q}(t)$ is continuously deformed along $\mathbf{Q}(s, t)$. This transformation is a continuous symmetry w.r.t. to $\mathcal{L}(\mathbf{q}(t), \mathbf{q}'(t), t)$ if

$$\frac{\partial}{\partial s} \mathcal{L}(\mathbf{Q}(s, t), \mathbf{Q}'(s, t), t) \bigg|_{s=0} = 0$$

This perhaps paints a more lucid picture of the idea of symmetry: The Lagrangian is conserved under a specific deformation of the path, which makes the family of the states of deformation akin to a "symmetric group".

Theorem 1.3 (Noether's Theorem)**.** *Suppose we have a Lagrangian $\mathcal{L}$ that exhibits infinitesimal symmetry. Then*

$$F = F((\mathbf{q}(t), \mathbf{q}'(t), t)) = \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial q_i'} \rho_i - f(\mathbf{q}(t), \mathbf{q}'(t), t)$$

is a conserved quantity.

Proof. By direct calculation, we get

$$\frac{dF}{dt} = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial q_i'}\right)\rho_i + \frac{\partial \mathcal{L}}{\partial q_i'}\rho_{i'} - \frac{d}{dt} f(\mathbf{q}(t), \mathbf{q}'(t), t) = 0$$

□

Remark. *From the previous alternative formulation, the conserved quantity F is equivelant to*

$$\sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial q_i}\right)\left(\frac{\partial Q_i}{\partial s}\right)\Bigg|_{s=0}$$

This is because by noticing that

$$\begin{aligned}
0 = \frac{\partial}{\partial s}\mathcal{L}(\mathbf{Q}(s,t), \mathbf{Q}'(s,t), t)\Big|_{s=0} &= \sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial Q_i}\frac{\partial Q_i}{\partial s}\Big|_{s=0} + \frac{\partial \mathcal{L}}{\partial Q_i'}\frac{\partial Q_i'}{\partial s}\Big|_{s=0}\right) \\
&= \sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial Q_i'}\right)\frac{\partial Q_i}{\partial s}\Big|_{s=0} + \frac{\partial \mathcal{L}}{\partial Q_i'}\frac{d}{dt}\left(\frac{\partial Q_i}{\partial s}\right)\Big|_{s=0}\right] \\
&= \sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial Q_i'}\frac{\partial Q_i}{\partial s}\Big|_{s=0}\right) = \frac{d}{dt}\sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial q_i}\right)\left(\frac{\partial Q_i}{\partial s}\right)\Big|_{s=0}
\end{aligned}$$

i.e. $\sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial q_i}\right)\left(\frac{\partial Q_i}{\partial s}\right)\Big|_{s=0}$ *is constant over time.*

2 Hamiltonian formalism (Chapter 1)

2.1 Symmetry to Lagrangian

In the Lagrangian formulation, the coordinates q and velocities $v = q'$ are treated quite differently. For a larger class of coordinate transformations known as canonical transformations, which occurs on the tangent bundle $T\mathcal{Q}$, the form of the Euler-Lagrange equations are not preserved. The Hamiltonian formulation rewrites the dynamical equations so that the variables appear on a more equal footing, and more importantly, gives us a set of equations that is invariant under canonical transformations.

Some discussion on commutative algebra, canonical quantization vs path integral quantization

Definition 2.1 (Hamiltonian)**.** *We define the Hamiltonian $\mathcal{H}(t, \mathbf{q}, \mathbf{p})$ to be the Legendre transformation of the Lagrangian. This gives us*

$$\mathcal{H}(t, \mathbf{q}, \mathbf{p}) = \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial q_i'} q_i' - \mathcal{L}(t, \mathbf{q}, \mathbf{q}') = \sum_{i=1}^{n} p_i q_i' - \mathcal{L}(t, \mathbf{q}, \mathbf{q}')$$

where

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}'} \iff p_i = \frac{\partial \mathcal{L}}{\partial q_i'}$$

The Legendre Transform essentially converts a function $F(x)$ to a function $G(x')$, with gradient as the input space, with a geometric meaning of $G(x)$ being the y-intercept of the tangent line at x. Intuitively, this has great implication in physics, as we can for example, provide an alternative functional description of a system in via its conjugate quantity, such as providing a description in momentum, when given a description in position.

Theorem 2.1 (Hamiltonian's Equation of Motion)**.** *The Euler-Lagrange Equation are equivalant to the following set of equations expressed via the Hamiltonian:*

$$\frac{\partial \mathcal{H}}{\partial \mathbf{p}} = \frac{d\mathbf{q}}{dt}$$
$$\frac{\partial \mathcal{H}}{\partial \mathbf{q}} = -\frac{d\mathbf{p}}{dt}$$

Proof. Using Definition 2.1, we have

$$\mathcal{H}(t, \mathbf{q}, \mathbf{p}) + \mathcal{L}(t, \mathbf{q}, \mathbf{q}') = \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial q_i'} q_i' = \langle \mathbf{p}, \mathbf{q}' \rangle$$

And since $\mathbf{q}$, $\mathbf{q}'$ and $\mathbf{q}$, $\mathbf{p}$ are independent variable pairs, we have

$$\frac{\partial \mathcal{H}}{\partial \mathbf{p}} = \mathbf{q}' = \frac{d\mathbf{q}}{dt}$$
$$\frac{\partial \mathcal{H}}{\partial \mathbf{q}} = -\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{q}'} = -\frac{d\mathbf{p}}{dt}$$

□

Using this set of equations of motion, we can give a further generalization on how observables(physical quantities like energy, momentum, etc.) evolve over time.

Theorem 2.2 (Evolution of an observable (Poisson Bracket))**.** *Let* $A(t, \mathbf{q}, \mathbf{p})$ *be an observable in generalized coordinates. Then, the evolution of* A *is governed by*

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, \mathcal{H}\}$$

where

$$\{A, \mathcal{H}\} = \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i}\right)$$

Proof. By chain rule, the total differential of the observable is given as the following:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i} q_i' + \frac{\partial A}{\partial p_i} p_i'\right)$$

By Theorem 2.1, this is equivalent to

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i}\right) = \frac{\partial A}{\partial t} + \{A, \mathcal{H}\}$$

□

Note that for an observable that is a constant of motion, i.e. invariant in time translations, we must have $\{A, \mathcal{H}\} = 0$. Furthermore, the poisson bracket operation is bilinear, antisymmetric, and satisfies the Jacobi identity, thus providing the observables with the structure of a Lie algebra.

For an infinite dimensional system, one could extend the poisson braket to something called a poisson structure.

2.2 Integrability in classical mechanics

The first integrals of a Hamiltonian system are frequently obtained from Noether's theorem: If the Lagrangian is invariant with respect to some continuous group of transformations, then the corresponding Hamiltonian system has first integrals of a certain type.

Theorem 2.3 (Arnold–Liouville theorem)**.**

A canonical transformation (where the poisson braket is invariant) that is particularly useful is called the action variables, where finding the first integral of the Hamiltonian in these variables gives us a lot more physical intuition for periodic motions. Specifically, in cases where we are only interested in the frequency of some motion and not it's trajectory, the action variables become immensely helpful.

Definition 2.2 (Action variables)**.** *Consider an integrable Hamiltonian system with* n *degrees of freedom, i.e.* $\dim(\mathcal{Q}) = n$*, and a* $2n$*-dimensional phase space* $M = T\mathcal{Q}$*. There exists a canonical transformation to new variables* $(I_1, \dots, I_n, \theta_1, \dots, \theta_n)$*, where the action variables* $I_1, \dots, I_n$ *are defined by*

$$I_i = \frac{1}{2\pi} \oint_{\gamma_i} p_i \, dq_i, \quad i = 1, 2, \dots, n,$$

γ_i is a closed path corresponding to the ith degree of freedom.

The conjugate angle angles $\theta_1, \ldots, \theta_n$ are periodic coordinates that parameterize the position on the invariant cycle defined by constant $I_1, \ldots, I_n$. In these coordinates, the Hamiltonian depends only on the actions:

$$H = H(I_1, \ldots, I_n),$$

and the equations of motion simplify to

$$\dot{I}_i = 0, \quad \dot{\theta}_i = \omega_i(I), \quad i = 1, \ldots, n.$$

Remark. *There is an elegant topological argument for the existence of such a canonical coordinate.*

Remark. *In the context of integrable systems, the action variables capture the "amount of motion" over one complete cycle in phase space.*

2.3 Sympletic Forms

Definition 2.3 (Sympletic two form)**.** *At each point $p \in M$, where M is a smooth manifold, the symplectic two-form restricted to p, $\omega|_p$ is an antisymmetric bilinear map:*

$$\omega|_p : T_pM \times T_pM \to \mathbb{R},$$

satisfying

$$\omega|_p(X, Y) = -\omega|_p(Y, X) \quad \textit{for all } X, Y \in T_pM.$$

A symplectic two-form ω is said to be *non-degenerate* if for every $p \in M$ and any vector $X \in T_pM$,

$$\omega|p(X, Y) = 0 \quad \text{for all } Y \in T_pM \quad \Longrightarrow \quad X = 0.$$

Equivalently, the map

$$\Phi_p : T_pM \to T_p^*M, \quad X \mapsto \omega|_p(X, \cdot)$$

is an isomorphism. Geometrically, this means that every nonzero tangent vector has a nontrivial pairing with at least one other tangent vector via ω.

Definition 2.4 (Sympletic manifold)**.** *A* symplectic manifold *is a smooth manifold M of dimension $2n$ equipped with a closed (i.e., $d\omega = 0$) and non-degenerate two-form ω.*

The symplectic manifold is what we use to model the phase space. For example, consider the cotangent bundle T^*Q of the configuration space Q. In canonical coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ on T^*Q, the standard symplectic form is given by

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

This encapsulates the geometric structure underlying Hamiltonian mechanics.

Given a smooth function $f : M \to \mathbb{R}$ (often the Hamiltonian), the *Hamiltonian vector field* X_f is uniquely defined by

$$X_f \lrcorner \omega = -df.$$

In canonical coordinates, express the vector field as

$$X_f = \sum_{i=1}^{n} \left(X_f^{q_i} \frac{\partial}{\partial q^i} + X_f^{p_i} \frac{\partial}{\partial p_i} \right).$$

The contraction of X_f with ω is given by

$$X_f \lrcorner \omega = \sum_{i=1}^{n} \left(X_f^{q_i}\, dp_i - X_f^{p_i}\, dq^i \right).$$

On the other hand, the differential of f is

$$df = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i}\, dq^i + \frac{\partial f}{\partial p_i}\, dp_i \right).$$

Setting

$$\sum_{i=1}^{n} \left(X_f^{q_i}\, dp_i - X_f^{p_i}\, dq^i \right) = -\sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i}\, dq^i + \frac{\partial f}{\partial p_i}\, dp_i \right),$$

and comparing coefficients, we obtain

$$X_f^{q_i} = -\frac{\partial f}{\partial p_i}, \quad X_f^{p_i} = \frac{\partial f}{\partial q^i}.$$

Thus, the Hamiltonian vector field is explicitly

$$X_f = \sum_{i=1}^{n} \left(-\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} + \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right).$$

We notice a striking similarity to poison brakets. In fact, The Poisson bracket of two functions $f, g \in C^\infty(M)$ is defined using Hamiltonian vector fields:

$$\{f, g\} = X_f(g).$$

In canonical coordinates, substituting the expression for X_f we obtain

$$\{f, g\} = \sum_{i=1}^{n} \left(-\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} + \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right).$$

Geometrically, this Poisson bracket measures the rate of change of the observable g along the flow generated by X_f; that is, it quantifies how g evolves under the dynamics induced by the Hamiltonian f.

3 Soliton equations and integrable systems(Chapter 2 & 3)

3.1 KdV equation and Sine-Gordon equation

Theorem 3.1 (KdV Equation)**.** *The KdV equation*

$$u_t - 6uu_x + u_{xxx} = 0$$

describe solitary non-linear waves which preserve their shape shape and speed during evolution (and interaction with other solitary waves) without dissipating or changing form.

Remark. *We can provide a physical derivation of the KdV. We start with a linear wave equation*

$$\Psi_{xx}(x,t) - \frac{1}{v^2}\Psi_{tt}(x,t) = 0$$

For a solution $u(x,t)$, we require

1. *$u(x,t) = u(x,-t)$ (no dissipation)*
2. *$u(x,t)^n$ terms are omitted (small amplitude of oscillation)*
3. *Constant group velocity (no dispersion).*

The solution is in the form of

$$\Psi_x(x,t) + \beta\Psi_{xxx}(x,t) + \frac{1}{v}\Psi_t(x,t) = 0$$

Definition 3.1 (Isometric Embedding)**.** *Let $U \subset \mathbb{R}^2$ be an open set equipped with a Riemannian metric*

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

An isometric embedding is a smooth map

$$X : U \to \mathbb{R}^3,$$

such that the pullback of the standard Euclidean metric by X is exactly ds^2. In other words, for any tangent vectors $\mathbf{v}, \mathbf{w} \in T_pU$,

$$\langle dX(\mathbf{v}), dX(\mathbf{w})\rangle_{\mathbb{R}^3} = ds^2(\mathbf{v}, \mathbf{w}).$$

When the surface is embedded in $\mathbb{R}^3$, the distances and angles measured on U via ds^2 are preserved.

Theorem 3.2 (Sine Gordon Equation)**.** *The Sine-Gordon equation*

$$\phi_{xx} - \phi_{tt} = \sin\phi.$$

locally describes the isometric embeddings of surfaces with constant negative Gaussian curvature in the Euclidean space $\mathbb{R}^3$

Proof. Consider a surface with constant Gaussian curvature $K = -1$. When the surface is parametrized in asymptotic coordinates (ρ, τ), the first fundamental form can be written as

$$ds^2 = d\tau^2 + 2\cos\phi\, d\rho\, d\tau + d\rho^2,$$

where $\phi = \phi(\rho, \tau)$ is the angle between the asymptotic directions.

For a surface with $K = -1$, the Gauss equation forces the function ϕ to satisfy

$$\phi_{\rho\tau} = \sin\phi.$$

This is the sine–Gordon equation in the coordinates (ρ, τ).

Next, we introduce the change of variables

$$x = \tau + \rho, \quad t = \tau - \rho.$$

Then the partial derivatives transform as follows:

$$\frac{\partial}{\partial\rho} = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial\tau} = \frac{\partial}{\partial x} - \frac{\partial}{\partial t}.$$

Thus, the mixed derivative becomes:

$$\phi_{\rho\tau} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)\phi = \phi_{xx} - \phi_{tt}.$$

So, the compatibility condition (which ensures that the metric corresponds to a surface with $K = -1$) transforms into:

$$\phi_{xx} - \phi_{tt} = \sin\phi.$$

□

3.2 The set-up: Scattering problem

An interesting way of solving soliton equations is by encoding it in a different physical problem, i.e. quantum mechanical wave systems. Consider the following 1-dimensional time-independent Schrodinger equation:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\Psi(x) = E\Psi(x)$$

The strategy is to encode the initial data of the soliton equation as the potential and consider a beam of free particles incident on that potential. By analyzing how these free particles scatter under this potential, we can recover information on the potential itself.

We use the following diagram to describe it formally:

$$
\begin{tikzcd}
u(x,0) \arrow[r, "\text{direct scattering}"] & S(\lambda,0) = \big(\{\kappa_n, C_j(0)\}_{j=1}^N,\, r(k,0),\, a(k,0)\big) \arrow[d, "\omega(k)\text{: dispersion relation via Lax pair}"] \\
u(x,t) & S(\lambda,t) = \big(\{\kappa_n, C_j(t)\}_{j=1}^N,\, r(k,t),\, a(k,t)\big) \arrow[l, "\text{inverse scattering}"]
\end{tikzcd}
$$

Given $u(x,0)$, we find the scattering data $S(\lambda,0)$ via scattering on $V(x) = u(x,0)$.

From the KdV equation, we find it's corresponding Lax pair, which is a decomposition of the evolution of $u(x,t)$ into space-dependent and time-dependent components under a compatibility condition. Using the time-dependent component, we evolve the scattering data $S(\lambda,0) \to S(\lambda,t)$.

We reconstruct $u(x,t)$ via $S(\lambda,t)$.

Remark. *There are theorems that ensure uniqueness of scattering and potential which we will not prove here.*

3.2.1 Direct Scattering at $t = 0$

To do this, we first investigate the direct scattering of particles described by simple waves on the soliton potential. Reformulate the Schrodinger equation by setting $-\frac{\hbar}{2m} = 1$ and $E = -k^2$, we get

$$L\phi = \left[\frac{d^2}{dx^2} + u(x,0)\right]\phi = -k^2\phi \tag{1}$$

We also assume that $u(x,0)$ approaches 0 "fast enough" at infinity, i.e. $|u(x)| \to 0$, and

$$\int_{\mathbb{R}} (1+|x|)|u(x,0)|dx < \infty \tag{2}$$

Remark. *In a sense, the Schrodinger Equation only provides a framework for scattering via a potential. The transmitted and reflected waves (scattering data) shall evolve according to the KdV equation and not Schrodinger Equation.*

At $|x| \to \infty$, (1) is equivalent to

$$\phi_{xx} + k^2\phi = 0$$

which has orthogonal solutions $\{e^{ikx}, e^{-ikx}\}$, which we interpret as waves traveling to the left and right, respectively.

Since this is an asymptotic behaviour at $\pm\infty$, it is natural to consider two sets of functions, the right basis

$\phi = \{\phi(x,k), \bar{\phi}(x,k)\}$ and the left basis $\psi = \{\psi(x,k), \bar{\psi}(x,k)\}$ to express solutions $+\infty$ and $-\infty$ respectively. We have

$$\begin{cases} \phi(x,k) \sim e^{-ikx} \\ \bar{\phi}(x,k) \sim e^{ikx} \end{cases} \quad \text{for } x \to -\infty \qquad \begin{cases} \psi(x,k) \sim e^{ikx} \\ \bar{\psi}(x,k) \sim e^{-ikx} \end{cases} \quad \text{for } x \to \infty \tag{3}$$

If we want to investigate the behaviour of a wave at $-\infty$ that is incoming from $+\infty$ and interacts with the potential $u(x,0)$, we expand ϕ in the basis of ψ, where

$$\phi(x,k) = a(k)\bar{\psi}(x,k) + b(k)\psi(x,k) \tag{4}$$

This expansion, along with the existing asymptotic given in (3), allows us to fully characterise the interaction of $\phi(x,k)$ with $u(x,0)$, where

$$\frac{\phi(x,k)}{a(k)} = \begin{cases} \dfrac{1}{a(k)}e^{-ikx} & \text{for } x \to -\infty \\ e^{-ikx} + \dfrac{b(k)}{a(k)}e^{ikx} & \text{for } x \to \infty \end{cases} \Leftrightarrow \tilde{\phi}(x,k) = \begin{cases} t(k)e^{-ikx} & \text{for } x \to -\infty \\ e^{-ikx} + r(k)e^{ikx} & \text{for } x \to \infty \end{cases}$$

This is chosen to match the physical set up of sending a beam of particles from $+\infty$, and going to $-\infty$ after intracting with the potential.

Lemma 3.3. *Given the transmission coefficient and reflection coefficient as $t(k) = \frac{1}{a(k)}$, $r(k) = \frac{b(k)}{a(k)}$, we have $\|t(k)\|^2 + \|r(k)\|^2 = 1$.*

Proof. Consider the following Wronskian with respect to x:

$$\begin{aligned} W(\phi(x,k), \bar{\phi}(x,k)) &= -a(k)\bar{a}(k)W(\psi(x,k), \bar{\psi}(x,k)) - a(k)\bar{b}(k)W(\psi(x,k), \psi(x,k)) \\ &\quad - b(k)\bar{a}(k)W(\bar{\psi}(x,k), \bar{\psi}(x,k)) + b(k)\bar{b}(k)W(\psi(x,k), \bar{\psi}(x,k)) \\ &= -[a(k)\bar{a}(k) - b(k)\bar{b}(k)]W(\psi(x,k), \bar{\psi}(x,k)) \end{aligned}$$

Also, for any f, g satisfying (1), we have

$$W_x(f,g) = fg_{xx} - gf_{xx} = (u+k^2)fg - (u+k^2)fg = 0$$

meaning $W(\phi(x,k), \bar{\phi}(x,k)) = C_1$, $W(\psi(x,k), \bar{\psi}(x,k)) = C_2$ for some constants C_1, C_2. Next, using the asymptotic behaviours of $\phi(x,k)$, $\psi(x,k)$ in (3), we have

$$W(\phi(x,k), \bar{\phi}(x,k)) = ik - (-ik) = 2ik$$

as $x \to -\infty$,

$$W(\psi(x,k), \bar{\psi}(x,k)) = -ik - (ik) = -2ik$$

as $x \to \infty$.

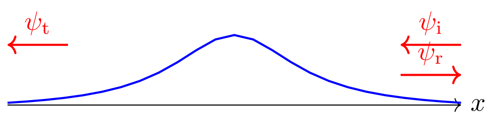
Substituting back, we get

$$b(k)\bar{b}(k) - a(k)\bar{a}(k) = -1$$

and equivalently

$$\|t(k)\|^2 + \|r(k)\|^2 = 1$$

$\square$



Soliton Potential $u(x)$

Next, we consider new sets of basis , $M = e^{ikx}\{\phi(x,k), \bar{\phi}(x,k)\}$ and $N = e^{ikx}\{\psi(x,k), \bar{\psi}(x,k)\}$, as these modified wavefunctions are easier to work with. Modifying (3), these satisfy the following asymptotic behaviors:

$$\begin{cases} M(x,k) \sim 1 \\ \bar{M}(x,k) \sim e^{2ikx} \end{cases} \quad \text{for } x \to -\infty \qquad \begin{cases} N(x,k) \sim e^{2ikx} \\ \bar{N}(x,k) \sim 1 \end{cases} \quad \text{for } x \to \infty \tag{5}$$

We also have the following useful relation:

Lemma 3.4. *The conjugates of the modified wave functions are related as follows:*

$$N(x,k) = \bar{N}(x,-k)e^{2ikx}$$

Proof. By (3), note that $\bar{\psi}(x,-k) \sim e^{-ikx}$ for $x \to -\infty$. Looking at the scattering problem (1),

$$L\psi(x,k) = k^2\psi(x,k) \implies L\bar{\psi}(x,k) = k^2\bar{\psi}(x,k) \implies L\bar{\psi}(x,-k) = (-k)^2\bar{\psi}(x,-k)$$

i.e. $\bar{\psi}(x,-k)$ is a solution to (1). since it's asymptotic behavior matches $\psi(x,k)$, in the basis of ψ we must have

$$\psi(x,k) = \bar{\psi}(x,-k)$$

This is equivalent to

$$N(x,k) = \bar{\psi}(x,-k)e^{-ikx}e^{2ikx} = \bar{N}(x,-k)e^{2ikx}$$

$\square$

Substituting the modified wave function into (4), and using Lemma 3.4, we arrive at our central equation describing the scattering behaviour.

$$\frac{M(x,k)}{a(k)} = \bar{N}(x,k) + r(k)e^{2ikx}\bar{N}(x,-k) \tag{6}$$

3.2.2 Scattering as a Riemann-Hilbert problem

The main reason we want to write the scattering problem in the form of (6), is to formulate it as a Riemann-Hilbert problem. This allows us to use powerful complex-analytical techniques.

To prove that (6) is indeed a Riemann-Hilbert problem, we first establish the following two lemmas:

Lemma 3.5. *M(x,k) can be analytically extended to* $\Im(k) > 0$ *and tends to unity as* $|k| \to \infty$*, while* $\bar{N}(x,k)$ *can be analytically extended to* $\Im(k) < 0$ *and tends to unity as* $|k| \to \infty$*.*

Proof. From (1), we make the substitution $v(x,k)$=$m(x,k)e^{-kx}$ to solve for the modified wavefunctions $M(x,k)$ and $N(x,k)$. Then $m(x,k)$ satisfies

$$\begin{aligned} m_{xx}(x,k) - 2ikm_x(x,k) &= -u(x)m(x,k) \quad \text{(N)} \\ m(x,k) &\to \mathbb{I} \text{ as} |k| \to \infty \quad \text{(BC)} \end{aligned} \tag{7}$$

Solving for $m(x,k)$ is equivalant to finding the Green's function $G(x,k)$ via the Delta function:

$$G_{xx}(x,k) - 2ikG_x(x,k) = -\delta(x) \tag{8}$$

Treating $-u(x)m(x,k)$ as the inhomogeneous part and using (BC), we can write out $m(x,k)$ as the follows:

$$m(x,k) = 1 + \int_{-\infty}^{\infty} G(x-\xi,k)u(\xi)m(\xi,k)d\xi \tag{9}$$

Apply the Fourier transform on (8), we get the following expression for $G(x,k)$:

$$\begin{aligned} (-p^2 + 2kp)\hat{G}(p,k) &= -1 \\ \iff \hat{G}(p,k) &= \frac{1}{p(p-2k)} \\ \iff G(x,k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ipx}}{p(p-2k)}dp \end{aligned}$$

Due to the poles at $p = 0$ and $p = 2k$, we consider ways to construct a contour that turns this into a residue problem, with the integral vanishing on most pieces except the real line. Specifically, we use C_+ and C_- as follows:

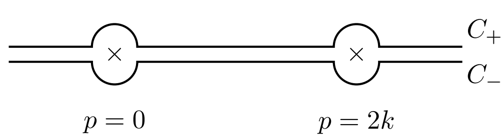


Figure 1: The contours C_+ and C_-.

And close $C+$ or $C-$ with a large semicircular arc $\Gamma_\pm$. Note that in order for the integral to vanish on $\Gamma_\pm$ for all x, we close the contour in the upper plane when $x > 0$, and close it in the lower plane when $x < 0$. This follows as Jordan's Lemma asserts

$$\int_{\Gamma_\pm} \frac{e^{ipx}}{p(p-2k)}dp \leq \frac{\pi}{x} \max_{\theta\in[0,2\pi]} \left| \frac{1}{Re^{i\theta}(Re^{i\theta}-2k)} \right|$$

Where the upper bound approaches 0 as $R \to \infty$. Below is an illustration of $\Gamma_\pm$ for the case of C_+:

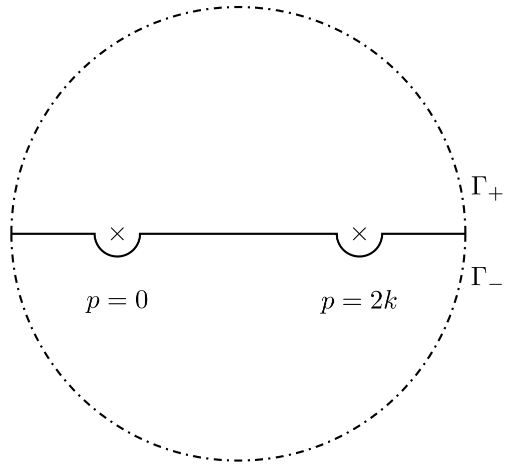


Figure 2: The contours $C_- \cup \Gamma_+$ and $C_- \cup \Gamma_-$.

Then, as $r \to 0$ for the inner arc and $R \to \infty$ for $\Gamma_\pm$ we have for $x > 0$:

$$\begin{aligned} G_+(x,k) &= \frac{1}{2\pi} \int_{C_-\cup\Gamma_+} \frac{e^{ipx}}{p(p-2k)}dp = i\sum_{i=1}^{2} \text{Res}\left(\frac{e^{ipx}}{p(p-2k)}, p = p_i \right) \\ &= i\left(\frac{-1}{2k} + \frac{e^{2ikx}}{2k} \right) = \frac{1}{2ik}(1 - e^{2ikx}) \end{aligned}$$

and for $x < 0$:

$$G_+(x,k) = \frac{1}{2\pi} \int_{C_-\cup\Gamma_-} \frac{e^{ipx}}{p(p-2k)}dp = i\sum_{i=1}^{2} \text{Res}\left(\frac{e^{ipx}}{p(p-2k)}, p = p_i \right) = 0$$

Repeat a similar procedure for $G_-(x,k)$ using $C_+ \cup \Gamma_\pm$, we get

$$\begin{aligned} G_+(x,k) &= \begin{cases} \frac{1}{2ik}(1 - e^{2ikx}) & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} &= \frac{1}{2ik}(1 - e^{2ikx})\mathcal{H}(x) \\ G_-(x,k) &= \begin{cases} 0 & \text{for } x > 0 \\ \frac{-1}{2ik}(1 - e^{2ikx}) & \text{for } x < 0 \end{cases} &= -\frac{1}{2ik}(1 - e^{2ikx})\mathcal{H}(-x) \end{aligned}$$

As two solutions to (8). To find out which solution corresponds to $M(x,k)$, $N(x,k)$ and their conjugates, we use the asymptotic behaviors described earlier in (5), and notice that $M(x,k) \to 1$ as $x \to -\infty$, matching the case where $G_+(x,k)$ is substituted into (9). Similarly, $\bar{N}(x,k)$ is associated with $G_-(x,k)$. We thus have

$$M(x,k) = 1 + \frac{1}{2ik}\int_{-\infty}^{x}[1 - e^{2ik(x-\xi)}]u(\xi)M(\xi,k)d\xi \tag{10}$$

$$\bar{N}(x,k) = 1 - \frac{1}{2ik}\int_{x}^{\infty}[1 - e^{2ik(x-\xi)}]u(\xi)\bar{N}(\xi,k)d\xi \tag{11}$$

Since $G_{\pm}(x,k)$ is analytic for all $\Im(k) > 0$ and $\Im(k) < 0$ respectively, and vanishes for $|k| \to \infty$, we must have both $M(x,k)$ and $\bar{N}(x,k)$ analytic on $\Im(k) > 0$ and $\Im(k) < 0$ respectively.

□

Remark. *The form $M(x,k)$ and $\bar{N}(x,k)$ take is called Volterra integral equations. Note that there is some form of involution in their definition. For a less hand-wavy proof of their analyticity, we express them in a uniformly convergent series, where the analyticty of the terms are much easier to prove. Here we only write out the procedure for $M(x,k)$, as $\bar{N}(x,k)$ takes an analogous proof.*

For $M(x,k)$, define the following Volterra operator $T_k[f](x)$, where:

$$T_k[f](x) = \frac{1}{2ik}\int_{-\infty}^{x}[1 - e^{2ik(x-\xi)}]u(\xi)f(\xi)d\xi$$

Then it is obvious that

$$M = 1 + T_k[M]$$

We are motivated to define the following series:

$$M_n(x,k) = \sum_{i=0}^{n} T_k^i[1](x)$$

Satisfying

$$M_n = 1 + T_k[M_{n-1}] \tag{12}$$

This series is known as a Neumann series. First, we aim to show that M_n uniformly convergent to some $M_\infty(x,k)$. Noticing that when $\Im(k) > 0$, $|1 - e^{2ik(x-\xi)}| \leq 2$, we can give an upper bound for $T_k^n[1](x)$, where

$$\begin{aligned}
\|T_k^n[1](x)\|_\infty &\leq \frac{1}{|k|^n}\int_{x\leq x_1\leq\cdots\leq x_n}\prod_{i=1}^{n}|u(x_i)|dx_i \\
&\leq \frac{1}{|k|^n}\frac{1}{n!}\left|\int_{-\infty}^{x}|u(\xi)|d\xi\right|^n \\
&\leq \frac{1}{n!}\left(\frac{\|u(x)\|_{\mathcal{L}_1}}{|k|}\right)^n
\end{aligned}$$

Where $\|u(x)\|_{\mathcal{L}_1}$ is finite from our initial assumption. Then $\forall k$ such that $\Im(k) > 0$, by the Weierstrass M-test, since

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\|u(x)\|_{\mathcal{L}_1}}{|k|}\right)^n = e^{\frac{\|u(x)\|_{\mathcal{L}_1}}{|k|}}$$

converges, we have $\lim_{n\to\infty} M_n = \sum_{n=1}^{\infty} T_k^n[1](x)$ *converges absolutely and uniformly. We declare its limit as* M_∞*. Then, by taking the limit of (12), and using the property of uniform convergence, we have*

$$\lim_{n\to\infty} M_n = 1 + \lim_{n\to\infty} T_k[M_n] = T_k\left[\lim_{n\to\infty} M_n\right] \iff M_\infty = 1 + T_k[M_\infty]$$

To show that $M_\infty = M$*, i.e. the solution is unique, we consider* $d(x,k) = M(x,k) - M_\infty(x,k)$*. Then on* $x \in (-\infty, t]$*, by Gronwall's inequality we have*

$$|d(x,k)| \leq \frac{1}{|k|}\int_{-\infty}^{x} u(\xi)d(\xi,k)d\xi \leq 0 \cdot e^{\int_{-\infty}^{x} u(\xi)d\xi} = 0$$

and hence $M_\infty = M$ *as required. Next, we prove analyticity of each term in the series via induction (at* $\Im(k) > 0$*). For the base case* $n = 0$*,* $T_k^0[1](x) = 1$ *is trivially analytic. Next for the inductive step, suppose* $T_k^m[1](x)$ *is analytic for some* $m \in \mathbb{Z}^+$*. Then,*

$$T_k^{m+1}[1](x) = T_k[T_k^m[1]](x) = \frac{1}{2ik}\int_{-\infty}^{x}[1 - e^{2ik(x-\xi)}]u(\xi)T_k^m[1](\xi)d\xi$$

Since $1 - e^{2ik(x-\xi)}$ *is entire, and* $T_k^m[1](x)$ *is analytic from inductive hypothesis, the entire integrand is analytic for* $\Im(k) > 0$*. Then,* $T_k^m[1](x)$ *is uniformly convergent, we have*

$$\frac{\partial}{\partial k}T_k^{m+1}[1](x) = \frac{1}{2ik}\int_{-\infty}^{x}\frac{\partial}{\partial k}\Big([1 - e^{2ik(x-\xi)}]u(\xi)T_k^m[1](\xi)\Big)d\xi$$

where the integrand is complex-differentiable implies $T_k^{m+1}[1](x)$ *is complex-differentiable, and hence analytic. Lastly, to show that the series itself is analytic, we consider an arbitary closed contour* γ *on* $\Im(k) > 0$*. B the uniform convergent property, in conjunction with Morera's theorem, we have*

$$\oint_\gamma M(x,k)dk = \oint_\gamma \lim_{n\to\infty}\left(\sum_{i=1}^{n} T_k^i[1](x)\right)dk = \lim_{n\to\infty}\sum_{i=1}^{n}\oint_\gamma T_k^i[1](x)dk = \lim_{n\to\infty}\sum_{i=1}^{n} 0 = 0$$

Hence, $M(x,k)$ *is analytic on* $\Im(k) > 0$*.*

Next, we establish analytic properties for $a(k)$:

Lemma 3.6. $a(k)$ *can be analytically extended to* $\Im(k) > 0$ *and tends to unity as* $|k| \to \infty$*.*

Proof. To derive this, first define

$$\Delta(x,k) = M(x,k) - a(k)\bar{N}(x,k)$$

Which has another representation using (6):

$$\Delta(x,k) = b(k)e^{2ikx}\bar{N}(x,-k)$$

Substituting (10), (11) into these representations respectively, we have

$$\Delta(x,k) = \left[1 - a(k) + \frac{1}{2ik}\int_{-\infty}^{\infty} u(\xi)M(\xi,k)d\xi - \frac{1}{2ik}\int_{x}^{\infty} u(\xi)\Delta(\xi,k)d\xi\right]$$
$$- e^{2ikx}\left[\frac{1}{2ik}\int_{-\infty}^{\infty} e^{-2ik\xi}u(\xi)M(\xi,k)d\xi + \frac{1}{2ik}\int_{x}^{\infty} e^{-2ik\xi}u(\xi)\Delta(\xi,k)d\xi\right]$$

and

$$\Delta(x,k) = \left[-\frac{1}{2ik}\int_{-\infty}^{\infty} u(\xi)M(\xi,k)d\xi\right]$$
$$+ e^{2ikx}\left[b(k) + \int_{x}^{\infty} e^{-2ik\xi}u(\xi)\Delta(\xi,k)d\xi\right]$$

Equating coefficients of 1 and e^{2ikx}, we have

$$a(k) = 1 + \frac{1}{2ik}\int_{-\infty}^{\infty} u(\xi)M(\xi,k)d\xi$$
$$b(k) = -\frac{1}{2ik}\int_{-\infty}^{\infty} u(\xi)M(\xi,k)e^{-2ik\xi}d\xi$$

The analytic property of $a(k)$ follows immediately from this integral representation. $a(k) \to \mathbb{I}$ as $|k| \to \infty$ follows from $M(x,k) \to \mathbb{I}$ as $|k| \to \infty$. □

Using these two lemmas, it is evident that finding the modified eigenfunctions $M(x,k)$, $\bar{N}(x,k)$ in (5) can be described as a Riemann-Hilbert problem, where one aims to find a function $G(x,k)$ that has two holomorphic parts, $G_+(x,k)$ and $G_-(x,k)$ on $\Im(k) > 0$ and $\Im(k) < 0$ respectively, while satisfying a jump condition on the real number line $\Im(k) = 0$, with

$$G_+(x,k) - G_-(x,k) = g(x,k) \tag{13}$$

where

$$G_+(x,k) = \frac{M(x,k)}{a(k)}, \quad G_-(x,k) = \bar{N}(x,k), \quad g(x,k) = r(k)e^{2ikx}\bar{N}(x,-k)$$

Remark. *equivalently, one can express this as a vector-valued homogeneous Riemann-Hilbert problem, i.e.*

$$\mathbf{G}_+(x,k) = \begin{pmatrix} 1 + r(k)\bar{r}(k) & r(k)e^{2ikx} \\ \bar{r}(k)e^{-2ikx} & 1 \end{pmatrix} \mathbf{G}_-(x,k)$$

This is a good generalization that gives a compact representation of the problem.

Next, we derive a very non-trivial result regarding the roots of $a(k)$ (i.e. the poles of $\frac{M(x,k)}{a(k)}$):

Lemma 3.7. *$a(k)$ has a finite number of zeros that lies on the imaginary axis on $\Im(k) > 0$.*

Proof. First, we show that all of $a(k)$'s zeros lie on the plane $\Im(k) > 0$.

$$W(\phi(x,k), \psi(x,k)) = \phi(x,k)\psi_x(x,k) - \psi(x,k)\phi_x(x,k) = 2ika(k)$$

Then, if $a(k_0) = 0$, we have $W(\phi(x,k_0), \psi(x,k_0))$ for all $x \in \mathbb{R}$, meaning $\phi(x,k_0)$ and $\psi(x,k_0)$ are linearly dependent, with $\phi(x,k_0) = \beta_0\psi(x,k_0)$ for some constant β_0. Set $k_0 = \xi_0 + i\kappa_0$, and returning to the asymptotic conditions in (3), we thus have

$$\begin{cases} \phi(x,k_0) \sim e^{-i\xi_0 x} & \text{as } x \to -\infty \\ \phi(x,k_0) \sim \beta_0 e^{i\xi_0 x} e^{-\kappa_0 x} & \text{as } x \to \infty \end{cases}$$

Since $\phi(x,k_0)$ must be a wavefunction and is thus square integrable, we must have $\phi(x,k_0) \to 0$ as $|x| \to \infty$, therefore $\kappa_0 > 0$ as required.

Next, we show that $\xi_0 = 0$, i.e. the zeros k_0 must lie on the imaginary axis. Since both $\phi(x,k)$ and it's conjugate $\bar{\phi}(x,k)$ satisfies (1), we have the following:

$$\phi_{xx} + [u(x) + k^2]\phi = 0 \tag{14}$$

$$\bar{\phi}_{xx} + [u(x) + \bar{k}^2]\bar{\phi} = 0 \tag{15}$$

Multiply (14) by $\bar{\phi}$ and (15) by ϕ, and take $\frac{\partial}{\partial x}$ on (14)-(15) gives

$$\frac{\partial}{\partial x} W(\phi, \bar{\phi}) + (\bar{k}^2 - k^2)\phi\bar{\phi} = 0 \tag{16}$$

Integrating (16) from $-\infty$ to ∞ then gives

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\partial}{\partial x} W(\phi, \bar{\phi}) dx + (\bar{k}^2 - k^2) \int_{-\infty}^{\infty} |\phi(x,k)|^2 dx = 0 \\ \iff\ & W(\phi, \bar{\phi})\Big|_{-\infty}^{\infty} + (\bar{k}^2 - k^2) \int_{-\infty}^{\infty} |\phi(x,k)|^2 dx = 0 \\ \iff\ & (\bar{k}^2 - k^2) \int_{-\infty}^{\infty} |\phi(x,k)|^2 dx = 0 \end{aligned}$$

By the asymtotic conditions in (3). Then, if $a(k_0) = 0$, the above is equivalent to $\xi_0\kappa_0 \left\|\phi(x,k_0)\right\|_2^2 = 0$. The $\mathcal{L}_2$-norm being non-zero for ϕ and $\kappa_0 > 0$ further implies $\xi_0 = 0$, o.e. the zeros of $a(k)$ lies on the imaginary axis.

Lastly, we show that these zeros are simple and finite.

$$\lim_{x\to\infty} \Big[\beta_0^2 W(\psi(x,i\kappa_0), \psi_k(x,i\kappa_0)) - W(\phi(-x,i\kappa_0), \phi_k(-x,i\kappa_0))\Big] = -2i\kappa_0 \int_{-\infty}^{\infty} \phi^2(x,i\kappa_0)dx - 2\kappa_0\beta_0 \frac{da}{dk}(i\kappa_0)$$

then we have

$$\frac{da}{dk}(i\kappa_0) = -\frac{i}{\beta_0} \int_{-\infty}^{\infty} \phi^2(x,i\kappa_0)dx$$

□

Furthermore, we codify the information that $a(k)$ has finite roots within the Riemann-Hilbert problem as follows

Lemma 3.8. *The Riemann-Hilbert problem given by (5) is equivalent to the following equation*

$$\mu_+(x,k) = \bar{N}(x,k) + r(k)e^{2ikx}\bar{N}(x,-k) - \sum_{j=1}^{N} \frac{C_j}{k - i\kappa_j} e^{-2i\kappa_j x}\bar{N}(x,-i\kappa_j) \tag{17}$$

Where $\mu_+(x,k)$ is analytic on the upper half plane, satisfying $\mu_+(x,k) \to 1$ as $|k| \to \infty$, C_j the normalization constants, and $i\kappa_k$ the roots.

Proof. From Lemma 3.7, suppose the roots of $a(k)$ are $\{i\kappa_j\}_{j\leq N}$. Then we can set

$$\frac{M(x,k)}{a(k)} = \mu_+(x,k) + \sum_{j=1}^{N} \frac{A_j(x)}{k - i\kappa_j}$$

Integrating this around all of $\{i\kappa_j\}_{j\leq N}$ and using (5), we have

expand the derivation

$$A_j = C_j e^{-2i\kappa_j x}\bar{N}(x,-i\kappa_j)$$

Substituting this back into (5), we arrive at the desired equality. □

Taking a step back, we have successfully expressed the direct scattering in terms of the data $S(\lambda,t) = \big(\{\kappa_n, C_j(t)\}_{j=1}^N,\, r(k,t),\, a(k,t)\big)$ in Lemma 3.8. Remarkbly, this data is sufficient for us to fully reconstruct the potential $u(x,t)$, which we will demonstrate in the following subsection.

Remark. *For a physical interpretation, one could treat the negative energies (From the complex roots, $k = -i\kappa_j \implies E = -\kappa_j^2$) as the soliton potential being flipped, and hence convex. Then, the discrete and finite energy levels can be thought of as levels in which the travelling waves are trapped in the potential well.*

3.2.3 Inverse scattering

We want to recover the potential $u(x,t)$ from the scattering data

$$S(\lambda,t) = \big(\{\kappa_n, C_j(t)\}_{j=1}^N,\, r(k,t),\, a(k,t)\big)$$

To do this, consider a contour $\mathcal{L}_0$ that passes through all roots of $a(k)$, and approaches 0 as $|x| \to \infty$.

Definition 3.2 (Projection Operator)**.** *The Projection operators $\mathcal{P}^{\pm}[\cdot](k)$ are defined as real line extension of the Cauchy transform on a function $f(k)$, where*

$$\mathcal{P}^{\pm}[f](k) = \lim_{\epsilon\to 0^+} \mathcal{F}[f](k) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(s)}{s-(k\pm i\epsilon)}ds$$

To conceptualize this, think of the Cauchy transform $\mathcal{F}[f](k)$ as "projecting" the data at the real line given by $f(k)$, into analytic functions in the upper and lower complex plane; the limit $\mathcal{P}^{\pm}[f](k) = \lim_{\epsilon\to 0^+}\mathcal{F}[f](k)$

then extends the analytic function to the boundary when $\Im(k) = 0$, which is the real line, where a pole exist at $s = k$. In fact, we can recover the original data via the projection operators:

$$
\begin{aligned}
\mathcal{P}^+[f](k) - \mathcal{P}^-[f](k) &= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \left(\int_{-\infty}^{\infty} \frac{f(s)}{s - (k + i\epsilon)} + \frac{f(s)}{s - (k - i\epsilon)} ds \right) \\
&= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \left(\int_{-\infty}^{\infty} f(s) \frac{2i\epsilon}{\epsilon^2 + (s-k)^2} ds \right) \\
&= \int_{-\infty}^{\infty} f(s) \left[\frac{1}{\pi} \lim_{\epsilon \to 0^+} \frac{2i\epsilon}{\epsilon^2 + (s-k)^2} \right] ds \\
&= \int_{-\infty}^{\infty} f(s) \delta(s-k) ds \\
&= f(k)
\end{aligned}
$$

A very useful property of the projection operator is that for functions $f^{\pm}(k)$ analytic on $\Im(k) > 0$ and $\Im(k) < 0$ respectively, we have

$$
\mathcal{P}^{\pm}[f^{\pm}](k) = \pm f^{\pm}(k) \tag{18}
$$

$$
\mathcal{P}^{\pm}[f^{\mp}](k) = 0 \tag{19}
$$

Remark. *For the first identity, by considering a semicircular contour Γ on the upper and lower plane respectively, and using the residue theorem, we have*

$$
\mathcal{P}^{\pm}[f^{\pm}](k) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\pm f^{\pm}(s)}{s - (k \pm i\epsilon)} ds = \lim_{\epsilon \to 0^+} Res\left(\frac{\pm f^{\pm}(k)}{s - (k \pm i\epsilon)}, k \pm i\epsilon \right) = \pm \lim_{\epsilon \to 0^+} f^{\pm}(k \pm i\epsilon) = \pm f^{\pm}(k)
$$

And the second identity follows trivially as the function $f^{\pm}$ as the pole is in the opposite half plane to the countour Γ, which gives a residue of 0.

Essentially, we are systematically separating out which piece of the boundary data belongs to which half-plane analytic function, using the projection operator. This gives us the ability to single out expressions for $G_+(x, k)$ and $G_-(x, k)$ respectively in (8).

Define the following:

$$
\begin{aligned}
M_j(x) &= M(x, i\kappa_j) \\
N_j(x) &= e^{-2i\kappa_j x} \bar{N}(x, -i\kappa_j)
\end{aligned}
$$

Note that

$$
u(x) = -\frac{\partial}{\partial x} \left[\frac{1}{\pi} \int_{-\infty}^{\infty} r(s) N(x, s) ds \right]
$$

$$u(x) = \frac{\partial}{\partial x}\left[2i\sum_{j=1}^{N} C_j N_j(x) - \frac{1}{\pi}\int_{-\infty}^{\infty} r(s)N(x,s)ds\right]$$

Remark (Gel'fand-Levitan-Marchenko Integral Equation)**.** *We can derive the GLM equation as a special case. Given that*

$$\phi(x, i\lambda_n) = \begin{cases} e^{\lambda_n x}, & \text{for } x \to -\infty, \\ b_n e^{-\lambda_n x}, & \text{for } x \to \infty. \end{cases}$$

The potential $u(x,t)$ is recovered by

$$u(x,t) = -2\,\frac{d}{dx}K(x,x)$$

where $K(x,y)$ satisfies

$$K(x,y) + F(x+y) + \int_{x}^{\infty} K(x,z)\,F(z+y)\,dz = 0, \quad y \geq x$$

and $F(x)$ is given by

$$F(x) = \sum_{n=1}^{N} \frac{b_n e^{-\lambda_n x}}{i\,a'(i\lambda_n)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k)e^{ikx}\,dk.$$

3.2.4 Time Evolution of scattering data

A sort of reformulation of the KdV that seem to have came out of nowhere, but extremely useful in solving its time evolution, is the Lax representation. It's central idea is rooted in the following lemma:

Lemma 3.9. *Suppose there exists a differential operator A such that*

$$\dot{L} = [L, A],$$

with

$$L = -\frac{d^2}{dx^2} + u(x,t).$$

Then the eigenvalue problem $Lf = Ef$ has a time independent spectrum.

Proof. Differentiating with respect to t yields

$$\dot{L}\phi + L\phi_t = E_t\phi + E\phi_t.$$

Using the Lax equation $\dot{L} = [L, A]$ and noting that $AL\phi = E \cdot A\phi$, one obtains

$$(L - E)(\phi_t + A\phi) = E_t\phi \tag{20}$$

Taking the inner product with ϕ and using the self-adjointness of L (i.e., $\langle \phi, (L-E)(\phi_t + A\phi)\rangle = \langle (L-$

$E)\phi, \phi_t + A\phi\rangle = \langle 0, \phi_t + A\phi\rangle\rangle = 0$, we find

$$E_t \|\phi\|^2 = 0$$

Thus, $E_t = 0$ and the eigenvalue E is independent of t.

□

The most important result we get out of this lemma is that given $E_t = 0$ and reexamining (3), we have

$$\phi_t + A\phi = k\phi$$

which essentially gives us a simpler DE for the time evolution of ϕ that satisfies $L\phi = E\phi$.

Below is a summary of the solution strategy:

1. **Direct Scattering:** Solve the Schrödinger equation for the potential $u(x, 0)$ to obtain the scattering data $\{r(k), \lambda_i\}$.
2. **Inverse Scattering:** Recover the potential $u(x, 0)$ by solving the GLM equation and using the relation $u(x, t) = -2\,\frac{d}{dx}K(x, x)$.
3. **Time Evolution:** Evolve the scattering data in time t via the Lax pair to recover $u(x, t)$.

4 Zero-curvature representation (Chapter 3)

An alternative way of tackling the inverse scattering problem that is much more general is to rewrite the Lax representation.

$$\begin{cases} L\phi = \lambda\phi \\ \phi_t = -A\phi - k\phi \end{cases}$$

The first equation governs evolution along x, while the second governs evolution along t, that is compatible with the first equation.

we would like to rewrite this into a more symmetric form, i.e.

$$\begin{cases} F_x = UF \\ F_t = VF \end{cases}$$

where F is a vector and U, V are matrices, with F depending on ϕ and U, V depending on λ.

For a solution F to exist, if $\dim F = n$, we have $2n$ equations for the n unknowns. This is an overdetermined system, in the sense that the solution F in $F_x = UF$ must be compatible with solutions to $F_t = VF$. We can express this compatibility condition as $F_{xt} = F_{tx}$, which gives us

$$U_t - V_x + [U, V] = 0 \tag{21}$$

This is known as the zero-curvature condition. The zero-curvature representation is extremely useful, as one could play around with the elements in L and A to get different integrable systems. For instance, the KdV system is given by

$$F = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}$$

$$\begin{cases} \dfrac{\partial}{\partial x} F = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix} F = \left[\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right] F \\ \dfrac{\partial}{\partial t} F = \begin{pmatrix} -u_x & 2(u + 2\lambda) \\ 2u^2 - u_{xx} + 2u\lambda - 4\lambda^2 & u_x \end{pmatrix} F = \left[\begin{pmatrix} -u_x & 2u \\ 2u^2 - u_{xx} & u_x \end{pmatrix} - \begin{pmatrix} 0 & 4\lambda \\ 2u\lambda - 4\lambda^2 & 0 \end{pmatrix} \right] F \end{cases}$$

It is easy to check that this choice of U and V satisfies the zero-curvature condition, and is equivalent to the KdV equation.

5 Hierarchies and finite-gap solutions

In this section we bring a lot of the ideas together and look at the conserved quantities in the soliton system.

Tau functions and family of solutions.

6 Gauge field theory (Chapter 5 & 6)