SRF Sypnosis

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1 introduction

Pricing financial derivatives is a difficult task. For complex financial derivatives, such as those modelled by certain variants of the Black-Scholes PDE, no closed form solution exist, and we can only solve them via numerical methods[FJO21]. For classical computers, there exist limitation as these methods are computationally intensive. The aim of this work is to explore quantum algorithms as an alternative to classical algorithms in pricing financial derivatives. Specifically, we apply a variational quantum algorithm to price a European option modelled by the Black-Scholes PDE with stochastic volatility, which does not admit a closed form.

2 Hull-White model

For an European Option with Volatility $\sqrt{y_t}$, and an underlying stock with price x_t, we have two stochastic processes:

$$dx_t = \mu x_t dt + \sqrt{y_t} x_t dW_1$$
$$dy_t = aydt + bydW_2$$

Where the correlation between the two Wiener process dW_1 and dW_2 is given by ρ. Let $u(x, y, t)$ be the value of the option on the underlying x at time t. Then, using portfolio replication with Ito's lemma, coupled with non-arbitrage arguments gives the following modified Black-Scholes PDE[HW87]:

$$\frac{\partial u}{\partial t} + \left(rx\frac{\partial}{\partial x} + \frac{1}{2}x^2 y\frac{\partial^2}{\partial x^2} + b\rho x y^{\frac{3}{2}}\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}b^2y^2\frac{\partial^2}{\partial y^2} + ay\frac{\partial}{\partial y} - r \right) u = 0 \tag{1}$$

To find the solution of the PDE using VarQITE, we can express this in a Schrodinger-type equation. First, we write

$$\frac{\partial u}{\partial t} = \mathfrak{G}u$$

With the infinitesimal generator

$$\mathfrak{G} = -\left(rx\frac{\partial}{\partial x} + \frac{1}{2}x^2y\frac{\partial^2}{\partial x^2} + b\rho xy^{\frac{3}{2}}\frac{\partial^2}{\partial x\partial y} + \frac{1}{2}b^2y^2\frac{\partial^2}{\partial y^2} + ay\frac{\partial}{\partial y} - r\right)$$

We consider the simplist case where a, b, r have no dependence on x, y, t and treat them as constants. For a numerical experiment, we set $a = b = r = 1$ and $\rho = 1/2$ where $\mathfrak{G}$ becomes

$$\mathfrak{G} = -\left(x\frac{\partial}{\partial x} + \frac{1}{2}x^2y\frac{\partial^2}{\partial x^2} + \frac{1}{2}xy^{\frac{3}{2}}\frac{\partial^2}{\partial x\partial y} + \frac{1}{2}y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} - 1\right) \tag{2}$$

The first step is to set $\tau = T - t$ since we are evolving backwards at the final time T. We then apply a wick rotation $\epsilon = -i\tau$. With the substitution, we get

$$-i\frac{\partial u}{\partial \epsilon} = \widetilde{\mathfrak{G}}u$$

We want to simulate this evolution in an equivalent quantum system, with

$$-i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

where

$$\hat{H} = \widetilde{\mathfrak{G}} = \left(x\frac{\partial}{\partial x} + \frac{1}{2}x^2y\frac{\partial^2}{\partial x^2} + \frac{1}{2}xy^{\frac{3}{2}}\frac{\partial^2}{\partial x\partial y} + \frac{1}{2}y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} - \mathbb{I}\right)$$

Since this $\hat{H}$ is not Hermitian, and along the imaginary axis, the corresponding evolution operator $\exp \hat{H}\tau$ is not unitary, it is difficult to directly calculate how $|\psi\rangle$ evolves under $\hat{H}$[FJO21]. Hence, we instead use a variational method to evolve a trial wavefunction $|\phi(\boldsymbol{\theta})\rangle$, parameterised by $\boldsymbol{\theta}$ that approximates $|\psi\rangle$.

3 Variational Quantum Imaginary Time Evolution

The trial wavefunction is constructed via a variational form, i.e. a sequence of unitary operations, where $|\phi(\boldsymbol{\theta}(t)\rangle = U(\theta_n(t)) \cdots U(\theta_2(t))U(\theta_1(t))\,|\phi(\boldsymbol{\theta_0})\rangle$. Essentially, it is a wavefunction living in a much smaller statespace. This stems from the intuition that states that are physically relavant are a subspace of the entire Hilbert space[McA+19]. We call $|\phi(\boldsymbol{\theta}(t))\rangle$ an ansatz.

Since we want

$$|\phi(\boldsymbol{\theta}(t))\rangle \approx |\psi(t)\rangle$$

We initialise $|\phi(\boldsymbol{\theta}(t))\rangle$ it by finding the parameter $\boldsymbol{\theta_0}$ that brings $|\phi(\boldsymbol{\theta}(0))\rangle$ as close to the initial state $|\psi(0)\rangle$, where

$$\boldsymbol{\theta_0} = \underset{\theta\in\mathbf{R}^n}{\arg\min}\, \||\phi(\boldsymbol{\theta})\rangle - |\psi(0)\rangle\| \tag{3}$$

This process is done classically, via some variant of gradient descent. In our investigation, we used the L-BFGS-B Algorithm in Scipy. The original wavefunction $|\psi(0)\rangle$ is prepared from the initial condition of the target function at $t = 0$, i.e. $u(x, y, 0) = \max\{x - K, 0\}$ where K is the strike price of the option. We simply match the amplitude of $|\psi(0)\rangle$ with $u(x, y, 0)$ on a qubit state grid, in the following form:

$$\sum_{i=0}^{2^n-1} \sum_{i=0}^{2^n-1} p_{i,j}(t) |i\rangle \qquad p_{i,j}(t) \equiv u(x_i, y_j, t) \tag{4}$$

Note that we require $\sum_{i=0}^{2^n-1} \sum_{i=0}^{2^n-1} p_{i,j}(t)^2 = 1$, where the quantum states are normalised under the l_2 norm, so we assert $p_{i,j}(t) = \dfrac{u(x_i, y_j, t)}{\sqrt{\sum_{i,j} u_(x_i, y_j, t)^2}}$.

Next, we need to decompose $\hat{H}$ in to $\sum \lambda_i h_i$, a sum of unitary operators[Alg+22], enabling us to apply McLachlan's Variational principle to evolve $\boldsymbol{\theta}$ in imaginary time. This is equivalent to solving the following system of ODEs[Yua+19], where

$$A(t)\boldsymbol{\theta(t)}' = C(t)$$

With the matrix $A(t)$, vector $C(t)$ defined as the following:

$$A_{i,j}(t) = \Re\left(\frac{\partial \langle\phi(\boldsymbol{\theta}(t))|}{\partial\theta_i} \frac{\partial |\phi(\boldsymbol{\theta}(t))\rangle}{\partial\theta_j}\right) \tag{5}$$

$$C_i(t) = \Re\left(-\sum_i \lambda_i \frac{\partial \langle\phi(\boldsymbol{\theta}(t))|}{\partial\theta_i} h_i |\phi(\boldsymbol{\theta}(t))\rangle\right) \tag{6}$$

We can then evolve the parameter $\boldsymbol{\theta}(0)$ with the forward Euler method, where:

$$\boldsymbol{\theta}(t + \delta t) \approx \boldsymbol{\theta}(t) + A^{-1}(t)C(t)\delta t$$

Usually, the matrix $A(t)$ is not well-conditioned, so instead, we solve for each t:

$$\underset{\theta(t)\in\mathbf{R}^n}{\arg\min} \|A(t)\boldsymbol{\theta}(t)' - C(t)\|$$

This updates the parameter $\boldsymbol{\theta}(0)$ forward in time, evolving the ansatz $|\boldsymbol{\theta}(0)\rangle$ and thus u.

4 Application of VarQITE on the Hull-White model

To carry this entire procedure out on our specific PDE, the first step is to discretize $\widetilde{\mathfrak{G}}$ with periodic boundary conditions, and then decompose $\widetilde{\mathfrak{G}}$ into unitary operations $\sum \lambda_i h_i$ that can be represented on a quantum circuit.

Consider a 16 by 16 grid of qubit states, achieved by using 8 qubits, which has the following states $\{|0\rangle, |1\rangle, \cdots |2^8 - 1\rangle\}$. We embed the solution u_{ij} into this grid, where the value of u_{ji} corresponds to the probability amplitude of the qubit state it is located at.

To discretize the generator $\widetilde{\mathfrak{G}}$, we use the stencil method, where we use the effect of $\widetilde{\mathfrak{G}}$ on the qubit grid to represent it:

$$\widetilde{\mathfrak{G}} = \sum_{i_1=0}^{2^4-1}\sum_{i_2=0}^{2^4-1}\sum_{j_1=0}^{2^4-1}\sum_{j_2=0}^{2^4-1}[\widetilde{\mathfrak{G}}]_{(i_1,i_2)\times(j_1,j_2)} \left|i_1, i_2\right\rangle \left\langle j_1, j_2\right|$$

First, express the first-order derivative operators via the following approximation:

$$\begin{aligned}\frac{\partial u}{\partial x} &\approx \frac{u_{i+1,j} - u_{ij}}{\Delta x}\\ \frac{\partial u}{\partial x} &\approx \frac{u_{i,j+1} - u_{ij}}{\Delta y}\end{aligned}$$

For second-order derivative operators, we have

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &\approx \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{\Delta x^2}\\ \frac{\partial^2 u}{\partial x^2} &\approx \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{\Delta y^2}\\ \frac{\partial^2 u}{\partial x \partial y} &\approx \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4\Delta x \Delta y}\end{aligned}$$

Then, by the methodology in [Alg+22] , $\widetilde{\mathfrak{G}}$ can be discretised as the following:

$$[\widetilde{\mathfrak{G}}]_{(i_1,i_2)\times(j_1,j_2)} = \begin{cases} \dfrac{x^4y^2}{\Delta x^2} + \dfrac{x^2y^3+y^4}{\Delta y^2} - 1 & \text{if } (i_1,i_2) = (j_1,j_2) \\ \dfrac{1}{2}\dfrac{x^4y^2}{\Delta x^2} \mp \dfrac{1}{2}\dfrac{x}{\Delta x} & \text{if } (i_1,i_2) = (j_1 \pm 1, j_2) \\ \dfrac{1}{2}\dfrac{x^2y^3+y^4}{\Delta y^2} \mp \dfrac{1}{2}\dfrac{y}{\Delta y} & \text{if } (i_1,i_2) = (j_1, j_2 \pm 1) \\ \dfrac{1}{4}\dfrac{x^3y^{5/2}}{\Delta x \Delta y} & \text{if } (i_1,i_2) = (j_1 \pm 1, j_2 \pm 1) \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

The next step is to use this discretization to express $\widetilde{\mathfrak{G}}$ as linear combination of unitary operators.

Notice that due to the stencil method, $\widetilde{\mathfrak{G}}$ only attains non zero value along some diagonal and its offset by 1. Hence, we write it in the following form:

$$
\begin{aligned}
\widetilde{\mathfrak{G}} = &\sum_{j_1=0}^{2^4-1}\sum_{j_2=0}^{2^4-1}\left(\frac{x^4y^2}{\Delta x^2}+\frac{x^2y^3+y^4}{\Delta y^2}-1\right)|j_1,j_2\rangle\langle j_1,j_2| \\
&+\sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2}\left(\frac{1}{2}\frac{x^4y^2}{\Delta x^2}-\frac{1}{2}\frac{x}{\Delta x}\right)|j_1+1,j_2\rangle\langle j_1,j_2| \\
&+\sum_{j_1=1}^{2^4-1}\sum_{j_2=1}^{2^4-1}\left(\frac{1}{2}\frac{x^4y^2}{\Delta x^2}+\frac{1}{2}\frac{x}{\Delta x}\right)|j_1-1,j_2\rangle\langle j_1,j_2| \\
&+\sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2}\left(\frac{1}{2}\frac{x^2y^3+y^4}{\Delta y^2}-\frac{1}{2}\frac{y}{\Delta y}\right)|j_1,j_2+1\rangle\langle j_1,j_2| \\
&+\sum_{j_1=1}^{2^4-1}\sum_{j_2=1}^{2^4-1}\left(\frac{1}{2}\frac{x^2y^3+y^4}{\Delta y^2}+\frac{1}{2}\frac{y}{\Delta y}\right)|j_1,j_2-1\rangle\langle j_1,j_2| \\
&+\sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2}\left(\frac{1}{4}\frac{x^3y^{5/2}}{\Delta x\Delta y}\right)|j_1\pm 1,j_2\pm 1\rangle\langle j_1,j_2|
\end{aligned}
$$

After the periodic boundary condition $|2^n\rangle = |0\rangle$, $|-1\rangle = |2^n-1\rangle$, and x,y are x_{j_1}, y_{j_2} respectively.

Next, we define the following Operators to express $\widetilde{\mathfrak{G}}$ where these operators can be easily decomposed into sums of unitary operators. These operators are:

$$
\begin{aligned}
V_+(n) &= \sum_{i=0}^{2^n-2}|i+1\rangle\langle i| = \mathrm{CycInc}(n)\frac{1}{2}(C^{n-1}Z+I^{\otimes n}) \\
V_-(n) &= \sum_{i=1}^{2^n-1}|i-1\rangle\langle i| = \frac{1}{2}(C^{n-1}Z+I^{\otimes n})\mathrm{CycDec}(n) \\
D(n) &= \sum_{i=0}^{2^n-1} i\,|i\rangle\langle i| = \frac{2^n-1}{2}I^{\otimes n} - \sum_{i=1}^{n}2^{n-i-1}Z_i
\end{aligned}
$$

Where $C^{n-1}Z$ is a (n-1)-qubit controlled pauli Z gate, I is the identity gate, and Z_i is the pauli Z gate acting on the i-th qubit.

Within the above definition, the following

$$
\begin{aligned}
\mathrm{CycInc}(n) &= \sum_{i=0}^{2^n-1}|i+1\rangle\langle i| \\
\mathrm{CycDec}(n) &= \sum_{i=}^{2^n-1}|i-1\rangle\langle i|
\end{aligned}
$$

are Cyclic Increment and Cyclic Decrement operations respectively. For $n = 4$ in our investigation, they are represented in the following way:

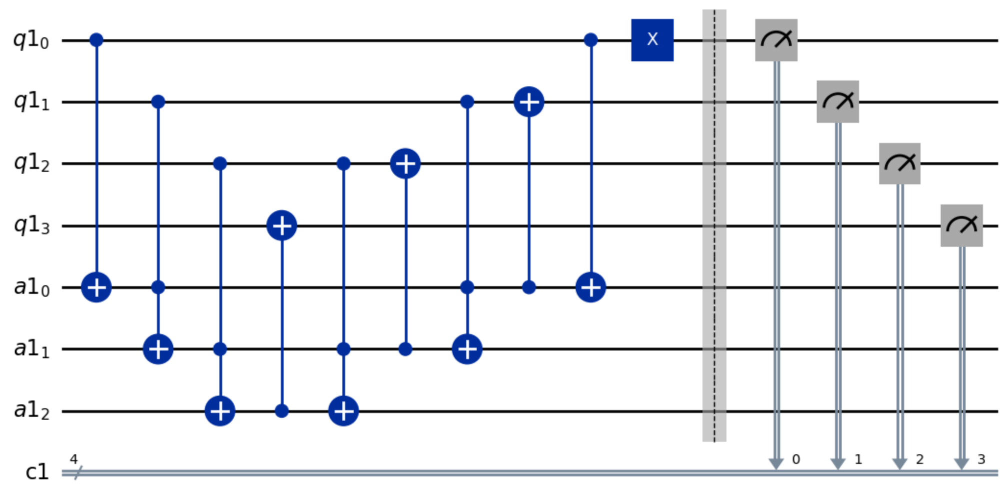

Figure 1: Cyclic Incrementer

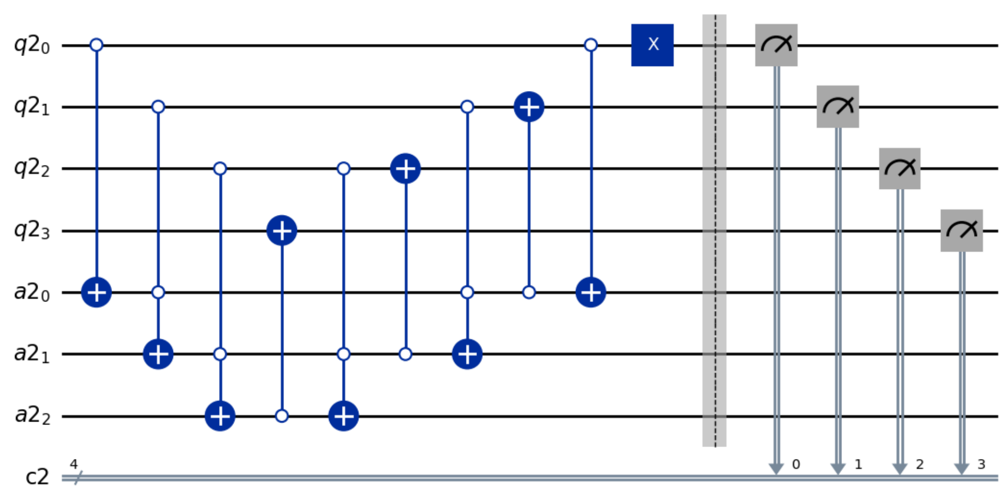

Figure 2: Cyclic Decrementer

Where both operators are composed of simple TIFFOLI, CNOT, (and their variant where $|0\rangle$ acts as control instead of $|1\rangle$ for the Decrementor) and X gates. With these unitary operations, we continue to build up the components that allows us to express $\widetilde{\mathfrak{G}}$:

$$V_{\pm}^{(k)}(n) = I^{\otimes k-1} \otimes V_{\pm}(n) \otimes I^{\otimes 2-k} \quad \text{k} = 1,\, 2$$
$$D^{(k)}(n) = I^{\otimes k-1} \otimes D(n) \otimes I^{\otimes 2-k} \quad \text{k} = 1,\, 2$$

Where I has dimension n. The following are examples of what these operators can be used to express in conjunction:

$$V_{+}^{(1)}(4)[D^{(1)}(4)]^m = \sum_{j_1=0}^{2^4-2} \sum_{j_2=0}^{2^4-2} j_1^m \,|j_1+1, j_2\rangle \langle j_1, j_2|$$
$$V_{-}^{(2)}(4)[D^{(2)}(4)]^m = \sum_{j_1=0}^{2^4-2} \sum_{j_2=0}^{2^4-2} j_2^m \,|j_1, j_2-1\rangle \langle j_1, j_2|$$

The purpose of the power m is to express $\widetilde{\mathfrak{G}}$ when we approximate the functions inside using taylor expansion at $(x, y) = (0, 0)$. For example, using second-order Taylor expansion, we have

$$\left(\frac{1}{2}\frac{x^4y^2}{\Delta x^2} - \frac{1}{2}\frac{x}{\Delta x}\right) = -\frac{1}{2\Delta x}x \approx -\frac{1}{2\Delta x} \cdot \Delta x j_1 \quad \forall (x, y) = (x_{j_1}, y_{j_2})$$

Although for our specific $\widetilde{\mathfrak{G}}$, Taylor expansion is not neccesary as all terms are polynomial of x and y, we still only consider terms up to the second order to decrease the overall complexity of $\widetilde{\mathfrak{G}}$ to carry the numerical experiment feasibly.

For this specific example, we can express it's corresponding term in $\widetilde{\mathfrak{G}}$ as

$$\sum_{j_1=0}^{2^4-2} \sum_{j_2=0}^{2^4-2} \left(\frac{1}{2}\frac{x^4y^2}{\Delta x^2} - \frac{1}{2}\frac{x}{\Delta x}\right) |j_1+1, j_2\rangle \langle j_1, j_2| \approx \sum_{j_1=0}^{2^4-2} \sum_{j_2=0}^{2^4-2} -\frac{1}{2\Delta x} \cdot \Delta x j_1 \,|j_1+1, j_2\rangle \langle j_1, j_2| = -\frac{1}{2}V_{+}^{(1)}(4)[D^{(1)}(4)]$$

We apply this technique to every term to express $\widetilde{\mathfrak{G}}$ in terms of sum of unitary operators, to get the following expression for $\widetilde{\mathfrak{G}}$:

$$\begin{aligned}
\widetilde{\mathfrak{G}} &\approx -\mathbb{I}^{\otimes 8} - \frac{1}{2}V_{+}^{(1)}(4)D^{(1)}(4) + \frac{1}{2}V_{-}^{(1)}(4)D^{(1)}(4) - \frac{1}{2}V_{+}^{(2)}(4)D^{(2)}(4) + \frac{1}{2}V_{-}^{(2)}(4)D^{(2)}(4) + 0 \\
&= -\mathbb{I}^{\otimes 8} + \frac{1}{2}[V_{-}^{(1)}(4) - V_{+}^{(1)}(4)]D^{(1)}(4) + \frac{1}{2}[V_{-}^{(2)}(4) - V_{+}^{(2)}(4)]D^{(2)}(4) \\
&= -\mathbb{I}^{\otimes 8} + \frac{1}{2}[V_{-}(4) - V_{+}(4)]D(4) \otimes \mathbb{I}^{\otimes 4} + \mathbb{I}^{\otimes 4} \otimes \frac{1}{2}[V_{-}(4) - V_{+}(4)]D(4)
\end{aligned}$$

Where we apply the same operation $\frac{1}{2}[V_{-}(4) - V_{+}(4)]D(4)$ on the 4 qubits that represent the dimension x and the 4 qubits that represent the dimension y. This is symmetrical because we took a second order approximation, which erased information of the higher order terms. Inspecting the original $\widetilde{\mathfrak{G}}$, the asymetry only appears at the fourth order terms.

We can further expand $\frac{1}{2}[V_-(4) + V_+(4)]D(4)$ as the following, from the previous definition:

$$\begin{aligned}\frac{1}{2}[V_-(4) - V_+(4)]D(4) = &\frac{1}{2}\left(\frac{1}{2}C^3Z\mathrm{CycDec}(4) + \frac{1}{2}\mathrm{CycDec}(4) - \frac{1}{2}\mathrm{CycInc}(4)C^3Z - \frac{1}{2}\mathrm{CycInc}(4)\right) \\ &\cdot\left(\frac{15}{2}\mathbb{I}^{\otimes 4} - 4Z_1 - 2Z_2 - Z_3 - \frac{1}{2}Z_4\right) \\ = &\frac{15}{8}H_1 - H_2 - \frac{1}{2}H_3 - \frac{1}{4}H_4 - \frac{1}{8}H_5 + \frac{15}{8}H_6 - H_7 - \frac{1}{2}H_8 - \frac{1}{4}H_9 - \frac{1}{8}H_{10} \\ &-\frac{15}{8}H_{11} + H_{12} + \frac{1}{2}H_{13} + \frac{1}{4}H_{14} + \frac{1}{8}H_{15} - \frac{15}{8}H_{16} + H_{17} + \frac{1}{2}H_{18} + \frac{1}{4}H_{19} + \frac{1}{8}H_{20}\end{aligned}$$

Where the $\mathbb{I}$ here acts on each qubit. Then, we have H_i's being the following:

$H_1 = C^3Z\mathrm{CycDec}(4)$	$H_2 = C^3Z\mathrm{CycDec}(4)Z_1$	$H_3 = C^3Z\mathrm{CycDec}(4)Z_2$	$H_4 = C^3Z\mathrm{CycDec}(4)Z_3$	$H_5 = C^3Z\mathrm{CycDec}(4)Z_4$
$H_6 = \mathrm{CycDec}(4)$	$H_7 = \mathrm{CycDec}(4)Z_1$	$H_8 = \mathrm{CycDec}(4)Z_2$	$H_9 = \mathrm{CycDec}(4)Z_3$	$H_{10} = \mathrm{CycDec}(4)Z_4$
$H_{11} = \mathrm{CycInc}(4)C^3Z$	$H_{12} = \mathrm{CycInc}(4)C^3Z(Z_1)$	$H_{13} = \mathrm{CycInc}(4)C^3Z(Z_2)$	$H_{14} = \mathrm{CycInc}(4)C^3Z(Z_3)$	$H_{15} = \mathrm{CycInc}(4)C^3Z(Z_4)$
$H_{16} = \mathrm{CycInc}(4)$	$H_{17} = \mathrm{CycInc}(4)Z_1$	$H_{18} = \mathrm{CycInc}(4)Z_2$	$H_{19} = \mathrm{CycInc}(4)Z_3$	$H_{20} = \mathrm{CycInc}(4)Z_4$

Table 1: Decomposition into implementable operator

5 Choice of ansatz

To carry out VarQITE using this decomposition of the Hamiltonian, we need to first construct the parameters $U(\theta_n(t))\cdots U(\theta_2(t))U(\theta_1(t))$ for our ansatz. We pick the RealAmplitude ansatz provided by qiskit, using 9 repetitions for each dimension separately in full entanglement mode.

This is because full entanglement ensures expressivity of the ansatz, while we assert a real amplitude for the evolution as our PDE only takes real values. We seperate the ansatz to each dimension to allow faster convergence during initialization, based on the intuition that the initional function $max\{x - K, 0\}$ is very asymmtrical between x and y. This also gives us the ability to evaluate the two ansatz seperately to reduce the size of the Hamiltonian. The following is the full picture of the ansatz, with 80 parameters in total:

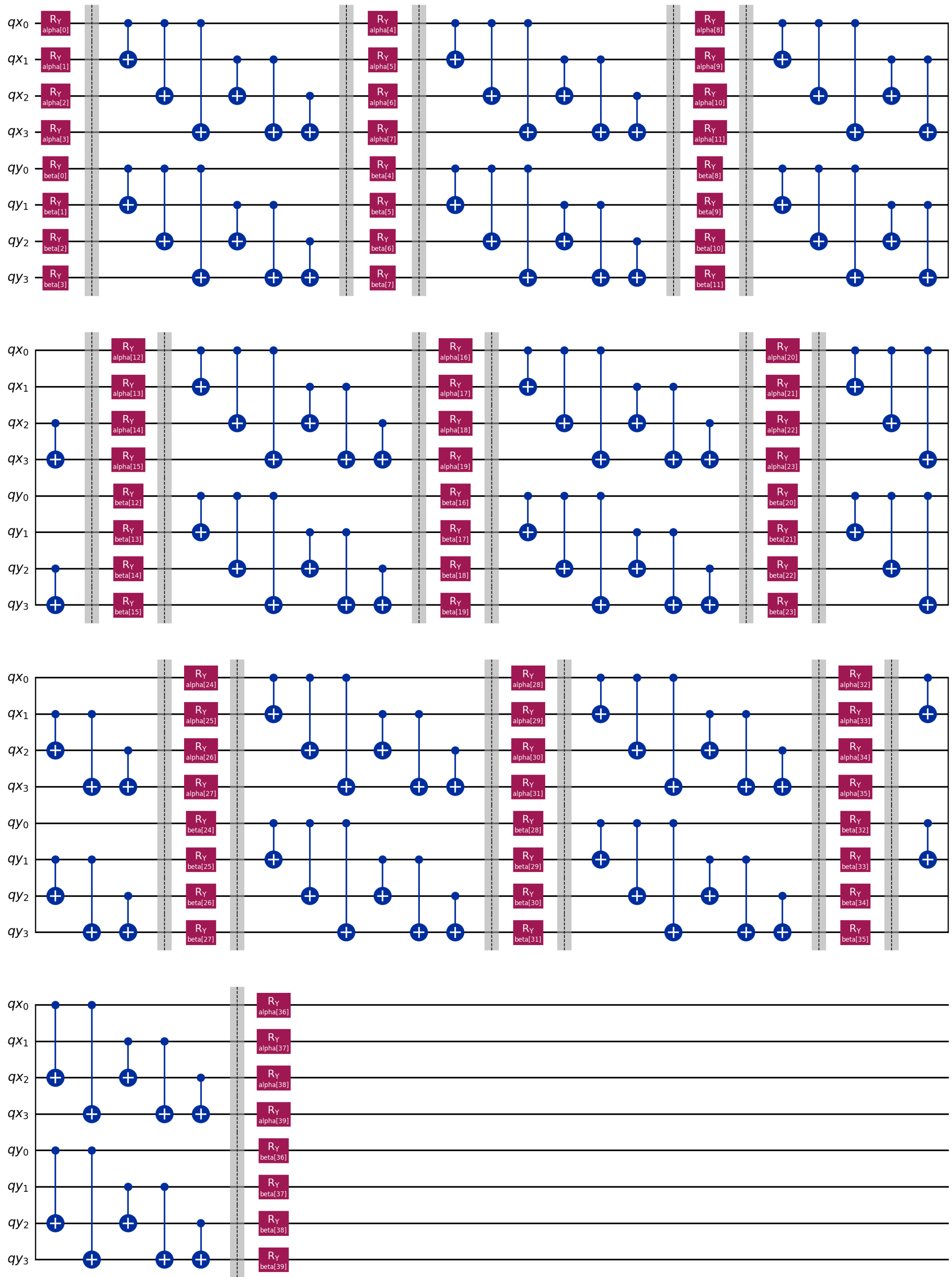

Figure 3: Problem specific ansatz

6 Numerical Results

We set the strike price $K = 1/2$, with the boundary conditions $0 \leq y \leq 1$, $0 \leq x \leq 1$. The total evolution time is $T = 0.1$ seconds, with timestep $\delta t = 0.001$ second. The following is the result of the evolution:

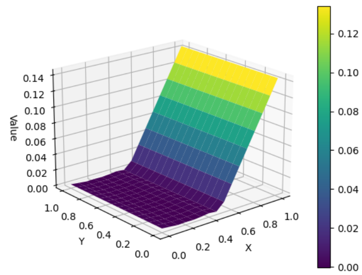

Figure 4: $u(x, y, 0)$

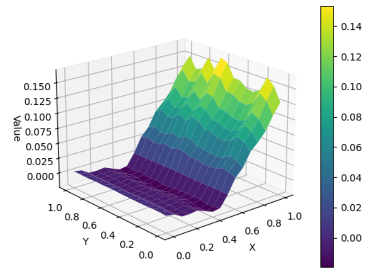

Figure 5: $u(x, y, 0.05)$

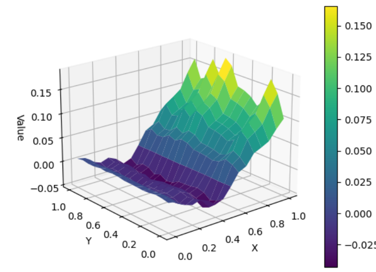

Figure 6: $u(x, y, 0.1)$

From a visual inspection, the simulation doesn't match existing results in literature that well[FJO21], and has evolved into negative values for some values of x, as we can see in Figure 6, which is incorrect. This is likely due to our approximation of the generator only up to the second order, which removed the effect of the higher order terms. We also need to rscale the probability distribution back into the value u, which we have not done for this investigation.

Another issue is with the expressivity of the ansatz. In Figure 4, we can see that the graph is not entirely flat when $x \leq 1/2$. This is due to our ansatz's number of parameter, which wasn't enough to fully express the initial condition of the wavefunction.

In future investigations, we can use higher-order approximation and more qubits for a more accurate, higher resolution result, and use deeper ansatz with more parameters and better design to establish a better initial condition.

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