Complex Differential Geometry

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Chapter 1

Complex Manifolds

1.1 Real manifolds

This short sections is just a reminder of the stuff which should be known. Even if you haven't seen abstract manifolds yet, think of submanifolds of a Euclidean space and convince yourself that they satisfy the conditions of the following definition.

Definition 1.1.1. A manifold of dimension n and class C^k, $k \geq 0$, is a Hausdorff topological space M with a countable basis of topology and a covering $\{U_i; i \in I\}$ by open sets such that

(i) each U_i is homeomorphic to an open subset of $\mathbb{R}^n$ via a $\phi : U_i \to \phi(U_i) \subset \mathbb{R}^n$;

(ii) if $U_i \cap U_j \neq \emptyset$, then $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is of class C^k.

The pairs $(U_i, \phi_i)_{i \in I}$ are called *charts*, their collection an *atlas*, and the maps $\phi_i \circ \phi_j^{-1}$ are *transition functions*. A manifold is *smooth* if the transition functions are smooth, and *analytic*, if transition functions are real-analytic.

Smooth functions, smooth maps between manifolds, etc. are defined by passing to the charts. A *tangent vector* v at a point m of a smooth manifold M can be defined either as

(i) an equivalence class of smooth curves $\gamma : (-\epsilon, \epsilon) \to M$, $\gamma(0) = m$, under the relation: $\gamma_1 \sim \gamma_2$ iff $(\phi_i \circ \gamma_1)'(0) = (\phi_i \circ \gamma_2)'(0)$ for some (or any) chart (U_i, ϕ_i) with $m \in U_i$; **or**

(ii) a linear map $L_v : C^\infty(U) \to \mathbb{R}$, U open and containing m, which satisfies the product rule: $L_v(fg) = f(m)L_v(g) + g(m)L_v(f)$.

Remark 1.1.2. Strictly speaking, in (ii) one needs to consider *germs of smooth functions* rather than functions. See any book on differential geometry for the precise definition.

The linear maps L_v are called *derivations* at m. The set of all tangent vectors at m is an n-dimensional vector space called the *tangent space* of M at m, denoted by T_m. The disjoint union $TM = \bigsqcup_{m \in M} T_m M$ has a natural structure of a smooth manifold of dimension $2n$ and the map $\pi : TM \to M$, $\pi(T_m M) = m$, makes it into a vector bundle[1]. Sections of TM, i.e. smooth maps $X : M \to TM$ such that $\pi \circ X = \mathrm{Id}_M$ are called *vector fields*. They can also be defined as *derivations* of the algebra $\mathbb{C}^\infty(M)$, i.e. $\mathbb{R}$-linear maps $L_X : \mathbb{C}^\infty(M) \to \mathbb{C}^\infty(M)$ which satisfy the product rule $L_X(fg) = fL_X(g) + gL_X(f)$.

1.2 Holomorphic Functions

Let V be an n-dimensional complex vector space. Then V can be regarded as a $2n$-dimensional real vector space and the multiplication by i gives a real linear endomorphism

$$J : V \to V \quad \text{with} \quad J^2 = -\mathrm{Id}.$$

If $(a_1, \ldots, a_n)$ is a complex basis of V, then $(a_1, \ldots, a_n, ia_1, \ldots, ia_n)$ is a real basis.

Conversely, given a $2n$-dimensional real vector space V, every real endomorphism $J : V \to V$ with $J^2 = -\mathrm{Id}$ makes V into a complex vector space via

$$(a + ib)v = av + bJ(v), \quad a, b \in \mathbb{R},\ v \in V.$$

Such a J is called a *complex structure*. $-J$ is also a complex structure called the *conjugate complex structure* and $(V, -J)$ is denoted by $\overline{V}$.

Example 1.2.1 (Standard example). $V = \mathbb{C}^n$ with basis $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$. Then

$$\mathbb{C}^n \simeq \mathbb{R}^{2n} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \mid x_i, y_i \in \mathbb{R}\}$$

and the complex structure

$$J(x_1, \ldots, x_n, y_1, \ldots, y_n) = (-y_1, \ldots, -y_n, x_1, \ldots, x_n).$$

We can generalise this example as follows:

Definition 1.2.2. Let E be an n-dimensional real vector space. The *complexification* of E is the real vector space $E^{\mathbb{C}} = E \oplus E$ together with the complex structure

$$J : E^{\mathbb{C}} \to E^{\mathbb{C}}, \quad J(v, w) = (-w, v).$$

$E^{\mathbb{C}}$ is equipped with the *conjugation*

$$C : E^{\mathbb{C}} \to E^{\mathbb{C}}, \quad C(v, w) = (v, -w).$$

Since $C \circ J = -J \circ C$, it is clear that C defines a complex isomorphism between $E^{\mathbb{C}}$ and $\overline{E^{\mathbb{C}}}$.

[1] If you haven't seen vector bundles yet, don't worry: they'll be discussed later.

Complexification of $\mathbb{R}^n$ is the complex n-space $\mathbb{C}^n$ identified with $\mathbb{R}^{2n}$ as above. In this case the conjugation is given by

$$C(z_1, \dots, z_n) = (\overline{z_1}, \dots, \overline{z_n}).$$

If $W = E^{\mathbb{C}} = E \oplus E$ is the complexification of a real vector space E, then the subspace

$$\mathrm{Re}(E) = \{(v, 0) \mid v \in E\}$$

is called the *real part* of W. It is canonically isomorphic to E and we can write $W = E \oplus iE$. An arbitrary complex vector space is the complexification in many different ways (non-canonically): just choose any complex basis B and define E as the real span of B.

Let (V, J) be a real vector space with a complex structure. We complexify V to $V^{\mathbb{C}}$ and extend J (uniquely!) to a complex linear endomorphism of $V^{\mathbb{C}}$:

$$J(v + iw) = J(v) + iJ(w)$$

We still have $J^2 = -\mathrm{Id}$, so the eigenvalues of J are $\pm i$. We set

$$V^{1,0} = \{z \in V^{\mathbb{C}} \mid J(z) = iz\}, \quad V^{0,1} = \{z \in V^{\mathbb{C}} \mid J(z) = -iz\}.$$

These are complex subspaces of $V^{\mathbb{C}}$. Their elements are called vectors of type (1,0) and (0,1) respectively.

Proposition 1.2.3. *The following identities hold:*

(i) $V^{1,0} = \{X - iJX \mid X \in V\}$ *and* $V^{0,1} = \{X + iJX \mid X \in V\}$*;*

(ii) $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ *(as a complex vector space sum);*

(iii) Complex conjugation defines a real linear isomorphism between $V^{1,0}$ *and* $V^{0,1}$.

Proof. Obvious. □

Let J be a complex structure on V. Then we obtain a complex structure on V^*:

$$(J\varphi)(v) = \varphi(Jv).$$

Definition 1.2.4. Let (V, J) be a real vector space with a complex structure. A differentiable function

$$f : V \underset{\text{open}}{\supset} U \longrightarrow \mathbb{C} \simeq (\mathbb{R}^2, i)$$

is called *holomorphic* if it's differential commutes with J, i.e.

$$df \circ J = i\, df.$$

Example 1.2.5. Let $V = \mathbb{R}^2$. Then $df|_p$ is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ which should commute with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A 2×2-matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ commutes with J iff $a_{12} = -a_{21},\ a_{11} = a_{22}$. Thus if $f = u + iv$, then $df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$ commutes with J iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the *Cauchy-Riemann equations.* If we introduce differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

then the Cauchy-Riemann equations can be rewritten as

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

Remark 1.2.6. A holomorphic $f : \mathbb{C}^n \to \mathbb{C}$ can be written locally as a convergent power series in $z_1, \ldots, z_n$ (no $\overline{z_1}, \ldots, \overline{z_n}$ occur).

Observe that $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ is a vector of type $(0,1)$ on $\mathbb{C}^2 = (\mathbb{R}^2)^{\mathbb{C}}$. In general, for an $f : V \to \mathbb{C}$ we can extend $df|_p$ linearly to $V^{\mathbb{C}}$, and then for any $Z = X + iJX \in V^{0,1}$ we have:

$$df|_p(X + iJX) = df|_p(X) + i\, df|_p(JX).$$

This is equal to 0 iff $df|_p(JX) = i\, df|_p(X)$. Thus:

Proposition 1.2.7. *A function $f : (V, J) \longrightarrow \mathbb{C}$ is holomorphic iff*

$$Z(f) = 0 \quad \forall\ Z \in V^{0,1}.$$

□

1.3 Complex manifolds

Definition 1.3.1. A complex manifold of (complex) dimension m is a topological manifold $(M, \mathcal{U})$ (with an atlas $\mathcal{U}$ consisting of charts $\varphi_i : U_i \to \mathbb{C}^m$) such that the transition functions $\varphi_i \circ \varphi_j^{-1}$ are holomorphic maps between open subsets of $\mathbb{C}^m$. In other words we have local complex coordinates on M.

Remark 1.3.2. Obviously a complex manifold of dimension m is smooth (real) manifold of dimension $2m$. We shall denote the underlying real manifold by $M_{\mathbb{R}}$.

Examples 1.3.3. 1) the complex projective space $\mathbb{C}\mathrm{P}^m$ is the set of complex lines in $\mathbb{C}^{m+1}$, i.e.

$$\mathbb{C}\mathrm{P}^m = \mathbb{C}^{m+1} \backslash \{0\}/\!\sim, \quad \text{where } z \sim w :\Longleftrightarrow \exists \alpha \in \mathbb{C}^* :\ z = \alpha w.$$

Similarly to $\mathbb{RP}^m$ we define an atlas

$$\begin{aligned} &U_i = \{[z_0, \dots, z_m] \mid z_i \neq 0\}, \quad i = 0, \dots, m, \\ &\varphi_i : \ U_i \longrightarrow \mathbb{C}^m, \quad [z_0, \dots, z_m] \longmapsto \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_m}{z_i} \right) \in \mathbb{C}^m. \end{aligned}$$

The transition functions are

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1}(w_1, \dots, w_m) &= \varphi_i([w_1, \dots, w_{j-1}, 1, w_{j+1}, \dots, w_m]) \\ &= \left(\frac{w_1}{w_i}, \dots, \widehat{\frac{w_i}{w_i}}, \dots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, \dots, \frac{w_m}{w_i} \right), \end{aligned}$$

hence holomorphic. $\mathbb{CP}^m$ is compact: We can restrict $\sim$ to the unit sphere $S^{2m+1} \subset \mathbb{C}^{m+1}$

$$S^{2m+1} = \left\{ z_i \in \mathbb{C}^{m+1} \middle| \sum_{i=0}^{m} |z_i|^2 = 1 \right\}.$$

A line $\{\alpha z \mid \alpha \in \mathbb{C}^*\}$ intersects S^{2m+1} in the set $\{\alpha \mid |\alpha|^2 = 1\}$, so in a circle S^1. Hence

$$\mathbb{CP}^m \simeq S^{2m+1}/S^1$$

as a real manifold (S^1 is viewed as a group acting on S^{2m+1}). E.g. for $m = 1$

$$S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$$

and S^1 acts via $\alpha(z, w) = (\alpha z, \alpha w)$. The quotient is S^2: notice that the following functions on $\mathbb{C}^2$ are invariant under the S^1-action: $a = |z|^2, b = |w|^2$ and $z\overline{w}$ and they satisfy the equation $c\bar{c} = ab$. Hence, if we write $x_1 = \operatorname{Re} c$, $x_2 = \operatorname{Im} c$, $x_3 = |z|^2$, then $x_1^2 + x_2^2 = x_3(1 - x_3)$, which describes a sphere. This projection $S^3 \to S^2$ is called the *Hopf fibration.*

2) More generally, the complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ is the set of all k-dimensional subspaces in $\mathbb{C}^n$. A basis of such a subspace can be written as a $k \times n$-matrix:

$$V = \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{kn} \end{pmatrix}.$$

Two such matrices define the same subspace if they are transformed into each other by an element $A \in GL_k(\mathbb{C})$ acting by the left multiplication. For each sequence of integers $\lambda = (\lambda_1, \dots, \lambda_k)$ with $1 \leq \lambda_1 < \dots < \lambda_k \leq n$ we can define a chart U_λ of $\mathrm{Gr}_k(\mathbb{C}^n)$ consisting of subspaces such that the columns with indices λ_i are linearly independent. In other words the minor V_λ consisting of columns with indices λ_i is invertible. The matrix $V_\lambda^{-1} V$ represents the same subspace and its λ_i-th column is e_i. Such a representation is unique. We define

$$\phi_\lambda : U_\lambda \to \mathbb{C}^{k(n-k)}$$

by associating to V the entries of the remaining $n-k$ columns of $V_\lambda^{-1}V$. Check that the transition functions are holomorphic.

Another construction of Grassmannians: $GL_n(\mathbb{C})$ acts transitively on the set of k-dimensional subspaces. The isotropy subgroup of a point, e.g. the subgroup which fixes $S_0 = \langle e_1, \dots, e_k \rangle$ is

$$H = \begin{array}{cc} \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) & \begin{array}{l} \}k \\ \}n-k \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{k} & \underbrace{\quad}_{n-k} \end{array} & \end{array}$$

Thus $\mathrm{Gr}_k(\mathbb{C}^n)$ is the coset space $GL_n(\mathbb{C})/H$. Both $GL_n(\mathbb{C})$ and H are complex Lie groups (open subsets of some $\mathbb{C}^N$) and as for real Lie groups and smooth manifolds one shows that the quotient space (complex Lie group)/(closed complex subgroup) is a complex manifold. As for $\mathbb{CP}^m$, $\mathrm{Gr}_k(\mathbb{C}^n)$ is compact: this time observe that we can choose unitary bases of subspaces, and then $\mathrm{Gr}_k(\mathbb{C}^n) \simeq U(n)/U(k) \times U(n-k)$.

3) As for smooth manifolds, level sets of submersions $f : \mathbb{C}^{m+1} \to \mathbb{C}$ are complex manifolds. If f is holomorphic and the holomorphic differential $df = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{m+1}}\right)$ does not vanish on $f^{-1}(c)$, then $f^{-1}(c)$ is a complex manifold. It is never compact - see homework.
On the other hand, if $p : \mathbb{C}^{m+1} \to \mathbb{C}$ is a homogeneous polynomial, then $v \in p^{-1}(0) \iff \alpha v \in p^{-1}(0)\ \forall \alpha \in \mathbb{C}^*$. Hence, if 0 is the only singular value of p, then we can consider

$$\big(p^{-1}(0)\backslash\{0\}\big)/\sim \quad \text{where} \quad v \sim w \Leftrightarrow \exists\ \alpha \in \mathbb{C}^* :\ v = \alpha w,$$

and we obtain a compact complex submanifold of $\mathbb{C}\mathrm{P}^m$.
Important examples of manifolds obtained in this way include the Fermat hypersurfaces $\{[z_0, \dots, z_m] \in \mathbb{C}\mathrm{P}^m \mid z_0^k + \cdots + z_m^k = 0\}$.

4) Let D be any lattice in $\mathbb{C}^m$, i.e. a discrete subgroup of the real translation group. Then $\mathbb{C}^m/D$ is a complex manifold, e.g. $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ is the torus.

5) *Hopf manifold*: Let $\lambda > 1$ be a real number. Consider the group $\Gamma \simeq \mathbb{Z}$ of transformations of $\mathbb{C}^m\backslash\{0\}$ given by

$$z \mapsto \lambda^n z, \quad n \in \mathbb{Z}.$$

This is a free and properly discontinuous action and $\mathbb{C}^m\backslash\{0\}/\Gamma$ is a complex manifold. We can identify it as a real manifold. First of all

$$\mathbb{C}^m\backslash\{0\} \simeq \mathbb{R}_{>0} \times S^{2m-1}, \quad z \mapsto \big(\|z\|, z/\|z\|\big).$$

In this representation λ (i.e. $1 \in \mathbb{Z}$) acts by $\lambda.(r,u) = (\lambda r, u)$, and so

$$\mathbb{C}^m\backslash\{0\}/\Gamma \simeq S^1 \times S^{2m-1}.$$

Definition 1.3.4. Let M be a complex manifold. A function $f : M \longrightarrow \mathbb{C}$ is called holomorphic iff for every local holomorphic chart (U, φ) on M, the function $f \circ \varphi^{-1}$ is holomorphic. More generally a map $\varphi : M \longrightarrow M'$ between complex manifolds is called holomorphic iff for every chart (U, φ) on M and (V, ψ) on M', the map $\psi \circ f \circ \varphi^{-1}$ is holomorphic.

We now want to define *holomorphic* tangent vectors. This time the definition in terms of derivations is much more suitable. First of all, for an open subset U of M set:

$$\mathrm{Hol}(U) := \{f : U \longrightarrow \mathbb{C} \mid f \text{ is holomorphic}\}.$$

We now define an *(holomorphic) tangent vector* at $p \in M$ to be a complex derivation of $\mathrm{Hol}(U)$, where U is any connected open neighbourhood of p, i.e. a map $\delta : \mathrm{Hol}(U) \to \mathbb{C}$, such that

$$\delta(\alpha f + \beta g) = \alpha\delta(f) + \beta\delta(g), \quad \forall\ \alpha, \beta \in \mathbb{C},$$
$$\delta(fg) = f(p)\delta(g) + \delta(f)g(p).$$

This time there is no need for germs, since a holomorphic function on a connected set is determined by its restriction to any open subset. In local complex coordinates $(z_1, \ldots, z_m)$ we can write such a tangent vector v as

$$v = \sum_{i=1}^{m} v_i \frac{\partial}{\partial z_i}.$$

The complex vector space of all holomorphic tangent vectors will be denoted by T_pM (not to be confused with $T_pM_{\mathbb{R}}$).

As for smooth manifolds, we consider the disjoint union

$$TM := \bigsqcup_{p \in M} T_pM.$$

This is again a complex manifold, called the *holomorphic tangent bundle*. The base map is $\pi : TM \to M$, $\pi(T_pM) = p$. A *holomorphic vector field* is a holomorphic map

$$X : M \longrightarrow TM \quad \text{s.t.} \quad \pi \circ X = id|_M.$$

A holomorphic map $F : M \longrightarrow N$ between holomorphic manifolds induces a holomorphic map between tangent bundles

$$F_* : TM \longrightarrow TN, \quad F_*(\delta)(f) = \delta(f \circ F).$$