Sheet 1	Name	Matrikelnummer
Complex Differential Geometry	Lukas Hantzko	10011048

Exercise 1. holomorphic functions

a) Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. The functions $u = \Re(f)$ and $v = \Im(f)$ are harmonic

With the Cauchy-Riemann-DGL:

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i} \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i} \quad \forall\, i \in \mathbb{N}$$

We get:

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\frac{\partial v}{\partial y_i} = \frac{\partial}{\partial y_i}\frac{\partial v}{\partial x_i} = \frac{\partial}{\partial y_i}\left(-\frac{\partial u}{\partial y_i}\right) = -\frac{\partial^2 u}{\partial y_i^2}$$
$$\frac{\partial^2 v}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(-\frac{\partial u}{\partial y_i}\right) = -\frac{\partial}{\partial y_i}\frac{\partial u}{\partial x_i} = -\frac{\partial}{\partial y_i}\left(\frac{\partial v}{\partial y_i}\right) = -\frac{\partial^2 v}{\partial y_i^2}$$

Therefore:

$$\triangle u = \sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2}\right) = 0$$
$$\triangle v = \sum_{i=1}^{n}\left(\frac{\partial^2 v}{\partial x_i^2} + \frac{\partial^2 v}{\partial y_i^2}\right) = 0$$

b) Let $U \subset \mathbb{C}^n$ be a connected open subset. Every non-constant harmonic function $f : U \to \mathbb{R}$ has no extremum on U

c) Show that any holomorphic function $f : M \to \mathbb{C}$, where M is a compact complex manifold, is constant

d) Deduce that $\mathbb{C}^n$ does not have any compact complex submanifolds of postive dimension.

Exercise 2. Hopf manifolds

Generalize the construction of the Hopf manifold as follows: Let $\lambda_1, \ldots, \lambda_n$ be complex numbers such that $|\lambda_i| > 1$ for each $i = 1, \ldots, n$. Consider the action of $\mathbb{Z}$ on $\mathbb{C}^n \setminus \{0\}$ given by

$$k \cdot (z_1, \ldots, z_n) = \left(\lambda_1^k z_1, \ldots, \lambda_n^k z_n\right)$$

where $k \in \mathbb{Z}$. Show that the quotient $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ is again diffeomorphic to $S^1 \times S^{2n-1}$. (Hint: Find a suitable map $\Phi : \mathbb{R} \times S^{2n-1} \to \mathbb{C}^n \setminus \{0\}$ such that for all $k \in \mathbb{Z}$ the relation $k \cdot \Phi(t, u) = \Phi(t + k, u)$ holds.)

Exercise 3. vector fields

For a complex manifold M denote by $H^0(M, TM)$ the space of holomorphic vector fields on M. Let M_1, M_2 be two compact complex manifolds and let $M = M_1 \times M_2$ be their product (as a complex manifold). Prove that

$$H^0(M, TM) \overset{?}{\cong} H^0(M_1, TM_1) \oplus H^0(M_2, TM_2)$$

Give examples to show that if the compactness assumption is dropped, then the result may or may not hold.