Advanced Quantum Theory
Exercise Sheet 1
11^{th} April 2019

Q1 Matrix properties:
Which of the following matrices are (a) unitary, (b) density matrices, (c) projections, (d) hermitian, or (e) normal? Note that the matrices may have more than one property.

$$\begin{aligned} M_1 &= \begin{pmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}, \\ M_3 &= \begin{pmatrix} 1 & 1+i & 1 \\ -1+i & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \; M_4 = \begin{pmatrix} 3/8 & -1/8+i/4 & 0 \\ -1/8-i/4 & 3/8 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}. \end{aligned} \tag{1}$$

Q2 Fourier transforms:
Find a normalized wavefunction $\psi(x) \in L^2(\mathbb{R})$ that has support on a small interval (so that ΔQ^2 is small) but has very large width in momentum space (ΔP^2 is very large). What is the function in momentum space (i.e., after a Fourier transform)?
Recall that, for any operator A,

$$\Delta A^2 = \langle A^2 \rangle_\psi - \langle A \rangle_\psi^2. \tag{2}$$

where $\langle B \rangle_\psi$ denotes the expectation value of the operator B in the state ψ. For example, for the position observable, we have

$$\langle Q \rangle_\psi = \int_{\mathbb{R}} \mathrm{d}x \, \psi^*(x) x \psi(x). \tag{3}$$

Q3 Pauli matrices:
The Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

(Sometimes people write X or σ_x instead of σ_1, and similarly $Y \equiv \sigma_y \equiv \sigma_2$ and $Z \equiv \sigma_z \equiv \sigma_3$.)

(i) Verify the relations

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k \tag{5}$$

for just three cases (e.g. $i = 1, j = 2$), where ϵ_{ijk} is the completely antisymmetric tensor, satisfying

$$\epsilon_{ijk} = \begin{cases} +1 \text{ if } (i,j,k) \text{ is } (1,2,3), \; (2,3,1) \text{ or } (3,1,2) \\ -1 \text{ if } (i,j,k) \text{ is } (3,2,1), \; (1,3,2) \text{ or } (2,1,3) \\ 0 \text{ otherwise.} \end{cases} \tag{6}$$

(ii) Use this to derive the relations

$$\begin{aligned} [\sigma_i, \sigma_j] &= 2i \epsilon_{ijk} \sigma_k \\ \{\sigma_i, \sigma_j\} &= 2 \delta_{ij} \mathbb{1}, \end{aligned} \tag{7}$$

where $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$.

(iii) Define $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\vec{n} = (n_1, n_2, n_3)$, where $n_i \in \mathbb{R}$. Show that

$$(\vec{n}.\vec{\sigma})^2 = (\vec{n}.\vec{n})\mathbb{1}. \tag{8}$$

(iv) If σ_i is the observable corresponding to spin along the i axis, what does $\vec{n}.\vec{\sigma}$ correspond to? (You can assume $\vec{n}.\vec{n} = 1$, and forget about factors of $\hbar/2$.)

Q4 Star Trek - The Penny-Flip Episode:

Starship Enterprise once again comes into a seemingly hopeless situation, when suddenly "Q" appears on the bridge and offers help - provided captain Picard can beat him in a very simple game: Q takes a coin and puts it Heads up in a box. Then Q, Picard and again Q (in this order) are allowed to access the box and choose whether to flip the coin or not. They must not look at the coin or watch the opponent's move. If at the end the coin is Heads up Q wins, if it is Tails up Picard wins. Picard calculates the odds as 50% and accepts the challenge. However, he gets quickly frustrated, because Q always wins, irrespective of the number of rounds they play.

(i) Obviously Q is cheating, but how? Assume that for Picard the coin is classical (as for any normal person), i.e., only the states "Heads" and "Tails" exist. On the other side Q can handle the coin as a quantum bit (it's his coin...). That means Q is allowed to perform any unitary on the state of the coin and can therefore prepare any superposition $\psi = a\,|H\rangle + b\,|T\rangle$. Find the unitaries U_1 and U_3 that Q has to perform on his first and third turn, that will make him win irrespective of Picard choosing to flip (operation $\sigma_1 = (|H\rangle\langle T| + |T\rangle\langle H|)$) or to pass (operation $\sigma_0 = \mathbb{1}$).

(ii) After a while Q starts to pity Picard (mainly because of the pending catastrophe) and offers him to swap the positions. Now Picard gets the first and third turn and Q the second. However, Q can still perform arbitrary unitary operations and Picard can only classically flip the coin. Can this change help Picard and level the odds? Explain.

(iii) Suppose Q insists on the original sequence of moves, but Picard suddenly remembers his quantum mechanics course at the Academy, and also plays quantum. Can he level the odds?