Leibniz-Universität Hannover
Insitut für Differentialgeometrie
Prof. Roger Bielawski, M. Sc Yannic Borchard Hannover, 20.04.2020

Sheet 1

Complex Differential Geometry

Sommer Semester 2020

Exercise 1. a) Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. Show that the functions $u, v : \mathbb{C}^n \to \mathbb{R}$ given by $u = \mathrm{Re}(f) = \frac{1}{2}(f + \bar{f})$ and $v = \mathrm{Im}(f) = \frac{1}{2i}(f - \bar{f})$ are harmonic, i.e.

$$\Delta u = 0 = \Delta v,$$

where $\Delta u := -\sum_{i=1}^{n} \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2} \right)$.

b) Let $U \subset \mathbb{C}^n$ be a connected open subset. Prove that a non-constant harmonic function $u : U \to \mathbb{R}$ has neither a maximum nor a minimum on U.

c) Show that any holomorphic function $f : M \to \mathbb{C}$, where M is a compact complex manifold, is constant.

d) Deduce that $\mathbb{C}^n$ does not have any compact complex submanifolds of positive dimension.

Exercise 2. Generalise the construction of the Hopf manifold as follows: Let $\lambda_1, \dots, \lambda_n$ be complex numbers such that $|\lambda_i| > 1$ for each $i = 1, \dots, n$. Consider the action of $\mathbb{Z}$ on $\mathbb{C}^n \setminus \{0\}$ given by

$$k \cdot (z_1, \dots, z_n) = (\lambda_1^k z_1, \dots, \lambda_n^k z_n),$$

where $k \in \mathbb{Z}$. Show that the quotient $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ is again diffeomorphic to $S^1 \times S^{2n-1}$. (Hint: Find a suitable map $\Phi : \mathbb{R} \times S^{2n-1} \to \mathbb{C}^n \setminus \{0\}$ such that for all $k \in \mathbb{Z}$ the relation $k \cdot \Phi(t, u) = \Phi(t + k, u)$ holds.)

Exercise 3. For a complex manifold M denote by $H^0(M, TM)$ the space of holomorphic vector fields on M. Let M_1, M_2 be two compact complex manifolds and let $M = M_1 \times M_2$ be their product (as a complex manifold). Prove that

$$H^0(M, TM) \cong H^0(M_1, TM_1) \oplus H^0(M_2, TM_2).$$

(Hint: Use the results proved in exercise 1.)

Give examples to show that if the compactness assumption is dropped, then the result may or may not hold.

Due Monday, 27.04.2020, 12:00 am.
Please send your solution to borchard@math.uni-hannover.de or via StudIP to Yannic Borchard.
For the Studienleistung you need to achieve 50% of the points of the homeworks.