Exercise 1: Coherent state (3 Points)

Recall the definition of coherent state that you have learned in the exercises. Show that as a function of the Fock states $|n\rangle$, the coherent state may be written in the form: $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$.

Exercise 2: Number operator (2.5 points)

Let $\hat{N} = \sum_i \hat{a}_i^\dagger \hat{a}_i$ be the particle number operator, and $\hat{H} = \sum_{i,j} t_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} \sum_{i,j,k,l} V_{ijkl} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_l \hat{a}_k$ the Hamilton operator. Show that $[\hat{N}, \hat{H}] = 0$ both for fermions and for bosons. What is the physical meaning of that?

Exercise 3: Fermionic operators (2.5 points)

The state of a system of identical fermions is given by the superposition of Fock states:

$$|\psi\rangle = \sum_{n_1,...,n_L} C(n_1, ..., n_L) |n_1, ..., n_L\rangle$$

Evaluate $\langle\psi|\hat{a}_i^\dagger \hat{a}_j|\psi\rangle$. (Hint: Be very careful with the minus signs coming from the anticommutation!).

Exercise 4: Hubbard model (3 points)

Let us consider interacting spin-less bosons in a periodic potential, which you can visualize as a lattice with lattice sites separated by potential barriers. The Hamiltonian of the interacting bosons can be written in the form:

$$\hat{H} = -\sum_{i,j} t_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} \sum_{i,j,k,l} V_{ijkl} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_l \hat{a}_k,$$

where a_i annihilates a particle at site i. The first term describes the motion of a particle from site j to site i, whereas the second is the interaction term, in which two particles in sites k and l end in sites i and j. Under some conditions, we can approximate that the motion is restricted to nearest neighbors, i.e. $t_{ij} = t$ if i and j are neighboring sites and zero otherwise. We can also approximate that the interaction just occurs for particles within the same site, i.e. $V_{ijkl} = U$ for $i = j = k = l$, and zero otherwise. In that case the Hamiltonian acquires a very simple form:

$$\hat{H} = -t \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_{i=1}^{N_{sites}} \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i,$$

where $\langle i,j \rangle$ indicates nearest neighbors, and N_{sites} is the number of sites. This Hamiltonian is the so-called Bose-Hubbard Hamiltonian.

(a) (1.5 Points) Show that the N-particle state

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} \left(\frac{1}{\sqrt{N_{sites}}} \sum_{i=1}^{N_{sites}} \hat{a}_i^\dagger \right)^N |0\rangle$$

is an eigen-state of $\hat{H}$ for $U = 0$. What is the eigen-energy? What is the physical meaning of $\frac{1}{\sqrt{N_{sites}}} \sum_{i=1}^{N_{sites}} \hat{a}_i^\dagger$? (Hint: Recall our discussion about the operators in momentum space.)

(b) (1.5 Points) Show that the state:

$$|\psi_1\rangle = \prod_{i=1}^{N_{sites}} \hat{a}_i^\dagger |0\rangle$$

is an eigen-state of $\hat{H}$ for $t = 0$. Which is the eigen-energy?