Advanced quantum theory, June 20th 2019

Born approximation

Let $H = H_0 + V$. During the lecture, the wave operators were defined

$$\Omega_\pm = \lim_{t\to\mp\infty} e^{iHt} e^{-iH_0 t} E_{ac}(H_0) \tag{1}$$

that obey the Lippmann-Schwinger equation

$$\Omega_\pm = \lim_{\epsilon\to 0} \int (\mathbb{1} + R_{H_0}(\omega \pm i\epsilon) V \Omega_\pm) E_{ac}(d\omega). \tag{2}$$

Problem 1 *Lippmann-Schwinger equation for 1D rectangular barrier.*

1. Iterate equation (2) to obtain a solution as a power series in $R_{H_0} V$.

2. From (2) obtain a Lippmann-Schwinger equation for wave functions in coordinate representation

   $$\langle x|\psi\rangle = \langle x|\phi\rangle + \int dy \langle x| \frac{1}{E - H_0 + i\epsilon} |y\rangle\langle y|V|\psi\rangle \tag{3}$$

   where $|\phi\rangle$ is a scattering state of the free Hamiltonian H_0 and $|\psi\rangle$ is the scattering state of H.

3. Compute the resolvent R_{H_0} for $H_0 = \frac{p^2}{2m}$ in coordinate representation. That is, show that (in units $\hbar = 1$)

   $$\langle x| \frac{1}{E - H_0 + i\epsilon} |y\rangle = \frac{1}{2\pi} \int dp \frac{e^{ip(x-y)}}{E - \frac{p^2}{2m} + i\epsilon}. \tag{4}$$

   (*Hint*: insert an identity $\mathbb{1} = \int dp |p\rangle\langle p|$).
   Close the integration contour and apply Jordan's lemma to get

   $$\langle x| \frac{1}{E - H_0 + i\epsilon} |y\rangle = -im \frac{e^{i\Sigma|x-y|}}{\Sigma} \tag{5}$$

   where $\Sigma^2 = 2mE$

4. Consider a potential $V(x)$ that is equal to U in the region $0 < x < a$ and 0 everywhere else. Observe that Lippmann-Schwinger equation yields

   $$\psi(x) = \phi(x) + \frac{imU}{\Sigma} \int_0^a dy e^{i\Sigma|x-y|} \psi(y). \tag{6}$$

   Take only the first term in the expansion form the subexercise 1 to get the *Born approximation* of the Lippmann-Schwinger equation

   $$\psi(x) = \phi(x) + \frac{imU}{\Sigma} \int_0^a dy e^{i\Sigma|x-y|} \phi(y). \tag{7}$$

   Take a scattering state $\phi(x) = \frac{e^{ikx}}{\sqrt{2\pi}}$ of $H_0 = \frac{p^2}{2m}$ to obtain the approximation for $\psi(x)$.

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Problem 2 *S-matrix for Yukawa potential*

1. During the lecture the S-matrix was defined as $S = \Omega_+^\dagger \Omega_-$. Let $|p\rangle$ and $|q\rangle$ be two eigenstates of H_0. Expand S to the first order in V to obtain

$$\langle p|S^{(1)}|q\rangle = \delta(p-q) + \langle p|V|q\rangle \lim_{\epsilon\to 0}\left(\frac{1}{E_p - E_q + i\epsilon} + \frac{1}{E_q - E_p + i\epsilon}\right). \tag{8}$$

2. Show that

$$\left(\frac{1}{E_p - E_q + i\epsilon} + \frac{1}{E_q - E_p + i\epsilon}\right) = -2\pi i \delta\left(E_p - E_q\right). \tag{9}$$

   (*Hint*: use Cauchy's integral formula.)
   Obtain the *Born approximation* to the S-matrix,

$$\langle p|S^{(1)}|q\rangle = \delta(p-q) - 2\pi i \delta\left(E_p - E_q\right)\langle p|V|q\rangle. \tag{10}$$

3. Let be a Yukawa potential $V = -\frac{\alpha}{r}e^{-\lambda r}$. This potential describes interaction of massive bosons. Calculate $\langle\vec{q}|V|\vec{p}\rangle$ where $|\vec{p}\rangle$ and $|\vec{q}\rangle$ are eigenvectors of $H_0 = \frac{\vec{p}^2}{2m}$ in 3D.

4. Calculate $S^{(1)}$ for Yukawa potential. Take a look at what happens if the mass of the bosons vanishes $\Leftrightarrow \lambda \to 0$.