Exercises for Advanced Quantum Theory, Summer 2019

For use in class May 16: problem 1 and recap for Problem 3,
Rest=Homework due May 23.

Problem 1: *Poisson Distribution*
Verify that the function

$$C(t) = \exp\bigl(m(e^{it} - 1)\bigr) \tag{1}$$

is the characteristic function of the Poisson distribution with mean m (look it up). Compute the second moment from the characteristic function.
Show, more generally, that the characteristic function of a distribution on the natural numbers has to be (a) periodic ($C(t + 2\pi) = C(t)$) and (b) analytic in e^{it}.

Problem 2: *Waiting for the bus*
In both cities, Clockville and Poissonville, the average time between two buses is 6 minutes. In Clockville, however, the buses are spaced exactly 6 minutes apart, in Poissonville buses arrive according to a Poisson process, i.e., the bus arrivals in disjoint time intervals are independent random variables.

What is the average wait for a bus after you arrive randomly (independently of the schedule) at the busstop in (i) Clockville, (ii) Poissonville?

Solve this by computing the waiting time distributions for these point processes. It may be helpful to discretize time, and recall from the lecture that for a poisson point process the number distributions in disjoint intervals are independent.

Problem 3: *Angular momentum via Fock space*
Consider the Bose Fock space over $\mathbb{C}^2$. Then $U \mapsto \Gamma_+(U)$ is a representation of SU(2).
(1) Our first aim is to show that the restriction of this representation to the N particle subspace is precisely the irreducible representation with spin $s = N/2$.

For the computation of L^2 we use the connection with $d\Gamma$ and $d^2\Gamma$. Let $\ell_\alpha = (1/2)\sigma_\alpha$ be the angular momentum operators for the defining representation, and $L_\alpha = d\Gamma(\ell_\alpha)$. For the square, show that

$$\sum_{\alpha=1}^{3} \ell_\alpha \otimes \ell_\alpha = \frac{1}{4}(2\mathbb{F} - \mathbb{1}), \tag{2}$$

where $\mathbb{F}$ denotes the flip operator $\mathbb{F}\phi\otimes\psi = \psi\otimes\phi$. In the Bose case we have $d^2\Gamma(\mathbb{F}) = d^2\Gamma(\mathbb{1})$. Use this and $d^2\Gamma(A \otimes A) = (d\Gamma(A)^2 - d\Gamma(A^2))/2$ to compute $L^2 = \sum_\alpha L_\alpha^2$, as a function of the particle number.
(2) Compare the dimension of $(\mathbb{C}^2)_+^{\otimes N}$ with $2s + 1$.
(3) Let us write the basis vectors of $\mathbb{C}^2$ as $|+\rangle$ and $|-\rangle$, and consider the corresponding creation and annihilation operators $a_\pm$ and $a_\pm^*$. Show that $a_+^* a_-$ acts as a ladder operator in the angular momentum sense.
(4) Introduce the antiunitary operator θ with $\theta|\pm\rangle = \mp|\mp\rangle$, and show that it commutes with all unitaries on $\mathbb{C}^2$ with determinant 1. Equivalently: It anticommutes with the Pauli matrices. Show that there is a antiunitary operator θ_s in the spin-s representation, which commutes with the representation and satisfies $\theta_s^2 = (-1)^{2s}$.