Advanced Quantum Theory
Homework Exercise Sheet 1
Due 18^{th} April 2019

Q1 Tensor products of vector spaces:
Recall that the tensor product of two vector spaces V_1 and V_2 is the vector space $V_1 \otimes V_2$, which is spanned by all vectors arising from the bilinear map (so it's linear in both arguments)

$$(v_1, v_2) \to v_1 \otimes v_2, \tag{1}$$

with $v_1 \in V_1$ and $v_2 \in V_2$.

(i) Let e_i with $i \in \{0, d_1 - 1\}$ be basis vectors for V_1 and f_i with $i \in \{0, d_2 - 1\}$ be basis vectors for V_2. Give a basis for $V_1 \otimes V_2$. What is the dimension of $V_1 \otimes V_2$?

(ii) Consider the vector

$$v = e_1 \otimes f_1 + e_2 \otimes f_2. \tag{2}$$

Show that this cannot be written as $w_1 \otimes w_2$, for any $w_1 \in V_1$ and $w_2 \in V_2$.

Q2 Tensor products of operators: Suppose we have operators A and B acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, i.e., $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. The operator $A \otimes B \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is defined via

$$(A \otimes B)(v_1 \otimes v_2) = (Av_1) \otimes (Bv_2). \tag{3}$$

(i) Calculate

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

(ii) For Hilbert spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$, the inner product of $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined by

$$\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is always conjugate linear in the first argument and linear in the second. Show that $(A \otimes B)^* = A^* \otimes B^*$.

(iii) Show that $A \otimes B$ is a positive operator if A and B are. Is it also true that A and B are positive operators if $A \otimes B$ is?

(iv) Suppose that $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$ have eigenvalues a_i and b_j respectively, with corresponding eigenvectors v_i and w_j respectively. What are the eigenvalues and eigenvectors of $A \otimes B$?

Q3 Partial Trace:
Let us look at how to find reduced states via the partial trace. This works as follows. Suppose Alice has a quantum system with Hilbert space $\mathcal{H}_A$ with dimension d_A and Bob has a quantum system with Hilbert space $\mathcal{H}_B$ with dimension d_B. The overall state of this quantum system is described by a density operator, which we will write as ρ_{AB}, which may be pure or mixed. The partial trace allows us to find the density operator that describes only the state of Alice's (or Bob's) system, which is called the reduced state ρ_A on Alice's system, and is defined by

$$\mathrm{tr}(\rho_A X) = \mathrm{tr}\big(\rho_{AB}(X \otimes \mathbb{1})\big). \tag{6}$$

(i) Show that ρ_A is given by the *partial trace* $\mathrm{tr}_B[\rho_{AB}]$, which is defined by

$$\langle\phi_A|\,\mathrm{tr}_B[\rho_{AB}]\,|\psi_A\rangle = \sum_{i=1}^{d_B} \langle\phi_A| \otimes \langle e_B^i|\, \rho_{AB}\, |e_B^i\rangle \otimes |\psi_A\rangle\,, \tag{7}$$

where $|\phi_A\rangle$ and $|\psi_A\rangle$ are states on $\mathcal{H}_A$ and $|e_B^i\rangle$ is *any* orthonormal basis of $\mathcal{H}_B$.

Suppose Alice and Bob each have a qudit (a quantum system with a d dimensional Hilbert space) with the overall state given by

$$|\Psi\rangle_{AB} = \sum_{i=1}^{d} c_i\, |\eta_i\rangle_A \otimes |\tau_i\rangle_B\,, \tag{8}$$

where $\langle\eta_i|\eta_j\rangle = \langle\tau_i|\tau_j\rangle = \delta_{ij}$ and $|\Psi\rangle_{AB}$ is normalized, so $\sum_{i=1}^{d} |c_i|^2 = 1$.

(ii) What is the reduced state on Alice's system? What is its spectrum? What is the reduced state on Bob's system?

(iii) A maximally entangled pure state can be defined by the following property: the reduced state of Alice or Bob's system is the maximally mixed state, meaning it is proportional to $\mathbb{1}$. If $|\Psi\rangle_{AB}$ is maximally entangled, what values can c_i take?

(iv) Suppose Bob applies a unitary V to his qudit. Now what is the reduced state of Alice's qudit?