Advanced quantum theory, May 30th 2019
Resolvents

Problem 1 *Krein formula*

Recall from the lecture that given the Hamiltonian H, the resolvent is

$$R(z) = (H - z)^{-1}. \tag{1}$$

If the Hamiltonian H can be written as $H = H_0 + V$, then

$$R(z) - R_0(z) = -R(z)VR_0(z). \tag{2}$$

In this exercise we will simplify (2) for the case of a rank-1 perturbation, that is,

$$H = H_0 + \lambda|\phi\rangle\langle\phi|, \quad \|\phi\| = 1. \tag{3}$$

1. Iterate (2) to obtain $R(z)$ as a power series in $VR_0(z)$.
2. Define $f_0 = \langle\phi, R_0(z)\phi\rangle$. Using the result from the previous subexercise, show that for $V = \lambda|\phi\rangle\langle\phi|$

$$R(z) = R_0(z) - \frac{\lambda}{1 + \lambda f_0(z)} R_0(z)|\phi\rangle\langle\phi|R_0(z) \tag{4}$$

Problem 2 *Eigenstates and poles of perturbed Hamiltonian*

1. Consider a free Hamiltonian $H_0 = p^2$ and $V = \lambda|\phi\rangle\langle\phi|$. From the time independent Schrödinger equation obtain

$$\psi(p) = -\lambda\langle\phi|\psi\rangle\frac{\phi(p)}{p^2 - \mu} \tag{5}$$

   for the eigenstate ψ with energy μ.
2. Take the inner product of (5) with $\langle\phi|$ to get

$$1 = -\lambda \int dp \frac{|\phi(p)|^2}{p^2 - \mu}. \tag{6}$$

   Observe that this condition is equivalent to $1 + \lambda f_0(\mu) = 0$ which gives poles in the expression (4).
3. Observe that the right hand side of (6) goes to 0 when $\mu \to -\infty$ and to infinity when $\mu \to p_0^2$ such that $\phi(p_0) \neq 0$. Conclude that for negative λ there exists a bound state.
4. Consider the Hamiltonian $H = H_0 + V = (\sum_n \epsilon_n|n\rangle\langle n|) + \lambda|\phi\rangle\langle\phi|$. Following the same logic as above, obtain

$$\psi_n = -\lambda\langle\phi|\psi\rangle\frac{\phi_n}{\epsilon_n - \mu} \tag{7}$$

   for the eigenvalue μ.

5. By taking the inner product, conclude that

$$1 = -\lambda \sum_n \frac{|\phi_n|^2}{\epsilon_n - \mu}. \tag{8}$$

Observe that the right hand side goes to $\pm\infty$ if μ approaches ϵ_n and the right hand side is monotone between ϵ_n and ϵ_{n+1}. Conclude that H has exactly one eigenvalue between ϵ_n and ϵ_{n+1}.

Problem 3 *Compactness of product*

Let Q be a position operator and P its conjugate momentum. We would like to show that $f(P)g(Q)$ is compact if f and g are sufficiently smooth operator-valued functions and have compact support.

1. Let $\hat{f}(x)$ be the Fourier transform of $f(p)$. Use that $\left(f(P)\psi\right)(x) = \int dy \hat{f}(x-y)\psi(y)$ to get the kernel $K(x,y)$ in

$$f(P)g(Q)\psi(x) = \int dy K(x,y)\psi(y). \tag{9}$$

2. Use smoothness to show that

$$tr(K^*K) = \|K\|_2^2 < \infty. \tag{10}$$

Conclude, that $f(P)g(Q)$ is compact.