Exercise 1: Bosonic operators

In the theory lecture we have discussed the creation and annihilation operators. For the case of bosons, we showed that the operators must fulfill the commutation relation: $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$. Let us practise a bit with the commutation relations for bosonic operators.

(a) Show that: $\left[\hat{a}_i, \left(\hat{a}_i^\dagger\right)^N\right] = N\left(\hat{a}_i^\dagger\right)^{N-1}$.

(b) Show that: $\hat{a}_i\hat{n}^m = (\hat{n}_i + 1)^m\hat{a}_i$, with $\hat{n}_i = \hat{a}_i^\dagger\hat{a}_i$.

(c) Let $\hat{H} = \sum_i \epsilon_i\hat{a}_i^\dagger\hat{a}_i$, where $\hat{a}_i$ and $\hat{a}_i^\dagger$ are the annihilation/creation operators for mode i. Show that: $\hat{a}_i(t) = e^{i\hat{H}t/\hbar}\hat{a}_i e^{-i\hat{H}t/\hbar} = e^{-i\epsilon_i t/\hbar}\hat{a}_i$. Do it in two different ways:

- Use the result of point (b) and the commutation relations.
- Solve the Heisenberg equation $i\hbar\partial_t\hat{a}_i = [\hat{a}_i, \hat{H}]$.

Exercise 2: Coherent states

Let $\hat{D}(\alpha) \equiv e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$. We define the coherent states as $|\alpha\rangle \equiv \hat{D}(\alpha)|0\rangle$. Coherent states play a crucial role in quantum mechanics (R. J. Glauber got in 2015 the Nobel Prize for his work on coherent states in quantum optics). We want to show in this exercise that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, i.e. the coherent states are eigenstates of the annihilation operator. For this we need:

(a) Show that $\hat{a}e^{\alpha\hat{a}^\dagger} = e^{\alpha\hat{a}^\dagger}(\hat{a} + \alpha)$.

(b) According to the Baker-Hausdorff formula, if two operators $\hat{A}$ and $\hat{B}$ fulfill that $[[\hat{A}, \hat{B}], \hat{A}] = [[\hat{A}, \hat{B}], \hat{B}] = 0$, then $e^{\hat{A}+\hat{B}} = e^{-[\hat{A},\hat{B}]/2}e^{\hat{A}}e^{\hat{B}}$. Use this result to show that: $\hat{D}(\alpha) = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}$ and $\hat{D}^\dagger(\alpha) = e^{|\alpha|^2/2}e^{\alpha^*\hat{a}}e^{-\alpha\hat{a}^\dagger}$.

(c) Using the results of points (a) and (b) show that $\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha$.

(d) Use the result of (c) to show that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.