Advanced quantum theory, 9th July 2019
Quantum channels

Problem 1 *Spin flips*

1. Consider an operation $\mathcal{E}_{sf}(\rho) = \sigma_x \rho \sigma_x$. Prove that $\mathcal{E}_{sf}(\cdot)$ is a channel.
2. Apply $\mathcal{E}_{sf}(\cdot)$ to a state $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$. By taking a convex combination with an identity channel, construct a channel that flips a spin with probability p.
3. Let there be N spins. How does a channel that flips any spin with probability p look like? How does a channel that flips that flips all the spins with probability p look like?

Problem 2 *Depolarizing channel*

In this problem we will study Kraus operators of a depolarizing channel $T(\rho) = tr(\rho)\frac{\mathbb{1}}{d}$ for a d-dimensional Hilbert space.

1. Let $|\psi_i\rangle \in \mathbb{C}^D,\ i = 1, \ldots, D$ and $\sum_i |\psi_i\rangle\langle\psi_i| = \mathbb{1}$. Prove that $|\psi_i\rangle$ are an orthonormal basis.
2. Apply Choi-Jamiolkowski isomorphism to the depolarizing channel, that is, compute $(T \otimes \mathbb{1})(|\Omega\rangle\langle\Omega|)$, where $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$. Use the result and previous subexercise to calculate the Kraus rank (the minimal number of Kraus operators) of T.
3. Find a Kraus decomposition of T using d^2 unitary operators.
4. What is the Kraus rank of $T_p(\rho) = p\rho + (1-p)tr(\rho)\frac{\mathbb{1}}{d}$?

Problem 3 *Stinespring's dilation*

In this exercise we will obtain an alternative form of Stinespring's dilation.

1. Suppose you would like to map a set of states state $\rho_1(t)$ to $\rho_2(t)$. Show that this operation can be written in a form
$$\rho_2(t) = \mathrm{tr}_2 \left[U \rho_1(t) \otimes \rho_2(t) U^\dagger \right] \tag{1}$$
where U is a unitary matrix.
2. Apply Choi-Jamiolkowski isomorphism to show that the map (1) can be written in a form $\rho_2(t) = \mathrm{tr}_2 \left[U \rho_1(t) \otimes |\phi\rangle\langle\phi| U^\dagger \right]$ for some vector $|\phi\rangle$.
3. Let us choose a vector $|G\rangle$ of a sufficiently large dimension. Prove that for any ρ_1 and ρ_2 there exists a unitary matrix U such that
$$\rho_2(t) = \mathrm{tr}_2 \left[U \rho_1(t) \otimes |G\rangle\langle G| U^\dagger \right]. \tag{2}$$
What is a sufficient dimension of $|G\rangle$?
4. What does the result from the previous subexercise imply about CPTP maps?