

1. UFF

The Person will be seeing a Force of $m_g \cdot g$ which is equal to $m_i \cdot a + F_{scale}$ meaning the Force, the scale and the person interact with. So the Force, the scale sees is $F_{scale} = -m_i \cdot a + m_g \cdot g$

2. energy conservation

The total momentum is given by $p = p_1 + p_2$. This is preserved, if $\frac{\partial p}{\partial t} = 0$:

$$\begin{aligned}\frac{\partial p_1}{\partial t} &= F_{1,2} = -G m_{1,p} m_{2,a} \frac{x_1 - x_2}{|x_1 - x_2|^3} \\ \frac{\partial p_2}{\partial t} &= F_{2,1} = -G m_{2,p} m_{1,a} \frac{x_2 - x_1}{|x_2 - x_1|^3}\end{aligned}$$

So that now we have equivalences:

$$\begin{aligned}\frac{\partial p}{\partial t} &= 0 \\ \Leftrightarrow F_{1,2} + F_{2,1} &= 0 \\ \Leftrightarrow -G \frac{1}{|x_2 - x_1|^3} (m_{1,p} m_{2,a} (x_1 - x_2) + m_{2,p} m_{1,a} (x_2 - x_1)) &= 0 \\ \Leftrightarrow m_{1,p} m_{2,a} &= m_{2,p} m_{1,a} \\ \Leftrightarrow \frac{m_{1,p}}{m_{1,a}} &= \frac{m_{2,p}}{m_{2,a}}\end{aligned}$$

3. gravitational field of a mass distribution

With the definition of M , we have the following:

$$\begin{aligned}M &:= \lim_{r \rightarrow \infty} \left\{ \frac{1}{4\pi G} \int_{S^2(r)} \nabla \phi \cdot \vec{n} \, d\sigma \right\} \\ &= \lim_{r \rightarrow \infty} \left\{ \frac{1}{4\pi G} \int_{B_0(r)} \nabla \cdot \nabla \phi \, dv \right\} \\ &= \lim_{r \rightarrow \infty} \left\{ \frac{1}{4\pi G} \int_{B_0(r)} \Delta \phi \, dv \right\} \\ &= \lim_{r \rightarrow \infty} \left\{ \frac{1}{4\pi G} \int_{B_0(r)} 4\pi G \rho \, dv \right\} \\ &= \lim_{r \rightarrow \infty} \left\{ \int_{B_0(r)} \rho \, dv \right\} \\ &= \int_{\mathbb{R}} \rho \, dv\end{aligned}$$

which needed to be shown.

To show the next identity, one just has to write out and use the definitions.

$$\begin{aligned}
 f_a &= -\nabla^b t_{ab} \\
 &= -\nabla^b \frac{1}{4\pi G} \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} \delta_{ab} \nabla_c \phi \nabla^c \phi \right) \\
 &= \frac{1}{4\pi G} \left(\nabla^b \nabla_a \phi \nabla_b \phi + \nabla_a \phi \nabla^b \nabla_b \phi - \frac{1}{2} \nabla^a \nabla_c \phi \nabla^c \phi - \frac{1}{2} \nabla_c \phi \nabla_a \nabla^c \phi \right) \\
 &= \frac{1}{4\pi G} (\nabla_a \phi \nabla^b \nabla_b \phi) \\
 &= \frac{1}{4\pi G} (\nabla_a \phi \Delta \phi) \\
 &= \rho \nabla_a
 \end{aligned}$$

the result, as was expected. The tensor t is symmetric, because all tensors in the definition of t are symmetric.

The total force vanishes because of the gaussian integral theorem:

$$\int_{\mathbb{R}} \nabla^b t_{ab} dv = \int_{\partial \mathbb{R}} t_{ab} n^b da = 0$$

Same for the total torque:

$$\int_{\mathbb{R}} \nabla^b t_{ab} x^a dv = \int_{\partial \mathbb{R}} t_{ab} x^a n^b da = 0$$

where in both cases the point being is, that $\partial \mathbb{R}$, the border of \mathbb{R} is empty.