Advanced quantum theory, April 25th 2019
Second quantization
For the discussions in class

Recall from the lecture that *creation* and *annihilation* operators obey

$$[a_\nu, a_\mu] = 0, \quad [a_\nu, a_\mu^\dagger] = \delta_{\mu\nu} \qquad \text{for bosons} \tag{1}$$
$$\{a_\nu, a_\mu\} = 0, \quad \{a_\nu, a_\mu^\dagger\} = \delta_{\mu\nu} \qquad \text{for fermions.} \tag{2}$$

Here $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$.
These operators act on Fock states as

$$a_i|\ldots, n_i, n_{i+1}, \ldots\rangle \;=\; \sqrt{n_i}|\ldots, n_i - 1, n_{i+1}, \ldots\rangle, \tag{3}$$
$$a_i^\dagger|\ldots, n_i, n_{i+1}, \ldots\rangle \;=\; \sqrt{n_i + 1}|\ldots, n_i + 1, n_{i+1}, \ldots\rangle. \tag{4}$$

Fock states form an orthonormal basis.

Problem 1 *Coherent states*

1. Write the *coherent state*

$$|\alpha\rangle = Ce^{\alpha a^\dagger}|0\rangle \tag{5}$$

   in the Fock basis for some complex numbers α and C.
   Find C such that the norm of $|\alpha\rangle$ is one.

2. Prove that

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha = \mathbb{1} \tag{6}$$

   Conclude, that coherent states form an overcomplete basis.

3. Express $a|\alpha\rangle$ in the coherent state basis.

Problem 2 *Commutation relations*

1. Let us define a number operator $N = \sum_i a_i^\dagger a_i$. Compute

$$[N, a_\mu] \qquad \text{for bosons} \tag{7}$$
$$\{N, a_\mu\} \qquad \text{for fermions.} \tag{8}$$

2. Let us call a product of operators *normal ordered* if all the creation operators are on the left and all the annihilation operators are on the right. Normal order the expression $a_\mu^\dagger a_\nu a_\kappa^\dagger a_\chi$.

3. Normal order the

$$a_\mu \left(a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_n}^\dagger a_{j_1} a_{j_2} \ldots a_{j_m}\right) \tag{9}$$

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4. Let us define a *field operator* $\psi(x) = \sum_i \phi_i(x) a_i$, where $\phi_i(x)$ is a complete orthonormal system. Find

$$[\psi(x), \psi^\dagger(x)] \qquad \text{for bosons} \tag{10}$$

$$\{\psi(x), \psi^\dagger(x)\} \qquad \text{for fermions.} \tag{11}$$

Problem 3 *Harmonic chain*

Suppose there is a chain of atoms with mass m that are connected via springs of stiffness K. The Hamiltonian is

$$H = \sum_{i=-\infty}^{+\infty} \left[\frac{p_i^2}{2M} + \frac{K}{2}(x_i - x_{i+1})^2 \right]. \tag{12}$$

1. Express the Hamiltonian in the Fourier transformed coordinates

$$x_l = \int_{-\pi}^{\pi} \tilde{x}_q e^{iql} \frac{dq}{2\pi}, \quad p_l = \int_{-\pi}^{\pi} \tilde{p}_q e^{iql} \frac{dq}{2\pi}. \tag{13}$$

2. Check that the commutation relations are

$$[\tilde{x}_l, \tilde{p}_q] = 2\pi i \delta(l+q) \tag{14}$$

3. Choose suitable $\omega(q)$ and define

$$\tilde{x}_l = \sqrt{\frac{1}{2m\omega(q)}} (a_{-l}^\dagger + a_l), \quad \tilde{p}_l = i\sqrt{\frac{m\omega(q)}{2}} (a_{-l}^\dagger - a_l). \tag{15}$$

   Observe that the Hamiltonian can be written as

$$H = \int_{-\pi}^{\pi} \omega(q) \left(a_q^\dagger a_q + \frac{1}{2} \right) \frac{dq}{2\pi}. \tag{16}$$