Exercise 1. holomorphic functions

a) Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. The functions $\Re(f)$ and $\Im(f)$ are harmonic

b) Let $U \subset \mathbb{C}^n$ be a connected open subset. Every non-constant harmonic function $f : U \to \mathbb{R}$ has no extremum on U

c) Show that any holomorphic function $f : M \to \mathbb{C}$, where M is a compact complex manifold, is constant

d) Deduce that $\mathbb{C}^n$ does not have any compact complex submanifolds of postive dimension.

Exercise 2. Hopf manifolds

Generalize the construction of the Hopf manifold as follows: Let $\lambda_1, \dots, \lambda_n$ be complex numbers such that $|\lambda_i| > 1$ for each $i = 1, \dots, n$. Consider the action of $\mathbb{Z}$ on $\mathbb{C}^n \setminus \{0\}$ given by

$$k \cdot (z_1, \dots, z_n) = \left(\lambda_1^k z_1, \dots, \lambda_n^k z_n\right)$$

where $k \in \mathbb{Z}$. Show that the quotient $\left(\mathbb{C}^n \setminus \{0\}\right)/\mathbb{Z}$ is again diffeomorphic to $S^1 \times S^{2n-1}$. (Hint: Find a suitable map $\Phi : \mathbb{R} \times S^{2n-1} \to \mathbb{C}^n \setminus \{0\}$ such that for all $k \in \mathbb{Z}$ the relation $k \cdot \Phi(t, u) = \Phi(t + k, u)$ holds.)

Exercise 3. vector fields

For a complex manifold M denote by $H^0(M, TM)$ the space of holomorphic vector fields on M. Let M_1, M_2 be two compact complex manifolds and let $M = M_1 \times M_2$ be their product (as a complex manifold). Prove that

$$H^0(M, TM) \overset{?}{\cong} H^0(M_1, TM_1) \oplus H^0(M_2, TM_2)$$

Give examples to show that if the compactness assumption is dropped, then the result may or may not hold.