1.4 Almost complex manifolds

Let M be a complex manifold of real dimension $2n$. Consider $TM_{\mathbb{R}}$ (the real tangent bundle).
Let (U,φ) be a holomorphic chart and define $J : T_pM_{\mathbb{R}} \longrightarrow T_pM_{\mathbb{R}},\ p \in U$ via

$$J(v) = (d\varphi)^{-1} \circ j_n \circ d\varphi(v),$$

where j_n is the standard linear complex structure on $\mathbb{R}^{2n}$

$$j_n(x_1,\ldots,x_n,y_1,\ldots,y_n) = (-y_1,\ldots,-y_n,x_1,\ldots,x_n).$$

If (V,ψ) is another holomorphic chart around p, then

$$\begin{aligned}(d\psi)^{-1} \circ j_n \circ d\psi(v) &= (d\psi)^{-1} \circ j_n \circ d\underbrace{(\psi \circ \varphi^{-1})}_{\text{holomorphic}} \circ d\varphi(v)\\ &= (d\psi)^{-1} \circ d(\psi \circ \varphi^{-1}) \circ j_n \circ d\varphi(v) = (d\varphi)^{-1} \circ j_n \circ d\varphi(v),\end{aligned}$$

so the definition does not depend on the chart. We obtain an endomorphism of the tangent bundle (i.e. a $(1,1)$-tensor)

$$J : TM_{\mathbb{R}} \longrightarrow TM_{\mathbb{R}}$$

satisfying $J^2 = -\mathrm{Id}$.

Definition 1.4.1. A $(1,1)$-tensor J on a smooth manifold M satisfying $J^2 = -\mathrm{Id}$ is called an *almost complex structure*. The pair (M,J) is then called an *almost complex manifold*.

A complex manifold is therefore canonically an almost complex manifold. We want to investigate the converse. Let (M,J) be an almost complex manifold. Complexify the tangent bundle $T^{\mathbb{C}}M$ (so complexify the vector space T_pM at every point) and consider the subbundles of vectors of type $(1,0)$ and $(0,1)$:

$$\begin{aligned}T^{1,0}M &= \{X - iJX \mid X \in TM\} \ \text{ – the } +i\text{-eigenbundle},\\ T^{0,1}M &= \{X + iJX \mid X \in TM\} \ \text{ – the } -i\text{-eigenbundle}.\end{aligned}$$

Suppose that J arises from complex coordinates $\{z_1,\ldots,z_n\}$ (i.e. (M,J) really is a complex manifold). Then the vectors

$$\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}, \quad \text{where} \quad \frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right)$$

are of type $(1,0)$ and

$$\frac{\partial}{\partial \bar{z}_1},\ldots,\frac{\partial}{\partial \bar{z}_n}, \quad \text{where} \quad \frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right)$$

of type $(0,1)$. They form bases of $T^{1,0}_pM$ and $T^{0,1}_pM$, respectively. If Z, W are two local sections of $T^{1,0}M$, i.e.

$$Z = \sum_{i=1}^{n} Z_i \frac{\partial}{\partial z_i}, \quad W = \sum_{j=1}^{n} W_i \frac{\partial}{\partial z_j},$$

then

$$[Z,W] = \sum_{i,j=1}^{n} \left(Z_i \frac{\partial W_j}{\partial z_i} - W_i \frac{\partial Z_j}{\partial z_i} \right) \frac{\partial}{\partial z_j}$$

is again a local section of $T^{1,0}M$. Similarly if Z, W are local sections of $T^{0,1}M$, then so is $[Z,W]$. Thus the condition[2] $[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$ is a necessary condition for the existence of complex coordinates inducing J. (Formally, this is similar to the *involutivity* required in the Frobenius theorem.)

It turns out that this necessary condition is also sufficient:

Theorem 1.4.2 (Newlander-Nirenberg)**.** *Let (M,J) be an almost complex manifold. The almost complex structure J arises from a holomorphic structure iff*

$$[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M.$$

One says then that J is ***integrable*** *and refers to J simply as* complex structure.

Let us work out what this condition means. Compute

$$[X + iJX, Y + iJY] = [X,Y] - [JX,JY] + i([JX,Y] + [X,JY]).$$

This should again be of the form $Z + iJZ$, which means that

$$[JX,Y] + [X,JY] = J([X,Y] - [JX,JY]).$$

Equivalently, the tensor[3]

$$N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$$

vanishes identically. N is called the *Nijenhuis tensor* (or the torsion of an almost complex manifold). Therefore an almost complex structure J arises from complex coordinates (i.e. (M,J)) is a complex manifold) iff the Nijenhuis tensor $N = N_J$ vanishes. The proof of the Newlander-Nirenberg theorem in full generality is much too long to present it here; next week I'll present a proof under the additional assumption that (M,J) is real-analytic. In the meantime, let us look at spheres.

Theorem 1.4.3 (Kirchhoff)**.** *If S^n admits an almost complex structure, then S^{n+1} has trivial tangent bundle.*

Proof. Let J be an almost complex structure on S^n. View S^n as the equator in S^{n+1}, which in turn is the unit sphere in $\mathbb{R}^{n+2}$. Set $e = (0,\ldots,0,1) \in \mathbb{R}^{n+2}$, so that every vector $x \in S^{n+1}$ can be written uniquely as $x = ae + by$, $b \geq 0$, $y \in S^n$. Consider

$$T_y S^n = \{z \in \mathbb{R}^{n+1} \mid z \perp y\},$$

and define $\sigma_x : \mathbb{R}^{n+1} \longrightarrow T_x S^{n+1}$ by

[2] $[T^{0,1}M, T^{0,1}M]$ is a shorthand for $\left[\Gamma(T^{0,1}M), \Gamma(T^{0,1}M)\right] \subset \Gamma(T^{0,1}M)$.

[3] In Homework 2 you are asked to show that this *is* a tensor.

$$\sigma_x(y) = ay - be$$
$$\sigma_x(z) = az + bJ_y(z), \quad \text{for} \quad z \in y^{\perp} = T_yS^n.$$

Let us check that this is in T_xS^{n+1}, i.e. that the right-hand side is orthogonal to $x = ae + by$. Obviously $\langle ay - be, ae + by \rangle = 0$. On the other hand $\sigma_x(z) \perp y$ by definition and, since $\sigma_x(z) \in \mathbb{R}^n$, $\sigma_x(z) \perp e$. Hence $\sigma_x(z) \perp x$. Thus we have a global map $S^n \times \mathbb{R}^{n+1} \longrightarrow TS^{n+1}$, $(x, v) \mapsto \sigma_x(v)$, linear for each x, and we only need to check that it is a bijection for each x. We show that $\mathrm{Ker}(\sigma_x) = 0$. Clearly $\sigma_x(y) \neq 0$. Suppose that $z \neq 0$ and $\sigma_x(z) = 0$. This means that $bJ_y(z) = -az$, and if $b \neq 0$, then z is an eigenvector of J_y with real eigenvalue, which is impossible. On the other hand, if $b = 0$, then $a = 0$, so $x = 0 \notin S^{n+1}$. □

Adams showed that TS^{n+1} is trivial iff $n+1 = 1, 3, 7$. Hence only S^2 and S^6 can admit an almost complex structure. For S^2 we already know this, since S^2 is diffeomorphic to $\mathbb{CP}^1$. Here is another description using *quaternions*, i.e. the algebra $\mathbb{H}$ consisting of pairs of complex numbers with coordinate-wise addition and multiplication given by

$$(z_1, z_2)(z_1', z_2') = (z_1 z_1' - z_2 \overline{z'}_2, z_1 z_2' + z_2 \overline{z'}_1).$$

This can be also interpreted by writing an element of $\mathbb{H}$ as $z_1 + z_2 j$, where $j^2 = -1$ and $ij = -ji$. The multiplication is then determined by these identities (plus the associativity and the distributivity). This multiplication is associative, but not commutative.

The quaternionic conjugate of $q = (z_1, z_2)$ is $\overline{q} = (\overline{z}_1, -z_2)$. We have $q\overline{q} = (z_1\overline{z}_1 + z_2\overline{z}_2, 0)$ and we define $|q|^2 = z_1\overline{z}_1 + z_2\overline{z}_2$. A quaternion is called *real* (resp. *purely imaginary*) if $q = \overline{q}$ (resp. $q = -\overline{q}$). q is purely imaginary iff $z_1 = -\overline{z}_1$, so these form a 3-dimensional subspace $\mathrm{Im}\,\mathbb{H}$. The scalar product on $\mathrm{Im}\,\mathbb{H} \simeq \mathbb{R}^3$ is given by $\langle q, q' \rangle = \mathrm{Re}(qq')$ and the vector product by $q \times q' = \mathrm{Im}(qq')$. Now:

$$S^2 = \{q \in \mathrm{Im}\,\mathbb{H} \mid |q| = 1\} \quad \text{and} \quad T_qS^2 = \{q' \in \mathrm{Im}\,\mathbb{H} \mid \langle q, q' \rangle = 0\}.$$

We define $J_q : T_qS^2 \to T_qS^2$ by

$$J_q(q') = q \times q'.$$

Then $J_q^2(q') \in T_qS^2$, since $q \times q' \perp q$. Moreover

$$J_q^2(q') = q \times (q \times q') = q \times (qq' - \mathrm{Re}\, qq') = q \times (qq') = \mathrm{Im}\, q(qq') = \mathrm{Im}\, q^2 q' = -q',$$

since any quaternion in S^2 satisfies $q^2 = -1$. Therefore J is an almost complex structure on S^2.

For S^6 we repeat the procedure. The algebra $\mathbb{O}$ of *Cayley numbers* (or *octonions*) is the set of pairs of quaternions with multiplication

$$(q_1, q_2)(q_1', q_2') = (q_1 q_1' - \overline{q_2'} q_2, q_2' q_2 + q_2 \overline{q_1'}).$$

This multiplication is not even associative. It does, however, satisfy the so-called *alternative law*:

$$x(xx') = (xx)x' \ , \quad (x'x)x = x'(xx),$$

i.e. associativity if two neighbouring factors are the same.

Again we have a conjugation:

$$\overline{(q_1, q_2)} = (\overline{q}_1, -q_2) \quad \text{with} \quad x\overline{x} = (q_1\overline{q}_1 + \overline{q}_2 q_2, 0),$$

and therefore a norm $|x|^2 = q_1\overline{q}_1 + q_2\overline{q}_2$. Again we can define real and purely imaginary Cayley numbers. The vector space of purely imaginary Cayley numbers is 7-dimensional, and it is equipped with a scalar product $\langle x, x'\rangle = -\operatorname{Re}(xx')$ and a vector product $x \times x' = \operatorname{Im}(xx')$. We have $x \times x' = -x' \times x$ and $\langle x \times x', x''\rangle = \langle x, x' \times x''\rangle$. Consider

$$S^6 = \{x \in \operatorname{Im}\mathbb{O} \mid |x| = 1\} \quad \text{and} \quad T_x S^6 = \{y \in \operatorname{Im}\mathbb{O} \mid \langle x, y\rangle = 0\}.$$

Define $J_x(y) = x \times y$. Again $J_x : T_xS^6 \to T_xS^6$ and again $J_x^2 = -\mathrm{Id}$ (observe that in the above calculation of J_q^2 for quaternions one needs exactly the alternative law). This almost complex structure on S^6 has $N \neq 0$, i.e. it is non-integrable. It is unknown whether S^6 admits a complex structure, i.e. whether S^6 is a complex manifold.

We have the following application:

Example 1.4.4. Let M be an oriented hypersurface in $\mathbb{R}^7$. For $m \in M$, consider the unit normal vector ν_m corresponding to the orientation. Then $T_mM \simeq \nu_m^\perp \simeq T_{\nu_m}S^6$. Therefore the almost complex structure on S^6 induces an almost complex structure on M. Thus every oriented hypersurface in $\mathbb{R}^7$ is an almost complex manifold.

1.5 Decomposition of the complexified exterior bundle

Let (M, J) be an almost complex manifold. We have seen that a complex structure on a vector space V induces a complex structure on V^*. Therefore we obtain a complex structure on each T^*_mM and consequently a decomposition of the complexified cotangent bundle

$$(T^*M)^{\mathbb{C}} = T^*M \otimes \mathbb{C}$$

into the $(1,0)$- and $(0,1)$-parts. For convenience, we shall write $\Lambda^1_{\mathbb{C}} = (T^*M)^{\mathbb{C}}$, $\Lambda^{1,0}M = \big((T^*M)^{\mathbb{C}}\big)^{(1,0)}$, and $\Lambda^{0,1}M = \big((T^*M)^{\mathbb{C}}\big)^{(0,1)}$. We have (see §1.2):

$$\Lambda^{1,0}M = \{\varphi - i\varphi \circ J \mid \varphi \in T^*M\}$$
$$\Lambda^{0,1}M = \{\varphi + i\varphi \circ J \mid \varphi \in T^*M\}.$$

Example 1.5.1. On $\mathbb{C}^n$ we have $(Jdx_i)(\frac{\partial}{\partial y_i}) = dx_i(J\frac{\partial}{\partial y_i}) = dx_i(-\frac{\partial}{\partial x_i}) = -1$, so $\Lambda^{1,0} = \{dx_i + idy_i\}$ and $Jdx_i = -dy_i$.

Lemma 1.5.2. *We have*

$$\Lambda^{1,0}M = \{\omega \in \Lambda^1_{\mathbb{C}}M \mid \omega(Z) = 0 \ \forall \ Z \in T^{0,1}M\}$$
$$\Lambda^{0,1}M = \{\omega \in \Lambda^1_{\mathbb{C}}M \mid \omega(Z) = 0 \ \forall \ Z \in T^{1,0}M\}$$

Proof. $\omega \in \Lambda^{1,0}M \iff \omega \circ J = i\omega \iff (\omega \circ J)(V) = i\omega(V) \ \forall \ V \in T^{\mathbb{C}}M$.
If we decompose $V = V^{1,0} + V^{0,1}$, then

$$(\omega \circ J)(V) = \omega(JV) = \omega(iV^{1,0} - iV^{0,1}) = i\omega(V^{1,0}) - i\omega(V^{0,1})$$

This is equal to $i\omega(V)$ iff $\omega(V^{0,1}) = 0$. Analogously for $\Lambda^{0,1}M$. □

We now decompose the k-th exterior power $\Lambda^k_{\mathbb{C}}M$ of $T^*M \otimes \mathbb{C}$:

$$\Lambda^k_{\mathbb{C}}M = \Lambda^k(\Lambda^{1,0}M \oplus \Lambda^{0,1}M) = \bigoplus_{p+q=k} \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M).$$

We shall write $\Lambda^{p,q}M = \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M)$. If $\varphi_1, \ldots, \varphi_n$ is a basis of $\Lambda^{1,0}_m M$, then $\overline{\varphi}_1, \ldots, \overline{\varphi}_n$ is a basis of $\Lambda^{0,1}_m M$, and the set of alternating forms

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p} \wedge \overline{\varphi}_{j_1} \wedge \cdots \wedge \overline{\varphi}_{j_q}, \quad \text{with } \ i_1 < \cdots < i_p \leq n, \ j_1 < \cdots < j_q \leq n,$$

is a basis of $\Lambda^{p,q}_m M$. Therefore the rank of $\Lambda^{p,q}M$ is $\binom{n}{p}\binom{n}{q}$.
Sections of $\Lambda^k_{\mathbb{C}}M$ are $\mathbb{C}$-valued differential forms; sections of $\Lambda^{p,q}M$ are called *differential forms of type (or degree) (p,q)* and their space is denoted by $\Omega^{p,q}(M)$.

Proposition 1.5.3.

$$d\Omega^{p,q} \subset \Omega^{p+2,q-1} \oplus \Omega^{p+1,q} \oplus \Omega^{p,q+1} \oplus \Omega^{p-1,q+2}.$$

Proof. Let $\omega \in \Omega^{p,q}(M)$. We can write it locally as

$$\omega = f\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p} \wedge \overline{\varphi}_{j_1} \wedge \cdots \wedge \overline{\varphi}_{j_q},$$

where $\varphi_1, \ldots, \varphi_n$ is a local frame of $(1,0)$-forms. We know that $df \in \Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$ and $d\varphi_s \in \Omega^2(M) = \Omega^{2,0}(M) \oplus \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$ and similarly for $\overline{\varphi}_s$. Applying d to ω decomposed as above proves the claim. □

For integrable almost complex structures this becomes much simpler, since we can choose a frame of the form $\varphi_i = dz_i$, where the z_i are local complex coordinates. Then $d(dz_i) = d(d\bar{z}_i) = 0$, and so

$$\begin{aligned} &d(f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}) \\ &= df \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q} \in \Omega^{p+1,q} \oplus \Omega^{p,q+1}, \end{aligned}$$

and also $df \in \Omega^{1,0} \oplus \Omega^{0,1}$. In fact we have:

Proposition 1.5.4. *For an almost complex manifold M, the following conditions are equivalent:*

a) If Z and W are complex vector fields of type $(1,0)$, then so is $[Z,W]$.

b) If Z and W are complex vector fields of type $(0,1)$, then so is $[Z,W]$.

c) $d\Omega^{1,0} \subset \Omega^{2,0} \oplus \Omega^{1,1}$ and $d\Omega^{0,1} \subset \Omega^{1,1} \oplus \Omega^{0,2}$.

d) $d\Omega^{p,q} \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1} \quad \forall p, q$.

e) the almost complex structure is integrable (i.e. $N = 0$).

Proof. Owing to the Newlander-Nirenberg theorem, we already know that a) $\iff$ b) $\iff$ e). Clearly d) $\implies$ c) and the argument in the proof of Proposition 1.5.3 implies that c) $\implies$ d). It remains to show that c) is equivalent to a) and b). Let ω be a 1-form of type $(0,1)$ and Z, W vector fields of type $(1,0)$. A well-known formula for the exterior derivative gives then

$$d\omega(Z,W) = Z(\underbrace{\omega(W)}_{=0}) - W(\underbrace{\omega(Z)}_{=0}) - \omega([Z,W]) = -\omega([Z,W]). \tag{1.5.1}$$

Observe that the 2nd formula in c) (denote it by c2)) is equivalent to $d\omega(Z,W) = 0$ for all $\omega \in \Omega^{0,1}$ and $(1,0)$ vector fields Z, W. Formula (1.5.1) implies that this is equivalent to $[Z,W]$ being of type $(1,0)$. Thus c2) $\iff$ a). Similarly c1) $\iff$ b). □

Given two manifolds M and M' and a smooth map $f : M \longrightarrow M'$, we can extend the differential f_* to a $\mathbb{C}$-linear mapping of $T^{\mathbb{C}}M$ to $T^{\mathbb{C}}M'$, which we still denote by f_*. Similarly[4] f^* maps complex differential forms on M' to complex differential forms on M.

Definition 1.5.5. A smooth map $f : (M,J) \longrightarrow (M',J')$ between almost complex manifolds is called *almost complex* if $f_* \circ J = J' \circ f_*$.

Note that for complex manifolds "almost complex map" is the same as "holomorphic map".

Proposition 1.5.6. *For a smooth map $f : (M,J) \longrightarrow (M',J')$ between almost complex manifolds the following conditions are equivalent:*

a) If Z is a complex tangent vector of type $(1,0)$ on M, then so is $f_*(Z)$ on M'.

b) If Z is a complex tangent vector of type $(0,1)$ on M, then so is $f_*(Z)$ on M'.

c) If ω is a complex differential form of type (p,q) on M', then $f^*\omega$ is a differential form of type (p,q) on M, for all p, q.

d) f is almost complex.

Proof. Homework. □

[4]Recall that the *pullback* $f^*\omega$ of a differential k-form is defined by $f^*\omega(X_1, \ldots, X_k) = \omega(f_*X_1, \ldots, f_*X_k)$.

Definition 1.5.7. An *infinitesimal automorphism* of an almost complex structure J on M is a vector field X such that $L_X J = 0$. (In other words, the local flow of X consists of (local) almost complex transformations.)

Proposition 1.5.8. *A vector field X is an infinitesimal automorphism of an almost complex structure J iff*

$$[X, JY] = J([X, Y]) \quad \forall\ Y \in \Gamma(TM).$$

Proof.

$$[X, JY] = L_X(JY) = (L_X J)Y + JL_X Y = (L_X J)Y + J([X, Y]).$$

□

Remark 1.5.9. If X is an infinitesimal automorphism of J, JX need not to be. In fact, the last proposition implies that if X is an infintesimal automorphism, then, for all vector fields Y,

$$N(X, Y) = [JX, JY] - J[JX, Y] - [X, Y] - J[X, JY] = [JX, JY] - J[JX, Y],$$

and so JX is also an infinitesimal automorphism iff $N(X, Y) = 0\ \forall\ Y$.
Conversely, it follows that if $N \equiv 0$, i.e. the almost complex structure J is integrable, then the Lie algebra $\mathfrak{a}$ of infinitesimal automorphisms of J is stable under J, and $[X, JY] = J[X, Y]\ \forall X, Y\ \in \mathfrak{a}$. Hence $\mathfrak{a}$ is a complex Lie algebra (possibly infinite-dimensional).

Proposition 1.5.10. *On a complex manifold M, the Lie algebra of infinitesimal automorphisms of the complex structure J is isomorphic to the Lie algebra of holomorphic vector fields, the isomorphism being given by*

$$X \mapsto Z = \frac{1}{2}(X - iJX).$$

Proof. Suppose that $X - iJX$ is holomorphic and $Y \in \Gamma(TM)$ is arbitrary. If f is a local holomorphic function, then

$$(X + iJX)(f) = 0 \implies (X - iJX)(f) = (2X - (X + iJX))(f) = 2X(f).$$

Hence $X(f)$ is holomorphic, which means that $(Y + iJY)(X(f)) = 0$ and of course $(Y + iJY)(f) = 0$. Therefore

$$[Y + iJY, X](f) = (Y + iJY)(X(f)) - X((Y + iJY)(f)) = 0.$$

On the other hand:

$$\begin{aligned}
[Y + iJY, X](f) = 0 &\iff [Y + iJY, X] \ \text{ is of type } (0,1) \\
&\iff \operatorname{Im}([Y + iJY, X]) = J \operatorname{Re}([Y + iJY, X]) \\
&\iff [JY, X] = J[Y, X] \iff X \in \mathfrak{a}.
\end{aligned}$$

Conversely, suppose that X is an infinitesimal automorphism of J. Due to Proposition 1.5.8, we know that $[JY, X] = J[Y, X]$, i.e. $[Y + iJY, X]$ is of type $(0, 1)$ for any vector field Y. Then (reversing the argument above) $[Y + iJY, X](f)$ for any local holomorphic function f, so $(Y + iJY)(X(f)) = 0$, which means that $X(f)$ is holomorphic, and hence $X - iJX$ is holomorphic.

Thus the map $\theta : X \mapsto \frac{1}{2}(X - iJX)$ is a linear isomorphism between infinitesimal automorphism of J and holomorphic vector fields on M. We need to check that this map is a Lie algebra homomorphism:

$$[\theta(X), \theta(Y)] = \frac{1}{4}[X - iJX, Y - iJY] = \frac{1}{2}\left([X, Y] - [JX, JY] - i[JX, Y] - i[X, JY]\right)$$
$$= \frac{1}{4}\left([X, Y] + [X, Y] - iJ[X, Y] - iJ[X, Y]\right) = \frac{1}{2}\left([X, Y] - iJ[X, Y]\right) = \theta([X, Y])$$

□

Definition 1.5.11. A real vector field X on a complex manifold is called *real-holomorphic* if $X - iJX$ is a holomorphic vector field.