Multi-curves

Variations on a theme

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Quantitative Research - OpenGamma

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Based on:

The irony in the derivatives discounting Part II: the Crisis. *Wilmott Journal*, 2: 301--316, 2010.

My future is not convex. *SSRN Working Paper series*, 2053657, May 2012.

Multi-curves: Variations on a Theme. *OpenGamma Quantitative Research n. 6*, October 2012. Available at http://developers.opengamma.com/quantitative-research/Multi-Curves-Variations-on-a-Theme-OpenGamma.pdf

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Varying theories of multi-curves

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Multi-curves variations.

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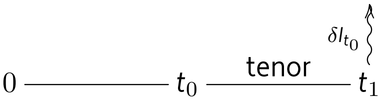
Introduction: one curve world



An Ibor coupons pays the Ibor rate at the end of the reference period.

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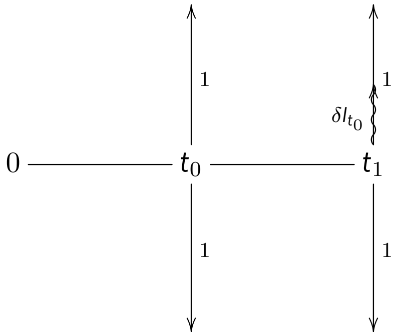
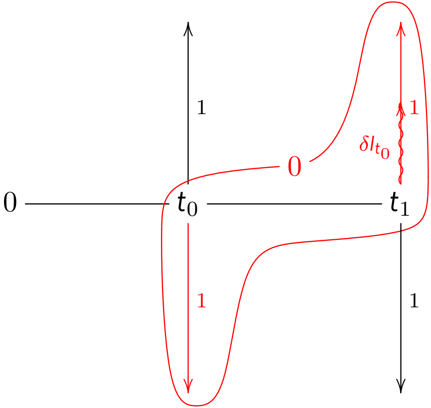
Introduction: one curve world



The notional is not exchanged in practice; the notional exchange is added to the figure to obtain a valuation mechanism.

Introduction: one curve world



The value of investing the notional at the start and receiving it back with the fair interest at the end of the reference period has a value of 0.

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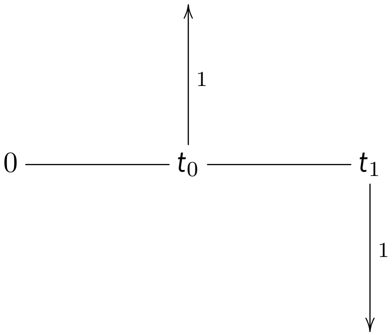
Introduction: one curve world



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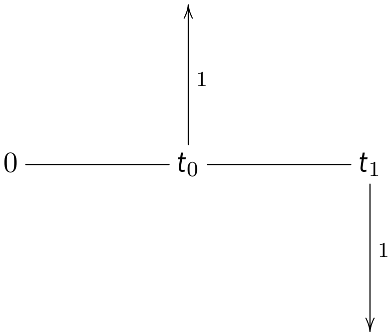

(Receiver) Ibor coupon value is

$$P(0, t_0) - P(0, t_1) = P(0, t_1) \left(\frac{P(0, t_0)}{P(0, t_1)} - 1 \right) = P(0, t_1)\delta F$$

Introduction

- The (nowadays standard) multi-curves framework recycles the one-curve formulas.
- There are potentially multiple coherent approaches to multi-curves discounting / estimation frameworks.
- Possible approaches are
  1. Ibor coupons from discount factor curve;
  2. Ibor coupons from direct forward rate curve;
  3. STIR futures from discount factor curve.

Cette histoire est vraie puisque je l'ai inventee...

Boris Vian

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Multi-curves variations.

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Discount factor curves: Hypothesis

Discounting: hypothesis common to all frameworks presented here.

D The instrument paying one unit in u is an asset for each u. It's value in t is denoted $P^D(t,u)$.

Ibor existence

$\mathrm{I}^{\mathrm{CPN}}$ The value of a j-Ibor floating coupon is an asset for each tenor and each fixing date.

In the presentation all asset prices are *continuous in time*.

Discount factor curves: Definitions

Definition (Pseudo-discount factors)

The forward curve $P^{\text{CDF},j}$ is the continuous function such that, $P^{\text{CDF},j}(t,t) = 1$, $P^{\text{CDF},j}(t,T)$ is an arbitrary function for $t < T < \text{Spot}(t) + j$ and, for $T \geq \text{Spot}(t)$,

$$P^D(t, T+j)\left(\frac{P^{\text{CDF},j}(t,T)}{P^{\text{CDF},j}(t,T+j)} - 1\right)$$

is the price in t of the j-Ibor coupon with start date T and maturity date $T + j$.

$$I^j_{t_0} = \frac{1}{\delta}\left(\frac{P^{\text{CDF},j}(t_0, \text{Spot}(t_0))}{P^{\text{CDF},j}(t_0, \text{Spot}(t_0) + j)} - 1\right)$$

Discount factor curves: Definitions

Definition (Forward rate)

The Ibor forward *rate over the period* $[t_{i-1}, t_i]$ *is given at time* t *by*

$$F_t^{\text{CDF},j}(t_{i-1}, t_i) = \frac{1}{\delta_i}\left(\frac{P^{\text{CDF},j}(t, t_{i-1})}{P^{\text{CDF},j}(t, t_i)} - 1\right).$$

With that definition the IRS price is

$$\sum_{i=1}^{\tilde{n}} c_i P^D(t, \tilde{t}_i) - \sum_{i=1}^{n} P^D(t, t_i)\delta_i F_t^{\text{CDF},j}(t_{i-1}, t_i).$$

Discount factor curves: Hypothesis

Definition (Spread)

The spread between a forward curve and the discounting curve is

$$\beta_t^{\mathrm{CDF},j}(u,u+j) = \frac{P_X^{\mathrm{CDF},j}(t,u)}{P_X^{\mathrm{CDF},j}(t,u+j)} \frac{P_X^D(t,u+j)}{P_X^D(t,u)}.$$

With that definition, a floating coupon price is

$$P^D(t,t_i)\left(\frac{P^{\mathrm{CDF},j}(t,t_{i-1})}{P^{\mathrm{CDF},j}(t,t_i)} - 1\right) = \beta_t^{\mathrm{CDF},j}(t_{i-1},t_i)P^D(t,t_{i-1}) - P^D(t,t_i)$$

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Discount factor curves: Hypothesis

Constant spread (to obtain simple formula for futures and FRA)

$S0^{CDF}$ The multiplicative coefficients between discount factor ratios, $\beta_t^{\text{CDF},j}(u, u+j)$, are constant through time: $\beta_t^{\text{CDF},j}(u, u+j) = \beta_0^{\text{CDF},j}(u, u+j)$ for all t and u.

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Discount factor curves: Results

Theorem (Futures price)

Let $0 \leq t \leq t_0 \leq t_1 \leq t_2$. In the one-factor Gaussian HJM model on the discounting curve under the hypotheses **D**, $\mathbf{I}^{\mathbf{CPN}}$ *and* $\mathbf{SO}^{\mathbf{CDF}}$, *the price of the futures fixing in t_0 for the period $[t_1, t_2]$ with accrual factor δ is given by*

$$
\begin{aligned}
\Phi_t^j &= 1 - \frac{1}{\delta}\left(\frac{P^{\mathrm{CDF},j}(t,t_1)}{P^{\mathrm{CDF},j}(t,t_2)}\gamma(t) - 1\right) \\
&= 1 - \gamma(t) F_t^{\mathrm{CDF},j} + \frac{1}{\delta}(1 - \gamma(t))
\end{aligned}
$$

where

$$
\gamma(t) = \exp\left(\int_t^{t_0} \nu(s,t_2)(\nu(s,t_2) - \nu(s,t_1))ds\right).
$$

Discount factor curves: Results

- Uses previously used formulas.
- Probably the multi-curves framework implemented in most software/financial institutions.
- But: maybe models the wrong thing (not a market instrument).

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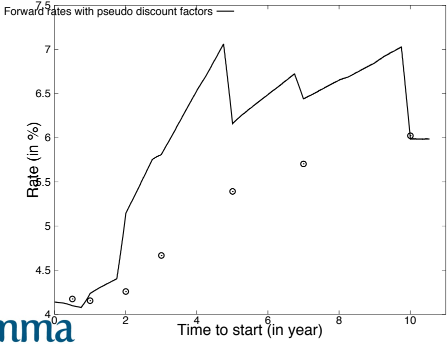

Multi-curves variations.

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Forward curves: Hypothesis

Why not modeling the forward rate directly (without the discount factor intermediary)?

Definition (Forward curve)

The forward curve $F^{\text{CFWD},j}$ is the continuous function such that,

$$P^D(t, t_2)\delta F_t^{\text{CFWD},j}(t_1)$$

is the price in t of the j-Ibor coupon with start date t_1 and maturity date t_2 ($t \leq t_0 \leq t_1 = \text{Spot}(t_0) < t_2$).

$$I_{t_0}^j = F_{t_0}^{\text{CFWD},j}(\text{Spot}(t_0)).$$

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Forward curves: Results

The value of a (fixed rate) receiver IRS is

$$\sum_{i=1}^{\tilde{n}} c_i P^D(t, \tilde{t}_i) - \sum_{i=1}^{n} P^D(t, t_i)\delta_i F_t^{\mathrm{CFWD},j}(t_{i-1}).$$

Definition (Spread)

The spread between a forward curve and the discounting curve is

$$\beta_t^{\mathrm{CFWD},j}(u, u+j) = (1 + \delta F_u^{\mathrm{CFWD},j})\frac{P^D(t, u+j)}{P^D(t, u)}.$$

A j-Ibor coupon price is

$$P^D(t, t_i)\delta_i F_t^{\mathrm{CFWD},j}(t_{i-1}) = \beta_t^{\mathrm{CFWD},j}(t_{i-1}, t_i)P^D(t, t_{i-1}) - P^D(t, t_i).$$

Forward curves: Results

$\mathrm{S0}^{\mathrm{CFWD}}$ The spreads $\beta_t^{\mathrm{CFWD},j}(u, u+j)$ are constant through time: $\beta_t^{\mathrm{CDF},j}(u, u+j) = \beta_0^{\mathrm{CDF},j}(u, u+j)$ for all t and u.

Theorem (Futures price)

Let $0 \leq t \leq t_0 \leq t_1 \leq t_2$. In the one-factor Gaussian HJM model on the discounting curve under the hypotheses **D**, **$\mathbf{I}^{\mathbf{CPN}}$** *and* **$\mathbf{S0}^{\mathbf{CFWD}}$**, *the price of the futures fixing in t_0 for the period $[t_1, t_2]$ with accrual factor δ is given by*

$$\Phi_t^j = 1 - \gamma(t) F_t^{\mathrm{CFWD},j} + \frac{1}{\delta}(1 - \gamma(t))$$

where

$$\gamma(t) = \exp\left(\int_t^{t_0} \nu(s, t_2)(\nu(s, t_2) - \nu(s, t_1)) ds \right).$$

Forward curves: Results

- Models directly the (forward) rate.
- Uses part of previously used formulas.

- Requires changes in libraries: forward curves do not provide discount factors anymore but only forward rate directly.
- Ibor discounting impossible: there is no discount factor linked to Ibor curves anymore.

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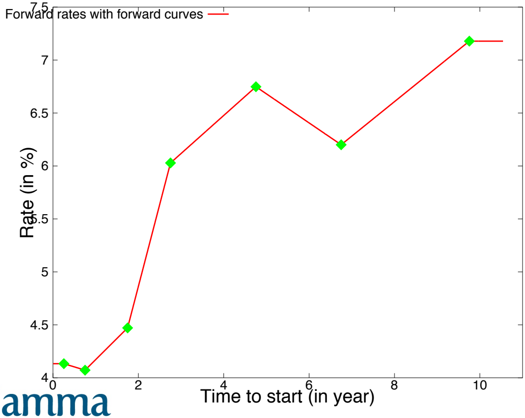

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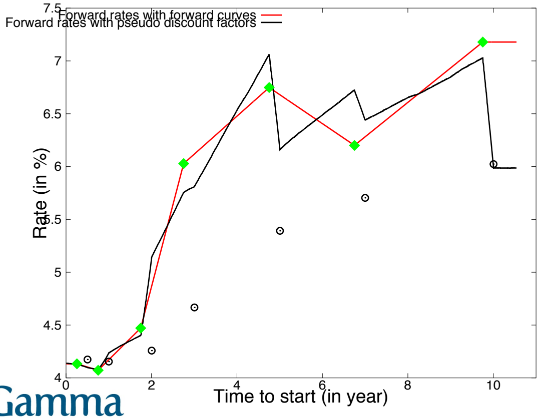

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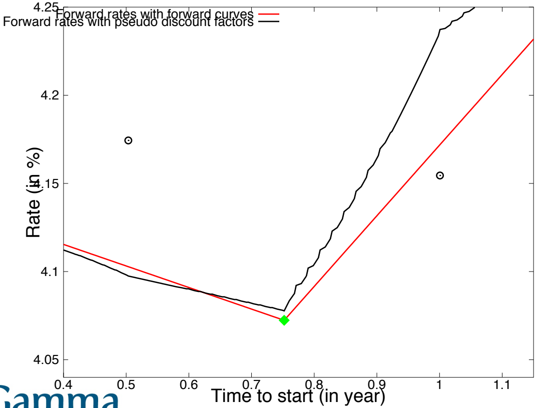

Multi-curves variations.

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Hypothesis

Hypothesis for the Ibor futures linked to futures price processes.

I^{FUT} The prices of the (j-Ibor) futures are futures price processes for each fixing date.

Definition (Futures pseudo-discount curves)

The forward curve $P^{\text{FDF},j}$ is the continuous function such that, $P^{\text{FDF},j}(t,t) = 1$, $P^{\text{FDF},j}(t,s)$ is an arbitrary function for $t \leq s < \text{Spot}(t) + j$, and for $t_0 \geq t$, $t_1 = \text{Spot}(t_0)$ and $t_2^f = t_1 + j$

$$\Phi_t^j(t_1) = 1 - \frac{1}{\delta}\left(\frac{P^{\text{FDF},j}(t,t_1)}{P^{\text{FDF},j}(t,t_2^f)} - 1\right).$$

Definitions

$$I^j_{t_0} = \frac{1}{\delta_f}\left(\frac{P^{\mathrm{FDF},j}(t_0, t_1)}{P^{\mathrm{FDF},j}(t_0, t^f_2)} - 1\right).$$

Definition (Futures rate)

The futures rate is given by

$$F^{\mathrm{FUT},j}_t(t_1) = 1 - \Phi^j_t(t_1).$$

Definition (Spread)

The variable $\beta^{\mathrm{FDF},j}_t(t_1, t_2)$ is defined as a ratio of discount factors ratios

$$\beta^{\mathrm{FDF},j}_t(t_1, t_2) = \frac{P^{\mathrm{FDF},j}(t, t_1)}{P^{\mathrm{FDF},j}(t, t^f_2)}\frac{P^D(t, t^p_2)}{P^D(t, t_1)} = (1 + \delta F^{\mathrm{FUT},j}_t(t_0))\frac{P^D(t, t^f_2)}{P^D(t, t_1)}.$$

Hypothesis

$\mathrm{SD}^{\mathrm{FDF}}$ The multiplicative coefficients between discount factor ratios, $\beta_t^{\mathrm{FDF},j}(t_1, t_2)$ are deterministic for all t_1.

This is the equivalent to the constant spread hypothesis $\mathbf{SO}^{\mathbf{CDF}}$ used in the coupon multi-curves framework.

An Ibor coupon pays the amount $\delta I_{t_0}^j$ in t_2. Its today's value is given by the following theorem.

Theorem (Coupon value)

In the futures multi-curves framework, under the hypothesis $\mathbf{I}^{\mathbf{FUT}}$ *and* $\mathbf{SD}^{\mathbf{FDF}}$*, the present value of the j-Ibor coupon fixing in* t_0 *for the period* $[t_1, t_2]$ *is given by*

$$P^D(0, t_1)\beta_{t_0}^{\mathrm{FDF},j}(t_1, t_2) - P^D(0, t_2).$$

Hypothesis

The quantity $\gamma(t)P^D(t,t_1)/P^D(t,t_2)$ is a N-martingale in the one-factor Gaussian HJM model. This is the base of the pricing of futures in the coupon framework. Our next hypothesis is coherent with that observation

HJM1 The quantities $\beta_t^{\text{FDF},j}(t_1,t_2)$ are such that

$$\beta_t^{\text{FDF},j} = \beta_0^{\text{FDF},j}\frac{\gamma(t)}{\gamma(0)}.$$

With the hypothesis we have

$$\frac{P^{\text{FDF},j}(t,t_1)}{P^{\text{FDF},j}(t,t_2)} = \frac{P^D(t,t_1)}{P^D(t,t_2)}\beta_t^{\text{FDF},j} = \frac{P^D(t,t_1)}{P^D(t,t_2)}\gamma(t)\frac{\beta_0^{\text{FDF},j}}{\gamma(0)}$$

Hypothesis

Theorem

In the futures multi-curves framework, under the hypothesis $\mathbf{I}^{\mathbf{FUT}}$ *and* **HJM1**, *in the one-factor Gaussian HJM model, the present value of the j-Ibor coupon fixing in* t_0 *for the period* $[t_1, t_2]$ *is given by*

$$P^D(0,t_1)\frac{\beta_0^{\mathrm{FDF},j}(t_1,t_2)}{\gamma(0)} - P^D(0,t_2) = P^D(0,t_2^p)\delta\frac{1}{\gamma(0)}\left(F_0^{\mathrm{FDF},j} + \frac{1}{\delta}(1-\gamma(0))\right).$$

Zero rate collateral

Standardization: There is a lot of discussion about CSA standardization. One of the proposal is to have payment of zero rate on the collateral.

With that proposal, the coupon *pays* $I^j_{t_0}$ in t_2 and is a futures price process. According to the general futures price theorem its value in 0 is

$$\mathrm{E}^N\left[I^j_{t_0}\right] = 1 - \mathrm{E}^N\left[1 - I^j_{t_0}\right] = 1 - \Phi^j_0(t_1).$$

When collateral rate is not the risk-free rate, a convexity adjustment is required between collateralized trade and risk free trades.

Multi-curves variations.

Conclusion

- Several multi-curves frameworks are possible.
- Differentiated by fundamental market instruments (coupons or futures) and by the way the forward curves are represented (pseudo-discount factors or direct market forward rates)
- Each approach has its advantages/drawbacks. The non-familiar one should not be disregarded immediately.
- The new approaches usually require library changes. From experience, the changes tend to clarify the distinction between financial meaning and the way the data is stored.

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