Cash-settled swaptions

How wrong are we?

Marc Henrard

Quantitative Research - OpenGamma

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Cash-settled swaptions: How wrong are we?
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Cash settled swaptions

Introduction

- Cash-settled swaptions are the most liquid swaptions in the interbank market for EUR and GBP.
- The standard pricing formulas used are not properly justified (they require untested *approximations*).
- The standard pricing formulas with standard smile description leads to arbitrage: see F. Mercurio, *Cash-settled swaptions and no-arbitrage*, Risk, 2008.
- Local coherence between Market pricing formula for cash-settled and physical delivery swaption has not been analyzed.

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Cash settled swaptions

Description

The analyses are done for receiver swaptions.
Swap start date t_0, fixed leg payment dates $(t_i)_{1 \leq i \leq n}$, coupon rate K and the accrual fraction $(\delta_i)_{1 \leq i \leq n}$.
Floating leg payment dates $(\tilde{t}_i)_{1 \leq i \leq \tilde{n}}$.
Swaption expiry date: $\theta \leq t_0$.

The cash annuity for a swap rate S is (for a frequency m)

$$C(S) = \sum_{i=1}^{n} \frac{\frac{1}{m}}{(1 + \frac{1}{m} S)^i}.$$

The cash-settled swaption payment is (in t_0)

$$C(S)(K - S)^+.$$

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Swap rate

We work in a multi-curve set-up. The discounting curve is denoted $P^D(s,t)$ and the forward curve is denoted $P^j(s,t)$ where j is the Libor tenor.

The physical annuity (also called PVBP or level) is

$$A_t = \sum_{i=1}^{n} \delta_i P^D(t, t_i).$$

The swap rate in t is

$$S_t = \frac{\sum_{i=1}^{\tilde{n}} P^D(t, \tilde{t}_i) \left(\frac{P^j(t,\tilde{t}_i)}{P^j(t,\tilde{t}_{i+1})} - 1 \right)}{A_t}.$$

Market formulas (physical delivery)

The value at expiry of the delivery swaption is

$$A_\theta (K - S_\theta)^+.$$

The generic price is

$$N_0 \, \mathrm{E}^N [N_\theta^{-1} A_\theta (K - S_\theta)^+]$$

With the numeraire A_t, S_t is a martingale and under log-normal model

$$dS_t = \sigma S_t dW_t,$$

the price can be computed explicitly

$$A_0 \, \mathrm{Black}(K, S_0, \sigma).$$

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Market formulas (cash-settled)

The value at expiry of the cash-settle swaption is

$$P^D(\theta, t_0)C(S_\theta)(K - S_\theta)^+.$$

With the numeraire $P^D(t, t_0)C(S_t)$,the price becomes

$$P^D(0, t_0)C(S_0)\,\mathrm{E}^C[(K - S_\theta)^+].$$

But S_θ is not a martingale anymore.
The market standard formula is to substitute C by A as numeraire and approximate the price by

$$\begin{aligned} P(0, t_0)C(S_0)\,\mathrm{E}^C[(K - S_\theta)^+] &\simeq P^D(0, t_0)C(S_0)\,\mathrm{E}^A[(K - S_\theta)^+] \\ &= P^D(0, t_0)C(S_0)\,\mathrm{Black}(K, S_0, \sigma). \end{aligned}$$

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Market formulas (cash-settled)

The value at expiry of the cash-settle swaption is

$$P^D(\theta, t_0) C(S_\theta)(K - S_\theta)^+.$$

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Market formulas (cash-settled)

The value at expiry of the cash-settle swaption is

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The market standard formula is to substitute C by A as numeraire and approximate the price by

$$\begin{aligned} P(0, t_0) C(S_0) \, \mathrm{E}^{C}[(K - S_\theta)^+] &\simeq P^D(0, t_0) C(S_0) \, \mathrm{E}^{A}[(K - S_\theta)^+] \\ &= P^D(0, t_0) C(S_0) \, \mathrm{Black}(K, S_0, \sigma). \end{aligned}$$

Market formulas (cash-settled)

The price is

$$A_0 \, \mathrm{E}^A \left[A_\theta^{-1} P^D(\theta, t_0) C(S_\theta)(K - S_\theta)^+ \right]$$

Consider that the annuity/annuity ratio

$$\frac{P^D(\theta, t_0) C(S_\theta)}{A_\theta}$$

has a low variance and it can be replaced by its initial value (initial freeze technique)

$$\begin{aligned} & A_0 \, \mathrm{E}^A \left[A_\theta^{-1} P^D(\theta, t_0) C(S_\theta)(K - S_\theta)^+ \right] \\ \simeq \; & A_0 \, \mathrm{E}^A \left[A_0^{-1} P^D(0, t_0) C(S_0)(K - S_\theta)^+ \right] \\ = \; & P^D(0, t_0) C(S_0) \, \mathrm{E}^A \left[(K - S_\theta)^+ \right]. \end{aligned}$$

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Market formulas (cash-settled)

The price is

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$$\begin{aligned}
& A_0 \, \mathrm{E}^A \left[A_\theta^{-1} P^D(\theta, t_0) C(S_\theta)(K - S_\theta)^+ \right] \\
\simeq \; & A_0 \, \mathrm{E}^A \left[A_0^{-1} P^D(0, t_0) C(S_0)(K - S_\theta)^+ \right] \\
= \; & P^D(0, t_0) C(S_0) \, \mathrm{E}^A \left[(K - S_\theta)^+ \right].
\end{aligned}$$

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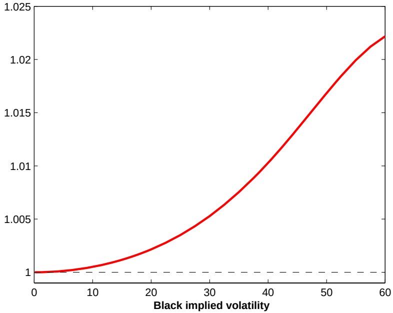
Mercurio's result

Mercurio presented no arbitrage conditions for cash-settled swaptions. One relates to the density deduced from option prices:

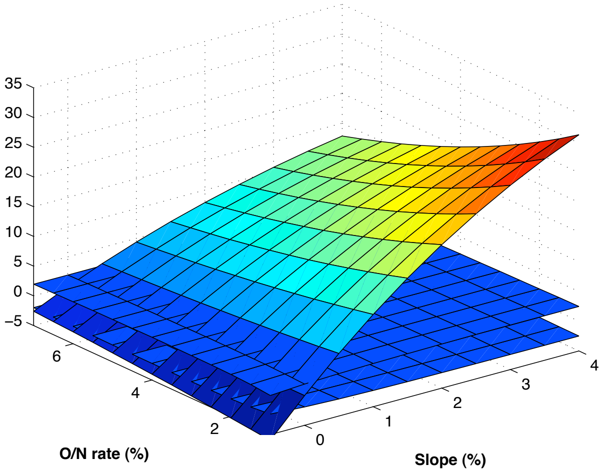
$$C(S_0)\int_0^{+\infty}\frac{\frac{\partial^2}{\partial x^2}\,\mathrm{Black}(x,S_0,\sigma)}{C(x)}dx=1.$$

If not a *digital cash-settled option* with strike 0 has a value different from its constant pay-off.



Cash-settle and physical swaptions

Difference between cash-settled and physical delivery swaptions for different curve environments.



Cash settled swaptions

Data

The tests are done for 5Yx10Y swaptions.
For all the model used, explicit formula exists for the delivery swaptions.
The market data are from 30-Sep-2010. The smile is obtained with a Hagan et al. SABR approach.
When a calibration is required, it is done on physical delivery swaption. At least the swaption of same maturity and strike is in the calibration basket (and calibrated perfectly).

Linear Terminal Swap Rate model

This is not a term structure model but an approximation technique to treat the swap rate as the only state variable for the yield curve. Applied in our case, the quantity $P^D(\theta, t_0)A_\theta^{-1}$ is approximated by a swap rate function $\alpha(S_\theta)$. It is an approximation technique more sophisticated than the initial freeze approach.
It is presented in Andersen, L. and Piterbarg, V. *Interest Rate Modeling*, Section 16.3.
The cash-settled swaptions price is

$$A_0 \, \mathrm{E}^A \left[\alpha(S_\theta) C(S_\theta)(K - S_\theta)^+ \right].$$

In the simple linear TSR, function is $\alpha(S) = \alpha_1 S + \alpha_2$ and the coefficients are

$$\alpha_2 = \frac{1}{\sum_{i=1}^n \delta_i} \quad \text{and} \quad \alpha_1 = \frac{1}{S_0} \left(\frac{P^D(0, t_p)}{A_0} - \alpha_2 \right).$$

Linear Terminal Swap Rate model

In this framework, the cash-settled swaption are priced by a replication formula

$$A_0 \left(k(K)\,\mathrm{Swpt}(S_0, K) + \int_K^{+\infty} (k''(x)(x-K) + 2k'(x))\,\mathrm{Swpt}(S_0, x) dx \right)$$

with $k(x) = (\alpha_1 x + \alpha_2) C(x)$.
Note that the linear TSR is such that $\alpha(S_0) = P^D(0, t_0) A_0^{-1}$ (the approximation is exact in 0 and ATM).

Linear Terminal Swap Rate model

Strike	Mrkt	D-C	TSR	Diff.	Vega
			Payer		
0.48	2318.84	19.40	2312.17	6.65	0.45
1.98	1308.67	10.95	1294.43	14.24	1.50
3.48	493.27	4.12	479.48	13.78	2.65
4.98	161.56	1.35	151.75	9.80	1.80
6.48	81.15	0.67	73.45	7.70	1.00
7.98	52.07	0.43	45.63	6.44	0.70
			Receiver		
0.48	39.53	0.33	39.84	-0.31	0.45
1.98	169.02	1.41	169.45	-0.43	1.50
3.48	493.27	4.12	490.03	3.23	2.65
4.98	1301.21	10.88	1277.24	23.97	1.80
6.48	2360.46	19.75	2279.45	81.01	1.00
7.98	3471.03	29.04	3287.18	183.84	0.70

Hull-White: mean reversion

Strike	**Mean reversion**				
	0.1%	1%	2%	5%	10%
			Payer		
1.39	5.50	3.97	2.26	-2.86	-11.32
2.89	2.04	1.02	-0.12	-3.50	-9.01
4.39	0.47	-0.03	-0.59	-2.26	-4.93
5.89	0.14	-0.10	-0.38	-1.19	-2.48
7.39	0.08	-0.07	-0.23	-0.72	-1.49
8.89	0.06	-0.04	-0.15	-0.48	-0.99
			Receiver		
1.39	0.36	0.59	0.85	1.61	2.80
2.89	0.64	1.03	1.47	2.74	4.74
4.39	1.24	1.89	2.61	4.72	8.02
5.89	3.40	4.65	6.02	10.00	16.17
7.39	8.38	10.54	12.89	19.69	30.08
8.89	16.73	20.02	23.59	33.79	49.12

Hull-White: mean reversion

Strike	Mean reversion				
	0.1%	1%	2%	5%	10%
			Payer		
1.39	5.50	3.97	2.26	-2.86	-11.32
2.89	2.04	1.02	-0.12	-3.50	-9.01
4.39	0.47	-0.03	-0.59	-2.26	-4.93
5.89	0.14	-0.10	-0.38	-1.19	-2.48
7.39	0.08	-0.07	-0.23	-0.72	-1.49
8.89	0.06	-0.04	-0.15	-0.48	-0.99
			Receiver		
1.39	0.36	0.59	0.85	1.61	2.80
2.89	0.64	1.03	1.47	2.74	4.74
4.39	1.24	1.89	2.61	4.72	8.02
5.89	3.40	4.65	6.02	10.00	16.17
7.39	8.38	10.54	12.89	19.69	30.08
8.89	16.73	20.02	23.59	33.79	49.12

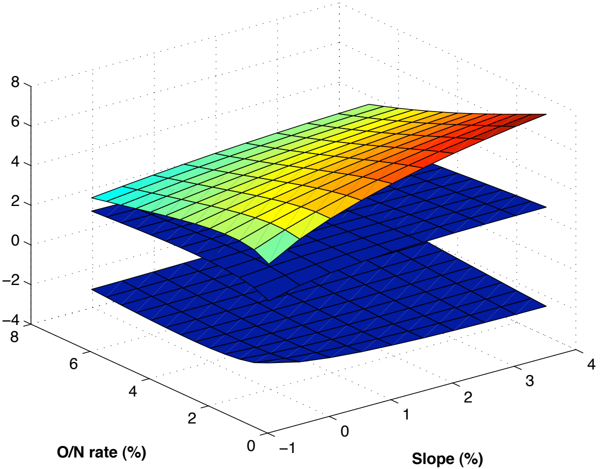
Hull-White: delta

Tenor	Delta DSC			Delta FWD		
	Mrkt	HW	Diff.	Mrkt	HW	Diff.
5Y	-5719.6	-5703.1	16.5	40450.3	40175.5	-274.8
6Y	-248.1	-309.8	-61.6	-1727.8	-1640.7	87.0
7Y	-140.2	-192.8	-52.6	-2184.7	-2088.9	95.8
8Y	-67.0	-105.9	-38.9	-2481.7	-2380.4	101.2
9Y	-27.9	-49.2	-21.3	-2732.0	-2627.7	104.2
10Y	49.3	48.2	-1.0	-2953.6	-2849.3	104.3
11Y	116.8	139.3	22.5	-3187.4	-3084.4	102.9
12Y	164.7	212.5	47.8	-3364.4	-3264.9	99.5
13Y	150.1	224.3	74.2	-3462.6	-3367.6	94.9
14Y	112.3	214.9	102.7	-3541.7	-3452.4	89.2
15Y	-828.9	-677.9	151.0	-82320.0	-82929.3	-609.3
Total	-6438.7	-6199.5	239.1	-67505.8	-67510.5	-4.7

Strike: ATM+1.5% – Notional: 100m.

Hull-White: curves

Differences for different curve shapes.



G2++: correlation

Calibration on one swaption, imposed mean reversions (1% and 30%), imposed volatility ratio (4). Pricing: numerical integration.

Strike	Correlation				
	-90%	-45%	0%	45%	90%
			Payer		
0.48	4.63	3.93	3.10	2.34	1.57
1.98	3.14	1.97	1.19	0.56	0.45
3.48	0.48	0.19	-0.18	-0.40	-0.40
4.98	-0.99	-0.81	-1.28	-1.10	-1.51
6.48	-1.02	-2.15	-2.22	-2.25	-1.92
7.98	-3.28	-3.16	-3.33	-3.37	-3.37
			Receiver		
0.48	0.47	0.54	0.68	0.61	0.92
1.98	1.24	1.47	1.66	3.02	2.01
3.48	1.43	1.93	2.24	2.54	3.05
4.98	2.86	3.82	4.27	5.35	6.00
6.48	8.35	9.87	11.27	12.76	14.17
7.98	18.62	21.23	23.78	26.21	28.05

G2++: term structure of volatility

Calibration to 10Y and 1Y swaptions. Missing numbers due to not perfect calibrations.

Strike	**Correlation**				
	-90%	-45%	0%	45%	90%
			Payer		
1.39	8.77				
2.89	3.91				
4.39		1.09	0.07	-0.09	0.03
5.89		-0.80	-1.31	-1.34	-1.57
7.39			-2.12	-1.97	-2.40
8.89		-2.96	-3.07	-2.73	-1.68
			Receiver		
1.39	2.92				
2.89	1.01				
4.39		0.79	1.77	2.10	2.35
5.89		1.45	4.04	5.08	5.28
7.39			8.34	9.76	10.22
8.89		10.14	16.22	17.52	17.74

LMM: multi-factor

Calibration to one swaptions (tenor 10Y); displacement is 10%.
Pricing by Monte-Carlo.

Strike	**Angle**				
	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
			Payer		
0.48	8.89	3.98	6.46	7.82	4.69
1.98	4.86	2.86	1.10	3.21	3.20
3.48	1.19	0.19	-0.39	0.95	1.52
4.98	-0.08	0.52	0.24	-0.16	0.21
6.48	-0.13	-0.67	0.03	-0.39	-0.29
7.98	-0.80	-0.85	-0.43	-0.84	-0.25
			Receiver		
0.48	-0.68	-0.10	-0.54	-0.29	-0.35
1.98	0.06	-0.01	0.61	-0.10	0.34
3.48	1.57	0.70	0.10	0.81	0.92
4.98	3.14	1.25	1.18	2.43	3.90
6.48	4.23	5.88	5.25	5.82	6.62
7.98	12.31	10.04	12.79	14.20	12.63

LMM: multi-factor

Calibration to swaptions with tenor 1Y to 10Y (10 calibrating instruments); displacement is 10%. Pricing by Monte-Carlo.

Strike	**Angle**				
	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
			Payer		
0.48	-0.04	-0.47	3.11	3.60	6.21
1.98	-7.60	-6.52	-7.01	-4.98	-3.10
3.48	-8.29	-7.69	-7.25	-7.78	-6.03
4.98	-5.00	-4.54	-3.22	-4.53	-3.67
6.48	-2.71	-3.41	-3.60	-3.14	-3.26
7.98	-4.55	-3.42	-3.49	-3.88	-2.81
			Receiver		
0.48	-0.10	-0.33	-0.47	-0.33	-0.39
1.98	1.78	1.98	1.53	1.36	0.53
3.48	7.93	7.78	7.18	6.94	6.72
4.98	19.43	19.81	18.09	18.55	17.60
6.48	32.98	33.05	33.31	29.43	28.80
7.98	51.66	47.57	46.64	46.91	43.42

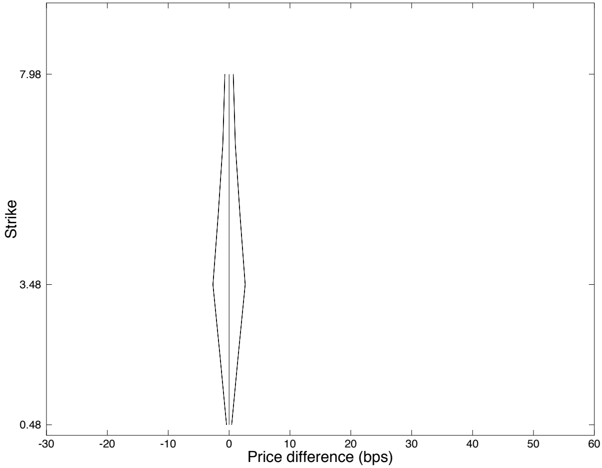
LMM: skew

Calibration to swaptions with tenor 10Y (one calibrating instrument); angle is $\pi/2$.

Strike	**Displacement**		
	0.05	0.10	1.00
	Payer		
0.48	10.21	4.47	6.65
1.98	4.54	2.12	1.56
3.48	1.69	0.46	1.14
4.98	0.49	0.04	0.07
6.48	0.19	0.06	0.17
7.98	0.37	-0.13	0.37
	Receiver		
0.48	0.54	-0.41	0.05
1.98	0.72	-0.10	0.43
3.48	1.85	-0.34	0.12
4.98	1.97	1.98	1.20
6.48	3.53	5.33	11.39
7.98	11.29	11.04	18.82

Multi-models: summary

Market data as of 30-Sep-2010.



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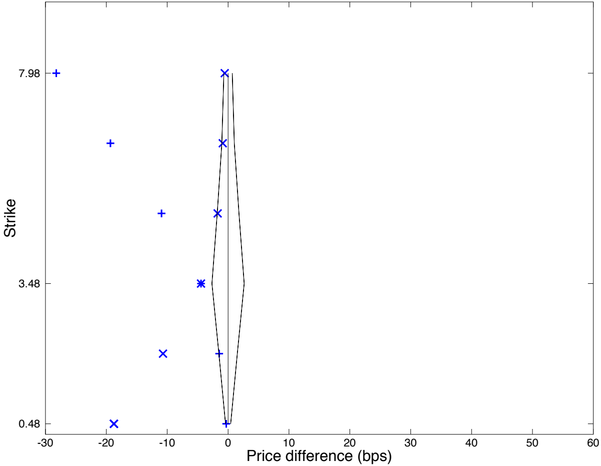
All figures computed with OpenGamma open source *OG-Analytics* library.

Multi-models: summary

Market data as of 30-Sep-2010.

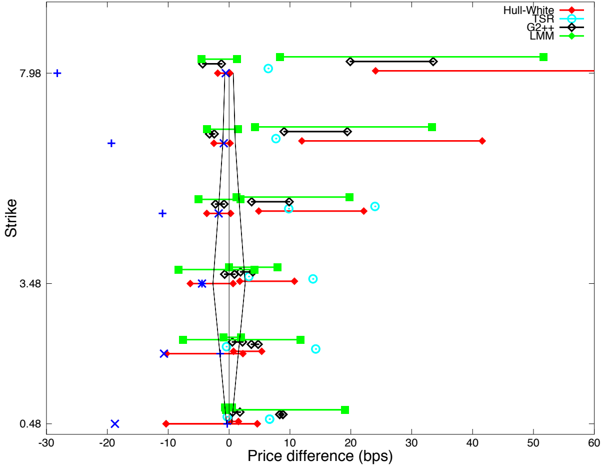


OpenGamma

All figures computed with OpenGamma open source *OG-Analytics* library.

Multi-models: summary

Market data as of 30-Sep-2010.



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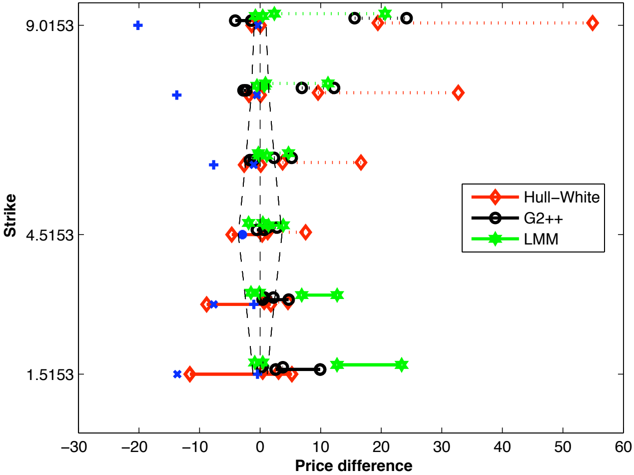
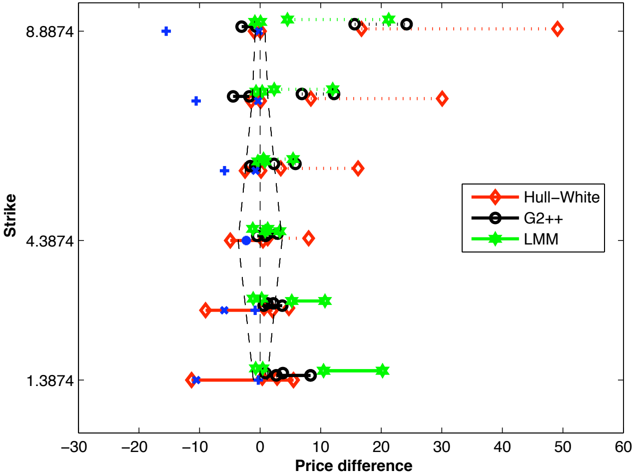
Multi-models: summary

Market data as of 23-Apr-2009.



Multi-models: summary

Market data as of 30-Apr-2010.



Cash settled swaptions

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Conclusion

- The standard market formula is arbitrable with Black or SABR smile (Mercurio result).
- We presented multi-models analyses of cash-settled swaption prices and delta.
- The model prices can be very far away from standard market formula prices.
- Within one model, the price range from non-calibrated parameters can be large.
- Risk uncertainty between different models for cash-settled swaption is not large but usually applies to non-netting positions.
- The cash-settled swaption are liquid but not as simple as they may look!

Conclusion

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Approximation in Hull-White model

In the case of the Hull-White model, it is possible to obtain an efficient approximated formula for the cash-settled swaptions.
All variable can be written as function of a normally distributed random variable X.
Exercise boundary: $S_\theta(X) < K$ which is equivalent to $X < \kappa$ where κ is defined by

$$S_\theta(\kappa) = K.$$

The rate S_θ is roughly linear in X and the annuity is also roughly linear. The result is more parabola shape than a straight line. A third order approximation is required to obtain a precise enough formula.
The pay-off expansion around a reference point X_0 is

$$C(S_\theta(X))(K - S_\theta(X)) \sim U_0 + U_1(X - X_0) + \frac{1}{2}U_2(X - X_0)^2 + \frac{1}{3!}U_3(X - X_0)^3.$$

Approximation in Hull-White model

Theorem

In the extended Vasicek model, the price of a cash-settled receiver swaption is given to the third order by

$$P(0,t_0)\,\mathrm{E}\left[\exp\left(-\tilde{\alpha}_0 X - \frac{1}{2}\tilde{\alpha}_0^2\right)\left(U_0 + U_1(X-X_0) + \frac{1}{2}U_2(X-X_0)^2 + \frac{1}{3!}U_3(X-X_0)^3\right)\right]$$

$$\simeq \left(U_0 - U_1\tilde{\alpha}_0 + \frac{1}{2}U_2(1+\alpha_0^2) - \frac{1}{3!}U_3(\tilde{\alpha}_0^3 + 3\alpha_0)\right)N(\tilde{\kappa})$$
$$+\left(-U_1 - \frac{1}{2}U_2(-2\tilde{\alpha}_0 + \tilde{\kappa}) + \frac{1}{3!}U_3(-3\tilde{\alpha}_0^2 + 3\tilde{\kappa}\tilde{\alpha}_0 - \tilde{\kappa}^2 - 2)\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\tilde{\kappa}^2\right).$$

where κ is given above, U_i are the pay-off expansion coefficients, $\tilde{\kappa} = \kappa + \alpha_0$ and $\tilde{\alpha}_0 = \alpha_0 + X_0$.

Approximation in Hull-White model

Strike	Int.	Appr. 2	Int-Ap. 2	Approx. 3	Int-Ap. 3
			Payer		
-2.50	1817.53	1811.42	6.11	1817.86	-0.33
-1.50	1199.82	1195.24	4.59	1200.00	-0.18
-0.50	662.30	658.99	3.31	662.43	-0.14
0.00	458.80	455.73	3.07	458.95	-0.15
0.50	318.27	316.16	2.11	318.38	-0.10
1.50	177.44	176.29	1.15	177.50	-0.06
2.50	117.96	117.10	0.86	118.01	-0.05
			Receiver		
-2.50	112.39	111.57	0.82	112.33	0.06
-1.50	175.30	174.21	1.10	175.23	0.07
-0.50	319.35	317.40	1.95	319.22	0.13
0.00	456.70	453.60	3.10	456.49	0.21
0.50	656.80	652.99	3.81	656.55	0.25
1.50	1195.97	1190.11	5.86	1195.61	0.37
2.50	1814.60	1805.32	9.28	1813.94	0.66