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# Introduction to 3-dimensional catesian, spherical polar & cylindrical co-ordinate system.

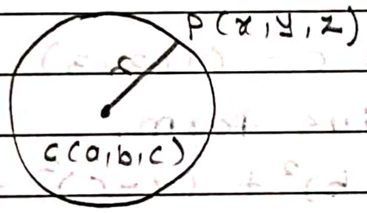
Definition of sphere: A sphere is a laws of points which moves in a space such that its distance from a fixed point is constant.

A fixed point is called centre of sphere and the constant distance is called radius of sphere.

# Equation of sphere in different forms

A] Centre - radius form:

consider a sphere with centre $C(a, b, c)$ and $P(x, y, z)$ be any point on the sphere as shown in figure, radius $= r$




then equation of sphere in centre-radius form is given by

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

if centre is at $(0,0,0)$

then, $$x^2 + y^2 + z^2 = r^2$$

B] General Form:

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

where,

centre $= C = (-u, -v, -w)$

& radius $= r = \sqrt{u^2 + v^2 + w^2 - d}$

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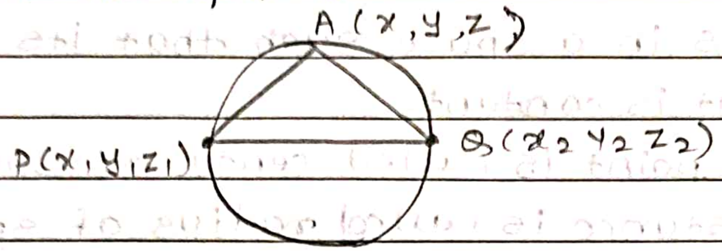
### c] Diameter form:

Let $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ be the end points of the diameter & let $A(x, y, z)$ be the any point on sphere




Then equation of sphere in diameter form is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

# Solved Examples:

Que.1 Find the equation of the sphere whose centre is $(1,2,3)$ & radius is $2$.

Soln: Given,

$$\text{centre} = C = (1,2,3) \quad \& \quad r = 2$$

$\therefore$ By centre-radius form,

$$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$$

$$(x-1)^2+(y-2)^2+(z-3)^2 = 2^2$$

$$x^2-2x+1+y^2-4y+4+z^2-6z+9=4$$

$$x^2+y^2+z^2-2x-4y-6z+1+4+9-4=0$$

$x^2+y^2+z^2-2x-4y-6z+10=0$ is the required equation.

Que.2 Find the equation of sphere with centre $(2,2,-3)$ and passing through $(7,-3,5)$.

Soln: Let, sphere passes through $P(7,-3,5)$

$$\text{centre} = C = (2,-2,-3)$$

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**Que.2** Find centre & radius of sphere

$$x^2+y^2+z^2-4x+6y-2z-11=0$$

**Soln:** Given equation,

$$x^2+y^2+z^2-4x+6y-2z-11=0$$

comparing with,

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

$\therefore$ $2u=-4$  $2v=6$  $2w=-2$  $d=-11$

$u=-2$  $v=3$  $w=-1$

$\therefore$ For general form,

centre $=(-u,-v,-w)=(2,-3,1)$

and

radius $= r = \sqrt{u^2+v^2+w^2-d}$

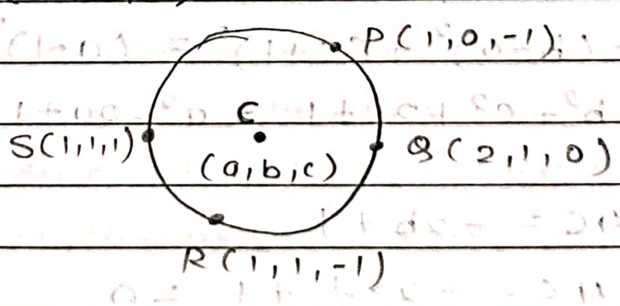
$= \sqrt{(-2)^2+(3)^2+(-1)^2-(-11)}$

$= \sqrt{4+9+1+11} = \sqrt{25} = 5$

$\therefore$ $C=(2,-3,1)$ & radius $=5$

**Que.3** Find equation of sphere passing through the points $(1,0,-1)$, $(2,1,0)$, $(1,1,-1)$, $(1,1,1)$.

**Soln:** Let $(a,b,c)$ be the centre of sphere and sphere passes through points $P(1,0,-1)$ $Q(2,1,0)$ $R(1,1,-1)$ $S(1,1,1)$




Note that all points are equidistant from the centre.

$\therefore$ $d(CP)=d(CQ)=d(CR)=d(CS)=$ Radius

We can find this by using distance formula,

Let,

$d(CP) = d(CQ)$

$\sqrt{(a-1)^2+(b-0)^2+(c+1)^2} = \sqrt{(a-2)^2+(b-1)^2+(c-0)^2}$

squaring on both sides,

$(a-1)^2+(b-0)^2+(c+1)^2 = (a-2)^2+(b-1)^2+(c-0)^2$

$a^2-2a+1+b^2+c^2+2c+1 = a^2-4a+4+b^2-2b+1+c^2$

$\therefore -2a+2c+2 = -4a-2b+5$

$\therefore (2a+2b+2c) = 3 \longrightarrow (1)$

similarly,

$d(CP) = d(CR)$

$\therefore \sqrt{(a-1)^2+(b)^2+(c+1)^2} = \sqrt{(a-1)^2+(b-1)^2+(c+1)^2}$

squaring,

$(a-1)^2+b^2+(c+1)^2 = (a-1)^2+(b-1)^2+(c+1)^2$

$a^2-2a+1+b^2+c^2+2c+1 = a^2-2a+1+b^2-2b+1+c^2+2c+1$

$0 = -2b+1$

$2b = 1 \Rightarrow \boxed{b = \frac{1}{2}}$

& $d(CP) = d(CS)$

$\therefore (a-1)^2+(b-0)^2+(c+1)^2 = (a-1)^2+(b-1)^2+(c-1)^2$

$a^2-2a+1+b^2+c^2+2c+1 = a^2-2a+1+b^2-2b+1+c^2-2c+1$

$4c = -2b+1$

$4c = -2\times\frac{1}{2}+1 = 0$

$\Rightarrow \boxed{c=0}$

$\therefore$ From (1),

$2a+2b+2c = 3$

$2a+2\times\frac{1}{2}+0 = 3 \Rightarrow \boxed{a=1}$

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$\therefore$ centre $= (a, b, c) = (1, \frac{1}{2}, 0)$

and radius is given by,

$$d(CP) = \sqrt{(a-1)^2 + (b-0)^2 + (c+1)^2}$$

$$= \sqrt{(1-1)^2 + (\frac{1}{2}-0)^2 + (0+1)^2}$$

$$= \sqrt{\frac{1}{4}+1} = \sqrt{\frac{5}{4}}$$

$\therefore$ Equation of sphere by centre - radius form is,

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

$$(x-1)^2 + (y-\frac{1}{2})^2 + (z-0)^2 = \left(\sqrt{\frac{5}{4}}\right)^2$$

$$x^2 - 2x + 1 + y^2 - y + \frac{1}{4} + z^2 = \frac{5}{4}$$

$$x^2 + y^2 + z^2 - 2x - y = \frac{5}{4} - \frac{1}{4} - 1$$

$x^2 + y^2 + z^2 - 2x - y = 0$ is the required equation.

Note: (1) Given equation of line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

the denominator a, b, c represents dr's

(2) If two lines are perpendicular to each other then product of their respective dr's is equal to zero.

(3) If two points are given then dr's $(x_2-x_1, y_2-y_1, z_2-z_1)$

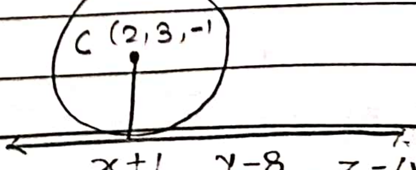
Que.4 Find the equation of sphere which has its centre at $C(2,3,-1)$ & touches the line $\frac{x+1}{-5} = \frac{y-8}{3} = \frac{z-4}{4}$

soln:

Let line touches sphere at point $P(x, y, z)$

eqn of line,

$$\frac{x+1}{-5} = \frac{y-8}{3} = \frac{z-4}{4} = k$$



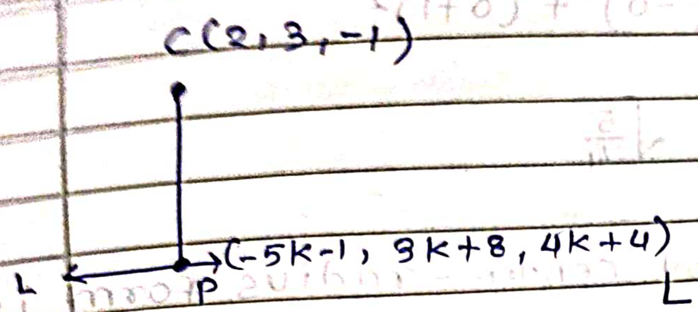

$$\frac{x+1}{-5} = \frac{y-8}{3} = \frac{z-4}{4}$$

$\therefore x+1 = -5k \Rightarrow x = -5k-1$

$y-8 = 3k \Rightarrow y = 3k+8$

$z-4 = 4k \Rightarrow z = 4k+4$

$\therefore P(x,y,z) = P(-5k-1, 3k+8, 4k+4)$




dr's of CP $= -5k-1-2,$
$= -5k-3$

$3k+8-3, \quad 4k+4+1$
$3k+5, \quad 4k+5$

Now, dr's of line ~~CP~~ L are $= -5, 3, 4$

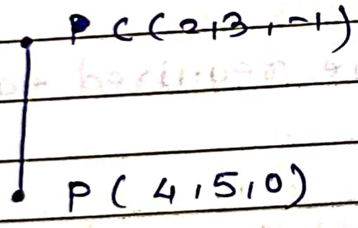
Now line CP is perpendicular to line L.

$-5(-5k-3) + 3(3k+5) + 4(4k+5) = 0$

$50k = -50$

$\boxed{k=-1}$

$\therefore P(x,y,z) = P(+5-1, -3+8, -4+4) = P(4,5,0)$




$\therefore d(CP) = \text{radius} = \sqrt{(2-4)^2+(3-5)^2+(-1-0)^2}$
$= \sqrt{4+4+1} = 3$

$\therefore c = (2,3,-1)$ & $r = 3$

$\therefore$ Equation of sphere by centre radius form,

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

$(x-2)^2 + (y-3)^2 + (z+1)^2 = 3^2$

$x^2+y^2+z^2-4x-6y+2z+5=0$ is required eqn.

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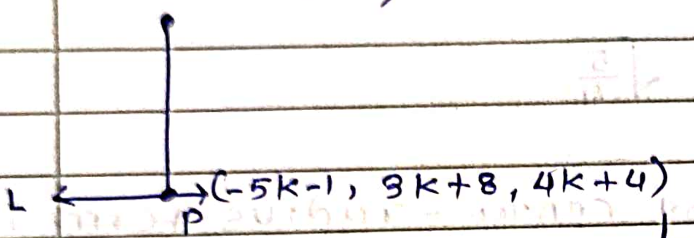
$\therefore x+1 = -5k \Rightarrow x = -5k-1$

$y-8 = 3k \Rightarrow y = 3k+8$

$z-4 = 4k \Rightarrow z = 4k+4$

$\therefore P(x,y,z) = P(-5k-1, 3k+8, 4k+4)$

$C(2,3,-1)$




dr's of CP $= -5k-1-2,$ $\quad 3k+8-3,$ $\quad 4k+4+1$

$= -5k-3$ $\quad 3k+5$ $\quad 4k+5$

L

Now, dr's of line ~~cp~~ are $= -5, 3, 4$

Now line cp is perpendicular to line L.

$2(-5k-3)+3(3k+5)+4(4k+5)=0$

$50k = -50$

$\boxed{k=-1}$

$\therefore P(x,y,z) = P(+5-1, -3+8, -4+4) = P(4,5,0)$

~~P~~ $C(2,3,-1)$

$P(4,5,0)$

$\therefore$ d(CP) = radius $= \sqrt{(2-4)^2+(3-5)^2+(-1-0)^2}$

$= \sqrt{4+4+1} = 3$

$\therefore c = (2,3,-1)$ & $r = 3$

$\therefore$ Equation of sphere by centre radius form,

$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$

$(x-2)^2+(y-3)^2+(z+1)^2 = 3^2$

$x^2+y^2+z^2-4x-6y+2z+5=0$ is required eq$^n$.

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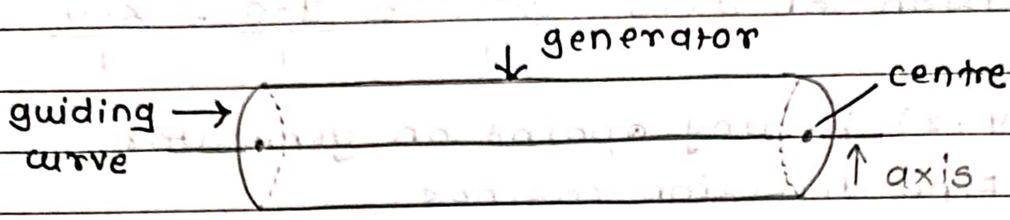
# Cylinder




## Direction cosines:

Line is given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

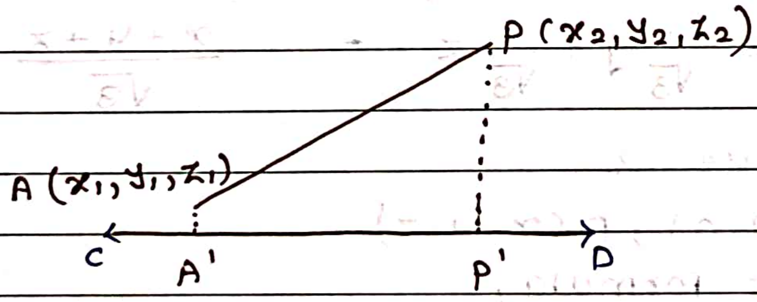
where $(x_1, y_1, z_1)$ points on line

$(a, b, c)$ dr's

then dc's (direction cosine) given by,

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

## Projection of segment of a line:

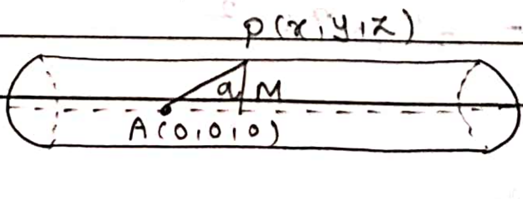



Projection of line AP on CD is A'P'

$\therefore A'P' = l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)$

## Solved Example:

Que.1 Find the equation of right circular cylinder of radius 2, whose axis passes through the origin & makes equal angle with co-ordinate axis $\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1}$



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**Sol$^n$:** **Step 1:** Represent the given data,

equation of axis, $\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1}$

$P(x, y, z)$ be any point on generator

**Step 2:** Find direction cosines,

dr's $= (1, 1, 1)$

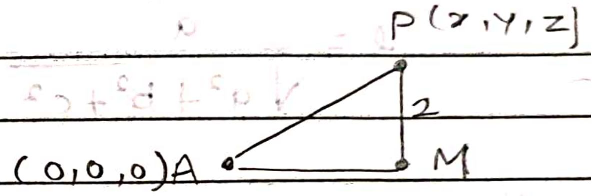
$\therefore$

$$l = \frac{1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}$$

$$m = \frac{1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}$$

$$n = \frac{1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}$$

$$\therefore l = m = n = \frac{1}{\sqrt{3}}$$




**Step 3:** Find AM, MP, AP

AM = projection of line AP on axis

$$AM = l(x-0) + m(y-0) + n(z-0)$$

$$AM = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{x+y+z}{\sqrt{3}}$$

MP = radius = 2

Also, $A(0,0,0)$, $P(x,y,z)$

$\therefore$ By distance formula,

$$AP = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$$

**Step 4:** Apply pythagoras theorem of $\Delta AMP$,

$$AM^2 + MP^2 = AP^2$$

$$\left(\frac{x+y+z}{\sqrt{3}}\right)^2 + 2^2 = \left(\sqrt{x^2+y^2+z^2}\right)^2$$

$$\frac{(x+y+z)^2}{3} + 4 = x^2+y^2+z^2$$

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$$(x+y+z)^2 + 12 = 3(x^2+y^2+z^2)$$

$$x^2+y^2+z^2+2xy+2yz+2xz+12 = 3x^2+3y^2+3z^2$$

$$-2x^2-2y^2-2z^2+2xy+2yz+2xz+12=0$$

multiply by $(-1)$,

$$2x^2+2y^2+2z^2-2xy-2yz-2xz-12=0$$

is the required equation.

**Que.2** Find the equation of right circular cylinder whose axis is $\frac{x-2}{2} = \frac{y-1}{1} = \frac{z}{3}$ & which passes through point ~~(0,0,3)~~ $(2,1,0)$ & radius is 3

**Soln:** **Step 1:** Represent given data,

equation of axis, $\frac{x-2}{2} = \frac{y-1}{1} = \frac{z}{3}$

$P(x,y,z)$ be any point on generator

**Step 2:** Find direction cosines,

dr's $= (2,1,3)$

$\therefore$

$$l = \frac{2}{\sqrt{2^2+1^2+3^2}} = \frac{2}{\sqrt{14}}$$

$$m = \frac{1}{\sqrt{2^2+1^2+3^2}} = \frac{1}{\sqrt{14}}$$

$$n = \frac{3}{\sqrt{2^2+1^2+3^2}} = \frac{3}{\sqrt{14}}$$

$$\therefore l = \frac{2}{\sqrt{14}} \quad m = \frac{1}{\sqrt{14}} \quad n = \frac{3}{\sqrt{14}}$$

**Step 3:** Find AM, MP, AP

AM = projection of line AP.

$$\therefore AM = l(x-2) + m(y-1) + n(z-0)$$

$$= \frac{2}{\sqrt{14}}(x-2) + \frac{1}{\sqrt{14}}(y-1) + \frac{3}{\sqrt{14}}(z)$$

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$$AM = \frac{2x-4+y-1+3z}{\sqrt{14}} = \frac{2x+y+3z-5}{\sqrt{14}}$$

MP = radius = 3

Also, A(2,1,0) & P(x,y,z)

$\therefore d(AP) = \sqrt{(x-2)^2+(y-1)^2+(z-0)^2}$

Step 4: Apply pythagoras theorem,

$$AM^2+MP^2=AP^2$$

$$\left(\frac{2x+y+3z-5}{\sqrt{14}}\right)^2+3^2=\left(\sqrt{(x-2)^2+(y-1)^2+z^2}\right)^2$$

$$\frac{(2x+y+3z-5)^2}{14}+9=(x-2)^2+(y-1)^2+z^2$$

$$(2x+y+3z+(-5))^2+126=14[(x-2)^2+(y-1)^2+z^2]$$

$$(2x)^2+y^2+(3z)^2+(-5)^2+2(2x)(y)+2(2x)(3z)+2(2x)(-5)+2(y)(3z)+2(y)(-5)+2(3z)(-5)+126 = 14[x^2-4x+4+y^2-2y+1+z^2]$$

$$4x^2+y^2+9z^2+25+4xy+12xz-20x+6yz-10y-30z+126 = 14x^2+14y^2+14z^2-56x-28y+70$$

$$-10x^2-13y^2-5z^2+36x+18y-30z+4xy+6yz+12zx+81=0$$

$\therefore 10x^2+13y^2+5z^2-36x-18y+30z-4xy-6yz-12xz-81=0$ is the required equation.

# Practice Example:

Que. 1 Find the equation of right circular cylinder of radius 2 whose axis passes through (1,2,3) & has dr's (2,-3,6)

Ans: $45x^2+40x^2+13z^2-42x-280y-126z+12xy+36yz-24xz+294=0$

# Unit 5: Multiple Integral

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- Introduction to 3-dimensional cartesian, spherical polar & cylindrical co-ordinate system.
- Double & triple integration
- Change the order of integration
- Application of double & triple integration in calculating area & volume.

## # Double Integration:

- When the limit of integration are constant.

## # Solved Example

① $\int_1^2 \int_3^4 dx\, dy$

Sol^n: $I = \int_1^2 \int_3^4 dx\, dy$

$= \int_1^2 (x)_3^4\, dy$

$= \int_1^2 (4-3)\, dy$

$= \int_1^2 1\, dy$

$= [y]_1^2$

$= 2-1$

$I = 1$

$\therefore \boxed{I = 1}$

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(2) $\int_1^2 \int_3^4 5 \cdot dx\, dy$

Soln: Let $I = \int_1^2 \int_3^4 5 \cdot dx\, dy$

$= \int_1^2 [5x]_3^4\, dy$

$= \int_1^2 [5\times4 - 5\times3]\, dy$

$= \int_1^2 (20-15)\, dy$

$= \int_1^2 5\, dy$

$= [5y]_1^2$

$= 5(2) - 5(1)$

$= 10 - 5$

$= 5$

$\therefore \boxed{I = 5}$

(3) $\int_0^1 \int_0^1 (x+y)\, dx\, dy$

Soln: Let, $I = \int_0^1 \int_0^1 (x+y)\, dx\, dy$

$= \int_0^1 \left[ \int_0^1 x\, dx + \int_0^1 y\, dx \right] dy$

$= \int_0^1 \left[ \frac{x^2}{2} + y \cdot x \right]_0^1 dy$

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$$= \int_0^1 \left[\frac{1}{2} + y - 0\right] dy$$

$$= \int_0^1 \left[\frac{1}{2} + y\right] dy$$

$$= \int_0^1 \frac{1}{2}\, dy + \int_0^1 y\, dy$$

$$= \left[\frac{1}{2}y + \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2}(1) + \frac{1}{2}$$

$$= 1$$

$$\therefore \boxed{I = 1}$$

(4) $\int_0^1 \int_0^2 (x+y)\, dx\, dy$

Soln: Let, $I = \int_0^1 \int_0^2 (x+y)\, dx\, dy$

$$= \int_0^1 \int_0^2 (x) dx + \int_0^2 y\, dx \quad dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + yx\right]_0^2 dy$$

$$= \int_0^1 \left(\frac{4}{2} + 2y - 0\right) dy$$

$$= \int_0^1 2 + 2y \quad dy$$

$$= \int_0^1 2dy + \int_0^1 2y\, dy$$

$$= \left[2y + 2y^2/2\right]_0^1 = \left[2y + y^2\right]_0^1$$

$$= 2 + 1 - 0$$

$$\boxed{I = 3}$$

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(5) $\int_0^1 \int_0^2 xy \, dx \, dy$

Sol^n:

$$I = \int_0^1 \left[ \int_0^2 xy \, dx \right] dy$$

$$= \int_0^1 \left[ y \int_0^2 x \, dx \right] dy$$

$$= \int_0^1 y \cdot \left[ \frac{x^2}{2} \right]_0^2 dy$$

$$= \int_0^1 y \left[ \frac{4}{2} - 0 \right] dy$$

$$= \int_0^1 y [2] \, dy$$

$$= 2 \int_0^1 y \, dy$$

$$= 2 \left[ \frac{y^2}{2} \right]_0^1$$

$$= [y^2]_0^1 = [1 - 0]$$

$$\boxed{I = 1}$$

- When limit of integration are variable.

(1) $\int_0^2 \int_0^y xy \, dx \, dy$

Sol^n: Let, $I = \int_0^2 \int_0^y xy \, dx \, dy$

Here limit of inner integral are in terms of 'y' therefore integrate first w.r.t. 'x'

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$$\therefore I = \int_0^2 \left[\int_0^y x\,dx\right] y\,dy$$

$$= \int_0^2 \left[\frac{x^2}{2}\right]_0^y \cdot y\,dy$$

$$= \int_0^2 \left[\frac{y^2}{2} - 0\right] \cdot y\,dy$$

$$= \int_0^2 \frac{y^3}{2}\,dy$$

$$= \frac{1}{2}\int_0^2 y^3\,dy$$

$$= \frac{1}{2} \cdot \left[\frac{y^4}{4}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{16}{4} - 0\right]$$

$$= \frac{1}{2} \times 4$$

$$\boxed{I = 2}$$

(2) $\int_0^2 \int_0^x xy\,dx\,dy$

Soln:

Let $I = \int_0^2 \int_0^x xy\,dx\,dy$

Here the limit of inner integral is in 'x' so integrate first w.r.t 'y'.

$$I = \int_0^2 \left[\int_0^x y\,dy\right] x\,dx$$

$$= \int_0^2 \left[\frac{y^2}{2}\right]_0^x \cdot x\,dx$$

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$$= \int_0^2 \left[\frac{x^2}{2} - 0\right] \cdot x \, dx$$

$$= \int_0^2 \frac{x^3}{2} \, dx$$

$$= \frac{1}{2} \int_0^2 x^3 \, dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4}\right]_0^2$$

$$= \frac{1}{2} \left[\frac{16}{4}\right]$$

$$= \frac{1}{2} \times 4$$

$$\boxed{I = 2}$$

(3) $$\int_{-2}^{1} \int_{x^2}^{2-x} y \, dx \, dy$$

Soln:

Let, $I = \int_{-2}^{1} \int_{x^2}^{2-x} y \, dx \, dy$

Here limit of integration are in terms of 'x' so integrate first w.r.t 'y'.

$$I = \int_{-2}^{1} \left[\int_{x^2}^{2-x} y \, dy\right] dx$$

$$= \int_{-2}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$= \int_{-2}^{1} \left[\frac{(2-x)^2}{2} - \frac{(x^2)^2}{2}\right] dx$$

$$= \frac{1}{2} \int_{-2}^{1} \left[(2-x)^2 - x^4\right] dx$$

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$$= \frac{1}{2}\int_{-2}^{1}\left[4 - 4x + x^2 - x^4\right] dx$$

$$= \frac{1}{2}\left[4x - 4\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1}$$

$$= \frac{1}{2}\left[\left(4 - \frac{4}{2} + \frac{1}{3} - \frac{1}{5}\right) - \left(-8 - \frac{16}{2} - \frac{8}{3} + \frac{32}{5}\right)\right]$$

$$= \frac{1}{2}\left[\frac{216}{15}\right)$$

$$\boxed{I = \frac{36}{5}}$$

(4) $\int_0^1 \int_0^{1-x} x \, dx dy$

Soln: Let,

$$I = \int_0^1 \int_0^{1-x} x \, dx dy$$

Here limit of integration are in terms of $x$ so integrate w.r.t 'y'

$$I = \int_0^1 \left[\int_0^{1-x} dy\right] x \, dx$$

$$= \int_0^1 [y]_0^{1-x} \cdot x \, dx$$

$$= \int_0^1 [(1-x) - 0] \, x \, dx$$

$$= \int_0^1 (1-x) \, x \, dx$$

$$= \int_0^1 x - x^2 \, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{6}$$

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# Practice Example:

# Evaluate:

① $\int_0^2 \int_0^2 (x+y)\,dx\,dy$ ANS: 8

② $\int_0^2 \int_0^2 xy\,dx\,dy$ ANS: 4

③ $\int_0^1 \int_0^y xy\,dx\,dy$ ANS: $\frac{1}{8}$

④ $\int_0^1 \int_{x^2}^{2-x} y\,dx\,dy$ ANS: $\frac{16}{15}$ or $\frac{32}{30}$

⑤ $\int_0^1 \int_0^{1-x} y\,dy\,dx$ ANS: $\cancel{\frac{13}{15}}$ $\frac{1}{6}$

# Area by using double integration

The most important application of double integration is to find area under the curve or area enclosed between the curve.

$\therefore$ Area of curve given by

$$A = \iint dx\,dy$$

# Solved Examples:

Que: 1 Find the area bounded by the curve $y = x^2$ & $y = x$

Q. Find area bounded by curve $y = x^2$ & $y^2 = x$

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Sol^n:

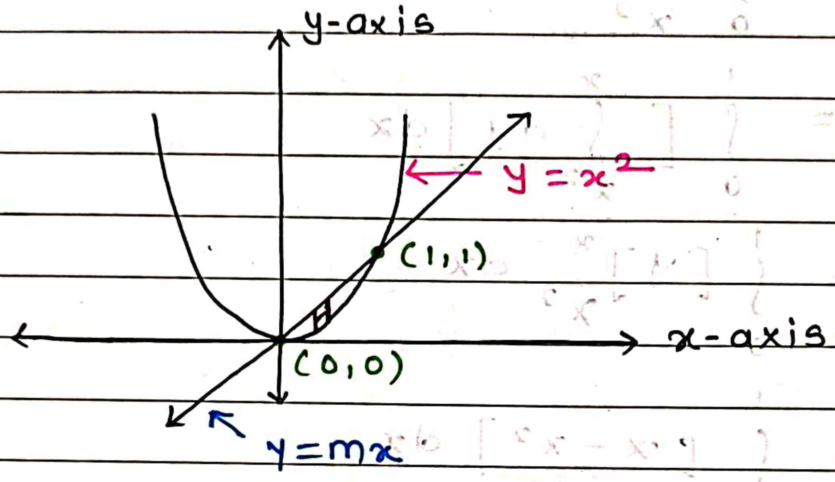
Step1: Area is given by

$$A = \iint dx\,dy$$

Step2: Diagram

$y = x^2$ parabola

$y = x$ straight line




Step3: point of intersection

$y = x^2$ & $y = x$

equating values of 'y',

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$\therefore x = 0$ or $x = 1$

| | |
|---|---|
| For $x = 0$ | $x = 1$ |
| $y = x$ | $y = x$ |
| $y = 0$ | $y = 1$ |
| $\therefore (0,0)$ | $(1,1)$ |

Step4: vertical strip

Bottom end of strip lies on parabola

$y = x^2$

To end of strip lies on line $y = x$

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To find limits of $x$ vary the strip from left to right

$\therefore$ $x$ varies from $x=0$ to $x=1$

& $y$ from $x^2$ to $x$

Step 5:

$$A = \iint dx\,dy$$

$$A = \int_0^1 \int_{x^2}^{x} dx\,dy$$

$$= \int_0^1 \left[ \int_{x^2}^{x} dy \right] dx$$

$$= \int_0^1 [y]_{x^2}^{x}\, dx$$

$$= \int_0^1 [x - x^2]\, dx$$

$$= \left. \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$A = \cancel{\tfrac{1}{6}}\ \frac{1}{6}$$

Que. 2 Find the area between the curve $y^2 = 4x$ & $2x - y - 4 = 0$

Soln:

Step 1: Area between the curve is given by

$$A = \iint dx\,dy$$

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step 2: diagram

$y^2 = 4x$ parabola

& $2x - y - 4 = 0$ straight line.

put $x = 0$

$0 - y - 4 = 0$

$\boxed{y = -4}$

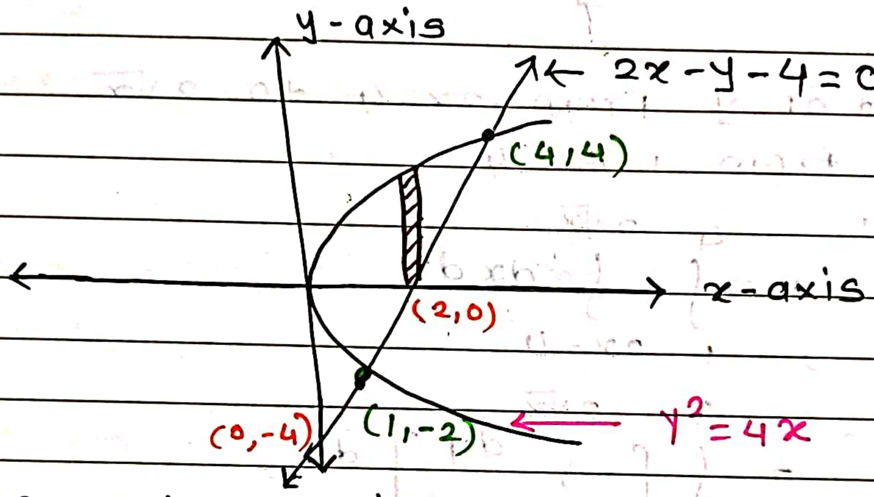
$\therefore (0, -4)$

$y = 0$

$2x - 4 = 0$

$2x = 4 \Rightarrow \boxed{x = 2}$

$(2, 0)$




step 3: point of intersection

$y^2 = 4x$

$x = y^2/4$

$2x - y - 4 = 0$

$\therefore 2\left(\frac{y^2}{4}\right) - y - 4 = 0$

$\frac{y^2}{2} - y - 4 = 0$

solving on calculator we get,

$y_1 = 4 \quad y_2 = -2$

Now,

for $y = 4$

$2x - y - 4 = 0$

$2x - 4 - 4 = 0$

$\boxed{x = 4}$

$\therefore (4, 4)$

$y = -2$

$2x - y - 4 = 0$

$2x + 2 - 4 = 0$

$2x - 2 = 0$

$\boxed{x = 1}$

$(1, -2)$

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Step4: Strip

vertical strip

Bottom end of the strip lies on line

$2x - y - 4 = 0$

$\therefore y = 2x - 4$

To end lies on $y^2 = 4x$

$y = \sqrt{4x}$

$y = 2\sqrt{x}$

$\therefore$ limit of $y$ from $2x-4$ to $2\sqrt{x}$

& $x$ from 1 to 4

Step 5:

$$A = \int_1^4 \int_{2x-4}^{2\sqrt{x}} dx\, dy$$

$$= \int_1^4 \left[ \int_{2x-4}^{2\sqrt{x}} dy \right] dx$$

$$= \int_1^4 [y]_{2x-4}^{2\sqrt{x}} dx$$

$$= \int_1^4 [2\sqrt{x} - (2x-4)] dx$$

$$= \int_1^4 2\sqrt{x} - 2x + 4 \, dx$$

$$= \frac{2x^{3/2}}{3/2} - \frac{2 \cdot x^2}{2} + 4x \Big|_1^4$$

$$= \left[ \frac{4}{3} x^{3/2} - x^2 + 4x \right]_1^4$$

$$= \left( \frac{4}{3}(4)^{3/2} - 16 + 16 \right) - \left( \frac{4}{3} - 1 + 4 \right)$$

$$A = \frac{19}{3}$$

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Que.3 Find the area between the curve $y^2 = 4x$ & $2x - 3y + 4 = 0$

Soln:

Step 1: Area between the curve is given by

$$A = \iint dx\,dy$$

Step 2: Diagram

$y^2 = 4x \rightarrow$ parabolla

& $2x - 3y + 4 = 0 \rightarrow$ straight line

Put $x = 0$

$0 - 3y + 4 = 0$

$\boxed{y = 4/3}$

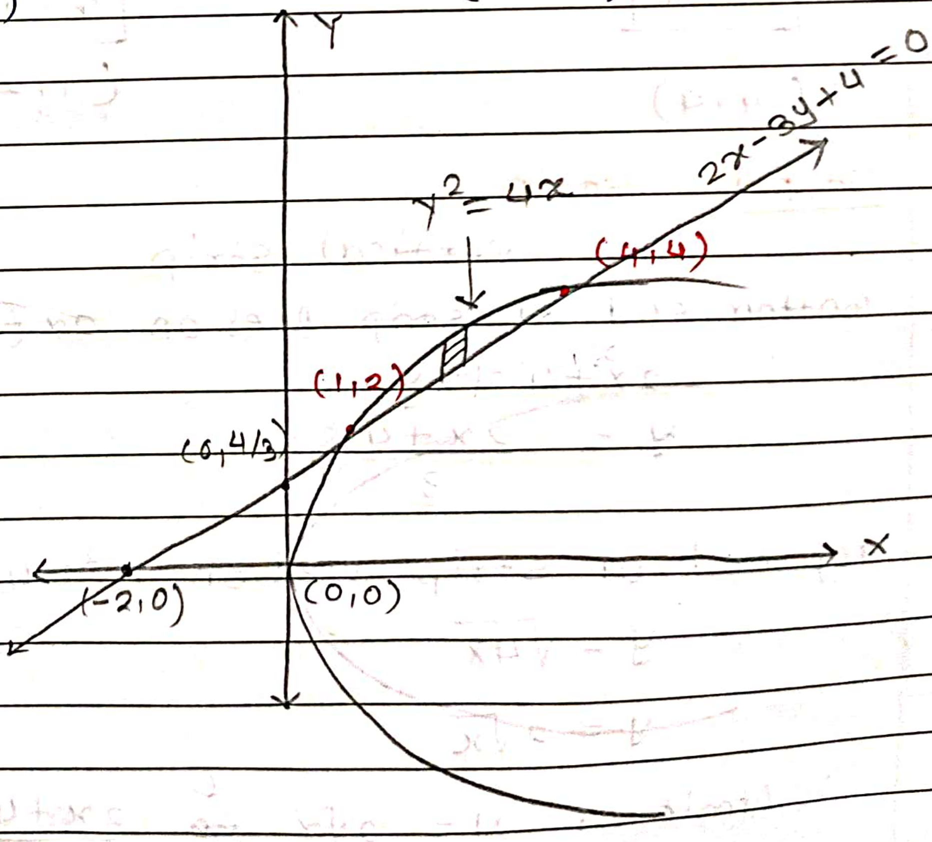
$\therefore (0, 4/3)$

Put $y = 0$

$2x + 4 = 0$

$\boxed{x = -2}$

$(-2, 0)$



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Que.3 Find the area between the curve $y^2 = 4x$ & $2x - 3y + 4 = 0$

soln:

Step1: Area between the curve is given by

$$A = \iint dx\,dy$$

Step2: Diagram

$y^2 = 4x \longrightarrow$ parabolla

& $2x - 3y + 4 = 0 \longrightarrow$ straight line

Put $x = 0$

$0 - 3y + 4 = 0$

$\boxed{y = 4/3}$

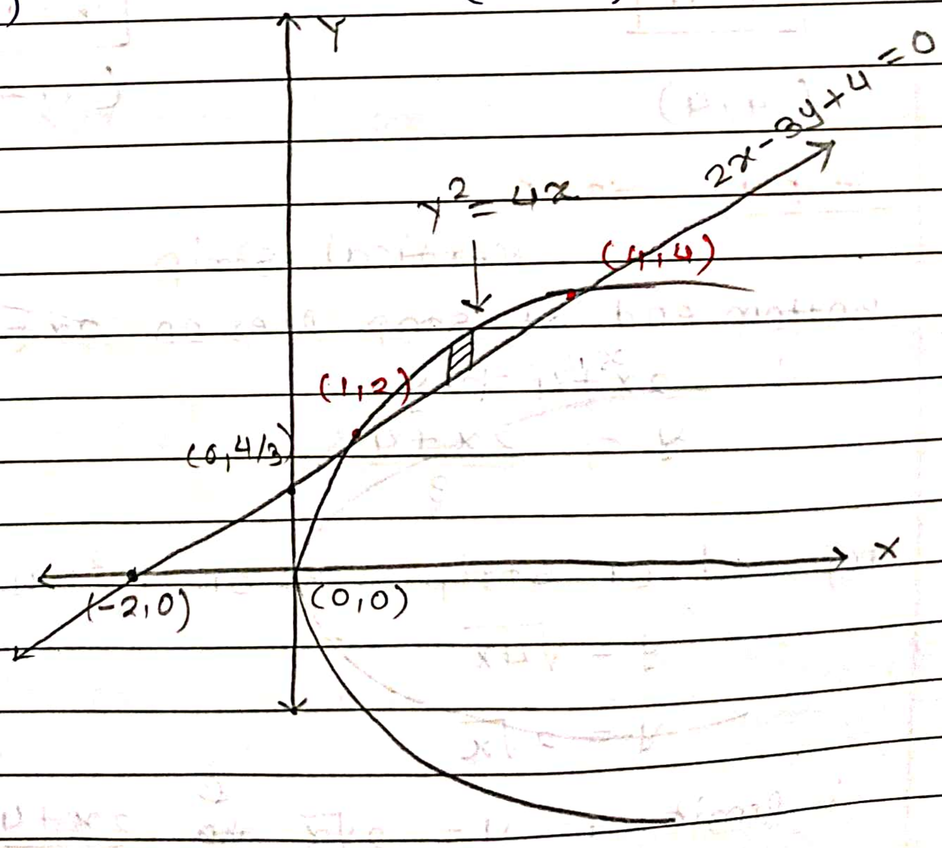
$\therefore (0, 4/3)$

Put $y = 0$

$2x + 4 = 0$

$\boxed{x = -2}$

$(-2, 0)$



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Step 3: Point of intersection

$y^2 = 4x$ & $2x - 3y + 4 = 0$

$\therefore \frac{y^2}{4} = x$

$\therefore 2\left(\frac{y^2}{4}\right) - 3y + 4 = 0$

$\frac{y^2}{2} - 3y + 4 = 0$

solving on calculator we get,

$y = 4$ $y = 2$

$\therefore$ for $y = 4$ | $y = 2$

$2x - 3(4) + 4 = 0$ | $2x - 3(2) + 4 = 0$

$2x = 8$ | $2x = 2$

$\boxed{x = 4}$ | $\boxed{x = 1}$

$(4, 4)$ | $(1, 2)$

Step 4: Strip

vertical strip

Bottom end of strip lies on $2x - 3y + 4 = 0$

$\therefore 2x + 4 = 3y$

$y = \frac{2x+4}{3}$

Top end of strip lies on $y^2 = 4x$

$y = \sqrt{4x}$

$y = 2\sqrt{x}$

$\therefore$ limit of $y = 2\sqrt{x}$ ~~to~~ & $\frac{2x+4}{3}$

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Now limit of 'x' vary the strip from left to right

$\therefore$ x varies from $x = 1$ to $x = 4$

Step 5: Apply limits

$$A = \iint dx\, dy$$

$$= \int_1^4 \int_{\frac{2x+4}{3}}^{2\sqrt{x}} dy\, dx$$

$$= \int_1^4 [y]_{\frac{2x+4}{3}}^{2\sqrt{x}} dx$$

$$A = \int_1^4 \left(2\sqrt{x} - \frac{2x+4}{3}\right) dx$$

$$A = \int_1^4 \frac{6\sqrt{x} - 2x - 4}{3} dx$$

$$= \frac{1}{3}\int_1^4 6x^{1/2} - 2x - 4\, dx$$

$$= \frac{1}{3}\left[\frac{6x^{3/2}}{3/2} - \frac{2x^2}{2} - 4x\right]_1^4$$

$$= \frac{1}{3}\left[4x^{3/2} - x^2 - 4x\right]_1^4$$

$$= \frac{1}{3}\left[\left(4(4)^{3/2} - 4^2 - 16\right) - \left(4(1)^{3/2} - 1 - 4\right)\right]$$

$$= \frac{1}{3}\left[0 - [-1]\right] = \frac{1}{3}$$

$$\therefore \boxed{A = \frac{1}{3}}$$

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# Change of order of integration:

we already know that, if limits of inner integral are in terms of $x$ i.e $\int \int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx$

then we integrate first w.r.t 'y'.

But sometimes integrating first w.r.t 'y' is not possible rather it is simpler to integrate first w.r.t 'x' And that is where concept of change of order of integration is useful.

# Solved Example:

Que.1 change the order of integration in the double integration $\int_0^5 \int_{2-x}^{2+x} f(x,y)\,dy\,dx$

Soln:

Step(1): Let, $I = \int_0^5 \int_{2-x}^{2+x} f(x,y)\,dy\,dx$

Note that, limits of inner integral are in terms of 'x'.

Therefore it represent y = lower limit & y = upper limit. that means vertical strip was used to find limits & we have to change it to horizontal strip.

$$\therefore I = \int_{x=0}^{x=5} \int_{y=2-x}^{y=2+x} f(x,y)\,dx\,dy \quad \text{---} (1)$$

Step2:

From (1),

$x=0 \longrightarrow$ y axis

$x=5 \longrightarrow$ line parallel to y-axis

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$y = 2 + x \rightarrow$ straight line passing through $(0,2)$ & $(-2,0)$

put $x = 0 \rightarrow y = 2$

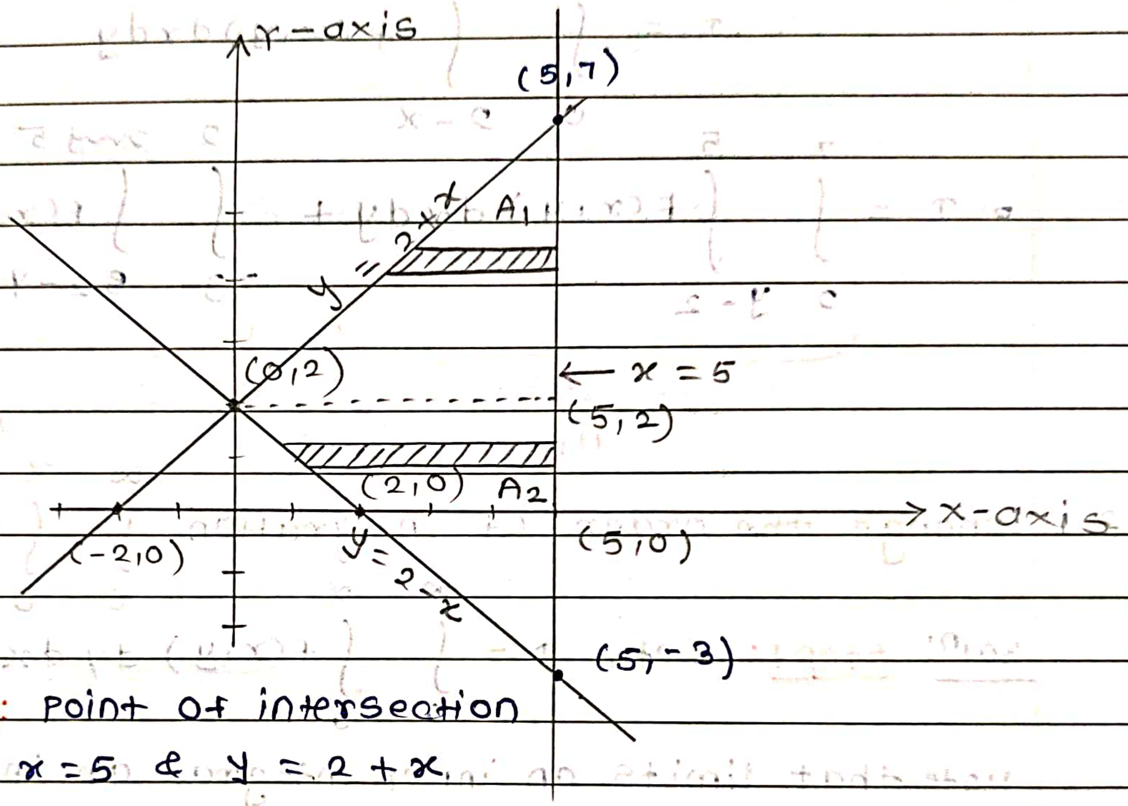
$y = 0 \rightarrow x = -2$ $\therefore (0,2)$ & $(-2,0)$

and,

$y = 2 - x \rightarrow$ straight line passing through $(0,2)$ & $(2,0)$

put $x = 0 \rightarrow y = 2$

$y = 0 \rightarrow x = 2$ $\therefore (0,2)$ & $(2,0)$




Step 3: Point of intersection

For $x = 5$ & $y = 2 + x$,

$y = 2 + 5 = 7$

i.e $(5,7)$

For $x = 5$ & $y = 2 - x$

$= 2 - 5$

$= -3$

i.e $(5,-3)$,

Step 4: Horizontal strip.

① For area $A_1$

left end of strip lies on $y = 2 + x$

$\therefore x = y - 2$

Right end on $x = 5$

To find the limit of y vary strip from bottom to top

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$\therefore y = 2$ to $y = 7$

(2) For area A2

left end of strip lies on $y = 2 - x$

$\therefore x = 2 - y$

& right end on $x = 5$

& y varies from $y = -3$ to $y = 2$

Step 5:

$$\therefore I = \int_{0}^{5} \int_{2-x}^{2+x} f(x,y)\,dx\,dy$$

$$I = \underbrace{\int_{2}^{7} \int_{y-2}^{5} f(x,y)\,dx\,dy}_{A_1} + \underbrace{\int_{-3}^{2} \int_{2-y}^{5} f(x,y)\,dx\,dy}_{A_2}$$

Que. 2 change the order of integration $\int_0^\infty \int_0^x f(x,y)\,dy\,dx$

Soln: Step 1: Let, $I = \int_0^\infty \int_0^x f(x,y)\,dy\,dx$

Note that limits on inner integral are in terms of $x$

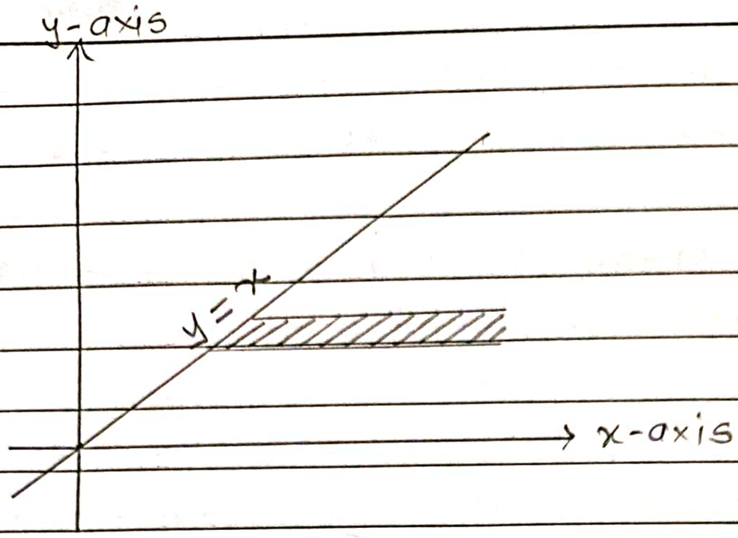
$\therefore$ It represent $y$ = lower & upper limit. i.e means vertical strip & we have to change it to horizontal strip.

$$\therefore I = \int_0^\infty \int_0^x f(x,y)\,dx\,dy \quad \text{—(1)}$$

Step 2: from (1),

$x = 0$ $\quad$ $y = 0$

$x = \infty$ $\quad$ $y = x$




Step 3: Horizontal strip,
left end of strip lies on $x = y$ to $x = \infty$
& y-varies from Bottom to top, $y = 0$ to $y = \infty$

Step 4:

$$\therefore \quad I = \int_{0}^{\infty} \int_{y}^{\infty} f(x, y) \, dx \, dy$$

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# Triple Integration

Basically we can say that, triple integrals are essentially the same thing as double integrals we just add a third dimension.

The notation for general triple integral is $\iiint f(x,y,z)\,dv$

where,

$$dv = dx\,dy\,dz$$

Evaluation of triple integral: In direct evaluation of triple integrals, if limits of inner integral are in terms of $x$ & $y$, then integrate w.r.t 'z' [similarly we can do this for other set of variables]

If the limits are not provided then integrate first w.r.t 'z' then use the same process as in double integration.

Solved Example

Que. 1 $\int_0^1 \int_0^1 \int_0^1 xyz\,dx\,dy\,dz$

Soln: Let, $I = \int_0^1 \int_0^1 \int_0^1 xyz\,dx\,dy\,dz$

$$= \int_0^1 \int_0^1 \int_0^1 xyz\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^1 xy\left[\frac{z^2}{2}\right]_0^1 dy\,dx$$

$$= \int_0^1 \int_0^1 xy\left[\frac{1}{2}-0\right] dy\,dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^2}{2}\right]_0^1 dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{1}{2}-0\right] dx = \frac{1}{4}\int_0^1 x\,dx$$

$= \frac{1}{4} \left[\frac{x^2}{2}\right]_0^1$

$= \frac{1}{4} \left[\frac{1}{2} - 0\right]$

$\boxed{I = \frac{1}{8}}$

Que. 2 $\int_0^1 \int_0^1 \int_0^1 (x+y+z)\, dz\, dy\, dx$

Soln: Let,

$I = \int_0^1 \int_0^1 \int_0^1 x+y+z \; dz\, dy\, dx$

$= \int_0^1 \int_0^1 \left[xz + yz + \frac{z^2}{2}\right]_0^1 dy\, dx$

$= \int_0^1 \int_0^1 \left(x + y + \frac{1}{2} - 0\right) dy\, dx$

$= \int_0^1 \left[xy + \frac{y^2}{2} + \frac{1}{2}y\right]_0^1 dx$

$= \int_0^1 \left[x + \frac{1}{2} + \frac{1}{2} - 0\right] dx$

$= \int_0^1 (x+1)\, dx$

$= \left[\frac{x^2}{2} + x\right]_0^1$

$= \frac{1}{2} + 1$

$\boxed{I = \frac{3}{2}}$

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Que.3 Evaluate $\int_0^a \int_0^b \int_0^c (x+y+z)\,dz\,dy\,dx$

Soln: Let, $I = \int_0^a \int_0^b \int_0^c (x+y+z)\,dz\,dy\,dx$

$$= \int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2}\right]_0^c dy\,dx$$

$$= \int_0^a \int_0^b \left(xc + yc + \frac{c^2}{2} - 0\right) dy\,dx$$

$$= \int_0^a \int_0^b xc + yc + \frac{c^2}{2}\,dy\,dx$$

$$= \int_0^a \left(xyc + \frac{y^2}{2}c + \frac{yc^2}{2}\right)_0^b dx$$

$$= \int_0^a xbc + \frac{b^2c}{2} + \frac{bc^2}{2}\,dx$$

$$= \left[\frac{x^2}{2}bc + \frac{b^2c}{2}x + \frac{bc^2}{2}x\right]_0^a$$

$$= \frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}$$

$$\boxed{I = \frac{abc}{2}(a+b+c)}$$

Que.4 Evaluate: $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x+y+z)\,dx\,dy\,dz$

Soln: Let,

$$I = \int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x+y+z)\,dx\,dy\,dz$$

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Here the limits of integration are in terms of $x$ & $z$ so integrate first w.r.t 'y'

$$\therefore I = \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dx\, dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z) - \left(x(x-z) + \frac{(x-z)^2}{2} + z(x-z)\right)\right] dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \left[x^2 + xz + \frac{x^2}{2} + xz + \frac{z^2}{2} + zx + z^2 - x^2 - xz + \frac{x^2}{2} - xz + \frac{z^2}{2} + zx - z^2\right] dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z} [4zx + 2z^2]\, dx\, dz$$

$$= \int_{-1}^{1} \left[4z\frac{x^2}{2} + 2z^2x\right]_0^z dz$$

$$= \int_{-1}^{1} 4z\cdot\frac{z^2}{2} + 2z^2\cdot z - 0 \; dz$$

$$= \int_{-1}^{1} (2z^3 + 2z^3)\, dz$$

$$= \int_{-1}^{1} 4z^3\, dz$$

$$= \left.\frac{4z^4}{4}\right|_{-1}^{1}$$

$$= [z^4]_{-1}^{1} = 1^4 - (-1)^4 = 1 - 1 = 0$$

Que.5 Evaluate: $\int_0^2 \int_0^x \int_0^{2x+2y} e^{x+y+z}\, dx\,dy\,dz$

Soln: Let,

$$I = \int_0^2 \int_0^x \int_0^{2x+2y} e^{x+y+z}\, dx\,dy\,dz$$

Here limit of inner integral are in terms of $x$ & $y$ so integrate first w.r.t $z$ first.

$$\therefore\ I = \int_0^2 \int_0^x \int_0^{2x+2y} e^{x+y+z}\, dz\,dy\,dx$$

$$= \int_0^2 \int_0^x \int_0^{2x+2y} e^x \cdot e^y \cdot e^z\, dz\,dy\,dx$$

$$= \int_0^2 \int_0^x e^x \cdot e^y \left[e^z\right]_0^{2x+2y} dy\,dx$$

$$= \int_0^2 \int_0^x e^x \cdot e^y \left[e^{2x+2y} - e^0\right] dy\,dx$$

$$= \int_0^2 \int_0^x e^x \cdot e^y \left[e^{2x+2y} - 1\right] dy\,dx$$

$$= \int_0^2 \int_0^x e^{3x} \cdot e^{3y} - e^x \cdot e^y\, dy\,dx$$

$$= \int_0^2 \left[e^{3x} \cdot \frac{e^{3y}}{3} - e^x\, e^y\right]_0^x dx$$

$$= \int_0^2 \left(e^{3x} \cdot \frac{e^{3x}}{3} - e^x \cdot e^x\right) - \left(e^{3x} \cdot \frac{e^0}{3} - e^x \cdot \frac{e^0}{3}\right) dx$$

$$= \int_0^2 \frac{e^{6x}}{3} - e^{2x} - \frac{e^{3x}}{3} + e^x\, dx$$

$$= \left[\frac{e^{6x}}{18} - \frac{e^{2x}}{2} - \frac{e^{3x}}{9} + e^x\right]_0^2$$

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$$= \frac{1}{18}e^{12} - \frac{1}{2}e^{4} - \frac{1}{9}e^{6} + e^{2} - \frac{1}{18} + \frac{1}{2} + \frac{1}{9} - 1$$

$$\boxed{I = \frac{1}{18}e^{12} - \frac{1}{2}e^{4} - \frac{1}{9}e^{6} + e^{2} - \frac{4}{9}}$$

# Practice Example

1. Evaluate: $\int_0^1 \int_0^2 \int_1^2 x^2 y z \, dz \, dy \, dx$ Ans: $I = 1$

2. Evaluate: $\int_1^2 \int_2^3 \int_0^3 x^2 y^2 z \, dz \, dy \, dx$ Ans: $I = \frac{195}{8}$

3. Evaluate: $\int_0^2 \int_2^3 \int_1^3 (x^2 y + z) \, dz \, dy \, dx$ Ans: $I = \frac{47}{3}$

4. Evaluate: $\int_0^1 \int_0^{1-x} \int_0^{x+y} z \, dz \, dy \, dx$

5. Evaluate: $\int_1^2 \int_2^3 \int_0^3 (x^2 + y^3 + z) \, dz \, dy \, dx$ Ans: $\frac{241}{4}$

6. Evaluate: $\int_0^{2a} \int_0^x \int_y^x xyz \, dz \, dy \, dx$

7. Evaluate: $\int_0^2 \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx$ Ans: $I = \frac{e^8 - 6e^4 + 8e^2 - 3}{8}$

8. Evaluate: $\int_0^a \int_0^a \int_0^a (yz + 2x + xy) \, dx \, dy \, dz$ Ans: $I = \frac{3}{4}a^5$

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# volume

As we learnt, double integration is used to find area, so its that we will use triple integration to find volume of solids.

volume of solids by triple integration is given by,

$$\therefore \quad V = \iiint dx\,dy\,dz$$

Note: First find limit of $z$ $(z_1, z_2)$ & then integrate w.r.t $z$,

i.e $$\iint \int_{z_1}^{z_2} dz\,dx\,dy$$

$$\iint (z_2 - z_1)\,dx\,dy$$

then solve by using process of double integration.

## # Cylindrical Polar system:

First integrate w.r.t $z$ & covert into double integral

$\iint f(x,y)\,dx\,dy$

then convert this double integral to polar form by using,

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$
$$dx\,dy = r\,dr\,d\theta$$
$$r = 0 \text{ to } a$$
$$\theta = 0 \text{ to } \theta_1$$

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Que. 1 Find the volume of cylinder $x^2+y^2=2ax$ intercepted between the paraboloid $x^2+y^2=2az$ and XY plane.

Solution:

Step 1: volume is given by,

$$V=\iiint dx\,dy\,dz$$

(xy plane represent $z=0$)

Now, here,

z varies from $z=0$ to

$$x^2+y^2=2az \Rightarrow z=\frac{x^2+y^2}{2a}$$

$$\therefore z=0 \text{ to } z=\frac{x^2+y^2}{2a}$$

$$\therefore V=\iint\int_{z=0}^{z=\frac{x^2+y^2}{2a}} dz\,dx\,dy$$

$$=\iint [z]_0^{\frac{x^2+y^2}{2a}} dx\,dy$$

$$V=\iint\left[\frac{x^2+y^2}{2a}-0\right]dx\,dy$$

$$V=\iint\frac{x^2+y^2}{2a}\,dx\,dy$$

converting double integral to polar by using,

$x^2+y^2=r^2$, $dx\,dy=r\,dr\,d\theta$

$$\therefore V=\iint\frac{r^2}{2a}\,r\,dr\,d\theta$$

$$V=\frac{1}{2a}\iint r^3\,dr\,d\theta$$

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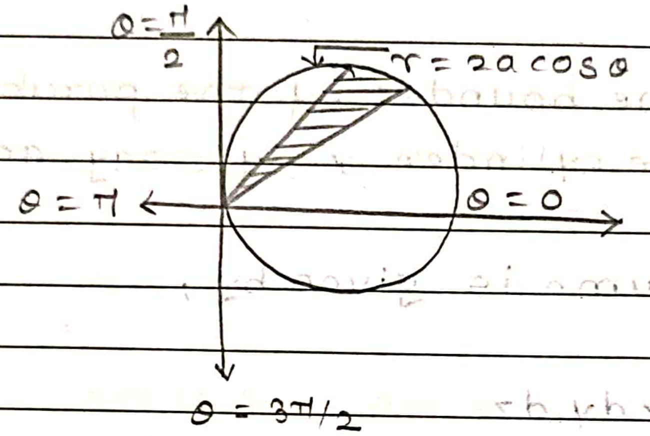
Step 2: diagram

$$x^2+y^2=2ax$$

converting to polar (Note $x=r\cos\theta$) & $x^2+y^2=r^2$

$$\therefore\ r^2 = 2a(r\cos\theta)$$

$$r = 2a\cos\theta \longrightarrow \text{circle.}$$




Step 3: strip

$$\therefore\ r=0 \text{ to } r=2a\cos\theta$$

$$\theta=0 \text{ to } \theta=\frac{\pi}{2}$$

Step 4: Apply limits & solve

$$\therefore\ V = \frac{1}{2a}\iint r^3\, dr\, d\theta$$

$$= \frac{1}{2a}\int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\, dr\, d\theta$$

$$= \frac{1}{2a}\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{2a}\int_0^{\pi/2}\left[\frac{(2a)^4\cos^4\theta}{4} - 0\right] d\theta$$

$$= \frac{1}{2a}\times\frac{16a^4}{4}\int_0^{\pi/2}\cos^4\theta\, d\theta = 2a^3\left[\frac{(4-1)(4-3)}{4(4-2)}\right]\frac{\pi}{2}$$

$$= 2a^3\cdot\frac{3\pi}{16}$$

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For symmetry,

$$V = 2 \times \frac{3\pi a^3}{8}$$

$$\boxed{V = \frac{3\pi a^3}{4}}$$

Que. 2 Find the volume bounded by the paraboloid $x^2+y^2 = az$, the cylinder $x^2+y^2 = 2ay$ and the plane $z=0$.

Soln: Step 1: volume is given by,

$$V = \iiint dx\,dy\,dz$$

Now,

Here $z$ varies from,

$z=0$ to $z = \frac{x^2+y^2}{a}$ $\qquad x^2+y^2 = az \Rightarrow z = \frac{x^2+y^2}{a}$

$$\therefore V = \iint \int_{z=0}^{\frac{x^2+y^2}{a}} dz\,dx\,dy$$

$$V = \iint [z]_0^{\frac{x^2+y^2}{a}} dx\,dy$$

$$V = \iint \frac{x^2+y^2}{a}\,dx\,dy$$

Converting into polar,

$x^2+y^2 = r^2$ & $dx\,dy = r\,dr\,d\theta$

$$\therefore V = \frac{1}{a}\iint r^2 . r\,dr\,d\theta$$

$$V = \frac{1}{a}\iint r^3\,dr\,d\theta.$$

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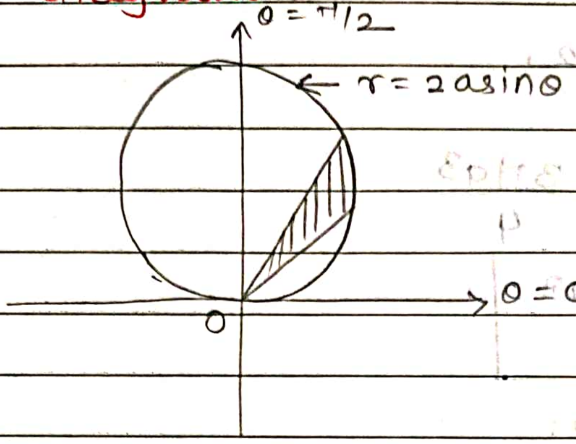
Step 2: diagram




Given,

$x^2 + y^2 = 2ay$

Note: $y = r\sin\theta$

& $x^2 + y^2 = r^2$

$\therefore r^2 = 2a(r\sin\theta)$

$\therefore r = 2a\sin\theta$

Step 3: Strip:

$r = 0$ to $r = 2a\sin\theta$

& $\theta = 0$ to $\theta = \pi/2$

Step 4: Apply New limits & solve

$$\therefore V = \frac{1}{a}\iint r^3\,dr\,d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\int_0^{2a\sin\theta} r^3\,dr\,d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2a\sin\theta} d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2} \frac{(2a)^4\sin^4\theta}{4} - 0\,d\theta$$

$$= \frac{(2a)^4}{4a}\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= \frac{16a^4}{4a}\left[\frac{(4-1)(4-3)}{4(4-2)}\cdot\frac{\pi}{2}\right]$$

$$V = 4a^3\cdot\frac{3\pi}{16}$$

$$V = \frac{3\pi a^3}{4}$$

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For symmetry,
Multiply by 2,

$$\therefore V = 2 \times \frac{3\pi a^3}{4}$$

$$\boxed{V = \frac{3\pi a^3}{2}}$$

# Practice example:

① Find the volume bounded by cylinder $x^2+y^2=4$ and plane $y+z=4$ & $z=0$ ANS: $V = 16\pi$