Unit 4: Tools of Integration

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Subpoints:

- Problems on Reduction formulae
- Beta, Gamma functions
- Differentiation under integral sign
- Error functions

Definite Integration

- Reduction formulae
- Gamma Function
- Beta Function

Reduction Formulae: are useful in evaluating integration of trignometric functions with angular interval which are not immediately solvable due to higher powers.

e.g $\int_0^{\pi/2} \sin^9 x \, dx$

without reduction formulae this would be difficult to integrate. However we can use reduction formula to reduce $\sin^9 x$ to something which we can integrate easily.

Reduction Formulae of some Standard Function:

① $$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \quad \text{If 'n' odd}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{If 'n' even}$$

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② $\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$ If 'n' even

$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1$ If 'n' is odd

③ $\int_0^{\pi/2} \sin^m x\cos^n x\,dx =$ If m = even, n = odd

$$\frac{[(n-1)(n-3)\cdots 3\cdot 1][(m-1)(m-3)\cdots 3\cdot 1]}{(m+n)(m+n-2)\cdots(m+3)(m+1)(m-1)(m-3)\cdots 3\cdot 1}$$

④ $\int_0^{\pi/2} \sin^m x\cos^n x\,dx =$ If m = odd, n = odd

$$\frac{[(n-1)(n-3)\cdots 4\cdot 2][(m-1)(m-3)\cdots 4\cdot 2]}{(m+n)(m+n-2)\cdots(m+3)(m+1)(m-1)(m-3)\cdots 4\cdot 2}$$

keypoints: Reduction formula can simply be calculated as $\frac{O}{AE}\,p$

where, O = substract odd numbers

A = As it is / Addition of powers

E = substract even ~~powers~~ Numbers

$p = \frac{\pi}{2}$ or $1 \longrightarrow$ if power is odd

if power is even

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e.g ① $\int_0^{\pi/2} \sin^6 x \, dx = \frac{0}{AE} p$

$= \frac{(6-1)(6-3)(6-5)}{(6)(6-2)(6-4)} \cdot \frac{\pi}{2}$

$= \frac{5 \times 3 \times 1}{6 \times 4 \times 2} \cdot \frac{\pi}{2}$

$= \frac{15}{96} \pi$

② $\int_0^{\pi/2} \cos^5 x \, dx = \frac{0}{AE} p$

$= \frac{(5-1)(5-3)}{5(5-2)(5-4)} \times 1$

$= \frac{4 \times 2}{5 \times 3 \times 1} = \frac{8}{15}$

③ $\int_0^{\pi/2} \sin^4 x \cos^3 x \, dx = \frac{0}{AE} p$

$= \frac{(4-1)(4-3).(3-1)}{(4+3)(7-2)(7-4)(7-6)} \times 1$

$= \frac{3 \times 1 \times 2}{7 \times 5 \times 3 \times 1} \times 1$

$= \frac{6}{105}$

Property of Definite integral:

① $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

② $\sin(\pi - x) = \sin x$ & $\cos(\pi - x) = -\cos x$

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List of conversion formulae

① $\int_0^{\pi} \sin^m x \cos^n x \, dx = 2 \int_0^{\pi/2} \sin^m x \cos^n x \, dx$, n even

$= 0$, n odd

Solved Example:

Que.1 Evaluate $\int_0^{\pi} x \cdot \sin^7 x \cdot \cos^4 x \, dx$

Sol^n: Given,

$$I = \int_0^{\pi} x \cdot \sin^7 x \cdot \cos^4 x \, dx \quad \text{——} ①$$

by difinite integral property,

$$\int_0^{a} f(x)\, dx = \int_0^{a} f(a-x)\, dx$$

$$\therefore I = \int_0^{\pi} (\pi - x) \cdot \sin^7(\pi - x) \cdot \cos^4(\pi - x)\, dx$$

but $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$

$$I = \int_0^{\pi} (\pi - x) \sin^7 x \cdot \cos^4 x \, dx$$

$$= \int_0^{\pi} \pi \sin^7 x \cdot \cos^4 x - \int x \cdot \sin^7 x \cdot \cos^4 x \, dx$$

$$I = \int_0^{\pi} \pi \sin^7 x \cdot \cos^4 x - \int_0^{\pi} x \sin^7 x \cdot \cos^4 x \, dx$$

$$I = \cancel{2} \; \pi \int_0^{\pi} \sin^7 x \cdot \cos^4 x \, dx - I$$

$$I + I = \pi \int_0^{\pi} \sin^7 x \cdot \cos^4 x \, dx$$

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$$2I = \pi \int_0^{\pi} \sin^7 x \cos^4 x \, dx \quad \text{—— (2)}$$

By conversion formula,

$$\int_0^{\pi} \sin^m x . \cos^n x \, dx = 2 \int_0^{\pi/2} \sin^m x . \cos^n x \, dx$$

$\therefore$ from (2),

$$\cancel{2}I = \pi . \cancel{2} \int_0^{\pi/2} \sin^7 x \cos^4 x \, dx$$

By Reduction formula,

$$I = \pi \left[\frac{(7-1)(7-3)(7-5)\ (4-1)(4-3)}{(11)(11-2)(11-4)(11-6)(11-8)(11-10)} \right]$$

$$\boxed{I = \frac{16\pi}{1155}}$$

Que.2 $\int_0^{\pi} x \sin^5 x . \cos^2 x \, dx$

soln: Given, $I = \int_0^{\pi} x . \sin^5 x . \cos^2 x \, dx$ —— (1)

by definite integral property,

$$\int_0^a f(x) = \int_0^a f(a-x) dx$$

$$\therefore I = \int_0^{\pi} (\pi - x) . \sin^5 (\pi - x) . \cos^2 (\pi - x) dx$$

but $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$

$$\therefore I = \int_0^{\pi} (\pi - x) . \sin^5 x \cos^2 x \, dx$$

$$= \int_0^{\pi} \pi . \sin^5 x . \cos^2 x \, dx - x \sin^5 x . \cos^2 x \, dx$$

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$$I = \int_0^{\pi} \pi \sin^5 x \cos^2 x \, dx - \int_0^{\pi} x \sin^5 x \cos^2 x \, dx$$

$$I = \pi \int_0^{\pi} \sin^5 x \cos^2 x \, dx - I$$

$$I + I = \pi \int_0^{\pi} \sin^5 x \cos^2 x \, dx$$

$$2I = \pi \int_0^{\pi} \sin^5 x \cos^2 x \, dx \quad \text{——— (2)}$$

by using conversion formula,

$$\int_0^{\pi} \sin^m x \cos^n x \, dx = 2 \int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

from (2)

$$2I = \pi \cdot 2 \int_0^{\pi/2} \sin^5 x \cos^2 x \, dx$$

$$I = \pi \cdot \left[\frac{(5-1)(5-3)(2-1)}{(7)(7-2)(7-4)(7-6)} \right]$$

$$\boxed{I = \frac{8\pi}{105}}$$

Practice Example:

(1) $\int_0^{\pi} x \cdot \sin^9 x \cdot \cos^2 x \, dx$

(2) $\int_0^{\pi} x \cdot \sin^5 x \cdot \cos^4 x \, dx$ Ans: $\frac{8\pi}{315}$

(3) $\int_0^{\pi} x \cdot \sin^7 x \cdot \cos^6 x \, dx$

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Gamma Function

Gamma function is useful in evaluating integrals with higher powers of 'x' multiplied by exponential functions with limit '0' to '∞'

e.g $\int_0^\infty e^{-x} \cdot x^{13} \, dx$

Definition: The definite integral $\int_0^\infty e^{-x} \cdot x^{n} \, dx$ denoted by the greek letter Γ is called gamma function & given by,

$$\Gamma(n+1) = \int_0^\infty e^{-x} \cdot x^{n} \, dx$$

The gamma function Γn (Read as Gamma of n) is also called as Euler's integral of second kind.

keypoints:

1. $\Gamma n = (n-1)!$

2. $\Gamma 1 = 1$

3. $\Gamma \frac{1}{2} = \sqrt{\pi}$

4. $\Gamma 0 = \infty$

5. $\Gamma(n+1) = n \Gamma n$

6. $\Gamma \frac{\text{odd no}}{2} = \frac{(\text{odd}-2)}{2} \times \frac{(\text{odd}-4)}{2} \times \frac{(\text{odd}-6)}{2} \times \cdots \frac{1}{2}\sqrt{\pi}$

e.g $\Gamma \frac{9}{2} = \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}$

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Solved Example:

Que. 1 Evaluate $\int_0^{\infty} e^{-t} \cdot t^{10}\, dt$

Soln: Given,

$$\int_0^{\infty} e^{-t}\, t^{10}\, dt$$

we have,

$$\int_0^{\infty} e^{-x} \cdot x^{n}\, dx = \Gamma(n+1)$$

$$\therefore \int_0^{\infty} e^{-t} \cdot t^{10}\, dt = \Gamma(11) = 10!$$

Que. 2 Evaluate $\int_0^{\infty} e^{-x} \cdot x^{5}\, dx$

Soln: Given,

$$\int_0^{\infty} e^{-x} \cdot x^{5}\, dx$$

we have,

$$\int_0^{\infty} e^{-x} \cdot x^{n}\, dx = \Gamma(n+1)$$

$$\therefore \int_0^{\infty} e^{-x} \cdot x^{5}\, dx = \Gamma(6) = 5!$$

Que: 3 Evaluate $\int_0^{\infty} e^{-x} \cdot x^{3/2}\, dx$

Soln: Given,

$$\int_0^{\infty} e^{-x} \cdot x^{3/2}\, dx$$

we have,

$$\int_0^{\infty} e^{-x} \cdot x^{n}\, dx = \Gamma(n+1)$$

$$\int_0^{\infty} e^{-x} \cdot x^{3/2} = \Gamma\left(\frac{3}{2}+1\right) = \Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$$

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Que.4 Evaluate $\int_0^\infty e^{-x} \cdot x^{7/2}\, dx$

Sol^n: Given $\int_0^\infty e^{-x} \cdot x^{7/2}\, dx$

we have, $\int_0^\infty e^{-x} \cdot x^{n}\, dx = \overline{|n+1}$

$$\int_0^\infty e^{-x} \cdot x^{7/2}\, dx = \overline{|\tfrac{7}{2}+1} = \overline{|\tfrac{9}{2}} = \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$$

$$= \frac{105}{16} \sqrt{\pi}$$

Practice Example:

① $\int_0^\infty e^{-t} \cdot t^{4}\, dt$

② $\int_0^\infty e^{-x} \cdot x^{3}\, dx$

③ $\int_0^\infty e^{-t} \cdot t^{5/2}\, dt$

5. $\boxed{0} = \infty$

6. $\boxed{1} = 1$

To understand the concept of calculation of gamma function with the help of video. Scan the QR code 3.2.

Gurukey

1. $\Gamma\left(\frac{\text{odd no.}}{2}\right) = \frac{(\text{odd}-2)}{2} \times \frac{(\text{odd}-4)}{2} \times \frac{(\text{odd}-6)}{2} \times \ldots\ldots \frac{1}{2}\sqrt{\pi}$

e.g. $\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

2. $\Gamma(5) = 4!$

$\Gamma(3) = 2!$

$\Gamma(1) = 0! = 1$

To understand how to solve Gamma and Beta function using 5 steps with the help of video. Scan the QR code 3.3.

Note: We will use 5 steps to solve Gamma and Beta Functions.

Step 1: Substitution

Step 2: Value of x

Step 3: Differentiating w.r.t. x

Step 4: Find new limits

Step 5: Convert the given integral in terms of t.

3.6 Solved Examples of the type $\int_0^\infty e^{-ax^n}\,dx$

Example 3.6.1 **Dec. 2014**

Evaluate $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Solution :

Given, $$I = \int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$$

$$\therefore \quad I = \int_0^\infty e^{-x^{1/2}}\, x^{1/4}\, dx \qquad \ldots(1)$$

Step 1: Substitution

Put $x^{1/2} = t$

Step 2: Value of x

$$x = t^2$$

Step 3: Differentiation

Differentiating w.r.t. x,

$$1 = 2t\frac{dt}{dx}$$

$$dx = 2t\,dt$$

Step 4: Limits

When x = 0	When x = ∞
$t = x^{1/2}$	$t = x^{1/2}$
$t = 0^{1/2}$	$t = \infty^{1/2}$
$t = 0$	$t = \infty$

Step 5: Convert the integral in terms of t.

$\therefore$ From Equation (1)

$$I = \int_0^\infty e^{-x^{1/2}}\, x^{1/4}\, dx$$

$$I = \int_0^\infty e^{-t}\,(t^2)^{1/4}\, 2t\, dt$$

$$\therefore \quad I = 2\int_0^\infty e^{-t}\, t^{1/2}\cdot t\, dt$$

$$\because (t^2)^{1/4} = t^{2/4} = t^{1/2}$$

$$I = 2\int_0^\infty e^{-t}\, t^{\frac{1}{2}+1}\, dt$$

$$I = 2\int_0^\infty e^{-t}\, t^{3/2}\, dt$$

Now, by definition of Gamma function

$$\therefore \quad I = 2\,\Gamma\left(\frac{3}{2}+1\right) \qquad \left[\because \int_0^\infty e^{-x}\, x^n\, dx = \Gamma(n+1)\right]$$

$$I = 2\Gamma\left(\frac{5}{2}\right)$$

$$I = 2\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$I = \frac{3\sqrt{\pi}}{2} \quad \text{...Ans.}$$

Example 3.6.2

Evaluate : $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$

Solution :

Given, $$I = \int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$$

$$\therefore \quad I = \int_0^\infty e^{-x^{1/3}}\, x^{1/2}\, dx \quad \text{...(1)}$$

Step 1 : Substitution

Put $x^{1/3} = t$

Step 2 : Value of x

$$x = t^3$$

Step 3 : Differentiation

$$x = t^3$$

Differentiating w.r.t. x

$$1 = 3t^2 \frac{dt}{dx}$$

$$dx = 3t^2\, dt$$

Step 4 : Limits

When $x = 0$	When $x = \infty$
$t = x^{1/3}$	$t = x^{1/3}$
$t = 0^{1/3}$	$t = \infty^{1/3}$
$t = 0$	$t = \infty$

Step 5 : Convert the integral in terms of t.

$\therefore$ From Equation (1),

$$I = \int_0^\infty e^{-x^{1/3}}\, x^{1/2}\, dx$$

$$I = \int_0^\infty e^{-t}\, (t^3)^{1/2} \cdot 3t^2\, dt$$

$$I = 3\int_0^\infty e^{-t}\, t^{3/2}\, t^2\, dt$$

$$I = 3\int_0^\infty e^{-t}\, t^{\frac{3}{2}+2}\, dt$$

$$I = 3\int_0^\infty e^{-t}\, t^{7/2}\, dt$$

Now, by definition of gamma functions

$$I = 3\Gamma\left(\frac{7}{2}+1\right) \qquad \left[\because \int_0^\infty e^{-x} x^n\, dx = \Gamma(n+1)\right]$$

$$I = 3\Gamma\left(\frac{9}{2}\right)$$

$$I = 3\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$I = \frac{315\sqrt{\pi}}{16} \quad \text{...Ans.}$$

Example 3.6.3 May 2012

Evaluate $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$

Solution :

Given, $$I = \int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$$

$$I = \int_0^\infty e^{-x^3}\, x^{1/2}\, dx$$

Step 1 : Substitution

Put $x^3 = t$

Step 2 : Value of x

$$\therefore \quad x = t^{1/3}$$

Step 3 : Differentiation

$$x = t^{1/3}$$

Differentiating w.r.t. x,

$$1 = \frac{1}{3} t^{\frac{1}{3}-1} \frac{dt}{dx}$$

$$\therefore \quad dx = \frac{1}{3} t^{-2/3}\, dt$$

Step 4 : Limits

When $x = 0$	When $x = \infty$
$t = x^3$	$t = x^3$
$t = 0^3$	$t = \infty^3$
$t = 0$	$t = \infty$

Step 5 : Convert the integral in terms of t.

$\therefore$ From Equation (1), we have,

$$I = \int_0^\infty e^{-x^3} x^{1/2} dx$$

$$\therefore \quad I = \int_0^\infty e^{-t} (t^{1/3})^{1/2} \frac{1}{3} t^{-2/3} dt$$

$$\therefore \quad I = \frac{1}{3} \int_0^\infty e^{-t} t^{1/6} t^{-2/3} dt$$

$$I = \frac{1}{3} \int_0^\infty e^{-t} t^{\frac{1}{6}-\frac{2}{3}} dt$$

$$I = \frac{1}{3} \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$$

Now, by definition of gamma function,

$$I = \frac{1}{3} \overline{\left| -\frac{1}{2}+1 \right.} \qquad \left[\because \int_0^\infty e^{-x} x^n dx = \overline{|n+1} \right]$$

$$I = \frac{1}{3} \overline{\left| \frac{1}{2} \right.} = \frac{1}{3}\sqrt{\pi} \qquad \text{...Ans.}$$

Example 3.6.4 **May 2017**

Evaluate $\int_0^\infty \sqrt{y}\, e^{-\sqrt{y}}\, dy$

Solution :

Given,

$$I = \int_0^\infty \sqrt{y}\, e^{-\sqrt{y}}\, dy$$

$$\therefore \quad I = \int_0^\infty e^{-y^{1/2}} y^{1/2} dy \qquad \text{...(1)}$$

Step 1 : Substitution

Put $y^{1/2} = t$

Step 2 : Value of y

$\therefore$ $y = t^2$

Step 3 : Differentiation

$$y = t^2$$

Differentiating w.r.t. y,

$$1 = 2t \frac{dt}{dy}$$

$$\therefore \quad dy = 2t\, dt$$

Step 4 : Limits

When $y = 0$	When $y = \infty$
$t = y^{1/2}$	$t = y^{1/2}$
$t = 0^{1/2}$	$t = \infty^{1/2}$
$t = 0$	$t = \infty$

Step 5 : Convert the integral in terms of t.

$\therefore$ From Equation (1), we have,

$$I = \int_0^\infty e^{-y^{1/2}} y^{1/2} dy$$

$$\therefore \quad I = \int_0^\infty e^{-t}\, t\, (2t)\, dt$$

$$I = 2 \int_0^\infty e^{-t} t^2 dt$$

Now, by definition of gamma function

$$I = 2 \overline{|2+1} \qquad \left[\because \int_0^\infty e^{-x} x^n dx = \overline{|n+1} \right]$$

$$I = 2 \overline{|3}$$

$$I = 2\,(2!)$$

$$I = 4 \qquad \text{...Ans.}$$

Example 3.6.5 **Dec. 2018, 2009**

[redacted]

...(1)

[redacted]

Beta Function

Beta function is useful in evaluating integrals with higher power of 'x' multiplied by higher power of (1-x) with limit 0 to 1

e.g $\int_0^1 x^9 (1-x)^{15} dx$

The beta function also known as Euler's integral of first kind.

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Definition: The definite integral $\int_0^1 x^m (1-x)^n dx$

$m>0, n>0$ is a function of m & n called as Beta function. It is denoted by $B(m,n)$

$$\therefore \int_0^1 x^m (1-x)^n \, dx = B(m+1, n+1)$$

Relation between Beta & Gamma Function:

$$B(m,n) = \frac{\Gamma m \; \Gamma n}{\Gamma m+n}$$

Solved Example:

Que.1 Find the value of ~~B(m,n)~~ $B(5/2, 3/2)$

Soln: $B(5/2, 3/2)$ given

by using formula,

$$B(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$$

$$= \frac{\Gamma 5/2 \; \Gamma 3/2}{\Gamma \frac{5}{2}+\frac{3}{2}}$$

$$= \frac{\left(\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}\right) \frac{1}{2} \sqrt{\pi}}{\Gamma 4}$$

$$B\left(\frac{5}{2}, 3/2\right) = \frac{\frac{3}{8}\pi}{3!} = \frac{3\pi}{48} = \frac{\pi}{16}$$

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Que.2 Find the value of $B\left(\frac{3}{2}, \frac{7}{2}\right)$

Sol^n: Given,

$$B\left(\frac{3}{2}, \frac{7}{2}\right)$$

By using formula, $B(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n}$

$$B\left(\frac{3}{2}, \frac{7}{2}\right) = \frac{\Gamma\frac{3}{2} \cdot \Gamma\frac{7}{2}}{\Gamma\left(\frac{3}{2}+\frac{7}{2}\right)}$$

$$= \frac{\frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{\Gamma 5}$$

$$= \frac{\frac{45}{16}\pi}{4!} = \frac{45}{16 \times 24}\pi$$

$$= \frac{15}{384}\pi = \frac{5}{128}\pi$$

$$\therefore B\left(\frac{3}{2}, \frac{7}{2}\right) = \frac{5}{128}\pi$$

Que.3 Find the value of $B(5/2, 3)$.

Sol^n:

by using formula,

$$B(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$$

$$= \frac{\Gamma 5/2 \cdot \Gamma 3}{\Gamma\left(\frac{5}{2}+3\right)}$$

$$= \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot 2!}{\Gamma\frac{11}{2}} = \frac{\frac{3}{2}\sqrt{\pi}}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

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$= \frac{16}{315}$

$\therefore B\left(\frac{5}{2}, 3\right) = \frac{16}{315}$

Differentiation under integral Sign

In these integrals in addition to the variable of integration one or more parameters involved. We consider only one parameter α, then the integral will take form

$$I(\alpha) = \int_a^b f(x, \alpha)\, dx$$

where, α - parameter

x - variable of integration

a & b - limits of integration.

Rule 1: Integral with constant limits.

$$\frac{d}{d\alpha} \int_a^b f(x, \alpha)\, dx = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha)\, dx$$

Solved Examples:

Que.1 show that $\int_0^1 \frac{x^\alpha - 1}{\log x}\, dx = \log(\alpha + 1)$

Soln: Step 1: Consider integral as $I(\alpha)$

$$\therefore I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x}\, dx$$

Step 2: Differentiate on both sides, w.r.t 'α'

$$I'(\alpha) = \frac{d}{d\alpha} \int_0^1 \frac{x^\alpha - 1}{\log x}\, dx$$

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Step 3: Apply DUIS

$$I'(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha} \frac{x^\alpha - 1}{\log x}\, dx$$

Step 4: Take the partial derivative.

$$I'(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha}\frac{x^\alpha}{\log x} - \frac{\partial}{\partial \alpha}\frac{1}{\log x}\, dx$$

$$= \int_0^1 \frac{x^\alpha \log x}{\log x} - 0\, dx$$

$$I'(\alpha) = \int_0^1 x^\alpha\, dx$$

Step 5: Integrate w.r.t 'x',

$$I'(\alpha) = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1$$

$$I'(\alpha) = \cancel{\frac{x^2}{2}}\ \frac{1^{\alpha+1}}{\alpha+1} = \frac{1}{\alpha+1}$$

Step 6: Integrate w.r.t 'α',

$$\int I'(\alpha) = \int \frac{1}{\alpha+1}$$

$$I(\alpha) = \log(\alpha+1) + c \quad \text{—— (1)}$$

Step 7: put value of 'α' to find c.

$\therefore$ put $\alpha = 0$

$$I(0) = \log(1) + c$$

$$0 = 0 + c \implies \boxed{c = 0} \quad \text{put in (1)}$$

$$I(\alpha) = \log(\alpha+1) + 0$$

$$= \log(\alpha+1)$$ Hence proved.

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Que.2 Show that $\int_0^\infty \left(\frac{1-e^{-\alpha x}}{x}\right) e^{-x}\, dx = \log(\alpha+1)$

Soln: Step 1: consider integral as $I(\alpha)$,

$$I(\alpha) = \int_0^\infty \left(\frac{1-e^{-\alpha x}}{x}\right) e^{-x}\, dx$$

Step 2: diff w.r.t α,

$$I'(\alpha) = \frac{d}{d\alpha}\int_0^\infty \left(\frac{1-e^{-\alpha x}}{x}\right) e^{-x}\, dx$$

Step 3:

$$I'(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha}\left(\frac{1-e^{-\alpha x}}{x}\right) e^{-x}\, dx$$

Step 4: Take partial derivative

$$I'(\alpha) = \int_0^\infty \left(\frac{\partial}{\partial\alpha}\frac{1}{x} - \frac{\partial}{\partial\alpha}\frac{e^{-\alpha x}}{x}\right) e^{-x}\, dx$$

$$= \int_0^\infty 0 - \frac{e^{-\alpha x}(-x)}{x} \cdot e^{-x}\, dx$$

$$= \int_0^\infty e^{-\alpha x} \cdot e^{-x}\, dx$$

$$I'(\alpha) = \int_0^\infty e^{-x(\alpha+1)}\, dx$$

Step 5: Integrate w.r.t 'x',

$$I'(\alpha) = \left[\frac{e^{-x(\alpha+1)}}{-(\alpha+1)}\right]_0^\infty$$

$$= \left[\frac{e^{-\infty(\alpha+1)}}{-(\alpha+1)} - \frac{e^{-0(\alpha+1)}}{-(\alpha+1)}\right] = 0 + \frac{1}{\alpha+1}$$

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Step 6: Integrate w.r.t 'α',

$$\int_0^\infty I'(\alpha)\,d\alpha = \int \frac{1}{\alpha+1}\,d\alpha$$

$$I(\alpha) = \log(\alpha+1) + c \quad \text{—— (1)}$$

Step 7: put the value of 'α' to find value of 'c'.

$\therefore$ put $\alpha = 0$

$\therefore I(0) = \log(0+1) + c$

$\Rightarrow \boxed{c=0}$ put in (1),

$$I(\alpha) = \log(\alpha+1)$$

Hence proved.

Practice Example:

(1) show that

$$\int_0^\infty \frac{e^{-x} - e^{-\alpha x}}{x \sec x}\,dx = \frac{1}{2}\log\left(\frac{\alpha^2+1}{2}\right)$$

Rule II: Integral with limits as function of parameter.

If $I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx$ then

$$\frac{dI}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x,\alpha)\,dx + f(b,\alpha)\cdot\frac{db}{d\alpha} - f(a,\alpha)\cdot\frac{da}{d\alpha}$$

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Error Functions

The error function of 'x' denoted by erf(x) & is defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

Solved Examples:

Que.1 Prove that $\frac{d}{dx}[\text{erf}(x)] = \frac{2}{\sqrt{\pi}} e^{-x^2}$

Soln: From the definition

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

diff w.r.t 'x' we get,

$$\frac{d}{dx}[\text{erf}(x)] = \frac{2}{\sqrt{\pi}} \frac{d}{dx} \int_0^x e^{-u^2}\, du$$

Here upper limit of integration is in 'x'

∴ Apply DUIS Rule - II

∴ By Rule II

$$\frac{d}{dx}[\text{erf}(x)] = \frac{2}{\sqrt{\pi}} \left\{ \int_0^x \frac{\partial}{\partial x} e^{-u^2}\, du + e^{-x^2} \cdot \left(\frac{dx}{dx}\right) - e^{-0} \cdot \frac{d}{dx}(0) \right\}$$

$$= \frac{2}{\sqrt{\pi}} \left\{ 0 + e^{-x^2}(1) - 0 \right\}$$

$$\frac{d}{dx}[\text{erf}(x)] = \frac{2}{\sqrt{\pi}} \cdot e^{-x^2}$$

Hence proved.

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Que.2 Show that $\frac{d}{dx}[\operatorname{erf}(ax)] = \frac{2ae^{-a^2x^2}}{\sqrt{\pi}}$.

Solⁿ: From the definition of error function,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$$

$$\therefore \operatorname{erf}(ax) = \frac{2}{\sqrt{\pi}}\int_0^{ax} e^{-u^2}\,du$$

diff w.r.t 'x',

$$\frac{d}{dx}[\operatorname{erf}(ax)] = \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^{ax} e^{-u^2}\,du$$

as upper limit is in terms of 'x' we used DUIS Rule - II

∴ By Rule - II

$$\frac{d}{dx}[\operatorname{erf}(ax)] = \frac{2}{\sqrt{\pi}}\left\{\int_0^{ax}\frac{\partial}{\partial x}e^{-u^2}\,du + e^{-(ax)^2}\cdot\frac{d}{dx}(ax) - e^{-0^2}\cdot\frac{d}{dx}0\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left\{0 + e^{-a^2x^2}\cdot(a) - 0\right\}$$

$$\frac{d}{dx}[\operatorname{erf}(ax)] = \frac{2a}{\sqrt{\pi}}\cdot e^{-a^2x^2}$$

Hence proved.

Practice Example:

① $\frac{d}{dx}[\operatorname{erf}(ax^n)] = \frac{2an}{\sqrt{\pi}}x^{n-1}e^{-a^2\cdot x^{2n}}$