Date: / /

subpoints:

- Tracing of curves in cartesian, parametric & polar co-ordinates.

Cartesian equation:
cartesian equation are equation in terms of x & y i.e (x, y).

Cartesian Equation
- Explicite equation
- Implicit Equation

Explicite Equations:
Explicite equations are the equations in which we can take terms of x on one side & terms of y on the other side

e.g $y^2 = \frac{2a - x}{x^2}$

$y^2 = x - 1$

Rules for curve tracing:

Rule 1: Symmetry

(A) About x-axis: If the given equation has even powers of y everywhere then curve is symmetrical about x-axis.

e.g $y^2 = x$

(B) About y-axis: If the given equation has even power of x everywhere, then curve is symmetrical about y-axis.

e.g $x^2 = y$

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(C) About the line $y = x$ Replace x by y & y by x, if the given equation remain unchange then curve is symmetrical about line $y = x$.

e.g $xy = c$

Rule 2: Origin

(A) Origin: put $x = 0$, if we get $y = 0$ i.e $(0,0)$ then curve passes through origin.

(B) Tangent at origin: If curve passes through origin then the tangent at origin can be obtained by equating to zero, the lowest degree terms taken together in given equation.

Rule 3: Intersection with axis

(A) Intersection with x-axis:

put $y = 0$ & find x.

(B) Intersection with Y-axis:

put $x = 0$ & find y

Rule 4: Tangent at point in Rule 3.

To find tangent at point differentiate given eqn w.r.t x i.e find $\frac{dy}{dx}$ & if

1. $\frac{dy}{dx} = 0$, tangent parallel to x-axis

2. $\frac{dy}{dx} = \infty$, tangent parallel to Y-axis.

Rule 5: Asymptotes

Asymptotes are tangent to the curve at infinity.

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(A) Asymptotes parallel to x-axis:
This can be obtained by equating heighest power of x taken together to zero.

(B) Asymptotes parallel to Y-axis:
This can be obtained by equating heighest power of y taken together to zero.

Rule 6: Region of Absence
Arrange the equation as $y^2 = f(x)$ OR $x^2 = f(y)$ then check for negative values of

1. odd power of x (OR y)
2. Addition
3. substraction.

e.g

$$x + a < 0$$

$$x < -a$$

i.e curve doesnot exist for value less than $-a$

Solved Example:

Que.1 Trace the curve $y^2(2a - x) = x^3$.

soln:

Rule 1: symmetry

$$y^2(2a - x) = x^3$$

$$y^2 2a - y^2 x = x^3$$

Even power of 'y' everywhere so symmetrical about x-axis:

Rule 2: Origin

(A) origin: $y^2(2a - x) = x^3$

put $x = 0$

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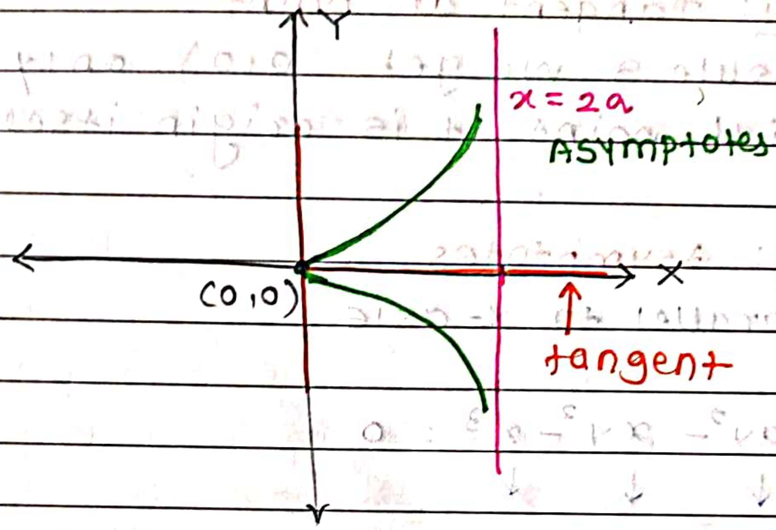
$y^2(2a-0)=0$

$2ay^2=0$

$y^2=0 \Rightarrow y=0$

i.e when $x=0$ we get $y=0$ i.e $(0,0)$

$\therefore$ curve passes through origin.



(B) Tangent at origin:

To find tangent at origin equate to zero lowest degree term taken together,

$y^2(2a-x)=x^3$

$2ay^2-xy^2=x^3$

$2ay^2-xy^2-x^3=0$

$\downarrow \quad \downarrow \quad \downarrow$

$2 \quad 3 \quad 3$ powers of x & y

$\therefore 2ay^2=0$

$y^2=0$

$\boxed{y=0} \longrightarrow$ x-axis

$\therefore$ x-axis is tangent at origin.

Rule 3: Intersection with axis

(A) intersection with x-axis

put $y=0$ in $y^2(2a-x)=x^3$

$x^3=0$

$\boxed{x=0}$ i.e $(0,0)$

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(B) intersection with Y-axis:

put $x=0$ in $y^2(2a-x)=x^3$

$\Rightarrow \boxed{y=0}$

i.e $(0,0) \longrightarrow$ origin

Rule 4: Tangent at point

In Rule 3 we get $(0,0)$ only is not special point it is origin itself.

Rule 5: Asymptotes

(A) parallel to X-axis.

$$2ay^2 - xy^2 - x^3 = 0$$

$\downarrow$ $\downarrow$ $\downarrow$

0 1 3 powers of x only

equate coefficient of heighest power of x to zero.

$$-1=0$$

Not possible.

$\therefore$ No assymptotes parallel to X-axis

(B) Parallel to Y-axis:

$$2ay^2 - xy^2 - x^3 = 0$$

$\downarrow$ $\downarrow$ $\downarrow$

2 2 0 powers of y only

$\therefore$ coefficient of heigest power of y

$$(2a-x)=0$$

$$\boxed{x=2a}$$ line / tangent / Assymptote

parallel to Y-axis

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Rule 6: Region of Absence

$$y^2(2a-x) = x^3$$

$$y^2 = \frac{x^3}{2a-x}$$

then we check for negative values

(i) odd power of x,

$\therefore x^3 < 0 \longrightarrow x < 0$

(ii) subtraction:

$2a - x < 0$

$2a < x$

i.e $x > 2a$

$\therefore \quad 2a < x < 0$

$\therefore$ curves lies between 0 to 2a

Que2 Trace the curve $y^2 = x^2(1-x)$

soln:

Rule 1: symmetry

given $y^2 = x^2(1-x)$

$y^2 = x^2 - x^3$

even power of y so curve is symmetrical about x-axis

Rule 2: Origin

(A) origin: $y^2 = x^2(1-x)$

put $x=0$

$y^2 = 0$

$\boxed{y=0}$

$\therefore$ for $x=0$ we get $y=0$ $\therefore (0,0)$.

curve passes through origin.

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Rule 2

(B) Tangent at origin:

To find tangent at origin,

$$y^2 = x^2(1-x)$$

$$y^2 = x^2 - x^3$$

$$y^2 - x^2 + x^3 = 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$2 \quad 2 \quad 3 \qquad \text{powers of } x \,\&\, y$$

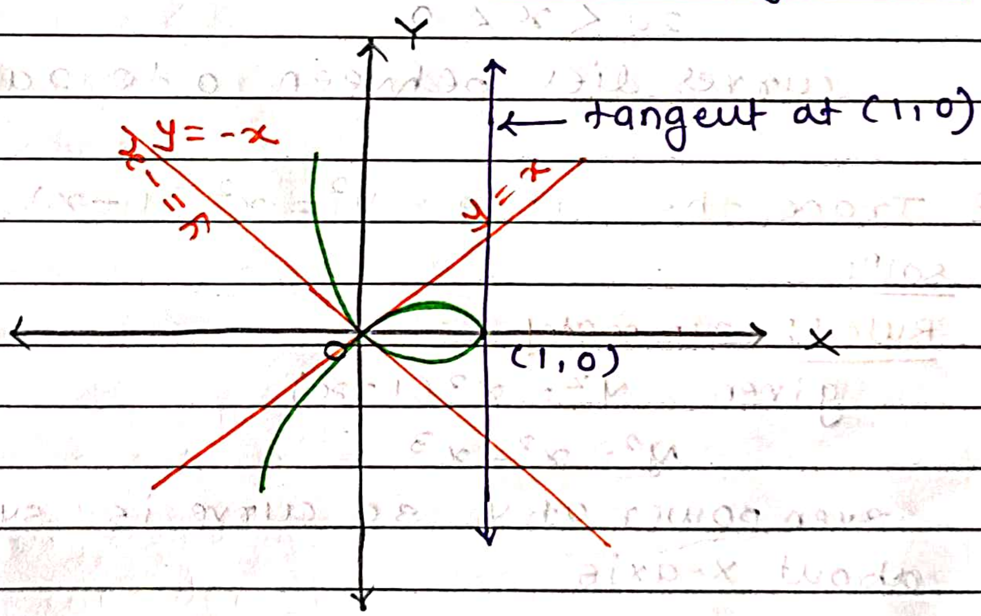
Lowest degree terms = 0

$$-x^2 + y^2 = 0$$

$$y^2 = x^2$$

$$y = \pm x$$

$\therefore$ lines $y = x$ $y = -x$ are tangents at origin.



Rule 3: Intersection with axis

(A) intersection with x-axis

$$y^2 = x^2(1-x)$$

put $y = 0$

$$0 = x^2(1-x)$$

$$x^2 = 0 \quad \text{or} \quad (1-x) = 0$$

$$x = 0 \qquad x = 1$$

then $y = 0$ $\qquad y = 0$

$\therefore$ (0,0) & (1,0) are the point of intersection with x-axis. out of which (1,0) is special point

(B) intersection with Y-axis

$y^2 = x^2(1-x)$

put $x=0$

$\Rightarrow y=0$

i.e (0,0) origin (No special point)

Rule 4: Tangent at a point.

Now,

$$y^2 = x^2(1-x)$$

$$y^2 = x^2 - x^3$$

$$y^2 - x^2 + x^3 = 0$$

diff w.r.t 'x',

$$2y\frac{dy}{dx} - 2x + 3x^2 = 0$$

$$2y\frac{dy}{dx} = 2x - 3x^2$$

$$\frac{dy}{dx} = \frac{2x-3x^2}{2y}$$

$$\left(\frac{dy}{dx}\right)_{(1,0)} = \frac{2-3}{0} = \frac{-1}{0} = \infty \longrightarrow \text{parallel to Y-axis.}$$

$\therefore$ at (1,0) tangent parallel to Y-axis.

Rule 5: Asymptotes

(A) Asymptotes parallel to x-axis

we have $y^2 - x^2 + x^3 = 0$

$\downarrow \quad \downarrow \quad \downarrow$

$0 \quad 2 \quad 3$ power of x only

coefficient of highest power = 0

$1 = 0$ Not possible.

$\therefore$ No Asymptotes parallel to x-axis.

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(B) Asymptotes parallel to Y-axis

$y^2 - x^2 + x^3 = 0$

$\downarrow \quad \downarrow \quad \downarrow$

$2 \quad 0 \quad 0$ Powers of y only

coefficient of heighest power of y = 0

$1 = 0$

No asymptotes parallel to y-axis also.

Rule 6: Region of absence

Arrange as, $y^2 = x^2(1-x)$

(i) odd power of x,

$x^3 < 0$

$x < 0$

(ii) subtraction;

$1 - x < 0$

$1 < x$

$1 < x < 0$

$\therefore$ value lies between 0 to 1.

Curves Given by Parametric Equations

Rule for curve tracing:

consider the parametric equations are given by $x = f(t)$ & $y = g(t)$ then the rules to trace the curves given by parametric equation as follows.

Rule: Symmetry

(i) If $x = f(t)$ is even function i.e [$f(-t) = f(t)$] & $y = g(t)$ is odd function i.e $g(-t) = -g(t)$ then curve is symmetrical about x-axis.

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(ii) If $x = f(t)$ is odd function & $y = g(t)$ is even function then curve is symmetrical about Y-axis.

Rule 2: Origin

If some value of t, both x & y are zero, then curve passes through origin. We can check it by using the process mentioned below.

Step 1: Put $x = 0$ to find 't'

Step 2: Put this value of 't' in y

Step 3: If we get $y = 0$ for some value of t then curve passes through origin.

Rule 3: Tangents at special point i.e $\frac{dy}{dx}$

Find $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Rule 4: Prepare a table of $t, x, y, \frac{dy}{dx}$

Step 1: Put $x = 0$ and find t

Step 2: Put this value of t in y

Step 3: If we get $y = 0$ for the same value of t, then the curve passes through origin

Rule 3: Tangents at special point i.e. $\frac{dy}{dx}$

$$\text{find } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Rule 4: Prepare a table of $t, x, y, \frac{dy}{dx}$.

5.10 Type 5 : Solved Examples on Curves given by Parametric Equations

Example 5.10.1 **May 10, 16**

Trace the cycloid $x = a(t + \sin t)$, $y = a(1 + \cos t)$

Solution :

Rule 1: Symmetry

Let $x = f(t) = a(t + \sin t)$

put $t = -t$

$$f(-t) = a[-t + \sin(-t)]$$

but $\sin(-\theta) = -\sin\theta$

$$f(-t) = a[-t - \sin t]$$

$$\therefore \quad f(-t) = -a(t + \sin t)$$

$$f(-t) = -f(t)$$

$$\therefore \quad x = f(t) \text{ is an odd function}$$

Also, $y = g(t) = a(1 + \cos t)$

put $t = -t$

$$g(-t) = a[1 + \cos(-t)]$$

but $\cos(-\theta) = \cos(\theta)$

$$g(-t) = a(1 + \cos t)$$

$$g(-t) = g(t)$$

$$\therefore \quad y = g(t) \text{ is an even function.}$$

Now as x is odd and y is even function, the given curve is symmetrical about Y-axis.

Rule 2 : **Origin**

Note : If we get $x = 0$, $y = 0$ for any value of t then curve passes through origin.

We have, $y = a(1 + \cos t)$

Put $y = 0$

$$0 = a(1 + \cos t)$$

$$\therefore \quad 1 + \cos t = 0$$

$$\cos t = -1$$

$$\cos t = \cos(180°)$$

$$\therefore \quad t = 180° = \pi$$

Now, put $t = 180 = \pi$ in

$$x = a(t + \sin t)$$

$$x = a(\pi + \sin \pi)$$

but $\sin \pi = 0$.

$$\therefore \quad x = a\pi$$

i.e for $t = \pi$ we do not get $x = 0$ and $y = 0$

$\therefore$ Curve does not pass through origin.

Rule 3 : **Find $\frac{dy}{dx}$**

we have, $y = a(1 + \cos t)$

Differentiation w.r.t. t

$$\frac{dy}{dt} = a[0 - \sin t]$$

$$\therefore \quad \frac{dy}{dt} = -a \sin t \qquad ...(1)$$

Also $x = a(t + \sin t)$

Differentiating w.r.t. 't'

$$\frac{dx}{dt} = a(1 + \cos t) \qquad ...(2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$\therefore$ from Equations (1) and (2)

$$\frac{dy}{dx} = \frac{-a \sin t}{a(1 + \cos t)}$$

$$\frac{dy}{dx} = -\frac{2 \sin\left(\frac{t}{2}\right) \cos\left(\frac{t}{2}\right)}{2 \cos^2\left(\frac{t}{2}\right)}$$

$$\frac{dy}{dx} = -\frac{\sin\left(\frac{t}{2}\right)}{\cos\left(\frac{t}{2}\right)}$$

$$\frac{dy}{dx} = -\tan\left(\frac{t}{2}\right)$$

Rule 4 : Prepare a table of t, x, y, $\frac{dy}{dx}$

t	0	$\frac{\pi}{2} = 90°$	$\pi = 180°$	$\frac{3\pi}{2} = 270°$	$2\pi = 360°$
x	0	$a\left(\frac{\pi}{2}+1\right)$	$a\pi$	$a\left(\frac{3\pi}{2}-1\right)$	$2a\pi$
y	2a	a	0	a	2a
$\frac{dy}{dx}$	0	–1	$-\infty$	–1	0

(i) For t = 0

A) $x = a(t + \sin t)$
$x = a(0 + \sin 0)$
$x = a(0 + 0)$
$x = 0$

B) $y = a(1 + \cos t)$
$y = a(1 + \cos 0)$
$y = a(1 + 1)$
$y = 2a$

C) $\frac{dy}{dx} = -\tan\left(\frac{t}{2}\right) = -\tan(0) = 0$

(ii) for $t = \frac{\pi}{2} = 90°$

A) $x = a(t + \sin t)$
$x = a\left(\frac{\pi}{2} + \sin 90\right)$
$x = a\left(\frac{\pi}{2} + 1\right)$

B) $y = a(1 + \cos t)$
$y = a(1 + \cos 90)$
$y = a(1 + 0)$
$y = a$

C) $\frac{dy}{dx} = -\tan\left(\frac{t}{2}\right) = -\tan\left(\frac{90}{2}\right)$
$\frac{dy}{dx} = -\tan(45) = -1$

(iii) For $t = \pi = 180°$

A) $x = a(t + \sin t)$
$x = a(\pi + \sin 180)$
$x = a(\pi + 0)$
$x = a\pi$

B) $y = a(1 + \cos t)$
$y = a(1 + \cos 180)$
$y = a(1 - 1)$
$y = 0$

C) $\frac{dy}{dx} = -\tan\left(\frac{t}{2}\right) = -\tan\left(\frac{180}{2}\right) = -\tan 90$
$\frac{dy}{dx} = -\infty$

(iv) for $t = \frac{3\pi}{2} = 270°$

A) $x = a(t + \sin t)$
$x = a\left(\frac{3\pi}{2} + \sin 270\right)$
$x = a\left(\frac{3\pi}{2} - 1\right)$

B) $y = a(1 + \cos t)$
$y = a(1 + \cos 270)$
$y = a(1 + 0)$
$y = a$

C) $\frac{dy}{dx} = -\tan\left(\frac{t}{2}\right) = -\tan\left(\frac{270}{2}\right)$
$\frac{dy}{dx} = -1$

Likewise other values can be calculated.

> **Gurukey**
>
> For any value of t, lets say $t = \pi$ put $t = 180°$ while calculating value of trigonometric functions as it is easy to calculate the value on calculator. Calculator is by default is degree (D) mode.

e.g $x = a(t + \sin t)$
$x = a(\pi + \sin \pi)$ use $\pi = 180$
$= a(\pi + \sin 180)$ easy to calculate

Note : $\frac{dy}{dx} = 0 \rightarrow$ tangent is parallel to X-axis.
$\frac{dy}{dx} = \pm\infty \rightarrow$ tangent is parallel to Y-axis

The sketch of the curve is as follows:

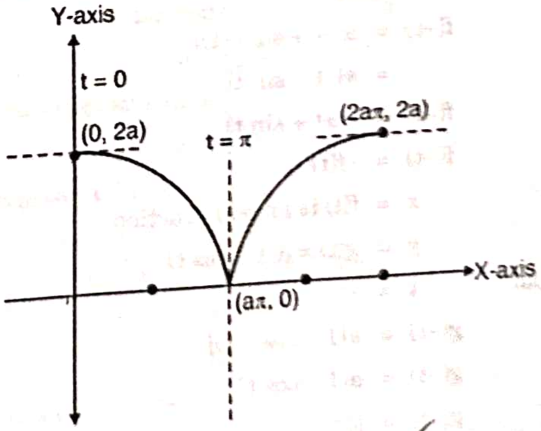

Fig. 5.32

But as the curve is symmetrical about Y-axis.

y = a(t) =

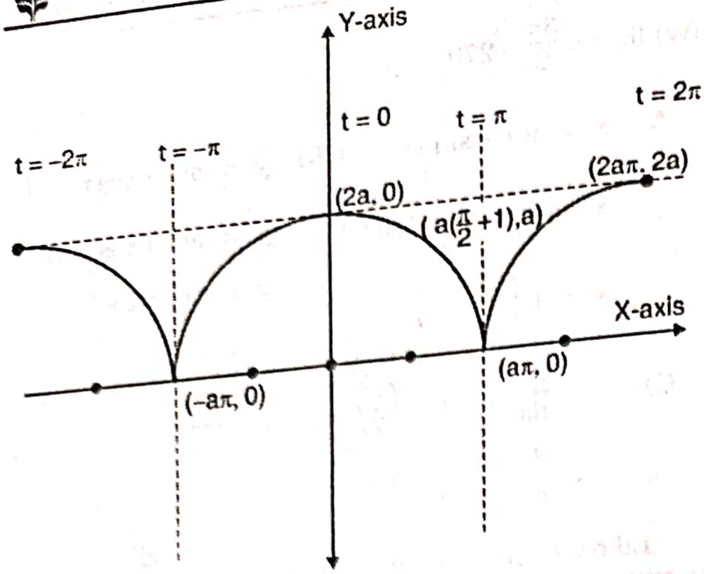

Fig. 5.33

Gurukey

When a circle rolls in a plane along given straight line, the locus traced out by a fixed point on the circumference of rolling circle is called as "cycloid".

For better understanding, check the link

https://youtu.be/vXNwYu1n9iw.

Example 5.10.2 **May 14, Dec. 16, 17**

Trace the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$

Solution :

Rule 1 : **Symmetry**

let $x = f(t) = a(t + \sin t)$

put $t = -t$

$$f(-t) = a(-t + \sin(-t))$$
$$= a[-t - \sin t]$$
$$f(-t) = -a(t + \sin t)$$
$$\therefore \quad f(-t) = -f(t)$$
$$\therefore \quad x = f(t) \text{ is an odd function}$$

Also $y = g(t) = a(1 - \cos t)$

put $t = -t$

$$g(-t) = a[1 - \cos(-t)]$$
$$g(-t) = a(1 - \cos t)$$
$$\therefore \quad g(-t) = g(t)$$
$$\therefore \quad y = g(t) \text{ is an even function.}$$

As x is odd and y is even function, the given curve is symmetrical about Y-axis.

Rule 2 : **Origin.**

we have, $y = a(1 - \cos t)$

put $y = 0$

$$0 = a(1 - \cos t)$$
$$0 = 1 - \cos t$$
$$\cos t = 1$$
$$\cos t = \cos 0$$
$$\therefore \quad t = 0$$

Now, put $t = 0$ in

$$x = a(t + \sin t)$$
$$x = a(0 + \sin 0)$$
$$x = a(0 + 0)$$
$$x = 0$$

$\therefore$ for $t = 0$, we get both $x = 0$ and $y = 0$

$\therefore$ Curve passes through origin

Rule 3 : **Find $\frac{dy}{dx}$**

Given $y = a(1 - \cos t)$

Differentiation w.r.t. 't'

$$\frac{dy}{dt} = a[0 + \sin t]$$
$$\frac{dy}{dt} = a \sin t \quad \text{...(1)}$$

Also, $x = a(t + \sin t)$

Differentiation w.r.t. t

$$\frac{dx}{dt} = a[1 + \cos t] \quad \text{...(2)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$\therefore$ from Equations (1) and (2)

$$\frac{dy}{dx} = \frac{a \sin t}{a(1 + \cos t)}$$
$$\frac{dy}{dx} = \frac{2 \sin\left(\frac{t}{2}\right) \cos\left(\frac{t}{2}\right)}{2 \cos^2\left(\frac{t}{2}\right)}$$
$$\frac{dy}{dx} = \tan\left(\frac{t}{2}\right)$$

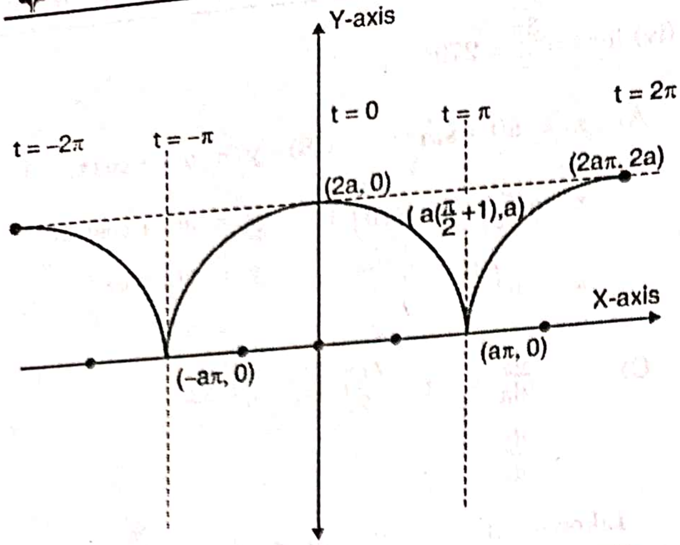

Fig. 5.33

Gurukey

When a circle rolls in a plane along given straight line, the locus traced out by a fixed point on the circumference of rolling circle is called as "cycloid".

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Example 5.10.2 May 14, Dec. 16, 17

Trace the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$

Solution :

Rule 1 : **Symmetry**

let $x = f(t) = a(t + \sin t)$

put $t = -t$

$$f(-t) = a(-t + \sin(-t))$$
$$= a[-t - \sin t]$$
$$f(-t) = -a(t + \sin t)$$
$$\therefore \quad f(-t) = -f(t)$$

$\therefore$ $x = f(t)$ is an odd function

Also $y = g(t) = a(1 - \cos t)$

put $t = -t$

$$g(-t) = a[1 - \cos(-t)]$$
$$g(-t) = a(1 - \cos t)$$
$$\therefore \quad g(-t) = g(t)$$

$\therefore$ $y = g(t)$ is an even function.

As x is odd and y is even function, the given curve is symmetrical about Y-axis.

Rule 2 : **Origin.**

we have, $y = a(1 - \cos t)$

put $y = 0$

$$0 = a(1 - \cos t)$$
$$0 = 1 - \cos t$$
$$\cos t = 1$$
$$\cos t = \cos 0$$
$$\therefore \quad t = 0$$

Now, put $t = 0$ in

$$x = a(t + \sin t)$$
$$x = a(0 + \sin 0)$$
$$x = a(0 + 0)$$
$$x = 0$$

$\therefore$ for $t = 0$, we get both $x = 0$ and $y = 0$

$\therefore$ Curve passes through origin

Rule 3 : **Find $\frac{dy}{dx}$**

Given $y = a(1 - \cos t)$

Differentiation w.r.t. 't'

$$\frac{dy}{dt} = a[0 + \sin t]$$
$$\frac{dy}{dt} = a \sin t \quad \text{...(1)}$$

Also, $x = a(t + \sin t)$

Differentiation w.r.t. t

$$\frac{dx}{dt} = a[1 + \cos t] \quad \text{...(2)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$\therefore$ from Equations (1) and (2)

$$\frac{dy}{dx} = \frac{a \sin t}{a(1 + \cos t)}$$
$$\frac{dy}{dx} = \frac{2 \sin\left(\frac{t}{2}\right) \cos\left(\frac{t}{2}\right)}{2 \cos^2\left(\frac{t}{2}\right)}$$
$$\frac{dy}{dx} = \tan\left(\frac{t}{2}\right)$$

Rule 4 : Prepare a table of t, x, y, $\frac{dy}{dx}$

t	0	$\frac{\pi}{2} = 90°$	$\pi = 180°$	$\frac{3\pi}{2} = 270°$	$2\pi = 360°$
x	0	$a\left(\frac{\pi}{2}+1\right)$	$a\pi$	$a\left(\frac{3\pi}{2}-1\right)$	$2a\pi$
y	0	a	2a	a	0
$\frac{dy}{dx}$	0	1	∞	−1	0

The sketch of the cycloid is as follows:

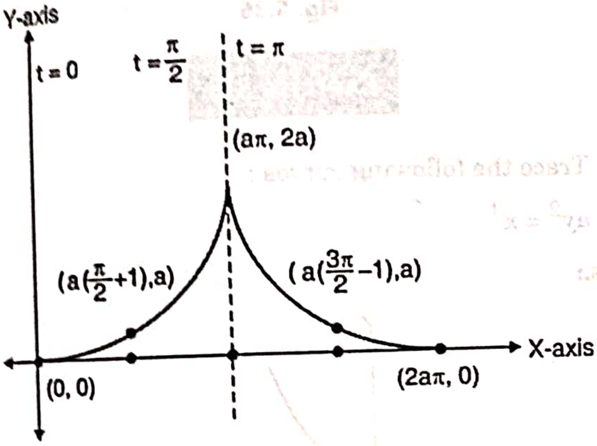

Fig. 5.34

But the curve is symmetrical about Y-axis.

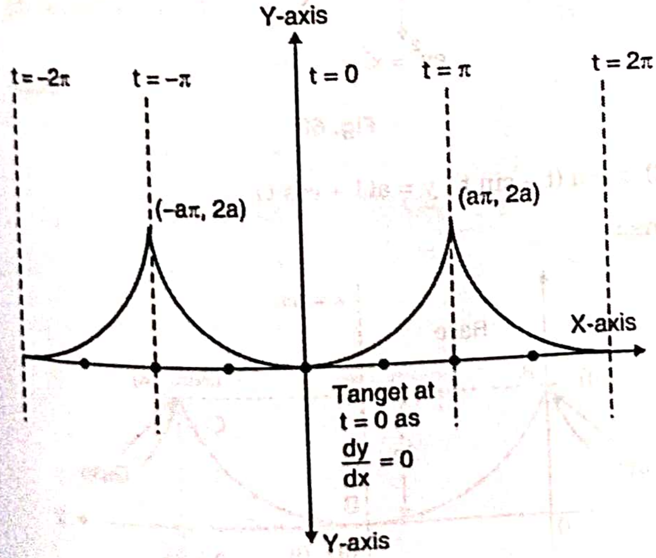

Fig. 5.35

Example 5.10.3 **Dec. 11**

Trace the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution :

This curve is known as Asteroid.

Also known as star shaped curve.

Parametric equations of the curve are given by,

$x = a\cos^3 t$ and $y = a\sin^3 t$

Rule 1: Symmetry

(i) Let $x = f(t) = a\cos^3 t$

Put $t = -t$

$f(-t) = a[\cos(-t)]^3$

but $\cos(-\theta) = \cos\theta$

$f(-t) = a\cos^3 t$

$f(-t) = f(t)$

$\therefore$ $x = f(t)$ is an even function

Also, $y = g(t) = a\sin^3 t$

put $t = -t$

$g(-t) = a[\sin(-t)]^3$

but $\sin(-\theta) = -\sin\theta$

$\therefore$ $g(-t) = a[-\sin t]^3 = -a\sin^3 t$

$\therefore$ $g(-t) = -g(t)$

$\therefore$ y is an odd function

As x is even and y is odd function, curve is symmetrical about X-axis.

(ii) If we don't get symmetry about Y-axis in Rule 1 then we can also check the symmetry about Y-axis by other rules as follows.

we have, $x = a\cos^3 t$

Replace t by $\pi - t$

$x = a[\cos(\pi - t)]^3$

but $\cos(\pi - t) = -\cos t$

$x = a[-\cos t]^3 = -a\cos^3 t$

$\therefore$ x has opposite value

Also $y = a\sin^3 t$

Replace t by $\pi - t$

$y = a[\sin(\pi - t)]^3$

but $\sin(\pi - t) = \sin t$

$y = a[\sin t]^3$

$y = a\sin^3 t$

$\therefore$ y has same value.

$\therefore$ After replacing t by $\pi - t$, x has opposite value and y has same value. Hence curve is symmetrical about Y-axis.

$(-t +$

$= a(-t - s$

Rule 2 : **Origin**

we have, $x = a\cos^3 t$

put $x = 0$

$0 = a\cos^3 t$

$\cos^3 t = 0$

$\cos t = 0 = \cos\frac{\pi}{2}$

$\therefore \quad t = \frac{\pi}{2} = 90°$

Now, substitute $t = \frac{\pi}{2} = 90°$ in

$y = a\sin^3 t$

$y = a[\sin 90]^3 = a(1)^3 = a$

$\therefore$ for $t = \frac{\pi}{2}$ $x = 0$ and $y = a$ i.e $(0, a)$

$\therefore$ Curve does not pass through origin.

Rule 3 : Find $\frac{dy}{dx}$

We have $y = a\sin^3 t$

Differentiating w.r.t. 't'

$$\frac{dy}{dt} = a \cdot 3\sin^2 t \cdot \cos t \quad \text{...(1)}$$

Also $x = a\cos^3 t$

Differentiating w.r.t. t

$$\therefore \quad \frac{dx}{dt} = a \cdot 3\cos^2 t(-\sin t)$$

$$\text{i.e.} \quad \frac{dx}{dt} = -a\,3\cos^2 t \cdot \sin t \quad \text{...(2)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$\therefore$ From Equations (1) and (2)

$$\frac{dy}{dx} = \frac{a \cdot 3\sin^2 t \cdot \cos t}{-a\,3\cos^2 t \cdot \sin t} = -\tan t.$$

Rule 4 : Prepare a table of t, x, y, $\frac{dy}{dx}$

t	0	$\frac{\pi}{2} = 90°$	$\pi = 180°$	$\frac{3\pi}{2} = 270°$	$2\pi = 360°$
x	a	0	–a	0	a
y	0	a	0	–a	0
$\frac{dy}{dx}$	0	$-\infty$	0	∞	0

The sketch of the curve is as follows :

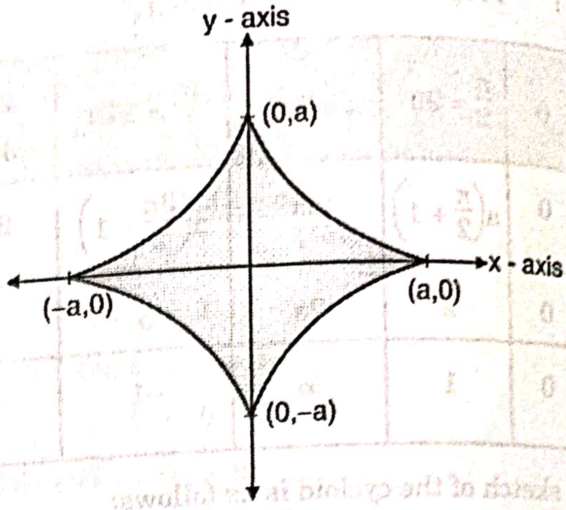

Fig. 5.36

Exercise 5.5

Q. Trace the following curves :

(1) $ay^2 = x^3$

Ans.:

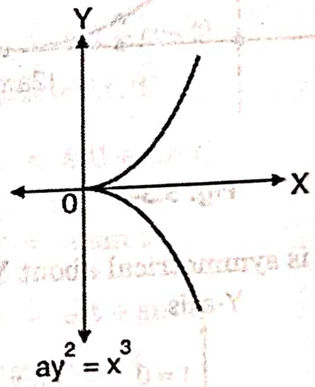

Fig. 68

(2) $x = a(t - \sin t), y = a(1 + \cos t)$

Ans.:

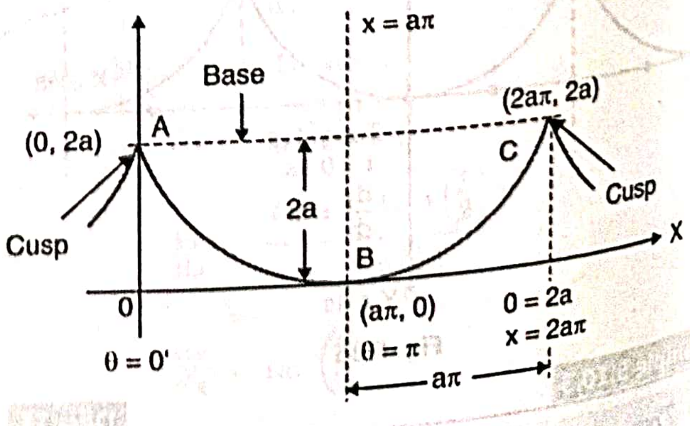

Fig. 69

Ans.:

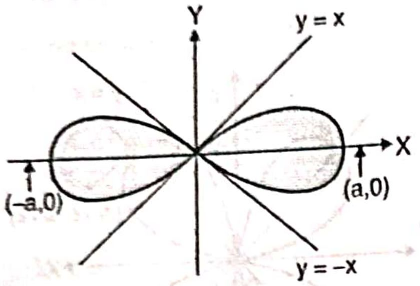

Fig. 38

5.5 Curves given by Polar Co-ordinates

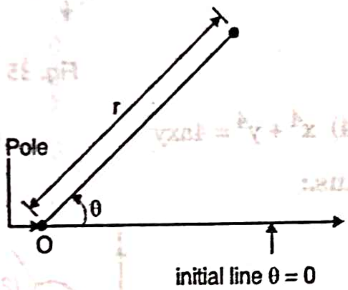

Fig. 5.15

To describe the position of a point in two dimensions, mainly two co-ordinate systems are used. One is Cartesian co-ordinates (x, y) and other is Polar Co-ordinates (r, θ). We are already familiar with Cartesian co-ordinate system. Now, In Polar co-ordinate system 'r' represents radius vector and θ represent angle measured positive in anticlockwise direction as shown in Fig. 5.15.

Guidelines for tracing a curve in polar co-ordinate are given below.

Note : In Polar Co-ordinates, the curve is often given by the equation $r = f(\theta)$.

Rule 1 : Symmetry

A) About Initial line (X-axis)

Replace θ by – θ, if the equation of the curve remains unchanged then curve is symmetrical about initial line (X-axis).

B) About the line $\theta = \frac{\pi}{2}$ (Y-axis)

(i) Replace θ by – θ and r by –r, if the equation of curve remains unchanged, then curve is symmetrical about the line through the pole, perpendicular to initial line (i.e. Y-axis).

(ii) The same symmetry also exists if the equation remains unchanged when θ is replaced by $\pi - \theta$.

Rule 2 : Pole (Origin)

The curve will pass through pole (origin) if for some value of θ, r becomes zero.

To check that, we put r = 0 and if we get a valid value of θ, the curve passes through pole.

Note : These values of θ are tangents at pole (origin)

Rule 3 : Table

θ	0	$90° = \frac{\pi}{2}$	$180° = \pi$	$270 = \frac{3\pi}{2}$
r				

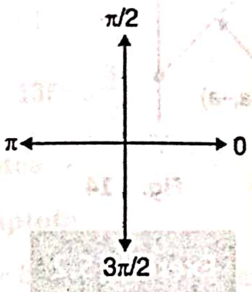

Fig. 5.16

Rule 4 : Angle between the radius vector and tangent [ϕ]: (Tangent at special point).

Use the formula $\tan\phi = r\frac{d\theta}{dr}$ and find ϕ.

Gurukey

Here ϕ is inclination of tangent at special point. For example for $r = a(1 + \cos\theta)$

when θ = 0 we get r = 2a

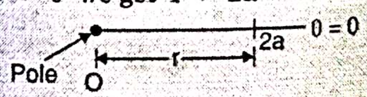

Fig. 5.17

and lets say we get $\phi = \frac{\pi}{2}$

Then $\phi = \frac{\pi}{2}$ is a line perpendicular to initial line at (r, θ) i.e (2a, 0).

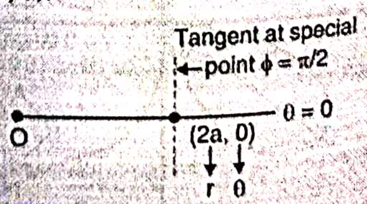

Fig. 5.18

Ans.:

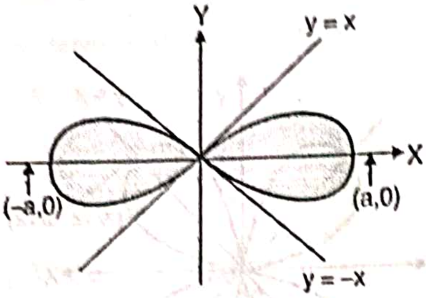

Fig. 38

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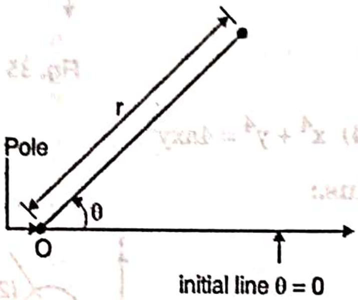

Fig. 5.15

Guidelines for tracing a curve in polar co-ordinate are given below.

> **Note :** In Polar Co-ordinates, the curve is often given by the equation $r = f(\theta)$.

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(ii) The same symmetry also exists if the equation remains unchanged when θ is replaced by π – θ.

Rule 2 : Pole (Origin)

The curve will pass through pole (origin) if for some value of θ, r becomes zero.

To check that, we put r = 0 and if we get a valid value of θ, the curve passes through pole.

> **Note :** These values of θ are tangents at pole (origin)

Rule 3 : Table

θ	0	$90° = \frac{\pi}{2}$	$180° = \pi$	$270 = \frac{3\pi}{2}$
r				

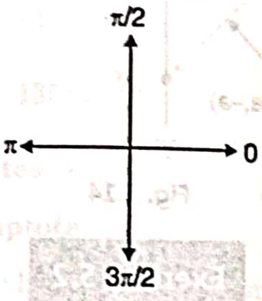

Fig. 5.16

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Use the formula $\tan\phi = r\frac{d\theta}{dr}$ and find ϕ.

Gurukey

Here ϕ is inclination of tangent at special point. For example for $r = a(1 + \cos\theta)$

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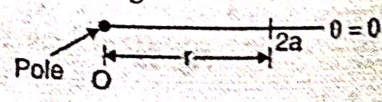

Fig. 5.17

and lets say we get $\phi = \frac{\pi}{2}$

Then $\phi = \frac{\pi}{2}$ is a line perpendicular to initial line at (r, θ) i.e (2a, 0).

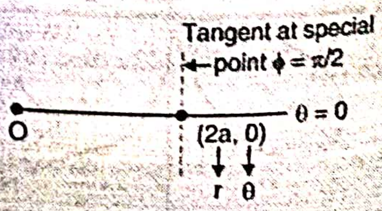

Fig. 5.18

5.6 Type 3: Solved Examples on Curves given by Polar Co-ordinates

Example 5.6.1 **Dec. 15, 12, May 17**

Trace the curve $r = a(1 + \cos\theta)$

Solution :

Rule 1: Symmetry :

A) About the initial line : (X-axis)

Given : $r = a(1 + \cos\theta)$

Replace θ by $-\theta$

$$r = a[1 + \cos(-\theta)]$$

but $\cos(-\theta) = \cos\theta$

$\therefore$ $r = a(1 + \cos\theta)$

$\therefore$ Equation remains unchanged.

$\therefore$ Curve is symmetrical about the initial line (X-axis).

B) About the line $\theta = \frac{\pi}{2}$ (Y-axis)

(i) **Given :** $r = a(1 + \cos\theta)$

Replace θ by $-\theta$ and r by $-r$

$\therefore$ $-r = a[1 + \cos(-\theta)]$

$-r = a(1 + \cos\theta)$.

$\therefore$ Equation changes.

$\therefore$ No symmetry about Y-axis.

(ii) **Given :** $r = a(1 + \cos\theta)$

Replace θ by $\pi - \theta$

$$r = a[1 + \cos(\pi - \theta)]$$

but $\cos(\pi - \theta) = -\cos\theta$

$\therefore$ $r = a(1 - \cos\theta)$

$\therefore$ Equation changes.

$\therefore$ No symmetry about Y-axis.

Rule 2 : Pole

Given: $r = a(1 + \cos\theta)$

put $r = 0$

$0 = a(1 + \cos\theta)$

$\therefore$ $1 + \cos\theta = 0$

$\cos\theta = -1$

$\theta = \cos^{-1}(-1)$

$\therefore$ $\theta = \pi, 3\pi, 5\pi \ldots$

but $\theta = 3\pi, 5\pi \ldots$ are values above 360°.

$\therefore$ we will neglect them

$\therefore$ $\theta = \pi$

$\therefore$ Curve passes through pole (origin).

As well as these values of θ represents tangents at pole (origin).

Rule 3 : Table

θ	0	90°	180°	270°
r	2a	a	0	a

(i) for $\theta = 0$

$r = a(1 + \cos\theta)$

$r = a(1 + \cos 0)$

$r = a(1 + 1)$

$r = 2a$

(ii) for $\theta = 90°$

$r = a(1 + \cos\theta)$

$r = a(1 + \cos 90)$

$r = a(1 + 0)$

$r = a$

(iii) for $\theta = 180°$

$r = a(1 + \cos\theta)$

$r = a(1 + \cos 180)$

$r = a(1 - 1)$

$r = 0$

(iv) for $\theta = 270°$

$r = a(1 + \cos\theta)$

$r = a(1 + \cos 270)$

$r = a(1 + 0)$

$r = a$

Rule 4 : Tangents at special points

Given : $r = a(1 + \cos\theta)$

Differentiating w.r.t. θ

$$\frac{dr}{d\theta} = a[-\sin\theta] = -a\sin\theta$$

On Reciprocal

$$\frac{d\theta}{dr} = \frac{1}{-a\sin\theta}$$

Multiplying by r on both sides,

$$r\frac{d\theta}{dr} = \frac{r}{-a\sin\theta}$$

but $\quad r = a(1+\cos\theta)$

$$r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\frac{1+\cos\theta}{\sin\theta}$$

1 + cos — = 2cos² — (Half Angle)

but

i.e $\boxed{1+\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)}$

sin — = 2sin — cos — (Half Angle)

and

$\therefore \quad \boxed{\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right)}$

$$\therefore \quad r\frac{d\theta}{dr} = \frac{-2\cos^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$\therefore \quad r\frac{d\theta}{dr} = -\frac{\cos\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta}{2}\right)} = \boxed{-\cot\left(\frac{\theta}{2}\right)}$$

but $\quad r\dfrac{d\theta}{dr} = \tan\phi$

$$\therefore \quad \tan\phi = -\cot\left(\frac{\theta}{2}\right) = \boxed{\tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)}$$

$$\therefore \quad \phi = \frac{\pi}{2}+\frac{\theta}{2}$$

$\therefore$ Table of r, θ and ϕ

θ	0°	90°	180°	270°
r	2a	a	0	a
ϕ	90°	135°	180°	225°

Explanation of r, θ and ϕ.

Note that, when $\theta = 0°$ we get $\phi = \frac{\pi}{2} \rightarrow$ i.e tangent at 90° with $\theta = 0$ at a distance $r = 2a$

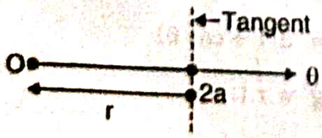

Fig. 5.19

$\therefore$ The sketch of the curve is as given below.

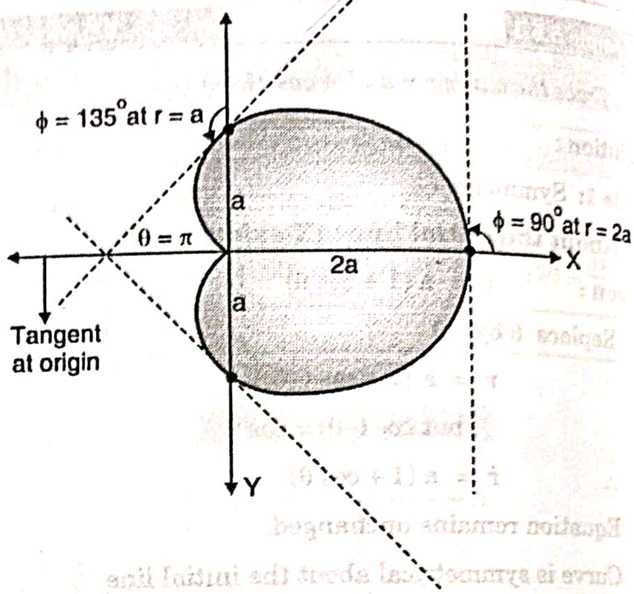

Fig. 5.20

Note : Use the symmetry to copy the curve from π to 2π.

Example 5.6.2 **Dec. 18**

Trace the curve $r = a(1+\sin\theta)$

Solution :

Rule 1 : Symmetry

A) About the initial line (X-axis)

Given : $\quad r = a(1+\sin\theta)$

Replace $\quad \theta$ by $-\theta$

$$r = a[1+\sin(-\theta)]$$

but $\sin(-\theta) = -\sin\theta$

$$\therefore \quad r = a(1-\sin\theta)$$

$\therefore$ Equation changes.

$\therefore$ No symmetry about initial line (X-axis).

B) About the line $\theta = \frac{\pi}{2}$ (Y-axis)

(i) Given : $\quad r = a(1+\sin\theta)$

Replace $\quad r$ by $-r$ and θ by $-\theta$

$$\therefore \quad -r = a[1+\sin(-\theta)]$$

$$-r = a(1-\sin\theta)$$

$\therefore$ Equation changes.

$\therefore$ No symmetry about Y-axis.

(ii) **Given :** $r = a(1+\sin\theta)$

Replace θ by $\pi-\theta$

$$r = a[1+\sin(\pi-\theta)]$$

but $\sin(\pi-\theta) = \sin\theta$

$\therefore$ $r = a(1+\sin\theta)$

$\therefore$ Equation remains unchanged.

$\therefore$ curve is symmetrical about Y-axis.

Rule 2 : Pole

Given : $r = a(1+\sin\theta)$

put $r = 0$

$$0 = a(1+\sin\theta)$$

$$\therefore\ 1+\sin\theta = 0$$

$$\sin\theta = -1$$

$$\therefore\ \theta = \sin^{-1}(-1)$$

$$\therefore\ \theta = 270^\circ = \frac{3\pi}{2}$$

$\therefore$ Curve passes through pole.

and $\theta = \frac{3\pi}{2}$ is tangent at origin

Rule 3 : Table

θ	0	90°	180°	270°
r	a	2a	a	0

(i) for $\theta = 0$

$$r = a(1+\sin\theta)$$
$$r = a(1+\sin 0)$$
$$r = a(1+0)$$
$$r = a$$

(ii) for $\theta = 90^\circ$

$$r = a(1+\sin\theta)$$
$$r = a(1+\sin 90)$$
$$r = a(1+1)$$
$$r = 2a$$

(iii) for $\theta = 180^\circ$

$$r = a(1+\sin\theta)$$
$$r = a(1+\sin 180)$$
$$r = a(1+0)$$

(iv) for

$$r = a$$
$$\theta = 270^\circ$$
$$r = a(1+\sin\theta)$$
$$r = a(1+\sin 270)$$
$$r = a(1-1)$$
$$r = 0$$

Rule 4 : Tangents at special points

Given : $r = a(1+\sin\theta)$

Differentiating w.r.t. θ $1 = a\cos\theta\,\frac{d\theta}{dr}$

$$\frac{dr}{d\theta} = a[0+\cos\theta] \qquad \frac{1}{a\cos\theta} = \frac{d\theta}{dr}$$

$$\frac{dr}{d\theta} = a\cos\theta$$

on Reciprocal

$$\frac{d\theta}{dr} = \frac{1}{a\cos\theta}$$

Multiplying by r on both sides,

$$r\frac{d\theta}{dr} = \frac{r}{a(1+\cos\theta)}$$

$$r\frac{d\theta}{dr} = \frac{a(1+\boxed{\sin\theta})}{a\cos\theta}$$

Note the adjustment in next step

$$\therefore\ r\frac{d\theta}{dr} = \frac{1+\boxed{\cos\left(\frac{\pi}{2}-\theta\right)}}{\sin\left(\frac{\pi}{2}-\theta\right)}$$

let $\frac{\pi}{2}-\theta = \alpha \rightarrow$ for simplification purpose.

$$\therefore\ r\frac{d\theta}{dr} = \frac{1+\cos\alpha}{\sin\alpha}$$

$$r\frac{d\theta}{dr} = \frac{2\cos^2\left(\frac{\alpha}{2}\right)}{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$r\frac{d\theta}{dr} = \cot\left(\frac{\alpha}{2}\right)$$

but $r\frac{d\theta}{dr} = \tan\phi$

$$\therefore\ \tan\phi = \boxed{\cot\left(\frac{\alpha}{2}\right)}$$

$$\tan\phi = \boxed{\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)}$$

$$\phi = \frac{\pi}{2}-\frac{\alpha}{2}$$

but $$\alpha = \frac{\pi}{2}-\theta$$

$\therefore$ $$\phi = \frac{\pi}{2}-\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)$$

$$\phi = \frac{\pi}{2}-\frac{\pi}{4}+\frac{\theta}{2}$$

$$\phi = \frac{\pi}{4}+\frac{\theta}{2}$$

ϕ gives angle (inclination) of tangent at special point

$$\phi = 45° + \frac{\theta}{2}$$

θ	0°	90°	180°	270°
r	a	2a	a	0
ϕ	45°	90°	135°	180°

The sketch of the curve as follows :

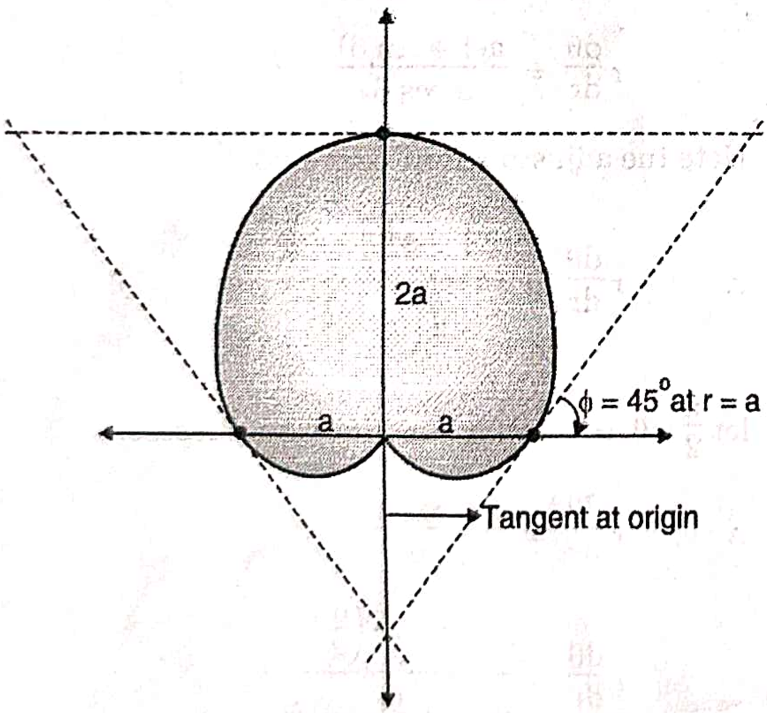

Fig. 5.21

Example 5.6.3