Unit 2: Application of Ordinary Differential Equation

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- Newton's law of cooling
- orthogonal Trajectories
- Electrical circuits
- 1-Dimensional Head conduction problems.

Newton's law of cooling:

This type of problem involves having a warm object in cooler surroundings such as a hot glass of water kept in freezer. This will give temperature difference between hot glass of water & freezer, resulting in energy flow into a system or energy flow from system to surrounding. Note that energy flow into the system leads to heating whereas energy flow from the system leads to cooling of object.

Statement:

Newton's law of cooling states that, the rate of temperature of the body is proportional to difference between temperature of the body & that of surrounding medium

i.e $$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\therefore \quad \frac{d\theta}{dt} = -k(\theta - \theta_0)$$

where, θ_0 = surrounding temperature

k = constant of proportionality

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Que.1 A body originally at 80°c cools to 60°c in 20 min the temperature of air being 40°c, what will be the temperature of body after 40 min from the original?

Soln: Given data,

Step 1: $\theta_1 = 80^\circ$ $t_1 = 0$ min

$\theta_2 = 60^\circ$ $t_2 = 20$ min

$\theta_3 = ?$ $t_3 = 40$ min

$\theta_0 = 40^\circ$

Step 2:

By Newton's law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\frac{d\theta}{\theta - \theta_0} = -k\,dt$$

on integration,

$$\int \frac{d\theta}{\theta - \theta_0} = \int -k\,dt$$

$$\int \frac{d\theta}{\theta - \theta_0} = -k\int dt$$

$$\log(\theta - \theta_0) = -kt + c \quad \text{--- (1)}$$

Step 3:

To find 'c'

when $t_1 = 0$ $\theta_1 = 80^\circ c$ $\theta_0 = 40^\circ$

$\therefore$ eqn (1) becomes

$$\log(80 - 40) = 0 + c$$

$\boxed{\log 40 = c}$, eqn (1) becomes

$$\log(\theta - \theta_0) = -kt + \log 40 \quad \text{--- (2)}$$

Step 4: To find '$-k$'

when $t_2 = 20$ min, $\theta_2 = 60^\circ$ $\theta_0 = 40^\circ$

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$\therefore$ Equation (2) becomes

$$\log(60-40) = -k(20) + \log 40$$

$$\log 20 = -k(20) + \log 40$$

$$\log 20 - \log 40 = -k(20)$$

$$\log\left(\frac{20}{40}\right) = -k(20)$$

$$\therefore \boxed{-k = \frac{1}{20}\log\left(\frac{1}{2}\right)}$$

put this value of $-k$ in eq^n (2),

$$\log(\theta-\theta_0) = \frac{1}{20}\log\left(\frac{1}{2}\right)t + \log 40 \quad \text{--- (3)}$$

Step 5: To find find Ans $\theta = ?$

when $t = 40$, $\theta = ?$ $\theta_0 = 40°C$

$\therefore$ eq^n (3) becomes,

$$\log(\theta-40) = \frac{1}{20}\log\left(\frac{1}{2}\right)^{40} + \log 40$$

$$= \frac{1}{2}\log\left(\frac{1}{2}\right) + \log 40$$

$$\log(\theta-40) = \log\left(\frac{1}{4}\right) + \log 40$$

$$= \log\left(\frac{1}{4}\times 40\right)$$

$$\log(\theta-40) = \log 10$$

$$\theta - 40 = 10$$

$$\boxed{\theta = 50°C}$$

Que.2 A body temperature is 100°c is placed in a room whose temperature is 20°c & cools to 60°c in 5 min. What will be its temperature after 10 min?

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soln: step 1: Given data,

$\theta_1 = 100°C$ $t_1 = 0$ min

$\theta_2 = 60°C$ $t_2 = 5$ min

$\theta = ?$ $t_3 = 10$ min

$\theta_0 = 20°C$

step 2: By Newton's law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\frac{d\theta}{\theta - \theta_0} = -k\,dt$$

on integrating,

$$\int \frac{d\theta}{\theta - \theta_0} = \int -k\,dt$$

$$\int \frac{d\theta}{\theta - \theta_0} = -k \int dt$$

$$\log(\theta - \theta_0) = -kt + c \quad \text{—— (1)}$$

step 3: To find 'c',

when $t_1 = 0$, $\theta_1 = 100°C$ $\theta_0 = 20°C$

$\therefore$ eqn (1) becomes,

$$\log(100 - 20) = -k(0) + c$$

$$\boxed{\log 80 = c}$$

$\therefore$ eqn (1) becomes,

$$\log(\theta - \theta_0) = -kt + \log 80 \quad \text{—— (2)}$$

step 4: To find '-k'

when, $t_2 = 5$ $\theta_2 = 60°C$ $\theta_0 = 20°C$

put all in eqn (2)

$$\log(60 - 20) = -k(5) + \log 80$$

$$\log 40 = -k(5) + \log 80$$

$$\log 40 - \log 80 = -k(5)$$

$$\log\left(\frac{40}{80}\right) = -k(5)$$

$$\log\left(\frac{1}{2}\right) = -k(5)$$

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$$\therefore \boxed{\frac{1}{5}\log\left(\frac{1}{2}\right) = -k}$$

$\therefore$ eqn (2) becomes,

$$\log(\theta-\theta_0) = \frac{1}{5}\log\left(\frac{1}{2}\right)t + \log 80 \quad \text{--- (3)}$$

Step 5:

To find final Ans θ

when $t_2 = 10$ min, $\theta = ?$ $\theta_0 = 20°C$

put in eqn (3),

$$\log(\theta-20) = \frac{1}{5}\log\left(\frac{1}{2}\right)(10) + \log 80$$

$$\log(\theta-20) = 2\log\left(\frac{1}{2}\right) + \log 80$$

$$= \log\left(\frac{1}{4}\right) + \log 80)$$

$$= \log\left(\frac{1}{4}\times 80\right)$$

$$\log(\theta-20) = \log 20$$

$$\theta - 20 = 20$$

$$\boxed{\theta = 40°C}$$

Practice Example:

(1) A body at temperature 100°C is placed in a room whose temperature is 20°C & cools to 60°C in 5 Min. Find temperature after a further 15 Min.

Ans: $\theta = 30°C$

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Orthogonal Trajectories

What is Trajectory :

A curve which cuts every member of a given family of curves according to some specified rule is called as trajectory of that family.

What is orthogonal Trajectory :

Two families of curve are said to be orthogonal trajectories of each other, if each curve in either family is orthogonal i.e perpendicular to every curve in other family.

Working Rule to find orthogonal Trajectories

Step 1 : Differentiate the given function w.r.t 'x' & find $\frac{dy}{dx}$

Step 2 : Eliminate constant if any by using given equation

Step 3 : Arrange D.E as $\frac{dy}{dx}$

Step 4 : Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

Step 5 : Solve D.E.

Solved Example

Que.1 Find the orthogonal trajectories of the family of straight lines $y = mx$

Soln: Given $y = mx$ —— (1)

diff w.r.t 'x'

$$\frac{dy}{dx} = m$$

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$\therefore$ From eqn (1)

$$m = \frac{y}{x}$$

$\therefore$ Equating values of 'm',

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{——— (2)}$$

$\therefore$ Replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ in (2),

$$\frac{-dx}{dy} = \frac{y}{x}$$

$$-x\,dx = y\,dy$$

on integrating

$$-\int x\,dx = \int y\,dy$$

$$-\frac{x^2}{2} = \frac{y^2}{2} + c$$

$$-x^2 = y^2 + 2c$$

$$-x^2 - y^2 = c_1$$

$$\boxed{x^2 + y^2 = c}$$

Que.2 Find the orthogonal trajectories of the family of $xy = c$

Soln: Given $xy = c$ ——— (1)

diff w.r.t 'x'

$$x\frac{dy}{dx} + y = 0$$

$$\therefore x\frac{dy}{dx} = -y$$

$$\therefore \frac{dy}{dx} = \frac{-y}{x} \quad \text{——— (2)}$$

Replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$

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$$-\frac{dx}{dy} = \frac{-y}{x}$$

$$x\,dx = y\,dy$$

on integrating,

$$\int x\,dx = \int y\,dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + c$$

$$x^2 = y^2 + 2c$$

$$\boxed{x^2 - y^2 = c}$$

Que.3 Find the orthogonal trajectory of family of $y^2 = 4ax$

Soln: Given, $y^2 = 4ax$ —— ①

diff w.r.t. 'x'

$$2y \cdot \frac{dy}{dx} = 4a$$

$\therefore$ From ①, $4a = \frac{y^2}{x}$

equating values of 4a,

$$2y\frac{dy}{dx} = \frac{y^2}{x}$$

$$\frac{dy}{dx} = \frac{y^2}{2xy}$$

$$\frac{dy}{y} = \frac{dx}{2x}$$

on integrating,

$$\int \frac{dy}{y} = \int \frac{dx}{2x}$$

$$\log y = \frac{1}{2} \log x$$

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Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{y}{2x}$$

$$-2x\,dx = y\,dy$$

on integrating,

$$-2\int x\,dx = \int y\,dy$$

$$-2 \cdot \frac{x^2}{2} = \frac{y^2}{2} + c$$

$$-x^2 = \frac{y^2}{2} + c$$

$$-2x^2 = y^2 + 2c$$

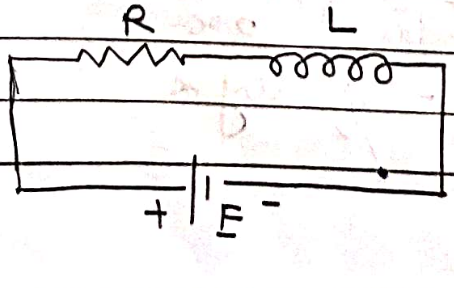
$$\boxed{2x^2 + y^2 = c}$$

Simple Electric Circuits

An electric circuit is a path or line through which an electric current flows. A simple electric circuit consists of a source (voltage or current) & passive elements. These 3 passive elements known as Resistance, Indutance & capacitence.

Differential Equations of Electrical Circuit :

① R - L circuit : consider a circuit containing resistance R & inductance L in series with voltage source E



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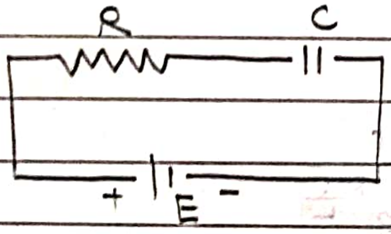
Differential equation for R-L circuit is

$$\boxed{\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}}$$

② R-C circuit: consider a circuit containing Resistance R & capacitance C in series with voltage source E



Differential Equation for R-C circuit is

$$\boxed{\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}}$$

Solved Example:

Que.1 A resistance of 250 Ω & an inductance of 640 H are connected in series with a battery of 500 V Find the current in the circuit if $i=0$ at $t=0$

Soln: $R = 250\,\Omega$, $L = 640\,H$, $E = 500\,V$ $i = ?$

The differential equation for R-L circuit is given by,

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

$$\therefore P = \frac{R}{L} \qquad Q = \frac{E}{L}$$

$$\therefore I.F = e^{\int P\,dt} = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$$

∴ General solution is given by

$$i(I.F) = \int Q \cdot (I.F)\,dt + C$$

$$i \cdot e^{\frac{Rt}{L}} = \int \frac{E}{L} e^{\frac{Rt}{L}} dt + c$$

$$i \cdot e^{\frac{Rt}{L}} = \frac{E}{L} \int e^{\frac{Rt}{L}} dt + c$$

$$= \frac{E}{L} \cdot \frac{e^{\frac{Rt}{L}}}{\frac{R}{L}} + c$$

$$i \cdot e^{\frac{Rt}{L}} = \frac{E}{R} \cdot e^{\frac{Rt}{L}} + c$$

$$i = \frac{E}{R} \cdot \frac{e^{\frac{Rt}{L}}}{e^{\frac{Rt}{L}}} + \frac{c}{e^{\frac{Rt}{L}}}$$

$$i = \frac{E}{R} + c \cdot e^{-\frac{Rt}{L}} \quad \text{——— (1)}$$

Now at $i=0$ at $t=0$

$$0 = \frac{E}{R} + c \cdot e^{-0}$$

$\boxed{c = -\frac{E}{R}}$ put in (1), $\because e^0 = 1$

$$i = \frac{E}{R} + \left(-\frac{E}{R}\right) e^{-\frac{Rt}{L}}$$

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

at $R = 250\,\Omega$, $L = 640H$, $E = 500V$

$$\therefore i = \frac{500}{250}\left(1 - e^{-\frac{250t}{640}}\right)$$

$$i = 2\left(1 - e^{-\frac{25}{64}t}\right)$$

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Que. A resistance of 150 Ω & inductance of 0.3 H are connected in series with a battery of 25 volts. Find the current in the circuit if $i=0$ at $t=0$.

Ans: $i = \frac{1}{6}(1 - e^{-500t})$

Que: A circuit consist of resistance R Ω & consider of C Farads connected e.m.f E volts. If $\frac{q}{c}$ is the voltage of condenser at time 't' after closing the circuit, show that

$$\frac{q}{c} = E(1 - e^{-t/RC})$$

Soln: Differential equation for R-C circuit is given by,

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$$

$$\therefore P = \frac{1}{RC} \qquad Q = \frac{E}{R}$$

$$\therefore I.F = e^{\int P dt} = e^{\int \frac{1}{RC} dt} = e^{\frac{1}{RC}\int dt}$$

$$I.F = e^{t/RC}$$

$\therefore$ General solution is given by,

$$q(I.F) = \int Q(I.F)\, dt + A$$

$$q.e^{t/RC} = \int \frac{E}{R} . e^{t/RC} . dt + A$$

$$= \frac{E}{R}\int e^{t/RC} dt + A$$

$$= \frac{E}{R} . \frac{e^{t/RC}}{\frac{1}{RC}} + A$$

$$q.e^{t/RC} = \frac{EC}{\cdot} . e^{t/RC} + A$$

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$$q = EC + \frac{A}{e^{t/RC}}$$

$$q = EC + A \cdot e^{-t/RC} \quad \text{——— (1)}$$

at $t = 0$ & $q = 0$ eqⁿ (1) becomes,

$$0 = EC + A(1)$$

$\boxed{A = -EC}$ put in (1),

$$q = EC - EC \cdot e^{-t/RC}$$

$$q = EC\,(1 - e^{-t/RC})$$

$$\frac{q}{c} = E\,(1 - e^{-t/RC})$$

Hence proved.

Heat Conduction Problems

Whenever there is a temperature difference there is heat transfer.

Fourier's Law of Heat conduction:

Fourier's law of heat conduction states that, the rate of heat conduction per unit area is directly proportional to temperature gradient

$$\therefore \boxed{q = -KA\frac{dT}{dr}}$$

q = rate of heat transfer

K = proportionality constant

A = Area of heat transfer ($2\pi r$)

$\frac{dT}{dr}$ = Temperature gradient

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Solved Example:

Que.1 A steam pipe 40 cm in diameter contains steam at 150°c & is protected with a covering 10 cm thick for which k = 0.0012. If the temperature of outer surface of covering is 30°c, find the temperature at a distance 25 cm from the centre of pipe under steady-state conditions.

Sol^n: Given,

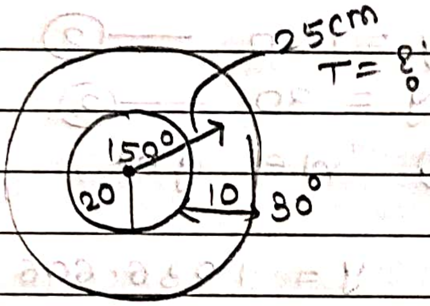
diameter (d) = 40 cm $T = 150°c$

$r = 20$ cm $T = 30°c$

$r = (20 + 10) = 30$ cm

$K = 0.0012$

$r = 25$ $T = ?$



∴ By Fourier's law of heat conduction,

$$q = -KA \frac{dT}{dr}$$

$A = 2\pi r$

$$\therefore \quad q = -K(2\pi r) \frac{dT}{dr}$$

$$\frac{q}{-K 2\pi r} dr = dT$$

on integration,

$$\frac{-q}{2\pi K} \int \frac{1}{r} dr = \int dT$$

$$\therefore \quad \frac{-q}{2\pi K} \log r + C = T \qquad \text{--- (1)}$$

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To find T, we will first find values of $\frac{-q}{2\pi K}$ & c as follows:

Now,

when $r = 20\ cm$ $T = 150°C$

eqⁿ (1) becomes,

$$\frac{-q}{2\pi K} \log(20) + c = 150° \quad \text{—— (2)}$$

when $r = 30cm$ $T = 30°C$

eqⁿ (1) becomes,

$$\frac{-q}{2\pi K} \log(30) + c = 30 \quad \text{—— (3)}$$

put $\frac{-q}{2\pi K} = x$ $c = y$ we get,

$$x \log 20 + y = 150 \quad \text{—— (2)}$$

$$x \log 30 + y = 30 \quad \text{—— (3)}$$

solving (2) & (3) we get,

$x = -295.956$ $y = 1036.606$

i.e $\frac{-q}{2\pi K} = -295.956$ $c = 1036.606$

put above values in (1),

$$-295.956 \log(r) + 1036.606 = T$$

when $r = 25$ $T = ?$

$$\therefore -295.956 \log(25) + 1036.606 = T$$

$$\boxed{T = 83.96°C}$$

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Que.2 A pipe 20 cm in diameter contain steam at 150°c & protected with covering 5 cm thick for which K = 0.0025. If the temperature of the outer surface of the covering is 40°c find temperature at a distance of 12.5 cm from the centre of pipe under steady - conditions.

Soln: Given,

$d = 20\,cm$

$r = 10\,cm$ $\quad T = 150°c$

$r = 10+5$

$= 15\,cm$ $\quad T = 40°C$

$r = 12.5\,cm$ $\quad T = ?$

$K = 0.0025$

∴ By Fourier's law of heat conduction,

$$q = -KA \frac{dT}{dr}$$

$A = 2\pi r$

$$q = -k(2\pi r) \cdot \frac{dT}{dr}$$

$$\frac{q}{-k\,2\pi r}\,dr = dT$$

on integration,

$$\frac{-q}{k 2\pi} \int \frac{1}{r}\,dr = \int dT$$

$$\frac{-q}{k 2\pi} \ln(r) + c = T \quad \text{—— (1)}$$

To find T, we will first find values of $\frac{-q}{2\pi k}$ & c

∴

when $r = 10\,cm$ $\quad T = 150°c$

eqn (1) becomes,

$$\frac{-q}{2\pi k} \ln(10) + c = 150 \quad \text{—— (2)}$$

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When $r = 15\,cm$ $T = 40^\circ C$

eqn (1) becomes,

$$\frac{-q}{2\pi K} \ln(15) + c = 40^\circ \quad \text{——— (3)}$$

put $\frac{-q}{2\pi k} = x$ & $c = y$ put in (2) & (3),

we get,

$$\ln(10) \cdot x + y = 150 \quad \text{——— (2)}$$

$$\ln(15) \cdot x + y = 40 \quad \text{——— (3)}$$

solving (2) & (3) we get,

$x = -271.293$ $y = 774.676$

$\therefore \frac{-q}{2\pi K} = -271.293$ $c = 774.676$

put value in (1) we get,

$$-271.293 \ln(r) + 774.676 = T$$

when $r = 12.5$

$$-271.293 \ln(12.5) + 774.676 = T$$

$$\boxed{T = 89.46^\circ C}$$

Practice Example:

Que: A steam pipe 20cm in diameter contain steam at $150^\circ C$ & is protected with covering 10cm thick for which $K = 0.0012$. If temperature of the outer covering is $30^\circ C$ find temperature at a distance of 15 cm from the centre.