Unit 1: First order Ordinary Differential Equation

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Subpoints:

- Introduction to first order D.E
- Exact D.E
- Linear D.E

First order differential Equation:

A first order D.E is a type of D.E that involves derivatives of first order of a function. It does not involve higher order derivatives.

It can generally be expressed in the form

$$\boxed{\frac{dy}{dx} = f(x, y)}$$

Example of First-order D.E:

① $\frac{dy}{dx} = 2x$

② $\frac{dy}{dx} = x + 11$

③ $\frac{dy}{dx} = 4x - 5y$

④ $\frac{dy}{dx} = \sin x + 2\cos y$

Types of First-order D.E:

① Exact Differential Equation

② Linear Differential Equation.

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① Exact Differential Equation:

A differential equation is said to be exact if it is of the form $Mdx + Ndy = 0$ & if there exists a function $u(x, y)$ such that $Mdx + Ndy = du$ then differential equation is called as an exact D.E.

The necessary & sufficient condition that $Mdx + Ndy = 0$ be exact is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

When condition of exactness is satisfied then general solution is given by

$$\underbrace{\int M\,dx}_{\text{Treat } y\text{-constant}} + \underbrace{\int N\,dy}_{\text{Terms free from } 'x'} = c$$

Solved Examples:

Que.1 Solve $(x+y)dx + (x-y)dy = 0$

Soln: The given D.E is

$$(x+y)dx + (x-y)dy = 0$$

comparing with $Mdx + Ndy = 0$ we get,

$M = x+y$ $N = x-y$

diff M w.r.t 'y',

$$\therefore \frac{\partial M}{\partial y} = 0+1 = 1$$

diff N w.r.t 'x',

$$\therefore \frac{\partial N}{\partial x} = 1-0 = 1$$

Thus we get $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ The given D.E is exact.

It's solution is obtained from,

$$\underbrace{\int M\,dx}_{\text{Treat } y\text{-constant}} + \underbrace{\int N\,dy}_{\text{Terms free from } 'x'} = c$$

$$\int (x+y)\,dx + \int (x-y)\,dy = c$$

$$\int x\,dx + \int y\,dx + \int x\,dy - \int y\,dy = c$$

$$\frac{x^2}{2} + y(x) + 0 - \frac{y^2}{2} = c$$

$\frac{x^2}{2} + xy - \frac{y^2}{2} = c$ is the required sol^n

Que.2 Solve $(3x^2 - y)\,dx - x\,dy = 0$

Sol^n: Given D.E is $(3x^2 - y)\,dx - x\,dy = 0$

comparing with $M\,dx + N\,dy = 0$ we get,

$M = 3x^2 - y \qquad N = -x$

diff. M w.r.t 'y'

$$\frac{\partial M}{\partial y} = 0 - 1 = -1$$

diff w.r.t 'x'

$$\frac{\partial N}{\partial x} = -1$$

Thus we get $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ The given D.E is exact.

Its solution is obtained from,

$$\underbrace{\int M\,dx}_{\text{Treat } y\text{-const}} + \underbrace{\int N\,dy}_{\text{Terms from from } 'x'} = c$$

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$\int (3x^2 - y)\,dx + \int -x\,dy = c$

$\int 3x^2 dx - \int y\,dx - \int x\,dy = c$

$\frac{3x^3}{3} - y(x) - 0 = c$

$x^3 - xy = c$ is the required solution

Que.3 solve $(x^2+y)\,dx + (y^3+x)\,dy = 0$

Soln: Given D.E is,

$(x^2+y)\,dx + (y^3+x)\,dy = 0$

comparing with $M\,dx + N\,dy = 0$ we get,

$M = x^2+y$ $\quad N = y^3+x$

diff M w.r.t 'y',

$\frac{\partial M}{\partial y} = 0+1 = 1$

diff N w.r.t 'x',

$\frac{\partial N}{\partial x} = 0+1 = 1$

Thus we get $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ The given D.E is exact

Its solution is given by,

$\underset{\text{treat } y\text{-const}}{\int M\,dx} + \underset{\text{terms free from } 'x'}{\int N\,dy} = c$

$\int x^2+y\,dx + \int y^3+x\,dy = c$

$\int x^2 dx + \int y\,dx + \int y^3 dy + \int x\,dy = c$

$\frac{x^3}{3} + y(x) + \frac{y^4}{4} + 0 = c$

∴ $\frac{x^3}{3} + xy + \frac{y^4}{4} = c$ is the required soln.

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Que.4 solve: $(3x^3-y)\,dx+(y-x)\,dy=0$

Soln: The given D.E $(3x^3-y)\,dx+(y-x)\,dy=0$

comparing with $M\,dx+N\,dy=0$

$M=3x^3-y$ $\quad N=y-x$

diff M w.r.t. 'y',

$$\frac{\partial M}{\partial y}=0-1=-1$$

diff N w.r.t 'x',

$$\frac{\partial N}{\partial x}=0-1=-1$$

Thus we get $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

Thus given D.E is exact.

Its solution is given by,

$$\underbrace{\int M\,dx}_{\text{Treat y-const}}+\underbrace{\int N\,dy}_{\text{Terms free from 'x'}}=C$$

$$\int(3x^3-y)\,dx+\int(y-x)\,dy=C$$

$$\int 3x^3\,dx-\int y\,dx+\int y\,dy-\int x\,dy=C$$

$$\frac{3x^4}{4}-y(x)+\frac{y^2}{2}-0=C$$

$\dfrac{3x^4}{4}-xy+\dfrac{y^2}{2}=C$ is the required soln.

Practice Example

1. solve $(x^3+y)\,dx+(y^2+x)\,dy=0$
2. solve $(4x^3-y)\,dx+(y^2-x)\,dy=0$
3. solve $(5x^2+y)\,dx+(y^3+x)\,dy=0$
4. solve $(5x^4-y)\,dx-x\,dy=0$
5. solve $(2xy+y^3)\,dx+(x^2+3xy^2)\,dy=0$

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Non-exact D.E Reducible to exact form:

Definition of integrating factor:

An integrating factor is the function of x & y which when multiplied to given non-exact DE will make it exact.

In simple words the purpose of I.F is to convert non-exact D.E to an exact DE

$$\text{Non exact D.E} \xrightarrow[\text{I.F}]{\text{Multiply by}} \text{Exact D.E}$$

Rules for finding intergrating factor:

Rule 1: The D.E is homogeneous & $Mx+Ny \neq 0$ then

$$I.F = \frac{1}{Mx+Ny}$$

Rule 2: The D.E is of form,

$y f_1(x,y)\,dx + x f_2(x,y)\,dy = 0$ & $Mx - Ny \neq 0$ then,

$$I.F = \frac{1}{Mx - Ny}$$

Rule 3: Rule 3 is generally applicable if we have $x\,dy$ & if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$

then, $I.F = e^{\int f(x)\,dx}$

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Rule 4: Rule 4 is applicable if we have $y\,dx$

$\therefore$ If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$ then,

$$I.F = e^{\int f(y)\,dy}$$

solved example on Rule I

Que.1 solve $x^2y\,dx - (x^3 + y^3)\,dy = 0$

Soln: Given D.E is $x^2y\,dx - (x^3 + y^3)\,dy = 0$ —(1)

comparing with $Mdx + Ndy = 0$ we get,

$M = x^2y$ $\quad N = -x^3 - y^3$

diff M w.r.t 'y'.

$$\frac{\partial M}{\partial y} = x^2$$

diff N w.r.t 'x'

$$\frac{\partial N}{\partial x} = -3x^2$$

so here $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ so it is not exact D.E

To make it exact will find integrating factor

$$I.F = \frac{1}{Mx + Ny} = \frac{1}{(x^2y)(x) + (-x^3 - y^3)y}$$

$$= \frac{1}{x^3y - x^3y - y^4}$$

$$I.F = \frac{-1}{y^4}$$

Now, Multiplying the equation (1) by I.F

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$$\frac{-1}{y^4}(x^2y)dx - \left(\frac{-1}{y^4}\right)(x^3+y^3)dy = 0$$

$$\left(\frac{-x^2}{y^3}\right)dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

so Now, this is exact D.E.

∴ General solution is given by,

$$\int_{y\text{-cont}} M\,dx + \int_{\substack{\text{Terms}\\ \text{free from } x}} N\,dy = c$$

$$\int \frac{-x^2}{y^3}\,dx + \int \frac{x^3}{y^4} + \frac{1}{y}\,dy = c$$

$$\frac{-1}{y^3}\int x^2\,dx + \int 0 + \frac{1}{y}\,dy = c$$

$$\frac{-1}{y^3}\cdot\frac{x^3}{3} + \log y = c$$

$\frac{-x^3}{3y^3} + \log y = c$ is the required so

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Que.3 solve $(xy-2y^2)\,dx-(x^2-3xy)\,dy=0$

Soln: Given D.E is $(xy-2y^2)\,dx-(x^2-3xy)\,dy$ ①

comparing with $M\,dx+N\,dy=0$ we get,

$M=xy-2y^2$ $\quad N=-x^2+3xy$

diff M w.r.t y, diff N w.r.t x

$\frac{\partial M}{\partial y}=x-4y$ $\quad \frac{\partial N}{\partial x}=-2x+3y$

Here $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ so it is not exact D.E

so will make it exact by I.F,

$$I.F=\frac{1}{Mx+Ny}=\frac{1}{x(xy-2y^2)+y(-x^2+3xy)}$$

$$=\frac{1}{\cancel{x^2y}-2xy^2-\cancel{x^2y}+3xy^2}$$

$$I.F=\frac{1}{xy^2}$$

Now Multiplying eqn ① by I.F we get,

$$\frac{1}{xy^2}(xy-2y^2)\,dx-\frac{1}{xy^2}(x^2-3xy)\,dy=0$$

$$\left(\frac{xy}{xy^2}-\frac{2y^2}{xy^2}\right)dx-\left(\frac{x^2}{xy^2}-\frac{3xy}{xy^2}\right)dy=0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$$

Now It is exact D.E

∴ General soln is,

$$\int M\,dx+\int N\,dy=c$$

$$\int\frac{1}{y}-\frac{2}{x}\,dx+\int-\frac{x}{y^2}+\frac{3}{y}\,dy=c$$

$$\int \frac{1}{y}\,dx - \int \frac{2}{x}\,dx + 0 + \int \frac{3}{y}\,dy = c$$

$$\frac{1}{y}(x) - 2\log x + 3\log y = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$ is the required soln.

Practice example:

① $(y^2 - xy)\,dx + x^2\,dy = 0$ $I.F = \frac{1}{xy^2}$

Ans: $\log x - \frac{x}{y} = c$

② $-y^2\,dx + (xy + x^2)\,dy = 0$ $I.F = \frac{1}{x^2 y}$

Ans: $\frac{y}{x} + \log y = c$

solved example on Rule 2:

ue.1 solve $y(1-xy)\,dx - x(\cancel{x}1+xy)\,dy = 0$

Soln: Given D.E is, $y(1-xy)\,dx - x(1+xy)\,dy = 0$ —— (1)

comparing with $Mdx + Ndy = 0$ we get,

$M = y(1-xy)$
$\quad = y - xy^2$

$N = -x(1+xy)$
$\quad = -x - x^2y$

diff M w.r.t y

$\frac{\partial M}{\partial y} = 1 - 2xy$

diff N w.r.t x,

$\frac{\partial N}{\partial x} = -1 - 2xy$

so, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so it is not exact

will make it exact by, $I.F = \frac{1}{Mx - Ny}$

$$I.F = \frac{1}{(y-xy^2)(x) - (-x-x^2y)(y)}$$

$$I.F = \frac{1}{xy - x^2y^2 + xy + x^2y^2} = \frac{1}{2xy}$$

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Multiplying eqn (1) by I.F we get,

$$\frac{1}{2xy}\, y(1-xy)\,dx - \frac{1}{2xy}\, x(1+xy)\,dy = 0$$

$$\left(\frac{1}{2x} - \frac{y}{2}\right)dx - \left(\frac{1}{2y} + \frac{x}{2}\right)dy = 0$$

Now it is exact D.E

$\therefore$ General soln is given by,

$$\int M\,dx + \int N\,dy = c$$

$$\int\left(\frac{1}{2x} - \frac{y}{2}\right)dx - \int\left(\frac{1}{2y} + \frac{x}{2}\right)dy = c$$

$$\int \frac{1}{2x}\,dx - \int \frac{y}{2}\,dx - \int \frac{1}{2y}\,dy - 0 = c$$

$\frac{1}{2}\log x - \frac{y}{2}(x) - \frac{1}{2}\log y = c$ is required solution.

Practice Example:

1. $y(x+y)\,dx + x(x-y)\,dy = 0$ Ans: $\frac{1}{2}\left[\frac{x}{y} + \log x - \log y\right] = c$
2. $y(x+1)\,dx + x(x)\,dy = 0$ Ans: $x + \log x = c$
3. $y(y)\,dx + x(1+y)\,dy = 0$ Ans: $-\left[y\log x + \log y + y\right] = c$

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Problems based on Rule 3:

Que. 1 Solve $(3y^2+4x)\,dx + 3xy\,dy = 0$

Soln: Given D.E is $(3y^2+4x)\,dx + 3xy\,dy = 0$ — (1)

comparing with $M\,dx + N\,dy = 0$ we get,

$M = 3y^2+4x$ | $N = 3xy$

diff M w.r.t y | diff N w.r.t x

$\frac{\partial M}{\partial y} = 6y$ | $\frac{\partial N}{\partial x} = 3y$

Here $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so it is not exact.

so will make it exact by,

$$f(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{6y-3y}{3xy} = \frac{3y}{3xy} = \frac{1}{x}$$

$$I.F = e^{\int f(x)\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

$\therefore I.F = x$

Now, Multiplying eqn (1) by I.F we get,

$$x(3y^2+4x)\,dx + 3x^2y\,dy = 0$$

Now it is exact D.E

$\therefore$ General solution is given by,

$$\int M\,dx + \int N\,dy = c$$

$$\int 3xy^2 + 4x^2\,dx + \int 3x^2y\,dy = c$$

$$\int 3xy^2\,dx + \int 4x^2\,dx + \int 3x^2y\,dy = c$$

$$3y^2\int x\,dx + 4\int x^2\,dx + 0 = c$$

$$3y^2\frac{x^2}{2} + \frac{4x^3}{3} = c$$ is required soln.

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Que.2 solve $(3xy - y^2)dx + (x^2 - xy)dy = 0$

soln: Given D.E is, $(3xy - y^2)dx + (x^2 - xy)dy = 0$ — ①

comparing with $Mdx + Ndy = 0$ we get,

$M = 3xy - y^2$ $\qquad$ $N = x^2 - xy$

diff M w.r.t y $\qquad$ diff N w.r.t x

$\frac{\partial M}{\partial y} = 3x - 2y$ $\qquad$ $\frac{\partial N}{\partial x} = 2x - y$

Here $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so it is not exact.

Now we will make it exact by,

$$f(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3x - 2y - 2x + y}{x^2 - xy}$$

$$f(x) = \frac{x - y}{x^2 - xy} = \frac{x - y}{x(x - y)} = \frac{1}{x}$$

$$\therefore I.F = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx}$$

$$= e^{\log x}$$

$$I.F = x$$

Multiplying eqn ① by I.F We get,

$$x(3xy - y^2)dx + x(x^2 - xy)dy = 0$$

This D.E is exact now

$\therefore$ General solution is given by,

$$\int Mdx + \int Ndy = c$$

$$\int 3x^2y - xy^2\, dx + \int x^3 - x^2y\, dy = c$$

$$\int 3x^2y\, dx - \int xy^2 dx + \int x^3 dy - \int x^2y\, dy = c$$

$$y\int 3x^2 dx - y^2\int x\, dx + 0 - 0 = c$$

$$y \cdot 3 \cdot \frac{x^3}{3} - y^2 \cdot \frac{x^2}{2} = c$$

$\therefore x^3y - \frac{x^2y^2}{2} = c$ is the required solution.

Example on Rule 4

Que.1 solve: $(x+4y^3)\,dy - y\,dx = 0$

soln: Given D.E is $(x+4y^3)\,dy - y\,dx = 0$ ——— ①

comparing with $M\,dx + N\,dy = 0$ we get,

$M = x+4y^3$ $\qquad\qquad$ $N = -y$

diff N w.r.t x $\qquad\qquad$ diff M w.r.t y

$\frac{\partial N}{\partial x} = 1$ $\qquad\qquad$ $\frac{\partial M}{\partial y} = -1$

so Here $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so it is not exact

To make it exact we will find,

$$f(y) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1-(-1)}{-y} = \frac{-2}{y}$$

$$\therefore I.F = e^{\int f(y)\,dy} = e^{\int \frac{-2}{y}\,dy} = e^{-2\log y} = e^{\log y^{-2}}$$

$$I.F = \frac{1}{y^2}$$

Now, Multiplying eqn ① by I.F we get,

$$\frac{1}{y^2}(x+4y^3)\,dy - \frac{1}{y^2}\cdot y\,dx = 0$$

$$-\frac{1}{y}\,dx + \left(\frac{x}{y^2} + 4y\right)dy = 0$$

Now it is exact

$\therefore$ General solution is,

$$\int M\,dx + \int N\,dy = c$$

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$$\int \frac{-1}{y} dx + \int \frac{x}{y^2} + 4y \, dy = c$$

$$\frac{-1}{y} \int dx + \int 0 \, dy + \int 4y \, dy = c$$

$$\frac{-1}{y} x + \frac{4y^2}{2} = c$$

$\frac{-x}{y} + 2y^2 = c$ is required solution.

practice example (Rule 4)

① $(y^4 + 2y) dx + (xy^3 + 2y^4 - 4x) dy = 0$

$$I.F = \frac{1}{y^3}$$

Ans: $xy + \frac{2x}{y^2} + y^2 = c$

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Linear Differential Equation

If P & Q are function of x or constant then $\frac{dy}{dx} + Py = Q$ is called linear D.E of first order.

Working Rule:

Step 1: compare D.E with $\frac{dy}{dx} + Py = Q$ to find value of P & Q.

Step 2: Find integrating factor (I.F) by using formula

$$I.F = e^{\int P\,dx}$$

Step 3: solution is given by

$$y\,(I.F) = \int Q\,(I.F)\,dx + c$$

Solved Examples:

(1) solve $\frac{dy}{dx} + \frac{y}{x} = x^3$

Sol^n: Given D.E is $\frac{dy}{dx} + \frac{y}{x} = x^3$ —— (1)

comparing with $\frac{dy}{dx} + Py = Q$ we get,

$$P = \frac{1}{x} \qquad Q = x^3$$

Now,

$$I.F = e^{\int P\,dx}$$

$$= e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

$$\because e^{\log_e f(x)} = f(x)$$

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Now its solution is given by,

$y(I.F) = \int Q(I.F)\,dx + c$

$y \cdot x = \int x^3 \cdot x \,dx + c$

$y \cdot x = \int x^4 \,dx + c$

$y \cdot x = \frac{x^5}{5} + c$ is required general sol^n

Que.2 Solve $\frac{dy}{dx} + y\cot x = \operatorname{cosec} x$

Sol^n:

Given D.E is $\frac{dy}{dx} + y\cot x = \operatorname{cosec} x$ — (1)

comparing eq^n (1) with $\frac{dy}{dx} + Py = Q$ we get,

$P = \cot x$ $\quad Q = \operatorname{cosec} x$

Now,

$I.F = e^{\int P dx}$

$= e^{\int \cot x}$

$= e^{\log \sin x}$

$I.F = \sin x$

Now, solution is given by,

$y(I.F) = \int Q(I.F)\,dx + c$

$y(\sin x) = \int \operatorname{cosec} x \cdot \sin x \,dx + c$

$= \int \frac{1}{\sin x} \cdot \sin x \,dx + c$

$= \int 1\,dx + c$

$y \cdot \sin x = [x + c]$ is required solution.

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Que.3 Solve $\frac{dy}{dx} + y\tan x = \cos^2 x$

Soln: Given D.E is $\frac{dy}{dx} + y\tan x = \cos^2 x$ —— (1)

comparing with $\frac{dy}{dx} + Py = Q$ we get,

$P = \tan x$ $Q = \cos^2 x$

Now,

$I.F = e^{\int P\,dx}$

$= e^{\int \tan x\,dx}$

$= e^{\log \sec x}$

$\boxed{I.F = \sec x}$

Now,

solution is given by,

$y.(I.F) = \int Q.(I.F).dx + c$

$y.\sec x = \int \cos^2 x . \sec x\, dx + c$

$= \int \cos^2 x . \frac{1}{\cos x} dx + c$

$= \int \cos x\, dx + c$

$y.\sec x = \sin x + c$ is required solution.

Que.4 solve $\frac{dy}{dx} - \frac{1}{x}y = x$

Soln: Given D.E is $\frac{dy}{dx} - \frac{1}{x}y = x$ —— (1)

comparing with $\frac{dy}{dx} + Py = Q$ we get,

$P = \frac{-1}{x}$ $Q = x$

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Now,

$$I.F = e^{\int P dx}$$

$$= e^{\int -\frac{1}{x} dx}$$

$$= e^{-\log x}$$

$$= e^{\log x^{-1}}$$

$$= x^{-1}$$

$$\boxed{I.F = \frac{1}{x}}$$

Now, solution is given by

$$y(I.F) = \int Q.(I.F)dx + c$$

$$y\left(\frac{1}{x}\right) = \int x \cdot \frac{1}{x} dx + c$$

$$\frac{y}{x} = \int 1 dx + c$$

$\frac{y}{x} = x + c$ is required solution.

Practice Examples

① solve $\frac{dy}{dx} + \frac{y}{x} = x^3$ Ans: $xy = \frac{x^4}{4} + c$

② Solve $\frac{dy}{dx} + \frac{y}{x} = x^4$ Ans: $xy = \frac{x^6}{6} + c$

③ solve $\frac{dy}{dx} - \frac{1}{x} y = x^2$ Ans: $\frac{y}{x} = \frac{x^2}{2} + c$

④ solve $\frac{dy}{dx} - \frac{y}{x+1} = e^x(x+1)$ Ans: $\frac{y}{x+1} = e^x + c$