



Ch5: Noise in Quantum Computation



Imperfect gates

Let's implement a computation in which the quantum gates U_1, U_2, \dots, U_T are applied sequentially to an initial state $|\psi_0\rangle$. The state prepared by our ideal quantum circuit is:

$$|\psi_T\rangle = U_T \dots U_2 U_1 |\psi_0\rangle$$

But in fact our gates do not have perfect fidelity – pulse timing error, *etc.* When we attempt to apply the unitary transformation U_t , we instead apply some “nearby” unitary transformation \tilde{U}_t :

$$\tilde{U}_1 |\psi_0\rangle = |\psi_1\rangle + |E_1\rangle$$

where:

$$|E_1\rangle = (\tilde{U}_1 - U_1) |\psi_0\rangle$$

is an unnormalized vector.



Coherent noise model

Now, if \tilde{U}_t denotes the actual gate applied at step t , $|\tilde{\psi}_t\rangle$ denotes the actual state after t steps, and $|\psi_t\rangle$ denotes the ideal state, then we may write:

$$\begin{aligned} |\tilde{\psi}_t\rangle &= \tilde{U}_t |\tilde{\psi}_{t-1}\rangle \\ &= U_t |\psi_{t-1}\rangle + (\tilde{U}_t - U_t) |\psi_{t-1}\rangle + \tilde{U}_t (|\tilde{\psi}_{t-1}\rangle - |\psi_{t-1}\rangle) \\ &= |\psi_t\rangle + |E_t\rangle + \tilde{U}_t (|\tilde{\psi}_{t-1}\rangle - |\psi_{t-1}\rangle) \end{aligned}$$

where $|E_t\rangle = (\tilde{U}_t - U_t) |\psi_{t-1}\rangle$. Hence:

$$\begin{aligned} |\tilde{\psi}_2\rangle &= \tilde{U}_2 |\tilde{\psi}_1\rangle = |\psi_2\rangle + |E_2\rangle + \tilde{U}_2 |E_1\rangle \\ |\tilde{\psi}_3\rangle &= \tilde{U}_3 |\tilde{\psi}_2\rangle = |\psi_3\rangle + |E_3\rangle + \tilde{U}_3 |E_2\rangle + \tilde{U}_3 \tilde{U}_2 |E_1\rangle \end{aligned}$$

and so forth, and after T steps we obtain:

$$|\tilde{\psi}_T\rangle = |\psi_T\rangle + |E_T\rangle + \tilde{U}_T |E_{T-1}\rangle + \tilde{U}_T \tilde{U}_{T-1} |E_{T-2}\rangle + \dots + \tilde{U}_T \tilde{U}_{T-1} \dots \tilde{U}_2 |E_1\rangle$$



Coherent noise model

Thus we have expressed the difference between $|\tilde{\psi}_t\rangle$ and $|\psi_t\rangle$ as a sum of T remainder terms. The worst case yielding the largest deviation of $|\tilde{\psi}_t\rangle$ from $|\psi_t\rangle$ occurs if all remainder terms line up in the same direction, so that the errors interfere constructively. Therefore, we conclude that:

$$\left\| |\tilde{\psi}_T\rangle - |\psi_T\rangle \right\| \leq \| |E_T\rangle \| + \| |E_{T-1}\rangle \| + \dots + \| |E_2\rangle \| + \| |E_1\rangle \|$$

where we have used the property $\|U|E_t\rangle\| = \| |E_t\rangle \|$ for any unitary U .



Coherent noise model

Let $\|A\|_{sup}$ denote the sup norm of the operator A – that is, the largest eigenvalue of $\sqrt{A^\dagger A}$. We then have:

$$\| |E_t\rangle \| = \left\| \left(\tilde{U}_t - U_t \right) |\psi_{t-1}\rangle \right\| \leq \left\| \tilde{U}_t - U_t \right\|_{sup}$$

(since $|\psi_{t-1}\rangle$ is normalized). Now suppose that, for each value of t , the error in our quantum gate is bounded by:

$$\left\| \tilde{U}_t - U_t \right\|_{sup} \leq \epsilon$$

then after T quantum gates are applied, we have:

$$\left\| |\tilde{\psi}_T\rangle - |\psi_T\rangle \right\| \leq T\epsilon$$

in this sense, the accumulated error in the state grows linearly with the length of the computation.



Beyond coherent noise

Incoherent noise arises from interaction between the system and its environment – electromagnetic interferences, *etc.* To properly describe incoherent noise, we first need to get familiar with the density operator formalism, a more general representation than the state vector.



Pure and mixed states

Suppose that we only now that:

$$\begin{aligned}Pr(state = \psi_1) &= \frac{1}{3} \\Pr(state = \psi_2) &= \frac{2}{3}\end{aligned}$$

in such situation we write system as an ensemble $\left\{ \left(\frac{1}{3}, \psi_1 \right), \left(\frac{2}{3}, \psi_2 \right) \right\}$.

More generally: $\left\{ (p_i, \psi_i) \right\}_{i=1}^n$ with $\sum_i p_i = 1$

→ $n = 1$: pure state

→ $n > 1$: mixed state

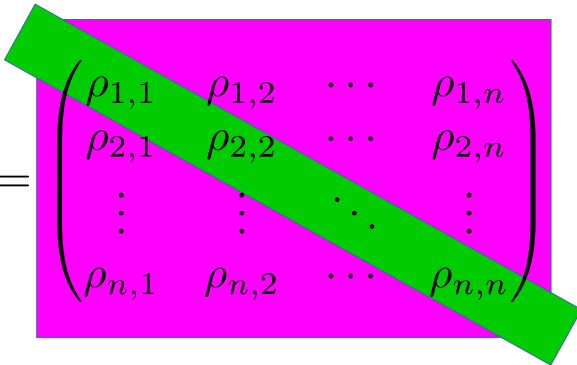


Density matrix

A density matrix is a matrix that describes the statistical distribution of quantum states in quantum mechanics:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

The diagonal elements determine the “populations” - the classical probability distribution of the states, while the off-diagonal elements determine the “coherence” - the quantum nature of the states:


$$\rho = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \cdots & \rho_{1,n} \\ \rho_{2,1} & \rho_{2,2} & \cdots & \rho_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n,1} & \rho_{n,2} & \cdots & \rho_{n,n} \end{pmatrix}$$

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \longleftarrow \left\{ \left(1, \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \right\}$$

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longleftarrow \left\{ \left(\frac{1}{2}, |0\rangle \right), \left(\frac{1}{2}, |1\rangle \right) \right\}$$

Properties

For an operator ρ to be a density operator, it must be a **positive operator** and have a **trace equal to one**.

Moreover, for pure states we have: $\rho^2 = \rho$

The average value of an operator A is given by: $\langle A \rangle = \text{Tr}\{\rho A\}$

Unitary transformation: $\rho \longrightarrow U\rho U^\dagger$

For the ensemble the probability of outcome m to occur is: $p(m) = \text{Tr}\{M^\dagger M \rho\}$

After measurement result m , if initially ρ then: $\rho_m = \frac{M\rho M^\dagger}{\text{Tr}\{M^\dagger M \rho\}}$



Composite systems

Subsystems are described by a reduced density operator. Suppose the system is composed of A and B , then the reduced density operator for subsystem A is:

$$\rho_A = \text{Tr}_B\{\rho_{AB}\}$$

Example:

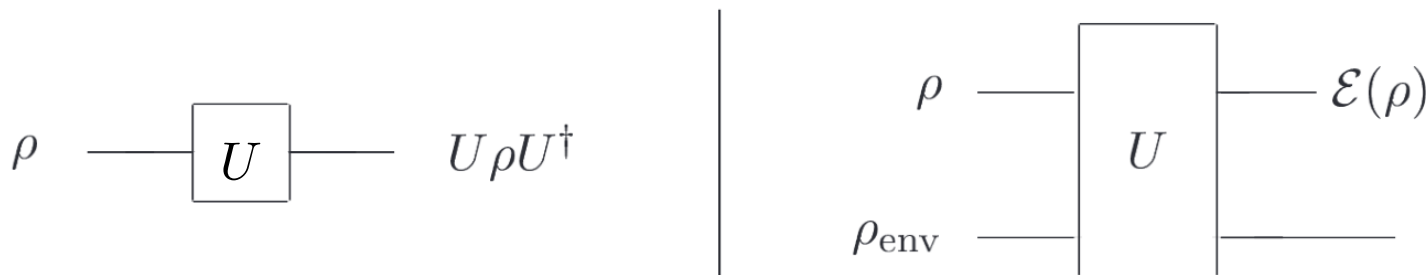
$$\begin{aligned}\rho_{AB} &= |\phi+\rangle\langle\phi+| = \frac{1}{2} (|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|) \\ \rho_A &= \text{Tr}_B\{\rho_{AB}\} = \sum_{i=0}^1 (I \otimes \langle i|) \rho_{AB} (I \otimes |i\rangle) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\end{aligned}$$

This is a statistical mixture of 0 and 1 (coin tossing) even though the composite system was pure ! Hallmark of entanglement.



Open quantum systems

An open quantum system consists of two parts, the principal system and an environment. Models of closed (left) and open (right) quantum systems:



where:

$$\mathcal{E}(\rho) = \text{Tr}_{\text{env}} \{ U(\rho \otimes \rho_{\text{env}}) U^\dagger \}$$

We assume that the system-environment input state is a product state – when an experimentalist prepares a quantum system in a specified state they undo all the correlations between that system and the environment.

Operator-sum representation

The operator-sum representation is essentially a re-statement of $\mathcal{E}(\rho)$ explicitly in terms of operators on the principal system's Hilbert space alone.

Let $|e_k\rangle$ be an orthonormal basis for the - finite dimensional - state space of the environment, and let $|e_0\rangle\langle e_0|$ be the initial state of the environment.

There is no loss of generality in assuming that the environment starts in a pure state, since if it starts in a mixed state we are free to introduce an extra system purifying the environment.

The main result is motivated by the following calculation: (see next slide)



Operator-sum representation

$$\begin{aligned}\mathcal{E}(\rho) &= \sum_k (I \otimes \langle e_k |) U (\rho \otimes |e_0\rangle\langle e_0|) U^\dagger (I \otimes |e_k\rangle) \\&= \sum_k (I \otimes \langle e_k |) U (\rho \otimes I) (I \otimes |e_0\rangle) (I \otimes \langle e_0|) U^\dagger (I \otimes |e_k\rangle) \\&= \sum_k (I \otimes \langle e_k |) U (\rho I) \otimes (I |e_0\rangle) (I \otimes \langle e_0|) U^\dagger (I \otimes |e_k\rangle) \\&= \sum_k (I \otimes \langle e_k |) U (I \rho) \otimes (|e_0\rangle 1) (I \otimes \langle e_0|) U^\dagger (I \otimes |e_k\rangle) \\&= \sum_k (I \otimes \langle e_k |) U (I \otimes |e_0\rangle) \rho (I \otimes \langle e_0|) U^\dagger (I \otimes |e_k\rangle) \\ \mathcal{E}(\rho) &\equiv \sum_k E_k \rho E_k^\dagger\end{aligned}$$

Reminder: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$



Operator-sum representation

The operator-sum representation describes the dynamics of the principal system without having to explicitly consider properties of the environment; all that we need to know is bundled up into the operators E_k , known as the Kraus operators, which act on the principal system alone.

The Kraus operators satisfy the completeness relation:

$$1 = \text{Tr}\{\mathcal{E}(\rho)\} = \text{Tr}\left\{\sum_k E_k \rho E_k^\dagger\right\} = \text{Tr}\left\{\sum_k E_k^\dagger E_k \rho\right\}$$

since this relationship is true for all ρ it follows that we must have:

$$\sum_k E_k^\dagger E_k = I$$



Bit-flip channel

The bit flip channel inverts the probability amplitudes of a qubit with probability $1-p$:

$$|\psi\rangle \longrightarrow \left\{ (p, |\psi\rangle), (1-p, X|\psi\rangle) \right\}$$

It has two Kraus operators:

$$E_0 = \sqrt{p}I \qquad E_1 = \sqrt{1-p}X$$

And thus:

$$\mathcal{E}(\rho) = p\rho + (1-p)X\rho X$$



Phase-flip channel

The phase flip channel inverts the phase of a qubit with probability $1-p$:

$$|\psi\rangle \longrightarrow \left\{ (p, |\psi\rangle), (1-p, Z|\psi\rangle) \right\}$$

It has two Kraus operators:

$$E_0 = \sqrt{p}I \qquad E_1 = \sqrt{1-p}Z$$

And thus:

$$\mathcal{E}(\rho) = p\rho + (1-p)Z\rho Z$$



Depolarizing channel

The depolarizing channel takes a single qubit, and with probability p that qubit is depolarized. That is, it is replaced by the completely mixed state $I/2$. With probability $1-p$ the qubit is left unchanged:

$$|\psi\rangle \longrightarrow \left\{ \left(\frac{p}{2}, |0\rangle \right), \left(\frac{p}{2}, |1\rangle \right), \left((1-p), |\psi\rangle \right) \right\}$$

Depolarizing can then be expressed with four Kraus operators:

$$\begin{aligned} \mathcal{E}(\rho) &= \frac{pI}{2} + (1-p)\rho \\ &= \frac{p}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z) + (1-p)\rho \\ &= \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z) \end{aligned}$$