Ch5: Noise in Quantum Computation

Imperfect gates

Let's implement a computation in which the quantum gates $U_1, U_2, ..., U_T$ are applied sequentially to an initial state $|\psi_0\rangle$. The state prepared by our ideal quantum circuit is:

$$|\psi_T\rangle = U_T ... U_2 U_1 |\psi_0\rangle$$

But in fact our gates do not have perfect fidelity – pulse timing error, *etc*. When we attempt to apply the unitary transformation U_t, we instead apply some “nearby” unitary transformation $\tilde{U}_t$:

$$\tilde{U}_1 |\psi_0\rangle = |\psi_1\rangle + |E_1\rangle$$

where:

$$|E_1\rangle = (\tilde{U}_1 - U_1)|\psi_0\rangle$$

is an unnormalized vector.

Coherent noise model

Now, if $\tilde{U}_t$ denotes the actual gate applied at step t, $|\tilde{\psi}_t\rangle$ denotes the actual state after t steps, and $|\psi_t\rangle$ denotes the actual state, then we may write:

$$\begin{aligned}|\tilde{\psi}_t\rangle &= \tilde{U}_t|\tilde{\psi}_{t-1}\rangle \\ &= U_t|\psi_{t-1}\rangle + (\tilde{U}_t - U_t)|\psi_{t-1}\rangle + \tilde{U}_t(|\tilde{\psi}_{t-1}\rangle - |\psi_{t-1}\rangle) \\ &= |\psi_t\rangle + |E_t\rangle + \tilde{U}_t(|\tilde{\psi}_{t-1}\rangle - |\psi_{t-1}\rangle)\end{aligned}$$

where $|E_t\rangle = (\tilde{U}_t - U_t)|\psi_{t-1}\rangle$. Hence:

$$\begin{aligned}|\tilde{\psi}_2\rangle &= \tilde{U}_2|\tilde{\psi}_1\rangle = |\psi_2\rangle + |E_2\rangle + \tilde{U}_2|E_1\rangle \\ |\tilde{\psi}_3\rangle &= \tilde{U}_3|\tilde{\psi}_2\rangle = |\psi_3\rangle + |E_3\rangle + \tilde{U}_3|E_2\rangle + \tilde{U}_3\tilde{U}_2|E_1\rangle\end{aligned}$$

and so forth, and after T steps we obtain:

$$|\tilde{\psi}_T\rangle = |\psi_T\rangle + |E_T\rangle + \tilde{U}_T|E_{T-1}\rangle + \tilde{U}_T\tilde{U}_{T-1}|E_{T-2}\rangle + ... + \tilde{U}_T\tilde{U}_{T-1}...\tilde{U}_2|E_1\rangle$$

Coherent noise model

Thus we have expressed the difference between $|\tilde{\psi}_t\rangle$ and $|\psi_t\rangle$ as a sum of T remainder terms. The worst case yielding the largest deviation of $|\tilde{\psi}_t\rangle$ from $|\psi_t\rangle$ occurs if all remainder terms line up in the same direction, so that the errors interfere constructively. Therefore, we conclude that:

$$\left\| |\tilde{\psi}_T\rangle - |\psi_T\rangle \right\| \leq \| |E_T\rangle \| + \| |E_{T-1}\rangle \| + ... + \| |E_2\rangle \| + \| |E_1\rangle \|$$

where we have used the property $\|U|E_t\rangle\| = \||E_t\rangle\|$ for any unitary U.

Coherent noise model

Let $\|A\|_{sup}$ denote the sup norm of the operator of the operator A – that is, the largest eigenvalue of $\sqrt{A^\dagger A}$. We then have:

$$\||E_t\rangle\| = \left\|\left(\tilde{U}_t - U_t\right)|\psi_{t-1}\rangle\right\| \leq \left\|\tilde{U}_t - U_t\right\|_{sup}$$

(since $|\psi_{t-1}\rangle$ is normalized). Now suppose that, for each value of *t*, the error in our quantum gate is bounded by:

$$\left\|\tilde{U}_t - U_t\right\|_{sup} \leq \epsilon$$

then after *T* quantum gates are applied, we have:

$$\left\||\tilde{\psi}_T\rangle - |\psi_T\rangle\right\| \leq T\epsilon$$

in this sense, the accumulated error in the state grows linearly with the length of the computation.

Beyond coherent noise

Incoherent noise arises from interaction between the system and its environment – electromagnetic interferences, *etc.* To properly describe incoherent noise, we first need to get familiar with the density operator formalism, a more general representation than the state vector.

Pure and mixed states

Suppose that we only now that:

$$Pr(state = \psi_1) = \frac{1}{3}$$
$$Pr(state = \psi_2) = \frac{2}{3}$$

in such situation we write system as an ensemble $\left\{ \left(\frac{1}{3}, \psi_1\right), \left(\frac{2}{3}, \psi_2\right) \right\}$.

More generally: $\{ (p_i, \psi_i) \}_{i=1}^{n}$ with $\displaystyle\sum_i p_i = 1$

$\rightarrow$ n $= 1$: pure state

$\rightarrow$ n > 1 : mixed state

Density matrix

A density matrix is a matrix that describes the statistical distribution of quantum states in quantum mechanics:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

The diagonal elements determine the “populations” - the classical probability distribution of the states, while the off-diagonal elements determine the “coherence” - the quantum nature of the states:

$$\rho = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \cdots & \rho_{1,n} \\ \rho_{2,1} & \rho_{2,2} & \cdots & \rho_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n,1} & \rho_{n,2} & \cdots & \rho_{n,n} \end{pmatrix}$$

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \longleftarrow \left\{ \left(1, \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \right\}$$

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longleftarrow \left\{ \left(\frac{1}{2}, |0\rangle \right), \left(\frac{1}{2}, |1\rangle \right) \right\}$$

Properties

For an operator ρ to be a density operator, it must be **a positive operator** and have **a trace equal to one**.

Moreover, for pure states we have: $\rho^2 = \rho$

The average value of an operator A is given by: $\langle A \rangle = Tr\{\rho A\}$

Unitary transformation: $\rho \longrightarrow U \rho U^\dagger$

For the ensemble the probability of outcome *m* to occur is: $p(m) = Tr\{M^\dagger M \rho\}$

After measurement result *m*, if initially ρ then: $\rho_m = \dfrac{M \rho M^\dagger}{Tr\{M^\dagger M \rho\}}$

Composite systems

Subsystems are described by a reduced density operator. Suppose the system is composed of A and B, then the reduced density operator for subsystem A is:

$$\rho_A = Tr_B\{\rho_{AB}\}$$

Example:

$$\rho_{AB} = |\phi+\rangle\langle\phi+| = \tfrac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|\right)$$

$$\rho_A = Tr_B\{\rho_{AB}\} = \sum_{i=0}^{1} \left(I \otimes \langle i|\right) \rho_{AB} \left(I \otimes |i\rangle\right) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

This is a statistical mixture of 0 and 1 (coin tossing) even though the composite system was pure ! Hallmark of entanglement.

Open quantum systems

An open quantum system consists of two parts, the principal system and an environment. Models of closed (left) and open (right) quantum systems:

where:

$$\mathcal{E}(\rho) = Tr_{env}\{U(\rho \otimes \rho_{env})U^\dagger\}$$

We assume that the system-environment input state is a product state – when an experimentalist prepares a quantum system in a specified state they undo all the correlations between that system and the environment.

Operator-sum representation

The operator-sum representation is essentially a re-statement of $\mathcal{E}(\rho)$ explicitly in terms of operators on the principal system's Hilbert space alone.

Let $|e_k\rangle$ be an orthonormal basis for the - finite dimensional - state space of the environment, and let $|e_0\rangle\langle e_0|$ be the initial state of the environment.

There is no loss of generality in assuming that the environment starts in a pure state, since if it starts in a mixed state we are free to introduce an extra system purifying the environment.

The main result is motivated by the following calculation: (see next slide)

Operator-sum representation

$$
\begin{aligned}
\mathcal{E}(\rho) &= \sum_k (I \otimes \langle e_k|)\, U\, (\rho \otimes |e_0\rangle\langle e_0|)\, U^\dagger\, (I \otimes |e_k\rangle) \\
&= \sum_k (I \otimes \langle e_k|)\, U\, (\rho \otimes I)\, (I \otimes |e_0\rangle)\, (I \otimes \langle e_0|)\, U^\dagger\, (I \otimes |e_k\rangle) \\
&= \sum_k (I \otimes \langle e_k|)\, U\, (\rho I) \otimes (I|e_0\rangle)\, (I \otimes \langle e_0|)\, U^\dagger\, (I \otimes |e_k\rangle) \\
&= \sum_k (I \otimes \langle e_k|)\, U\, (I\rho) \otimes (|e_0\rangle 1)\, (I \otimes \langle e_0|)\, U^\dagger\, (I \otimes |e_k\rangle) \\
&= \sum_k (I \otimes \langle e_k|)\, U\, (I \otimes |e_0\rangle)\, \rho\, (I \otimes \langle e_0|)\, U^\dagger\, (I \otimes |e_k\rangle) \\
\mathcal{E}(\rho) &\equiv \sum_k E_k \rho E_k^\dagger
\end{aligned}
$$

Reminder: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

Operator-sum representation

The operator-sum representation describes the dynamics of the principal system without having to explicitly consider properties of the environment; all that we need to know is bundled up into the operators E_k, known as the Kraus operators, which act on the principal system alone.

The Kraus operators satisfy the completeness relation:

$$1 = Tr\{\mathcal{E}(\rho)\} = Tr\left\{\sum_k E_k \rho E_k^\dagger\right\} = Tr\left\{\sum_k E_k^\dagger E_k \rho\right\}$$

since this relationship is true for all ρ it follows that we must have:

$$\sum_k E_k^\dagger E_k = I$$

Bit-flip channel

The bit flip channel inverts the probability amplitudes of a qubit with probability 1-*p:*

$$|\psi\rangle \longrightarrow \Big\{ \big(p, |\psi\rangle\big), \big(1-p, X|\psi\rangle\big) \Big\}$$

It has two Kraus operators:

$$E_0 = \sqrt{p} I \qquad\qquad E_1 = \sqrt{1-p} X$$

And thus:

$$\mathcal{E}(\rho) = p\rho + (1-p) X \rho X$$

Phase-flip channel

The phase flip channel inverts the phase of a qubit with probability 1-*p:*

$$|\psi\rangle \longrightarrow \Big\{\big(p, |\psi\rangle\big), \big(1-p, Z|\psi\rangle\big)\Big\}$$

It has two Kraus operators:

$$E_0 = \sqrt{p}I \qquad\qquad E_1 = \sqrt{1-p}Z$$

And thus:

$$\mathcal{E}(\rho) = p\rho + (1-p)Z\rho Z$$

Depolarizing channel

The depolarizing channel takes a single qubit, and with probability p that qubit is depolarized. That is, it is replaced by the completely mixed state $I/2$. With probability 1-p the qubit is left unchanged:

$$|\psi\rangle \longrightarrow \left\{\left(\tfrac{p}{2}, |0\rangle\right), \left(\tfrac{p}{2}, |1\rangle\right), \left((1-p), |\psi\rangle\right)\right\}$$

Depolarizing can then be expressed with four Kraus operators:

$$\begin{aligned}
\mathcal{E}(\rho) &= \frac{pI}{2} + (1-p)\rho \\
&= \frac{p}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z) + (1-p)\rho \\
&= \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z)
\end{aligned}$$