우림 8-3

2019년 8월 14일 수요일 오후 5:19

$$\begin{cases} \dfrac{e^{ix}+e^{iy}}{2} = e^{\frac{i(x+y)}{2}} \left(\dfrac{e^{\frac{i(x-y)}{2}}+e^{-\frac{i(x-y)}{2}}}{2} \right) \\ \dfrac{e^{\frac{i(x-y)}{2}}+e^{-\frac{i(x-y)}{2}}}{2} = \cos\dfrac{(x-y)}{2} \end{cases}$$

i.e. $$\frac{e^{i\cdot 0}+e^{i\cdot\frac{x}{2^n}}}{2} = e^{i\frac{x}{2^{n+1}}}\cos\frac{x}{2^{n+1}},$$

$$\frac{(1+\cos x+i\sin x)}{2}\,\frac{(1+\cos\frac{x}{2}+i\sin\frac{x}{2})}{2}\cdots\frac{(1+\cos\frac{x}{2^n}+i\sin\frac{x}{2^n})}{2}$$

$$= e^{i\frac{x}{2}}e^{i\frac{x}{4}}\cdots e^{i\frac{x}{2^{n+1}}}\cdot\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^{n+1}}$$

$$2^{n+1}\left(\cos\frac{x}{2}\cdots\cos\frac{x}{2^{n+1}}\right)\cdot\sin\frac{x}{2^{n+1}} = \sin x$$

(By double-angle formula)

$$\cos\frac{x}{2}\cdots\cos\frac{x}{2^{n+1}} = \frac{\sin x}{2^{n+1}\sin\frac{x}{2^{n+1}}}$$

$$\lim_{n\to\infty}\frac{\sin\frac{x}{2^{n+1}}}{\frac{x}{2^{n+1}}} = 1 \to \lim_{n\to\infty}\cos\frac{x}{2}\cdots\cos\frac{x}{2^{n+1}} = \frac{\sin x}{x}$$

Finally, the given limit $= e^{ix}\dfrac{\sin x}{x}$ for any x.

(For $x=0$ (exceptionally) it's 1.)