Notes on anisotropic propatation matrix

I. CONSITITUITIVE RELATION

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{1}$$

The permutation symmetry relationship always holds:

$$\begin{aligned} C_{ijkl} &= C_{ijlk} && (2)\\ C_{ijkl} &= C_{jikl} && (3)\\ C_{ijkl} &= C_{klij} && (4)\\ & && (5) \end{aligned}$$

With all those relationship, there are totally 21 free tensor elements. The constituitive relationship can be rewritten by Voigt notation:

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{pmatrix} = \begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} \tag{6}$$

$$\begin{pmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{23}\\ 2\epsilon_{13}\\ 2\epsilon_{12}\end{pmatrix} = \begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{pmatrix} \tag{7}$$

The elastic tensor can be written as a 6×6 matrix under Voigt notation.

II. GREEN-CHRISTOFFEL EQUATION

The constituitive relationship combines with Newton's second law, we will have

$$\begin{aligned} \rho\ddot{u}_i(\mathbf{x}, t) &= \sigma_{ij,j}(\mathbf{x}, t) + F && (8)\\ \sigma_{ij}(\mathbf{x}, t) &= C_{ijkl}u_{k,l}(\mathbf{x}, t) && (9)\\ & && (10) \end{aligned}$$

The wave equation with general elastic tensor is:

$$\rho\ddot{u}_i(\mathbf{x}, t) = C_{ijkl}u_{k,lj}(\mathbf{x}, t) \tag{11}$$

We assume the wave is a time-harmonic plane wave.

$$u_i(\mathbf{x}, t) = At_i \exp(i\mathbf{k} \cdot \mathbf{x}) \exp(-i\omega t) \tag{12}$$

t_i is the polarization vector $\mathbf{t} = [\phi_1, \phi_2, \phi_3]$ represents the wave amplitude on each direction. We have the Green-Christoffel equation

$$(\rho\omega^2\delta_{ik} - C_{ijkl}k_jk_l)t_i = 0 \tag{13}$$

In general anisotropic medium, we have totally 3 types of wave: P wave, slow S wave, fast S wave. In the layered medium, we can decompose the wavefield as up-going and down-going waves. Therefore, we have all 6 polarization vectors, each of them is the solution of Green-Christoffel equation.

For practical consideration, we use Mathematica for the tensor product and contract. See GC_equation.nb.

III. PROPAGATION MATRIX

When the wave goes across the boundary of different elactic medium, the displacement and the traction across the two sides of boundary should be the same. Here we only consider the plane incident wave in homogeneous layered medium. In this case, the wavefield can be calculated by the propagator matrix method. In one layer, the wave propagation process can be decomposed as two processes: 1. Wave propagation within the homogeneous layer 2. Wave goes across the boundary which must satisfy the constraints of continuous displacement-traction vector .

So forth, the one-layer propagator matrix basically is the multiplication of three matrces:

- Phase-shift matrix $\mathbf{P_s}$, propagate 3 upgoing 3 downgoing matrix within the layer space.
- Compose matrix $\mathbf{C}$, combine all upgoing and downgoing P, slow S, fast S wavefield to the 6 component displacment-traction vector.
- Decompose matrix $\mathbf{D}$, decompose the displacement-traction vector back into the upgoing-downgoing P, slow S, fast S wavefield according to the constituitive relationship in the next layer.

The ith-layer propagator matrix $\mathbf{P}^i = \mathbf{C}^i \mathbf{P_s}^i \mathbf{D}^i$. The total n layer propagator matrix can be written as $\mathbf{P} = \prod_{i=1}^{n} \mathbf{P}^i$.

A. $\mathbf{P_s}$ matrix

For the $\mathbf{P_s}$ matrix, we have a diagonal matrix

$$\mathbf{P_s} = diag(e^{ik_z^{UP}\cdot h}, e^{ik_z^{USs}\cdot h}, e^{ik_z^{USf}\cdot h}, e^{ik_z^{DP}\cdot h}, e^{ik_z^{DSs}\cdot h}, e^{ik_z^{DSf}\cdot h}) \tag{14}$$

B. $\mathbf{C}, \mathbf{D}$ matrix

We introduce notation for deduction, superscript ^{n} is the up-going/down-going wave index, $n = 1, 2, 3, 4, 5, 6$, subscript $_i$ is the component index, $i = 1, 2, 3$. For each component, the displacement can be written as:

$$u_i = \sum_n A^n t_i^n \tag{15}$$

$$A^n = a^n \exp(i\mathbf{k}^n \cdot \mathbf{x} - i\omega t) \tag{16}$$

$$\tag{17}$$

Write it as matrix form is

$$u_i = U_{ij} A^j \tag{18}$$

$$U_{ij} = \begin{pmatrix} t_1^1 & \cdots & t_1^6 \\ t_2^1 & \cdots & t_2^6 \\ t_3^1 & \cdots & t_3^6 \end{pmatrix} \tag{19}$$

The constituitive relationship is $\sigma_{pq} = C_{pqrs}\epsilon_{rs}$. We first write down the ϵ_{rs}

$$\epsilon_{rs} = \sum_n A_n \frac{1}{2}(it_r^n k_s^n + it_s^n k_r^n) \tag{20}$$

With the Voigt notation, $4^{th} - 6^{th}$ components need to be doubled, the strain can be written as a matrix:

$$\epsilon_i = E_{ij} A^j \tag{21}$$

$$E = \begin{pmatrix} it_1^1 k_1^1 & \cdots & it_1^6 k_1^6 \\ it_2^1 k_2^1 & \cdots & it_2^6 k_2^6 \\ it_3^1 k_3^1 & \cdots & it_3^6 k_3^6 \\ i(t_2^1 k_3^1 + t_3^1 k_2^1) & \cdots & i(t_2^6 k_3^6 + t_3^6 k_2^6) \\ i(t_1^1 k_3^1 + t_3^1 k_1^1) & \cdots & i(t_1^6 k_3^6 + t_3^6 k_1^6) \\ i(t_2^1 k_1^1 + t_1^1 k_2^1) & \cdots & i(t_1^6 k_2^6 + t_2^6 k_1^6) \end{pmatrix} \tag{22}$$

The relation between C_{pqrs} to C_{ij} is

$$C_{ij} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1222} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{pmatrix} \tag{23}$$

The stress vector can be written as:

$$\sigma_i = \sum_j \sum_k C_{ij} * E_{jk} * A^k \tag{24}$$

The traction along $[0, 0, 1]$ direction is $[\sigma_{13}, \sigma 23, \sigma_{33}]$, which needs a window matrix on the σ_i, we call it $\mathbf{w}$

$$w_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{25}$$

The final $\mathbf{C}$ matrix has the form

$$\mathbf{C} = \begin{bmatrix} \mathbf{U} \\ \mathbf{W} \cdot \mathbf{C} \cdot \mathbf{E} \end{bmatrix} \tag{26}$$

The $\mathbf{D}$ matrix is the inverse of $\mathbf{C}$.

$$\mathbf{D} = \mathbf{C}^{-1} \tag{27}$$

IV. WORK-FLOW

V. RESULT

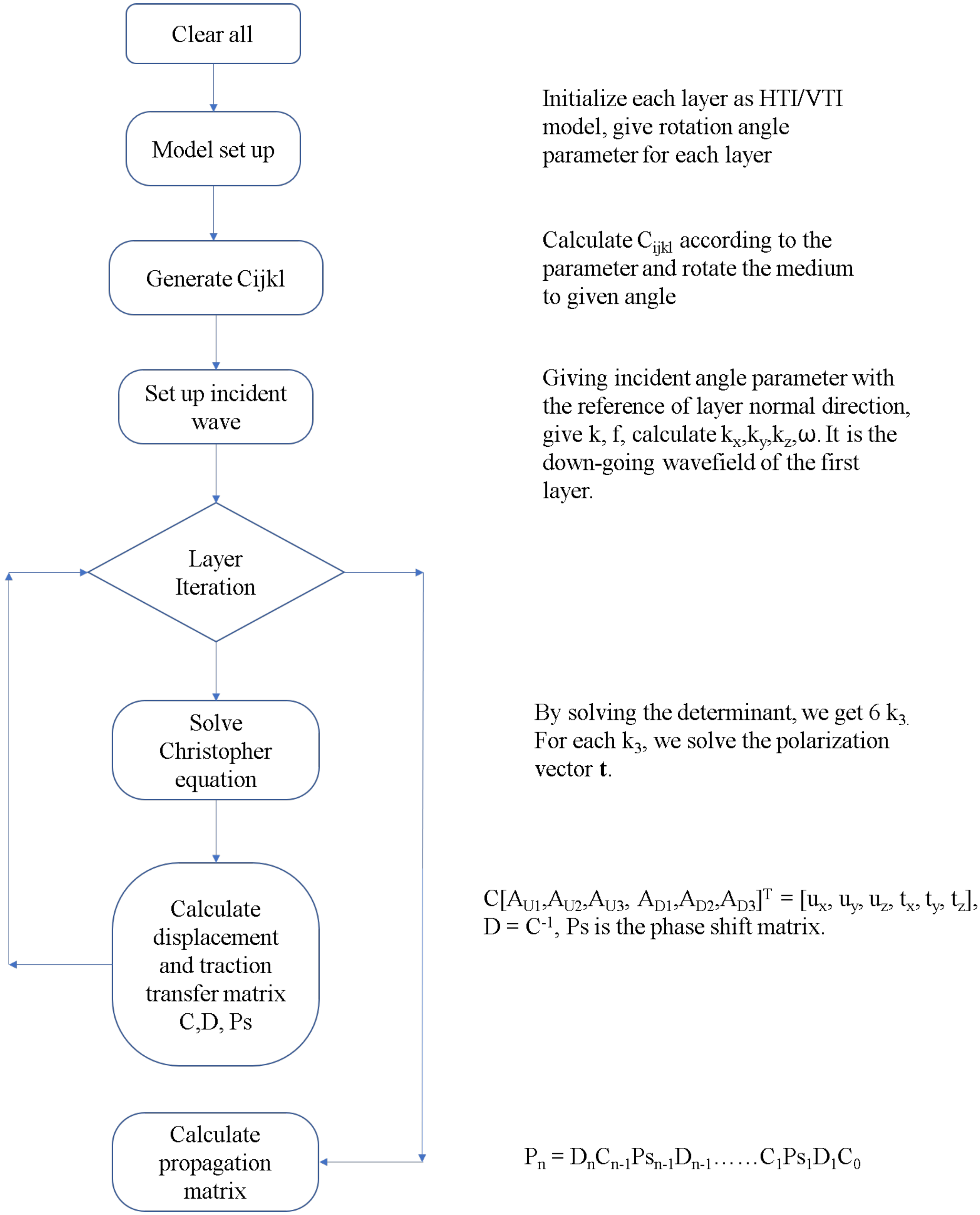

FIG. 1. The workflow of the anisotropical propagation matrix method

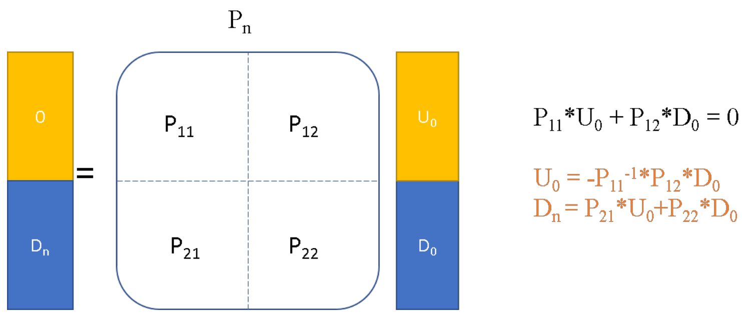

FIG. 2. Schematic to show how to solve the up going wave in the first layer and the down going wave in the bottom layer