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*Research Article*

# The Finite-Dimensional Uniform Attractors for the Nonautonomous g-Navier-Stokes Equations

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We consider the uniform attractors for the two dimensional nonautonomous g-Navier-Stokes equations in bounded domain $\Omega$. Assuming $f = f(x,t) \in L^2_{\text{loc}}$, we establish the existence of the uniform attractor in $L^2(\Omega)$ and $D(A^{1/2})$. The fractal dimension is estimated for the kernel sections of the uniform attractors obtained.



## 1. Introduction

In this paper, we study the behavior of solutions of the nonautonomous 2D g-Navier-Stokes equations. These equations are a variation of the standard Navier-Stokes equations, and they assume the form,

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \text{ in } \Omega, \\ &\frac{1}{g}(\nabla\cdot gu) = \frac{\nabla g}{g}\cdot u + \nabla\cdot u = 0 \text{ in } \Omega, \end{aligned} \tag{1.1}$$

where $g = g(x_1, x_2)$ is a suitable smooth real-valued function defined on $(x_1, x_2) \in \Omega$ and $\Omega$ is a suitable bounded domain in $\mathbb{R}^2$. Notice that if $g(x_1, x_2) = 1$, then (1.1) reduce to the standard Navier-Stokes equations.

In addition, we assume that the function $f(\cdot, t) =: f(t) \in L^2_{\text{loc}}(\mathbb{R}; E)$ is translation bounded, where $E = L^2(\Omega)$ or $H^{-1}(\Omega)$. This property implies that

$$\|f\|^2_{L^2_b} = \|f\|^2_{L^2_b(\mathbb{R};E)} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s)\|^2_E \, ds < \infty. \tag{1.2}$$

We consider this equation in an appropriate Hilbert space and show that there is an attractor $\mathfrak{A}$ which all solutions approach as $t \to \infty$. The basic idea of our construction, which is motivated by the works of [1, 2].

In [1, 2] the author established the global regularity of solutions of the g-Navier-Stokes equations. For the boundary conditions, we will consider the periodic boundary conditions, while same results can be got for the Dirichlet boundary conditions on the smooth bounded domain. For many years, the Navier-Stokes equations were investigated by many authors and the existence of the attractors for 2D Navier-Stokes equations was first proved by Ladyzhenskaya [3, 4] and independently by Foias and Temam [5]. The finite-dimensional property of the global attractor for general dissipative equations was first proved by Mallet-Paret [6]. For the analysis on the Navier-Stokes equations, one can refer to [7] and specially [8] for the periodic boundary conditions.

The book in [9] considers some special classes of such systems and studies systematically the notion of uniform attractor parallelling to that of global attractor for autonomous systems. Later on, [10] presents a general approach that is well suited to study equations arising in mathematical physics. In this approach, to construct the uniform (or trajectory) attractors, instead of the associated process $\{U_\sigma(t,\tau),\ t \geq \tau,\ \tau \in \mathbb{R}\}$ one should consider a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ in some Banach space $E$, where the functional parameter $\sigma_0(s)$, $s \in \mathbb{R}$ is called the symbol and $\Sigma$ is the symbol space including $\sigma_0(s)$. The approach preserves the leading concept of invariance which implies the structure of uniform attractor described by the representation as a union of sections of all kernels of the family of processes. The kernel is the set of all complete trajectories of a process.

In the paper, we study the existence of compact uniform attractor for the non-autonomous the two dimensional g-Navier-Stokes equations in the periodic boundary conditions $\Omega$. We apply measure of noncompactness method to nonautonomous g-Navier-Stokes equations equation with external forces $f(x,t)$ in $L^2_{\mathrm{loc}}(\mathbb{R}; E)$ which is normal function (see Definition 4.2). Last, the fractal dimension is estimated for the kernel sections of the uniform attractors obtained.

## 2. Functional Setting

Let $\Omega = (0,1) \times (0,1)$ and we assume that the function $g(x) = g(x_1, x_2)$ satisfies the following properties:

(1) $g(x) \in C^\infty_{\mathrm{per}}(\Omega)$ and

(2) there exist constants $m_0 = m_0(g)$ and $M_0 = M_0(g)$ such that, for all $x \in \Omega$, $0 < m_0 \leq g(x) \leq M_0$. Note that the constant function $g \equiv 1$ satisfies these conditions.

We denote by $L^2(\Omega, g)$ the space with the scalar product and the norm given by

$$(u,v)_g = \int_\Omega (u \cdot v) g\, dx, \qquad |u|_g^2 = (u,u)_g, \tag{2.1}$$

as well as $H^1(\Omega, g)$ with the norm

$$\|u\|_{H^1(\Omega,g)} = \left[ (u,u)_g + \sum_{i=1}^{2} (D_i u, D_i u)_g \right]^{1/2}, \tag{2.2}$$

where $\partial u / \partial x_i = D_i u$.

Then for the functional setting of (1.1), we use the following functional spaces

$$H_g = Cl_{L^2_{\text{per}}(\Omega,g)}\left\{u \in C^\infty_{\text{per}}(\Omega) : \nabla \cdot gu = 0,\ \int_\Omega u\, dx = 0\right\},$$
$$V_g = \left\{u \in H^1_{\text{per}}(\Omega, g) : \nabla \cdot gu = 0, \int_\Omega u\, dx = 0\right\}, \tag{2.3}$$

where $H_g$ is endowed with the scalar product and the norm in $L^2(\Omega, g)$, and $V_g$ is the spaces with the scalar product and the norm given by

$$((u, v))_g = \int_\Omega (\nabla u \cdot \nabla v) g\, dx, \qquad \|u\|_g = ((u, u))_g. \tag{2.4}$$

Also, we define the orthogonal projection $P_g$ as

$$P_g : L^2_{\text{per}}(\Omega, g) \longrightarrow H_g \tag{2.5}$$

and we have that $Q \subseteq H_g^\perp$, where

$$Q = Cl_{L^2_{\text{per}}(\Omega,g)}\left\{\nabla\phi : \phi \in C^1\left(\overline{\Omega}, \mathbb{R}\right)\right\}. \tag{2.6}$$

Then, we define the $g$-Laplacian operator

$$-\Delta_g u \equiv \frac{1}{g}(\nabla \cdot g\nabla)u = -\Delta u - \frac{1}{g}(\nabla g \cdot \nabla)u \tag{2.7}$$

to have the linear operator

$$A_g u = P_g\left[-\frac{1}{g}(\nabla \cdot (g\nabla u))\right]. \tag{2.8}$$

For the linear operator $A_g$, the following hold (see [1, 2]):

(1) $A_g$ is a positive, self-adjoint operator with compact inverse, where the domain of $A_g$, $D(A_g) = V_g \cap H^2(\Omega, g)$.

(2) There exist countable eigenvalues of $A_g$ satisfying

$$0 < \lambda_g \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots, \tag{2.9}$$

where $\lambda_g = 4\pi^2 m/M$ and $\lambda_1$ is the smallest eigenvalue of $A_g$. In addition, there exists the corresponding collection of eigenfunctions $\{e_1, e_2, e_3, \ldots\}$ which forms an orthonormal basis for $H_g$.

Next, we denote the bilinear operator $B_g(u,v) = P_g(u\cdot\nabla)v$ and the trilinear form

$$b_g(u,v,w) = \sum_{i,j=1}^{n}\int_\Omega u_i(D_iv_j)w_jg\,dx = \left(P_g(u\cdot\nabla)v,w\right)_g, \tag{2.10}$$

where $u,v,w$ lie in appropriate subspaces of $L^2(\Omega,g)$. Then, the form $b_g$ satisfies

$$b_g(u,v,w) = -b_g(u,w,v) \quad \text{for } u,v,w\in H_g. \tag{2.11}$$

We denote a linear operator $R$ on $V_g$ by

$$Ru = P_g\left[\frac{1}{g}(\nabla g\cdot\nabla)u\right] \quad \text{for } u\in V_g, \tag{2.12}$$

and have $R$ as a continuous linear operator from $V_g$ into $H_g$ such that

$$|(Ru,u)| \le \frac{|\nabla g|_\infty}{m_0}\|u\|_g|u|_g \le \frac{|\nabla g|_\infty}{m_0\lambda_g^{1/2}}\|u\|_g \quad \text{for } u\in V_g. \tag{2.13}$$

We now rewrite (1.1) as abstract evolution equations,

$$\frac{du}{dt} + \nu A_gu + B_gu + \nu Ru = P_gf, \qquad u(\tau) = u_\tau. \tag{2.14}$$

Hereafter $c$ will denote a generic scale invariant positive constant, which is independent of the physical parameters in the equation and may be different from line to line and even in the same line.

## 3. Abstract Results

Let $E$ be a Banach space, and let a two-parameter family of mappings $\{U(t,\tau)\} = \{U(t,\tau) \mid t\ge\tau,\ \tau\in\mathbb{R}\}$ act on $E$:

$$U(t,\tau): E \longrightarrow E, \quad t\ge\tau,\ \tau\in\mathbb{R}. \tag{3.1}$$

*Definition 3.1.* A two-parameter family of mappings $\{U(t,\tau)\}$ is said to be a process in $E$ if

$$\begin{gathered} U(t,s)U(s,\tau) = U(t,\tau), \quad \forall t\ge s\ge\tau,\ \tau\in\mathbb{R}, \\ U(\tau,\tau) = Id, \quad \tau\in\mathbb{R}. \end{gathered} \tag{3.2}$$

By $\mathcal{B}(E)$ we denote the collection of the *bounded* sets of $E$. We consider a family of processes $\{U_\sigma(t,\tau)\}$ depending on a parameter $\sigma\in\Sigma$. The parameter $\sigma$ is said to be the *symbol* of the process $\{U_\sigma(t,\tau)\}$ and the set $\Sigma$ is said to be the *symbol space*. In the sequel $\Sigma$ is assumed to be a complete metric space.

A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is said to be *uniformly* (with respect to (*w.r.t.*) $\sigma \in \Sigma$) *bounded* if for any $B \in \mathcal{B}(E)$ the set

$$\bigcup_{\sigma\in\Sigma}\bigcup_{\tau\in R}\bigcup_{t\geq\tau} U_\sigma(t,\tau)B \in \mathcal{B}(E). \tag{3.3}$$

A set $B_0 \subset E$ is said to be *uniformly* (*w.r.t.* $\sigma \in \Sigma$) *absorbing* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if for any $\tau \in R$ and every $B \in \mathcal{B}(E)$ there exists $t_0 = t_0(\tau, B) \geq \tau$ such that $\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subseteq B_0$ for all $t \geq t_0$.

A set $P \subset E$ is said to be *uniformly* (*w.r.t.* $\sigma \in \Sigma$) *attracting* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if for an arbitrary fixed $\tau \in R$,

$$\lim_{t\to+\infty}\left(\sup_{\sigma\in\Sigma} \operatorname{dist}_E(U_\sigma(t,\tau)B, P)\right) = 0. \tag{3.4}$$

A family of processes possessing a compact uniformly absorbing set is called *uniformly compact* and a family of processes possessing a compact uniformly attracting set is called *uniformly asymptotically compact*.

*Definition 3.2.* A closed set $\mathcal{A}_\Sigma \subset E$ is said to be the uniform (*w.r.t.* $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if it is uniformly (*w.r.t.* $\sigma \in \Sigma$) attracting and it is contained in any closed uniformly (*w.r.t.* $\sigma \in \Sigma$) attracting set $\mathcal{A}'$ of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ : $\mathcal{A}_\Sigma \subseteq \mathcal{A}'$.

A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ acting in $E$ is said to be $(E \times \Sigma, E)$-*continuous*, if for all fixed $t$ and $\tau$, $t \geq \tau$, $\tau \in \mathbb{R}$ the mapping $(u,\sigma) \mapsto U_\sigma(t,\tau)u$ is continuous from $E \times \Sigma$ into $E$.

A curve $u(s)$, $s \in \mathbb{R}$ is said to be *a complete trajectory* of the process $\{U(t,\tau)\}$ if

$$U(t,\tau)u(\tau) = u(t), \quad \forall t \geq \tau,\ \tau \in \mathbb{R}. \tag{3.5}$$

The *kernel* $\mathcal{K}$ of the process $\{U(t,\tau)\}$ consists of all bounded complete trajectories of the process $\{U(t,\tau)\}$:

$$\mathcal{K} = \left\{u(\cdot) \mid u(\cdot) \text{ satisfies (3.6)},\ \|u(s)\|_E \leq M_u \text{ for } s \in \mathbb{R}\right\}. \tag{3.6}$$

The set

$$\mathcal{K}(s) = \{u(s) \mid u(\cdot) \in \mathcal{K}\} \subseteq E \tag{3.7}$$

is said to be the *kernel section* at time $t = s$, $s \in \mathbb{R}$.

For convenience, let $B_t = \bigcup_{\sigma\in\Sigma}\bigcup_{s\geq t} U_\sigma(s,t)B$, the closure $\overline{B}$ of the set $B$ and $\mathbb{R}_\tau = \{t \in \mathbb{R} \mid t \geq \tau\}$. Define the uniform ($w.r.t.$ $\sigma \in \Sigma$) $\omega$-limit set $\omega_{\tau,\Sigma}(B)$ of $B$ by $\omega_{\tau,\Sigma}(B) = \bigcap_{t\geq\tau}\overline{B}_t$ which can be characterized, analogously to that for semigroups, the following:

$$y \in \omega_{\tau,\Sigma}(B) \Longleftrightarrow \text{there are sequences } \{x_n\} \subset B,\ \{\sigma_n\} \subset \Sigma,\ \{t_n\} \subset \mathbb{R}_\tau \\ \text{such that } t_n \longrightarrow +\infty,\ U_{\sigma_n}(t_n,\tau)x_n \longrightarrow y\ (n \longrightarrow \infty). \tag{3.8}$$

We recall characterize the existence of the uniform attractor for a family of processes satisfying (3.8) in term of the concept of measure of noncompactness that was put forward first by Kuratowski (see [11, 12]).

Let $B \in \mathcal{B}(E)$. Its Kuratowski measure of noncompactness $\kappa(B)$ is defined by

$$\kappa(B) = \inf\{\delta > 0 \mid B \text{ admits a finite covering by sets of diameter} \leq \delta\}. \tag{3.9}$$

*Definition 3.3.* A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is said to be uniformly ($w.r.t.$ $\sigma \in \Sigma$) $\omega$-limit compact if for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(E)$ the set $B_t$ is bounded for every $t$ and $\lim_{t\to\infty}\kappa(B_t) = 0$.

We present now a method to verify the uniform ($w.r.t.$ $\sigma \in \Sigma$) $\omega$-limit compactness (see [13, 14]).

*Definition 3.4.* A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is said to satisfy uniformly ($w.r.t.$ $\sigma \in \Sigma$) Condition (C) if for any fixed $\tau \in \mathbb{R}$, $B \in \mathcal{B}(E)$ and $\varepsilon > 0$, there exist $t_0 = t(\tau, B, \varepsilon) \geq \tau$ and a finite-dimensional subspace $E_1$ of $E$ such that

(i) $P(\bigcup_{\sigma\in\Sigma}\bigcup_{t\geq t_0} U_\sigma(t,\tau)B)$ is bounded; and

(ii) $\|(I-P)(\bigcup_{\sigma\in\Sigma}\bigcup_{t\geq t_0} U_\sigma(t,\tau)x)\| \leq \varepsilon$, $\forall x \in B$,

where $P : E \to E_1$ is a bounded projector.

Therefore we have the following results.

**Theorem 3.5.** *Let $\Sigma$ be a metric space and let $\{T(t)\}$ be a continuous invariant semigroup $T(t)\Sigma = \Sigma$ on $\Sigma$. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ acting in $E$ is $(E \times \Sigma, E)$-(weakly) continuous and possesses the compact uniform ($w.r.t.$ $\sigma \in \Sigma$) attractor $A_\Sigma$ satisfying*

$$\mathcal{A}_\Sigma = \omega_{0,\Sigma}(B_0) = \omega_{\tau,\Sigma}(B_0) = \bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma(0), \quad \forall\tau \in \mathbb{R}, \tag{3.10}$$

*if it*

(i) *has a bounded uniformly ($w.r.t.$ $\sigma \in \Sigma$) absorbing set $B_0$, and*

(ii) *satisfies uniformly ($w.r.t.$ $\sigma \in \Sigma$) Condition (C)*

*Moreover, if $E$ is a uniformly convex Banach space then the converse is true.*

## 4. Uniform Attractor of Nonautonomous g-Navier-Stokes Equations

This section deals with the existence of the attractor for the two-dimensional nonautonomous g-Navier-Stokes equations with periodic boundary condition (see [1, 2]).

It is similar to autonomous case that we can establist the existence of solution of (2.14) by the standard Faedo-Galerkin method.

In [1, 2], the authors have shown that the semigroup $S(t) : H_g \to H_g$ $(t \geq 0)$ associated with the autonomous systems (2.14) possesses a global attractor. The main objective of this section is to prove that the nonautonomous system (2.14) has uniform attractors in $H_g$ and $V_g$.

To this end, we first state some the following results of existence and uniqueness of solutions of (2.14).

**Proposition 4.1.** *Let $f \in V'$ be given. Then for every $u_\tau \in H_g$ there exists a unique solution $u = u(t)$ on $[0, \infty)$ of (2.14), satisfying $u(\tau) = u_\tau$. Moreover,one has*

$$u(t) \in C\left[0, T; H_g\right) \cap L^2\left(0, T; V_g\right), \quad \forall T > 0. \tag{4.1}$$

*Finally, if $u_\tau \in V_g$, then*

$$u(t) \in C\left[0, T; V_g\right) \cap L^2\left(0, T; D\left(A_g\right)\right), \quad \forall T > 0. \tag{4.2}$$

*Proof.* The Proof of Proposition 4.1 is similar to autonomous in [1, 15]. □

Now we will write (2.14) in the operator form

$$\partial_t u = A_{\sigma(t)}(u), \quad u|_{t=\tau} = u_\tau, \tag{4.3}$$

where $\sigma(s) = f(x, s)$ is the symbol of (4.3). Thus, if $u_\tau \in H_g$, then problem (4.3) has a unique solution $u(t) \in C([0, T]; H_g) \cap L^2([0, T]; V_g)$. This implies that the process $\{U_\sigma(t, \tau)\}$ given by the formula $U_\sigma(t, \tau)u_\tau = u(t)$ is defined in $H_g$.

Now recall the following facts that can be found in [13].

*Definition 4.2.* A function $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}; E)$ is said to be *normal* if for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sup_{t \in \mathbb{R}} \int_t^{t+\eta} \|\varphi(s)\|_E^2 ds \leq \varepsilon. \tag{4.4}$$

*Remark 4.3.* Obviously, $L^2_n(\mathbb{R}; E) \subset L^2_b(\mathbb{R}; E)$. Denote by $L^2_c(\mathbb{R}; E)$ the class of translation compact functions $f(s)$, $s \in \mathbb{R}$, whose family of $\mathscr{H}(f)$ is precompact in $L^2_{\mathrm{loc}}(\mathbb{R}; E)$. It is proved in [13] that $L^2_n(\mathbb{R}; E)$ and $L^2_c(\mathbb{R}; E)$ are closed subspaces of $L^2_b(\mathbb{R}; E)$, but the latter is a proper subset of the former (for further details see [13]).

We now define the *symbol space* $\mathscr{H}(\sigma_0)$ for (4.3). Let a fixed symbol $\sigma_0(s) = f_0(s) = f_0(\cdot, s)$ be normal functions in $L^2_{\mathrm{loc}}(\mathbb{R}; E)$; that is, the family of translation $\{f_0(s + h), h \in \mathbb{R}\}$

forms a normal function set in $L^2_{\text{loc}}([T_1, T_2]; E)$, where $[T_1, T_2]$ is an arbitrary interval of the time axis $\mathbb{R}$. Therefore

$$\mathscr{H}(\sigma_0) = \mathscr{H}(f_0) = [f_0(x, s+h) \mid h \in \mathbb{R}]_{L^2_{\text{loc}}(\mathbb{R};E)}. \tag{4.5}$$

Now, for any $f(x,t) \in \mathscr{H}(f_0)$, the problem (4.3) with $f$ instead of $f_0$ possesses a corresponding process $\{U_f(t,\tau)\}$ acting on $V_g$. As is proved in [10], the family $\{U_f(t,\tau) \mid f \in \mathscr{H}(f_0)\}$ of processes is $(V_g \times \mathscr{H}(f_0); V_g)$-continuous.

Let

$$\mathscr{K}_f = \{u_f(x,t) \text{ for } t \in \mathbb{R} \mid u_f(x,t) \text{ is solution of (4.3) satisfying } \|u_f(\cdot,t)\|_H \le M_f \ \forall t \in \mathbb{R}\} \tag{4.6}$$

be the so-called kernel of the process $\{U_f(t,\tau)\}$.

**Proposition 4.4.** *The process $\{U_f(t,\tau)\} : H_g \to H_g\,(V_g)$ associated with the* (4.3) *possesses absorbing sets*

$$\mathcal{B}_0 = \{u \in H_g \mid |u|_g \le \rho_0\}, \qquad \mathcal{B}_1 = \{u \in V_g \mid \|u\|_g \le \rho_1\} \tag{4.7}$$

*which absorb all bounded sets of $H_g$. Moreover $\mathcal{B}_0$ and $\mathcal{B}_1$ absorb all bounded sets of $H_g$ and $V_g$ in the norms of $H_g$ and $V_g$, respectively.*

*Proof.* The proof of Proposition 4.4 is similar to autonomous g-Navier-Stokes equation. We can obtain absorbing sets in $H_g$ and $V_g$ the following from [1] and the proof of the main results as follow. □

The main results in this section are as follows.

**Theorem 4.5.** *If $f_0(x,s)$ is normal function in $L^2_{loc}(\mathbb{R}; V_g')$, then the processes $\{U_{f_0}(t,\tau)\}$ corresponding to problem* (2.14) *possess compact uniform* (*w.r.t.* $\tau \in \mathbb{R}$) *attractor $\mathfrak{A}_0$ in $H_g$ which coincides with the uniform* (*w.r.t.* $f \in \mathscr{H}(f_0)$) *attractor $\mathfrak{A}_{\mathscr{H}(f_0)}$ of the family of processes $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$:*

$$\mathfrak{A}_0 = \mathfrak{A}_{\mathscr{H}(f_0)} = \omega_{0,\mathscr{H}(f_0)}(\mathcal{B}_0) = \bigcup_{f \in \mathscr{H}(f_0)} \mathscr{K}_f(0), \tag{4.8}$$

*where $\mathcal{B}_0$ is the uniformly* (*w.r.t.* $f \in \mathscr{H}_g(f_0)$) *absorbing set in $H_g$ and $\mathscr{K}_f$ is the kernel of the process $\{U_f(t,\tau)\}$. Furthermore, the kernel $\mathscr{K}_f$ is nonempty for all $f \in \mathscr{H}(f_0)$.*

*Proof.* As in the previous section, for fixed $N$, let $H_1$ be the subspace spanned by $w_1; \ldots; w_N$, and $H_2$ the orthogonal complement of $H_1$ in $H_g$. We write

$$u = u_1 + u_2; \quad u_1 \in H_1,\ u_2 \in H_2 \text{ for any } u \in H_g. \tag{4.9}$$

Now, we only have to verify Condition (C). Namely, we need to estimate $|u_2(t)|_2$, where $u(t) = u_1(t) + u_2(t)$ is a solution of (2.14) given in Proposition 4.1.

Multiplying (2.14) by $u_2$, we have

$$\left(\frac{du}{dt}, u_2\right)_g + (\nu A_g u, u_2)_g + (B(u,u), u_2)_g = (f, u_2)_g - (Ru, u_2)_g. \tag{4.10}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}|u_2|_g^2 + \nu\|u_g\|_g^2 \le \left|(B(u,u), u_2)_g\right| + \left|(f, u_2)_g\right| + (Ru, u_2)_g. \tag{4.11}$$

Since $b_g$ satisfies the following inequality (see [15]):

$$\left|b_g(u,v,w)\right| \le c|u|_g^{1/2}\|u\|_g^{1/2}\|v\|_g|w|_g^{1/2}\|w\|_g^{1/2}, \quad \forall u, v, w \in V_g, \tag{4.12}$$

thus,

$$\begin{aligned}\left|(B(u,u), u_2)_g\right| &\le c|u|_g^{1/2}\|u\|_g^{3/2}|u_2|_g^{1/2}\|u_2\|_g^{1/2} \\ &\le \frac{c}{\lambda_{m+1}}|u|_g^{1/2}\|u\|_g^{3/2}\|u_2\|_g \\ &\le \frac{\nu}{6}\|u_2\|_g^2 + c\rho_0\rho_1^3.\end{aligned} \tag{4.13}$$

Next, the Cauchy inequality,

$$\begin{aligned}\left|(Ru, u_2)_g\right| &= \left|\left(\frac{\nu}{g}(\nabla g \cdot \nabla)u, u_2\right)_g\right| \\ &\le \frac{\nu}{m_0}|\nabla g|_\infty\|u\|_g|u_2|_g \\ &\le \frac{\nu}{6}\|u_2\|_g^2 + \frac{3\nu\rho_1^2|\nabla g|_\infty^2}{2m_0^2\lambda_g\lambda_{m+1}}.\end{aligned} \tag{4.14}$$

Finally, we have

$$\left|(f, u_2)_g\right| \le |f|_{V_g'}\|u_2\| \le \frac{\nu}{6}\|u_2\|_g^2 + \frac{3}{2\nu}|f|_{V_g'}^2. \tag{4.15}$$

Putting (4.13)–(4.15) together, there exist constant $M_1 = M_1(m_0, |\nabla g|_\infty, \rho_0, \rho_1)$ such that

$$\frac{1}{2}\frac{d}{dt}|u_2|_g^2 + \frac{1}{2}\nu\|u_2\|_g^2 \le \frac{3|f|_{V_g'}}{2\nu} + M_1 \cdot \tag{4.16}$$

Therefore, we deduce that

$$\frac{d}{dt}\left|u_2\right|_g^2 + \nu\lambda_{m+1}\left|u_2\right|_2^2 \le 2M_1 + \frac{3}{\nu}|f|_{V_g'}^2. \tag{4.17}$$

Here $M_1$ depends on $\lambda_{m+1}$, is not increasing as $\lambda_{m+1}$ increasing.

By the Gronwall inequality, the above inequality implies

$$\left|u_2(t)\right|_g^2 \le \left|u_2(t_0+1)\right|_2^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))} + \frac{2M_1}{\nu\lambda_{m+1}} + \frac{3}{\nu}\int_{t_0+1}^{t} e^{-\nu\lambda_{m+1}(t-s)}|f|_{V'}^2 ds. \tag{4.18}$$

Applying (4.4) for any $\varepsilon$

$$\frac{3}{\nu}\int_{t_0+1}^{t} e^{-\nu\lambda_{m+1}(t-s)}|f|_{V_g'}^2 ds < \frac{\varepsilon}{3}. \tag{4.19}$$

Using (2.9) and let $t_1 = t_0 + 1 + (1/\nu\lambda_{m+1})\ln(3\rho_0^2/\varepsilon)$, then $t \ge t_1$ implies

$$\begin{gathered}\frac{2M}{\nu\lambda_{m+1}} < \frac{\varepsilon}{3};\\ \left|u_2(t_0+1)\right|_2^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))} \le \rho_0^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))/2} < \frac{\varepsilon}{3}.\end{gathered} \tag{4.20}$$

Therefore, we deduce from (4.18) that

$$\left|u_2\right|_2^2 \le \varepsilon, \quad \forall t \ge t_1,\ f \in \mathscr{H}(f_0), \tag{4.21}$$

which indicates $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$ satisfying uniform ($w.r.t.$ $f \in \mathscr{H}(f_0)$) Condition (C) in $H_g$. Applying Theorem 3.5 the proof is complete. □

**Theorem 4.6.** *If $f_0(x,s)$ is normal function in $L^2_{loc}(\mathbb{R}; H_g)$, then the processes $\{U_{f_0}(t,\tau)\}$ corresponding to problem (2.14) possesses compact uniform ($w.r.t.$ $\tau \in \mathbb{R}$) attractor $\mathfrak{A}_1$ in $V_g$ which coincides with the uniform ($w.r.t.$ $f \in \mathscr{H}(f_0)$) attractor $\mathfrak{A}_{\mathscr{H}(f_0)}$ of the family of processes $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$:*

$$\mathfrak{A}_1 = \mathfrak{A}_{\mathscr{H}(f_0)} = \omega_{0,\mathscr{H}(f_0)}(\mathcal{B}_1) = \bigcup_{f\in\mathscr{H}(f_0)} \mathcal{K}_f(0), \tag{4.22}$$

*where $\mathcal{B}_1$ is the uniformly ($w.r.t.$ $f \in \mathscr{H}(f_0)$) absorbing set in $V_g$ and $\mathcal{K}_f$ is the kernel of the process $\{U_f(t,\tau)\}$. Furthermore, the kernel $\mathcal{K}_f$ is nonempty for all $f \in \mathscr{H}(f_0)$.*

*Proof.* Using Proposition 4.4, we have the family of processes $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$ corresponding to (4.3) possesses the uniformly ($w.r.t.$ $f \in \mathscr{H}(f_0)$) absorbing set in $V_g$.

Now we prove the existence of compact uniform ($w.r.t.$ $f \in \mathscr{H}(f_0)$) attractor in $V_g$ by applying the method established in Section 3, that is, we testify that the family of processes

$\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$ corresponding to (4.3) satisfies uniform ($w.r.t.$ $f \in \mathscr{H}(f_0)$) Condition (C).

Multiplying (2.14) by $A_g u_2(t)$, we have

$$\left(\frac{dv}{dt}, A_g u_2\right) + (\nu A_g u, A_g u_2) + \left(B_g(u,u), A_g u_2\right)_g = (f, A_g u_2) - \left(Ru, A_g u_2\right)_g. \tag{4.23}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|u_2\|_g^2 + \nu\left|A_g u_2\right|_g^2 \leq \left|\left(B_g(u,u), A_g u_2\right)_g\right| + \left|(f, A_g u_2)_g\right| + \left|(Ru, A_g u_2)_g\right|. \tag{4.24}$$

To estimate $(B_g(u,u), Au_2)_g$, we recall some inequalities [16]: for every $u, v \in D(A_g)$:

$$\left|B_g(u,v)\right| \leq c \begin{cases} |u|_g^{1/2}\|u\|_g^{1/2}\|v\|_g^{1/2}\left|A_g v\right|_g^{1/2}, \\ |u|_g^{1/2}\left|A_g u\right|_g^{1/2}\|v\|_g \end{cases} \tag{4.25}$$

(see [16])

$$|w|_{L^\infty(\Omega)^2} \leq c\|w\|_g\left(1 + \log\frac{\left|A_g w\right|}{\lambda_g\|w\|_g^2}\right)^{1/2} \tag{4.26}$$

from which we deduce that

$$\left|B_g(u,v)\right| \leq c|u|_{L^\infty(\Omega)}|\nabla v|_g|u|_g|\nabla v|_{L^\infty(\Omega)}, \tag{4.27}$$

and using, (4.26)

$$\left|B_g(u,v)\right| \leq c \begin{cases} \|u\|_g\|v\|_g\left(1 + \log\dfrac{|A_g u|^2}{\lambda_g\|w\|_g^2}\right)^{1/2}, \\ |u|_g\left|A_g v\right|_g\left(1 + \log\dfrac{\left|A_g^{3/2} v\right|^2}{\lambda_g\left\|A_g v\right\|_g^2}\right)^{1/2}. \end{cases} \tag{4.28}$$

Expanding and using Young's inequality, together with the first one of (4.28) and the second one of (4.25), we have

$$\begin{aligned} \left|\left(B_g(u,u), A_g u_2\right)\right| &\leq \left|\left(B_g(u_1, u_1 + u_2), A_g u_2\right)\right| + \left|\left(B_g(u_2, u_1 + u_2), A_g u_2\right)\right| \\ &\leq cL^{1/2}\left\|u_1\right\|_g\left|A_g u_2\right|_g\left(\left\|u_1\right\|_g + \left\|u_2\right\|_g\right) + c\left|u_2\right|_g^{1/2}\left|A_g u_2\right|_g^{3/2} \\ &\leq \frac{\nu}{6}\left|A_g u_2\right|_g^2 + \frac{c}{\nu}\rho_1^4 L + \frac{c}{\nu^3}\rho_0^2\rho_1^4, \quad t \geq t_0 + 1, \end{aligned} \tag{4.29}$$

where we use

$$|A_g u_1|_g^2 \le \lambda_m \|u_1\|_g^2 \tag{4.30}$$

and set

$$L = 1 + \log \frac{\lambda_{m+1}}{\lambda_g}. \tag{4.31}$$

Next, using the Cauchy inequality,

$$\begin{aligned} |(Ru, A_g u_2)_g| &= \left|\left(\frac{\nu}{g}(\nabla g \cdot \nabla)u, A_g u_2\right)_g\right| \\ &\le \frac{\nu}{m_0}|\nabla g|_\infty \|u\|_g |A_g u_2|_g \\ &\le \frac{\nu}{6}|A_g u_2|_g^2 + \frac{3\nu}{2}|\nabla g|_\infty^2 \rho_1^2. \end{aligned} \tag{4.32}$$

Finally, we estimate $|(f, A_g u_2)|$ by

$$|(f, A_g u_2)| \le |f|_g |A_g u_2|_2 \le \frac{\nu}{6}|A_g u_2|_g^2 + \frac{3}{2\nu}|f|_g^2. \tag{4.33}$$

Putting (4.29)–(4.33) together, there exists a constant $M_2$ such that

$$\frac{d}{dt}\|u_2\|_g^2 + \nu\lambda_{m+1}\|u_2\|_g^2 \le \frac{3}{\nu}|f|_g + M_2. \tag{4.34}$$

Here $M_2 = M_2(\rho_0, \rho_1, L, \nu, |\nabla g|)$ depends on $\lambda_{m+1}$, is not increasing as $\lambda_{m+1}$ increasing. Therefore, by the Gronwall inequality, the above inequality implies

$$\|u_2\|_g^2 \le \|u_2(t_0+1)\|_g^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))} + \frac{2M_2}{\nu\lambda_{m+1}} + \frac{3}{\nu}\int_{t_0+1}^t e^{-\nu\lambda_{m+1}(t-s)}|f|_g^2 ds. \tag{4.35}$$

Applying (4.4) for any $\varepsilon$

$$\frac{3}{\nu}\int_{t_0+1}^t e^{-\nu\lambda_{m+1}(t-s)}|f|_g^2 ds < \frac{\varepsilon}{3}. \tag{4.36}$$

Using (2.9) and let $t_1 = t_0 + 1 + (1/\nu\lambda_{m+1})\ln(3\rho_1^2/\varepsilon)$, then $t \geq t_1$ implies

$$\frac{2M_2}{\nu\lambda_{m+1}} < \frac{\varepsilon}{3};$$
$$\|u_2(t_0+1)\|_g^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))} \leq \rho_1^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))} < \frac{\varepsilon}{3}. \tag{4.37}$$

Therefore, we deduce from (4.35) that

$$\|u_2\|_g^2 \leq \varepsilon, \quad \forall t \geq t_1,\ f \in \mathscr{H}(f_0), \tag{4.38}$$

which indicates $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$ satisfying uniform $(w.t.r.\ f \in \mathscr{H}(f_0))$ Condition (C) in $V_g$. □

## 5. Dimension of the Uniform Attractor

In this section we estimate the fractal dimension (for definition see, e.g., [2, 10, 15]) of the kernel sections of the uniform attractors obtained in Section 4 by applying the methods in [17].

Process $\{U(t,\tau)\}$ is said to be uniformly quasidifferentiable on $\{\mathcal{K}(s)\}_{\tau\in\mathbb{R}}$, if there is a family of bounded linear operators $\{L(t,\tau;u) \mid u \in \mathcal{K}(s),\ t \geq \tau,\ \tau \in \mathbb{R}\}$, $L(t,\tau;u) : E \to E$ such that

$$\limsup_{\varepsilon\to 0\,\tau\in\mathbb{R}} \sup_{\substack{u,v\in\mathcal{K}(s)\\ 0<|u-v|\leq\varepsilon}} \frac{|U(t,\tau)v - U(t,\tau)u - L(t,\tau;u)(v-u)|}{|v-u|} = 0. \tag{5.1}$$

We want to estimate the fractal dimension of the kernel sections $\mathcal{K}(s)$ of the process $\{U(t,\tau)\}$ generated by the abstract evolutionary (2.14). Assume that $\{L(t,\tau;u)\}$ is generated by the variational equation corresponding to (2.14)

$$\partial_t w = F'(u,t)w, \quad w|_{t=\tau} = w_\tau \in E,\ t \geq \tau,\ \tau \in \mathbb{R}, \tag{5.2}$$

that is, $L(t,\tau;u_\tau)w_\tau = w(t)$ is the solution of (5.2), and $u(t) = U(t,\tau)u_\tau$ is the solution of (2.14) with initial value $u_\tau \in \mathcal{K}(\tau)$. For natural number $j \in N$, we set

$$\tilde{q}_j = \lim_{T\to+\infty} \sup_{\tau\in R} \sup_{u_\tau\in\mathcal{K}(\tau)} \left(\frac{1}{T}\int_\tau^{\tau+T} \operatorname{Tr}(F'(u(s),s))\,ds\right), \tag{5.3}$$

where Tr is trace of the operator.

We will need the following Theorem VIII.3.1 in [10] and [2].

**Theorem 5.1.** *Under the assumptions above, let us suppose that $U_{\tau\in\mathbb{R}}\mathcal{K}(\tau)$ is relatively compact in E, and there exists $q_j$, $j = 1, 2, \ldots$, such that*

$$\begin{gathered}\widetilde{q}_j \leq q_j, \quad \text{for any } j \geq 1, \\ q_{n_0} \geq 0, \quad q_{n_0+1} < 0, \quad \text{for some } n_0 \geq 1, \\ q_j \leq q_{n_0} + \left(q_{n_0} - q_{n_0+1}\right)\left(n_0 - j\right), \quad \forall j = 1, 2, \ldots.\end{gathered} \tag{5.4}$$

*Then,*

$$d_F(\mathcal{K}(\tau)) \leq d_0 := n_0 + \frac{q_{n_0}}{q_{n_0} - q_{n_0+1}}, \quad \forall \tau \in \mathbb{R}. \tag{5.5}$$

We now consider (2.14) with $f \in L_n^2(\mathbb{R}; V_g')$. The equations possess a compact uniform ($w.r.t.$ $f \in \mathscr{H}(f)$) attractor $\mathfrak{A}_{\mathscr{H}(f)}$ and $\bigcup_{\tau\in\mathbb{R}}\mathcal{K}_f(\tau) \subset \mathfrak{A}_{\mathscr{H}(f)}$. By [2, 10, 15], we know that the associated process $\{U_f(t,\tau)\}$ is uniformly quasidifferentiable on $\{\mathcal{K}_f(\tau)\}_{\tau\in\mathbb{R}}$ and the quasidifferential is Hölder-continuous with respect to $u_\tau \in \mathcal{K}_f(\tau)$. The corresponding variational equation is

$$\partial_t w = -\nu A_g u - B_g u - \nu R u + P_g f \equiv F'(u(t), t)w, \quad w|_{t=\tau} = w_\tau \in E, \ \tau \in \mathbb{R}. \tag{5.6}$$

We have the main results in this section.

**Theorem 5.2.** *Suppose that $f(t)$ satisfies the assumptions of Theorem 4.5. Then, if $\gamma = 1 - (2|\nabla g|_\infty)/(m_0\lambda_g^{1/2}) > 0$, the Uniform attractor $\mathfrak{A}_0$ defined by (4.8) satisfies*

$$d_F(\mathfrak{A}_0) \leq \sqrt{\frac{\beta}{\alpha}}, \tag{5.7}$$

*where*

$$\begin{aligned}\alpha &= \frac{c_2 \nu m_0 \lambda_1' \gamma}{2M_0}, \\ \beta &= \frac{c_1 d_1}{2\nu^3 m_0 \gamma} \sup_{\substack{\varphi_j \in H_g \, |\varphi_j| \leq 1 \\ j=1,2,\ldots,m}} \frac{1}{T}\int_\tau^{\tau+T} \|f(s)\|_{V_g'}^2 ds,\end{aligned} \tag{5.8}$$

*the constant $c_1$, $c_2$ of (3.29) and (3.32) of Chapter VI in [15] and [2], $\lambda_1'$ is the first eigenvalue of the Stokes operator and $d_1 = |\nabla g|_\infty^2/4m_0 + |\nabla g|_\infty + M_0$.*

*Proof.* With Theorem 4.5 at our disposal we may apply the abstract framework in [2, 10, 15, 17].

For $\xi_1, \xi_2, \ldots, \xi_m \in H_g$, let $v_j(t) = L(t, u_\tau) \cdot \xi_j$, where $u_\tau \in H_g$. Let $\{\varphi_j(s); j = 1, 2, \ldots, m\}$ be an orthonormal basis for span $\{v_j; j = 1, 2, \ldots, m\}$. Since $v_j \in V_g$ almost everywhere $s \geq \tau$, we can also assume that $\varphi_j(s) \in V_g$ almost everywhere $s \geq \tau$. Then, similar to the Proof process of Theorems 4.5 and 4.6, we may obtain

$$\begin{aligned}\sum_{i=1}^{m}\left(F'(U(s,\tau)u_\tau,s)\varphi_i,\varphi_i\right)_g &= -\nu\sum_{i=1}^{m}\|\varphi_j\|_g^2 - \sum_{i=1}^{m} b_g\left(\varphi_j, U(s,\tau)u_\tau,\varphi_j\right) \\ &\quad - \sum_{i=1}^{m}\left(\frac{\nu}{g}(\nabla g\cdot\nabla)\varphi_j,\varphi_j\right)_g,\end{aligned} \tag{5.9}$$

almost everywhere $s \geq \tau$. From this equality, and in particular using the Schwarz and Lieb-Thirring inequality (see [2, 10, 15, 17]), one obtains

$$\begin{aligned}\sum_{i=1}^{m}\|\varphi\|_g^2 &\geq \lambda_1+\cdots+\lambda_m \geq \frac{m_0}{M_0}\left(\lambda_1'+\cdots+\lambda_m'\right) \geq \frac{m_0}{M_0}c_2\lambda_1' m^2, \\ \mathrm{Tr}_j\left(F'(U(s,\tau)u_\tau,s\right)_g &\leq -\nu\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\sum_{i=1}^{m}\|\varphi_j\|_g^2 + \|U(s,\tau)u_\tau\|_g\left(\frac{c_1 d_1}{m_0}\sum_{i=1}^{m}\|\varphi_j\|_g^2\right)^{1/2} \\ &\leq -\frac{\nu}{2}\left(1-\frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\sum_{i=1}^{m}\|\varphi_j\|_g^2 + \frac{c_1 d_1}{2\nu m_0}\|U(s,\tau)u_\tau\|_g^2 \\ &\leq -\frac{\nu m_0}{2M_0}\left(1-\frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)c_2\lambda_1' m^2 + \frac{c_1 d_1}{2\nu m_0}\|U(s,\tau)u_\tau\|_g^2,\end{aligned} \tag{5.10}$$

on the other hand, we can deduce (2.14) that

$$\frac{d}{dt}\left|U(s,\tau)u_\tau\right|_g^2 + \nu\|U(s,\tau)u_\tau\|_g^2 \leq \frac{\|f\|_{V_g'}^2}{\nu} + \frac{2\nu}{m_0\lambda_g^{1/2}}|\nabla g|_\infty\|U(s,\tau)u_\tau\|_g^2 \tag{5.11}$$

for $\lambda_g = 4\pi^2 m_0/M_0$, and then

$$\int_\tau^t \|U(s,\tau)u_\tau\|_g^2 ds \leq \left(\frac{1}{\nu^2}\int_\tau^t\|f(s)\|_{V_g'}^2 ds + \frac{|u_\tau|^2}{\nu}\right)\left(1-\frac{2|\nabla g|_\infty}{m_0\lambda_g^{1/2}}\right)^{-1}, \quad t \geq \tau. \tag{5.12}$$

Now we define

$$q_m = \sup_{\substack{\varphi_j\in H_g\, |\varphi_j|\leq 1\\ j=1,2,\ldots,m}}\left(\frac{1}{T}\int_\tau^{\tau+T}\mathrm{Tr}_j\left(F'\left(U(s,\tau)u_\tau,s\right)ds\right)\right)_g, \tag{5.13}$$

Using Theorem 5.1, we have

$$\begin{aligned}
\widetilde{q}_m &\leq -\frac{\nu m_0}{2M_0}\left(1-\frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)c_2\lambda_1' m^2 + \frac{c_1 d_1}{2\nu m_0}\left(\sup_{\substack{\varphi_j\in H_g\ |\varphi_j|\leq 1\\ j=1,2,\ldots,m}}\left(\frac{1}{T}\int_\tau^{\tau+T}\left\|U(s,\tau)u_\tau\right\|_g^2 ds\right)\right)\\
&\leq -\frac{\nu m_0}{2M_0}\left(1-\frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)c_2\lambda_1' m^2\\
&\quad + \frac{c_1 d_1}{2\nu m_0}\left(\frac{1}{\nu^2}\sup_{\substack{\varphi_j\in H_g\ |\varphi_j|\leq 1\\ j=1,2,\ldots,m}}\left(\frac{1}{T}\int_\tau^{\tau+T}\|f(s)\|_{V_g'}^2 ds + \frac{|u_\tau|^2}{\nu T}\right)\left(1-\frac{2|\nabla g|_\infty}{m_0\lambda_g^{1/2}}\right)^{-1}\right),\\
q_m &= \limsup_{T\to\infty}\widetilde{q}_m \leq -\alpha m^2+\beta,
\end{aligned} \tag{5.14}$$

Hence

$$\dim_F \mathcal{A}_0(\tau) \leq \sqrt{\frac{\beta}{\alpha}}. \tag{5.15}$$

□


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## References


[1] J. Roh, "Dynamics of the $g$-Navier-Stokes equations," *Journal of Differential Equations*, vol. 211, no. 2, pp. 452–484, 2005.

[2] M. Kwak, H. Kwean, and J. Roh, "The dimension of attractor of the 2D $g$-Navier-Stokes equations," *Journal of Mathematical Analysis and Applications*, vol. 315, no. 2, pp. 436–461, 2006.

[3] O. A. Ladyzhenskaya, "Über ein dynamisches System, das durch die Navier-Stokesschen Gleichungen erzeugt wird," *Zapiskii of Nauchnich, Seminarov LOMI*, vol. 27, pp. 91–115, 1972.

[4] O. A. Ladyzhenskaya, "A dynamical system generated by the Navier-Stokes equations," *Journal of Mathematical Sciences*, vol. 3, no. 4, pp. 458–479, 1975.

[5] C. Foiaş and R. Temam, "Finite parameter approximative structure of actual flows," in *Nonlinear Problems: Present and Future*, A. R. Bishop, D. K. Campbell, and B. Nicolaenko, Eds., vol. 61 of *North-Holland Mathematics Studies*, pp. 317–327, North-Holland, Amsterdam, The Netherlands, 1982.

[6] J. Mallet-Paret, "Negatively invariant sets of compact maps and an extension of a theorem of Cartwright," *Journal of Differential Equations*, vol. 22, no. 2, pp. 331–348, 1976.

[7] P. Constantin and C. Foiaş, *Navier-Stokes Equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, Ill, USA, 1988.

[8] R. Temam, *Navier-Stokes Equations*, vol. 2 of *Studies in Mathematics and Its Applications*, North-Holland, Amsterdam, The Netherlands, 3rd edition, 1984.

[9] A. Haraux, *Systèmes Dynamiques Dissipatifs et Applications*, vol. 17 of *Recherches en Mathématiques Appliquées*, Masson, Paris, France, 1991.

[10] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, vol. 49 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, USA, 2002.

[11] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, vol. 25 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 1988.

[12] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations*, vol. 143 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2002.

[13] S. S. Lu, H. Q. Wu, and C. K. Zhong, "Attractors for non-autonomous 2D Navier-Stokes equations with normal external forces," *Discrete and Continuous Dynamical Systems*, vol. 8, pp. 585–597, 2005.

[14] Q. Ma, S. Wang, and C. Zhong, "Necessary and sufficient conditions for the existence of global attractors for semigroups and applications," *Indiana University Mathematics Journal*, vol. 51, no. 6, pp. 1541–1559, 2002.

[15] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, vol. 68 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1997.

[16] C. Foiaş, O. P. Manley, R. Temam, and Y. M. Treve, "Asymptotic analysis of the Navier-Stokes equations," *Physica D*, vol. 9, no. 1-2, pp. 157–188, 1983.

[17] V. V. Chepyzhov and A. A. Ilyin, "On the fractal dimension of invariant sets: applications to Navier-Stokes equations," *Discrete and Continuous Dynamical Systems. Series A*, vol. 10, no. 1-2, pp. 117–135, 2004.