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A Closed-form Model-free Implied Volatility Formula through Delta Sequences

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ABSTRACT

In this paper, we derive a closed-form explicit model-free formula for the (Black-Scholes) implied volatility. The method is based on the novel use of the Dirac Delta function, corresponding delta sequences, and the change of variable technique. The formula is expressed through either a limit or as an infinite series of elementary functions, and we establish that the proposed formula converges to the true implied volatility value. In numerical experiments, we verify the convergence of the formula, and consider several benchmark cases, for which the data generating processes are respectively the stochastic volatility inspired (SVI) model, the stochastic alpha beta rho (SABR) model. We also establish an explicit formula for the implied volatility expressed directly in terms of respective model parameters, and use the Heston model to illustrate this idea. The delta sequence and change of variable technique that we develop is of independent interest and can be used to solve inverse problems arising in other applications.

JEL Classification: G12, G13, G14, C58

Keywords: Dirac Delta function, delta sequence, implied volatility, model-free, SVI, SABR, Heston

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1 Introduction

Options are financial derivatives that are used for investing, speculation and hedging purposes. Contracts similar to options have been used for millennia and related *real option* contracts have found applications in many areas of decision making, c.f. [37]. Options provide forward-looking information about security price distributions, as they are market-based estimates of the expectation and views of individual investors who participate in the option market. Model-free methods have been developed to extract information from observable option prices which in turn may be used to predict market returns, c.f. [24], and guiding portfolio choices, c.f. [11]. Options markets are among the most mature and liquid markets and exist for several asset classes including equity, fixed income and foreign exchange markets. Moreover, options trading volumes continue to grow; as of September 2019 there was approximately \$72.8 trillion in open interest across all major global exchange traded options markets and \$12.8 trillion of notional outstanding in analogous over-the-counter markets according to the Bank of International Settlements [5]. This has generated continued interests in developing efficient option pricing models and methods in the academic literature. The option pricing theory is one of the fundamental pillars of finance, with one notable example of the Nobel prize winning Black-Scholes-Merton option pricing theory, which assumes that the underlying risky asset follows a geometric Brownian motion process. There are continued interests in theoretical developments of stochastic models suitable for options pricing beyond the Black-Scholes model, such as jump diffusion models, [26], exponential Lévy processes, [6], stochastic volatility models, [22], stochastic local volatility models, [19], and rough volatility models [1, 13].

The fact that asset price distributions are seldom lognormal in reality has resulted in the development of many extensions of the Black-Scholes-Merton framework. Although the underlying model assumption of the Black-Scholes-Merton framework lacks fidelity, the associated European call option pricing formula still provides a useful mapping device when applied in the reverse direction. More specifically, we map the market-observed option price to a single positive real number, named the *implied volatility*, by inverting the Black-Scholes formula. This (Black-Scholes) implied volatility is a fundamental building block in computational finance, and is used for quoting, hedging and model calibration of options. Typically in options trading, prices are quoted in terms of implied volatilities rather than absolute prices. For example, such quoting conventions are common for equity, index, and commodity options whereas foreign exchange options are quoted in terms of the Black-Scholes-Merton delta. This allows for a convenient comparison between options with different underlyings across different markets, e,g, equity, foreign exchange, or commodities.

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Current industry practice involves numerically inverting the Black-Scholes-Merton formula to compute the implied volatility. Given the large number of distinct actively traded options, this procedure involves a large number of inversions. For example, in the case of high frequency trading where a considerable number of option prices need to be converted to implied volatilities in real-time, this issue becomes especially prominent, c.f. [3]. The implied volatility also plays an important role in model calibration as it often appears in the objective function of the related optimization problems. For example, in the calibration of the Heston model, in practice one typically seeks to minimize the *implied volatility root mean-squared error (IVRMSE)*, see for example [8, 2]. Thus it is appealing to construct a closed-form explicit expression for the implied volatility with the purpose of applying gradient-based optimization routines.

It has long been believed that there is no explicit closed-form formula for the Black-Scholes implied volatility, see for example the discussions in [16], where the author shows that the implied volatility does not belong to a certain class of D-finite functions. Thus, in practice, the implied volatility is usually determined from an iterative numerical root solving algorithm, such as the Dekker-Brent (i.e. bisection) method or a gradient style method such as the Newton-Raphson technique. Although these existing numerical methods are simple to apply and highly efficient given that there are many robust packages developed across different programming languages, it is still of interest to develop the explicit closed-form formula, given its theoretical elegance. This issue constitutes the primary motivation for this work, which is to propose an explicit link between the implied volatility and market-observable option prices. The proposed formula is model-free in nature, and can be used to invert the option price surface to obtain the implied volatility surface.

In previous literature, there are mainly two lines of research: the first focuses on improving the numerical computation of the implied volatility. The central issue is the choice of the initial value (see [29]) and how to speed up the convergence of the iterations (see [23]). The second strand of literature proposes to develop analytical (non-iterative) approximations to the implied volatility, and present a stand-alone explicit formula for the implied volatility without any intermediate recursive calculations. Within the second category, there are three representative approaches:

1. Series expansions, of which most are *asymptotic* in nature and only hold accurate in certain limiting regions. Such techniques comprise the largest body of literature, and there are many distinct methods developed along these lines. Non-exhaustive examples include: expansions based on fast-varying and slow-varying volatility, see [14]; short-maturity expansions, see [12, 25, 30, 32]; singular perturbation theory and

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partial differential equation expansion techniques, see [4]; series expansion based on the Lagrange inversion theorem, see [39]. Our proposed formula can also be written as closed-form series (see equation (14)), and it distinguishes from previous literature in that it is exact and non-asymptotic, i.e. it works for all possible parameter ranges.

2. Interpolation methods, and the range of possible implied volatilities are first classified into non-overlapping *interpolating regions*, and then different interpolation strategies are employed on these different regions. Examples include: rational function approximation, see [28]; the Chebyshev interpolation method, see [17].

3. Characterizations through partial differential equations (PDEs). This is a relatively more recent approach, which derives the PDE satisfied by the implied volatility as a function of strike and time to maturity. Note that the PDE is still an *implicit* characterization, but this method can be combined with the existing knowledge on the approximation of solutions to PDEs (e.g. series expansions of solutions of PDEs), and yields explicit approximate expressions. For related literature, c.f. [7, 31].

In order to achieve an explicit closed-form formula, we take a distinct approach from the current literature. The method we utilize, i.e. the theory of delta sequences, is of independent interest as it may extend to other potential applications. More specifically, we rely heavily on the theory of the Dirac Delta function and its associated delta sequence. To the best of the authors' knowledge, this is the first time an explicit non-asymptotic and exact formula for the implied volatility is obtained. Note that the delta sequence method has been previously applied in the nonparametric estimation of probability densities, c.f. [35, 38]. However, this is the first time that the method of the delta sequence is applied to the study of implied volatility, and we believe that this delta sequence method can find other potential applications in finance or related contextual areas such as operations research, where inverse problems are often encountered.

The contributions of our paper are as follows:

1. To the best of our knowledge, we offer the first construction of an exact explicit closed-form non-asymptotic formula for the Black-Scholes implied volatility. This is achieved through the novel application of the properties of the Dirac Delta function, delta sequence, and the change of variable technique. The method of derivation is novel and of independent interest with the potential to be applied to other inverse problems arising in related areas.

2. The implied volatility formula is fully explicit and does not involve iterative numerical procedures, recursions, or asymptotic expansions, and works for all possible model

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parameter ranges. Under some technical conditions, the formula is proven to converge to the true implied volatility. The formula can be written either as a limit (in (11)) or as a closed-form infinite series (in (14)) in terms of elementary functions.

The remainder of the paper is organized as follows. Section 2 presents the main theoretical results on a model-free closed-form formula for the implied volatility in terms of market-observable option prices. The corresponding formula under Black's formulation is also presented. The theoretical convergence of our formula to the true implied volatility is established. Section 3 discusses the numerical implementation, and benchmarks our formula to existing results in the literature. The numerical convergence of the proposed formula is verified. We also consider data generating processes based on some popular stochastic models in the literature, and benchmark our formula. Section 4 concludes the paper.

2 Main Results

We first review the definition of the Black-Scholes implied volatility as well as associated bounds on its value which are determined as a function of model and option parameters. We then review delta sequences which offer multiple ways to approximate the Dirac delta function. Next, we prove a relationship that demonstrates how to compute implied volatility through delta sequences and then develop an explicit formula for the Black-Scholes implied volatility in the case of a Gaussian delta series. This analysis is then repeated for Black's formulation in order to develop a framework for fixed income examples.

2.1 Black-Scholes implied volatility

In the Black-Scholes Merton framework, under the risk-neutral measure, the risky asset S_t is assumed to follow a geometric Brownian motion:

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t.$$

We recall the Black-Scholes formula $f_{BS}(\sigma)$ for the European option price, as a function

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of the unobservable volatility σ is given by:

$$f_{BS}(\sigma) \equiv f_{BS}(\sigma, r, q, K, S_0, T) = \begin{cases} S_0 e^{-qT}\mathcal{N}(d_1(\sigma)) - Ke^{-rT}\mathcal{N}(d_2(\sigma)), & \text{call option}, \\ Ke^{-rT}\mathcal{N}(-d_2(\sigma)) - S_0 e^{-qT}\mathcal{N}(-d_1(\sigma)), & \text{put option}, \end{cases} \tag{1}$$

where $r \in \mathbb{R}^+$ is the risk-free interest rate, $q \in \mathbb{R}^+$ is the dividend (convenience) yield, and

$$d_1(\sigma) := \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2(\sigma) := d_1(\sigma) - \sigma\sqrt{T}, \tag{2}$$

with $\mathcal{N}(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. We can derive the Vega formula by differentiating with respect to volatility holding all other parameters constant by

$$f'_{BS}(\sigma) = S_0\sqrt{T}\mathcal{N}'(d_1(\sigma)) > 0, \tag{3}$$

hence $f_{BS}(\sigma)$ is monotonically increasing in σ. Given an observed market price C of an European option, we aim to solve for the root of the following equation

$$f_{BS}(\sigma) = C. \tag{4}$$

This root exists and is unique due to the monotonicity of $f_{BS}(\cdot)$, and we denote it by $\sigma_{BS}(C) = f_{BS}^{-1}(C)$, and call it the (Black-Scholes) implied volatility. There are many methods proposed in the literature to compute the implied volatility, but it remains an open problem to provide a closed-form solution to equation (4). Closed-form *approximate* solutions have been proposed, and we note that they either do not converge to the true implied volatility or they only work for certain parameters falling within asymptotic regimes. In this section, we shall utilize the distributional property of the Dirac Delta function to provide a model-free closed-form solution to the implied volatility. To do this, we have to determine the proper integration regions and thus need to bound the implied volatility. We first recall the following model-independent bounds for the implied volatility, which are derived in [36].

Lemma 1. *(Implied Volatility Bounds, [36]) Let*

$$c := \frac{C}{e^{-qT}S_0}, \quad k := \ln\frac{K}{S_0 e^{(r-q)T}}.$$

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Assume that $(1-e^k)^+ \leq c < 1$. For $k \geq 0$ the following inequalities hold:

$$\frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{2}\right) \leq \sigma(C) \leq \frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{1+e^k}\right),$$

and for $k < 0$ we have

$$\frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{2e^k}\right) \leq \sigma(C) \leq \frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{1+e^k}\right).$$

From Lemma 1, there exist two finite bounds $\sigma_l, \sigma_u \in (0, \infty)$ such that

$$\sigma_l \leq \sigma(C) \leq \sigma_u,$$

and more specifically, we have

$$\sigma_u := \frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{1+e^k}\right), \quad \sigma_l := \begin{cases} \frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{2}\right) & \text{if } k \geq 0, \\ \frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{2e^k}\right) & \text{if } k < 0. \end{cases} \tag{5}$$

This result will be helpful to us for restricting the integration region when we calculate the coefficients of our proposed formula.

2.2 Closed-Form Implied Volatility via delta sequences

For an open subset $\Omega \subset \mathbb{R}$, let $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions on Ω having compact support.

Definition 1. *A sequence of functions $\delta_\varepsilon \in L^\infty(\Omega \times \Omega)$ is said to be a delta sequence on Ω if for each $\varphi \in C_0^\infty(\Omega)$ and $x \in \Omega$, the following holds*

$$\lim_{\varepsilon \to 0} \int_\Omega \delta_\varepsilon(x,y)\varphi(y)dy = \varphi(x).$$

Table 1 lists several common examples of delta sequences which can be applied in our framework. For convenience, we will utilize the following Gaussian form

$$\delta_\varepsilon(x,y) = \frac{1}{2\sqrt{\pi\varepsilon}}e^{-\frac{(x-y)^2}{4\varepsilon}}, \tag{6}$$

which resembles the probability density function (pdf) of a normal random variable.

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Types	delta sequences $\delta_\varepsilon(x,y)$
Normal density	$\frac{1}{2\sqrt{\pi\varepsilon}}e^{-\frac{(x-y)^2}{4\varepsilon}}$
Lorentzian type	$\frac{1}{\pi}\frac{\varepsilon}{(x-y)^2+\varepsilon^2}$
Fourier integral	$\frac{1}{2\pi}\int_{-1/\varepsilon}^{1/\varepsilon} e^{i(x-y)t}dt$
Trigonometric function	$\frac{1}{2\pi}\frac{\sin\left[\left(\frac{1}{\varepsilon}+\frac{1}{2}\right)(x-y)\right]}{\sin\left(\frac{1}{2}(x-y)\right)}$

Table 1: Different types of delta sequences.

Next, we recall the following definition from [35]:

Definition 2. *A delta sequence $\{\delta_\varepsilon\}$ on $\mathbb{R}$ is said to be of positive type if $\delta_\varepsilon \geq 0$ holds and for each $x \in \mathbb{R}$,*

1. $\displaystyle\int_{\mathbb{R}} \delta_\varepsilon(x,y)dy = 1,$
2. $\sup_{r>0}\left\{\displaystyle\int_{|x-y|>r} \delta_\varepsilon(x,y)dy\right\} = \mathcal{O}(\varepsilon),$
3. $||\delta_\varepsilon(x,.)||_\infty = \mathcal{O}(\varepsilon),$
4. *For each* $\eta > 0$, $\sup\{\delta_\varepsilon(x,y)||x-y| > \eta\} \to 0.$

We have the following results regarding the delta sequence when applied to $f_{BS}^{-1}(\sigma)$.

Proposition 3. *Let $C_l = f(\sigma_l)$ and $C_u = f(\sigma_u)$, where σ_l and σ_u are defined in Lemma 1.*

1. $f_{BS}^{-1}(C) \in L^p(C_l, C_u)$ *for any* $p \geq 0$.
2. $|f_{BS}^{-1}(C_1) - f_{BS}^{-1}(C_0)| \leq M_1|C_1 - C_0|^\gamma$ *for some* $M_1 > 0$, *and for any* $\gamma \in (0,1]$.
3. *For a positive type $\{\delta_\varepsilon\}$ sequence,*

$$\Big|\int_{C_l}^{C_u} \delta_\varepsilon(x,y)f_{BS}^{-1}(y)dy - f_{BS}^{-1}(x)\Big| = \mathcal{O}(\varepsilon).$$

This, in particular, implies that

$$f_{BS}^{-1}(x) = \lim_{\varepsilon\to 0}\int_{C_l}^{C_u} \delta_\varepsilon(x,y)f_{BS}^{-1}(y)dy.$$

which establishes the convergence of $\int_{C_l}^{C_u} \delta_\varepsilon(x,y)f_{BS}^{-1}(y)dy$ to the true implied volatility value when we substitute $x = C$, with C being the market-observable price.

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Proof. For the first part, from the change of variable formula and using the fact $f'_{BS}(\sigma) = S_0\sqrt{T}\mathcal{N}'(d_1(\sigma))$, we have

$$\begin{aligned}\int_{C_l}^{C_u} [f_{BS}^{-1}(C)]^p dC &= \int_{\sigma_l}^{\sigma_u} \sigma^p f'_{BS}(\sigma) d\sigma \\ &= \frac{S_0\sqrt{T}}{\sqrt{2\pi}} \int_{\sigma_l}^{\sigma_u} \sigma^p \exp\left(-\frac{1}{2\sigma\sqrt{T}}(\ln(S_0/K) + (r - q + \sigma^2/2)T)\right) d\sigma < \infty.\end{aligned}$$

Next, similar to [36], consider the following change of variables:

$$c = \frac{C}{S_0 e^{-qT}}, \quad k = \ln\left(\frac{K}{S_0 e^{(r-q)T}}\right).$$

Let $C_{BS} : \mathbb{R} \times [0, \infty) \to [0, 1)$ be defined as

$$C_{BS}(k, \sigma) = \mathcal{N}\left(-\frac{k}{\sigma} + \frac{\sigma}{2}\right) - e^{-k}\mathcal{N}\left(-\frac{k}{\sigma} - \frac{\sigma}{2}\right).$$

It is easy to see that

$$f_{BS}(\sigma, r, K, S_0, T) = S_0 e^{-qT} C_{BS}\left(\ln\left(\frac{K}{S_0 e^{(r-q)}T}\right), \sigma\sqrt{T}\right).$$

Let

$$\Sigma(c, \cdot) = C_{BS}^{-1}(c, \cdot).$$

This is equivalent to

$$c = C_{BS}(\Sigma(c)) = \mathcal{N}\left(-\frac{k}{\Sigma(c)} + \frac{\Sigma(c)}{2}\right) - e^{-k}\mathcal{N}\left(-\frac{k}{\Sigma(c)} - \frac{\Sigma(c)}{2}\right). \tag{7}$$

Recall the definition of cumulative normal distribution function $\mathcal{N}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$, and as in [31] it can be checked that

$$\mathcal{N}'\left(-\frac{k}{x} + \frac{x}{2}\right) = e^k \mathcal{N}'\left(-\frac{k}{x} - \frac{x}{2}\right) \quad \text{and} \quad \mathcal{N}''(x) = -x\mathcal{N}'(x). \tag{8}$$

Taking derivative of (7) with respect to c on both sides and using the relation (8), we have

$$1 = e^k \mathcal{N}'\left(-\frac{k}{\Sigma(c)} - \frac{\Sigma(c)}{2}\right) \frac{\partial \Sigma(c)}{\partial c}.$$

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This implies that

$$\frac{\partial \Sigma(c)}{\partial c} = \sqrt{2\pi} e^{\frac{1}{2}\left(\frac{\Sigma(c)}{2} - \frac{k}{\Sigma(c)}\right)^2} < \infty.$$

Therefore $\Sigma(c)$ is Lipschitz in c which implies $f^{-1}(C)$ is Lipschitz in C. As a result, there exists $M > 0$ such that

$$|f_{BS}^{-1}(C_1) - f_{BS}^{-1}(C_0)| \leq M|C_1 - C_0|.$$

Let $\gamma \in (0,1)$, we have

$$\begin{aligned}
|f_{BS}^{-1}(C_1) - f_{BS}^{-1}(C_0)| &\leq M|C_1 - C_0| \\
&\leq M|C_1 - C_0|^{1-\gamma}|C_1 - C_0|^{\gamma} \\
&\leq M_1|C_1 - C_0|^{\gamma},
\end{aligned}$$

where $M_1 := M \max\limits_{C_1, C_1 \in [C_l, C_u]} \{|C_1 - C_0|^{1-\gamma}|\}$.

Next, we have

$$\begin{aligned}
\Big| \int_{C_l}^{C_u} \delta_\varepsilon(x,y) f_{BS}^{-1}(y) dy - f_{BS}^{-1}(x) \Big| &= \Big| \int_{C_l}^{C_u} \delta_\varepsilon(x,y) \left(f_{BS}^{-1}(y) - f_{BS}^{-1}(x)\right) dy \Big| \\
&= \int_{C_l}^{C_u} |\delta_\varepsilon(x,y) \left(f_{BS}^{-1}(y) - f_{BS}^{-1}(x)\right)| dy \\
&\leq \int_{C_l}^{C_u} |\delta_\varepsilon(x,y) M_1 |x-y| dy \\
&\leq \int_{C_l}^{C_u} ||\delta_\varepsilon(x,.)||_\infty M_1 |x-y| dy \\
&= ||\delta_\varepsilon(x,.)||_\infty \int_{C_l}^{C_u} M_1 |x-y| dy \\
&\leq \frac{1}{2} \mathcal{O}(\varepsilon) M_1 |C_u - C_l|^2 = \mathcal{O}(\varepsilon).
\end{aligned}$$

This completes the proof. □

Remark 4. *Note that the expression* $\int_{C_l}^{C_u} \delta_\varepsilon(x,y) f_{BS}^{-1}(y) dy$ *is still unknown to us given that we do not know* $f_{BS}^{-1}(\cdot)$. *However, from the change of variable technique, it is possible to rewrite this integral into a fully explicit closed-form formula, which is given by the following Proposition 5.*

Recall that we have a model-free lower bound C_l and a model-free upper bound C_u for the call option price, and the corresponding (model-free) lower and upper bounds for the

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implied volatility are respectively denoted as σ_l and σ_u.

Proposition 5. *Assume that $\{\delta_\varepsilon\}$ is a delta sequence of positive type, then*

$$f_{BS}^{-1}(x) = \lim_{\varepsilon \to +0} \int_{\sigma_l}^{\sigma_u} \delta_\varepsilon(f_{BS}(s), x) \cdot s f'_{BS}(s) ds =: \lim_{\varepsilon \to +0} \sigma_{BS,\varepsilon}(x). \tag{9}$$

Proof. By the bounded convergence theorem, we have

$$\begin{aligned} f_{BS}^{-1}(x) &= \int_{C_l}^{C_u} f_{BS}^{-1}(u) \delta(u, x) du \\ &= \int_{C_l}^{C_u} f_{BS}^{-1}(u) \left(\lim_{\varepsilon \to +0} \delta_\varepsilon(u, x) \right) du \\ &= \lim_{\varepsilon \to +0} \int_{C_l}^{C_u} f_{BS}^{-1}(u) \delta_\varepsilon(u, x) du \\ &= \lim_{\varepsilon \to +0} \int_{\sigma_l}^{\sigma_u} f_{BS}^{-1}(f_{BS}(s)) \delta_\varepsilon(f_{BS}(s), x) d f_{BS}(s) \\ &= \lim_{\varepsilon \to +0} \int_{\sigma_l}^{\sigma_u} \delta_\varepsilon(f_{BS}(s), x) \cdot s f'_{BS}(s) ds, \end{aligned} \tag{10}$$

where in the fourth equality sign we have utilized the change of variable $u = f_{BS}(s)$, which maps the domain $u \in [C_l, C_u]$ to the domain $s \in [\sigma_l, \sigma_u]$. This completes the proof. □

Next, by taking a specific delta sequence that mirrors the normal density, we have the following closed-form integral representation of Black-Scholes implied volatility.

Corollary 6. *The Black-Scholes implied volatility for the market-observable option price C admits the following closed-form representation*

$$\sigma_{BS}(C) = \lim_{\varepsilon \to 0} \frac{S_0 \sqrt{T}}{2\pi \sqrt{2\varepsilon}} \int_{\sigma_l}^{\sigma_u} s \cdot \exp\left(-\frac{(f_{BS}(s) - C)^2}{4\varepsilon} - \frac{d_1^2(s)}{2} \right) ds. \tag{11}$$

Proof. Consider the delta sequence

$$\delta_\varepsilon(x, y) = \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon}}. \tag{12}$$

Using the fact that

$$f'_{BS}(\sigma) = S_0 \sqrt{T} \mathcal{N}'(d_1(\sigma)),$$

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and Proposition 5, we have that

$$\begin{aligned}\sigma_{BS}(C) &= \lim_{\varepsilon\to 0} S_0\sqrt{T}\int_{\sigma_l}^{\sigma_u} \frac{s}{2\sqrt{\pi\varepsilon}} e^{-\frac{(f_{BS}(s)-C)^2}{4\varepsilon}} \mathcal{N}'(d_1(s))ds \\ &= \lim_{\varepsilon\to 0} \frac{S_0\sqrt{T}}{2\pi\sqrt{2\varepsilon}} \int_{\sigma_l}^{\sigma_u} s\cdot \exp\left(-\frac{(f_{BS}(s)-C)^2}{4\varepsilon} - \frac{d_1^2(s)}{2}\right) ds,\end{aligned}$$

and this completes the proof. □

Remark 7. *In its present form, equation* (11) *is given as a limit when* $\varepsilon \to 0$*, and in actual implementations we shall take a very small value of* ε *and then evaluate the formula. Sometimes, it is more convenient to rewrite this formula into an equivalent infinite series formula, be which is considered to be a closed-form formula. Recall that the Black-Scholes formula is unambiguously regarded as a "closed-form formula", but it is also inherently an infinite series at the core. The reason is that the cumulative normal distribution function which is part of this formula is typically implemented as a truncated infinite series. More specifically, the cumulative normal distribution function is first linked to the error function, and then the well-known infinite series expression for the error function is utilized for the computation.*

Given the above Remark 7, we rewrite the formula (11) into its equivalent series form. Let $\varepsilon = \frac{1}{n}$ for an integer $n \in \mathbb{N}$; then it is clear that $\varepsilon \to 0$ as $n \to \infty$ and we have

$$\begin{aligned}&\frac{S_0\sqrt{T}}{2\pi\sqrt{2\varepsilon}} \int_{\sigma_l}^{\sigma_u} s\cdot \exp\left(-\frac{(f_{BS}(s)-C)^2}{4\varepsilon} - \frac{d_1^2(s)}{2}\right) ds \\ &\quad = \frac{S_0\sqrt{nT}}{2\pi\sqrt{2}} \int_{\sigma_l}^{\sigma_u} s\cdot \exp\left(-\frac{n(f_{BS}(s)-C)^2}{4} - \frac{d_1^2(s)}{2}\right) ds =: I_n. \qquad (13)\end{aligned}$$

Through the use of telescopic series, we can rewrite the formula (11) in the following equivalent way:

$$\sigma_{BS}(C) = \lim_{n\to\infty} I_n = \sum_{j=1}^{\infty} (I_j - I_{j-1}), \qquad (14)$$

and note that $I_0 = 0$ holds. The formula (14) expresses the Black-Scholes implied volatility as a closed-form infinite series. To summarize, it is always possible to rewrite a limiting expression into the corresponding infinite series through the technique of telescopic sequences. We can use the two expressions (11) and (14) interchangeably since they are equivalent. Given the above Remark 7, it is accurate to call the expression in (14) as a "closed-form"

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formula if we agree that the Black-Scholes formula itself is a closed-form formula.

2.3 Black's Formulation

Sometimes it is more convenient to use Black's formulation, especially when one wants to express the implied volatility in terms of the forward log moneyness. Define $F_0 = \mathbb{E}[S_T|\mathcal{F}_0]$ as the forward price, and recall Black's option pricing formula

$$f_{BK}(\sigma) = f_{BK}(\sigma; K) = \begin{cases} e^{-rT}\left(F_0\mathcal{N}(d_1(\sigma)) - K\mathcal{N}(d_2(\sigma))\right), & call \\ e^{-rT}\left(K\mathcal{N}(-d_2(\sigma)) - F_0\mathcal{N}(-d_1(\sigma))\right), & put \end{cases} \tag{15}$$

where

$$d_1(\sigma) = \frac{\ln(F_0/K) + (\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}. \tag{16}$$

Corollary 8. *Black's implied volatility for market–observable option price C admits the following closed-form representation*

$$\begin{aligned} \sigma_{BK}(C) &:= f_{BK}^{-1}(C) \\ &= \lim_{\varepsilon\to+0} \frac{e^{-rT}F_0\sqrt{T}}{2\sqrt{2\varepsilon}\pi}\int_0^\infty s\cdot\exp\left(-\frac{(f_{BK}(s)-C)^2}{4\varepsilon} - \frac{d_1^2(s)}{2}\right)ds. \end{aligned} \tag{17}$$

Define the log moneyness of the option as $k := \ln(K/F_0)$. The Black-Scholes formula can be written using these new notations as the following bivariate function in terms of the log moneyness and the volatility:

$$C^{BK}(k,\sigma) := e^{-rT}F_0\left(\mathcal{N}\left(d_+\right) - e^k\mathcal{N}\left(d_-\right)\right), \quad d_\pm(k,\sigma) := \frac{-k}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}. \tag{18}$$

Then we have the following alternative representation of Black's implied volatility, where the dependence of the implied volatility and the market-observable option prices on the log moneyness is made explicit:

$$\sigma_{BK}(C(k);k) = \lim_{\varepsilon\to+0}\frac{e^{-rT}F_0\sqrt{T}}{2\sqrt{2\varepsilon}\pi}\int_0^\infty s\cdot\exp\left(-\frac{(C^{BK}(k,s)-C(k))^2}{4\varepsilon} - \frac{d_-^2(k,s)}{2}\right)ds. \tag{19}$$

Similarly as in (14), it is possible to express $\sigma_{BK}(C(k);k)$ in terms of a closed-form infinite

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series.

3 Numerical Examples

We now turn to numerical examples and associated error analyses to demonstrate the previously developed theoretical ideas. We first study the convergence behavior of implied volatility formulas as a function of ϵ where here the essential tradeoff is that the approximation will be more accurate as $\epsilon \to 0$; however, in this limit numerical quadrature methods become increasingly unstable. We then apply this technique to three widely studied implied volatility models, namely SVI, SABR, and the Heston model and compare its accuracy to a Taylor expansion of the associated implied volatility formulas about the at-the-money forward values.

3.1 Implementation and Convergence

In general, we can use the quadrature method to implement the integral term that arises in (9). We obtain he following closed form approximation to the implied volatility, for a fixed $\varepsilon > 0$,

$$f_{BS}^{-1}(x) \approx \sum_{j=1}^{J} \delta_\varepsilon(f_{BS}(\sigma_j), x) \cdot \lambda_j, \tag{20}$$

and here $\{(\sigma_j, w_j)\}_{j=1}^J$ is a set of quadrature nodes and weights, and

$$\lambda_j := w_j \sigma_j f'_{BS}(\sigma_j),$$

for which we find a simple trapezoidal approximation works well as shown below.

In our implementation, we utilize the Gaussian delta sequence $\delta_\varepsilon(x,y) = \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon}}$. We next illustrate numerically the convergence of the integral formula (11) for computing the Black-Scholes implied volatility. We fix $\sigma \in \{0.1, \ldots, 0.5\}$, which represents a common range of volatility levels as observed in the financial markets, and plot the convergence results in Figure 1 of the error defined as $err := |f_{BS}^{-1}(C_i) - f_{BS,\varepsilon}^{-1}(C_i)| = |\sigma_i - f_{BS,\varepsilon}^{-1}(C_i)|$. Here, $C_i := f_{BS}(\sigma_i)$, the Black-Scholes price of a call option with volatility σ_i. We can observe that our formula converges well to the true volatility value, as expected from Proposition 3.

In addition, we have conducted similar studies for the other types of delta sequences

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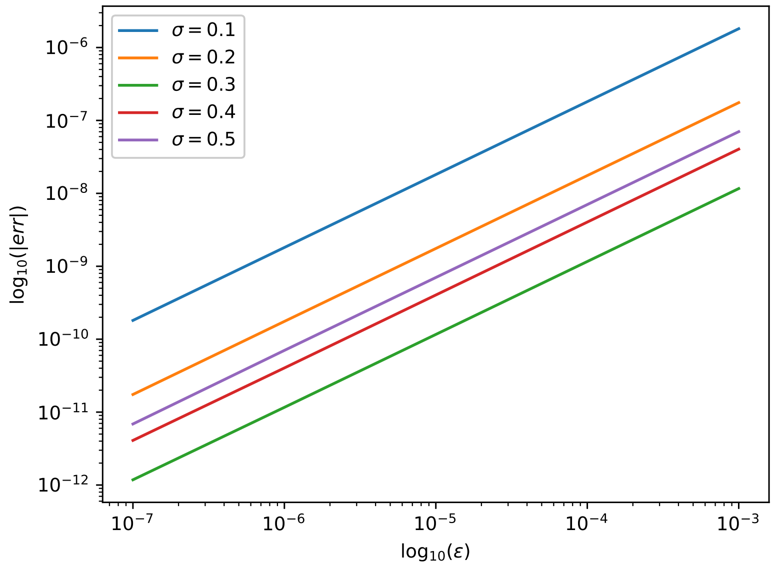

Figure 1: Implied volatility convergence of formula (11) as a function of ε for Black-Scholes call option prices, with $\sigma \in \{0.1, \ldots, 0.5\}$, $T = 0.5$, $r = 0.05$, $K = S_0 = 100$

including the Lorentzian, trigonometric, and sinc function type, as documented in Table 1. In each case, we found the overall error to be several orders of magnitude higher than when one uses the Gaussian-type delta sequence approximation. Moreover we found that all of the trapezoid rule, Romberg integration, and adaptive quadrature methods are less stable in the case of non-Gaussian delta sequences in the sense that the error is not necessarily monotonically decreasing as one decreases ε. Thus we stick to the Gaussian-type delta sequence in our implementations and leave the further investigation of other delta sequences to future research.

Next, we shall illustrate our proposed closed-form formula in computing the implied volatility value when the data generating processes (DGP) correspond to some commonly-used models in finance. We shall first use the DGP below to generate the option prices, and then fit the prices into our formula to compute the implied volatility values.

3.2 Example: Stochastic Volatility Inspired (SVI)

As a popular example, we consider the SVI model of [15] (see also its generalizations in [18, 21]), which models total variance directly. Specifically, SVI(a, b, ρ, m, σ) is formulated

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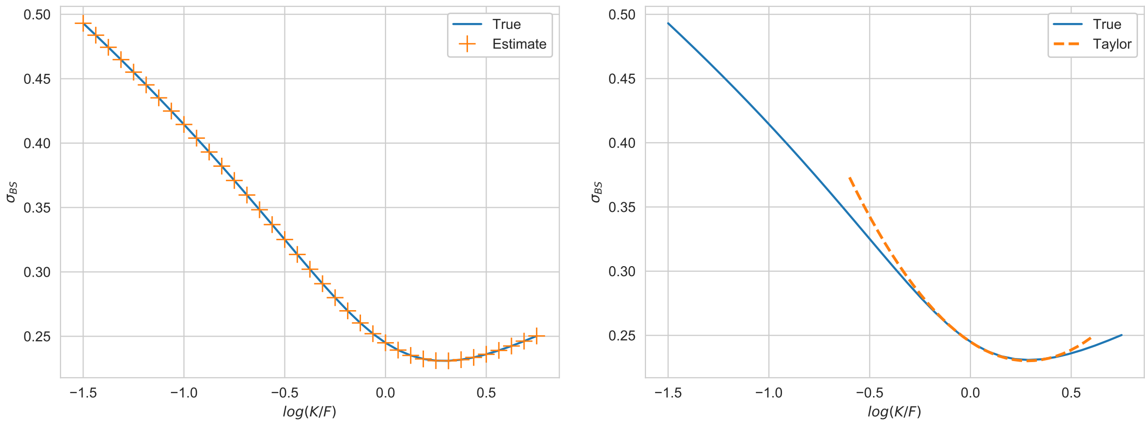

Figure 2: Implied volatility fit (Left) to SVI model using (11), $a = 0.01, b = 0.1, \rho = -0.5, m = 0, \sigma = 0.5$, $T = 0.5$, $r = 0.00$, $S_0 = 100$. Right: Taylor expansion around at-the-money forward (ATMF) using the delta formula.

as

$$\begin{aligned} w(k,T) &= T \cdot \sigma_{BS}^2(k,T) \\ &= T \cdot \left(a + b\left\{\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right\}\right). \end{aligned}$$

The time-dependent $\sigma_{BS}^2(k,T)$ highlights that in general, these parameters are chosen (calibrated) specifically for each time slice T. We calibrate $w(k,T)$ to observed market implied volatility (which we calculate using our formulas, below). Then

$$C^{BS}(k, w(k,T)) = S_0\left(\mathcal{N}(d_+) - e^k \mathcal{N}(d_-)\right), \quad d_\pm(k) := \frac{-k}{\sqrt{w(k,T)}} \pm \frac{\sqrt{w(k,T)}}{2}. \tag{21}$$

Similarly,

$$C^{BK}(k, w(k,T)) = e^{-rT} F_0\left(\mathcal{N}(d_1) - e^k \mathcal{N}(d_2)\right), \tag{22}$$

In Figure 2, we demonstrate the proposed formula (11) for an SVI model with parameters $a = 0.01, b = 0.1, \rho = -0.5, m = 0, \sigma = 0.5$, typically of equity indices. It can be seen that the fit of our formula to the true implied volatility curve is very good, as compared to the Taylor expansion approximation.

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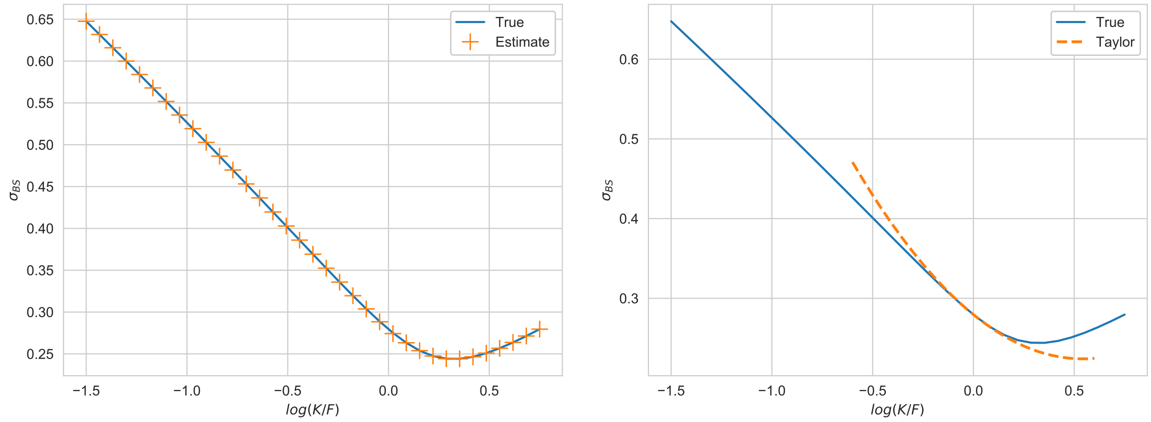

Figure 3: Implied volatility fit (Left) to SABR model using (17), $\alpha = 0.7, \beta = 0.8, \rho = -0.5, \nu = 0.7$, $T = 0.5$, $r = 0.00$, $F_0 = 100$. Right: Taylor expansion around ATMF using delta formula

3.3 Example: SABR

Another prominent example is the Stochastic Alpha Beta Rho (SABR) model of [20], which is especially popular in interest rates and foreign exchange markets, and has received substantial interest in the literature [9, 10, 27, 33]. This model is given by the following forward price dynamics

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} dW_t, \\ d\alpha_t = \nu \alpha_t dZ_t, \end{cases} \tag{23}$$

where $\alpha_0 = \alpha$. A closed form asymptotic expansion for implied volatility is provided in [20]. The correction due to [33], which yields an expansion of order of ϵ^2, is given by

$$\sigma_{imp} = \alpha \frac{\log\left(\frac{F_0}{K}\right)}{D(\zeta)} \left\{ 1 + \left[\frac{(\beta-1)^2}{24} \left(\frac{v_0}{\alpha}\right)^2 (S_0 K)^{(\beta-1)} + \frac{\rho\beta}{4} \frac{v_0}{\alpha} (S_0 K)^{\frac{\beta-1}{2}} + \frac{2-3\rho^2}{24} \right] \xi \right\}, \tag{24}$$

where $\zeta = \frac{\alpha}{v_0(1-\beta)}(S_0^{(1-\beta)} - K^{(1-\beta)})$, $D(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}+\zeta-\rho}{1-\rho}\right)$, and $\xi = \alpha^2 T$.

Figure 3 illustrates the implied volatility fit using our formula (17) cast into Black's formulation, as well as the second order Taylor expansion around the ATMF point in the right panel. This formula performs well across different moneyness levels on the implied volatility smile.

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3.4 Implied volatility from parametric models: Heston model

Previously, we assume that option prices are directly observable from the market, and we established a link of the implied volatility only up to the market-observable options prices. In this section, we shall consider linking the implied volatility directly to the model parameters in a parametric model setting. The main idea is very intuitive: the representation (11) links the implied volatility to the call option price C. If we assume a parametric stochastic model for the underlying asset, we can derive analytical expressions for the call option price in terms of model parameters. Thus we can combine the two expressions above and directly link the implied volatility to the model parameters.

This allows us to calibrate the parameters of these stochastic models directly to the market-observable implied volatility surface, which is also calculated from our method. To summarize, there are two main steps:

1. First, we use the formula (11) to translate the call option price surface to the equivalent implied volatility surface.

2. Second, we utilize the representation of the implied volatility in terms of model parameters, to set up the objective function in an optimization routine to calibrate the parametric stochastic model to the implied volatility surface.

As an illustrative example, we consider a representative stochastic volatility model, the Heston model. Assume that the current stock price level S_t and variance level v_t, the strike is K and the maturity is T, and the time to maturity is denoted as $\tau := T - t$. The dynamics of the risky asset price and the stochastic variance level follow the following stochastic differential equations system:

$$\begin{cases} dS_t = (r-q)S_t dt + \sqrt{v_t} S_t dW_t, \\ dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dZ_t. \end{cases} \tag{25}$$

The log stock price is given by $x_t := \ln S_t$. The call option price is given in closed-form, see [22], as below:

$$C\left(t, S_t, v_t, K, T\right) = S_t P_1 - K e^{-r\tau} P_2, \tag{26}$$

where

$$\begin{aligned} P_j\left(x_t, v_t; \ln K\right) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\phi \ln K} f_j(\phi; x_t, v_t)}{i\phi}\right) d\phi \\ f_j\left(\phi; x_t, v_t\right) &= \exp\left[C_j(\tau,\phi) + D_j(\tau,\phi)v_t + i\phi x_t\right] \end{aligned} \tag{27}$$

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and

$$
\begin{aligned}
&C_j(\tau,\phi) = (r-q)i\phi\tau + \tfrac{a}{\sigma^2}\left((b_j - \rho\sigma i\phi + d_j)\,\tau - 2\ln\tfrac{1-g_j e^{d_j\tau}}{1-g_j}\right)\\
&D_j(\tau,\phi) = \tfrac{b_j-\rho\sigma i\phi+d_j}{\sigma^2}\left(\tfrac{1-e^{d_j\tau}}{1-g_j e^{d_j\tau}}\right)\\
&g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j}\\
&d_j = \sqrt{(b_j - \rho\sigma i\phi)^2 - \sigma^2\left(2iu_j\phi - \phi^2\right)}\\
&u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = \kappa\theta, b_1 = \kappa + \lambda - \rho\sigma, b_2 = \kappa + \lambda, i^2 = -1.
\end{aligned}
\tag{28}
$$

Next we combine the above option price formula (26) with (11) to arrive at the following fully-parametric formula for the implied volatility under the Heston model

$$
\sigma_{Hes}(\kappa,\theta,\sigma) = \lim_{\varepsilon\to 0}\frac{S_0\sqrt{T}}{2\sqrt{2\varepsilon\pi}}\int_0^\infty s\cdot\exp\left(-\frac{(f_{BS}(s) - C\left(t,s,v_t,K,T\right))^2}{4\varepsilon} - \frac{d_1^2(s)}{2}\right)ds. \tag{29}
$$

Note that in (29), the implied volatility is explicitly expressed through the model parameters, i.e. κ, θ, σ. Thus we can utilize this expression to build a calibration optimization routine to calibrate the model parameters (κ, θ, σ) from the observable implied volatility surface.

We now consider an example of computing and displaying an implied volatility surface for the Heston model given by equation (25). Specifically, we approximate the associated implied volatility in equation (29) with the example given in Figure 4. Here we numerically evaluate the integral in equation (29) using the adaptive numerical quadrature procedure of [34]. We further note that one may extend these results to calibrating market implied volatility surfaces by minimizing a mean squared error objective function that minimizes the differences between model implied volatilities given by equation (29) and the associated option market implied volatilities, which we leave for future work.

4 Conclusion

In this paper, we derive for the first time an explicit closed-form formula for the Black-Scholes implied volatility, which is baed on the novel use of the theory of the Dirac Delta function and delta sequences. The proposed formula is easy to implement and agrees with existing methods and benchmarks in the literature. The delta sequence method that we consider is of independent interest and can potentially find applications in other contextual areas where inverse problems are considered.

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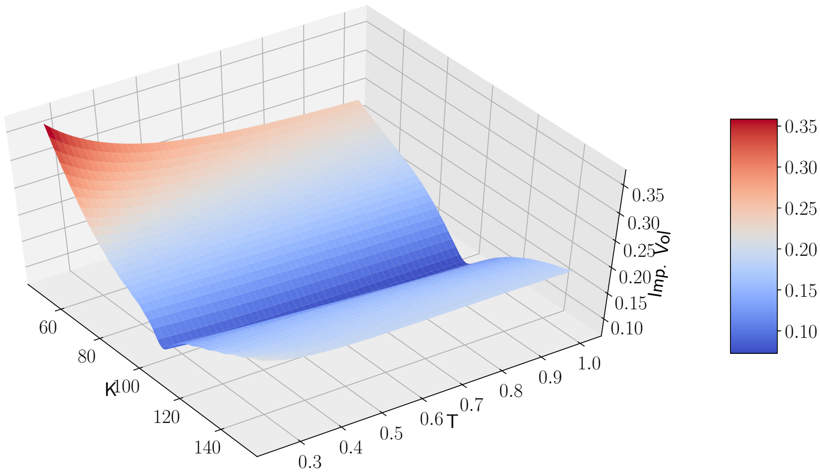

Figure 4: Heston model implied volatility surface computed from equation (29) with parameters $S_0 = 100$, $r = 3.19\%$, $q = 0\%$, $v_0 = 0.02$, $\rho = -0.1$, $\kappa = 0.15$, $\theta = 0.05$, $\sigma = 1.3$, and $\epsilon = 0.01$.

References

[1] E. Abi Jaber and O. El Euch. Multifactor approximation of rough volatility models. *SIAM J. Financial Math.*, 10(2):309–349, 2019.

[2] Alexandru Badescu, Zhenyu Cui, and Juan-Pablo Ortega. Non-affine GARCH option pricing models, variance-dependent kernels, and diffusion limits. *Journal of Financial Econometrics*, 15(4):602–648, 2017.

[3] J Baumeister. Inverse problems in finance. In *Recent developments in computational finance: Foundations, algorithms and applications*, pages 81–157. World Scientific, 2013.

[4] Henri Berestycki, Jérôme Busca, and Igor Florent. Computing the implied volatility in stochastic volatility models. *Communications on Pure and Applied Mathematics*, 57(10):1352–1373, 2004.

[5] BIS. *Bank for International Settlements Statistical Bulletin*. 2020.

[6] Peter Carr, Hélyette Geman, Dilip Madan, and Marc Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, 13(3):345–382, 2003.

[7] Peter Carr, Andrey Itkin, and Sasha Stoikov. A model-free backward and forward nonlinear PDEs for implied volatility. *arXiv preprint arXiv:1907.07305*, 2019.

[8] Peter Christoffersen and Kris Jacobs. The importance of the loss function in option valuation. *Journal of Financial Economics*, 72(2):291–318, 2004.

[9] Z. Cui, J.L. Kirkby, and D. Nguyen. A general valuation framework for SABR and stochastic local volatility models. *SIAM Journal on Financial Mathematics*, 9(2):520–563, 2018.

[10] Z. Cui, J.L. Kirkby, and D. Nguyen. Full-fledged SABR through Markov chains. *Wilmott*, 102, 2019.

Electronic copy available at: https://ssrn.com/abstract=3573239

[11] Victor DeMiguel, Yuliya Plyakha, Raman Uppal, and Grigory Vilkov. Improving portfolio selection using option-implied volatility and skewness. *Journal of Financial and Quantitative Analysis*, 48(6):1813–1845, 2013.

[12] O. El Euch, M. Fukasawa, J. Gatheral, and M. Rosenbaum. Short-term at-the-money asymptotics under stochastic volatility models. *SIAM J. Financial Math.*, 10(2):491–511, 2019.

[13] Omar El Euch and M. Rosenbaum. The characteristic function of rough Heston models. *Mathematical Finance*, 29(1), 2018.

[14] Jean-Pierre Fouque, Matthew Lorig, and Ronnie Sircar. Second order multiscale stochastic volatility asymptotics: stochastic terminal layer analysis and calibration. *Finance and Stochastics*, 20(3):543–588, 2016.

[15] J. Gatheral and A. Jacquier. Arbitrage-free SVI volatility surfaces. *Quantitative Finance*, 14(1):59–71, 2014.

[16] Stefan Gerhold. Can there be an explicit formula for implied volatility. *Appl. Math. E-Notes*, 13:17–24, 2013.

[17] Kathrin Glau, Paul Herold, Dilip B Madan, and Christian Pötz. The Chebyshev method for the implied volatility. *Journal of Computational Finance*, 23(3), 2019.

[18] G. Guo, A. Jacquier, C. Martini, and L. Neufcourt. Generalized arbitrage-free SVI volatility surfaces. *SIAM J. Financial Mathematics*, 7:619–641, 2016.

[19] Patrick Hagan, Deep Kumar, Andrew Lesniewski, and Diana Woodward. Arbitrage-free SABR. *Wilmott*, 2014(69):60–75, 2014.

[20] P.S. Hagan, D. Kumar, A. Lesniewski, and D.E. Woodward. Managing smile risk. *Wilmott*, 1:84–108, 2001.

[21] S. Hendriks and C. Martini. The extended SSVI volatility surface. *J. Computational Finance*, 22(5):25–39, 2019.

[22] Steven Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

[23] Peter Jäckel. By implication. *Wilmott*, 26:60–66, 2006.

[24] Jens Carsten Jackwerth and Mark Rubinstein. Recovering probability distributions from option prices. *The Journal of Finance*, 51(5):1611–1631, 1996.

[25] A. Jacquier and F. Shi. The randomized Heston model. *SIAM J. Finan. Math.*, 10(1):89–129, 2019.

[26] Steven Kou. A jump-diffusion model for option pricing. *Management Science*, 48(8):1086–1101, 2002.

[27] A. Leitao, L. Grzelak, and C.W. Oosterlee. On a one time-step Monte Carlo simulation approach of the SABR model: Application to European options. *Applied Math. and Computation*, 293:461–479, 2017.

[28] Minqiang Li. Approximate inversion of the Black–Scholes formula using rational functions. *European Journal of Operational Research*, 185(2):743–759, 2008.

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[29] Minqiang Li and Kyuseok Lee. An adaptive successive over-relaxation method for computing the Black–Scholes implied volatility. *Quantitative Finance*, 11(8):1245–1269, 2011.

[30] Matthew Lorig, Stefano Pagliarani, and Andrea Pascucci. Explicit implied volatilities for multifactor local-stochastic volatility models. *Mathematical Finance*, 27(3):926–960, 2017.

[31] Ivan Matić, Radoš Radoičić, and Dan Stefanica. A PDE method for estimation of implied volatility. *Quantitative Finance*, pages 1–16, 2019.

[32] Alexey Medvedev and Olivier Scaillet. Approximation and calibration of short-term implied volatilities under jump-diffusion stochastic volatility. *Review of Financial Studies*, 20(2):427–459, 2006.

[33] Jan Oblój. Fine-tune your smile: Correction to Hagan et al. *Wilmott Magazine, May 2008*, 2008.

[34] Robert Piessens, Elise deDoncker Kapenga, Christian Ueberhuber, and David Kahaner. *QUADPACK: A Subroutine Package for Automatic Integration*. Springer, 1983.

[35] V Susarla and G Walter. Estimation of a multivariate density function using delta sequences. *The Annals of Statistics*, 9(2):347–355, 1981.

[36] Michael R Tehranchi. Uniform bounds for Black–Scholes implied volatility. *SIAM Journal on Financial Mathematics*, 7(1):893–916, 2016.

[37] Lenos Trigeorgis and Andrianos E Tsekrekos. Real options in operations research: A review. *European Journal of Operational Research*, 270(1):1–24, 2018.

[38] G Walter and J Blum. Probability density estimation using delta sequences. *Annals of Statistics*, 7(2):328–340, 1979.

[39] Yuxuan Xia and Zhenyu Cui. An exact and explicit implied volatility inversion formula. *International Journal of Financial Engineering*, 5(03):1850032, 2018.

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