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An exact and explicit implied volatility inversion formula

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Abstract

In this paper, we develop an exact and explicit (model-independent) Taylor series representation of the implied volatility based on the novel applications of an extended Faà di Bruno formula under the operator calculus setting, and the Lagrange inversion theorem. We rigorously establish that our formula converges to the true implied volatility as the truncation order increases. Numerical examples illustrate the remarkable accuracy and efficiency of the formula. The formula distinguishes from previous literature as it converges to the true exact implied volatility, is a closed-form formula whose coefficients are explicitly determined and do not involve numerical iterations.

Keywords: Implied volatility; Taylor series; arbitrary greeks; Lagrange inversion theorem; operator calculus.

JEL Subject Classifications: 91G80, 93E11, 93E20

1. Introduction

As late Steve Ross wrote in the Palgrave Dictionary of Economics, " ... option pricing theory is the most successful theory not only in finance, but in all of economics." The Black–Scholes–Merton theory is one of the most important and influential in the option pricing theory. Due to the observed volatility fluctuations, the Black–Scholes formula is rarely used in the original direction, but rather is often used in the opposite direction to quote option prices through the "implied volatility". The implied volatility is a single number that summarizes the essential

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features of the option contract, and it allows traders to compare option prices on different underlying stock prices and with different strikes and maturities, etc. In the practice of financial engineering, one of the most frequently executed tasks is to calculate the implied volatility for European call and put options.

It is long-believed that there is no closed-form formula for the implied volatility. Thus, in practice, the implied volatility is usually determined from an iterative numerical root solving algorithm, such as the Newton–Raphson method or the Dekker–Brent (i.e., bisection) method. It is documented that the numerical root-solving methods suffer from slow speed, divergence issues, and biases that may lead to artificial fake smiles. See the recent interesting empirical evidence in Chance *et al.* (2017), where the authors found that even if options are traded following the Black–Scholes model, there are still observed smiles, skews and smirks due to inaccuracies of the numerical inversion algorithms employed. Given that option prices are observed with plausible errors, especially for those away-from-the-money options, small changes in the option price characteristics can produce large errors in the implied volatility estimates. There are extensive empirical studies documenting this phenomenon in Hentschel (2003), where the author showed that "Implied volatilities can also show pronounced smiles or smirks, even if all of the Black–Scholes assumptions hold but prices are measured with plausible errors."

From the applications viewpoint, the computation of implied volatility across a wide range of option price surfaces is an important component of any financial toolbox, and is central to pricing and hedging, risk management and model calibrations involving option prices. On a particular trading day, millions of real-time option prices have to be converted to the implied volatilities at the same time. Thus, there is a need for a quick evaluation method for implied volatilities. There is a deep literature on developing efficient noniterative approximations to the implied volatility in order to speed up the calibration process. We roughly classify this strand of the literature into two categories:

(i) The first category is based on direct approximation of the Black–Scholes formula, and then carrying out explicit or semi-explicit inversion of the approximate formula. The resulting implied volatility approximation usually does not converge to the true implied volatility and there is an associated bias which is hard to quantify or bound. Some recent literature within this category includes Li (2008); Tehranchi (2016); Stefanica and Radoičić (2017); Gatheral *et al.* (2017); Matić *et al.* (2017), etc.
(ii) The second category aims at developing *asymptotic* expansion of the implied volatility either in some extreme parameter regimes (i.e., small maturity or large strike) or through perturbations of the associated partial differential equations (PDE) or estimates of semi-groups. Usually only the first few

leading order terms are explicitly calculated, and these asymptotic expansion methods may not work well outside those extreme parameter regimes. Some recent literature includes Berestycki *et al.* (2004); Medvedev and Scaillet (2006); Gao and Lee (2014), etc.

The above theoretical and practical needs motivate us to develop a closed-form exact and explicit formula for the implied volatility. The contributions of our paper are as follows:

(1) To the best of our knowledge, it is the first time that an exact explicit closed-form formula for the Black–Scholes implied volatility is obtained. This is achieved through the novel applications of the operator calculus and the Lagrange inversion theorem.
(2) The implied volatility formula is given as a real-valued Taylor series, which is well-defined, and with all the expansion coefficients explicitly expressed through known functions and constants.
(3) The implied volatility formula does not involve iterative numerical procedures, and is guaranteed to converge to the true implied volatility.

It is important to distinguish our paper from a recent important ground-breaking work of Aït-Sahalia *et al.* (2017). They make a remarkable breakthrough in establishing the direct explicit link between the implied volatility and the model parameters in a general class of stochastic volatility models. Our work is similar to theirs in the sense that explicit and exact formulas are developed. However, there are two main differences between our work and theirs:

(1) The main objective of this paper is to establish an exact Taylor series representation of the implied volatility in terms of put/call option prices, which is a model-independent and nonparametric formula. The focus of Aït-Sahalia *et al.* (2017) is on parametric models, i.e., stochastic volatility models, and seek to represent implied volatility in terms of model parameters rather than market-observed option prices.
(2) Note that the problem of explicitly inverting the Black–Scholes formula in terms of market-observed option prices, which is achieved in this paper, has not been solved in Aït-Sahalia *et al.* (2017). One key equation established in their paper is Eq. (A.9) on p. 37, which presents a closed-form representation of the *sensitivity* of the inverse Black–Scholes formula. The footnote on p. 37 of their paper states that:

"*Though P_{BS} is in closed-form as shown in the Black–Scholes formula* (4)*, its inverse function P_{BS}^{-1} is* ***implicit****. Thus, it is* ***impossible*** *to express the explicit dependence of its derivative $P_{\mathrm{BS}}^{-1,(l_1,l_2,l_3)}$ on P*"

From this quote, it is clearly seen that the problem of *explicitly* inverting the Black–Scholes formula (i.e., explicit expression for P_{BS}^{-1}) has not been solved in their paper. However, in order to achieve the goal of linking the implied volatility to model parameters, they cleverly bypass this by employing Eq. (A.9). Note that their Eq. (A.9) can not yield (as indicated in the quote above) an **explicit** expression of P_{BS}^{-1}, which is the main contribution of this paper. It can also be alternatively seen from the first sentence of Sec. 5.2 on p. 19 quoted here: "…*by comparing them to the* ***numerically*** *computed implied volatilities*…" It is clear that they employed *numerical* procedures in evaluating P_{BS}^{-1}, while we *analytically* express P_{BS}^{-1} in terms of explicit closed-form series of market-observed option prices.

To summarize, the focus of Aït-Sahalia *et al.* (2017) is to express the implied volatility in closed-form in terms of the model parameters of a parametric stochastic volatility model, while this paper expresses the implied volatility in a nonparametrical way in terms of market-observed option prices. The techniques in Aït-Sahalia *et al.* (2017) do not cover our problem as can be seen from the above quote from their paper. Furthermore, based on the explicit inversion of the Black–Scholes formula achieved in this paper, by taking derivatives with respect to the coefficients in the closed-form explicit series expression, it paves the way to "express the explicit dependence of $P_{\mathrm{BS}}^{-1,(l_1,l_2,l_3)}$ on P". Thus, our explicit inversion of P_{BS}^{-1} can replace their Eq. (A.9), and facilitate the task of linking implied volatility to model parameters if we assume, for example a parametric underlying stochastic volatility model. This is left for future research.

The structure of the paper is organized as follows: Section 2 describes the general theoretical framework, and presents the main result: an explicit power series expansion of the implied volatility in terms of market observable option prices. Section 3 conducts detailed numerical studies that confirm the convergence and accurateness of our explicit formula. Section 4 concludes the paper.

2. Arbitrary Greeks and Explicit Implied Volatility Series

Consider a vanilla European call option on an underlying asset price which is log-normally distributed, i.e., the risky asset price evolves according to the following stochastic differential equation:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t, \tag{1}$$

then the Black–Scholes implied volatility (denoted as σ_{imp}) is the unique positive solution to the following equation:

$$\mathrm{BS}(\sigma_{\mathrm{imp}}) = V_{\mathrm{market}},$$

where V_{market} is the price of a call option, and the left-hand side is the Black–Scholes formula:

$$\mathrm{BS}(\sigma) := V(S,T;\sigma,r,q,K) = SN(d_1) - Ke^{-rT}N(d_2), \tag{2}$$

where d_1, d_2 are defined as

$$d_1 = \frac{\ln(S/K) + rT + \sigma^2 T/2}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln(S/K) + rT - \sigma^2 T/2}{\sigma\sqrt{T}}.$$

The case for the put option can be similarly defined, and we also use the notation V_{market} to denote the put option price when the context is clear.

Our analysis consists of the following four steps:

Step 1. We first obtain the Taylor series expansion of the Black–Scholes call/put option price with respect to σ around a positive initial value σ_0. Note that the usual Taylor series of call/put option prices in terms of σ is singular at $\sigma = 0$, and by expanding around a positive level σ_0, the resulting Taylor series is well-defined. The coefficients of all orders are calculated explicitly through operator calculus techniques inspired by Carr (2001).

Step 2. For any given positive initial value σ_0, we invoke the Lagrange inversion theorem, and carry out the series inversion. We obtain the power series of the implied volatility in terms of the option prices. The coefficients of the inverted power series can be explicitly calculated up to any arbitrary order in terms of explicit coefficients of the Taylor series in step 1.

Step 3. Furthermore, we investigate the convergence radius of the power series representation of the implied volatility correspondent to the initial point σ_0.

Step 4. Finally, we choose an appropriate σ_0, which is within the convergence radius of the power series in Step 3.

In order to obtain a Taylor series expansion of the Black–Scholes option price with respect to σ, we need the explicit formulae for the higher-order derivatives of the option price with respect to σ, i.e., we need arbitrary order Greeks, of the option price with respect to the volatility. Note that the first order derivative is the vega. Thus, in this section, we first consider the problem of obtaining arbitrary Greeks. The direct calculation is intense in computation and hardly leads to explicit formulae. Inspired by Carr (2001), we first link all Greeks to the spatial derivatives using the operator calculus.

Denote the log stock price as $x = \ln S$, then the Black–Scholes formula can be reexpressed as:

$$V(S,T) = U(x,T) = e^{x-qT}N(d_1) - Ke^{-rT}N(d_2),$$

where

$$d_1 = \frac{x + (r-q)T - \ln K + \sigma^2 T/2}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma^2 T.$$

In the following, we shall use the two equivalent notations $V(S,T)$ and $U(x,T)$ interchangeably. There are two desirable properties associated with using the log stock price as the variable:

(i) The component of the partial derivative operator corresponding to x commute to each other (see Appendix A.2).
(ii) By transformation from S to x, the Black–Scholes formula bears no singularity in x (see Appendix B).

Lemma 1 (Spatial derivatives of the Black–Scholes formula). *For $n = 1, 2, \ldots,$ the nth order derivative of the European option price with respect to x is given by*

$$\begin{aligned}\partial_x^n \mathrm{Call}(S,T) &= e^{x-qT}\frac{1}{2}(1+\mathrm{erf}(g(x))) \\ &\quad + \frac{e^{x-qT}}{\sqrt{\pi}}\sum_{j=1}^{n}\binom{n-1}{j}(-1)^{j-1}(\sqrt{2}\sigma\sqrt{T})^{-j}(e^{-g^2(x)}H_{j-1}(g(x))),\end{aligned} \tag{3}$$

$$\begin{aligned}\partial_x^n \mathrm{Put}(S,T) &= -e^{x-qT}\frac{1}{2}(1+\mathrm{erf}(g(x))) \\ &\quad - \frac{e^{x-qT}}{\sqrt{\pi}}\sum_{j=1}^{n}\binom{n-1}{j}(-1)^{j-1}(-\sqrt{2}\sigma\sqrt{T})^{-j}(e^{-g^2(x)}H_{j-1}(g(x))),\end{aligned} \tag{4}$$

where

$$g(x) := \begin{cases} \dfrac{x - \ln K + (r-q)T + \sigma^2 T/2}{\sqrt{2}\sigma\sqrt{T}}, & \text{for the call option,} \\ -\dfrac{x - \ln K + (r-q)T + \sigma^2 T/2}{\sqrt{2}\sigma\sqrt{T}}, & \text{for the put option,} \end{cases}$$

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

Here, $H_n(\cdot)$ are the Hermite polynomials given by

$$H_n(x) := (-1)^n e^{x^2}\left(\frac{d^n}{dx^n}e^{-x^2}\right).$$

The proof of Lemma 1 is given in Appendix A.1.

The next step is to calculate the other Greeks by linking them to the calculation of spatial derivatives, and this is achieved through operator calculus. In the following theorem, we present these formulae explicitly. For more details about the calculation, one can refer to the Appendix A. Note that in Appendix A.2, we novelly generalize the Faà di Bruno formula in the context of operator calculus in order to facilitate the derivation of arbitrary Greeks.

Theorem 1 (Arbitrary Greeks). *For $n = 1, 2, \ldots,$ the nth order derivative of the European option price with respect to σ can be represented in terms of the security's spatial derivatives as*:

$$\partial_\sigma^n V(S,T) = \sum_{k=1}^{n} \frac{n!}{(2k-n)!(n-k)!} \cdot \frac{T^k \sigma^{2k-n}}{2^{n-k}} \sum_{j=0} k \binom{k}{j} (-1)^{k-j} \partial_x^{k+j} V(S,T), \tag{5}$$

$$\partial_r^n V(S,T) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \partial_x^k V(S,T), \tag{6}$$

$$\partial_q^n V(S,T) = (-1)^n \partial_x^n V(S,T), \tag{7}$$

where the spatial derivatives $\partial_x^k V$ are given in Lemma 1.

The proof of Theorem 1 is given in Appendix A.2. Note that the formula (5) applies to both call and put options. In the following, we focus on the higher-order Greeks with respect to σ, as provided in Eq. (5) of Theorem 1.

Proposition 2. *The nth order Taylor series expansion of the Black–Scholes formula with respect to σ around a positive value σ_0 is given by*

$$\tilde{V}_n(S,T;\sigma) = V(S,T;\sigma_0) + \sum_{k=1}^{n} \frac{\partial_\sigma^k V(S,T;\sigma_0)}{k!} (\sigma - \sigma_0)^k, \tag{8}$$

where the coefficients $\partial_\sigma^k V$ is determined by Eq. (5) *in Theorem* 1 *combined with either Eq.* (3) *or Eq.* (4) *in Lemma* 1 *depending on whether we are considering a call or a put option. The representation $\tilde{V}_n(S,T;\sigma)$ converges to the true value $V(S,T;\sigma)$ as $n \to \infty$, if $0 < \sigma < 2\sigma_0$ and diverges if $\sigma > 2\sigma_0 > 0$.*

The proof of Proposition 2 follows from the classical Taylor expansion theorem, and the determination of the convergence radius is given in Appendix B.

Given a call/put option price V_{market}, denote the Black–Scholes implied volatility as $\Sigma_{\text{imp}}^{\text{BS}}(V_{\text{market}})$, then by definition, we have

$$V_{\text{market}} = V(S,T;\Sigma_{\text{imp}}^{\text{BS}}(V_{\text{market}})) = \tilde{V}_\infty(S,T;\Sigma_{\text{imp}}^{\text{BS}}(V_{\text{market}})), \tag{9}$$

where the right-hand side has a Taylor series representation given in (8). Intuitively, given a power series representation of the call/put option prices V_{market} in terms of $\Sigma^{\text{BS}}_{\text{imp}}(V_{\text{market}}) - \sigma_0$, a natural idea is to invert the power series and obtain a representation of $\Sigma^{\text{BS}}_{\text{imp}}(V_{\text{market}}) - \sigma_0$ in terms of the inverse power series of the call/put option prices. Then, after adding σ_0 to both sides, we obtain a power series representation of $\Sigma^{\text{BS}}_{\text{imp}}(V_{\text{market}})$ in terms of option prices. Thus in the next step, we aim to invert the Taylor series expansion $V(S,T;\sigma) = \tilde{V}_\infty(S,T;\sigma)$. The existence and well-definedness of the inverted series are guaranteed by the Lagrange inversion theorem (see for example Whittaker and Watson (1996)), which we recall as follows.

Theorem 3 (Lagrange Inversion Theorem). *Suppose that a function $f(\cdot)$ is analytic at a point x_0, and $f'(x_0) \neq 0$, then we can express the function $f(\cdot)$ in its Taylor series. Moreover, the inverse function $g(\cdot) = f^{-1}(\cdot)$ can also be represented as a formal power series which has a nonzero convergence radius.*

It is straightforward to verify that $f(\sigma) := V(S,T;\sigma)$ is analytic at the positive level σ_0, and that $f'(\sigma_0) \neq 0$. The corresponding Taylor series expansion of $f(\cdot)$ is explicitly given in Eq. (8). Thus the Lagrange inversion theorem guarantees that the inverse series exists with a nonzero convergence radius. After establishing the existence of the inverse series, the next task is to calculate the coefficients of the inverse series. In general, if we are given a power series $y = \sum_{i=0}^{\infty} a_i x^i$, then the inverse series has the following form: $x = \sum_{j=0}^{\infty} A_j y^j$, where the coefficients are explicitly given by the following formula (see for example Morse and Feshbach (1954)):

$$A_n := \frac{1}{n a_1^n} \sum_{s \neq \mu} (-1)^{s+t+u+\cdots} \frac{n(n+1)\cdots(n-1+s+t+u+\cdots)}{s!t!u!\cdots} \times \left(\frac{a_2}{a_1}\right)^s \left(\frac{a_3}{a_1}\right)^t \cdots, \tag{10}$$

where

$$s + 2t + 3u + \cdots = n - 1.$$

The first few inversion coefficients are explicitly calculated as below:

$$A_1 := 1/a_1,$$
$$A_2 := -a_1^{-3} a_2,$$

$$
\begin{aligned}
A_3 &:= a_1^{-5}(2a_2^2 - a_1a_3),\\
A_4 &:= a_1^{-7}(5a_1a_2a_3 - a_1^2a_4 - 5a_2^3),\\
A_5 &:= a_1^{-9}(6a_1^2a_2a_4 + 3a_1^2a_3^2 + 14a_2^4 - a_1^3a_5 - 21a_1a_2^2a_3),\\
A_6 &:= a_1^{-11}(7a_1^3a_2a_5 + 7a_1^3a_3a_4 + 84a_1a_2^3a_3 - a_1^4a_6 - 28a_1^2a_2a_3^2\\
&\quad -42a_2^5 - 28a_1^2a_2^2a_4),\\
A_7 &:= a_1^{-13}(8a_1^4a_2a_6 + 8a_1^4a_3a_5 + 4a_1^4a_4^2 + 120a_1^2a_2^3a_4 + 180a_1^2a_2^2a_3^2 + 132a_2^6\\
&\quad -a_1^5a_7 - 36a_1^3a_2^2a_5 - 72a_1^3a_2a_3a_4 - 12a_1^3a_3^3 - 330a_1a_2^4a_3),
\end{aligned}
$$

Alternative to the expression (10), the inversion coefficients can be expressed through the Bell polynomials:

$$A_n = \frac{1}{a_1^n}\sum_{k=1}^{n-1}(-1)^k n^{(k)} B_{n-1,k}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n-k}), \quad n \geq 2,$$

where

$$\hat{a}_k = \frac{a_{k+1}}{(k+1)a_1} \quad A_1 = \frac{1}{a_1} \quad \text{and} \quad n^{(k)} := n(n+1)\ldots(n+k-1).$$

Here, $B_{n-1,k}$ denotes the $(n-1,k)$-th partial exponential Bell polynomial, which is defined as:

$$B_{n-1,k}(x_1, x_2, \ldots, x_{n-k}) := \sum \frac{(n-1)!}{j_1!j_2!\cdots j_{n-k}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2}\cdots\left(\frac{x_{n-k}}{(n-k)!}\right)^{j_{n-k}},$$

where the summation is taken over all sequences $j_1, j_2, j_3, \ldots, j_{n-k}$ of nonnegative integers such that the following two conditions are satisfied:

$$j_1 + j_2 + \cdots + j_{n-k} = k,$$

$$j_1 + 2j_2 + \cdots + (n-k)j_{n-k} = n-1.$$

The Bell polynomial is a commonly used function in science and engineering. It is a built-in function in many scientific computing languages such as Mathematica, Maple, Matlab and in some packages such as *sympy* in Python. Next, we present the main result of the paper summarized in the following theorem.

Theorem 4 (The Lagrange Inversion Series of Implied Volatility: An Explicit Taylor Series). *Denote $V(\cdot)$ and V_{market}, respectively, as the Black–Scholes formula and the prices for call/put options, then the Black–Scholes implied* *volatility* $\Sigma_{\text{imp}}^{\text{BS}}(V_{\text{market}})$ *has the following series representation*:

$$\Sigma_{\text{imp}}^{\text{BS}}(V_{\text{market}}) = \sigma_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{\partial^n \Sigma_{\text{imp}}^{\text{BS}}}{\partial V^n}(V(\sigma_0)) \cdot (V_{\text{market}} - V(\sigma_0))^n. \tag{11}$$

The coefficients of (11) *are explicitly represented by*:

(i) *For* $n = 1$,

$$\frac{\partial \Sigma_{\text{imp}}^{\text{BS}}}{\partial V}(V(\sigma_0)) = \frac{1}{V_1};$$

(ii) *For* $n \geq 2$,

$$\frac{1}{n!} \cdot \frac{\partial^n \Sigma_{\text{imp}}^{\text{BS}}}{\partial V^n}(V(\sigma_0)) = \frac{1}{V_1^n} \sum_{k=1}^{n-1} (-1)^k \frac{(n+k-1)!}{(n-1)!} B_{n-1,k}(\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_{n-k}),$$

where

$$V_k := \partial_\sigma^k V(\sigma_0), \quad \hat{V}_k := \frac{V_{k+1}}{(k+1)V_1}.$$

Here, $V(\sigma_0) := V(S, T; \sigma_0)$ *and* $\partial_\sigma^k V(\sigma_0)$ *are explicitly given in Theorem* 1 *and Lemma* 1.

The proof of Theorem 4 follows straightforwardly from Lemma 1, Theorem 1, Proposition 2 and the Lagrange inversion theorem.

The next result establishes the convergence of the Lagrange inversion series given in (11).

Proposition 5 (Convergence of the Lagrange Inversion Series of Implied Volatility). *The convergence of the series* (11) *around a nondegenerate neighborhood centered at* $V(\sigma_0)$ *is guaranteed by both the original Black–Scholes Taylor series expansion with respect to the volatility at the fixed point* σ_0 *and the Lagrange inversion theorem.*

The proof of Proposition 5 follows from the Taylor's theorem (see for example Apostol (1967)) and the Lagrange inversion theorem (see Theorem 4). Note that the classical Lagrange inversion theorem only guarantees the convergence within a nondegenerate convergence interval, but does not specify the length of the convergence radius. In applications of inverting the implied volatility, we carry out further in-depth study of the convergence radius. The details of discussions are given in Appendix B.

Given that the original Taylor series is convergent (i.e., (8) is convergent), the Lagrange inversion theorem *only* guarantees that the inverted Taylor series is

Table 1. Summary of properties of the two Taylor series.

Properties	Taylor series 8	Taylor series 11
Convergence radius	σ_0	some $R > 0$ (numerically determined)
Convergence interval	$(0, 2\sigma_0)$	$(V(\sigma_0) - R, V(\sigma_0) + R)$

convergent around a *nondegenerate* neighborhood centered at the value we expand around (i.e., $V(\sigma_0)$), but it does not provide an explicit theoretical formula for the determination of the *convergence radius*. Thus, in the numerical experiments, we shall numerically determine the actual convergence radius of the series (11). A summary of the properties of the two Taylor series is given in Table 1.

In choosing the initial value σ_0, we base our analysis on several recent literature on uniform bounds on the Black and Scholes implied volatility, and in particular, the results of Stefanica and Radoičić (2017) and Tehranchi (2016). For most parameter ranges, one can use the tight approximation results in Stefanica and Radoičić (2017), but this approximation fails to be real numbers in some extreme parameter ranges. Thus, we choose to set the fixed point σ_0 as the model-independent upper bound in Tehranchi (2016). We prefer an upper bound because it guarantees the convergence of the Taylor series (8), since the true implied volatility is within $(0, \sigma_0)$, which falls into the convergence interval as established in Proposition 5 (see also Table 1).

Given the initial stock price S, the strike K, the maturity T, the continuously compounded interest rate r and dividend rate q, denote V^c_{market} as the price for a European call option, and V^p_{market} for an European put option. Introduce the notations:

$$c := \frac{V^c_{\text{market}}}{e^{-qT}S}, \quad p = \frac{V^p_{\text{market}}}{e^{-qT}S}, \quad k = \ln \frac{K}{Se^{(r-q)T}}.$$

Recall the put–call parity:

$$c - p = 1 - e^k. \tag{12}$$

Recall the following result in Tehranchi (2016):

Proposition 6 (Uniform Bounds for the Black–Scholes Implied Volatility). *Fix $k \in \mathbb{R}$ and $(1 - e^k)^+ \le c < 1$. For $k \ge 0$, the following inequalities hold*:

$$-2N^{-1}\left(\frac{1-c}{2}\right) \le \Sigma^{\text{BS}}_{\text{imp}}(V_{\text{market}})\sqrt{T} \le -2N^{-1}\left(\frac{1-c}{1+e^k}\right),$$

and for $k < 0$ we have

$$-2N^{-1}\left(\frac{1-c}{2e^k}\right) \le \Sigma^{\text{BS}}_{\text{imp}}(V_{\text{market}})\sqrt{T} \le -2N^{-1}\left(\frac{1-c}{1+e^k}\right).$$

For any $k \in \mathbb{R}$, from the put–call parity 12, the upper bound of implied volatility is given by

$$\Sigma_{\text{imp}}^{\text{BS}}(V_{\text{market}}) \le -\frac{2}{\sqrt{T}} N^{-1}\left(\frac{1-c}{1+e^k}\right) = -\frac{2}{\sqrt{T}} N^{-1}\left(\frac{e^k - p}{1+e^k}\right) =: \sigma_0, \tag{13}$$

and the two equivalent expressions on the right-hand side are for cases when call or put option prices are available. In the last equality above, we set the initial level σ_0 as the upper bound for both call and put options.

3. Numerical Implementation

In this section, we implement the exact explicit closed-form formula for the Black–Scholes implied volatility as given in (11). The numerical experiment is designed as follows: we use call and put option prices generated from a Black–Scholes model, and compare the inverted implied volatility value σ_{imp} with the true value σ that we start with. In this way, we have the exact benchmark value for the true implied volatility, and can precisely analyze the convergence behavior of the series depending on the choice as initial value σ_0, and also study the accuracy improvement of the series when we increase the truncation order. We report the log errors for three cases: at-the-money (ATM), in-the-money (ITM) and out-of-the-money (OTM). All numerical computations are conducted using Mathematica version 11.2.0.0 on a personal computer with an Intel Core i5 1.60 GHZ processor. The source codes and associated numerical figures for all numerical experiments are available upon request.

3.1. Numerical experiments with the convergence radius

From Table 1, it is clear that the choice of the positive initial value σ_0 is of crucial importance for the convergence of the two Taylor series (8) and (11). In order for the Taylor series (8) to be convergent, we need to choose σ_0 such that the true implied volatility falls within the convergence interval, i.e., $0 < \Sigma_{\text{imp}}^{\text{BS}}(V_{\text{market}}) < \sigma_0$. A naive choice of the initial value is for example to set $\sigma_0 = 1$, since in practice the implied volatility is never above 100% (e.g., it is around 70% even in the most extreme market condition during the 2008 financial crisis). In this way, we can guarantee that the Taylor series (8) is convergent. Note that if it happens that $\Sigma_{\text{imp}}^{\text{BS}}(V_{\text{market}}) > 2\sigma_0$, or equivalently $\sigma_0 < \Sigma_{\text{imp}}^{\text{BS}}(V_{\text{market}})/2$, then the theoretical analysis (Proposition 2) indicates that the Taylor series (8) is divergent, and consequently the inverted Taylor series (11) should also be divergent.

In order to analyze the sensitivity of the inverted Taylor series with respect to σ_0, and also to substantiate the above discussions, we test the explicit implied volatility formula given in Eq. (11) for different values of σ_0, using option prices generated from a Black and Scholes model. We start from a true value of $\sigma = 0.3$, and assume an initial stock price $S = 100$, the time to maturity $T = 1$, the interest rate $r = 0.05$, and that no dividend is paid. We generate the "market" option price using the Black–Scholes formula. Then we plug the generated "market" option prices into our formula (11) and compare the inverted implied volatility value with the true value $\sigma = 0.3$ that we start with. We consider four different truncation levels, i.e., $n = 1, 4, 8, 16$. For practically relevant implied volatility values, here we take different initial values of σ_0 within the range $(0, 1)$, and plot the values from the formula for different truncation orders. Note that the true implied volatility can not be practically larger than 100%, even in the most extreme market conditions in the 2008 financial crisis.

The numerical results are summarized in Fig. 1. There are two red dashed lines in each of the five sub-graphs: the upper one represents the case when the absolute error equals 10^{-2}, and the lower one corresponds to the case when the absolute error is at 10^{-4}. We justify the results by these two benchmark lines. For error curves below the upper dashed red line, we categorize it as a *feasible* estimation. For error curves below the lower dashed red line, we categorize it as a *good* estimation. We make the following observations:

(a) Fixing the level of the truncation order n, the absolute error of the implied volatility formula converges to 0 rapidly as the initial point approaches the true value ($\sigma_0 \to \sigma = 0.3$ in this case) either from the left or from the right.
(b) Fixing the level of σ_0, and conditional on that the series is convergent with this chosen σ_0 level, the implied volatility formula is converging to the true value (i.e., $\sigma = 0.3$) as the truncation order increases.
(c) The implied volatility formula exhibits stable convergence patterns when σ_0 is residing on the right-hand side of the true implied volatility value, i.e., $\sigma_0 > 0.3$.
(d) The implied volatility formula is divergent when the initial value σ_0 is taken to be smaller than half of the true value (i.e., $\sigma_0 < \frac{1}{2}\sigma = 0.15$). This phenomenon is consistent with the theoretical findings in Proposition 5 and Table 1. Note that in this case, the original Taylor series (8) is divergent, thus the corresponding inverted Taylor series (11) is also divergent.
(e) For ATM case, a truncation order $n = 4$ is good enough for the log error to be below the lower dashed red line, i.e., a *good* estimation. For extreme OTM cases, in order to achieve a *good* estimation, a truncation order of $n = 8$ is required. For extreme ITM cases, a truncation order of $n = 16$ is needed.

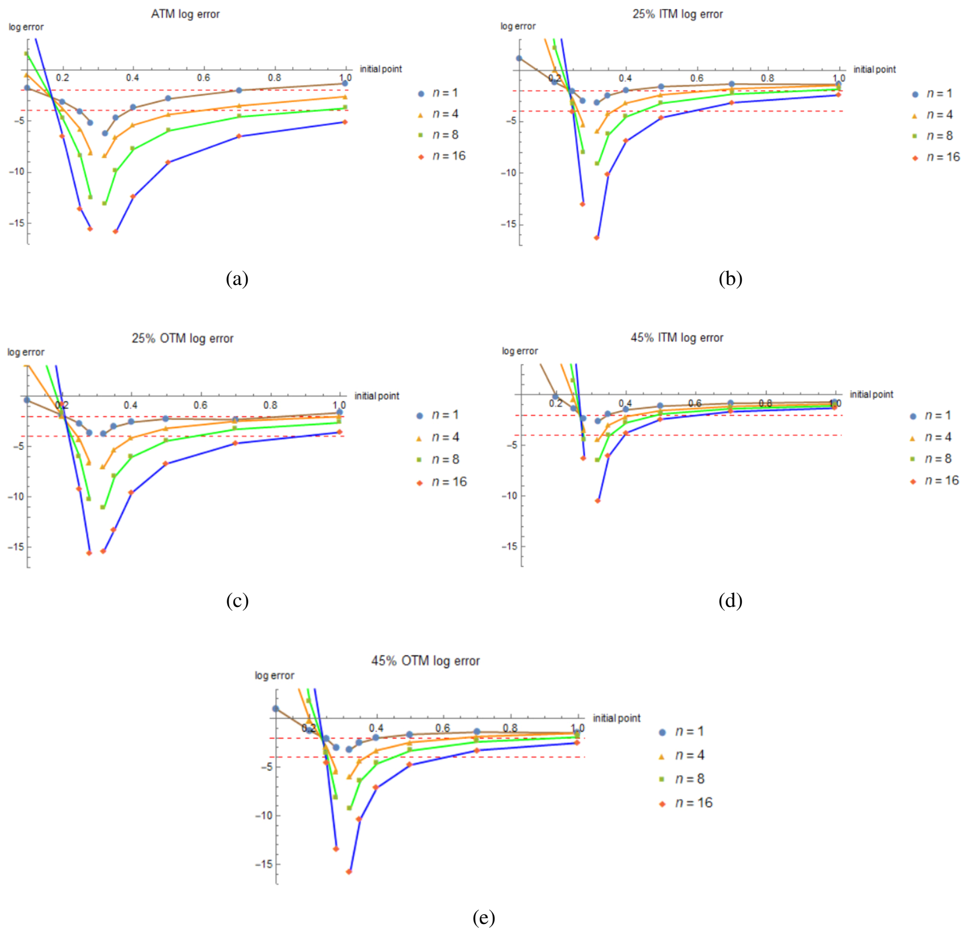


Fig. 1. The log error corresponding to different initial values of σ_0.

In practice, one can choose the initial value σ_0 independent from any model parameter (e.g., choose $\sigma_0 = 1$ in any case) and our stationary test shows that even this naive choice leads to stable (although slow) convergence to the true exact value as the truncation order increases. One natural extension is to use some reasonable estimate based on information of the option that are independent from the underlying stochastic model assumed, i.e., the strike, the maturity and the interest rate etc. In the following, we shall illustrate the significant gain in accuracy if we choose σ_0 based on the above basic information of an option.

From now on, we set σ_0 to be equal to the upper bound (see Proposition 6) given in Tehranchi (2016). In the following Table 2, we report the log errors of the explicit implied volatility formula as compared to the exact value. To report one entire row (ten entries) in Table 2, it takes the following CPU times: 0.01 s ($n = 1$);

Table 2. Log error corresponding to different truncation orders without iteration.

Strike	60	70	80	90	100	110	120	130	140	150
$n = 1$	−0.930977	−1.36495	−2.06819	−3.3251	−6.90561	−8.14309	−3.72417	−2.59724	−1.97486	−1.58266
$n = 5$	−1.34988	−2.15033	−3.67624	−6.66223	−13.5289	−14.1285	−7.6069	−4.92013	−3.46269	−2.59865
$n = 10$	−1.65706	−2.84722	−5.35769	−10.5755	−15.6536	−16.2556	−12.2517	−7.50653	−4.99536	−3.56167
$n = 15$	−1.89052	−3.44412	−6.91698	−14.3806	−15.6536	−16.2556	−15.6536	−9.96422	−6.40771	−4.4155

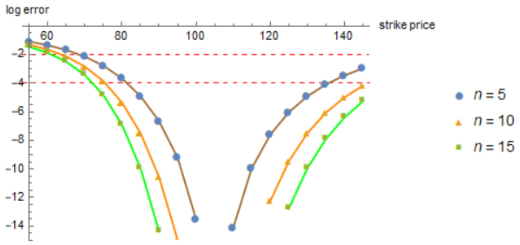


Fig. 2. Lagrange Inversion Expansion combined with upper bound in Tehranchi (2016).

0.4 s ($n = 5$); 7.4 s ($n = 10$); 52.9 s ($n = 15$). We not that most of the CPU time is spent on the calculation using the Faà di Bruno formula, and it is possible to carry out these related symbolic formulae in an offline fashion to further save on the computing time. Furthermore, our implementation is conducted in Mathematica for its convenience of handling symbolic computation, and we believe that the implementation procedure of the explicit closed-form formula in (11) and the choice of programming language can be further optimized.

To summarize, using our volatility representation formula, combined with Tehranchi (2016)'s upper bound as the initial point σ_0, the results (see Fig. 2) are uniformly accurate from very extreme ITM regime to the extreme OTM regime across a wide range of strikes. The formula converges to the true implied volatility as the truncation order increases.

3.2. A hybrid iteration procedure based on Lagrangian inversion series

Moreover, in order to make this method more accurate and efficient, we introduce the following hybrid scheme:

Step 1. Start with σ_0 set as the upper bound in Tehranchi (2016), with a fixed truncation order n, we obtain the first-step estimate the implied volatility $\sigma^{(1)}_{\text{imp}}$.

Step 2. Then we set $\sigma_0 := \sigma^{(1)}_{\text{imp}}$, and then apply the same formula again with the same fixed truncation order n, and obtain the second-step estimate $\sigma^{(2)}_{\text{imp}}$. (Repeat Step 2 for more iterations).

The values of $\sigma^{(1)}_{\text{imp}}$ and $\sigma^{(2)}_{\text{imp}}$ are, respectively, reported in Tables 2 and 3. We can apply the above iteration steps twice, and obtain $\sigma^{(3)}_{imp}$, which is reported in Table 4. From Table 3, we conclude that even with the truncation order equal to 10, one can obtain a remarkably good accuracy for any moneyness including very extreme situations. For the two-step iteration method, Table 4 shows that the

Table 3. Log error corresponding to different truncation orders with 1-step iteration.

Strike	60	70	80	90	100	110	120	130	140	150
$n = 1$	−1.49323	−2.3975	−4.03645	−7.03956	−15.9546	−15.9546	−7.99151	−5.30098	−3.81427	−2.89237
$n = 5$	−3.8121	−8.74515	−15.6536	−15.3014	−15.7785	−16.2556	−16.2556	−15.1764	−15.3014	−11.7103
$n = 10$	−7.96287	−15.3014	−15.9546	−15.9546	−15.6536	−16.2556	−16.2556	−15.4775	−15.9546	−15.7785
$n = 15$	−13.6536	−15.4105	−15.1095	−15.4105	−15.6536	−16.2556	−16.2556	−15.7785	−15.5566	−15.7785

Table 4. Log error corresponding to different truncation orders with 2-steps iteration.

Strike	60	70	80	90	100	110	120	130	140	150
$n = 1$	−2.35705	−4.32938	−7.94478	−14.4105	−15.9546	−16.2556	−15.7785	−10.7002	−7.45814	−5.42719
$n = 5$	−14.6758	−15.4105	−15.9546	−15.5566	−15.7785	−16.2556	−16.2556	−15.2556	−15.3014	−15.3014
$n = 10$	−14.9546	−15.2142	−15.9546	−15.4775	−15.6536	−16.2556	−16.2556	−15.3525	−15.9546	−15.7785
$n = 15$	−14.7785	−15.0252	−15.7785	−16.2556	−15.6536	−16.2556	−16.2556	−15.3014	−15.3014	−15.7785

absolute errors for any claims are around 10^{-15} and lower bound is around $10^{-16.2556}$, which reaches the machine precision in Mathematica.

4. Conclusion

In this paper, we obtain for the first time an explicit and exact Taylor series representation of the implied volatility. We establish the convergence of the formula to the true exact implied volatility. We utilize a model-independent upper bound of the true implied volatility as the initial expansion point σ_0, which is within the convergence radius. The formula performs extremely well when the options are close to the money, and even in extreme cases (e.g., deep in the money and deep out of the money), the accuracy is remarkably good. We also propose a hybrid scheme where we iteratively evaluate our formula, and numerical experiments demonstrate the improved accuracy across all parameter ranges, and in particular significant improvement in accuracy for extreme moneyness cases. We foresee that the proposed formula can be highly valuable to both the theory and practice of financial engineering.

Appendix A. Operator Calculus and Proofs of Main Results

In this section, we mainly discuss about how to utilize operator calculus to decouple different model parameters. An alternative application is to build bridge between different parameters, and the details are discussed in Appendix A.2.

A.1. Proof of Lemma 1

Recall the following well-known identity:

$$FN'(d_1)\frac{\partial d_1}{\partial S} - KN'(d_2)\frac{\partial d_2}{\partial S} \equiv 0,$$

where $F := Se^{(r-q)T}$ is the forward stock price and K is the strike price. The spatial derivative with respect to $x = \ln S$ is

$$\begin{aligned}\frac{\partial V}{\partial x} &= c \cdot e^{x-qT} N(c \cdot d_1)\\ &= c \cdot e^{x-qT}\frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{c \cdot d_1}{\sqrt{2}}\right)\right),\end{aligned}$$

where c is the option type indicator, i.e., $c = 1$ for the call option, and $c = -1$ for the put option.

We denote the jth partial differential operator with respect to x as ∂_x^j. To simplify the notation, we denote the term in the error function as $g(x)$, i.e., $g(x) := \frac{c \cdot d1}{\sqrt{2}}$, or equivalently

$$g(x) := c \cdot \frac{x + (r-q)T - \ln(K) + \sigma^2 T/2}{\sigma\sqrt{T}}.$$

We first recall the following result on the derivatives of the error function.

Lemma A.1 (*n*th derivatives of the error function).

$$\text{erf}^{(j)}(y) = \frac{2}{\sqrt{\pi}} e^{-y^2} H_{j-1}(y)(-1)^{j-1}, \tag{A.1}$$

where function $H_n(\cdot)$ are Hermite polynomials given by

$$H_n(x) = (-1)^n e^{x^2}\left(\frac{d^n}{dx^n} e^{-x^2}\right) = \left(2x - \frac{d}{dx}\right)^n \cdot 1. \tag{A.2}$$

One can utilize these formulas to obtain the derivatives of the option value:

$$\begin{aligned}
\frac{\partial^{n+1} V}{\partial x^{n+1}} &= \frac{e^{x-qT}}{2} + \frac{1}{2}\frac{\partial^n}{\partial x^n}(e^{x-qT}\text{erf}(g(x))) && \text{(A.3)}\\
&= \frac{e^{x-qT}}{2} + \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j} e^{x-qT}\partial_x^j \text{erf}(g(x)) && \text{(A.4)}\\
&= \frac{e^{x-qT}}{2}(1+\text{erf}(g(x))) + \frac{e^{x-qT}}{2}\sum_{j=1}^{n}\binom{n}{j}\partial_x^j \text{erf}(g(x)) && \text{(A.5)}\\
&= \frac{e^{x-qT}}{2}(1+\text{erf}(g(x)))\\
&\quad + \frac{e^{x-qT}}{2}\sum_{j=1}^{n}\binom{n}{j}\frac{2}{\sqrt{\pi}}(-1)^{j-1}(\sqrt{2}\sigma\sqrt{T})^{-j}(\exp(-g(x)^2 H_{j-1}(g(x))), && \text{(A.6)}
\end{aligned}$$

where $n = 0, 1, 2, \ldots$. This completes the proof.

A.2. Proof of Theorem 1

In this section, we present a formal derivation of the arbitrary higher-order Greeks of the Black–Scholes model, which is based on operator calculus and the Faà di Bruno formula. To simplify our analysis, we denote the spatial parameter as $x = \ln S$. Our goal is to express higher-order derivatives with respect to the volatility σ, interest rate r, the dividend rate q in terms of the spatial derivatives x

derived in Appendix A.1. Below, we first consider the Black–Scholes equation and derive its formal solution.

Definition A.1. The Black–Scholes price $U(x,\tau)$ is determined by the evolution problem:

$$\frac{\partial U(x,\tau)}{\partial \tau} = \mathcal{L} \circ U(x,\tau), \quad 0 < \tau < T \tag{A.7}$$

$$U(x,0) = f(x), \tag{A.8}$$

where τ denotes the time to maturity, and the parameterized linear operator $\mathcal{L} = \mathcal{L}(\sigma, r, q)$ is given by

$$\begin{aligned}\mathcal{L} &:= \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} + (r-q)S\frac{\partial}{\partial S} - r\mathcal{I} \\ &= \frac{\sigma^2}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) + (r-q)\frac{\partial}{\partial x} - r\mathcal{I} \\ &= \frac{\sigma^2}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) - q\frac{\partial}{\partial x} + r\left(\frac{\partial}{\partial x} - \mathcal{I}\right).\end{aligned} \tag{A.9}$$

Operator calculus treats Eq. (A.7) as an ordinary differential equation (ODE) in τ. If we further assume that the linear operator $\mathcal{L}$ is time-independent, then we can treat it as a constant. Therefore, the formal solution of Eq. (A.7) subject to the terminal condition (A.8) is given by

$$U(x,\tau) = \exp\{\tau\mathcal{L}\} \circ f(x), \tag{A.10}$$

where $\exp\{\tau\mathcal{L}\}$ is itself a linear operator with the following series representation:

$$\exp\{\tau\mathcal{L}\} = \sum_{k=1}^{\infty} \frac{(\tau\mathcal{L})^k}{k!}. \tag{A.11}$$

Remark A.2. Recall that two operators $\mathcal{A}, \mathcal{B}$ are said to *commute* if $\mathcal{AB} \equiv \mathcal{BA}$ holds. To simplify our calculation, we decompose the linear operator $\mathcal{L}$ into three commuting operators:

$$\mathcal{L} = \mathcal{L}_\sigma + \mathcal{L}_q + \mathcal{L}_r, \tag{A.12}$$

where

$$\mathcal{L}_\sigma(\sigma) := \frac{\sigma^2}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right), \tag{A.13}$$

$$\mathcal{L}_q(q) := -q\frac{\partial}{\partial x}, \tag{A.14}$$

$$\mathcal{L}_r(r) := r\left(\frac{\partial}{\partial x} - \mathcal{I}\right). \tag{A.15}$$

Remark A.3. In Definition A.1, $\mathcal{L}$ is parameterized by (σ, r, q), which are the model parameters. We have that any derivative with respect to a model parameter z will have the following representation:

$$\begin{aligned}\frac{\partial^n U(x,\tau)}{\partial z^n} &= \frac{\partial^n}{\partial z^n}(\exp\{\tau\mathcal{L}(z)\} \circ f(x)) \\ &= \frac{\partial^n \exp\{\tau\mathcal{L}(z)\}}{\partial z^n} \circ f(x).\end{aligned} \tag{A.16}$$

To begin our derivation, we briefly recall the well-known Faà di Bruno formula.

Theorem A.4 (Faà di Bruno formula). *Given a composite function* $f(g(x))$, *the following equation holds*,

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)), \tag{A.17}$$

where $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ *are the Bell polynomials defined by*

$$\begin{aligned}&B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \\ &\quad = \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}.\end{aligned} \tag{A.18}$$

Here, the summation is taken over all sequences $j_1, j_2, j_3, \ldots, j_{n-k+1}$ *of nonnegative integers, such that*

$$j_1 + j_2 + j_3 + \cdots + j_{n-k+1} = k,$$

$$j_1 + 2j_2 + 3j_3 + \cdots + (n-k+1)j_{n-k+1} = n.$$

The above result is valid in the context of real-valued C^∞ function f and g. To obtain an extended Faà di Bruno formula suitable for calculating (A.16), we provide an operator extension as follows:

Proposition A.5 (Extended Faà di Bruno formula). *Given a function represented through the evolution equation* (A.10), *the higher-order derivatives* *of $U(x,\tau)$ with respect to the model parameter z are given by*

$$
\begin{aligned}
\partial_z^n[\exp(\mathcal{G}(z))f(x)] &= (\partial_z^n \exp(\tau\mathcal{L}(z)))\cdot f(x)\\
&= \sum_{k=1}^{n} B_{n,k}(\mathcal{G}'(z),\mathcal{G}''(z),\ldots,\mathcal{G}^{(n-k+1)}(z))\exp^{(k)}(\mathcal{G}(z))f(x)\\
&= \sum_{k=1}^{n} B_{n,k}(\mathcal{G}'(z),\mathcal{G}''(z),\ldots,\mathcal{G}^{(n-k+1)}(z))\\
&\quad\times\exp(\mathcal{G}(z))f(S)(\mathcal{G}(z))f(x),
\end{aligned}
\tag{A.19}
$$

where

$$\mathcal{G} = \tau\mathcal{L} = \tau(\mathcal{L}_\sigma + \mathcal{L}_r + \mathcal{L}_q). \tag{A.20}$$

We denote the above method as the "extended Faà di Bruno formula" since it is a straightforward extension under operator calculus. In addition, derivatives with respect to model parameters satisfy:

$$
\begin{aligned}
\partial_\sigma^n\mathcal{G} &= \tau\cdot\partial_\sigma^n\mathcal{L}_\sigma\\
&= \begin{cases}
\tau\cdot\sigma\left(\dfrac{\partial^2}{\partial x^2}-\dfrac{\partial}{\partial x}\right), & n=1\\[2ex]
\tau\cdot\left(\dfrac{\partial^2}{\partial x^2}-\dfrac{\partial}{\partial x}\right), & n=2\\[2ex]
0, & n\geq 3
\end{cases},
\end{aligned}
\tag{A.21}
$$

$$
\begin{aligned}
\partial_q^n\mathcal{G} &= \tau\cdot\partial_q^n\mathcal{L}_q\\
&= \begin{cases}
-\tau\cdot\dfrac{\partial}{\partial x}, & n=1\\[2ex]
0, & n\geq 2
\end{cases},
\end{aligned}
\tag{A.22}
$$

$$
\begin{aligned}
\partial_r^n\mathcal{G} &= \tau\cdot\partial_r^n\mathcal{L}_r\\
&= \begin{cases}
\tau\cdot\left(\dfrac{\partial}{\partial x}-\mathcal{I}\right), & n=1\\[2ex]
0, & n\geq 2
\end{cases}.
\end{aligned}
\tag{A.23}
$$

From the above calculations, it can be seen that the higher-order derivatives of operators $\mathcal{L}$ vanish to zero, which brings convenience in computing the Greeks of the Black–Scholes formula. We have

$$\begin{aligned}\partial_\sigma^n U(x,\tau) &= (\partial_\sigma^n \exp(\tau\cdot\mathcal{L})f(x)\\ &= \sum_{k=1}^{n} B_{n,k}(\mathcal{G}'(\sigma),\mathcal{G}''(\sigma),\ldots,\mathcal{G}^{(n-k+1)}(\sigma))\exp^{(k)}(\tau\cdot\mathcal{L})f(x) \qquad \text{(A.24)}\\ &= \underbrace{0+\cdots+0}_{k\text{–}2} + B_{n,n-1}(\mathcal{G}'(\sigma),\mathcal{G}''(\sigma))\exp(\tau\cdot\mathcal{L})f(x)\\ &\quad + B_{n,n}(\mathcal{G}'(\sigma))\exp(\tau\cdot\mathcal{L})f(x), \qquad \text{(A.25)}\end{aligned}$$

where

$$B_{n,n}(x) = x^n,$$
$$B_{n,n-1}(x,y) = \frac{n!}{2(n-2)!}x^{n-2}y.$$

By substitution and simplification, we have

$$\partial_\sigma^n U(x,\tau) = \sum_{k=1}^{n}\frac{n!}{(2k-n)!(n-k)!}\frac{T^k\sigma^{2k-n}}{2^{n-k}}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\partial_x^{k+j}f(x).$$

This corresponds to (5), which is one of the key formulae in Theorem 1. To obtain higher-order derivatives of the option price with respect to other model parameters such as r and q, one can substitute (A.14) and (A.15) into (A.19), and follow similar calculations. This completes the proof.

Appendix B. Convergence of the Power Series

In this section, we discuss in more details the radius of convergence of the Taylor series 8 and 11. For illustration purpose, we quote several well established results in the theory of power series of analytic functions.

B.1. Convergence of the Taylor Series in (8)

First, we review some basic results on analytic functions in complex analysis. Recall the following basic property about Taylor series.

Proposition B.1 (Singularities and the Convergence Radius of Taylor Series). *Given the singularities of a function as the points at which the function is not analytic, let $f(x)$ be analytic everywhere except at a finite number of singularities. Denote $R \in \mathbb{R}_+$ as the distance from x_0 to the closest singularity of f. Then, the*

Taylor series of f about x_0 converges to f at every point $x \in \mathbb{R}$ satisfying $|x - x_0| < R$.

In the current study, the only singularity of the Black–Scholes formula (2) as a function of σ is at the origin $\sigma = 0$, which is an *essential singularity*, i.e., it is neither a pole or a removable singularity. Thus the Taylor series 8 converges in a disc with radius being the distance from σ_0 to the origin. The radius of convergence is clearly $|\sigma_0 - 0| = \sigma_0$ and the series is divergent beyond this radius. In the real line, the power series (8) converges uniformly to the Black–Scholes formula as the truncation order $n \to \infty$ when $0 < \sigma < 2\sigma_0$ and diverges when $\sigma > 2\sigma_0$.

B.2. Convergence of the Taylor Series in (11)

The Lagrange inversion theorem (i.e., Theorem 3) ensures that the Taylor series 11 has a nonzero radius of convergence, however, it is not easy to find the explicit radius of convergence. This motivates us to test the actual radius in a numerical experiment. First, we recall Theorem 94 in chapter 18 of Knopp (1990).

Theorem B.2 (The Radius of Convergence of Power Series). *If the power series $\sum_{j=1}^{\infty} a_j(x - x_0)^j$ is given and μ denotes the upper limit of the (positive) sequence of numbers $|a_1|$, $\sqrt{|a_2|}$, $\sqrt[3]{|a_3|}, \ldots,$ $\sqrt[n]{|a_n|}$, i.e., $\mu = \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}$, then*

(1) *if $\mu = 0$, the power series is everywhere convergent;*
(2) *if $\mu = +\infty$, the power series is nowhere convergent;*
(3) *if $0 < \mu < \infty$, the power series*
   (a) *converges absolutely for every $|x - x_0| < \frac{1}{\mu}$,*
   (b) *but diverges for every $|x - x_0| > \frac{1}{\mu}$.*

thus, with the suitable interpretation,

$$R = \frac{1}{\mu} = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}}$$

is the radius of convergence of the given power series.

By the merit of the simplicity of this theorem, a standard numerical method to estimate the radius of convergence is given by follows:

$$R_n = \frac{1}{\sqrt[n]{|a_n|}}. \tag{B.1}$$

In the following, we first investigate the convergence property of R_n, and then estimate it to see whether our representation is within the radius of convergence. The test is carried out under different sets of parameters reflecting different scenarios. Denote the initial stock price as S, the time to maturity as T, the strike price as K, the interest rate as r, the dividend rate as q, and the true volatility as σ. Assume that the market price V is generated from plugging σ into the Black–Scholes formula.

In testing the asymptotic property of R_n, we fix our initial point using the Tehranchi (2016)'s upper bound, i.e., $\sigma_0 := \sigma_{\mathrm{Teh}}(V, S, K, T, r, q)$. We vary the values of the other model parameters in some extreme cases and evaluate R_n from $n = 1$ to a sufficient large number N. Figure B.1 showcases the sequence R_n against the index n under different scenarios. Note that the ten test cases (cases (a)–(i)) considered here reflect situations of normal market conditions as well as very extreme market situations, e.g., deep ITM, deep OTM, high and low volatility levels, high and low interest rate environments, and short and long maturities. We believe that these ten cases are representative of actual market situations.

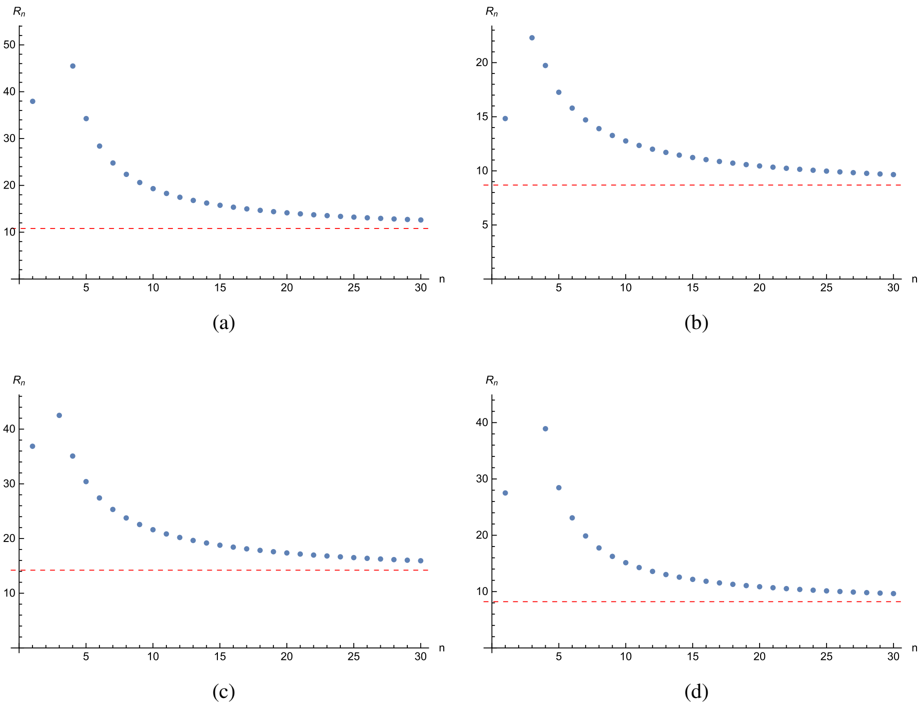
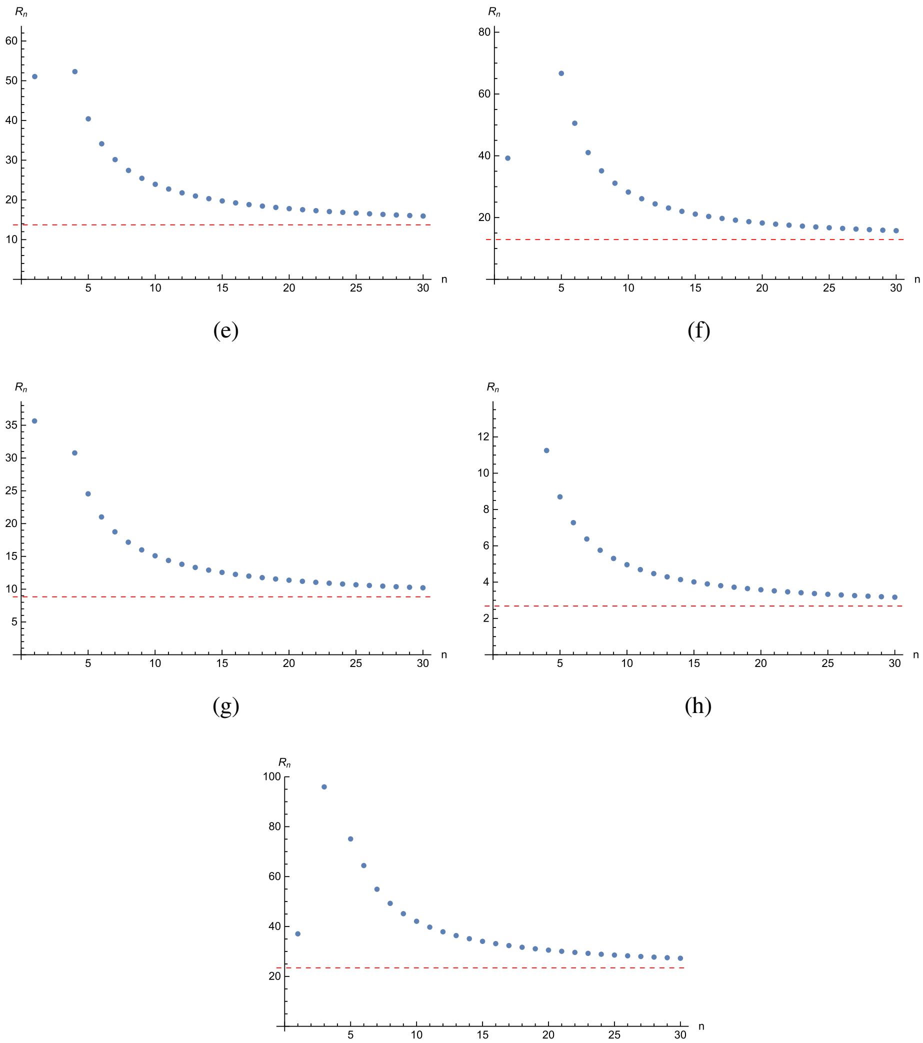


Fig. B.1. The convergence of R_n under different model parameters. (a) $S = 100$, $K = 100$, $T = 1$, $r = 0.05$, $q = 0$, $\sigma = 0.3$; (b) $S = 100$, $K = 40$, $T = 1$, $r = 0.05$, $q = 0$, $\sigma = 0.3$; (c) $S = 100$, $K = 200$, $T = 1$, $r = 0.05$, $q = 0$, $\sigma = 0.3$; (d) $S = 100$, $K = 100$, $T = 0.5$, $r = 0.05$, $q = 0$, $\sigma = 0.3$; (e) $S = 100$, $K = 100$, $T = 2$, $r = 0.05$, $q = 0$, $\sigma = 0.3$; (f) $S = 100$, $K = 100$, $T = 1$, $r = 0.01$, $q = 0$, $\sigma = 0.3$; (g) $S = 100$, $K = 100$, $T = 1$, $r = 0.1$, $q = 0$, $\sigma = 0.3$; (h) $S = 100$, $K = 100$, $T = 1$, $r = 0.05$, $q = 0$, $\sigma = 0.1$; (i) $S = 100$, $K = 100$, $T = 1$, $r = 0.05$, $q = 0$, $\sigma = 0.6$.



(e) (f) (g) (h) (i)

Fig. B.1. (*Continued*)

From Fig. B.1, it is clear that in all ten scenarios the sequence $\{R_n\}$ is convergent to some positive number R_∞. Moreover, the observed monotonically decreasing property of this sequence after some sufficient large number further guarantees that

$$R_\infty = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}} = R.$$

Therefore, R_N can be viewed as the Nth order approximation for the true (unknown) radius of convergence R.

Next, we employ this numerical approximation of the radius of convergence, and compare it with the absolute price deviation $|V_{\text{market}} - V_{\sigma_0}|$ by plotting them onto the same graph. Here, V_{σ_0} is obtained by substituting the Tehranchi (2016)'s upper bound (13) into the Black–Scholes formula. The results of the test under different parameter settings are summarized in Fig. B.2, respectively.

Figure B.2 demonstrates that the absolute price deviation is always within the approximate radius of convergence R_{30}, i.e., $|V_{\text{market}} - V_{\sigma_0}| < R_{30}$. Note that R_{30} is already a very precise estimate of the true radius of convergence R, thus we can conclude that $|V_{\text{market}} - V_{\sigma_0}| < R$ most of the time if not always. This means that our representation $\Sigma^{\text{BS}}_{\text{imp}}(V, S, K, T, r, q)$ defined by (11) converges to the true

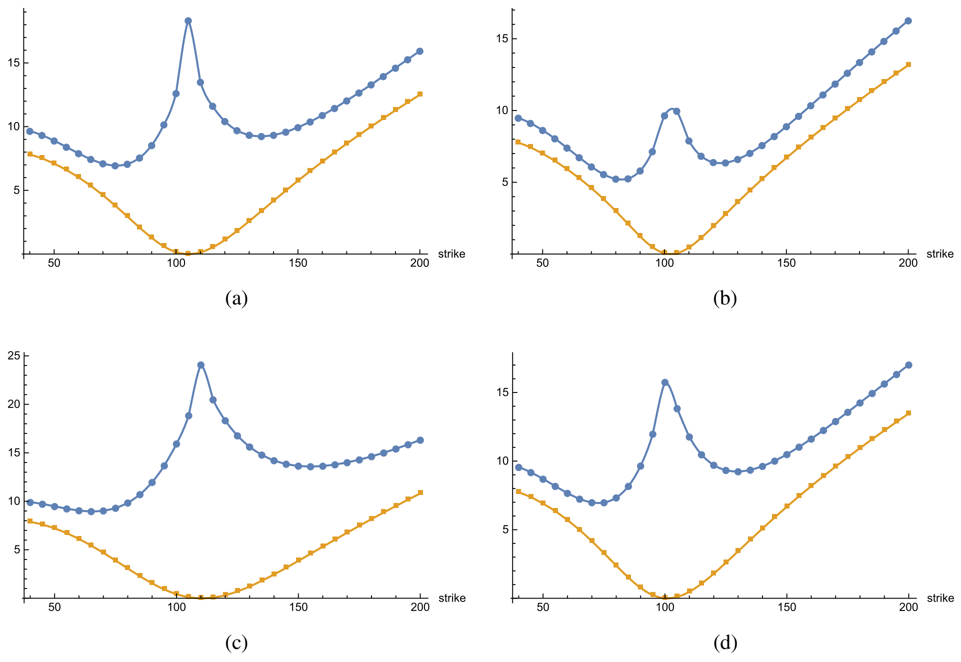
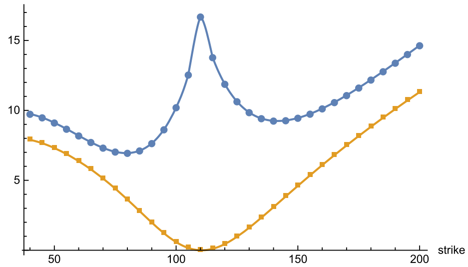
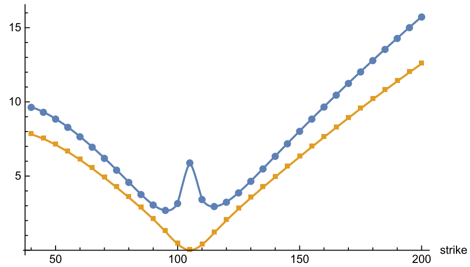


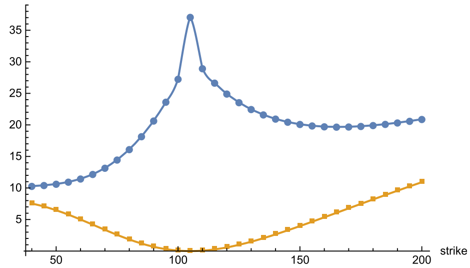
Fig. B.2. (Color online) The convergence radius and the price deviation are plotted as a function of the strike for different maturities T, interest rates r and true volatilities σ. We first fix the initial point σ_0 as defined by (13). The blue line corresponds to the radius of convergence of the Lagrange inversion series 11, obtained by Eq. (B.1) combined with Theorem 4, and then numerically solved to the 30th order approximation R_{30}. The dark yellow line corresponds to the deviation of the price to the center point $|V_{\text{market}} - V_{\sigma_0}|$. The parameters are: (a) $S = 100$, $40 \leq K \leq 200$, $T = 1$, $r = 0.05$, $q = 0$, $\sigma = 0.3$; (b) $S = 100$, $40 \leq K \leq 200$, $T = 0.5$, $r = 0.05$, $q = 0$, $\sigma = 0.3$; (c) $S = 100$, $40 \leq K \leq 200$, $T = 2$, $r = 0.05$, $q = 0$, $\sigma = 0.3$; (d) $S = 100$, $40 \leq K \leq 200$, $T = 1$, $r = 0.01$, $q = 0$, $\sigma = 0.3$; (e) $S = 100$, $40 \leq K \leq 200$, $T = 1$, $r = 0.1$, $q = 0$, $\sigma = 0.3$; (f) $S = 100$, $40 \leq K \leq 200$, $T = 1$, $r = 0.05$, $q = 0$, $\sigma = 0.1$; (g) $S = 100$, $40 \leq K \leq 200$, $T = 1$, $r = 0.05$, $q = 0$, $\sigma = 0.6$.



(e)



(f)



(g)

Fig. B.2. (*Continued*)

implied volatility since it is within the radius of convergence. On the other hand, the graphs have a hump in the radius of convergence around $K = F = e^{(r-q)T}S$, whereas the price deviation approaches its minimum within the same area. In fact, it reflects that the Lagrange inversion series converges more rapidly when $K = F = e^{(r-q)T}S$, i.e., when the option is very close to the money from the right or from the left. Thus, we shall expect that the performance of the inversion series is extremely well for ATM options. This observation is consistent with numerical evidence exhibited in Tables 2–4 by comparing the columns in the middle to those in the left and right ends.

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