CSWP 22-23: Simple 2D BEM Implementation

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In this exercise you will implement a simple boundary element solver in 2D. We want to solve the equation

$$\int_\Gamma G_\kappa(x,y)q(y)dy = -e^{inc}(x), \quad \forall x \in \Gamma \tag{1}$$

Here Γ is the segment stretching from $(-1,0)$ to $(+1,0)$. Remember that the Green function for the 2D wave equation is given by

$$G_\kappa(x,y) = -\frac{i}{4}H_0^{(2)}(k|x-y|) \tag{2}$$

with $H_0^{(2)}$ the Hankel function of order 0 and and kind 2 (matlab: `besselh`, python: `scipy.special.hankel2`).

The discretised weak form of this equation reads: find $q(x) = \sum_{i=1}^N q_i p_i(x)$, such that, for all $j = 1, ..., N$ it holds that

$$\sum_{i-1}^{N} \left(\int_\Gamma dx \int_\Gamma dy p_j(x) G_\kappa(x,y) p_i(y) \right) q_j = -\int_\Gamma dx p_j(x) e^{inc}(x), \quad j = 1, ..., N. \tag{3}$$

or $Aq = b$ in short. The unknown function is expanded in pulse functions $p_i(x)$, which take on the value 1 on segment i and zero elsewhere. A generic element of the system matrix thus takes on the form:

$$A_{ij} = \int_{S_i} dx \int_{S_j} dy G_\kappa(x,y) \tag{4}$$

where S_i, S_j are the i-th and j-th segment, respectively. We will use the midpoint rule to approximate the integral over x, and the trapezoidal rule the approximate the integral over y, resulting in

$$A_{ij} \approx \frac{1}{2} l_i l_j \left(G_\kappa(x_{i,C}, y_{j,L}) + G_\kappa(x_{i,C}, y_{j,R}) \right) \tag{5}$$

where $x_{i,C}$ is the centre point of S_i, and $y_{j,L}, y_{j,R}$ are the left and right points of segment S_j.

The incident field is taken to be a plane wave:

$$e^{inc}(x) = \exp(-ikd \cdot x) \tag{6}$$

with d the unit vector corresponding to the direction the wave travels in.

Assignment

- Construct the matrices A and b and solve for q. Plot the values of q along the segment Γ.
- You will see that there are unexpected oscillations superimposed on the solution q. It turns out that these can be traced back to the lack of accuracy of some of the integrals appearing in the expression of A_{ij}. Make a note in the comments about which elements you expect to be accurately approximated and which require a more careful treatment.
- Fix the problem you diagnosed by using the small argument approximation of the Green function:

$$G_\kappa(x, y) \approx -\frac{i}{4} - \frac{1}{2\pi}\left(\gamma + \log\frac{k|x-y|}{2}\right) \tag{7}$$

  with γ the Euler constant.
- Write a post-processing routine that computes the electric field on a regular grid of points encompassing the scatterer. This is completely similar to building the system matrix A but now, instead of having the segments interact amongst themselves, you will need to compute the interaction of a segment and a point on the grid. Use your routine to create a visual representation of the field near the segment (matlab: `surf`, python: `matplotlib.pyplot.imshow`)

The solution to this exercise should be submitted as a single script to UFORA. Answers to non-computing questions should be included as comments in your script.