CSWP 22-23: Mathematical Techniques for the Solution of Wave Equations

October 24, 2024

The solution to this exercise should be submitted as a single script to UFORA. Answers to non-computing questions should be included as comments in your script.

The Nystrom method

Consider the integral equation:

$$\phi(x) = x + \lambda \int_0^1 \sqrt{x} s^2 \phi(s) ds \tag{1}$$

$$\tag{2}$$

Fredholm equations of the 2nd kind like thi can be solved approximately by application of the so-called Nystrom method. In the Nystrom method, the unknowns are the function values of ϕ in a set of quadrature nodes used to approximate the integral that appears in the equation. Demanding that the equation holds in the same set of quadrature points results in a square linear system that can be solved using a direct or iterative solution method.

Most scripting languages for scientific computing provide a means to generate the nodes and weights:

- python: documentation for numpy supplied Gauss-Legendre rule.
- matlab: Community supplied script for the computation of Gauss-Legendre quadrature rule.
- julia: FastGaussQuadrature.jl package for the compuation of Gaussian quadrature rules.

It turns out that this integral equation can also be solved analytically (the right hand side provides a one-parameter representation for the solution!).

- Compute the analytic solution for a generic value for λ and provide the expression as a comment in your script.

- Use either the conjugate gradient or GMRES iterative method to solve the linear system resulting upon application of the Nystrom method. Motivate your choice in the comments in your script.
- Have your solution script produce a plot containing both the analytical and numerical solution at $\lambda = 1$ using 100 quadrature points.
- The solution breaks down when λ attains an eigenvalue for the second kind operator. Make a note of this value in the comments in your script.
- Compute the condition number of the system matrix as λ progresses through the range $[1, 4]$ with increments of 0.05. Make a plot and explain what you see.

Numerical quadrature for definite integrals

Consider the integral

$$I = pv \int_{-2}^{1} \frac{1 + |x|}{\tan(\exp x - 1)} dx \tag{3}$$

Here, pv denotes that the integral should be interpreted as Cauchy principal value, i.e.

$$pv \int_{a}^{b} f(x)dx = \lim_{\epsilon \to 0} \left(\int_{a}^{-\epsilon} f(x)dx + \int_{\epsilon}^{b} f(x)dx \right) \tag{4}$$

Compute a numerical approximation of this integral, exact to up 10 decimal digits. The task can be reduced to computing an ordinary Riemann integral by subtracting a judiciously chosen integrand that (i) can be treated analytically, and (ii) such that the remainder does not contain a singularity at $x = 0$.