EXERCISE: Accuracy of 2D-FDTD scheme staggered versus collocated

An easy method to determine the accuracy of a numerical scheme consists in comparing numerical results with a known analytical solution.

The acoustical field generated by a (coherent) line source takes the following form in frequency domain:

$$p(r) = i\pi H_0^1(kr)$$

with *p* the acoustic pressure, H_0^1 is the Hankel function of the first kind and zeroth order, *k* is the wave number and *r* the distance between source and receiver. A (coherent) line source can easily be simulated as a point source in a two-dimensional simulation domain. Note that the two-dimensional electromagnetic problem has exactly the same solution.

Numerical accuracy can be subdivided in amplitude and phase errors. The ratio of amplitude and difference in phase between two locations exposed to the cylindrical wave, calculated using FDTD, can be compared to the theoretical amplitude ratio and phase difference. Using two receivers (observation points) allows reducing dependence on the details of the implementation of the source. In this exercise we consider two directions, one along the coordinate axis, one along the diagonal of the Cartesian grid.

EXERCISE

1. Implement a 2D-FDTD scheme on a grid that is staggered-in-time and staggered-in-space for a lossless and homogeneous propagation medium:

$$\frac{\partial p}{\partial t} + c^2 \nabla \cdot \mathbf{o} = 0$$
$$\frac{\partial \mathbf{o}}{\partial t} + \nabla p = 0$$

To avoid reflections on the edges of the simulation domain, specific boundary conditions are needed. However, if the simulation is stopped quickly enough the boundary does not influence the impulse response. Hence one can for example implement a simple cyclic boundary condition, assuming that the simulation domain is repeated over and over again.

[Note : Only the function *step_SIT_SIP in python* OR the function defined in the file *step_SIT_SIP_cyclic_noflow.m* needs to be written. The main program with definition of simulation parameters and grid and time stepping loop, are given as well as a function for post-processing the results. Study this code beforehand.]

2. Now implement a 2D FDTD scheme in a collocated grid in space and in time. The same type of boundary condition can be used.

3. In both of the implementations above, choose the value for CN that results in the lowest numerical error and compare amplitude and phase errors based on the numerical results.

4. Now calculate the theoretical phase error, plot it as a function of k dx and compare with the numerical results obtained in 3.

5. Bonus: try to implement the 7^{th} order collocated DRP scheme described in the course and analyze the numerical error.

PRACTICAL

By the end of lecture, each student needs to submit a brief **report** (pdf file, WORD file, ...), and the **program files** via Ufora. Include a brief derivation of the **discretised propagation equations** and the **derivation of the theoretically expected numerical error**.