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Equal Risk Contribution Portfolios

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Abstract

Decades since the introduction of modern portfolio theory by Harry Markowitz in 1952, portfolio optimization remains an actively studied research problem. There exist any number of schemes for constructing the "optimal" portfolio, which run the gamut from simple rules of thumb to highly technical approaches founded on (e.g.) stochastic control theory. In recent years, a class of purely risk-based allocation programs have become popular. In this note, we review one method in particular: the equal risk contributions (ERC) portfolio. ERC is a robust option not only for building portfolios of assets, but also for combining trading models in a multi-strategy fund. We attempt to elucidate the ERC methodology and give some intuition for its behavior.

Keywords

Portfolio Construction, Equal Risk Contributions, Minimum Variance, Mean Variance

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1. Introduction

The mathematical problem of portfolio optimization has been studied extensively since the mean-variance framework was first put forth by Markowitz (1952). The basic Markowitz portfolio finds an allocation vector $\mathbf{w}$ which minimizes risk while maximizing an expected return (effectively maximizing the expected Sharpe Ratio):

$$\max_{\mathbf{w}} \; \mathbf{f}^T\mathbf{w} - \frac{1}{2}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w},$$

where f represents an estimate of the expected returns and $\boldsymbol{\Sigma}$ is the covariance matrix.

The classical Markowitz solution is not only important theoretically; it continues to be the ubiquitous baseline approach to portfolio building. Nevertheless, successful application of the method has its subtleties. Significant attention has been devoted to the issue of regularizing $\boldsymbol{\Sigma}$ (Ledoit and Wolf, 2003; Karoui, 2008). Specification of $\mathbf{f}$ requires some forecast of future expected returns, to which the final solution $\mathbf{w}$ can be quite sensitive.

More recently, there is growing interest in allocation models which consider only the risk term,

$$\sigma(\mathbf{w}) = \sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}} \quad \text{(portfolio risk)}, \tag{1}$$

some examples being the minimum variance and ERC portfolios (Maillard et al., 2010). Our focus in this note is on the latter. We start by trying to build some intuition for how and why ERC makes the allocations it does. Next, we formalize the problem and give the brief mathematical description of the approach. Finally, we draw some connections to minimum variance and describe when and why ERC might be an appropriate choice.

2. The Intuition Behind ERC

As we've alluded, the ERC portfolio does not require the investor to express a view on future expected returns[1]. Of course, in that sense, it is just one of many such approaches, the simplest of all being the naive $1/n$ equal weighting rule. The equally weighted portfolio invests equal dollar amounts in each constituent. ERC, on the other hand, tries to allocate *risk* equally. Essentially, it postulates that without any strong *a priori* expectation about returns, the robust choice is to let each constituent[2] contribute equally to the portfolio's overall volatility.

It's helpful to think about how the ERC portfolio differs from the simple equal weighting scheme. The fundamental difference arises in ERC's use of the covariance matrix, which encodes information about both the volatility and the correlation structure among assets. That volatility is important is not surprising. Intuitively, investors understand that an equal dollar allocation to stocks and bonds does not correspond to an equal *risk* allocation, since historically stock markets have been much more volatile than government bonds.

Correlation effects are more subtle but just as important. Say we have three assets; the first two are perfectly correlated and the third is uncorrelated with either of those two. Figure 1 illustrates the situation, showing quite simply how minding the correlation can lead to a divergence between the equal weights and ERC portfolios.

While these simple examples might shed some light, real

[1] Any view, that is, other than the implicit assumption that they are positive in the long run.

[2] Henceforth we'll generically refer to the constituents of the portfolio as "assets," with the understanding that these constituents could be financial instruments such as futures contracts, *or* strategies within a multi-strategy fund. ERC is one solution to a more general allocation problem, for which the targets of allocation are not necessarily traditional investable assets.

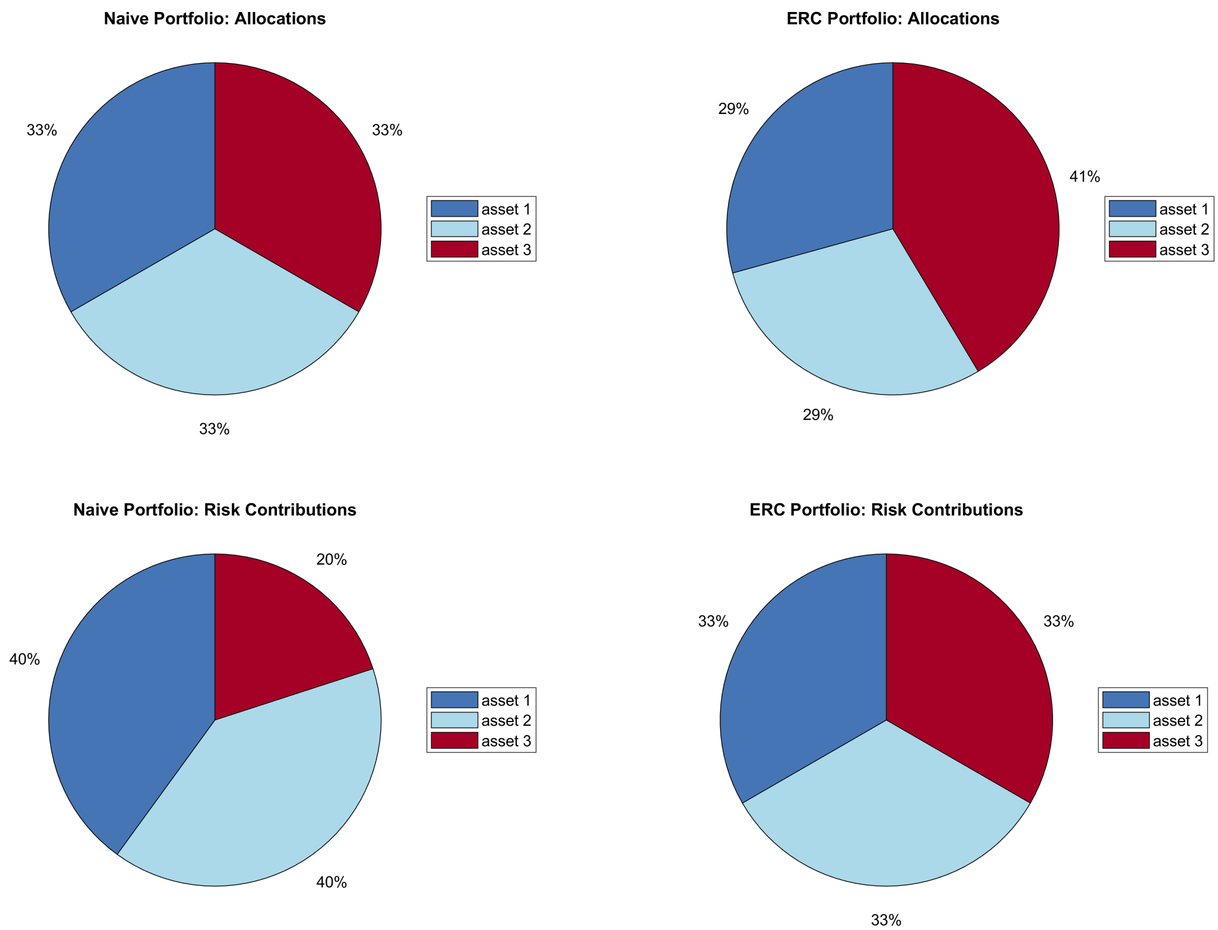


Figure 1. Consider a portfolio of three assets with identical volatility, where the first two are perfectly correlated and the third is uncorrelated with either. The naive $1/n$ portfolio gives equal allocations to each asset, but the sources of risk are not balanced. The ERC portfolio on the other hand finds allocations which ensure that the risk contributions are equalized.

market data is more complicated and much more interesting. Figure 2 shows a set of results for a simulated ERC portfolio of futures contracts on three underlying instruments: the US 30 Year Bond, the S&P 500, and the US Dollar Index. The figure shows rolling volatilities, correlations, and ERC allocations as a fraction of gross portfolio value. The allocations are nonlinear functions of volatility and correlation, and while it is not always obvious from visual inspection how they are being determined, the basic idea is that ERC down-weights positively correlated and/or highly volatile assets (and vice versa). It becomes even more difficult to appreciate the behavior of the solution for portfolios of realistic dimension (e.g. dozens of constituents), but the same general principles will apply.

3. The Mathematics of ERC

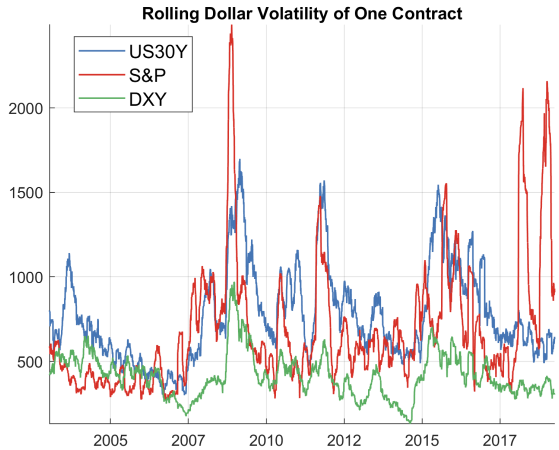
Here we'll try to formalize the concepts described thus far and give a concise mathematical description of the ERC problem. In order to do this, we first need to define what we really mean when we talk about the "risk contribution" of an asset. It turns out that the risk (1) has a special property which allows us to decompose it via the following sum[3]:

$$\sigma(\mathbf{w}) = \sum_i w_i \frac{\partial \sigma(\mathbf{w})}{\partial w_i}.$$

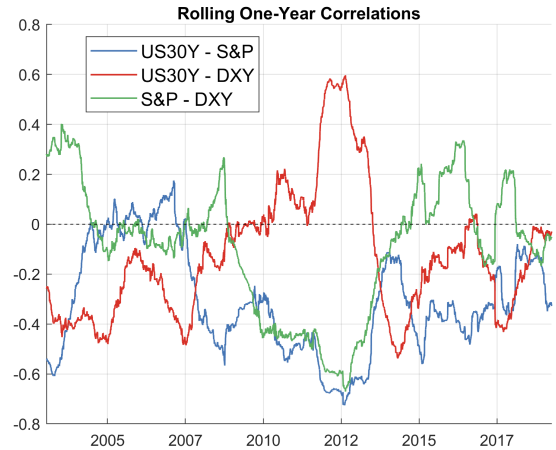
This says that the total risk of a portfolio can be written as a simple sum over the constituents. The derivative $\partial\sigma/\partial w_i$ is known as the "marginal risk contribution." It represents the increase in portfolio risk given an infinitesimal increase in the allocation to asset i. Multiplying the marginal contribution of i by its current portfolio weight gives the full risk contribution for that asset. We'll denote these contributions

$$\sigma_i \equiv w_i \frac{\partial \sigma(\mathbf{w})}{\partial w_i}.$$

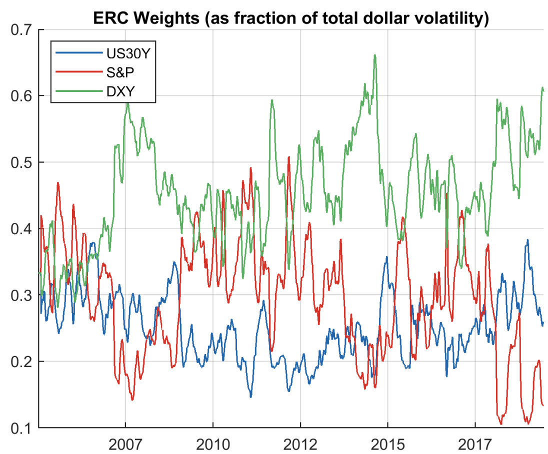
[3]The risk is a homogeneous function of degree one, meaning that $\sigma(c \cdot \mathbf{w}) = c \cdot \sigma(\mathbf{w})$. By Euler's theorem we can write any such function as an inner product $\mathbf{w}^T \nabla \sigma(\mathbf{w})$.



(a) Rolling dollar volatility for each of the three futures contracts.



(b) Rolling correlations among each of the three pairs of instruments.



(c) Rolling ERC allocations shown as a fraction of the dollar volatility of the portfolio (so that they sum to one on any given day).

Figure 2. A hypothetical ERC portfolio consisting of futures contracts on the US 30 Year Bond, S&P 500, and the US Dollar Index. Notice the effects on allocations during periods when volatility spikes, as it does for the S&P in 2018, or when correlations become substantially positive or negative, as they did in 2012.

The goal of ERC is to set $\sigma_i = \sigma_j$ for all assets i and j in the portfolio; that is, we want the risk contribution from all assets to be equal. Taking the derivative of σ with respect to $\mathbf{w}$ and simplifying, this condition reduces to the requirement that

$$w_i \left(\boldsymbol{\Sigma}\mathbf{w}\right)_i = w_j \left(\boldsymbol{\Sigma}\mathbf{w}\right)_j \text{ for all } i \text{ and } j, \tag{2}$$

where by $\left(\boldsymbol{\Sigma}\mathbf{w}\right)_i$ we mean the i^{th} component of the vector $\boldsymbol{\Sigma}\mathbf{w}$. The solution is found by numerically solving

$$\min_{\mathbf{w}} \sum_{i,j} \left[w_i \left(\boldsymbol{\Sigma}\mathbf{w}\right)_i - w_j \left(\boldsymbol{\Sigma}\mathbf{w}\right)_j \right]^2 .$$

Of course, we need to put some constraints on the solution (we can trivially equalize the risk contributions by setting $\mathbf{w} = \mathbf{0}$), so it's typical to require $w_i \geq 0$ and $\sum_i w_i = 1$.

3.1 Further Connections

The ERC portfolio is closely related to another well-known model called the "minimum variance" (MV) portfolio, which simply tries to minimize the overall risk without consideration for individual allocations. Specifically, the problem is to solve

$$\min_{\mathbf{w}} \mathbf{w}^T \boldsymbol{\Sigma}\mathbf{w}, \text{ subject to } \sum_i w_i = 1. \tag{3}$$

Whereas ERC equalizes the risk contributions among assets, it turns out that the MV portfolio actually equalizes the *marginal* risk contributions introduced earlier. Recall, for asset i the marginal contribution is $\partial\sigma/\partial w_i = (\boldsymbol{\Sigma}\mathbf{w})_i$.

To see that this is true, consider that the Lagrangian for

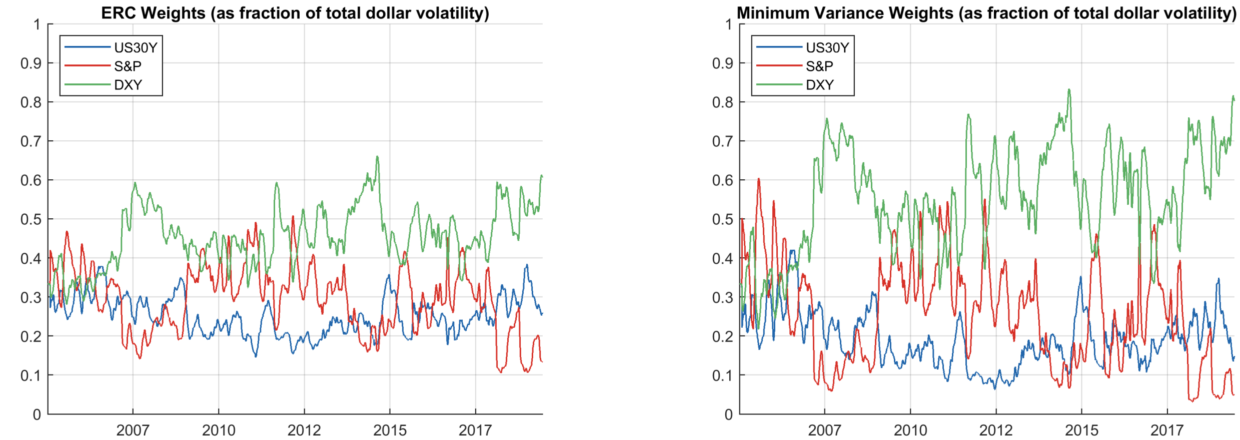


Figure 3. ERC allocations versus minimum variance allocations for the hypothetical portfolio described previously. Notice that while the two solutions are generally very similar, the swings of the minimum variance method are more exaggerated. ERC tends not to let the allocation to any one constituent fall too low as a fraction of the overall portfolio.

(3) is

$$\mathcal{L} = \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} + \lambda \left(1 - \sum_i w_i\right).$$

Differentiating with respect to $\mathbf{w}$ and equating to zero, we find

$$\mathbf{\Sigma w} = \frac{1}{2}\lambda \mathbf{1},$$

which says that each component of the vector $\mathbf{\Sigma w}$ must be equal to the same constant $\lambda/2$ for some λ.

How does ERC differ from the MV solution? As it turns out, we can think of ERC as generally falling somewhere in between the simple equal weights portfolio on the one end, and the MV portfolio on the other (Maillard et al., 2010). Minimum variance can often give assets very little or even zero weight, whereas ERC will only do so in extreme situations (e.g. when the correlation between a subset is exactly equal to -1). In this way, ERC can be thought to have a "minimum diversification" requirement; it tends to ensure that there is at least some nontrivial amount invested in each asset. Hence ERC can be more robust from the perspective of not wanting to concentrate risk in any one constituent or subset of the constituents. Figure (3) shows the two methods side-by-side.

Finally, the investor must think carefully about her various and possibly conflicting goals when deciding on a portfolio layer. It's typical to see both the MV and ERC portfolios written with a "fully invested" constraint, i.e. $\sum_i w_i = 1$. This requirement may or may not make sense; for a long-short futures portfolio, for example, it is not appropriate. Here, the investor might instead prefer to bound overall portfolio risk. In this case, the standard portfolio objectives become the following:

- *Mean-Variance*

$$\max_{\mathbf{w}} \mu^T \mathbf{w} \text{ subject to } \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} \leq 1$$

- *Minimum Variance*

$$\max_{\mathbf{w}} \sum_i w_i \text{ subject to } \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} \leq 1$$

- *Equal Risk Contributions*

$$\max_{\mathbf{w}} \sum_i \log w_i \text{ subject to } \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} \leq 1$$

Writing them this way provides further insight. For example, we see that the minimum variance portfolio is the classical mean-variance portfolio with identical expected returns ($\mu = \mathbf{1}$), and the ERC portfolio is minimum variance with a "damped" utility function. To efficiently solve any of the above problems, we can re-formulate them using second-order cone programming (SOCP), the details of which we omit here.

4. Conclusion

Weighting the constituents of a portfolio based on their risk contributions is a robust choice in the absence of strong expectations about future returns. We've briefly shown the mathematical formalism behind ERC along with some simple examples. While ERC is just one of many options for making allocation decisions, it is an eminently reasonable one under a broad set of assumptions. In fact, ERC is a powerful framework even beyond the generic setting of a long-only portfolio comprised of investable financial assets. For example, consider allocating to trading strategies within a multi-strategy fund. The long-only constraint makes perfect sense here[4]. Understanding and controlling for periods when strategies get too volatile, or become highly correlated, is extremely important. And if the investor has a prior that each strategy has equal positive expected value, the right thing to do is to be equally invested in terms of risk.

[4] Except under very unusual circumstances, it would be hard to justify shorting a *strategy* that you otherwise expect to be long-run profitable.

References

N. E. Karoui. Spectrum estimation for large dimensional covariance matrices using random matrix theory. *Annals of Statistics*, 36(6):2757–2790, 2008.

O. Ledoit and M. Wolf. Improved estimation of the covariance matrix of stock returns with an application to portfolio selection. *Journal of Empirical Finance*, 10:603–621, 2003.

S. Maillard, T. Roncalli, and J. Teïletche. The properties of equally weighted risk contribution portfolios. *The Journal of Portfolio Management*, 36(4):60–70, 2010.

H. Markowitz. Portfolio selection. *The Journal of Finance*, 7 (1):77–91, 1952.

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