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MA's Dynamic Factor Model

Our GDP tracking system approximates, at a high level, the add-up to GDP as reported in BEA's methodology materials. That is, we employ much of the same monthly source data as BEA and translate them into NIPA GDP components as indicated in the NIPA Handbook.[1] For much of the GDP tracking period for a given quarter, we have only a subset of the monthly data that will ultimately be incorporated into GDP. To fill in the missing months, we employ a suite of statistical models that capture behavioral relationships among key data series. By employing these models, there have always been sensible projections for and internal consistency among key monthly data in our tracking system. Our newly developed dynamic factor model, however, takes this model-based discipline to a new level. In what follows, we outline the development and use of our dynamic factor model.

The Model

A dynamic factor model (DFM) is a statistical representation of a panel of data in which each data series is assumed to be a function of a relatively small number of latent common factors and an idiosyncratic error term. The dynamics of the system are captured in the dynamics of the common factors, which are typically assumed to evolve as a vector autoregression.[2] Because the number of common factors is relatively small, their dynamics — and hence the dynamics of the entire panel of observed data — can be represented with a relatively small model. That is, a DFM leverages the comovement in macroeconomic data to express the time-series properties of a relatively large data set with a relatively small model.

An individual data series, which (in our application) has been transformed to be stationary and normalized to have a mean of zero and standard deviation of unity, is expressed as

(1) $$z_{i,t} = \phi_{i,1} f_{1,t} + \phi_{i,2} f_{2,t} + \cdots + \phi_{i,p} f_{p,t} + e_{i,t}$$

($i = 1 \ldots n$), where $z_{i,t}$ is the normalized observed data series, $f_{1,t} \ldots f_{p,t}$ are the latent common factors, $e_{i,t}$ is the idiosyncratic error term, and $\phi_{i1} \ldots \phi_{i,p}$ are parameters to be estimated. These equations are typically referred to as "observation equations," because they explain the observed data. In matrix notation, collecting all *n* observation equations,

(2) $$z_t = \Phi f_t + e_t$$

where z_t is an nx1 vector of observed data, f_t is a px1 vector of common factors, e_t is an nx1 vector of idiosyncratic error terms, and ϕ is an nxp matrix of parameters. We assume that the idiosyncratic error terms are mean zero, have a constant variance, and are uncorrelated across equations and over time. In other words:

(3) $$E(e_t) = 0$$

(4) $$E(e_t e_s') = \begin{cases} R, & t = s \\ 0, & t \neq s \end{cases}$$

where R is an nxn diagonal matrix whose diagonal elements hold the variances of the idiosyncratic error terms.

[1] BEA's methodology materials can be found at https://www.bea.gov/methodologies/index.htm#national_meth.

[2] A good overview of dynamic factor models can be found in James Stock and Mark Watson, "Dynamic Factor Models," *The Oxford Handbook of Economic Forecasting*, edited by Michael Clements and David Hendry, July 2011.

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The common factors evolve according to

(5) $$f_t = A_1 f_{t-1} + A_2 f_{t-2} + \cdots + A_q f_{t-q} + \nu_t$$

where A_i ($i = 1 \ldots q$) are pxp matrices of parameters, and the properties of ν_t are given by

(6) $$E(\nu_t) = 0$$

(7) $$E(\nu_t \nu_s{}') = \begin{cases} \Omega, & t = s \\ 0, & t \neq s \end{cases}$$

The equations that characterize the evolution of the common factors are typically referred to as the "transition equations." The covariance matrix Ω is not restricted to be diagonal. That is, we allow for the possibility that the common factors are contemporaneously correlated.

The Data

The panel of data we have chosen comprises 36 indicator series covering business and consumer sentiment, financial indicators, inflation, labor markets, consumption, orders and shipments, housing and construction, inventories, trade, and production. Most of the indicator series are currently included in our GDP tracking system, and roughly half of them are source data that feed directly into our GDP estimates. There is a fair amount of overlap between the indicators in our DFM and those included in the New York Fed's Nowcast model.[3] Table 1 shows the data series included in our model and the transformations we employ to make them stationary. In addition to transforming the data to be stationary, we normalize them by subtracting their means and dividing by their standard deviations.

Specifying and Estimating the Model

Because the common factors are not observed, we cannot directly estimate the parameters of the model — ϕ, R, A_i, and Ω — by running regressions on the factors. However, it has been shown that the common factors can be consistently estimated with principal components on the observed data.[4] Therefore, following Giannone, Reichlin, and Small (2008), our approach is to extract principal components from the normalized data and estimate the parameters of the DFM with OLS after first replacing the common factors with the principal components.[5]

Of course, before we can estimate the model, we must first specify it. The choices we need to make are (i) the number of common factors, and (ii) the number of lags in the transition equations. To choose the number of common factors, we use two guiding principles. First, we want the set of factors to account for a large share of the comovement in the data. Second, for a subset of indicators that are critical for our GDP tracking, we want good observation equations (mainly a good fit). With some exploration, we found that 10 factors satisfied both of these objectives. In particular, the first 10 principal components account for about ¾ of the covariance in the data, and none of the key observation equations has an R^2 less than 0.50.

To determine the number of lags in the transition equations, we recorded the Akaike information criterion (AIC) for ten-variable VAR's of various lag lengths and chose the lag length that minimized the AIC, which was a lag length of three.

The right two columns of Table 1 list two key properties of the observation equations. For each regression of the normalized data on the first ten principal components, we show the r-squared and Durbin-Watson (DW) statistic. The lowest r-squared is 0.48, and the

[3] Information on the New York Fed's Nowcast model can be found at https://www.newyorkfed.org/research/policy/nowcast.

[4] See, for example, Domenico Giannone, Lucrezia Reichlin, and David Small, "Nowcasting: The real-time informational content of macroeconomic data," Journal *of Monetary Economics* 55 (2008) pp. 665 – 676.

[5] To be clear, to estimate the coefficients in the observation equations, we regress the normalized observed data on principal components. We use the OLS residuals from those regressions to estimate the elements of R. To estimate the coefficients in the transition equations, we regress principal components on lags of principal components, and we use the OLS residuals from those regressions to estimate the elements of Ω.

highest is 0.90. Eighty percent of the regressions have an r-squared greater than 0.60, and one-third have an r-squared greater than 0.75. An area of concern is the Durbin-Watson statistics. In more than half of the equations, the DW statistic is either less than 1.5 or greater than 2.5, indicating serial correlation in the error terms for these equations. Indeed, examination of the correlograms of the regression residuals indicates the presence of moving-average error processes in many of the observation equations. We will be exploring refinements to our DFM to account generally for ARMA error processes in the observation equations.

Table 2 illustrates several key properties of the transition equations. Each row shows the r-squared and the Durbin-Watson statistic from the associated regression and the results of several hypothesis tests. Recall that the parameters of each transition equation are estimated by regressing a principal component on three lags of each of the 10 principal components. The first column of Table 2 identifies the dependent variable. The second and third columns show the fit and the DW statistics. The r-squared ranges from 0.15 to 0.74, and the DW statistics are all close to 2.0.

The remainder of Table 2 identifies key linkages across the principal components through a series of F-test. Each cell reports the p-value from an F-test of the null hypothesis that a group of coefficients are jointly equal to zero. Cells with p-values less than 0.01 are shaded dark gray, and cells with p-values between 0.01 and 0.05 are shaded light gray. The first panel of p-values (columns 4 – 6) are associated with the null hypothesis that all 10 coefficients of the i^{th} lag are jointly equal to zero, for i = 1, 2, 3. These F-tests are intended to reveal the importance of lagged relationships in general for each of the principal components. For example, for the first principal component (the first transition equation), we reject the null that each group of lags is equal to zero, indicating the importance of lagged data in projecting values for the first principal component. For the second and third transition equations, lags longer than one month appear not to be important.

The second panel of p-values (columns 7 – 16) are associated with the null hypothesis that the coefficients on all 3 lags of the j^{th} principal component are jointly equal to zero, for j = 1, 2, ... , 10. These F-tests are intended to reveal interdependencies across principal components regardless of lag length. For example, reading across the first row reveals that the first principal component is significantly affected by its own lags and lags of the second, sixth, and eighth components. Reading down the first column of this panel reveals that the first principal component has a significant impact on its own subsequent values as well as future values of the second, third, fourth, and ninth components.

The principal components themselves, and hence the common factors, are hard to interpret. This is because they summarize the comovement among several seemingly disparate variables. What Table 2 is intended to communicate is the rich and statistically significant linkages among them, with the goal of giving us confidence that the transition equations are a good forecasting model underlying the panel of observed data.

State Space Representation and the Kalman Filter

With estimates of the model parameters in hand, we can estimate the mean and covariance of the common factors with the Kalman filter. However, before we do that, we must recast the model in state-space form. To this end, we define the following

$$F_t \equiv \begin{bmatrix} f_t \\ f_{t-1} \\ f_{t-2} \end{bmatrix} \quad A \equiv \begin{bmatrix} A_1 & A_2 & A_3 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \quad v_t \equiv \begin{bmatrix} v_t \\ 0 \\ 0 \end{bmatrix}$$

$$\Lambda \equiv \begin{bmatrix} \Phi & 0 & 0 \end{bmatrix} \qquad Q \equiv \begin{bmatrix} \Omega & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

With these definitions, we can express the model in state-space form as

$$z_t = \Lambda F_t + e_t \tag{8}$$

$$F_t = AF_{t-1} + v_t \tag{9}$$

$$E(e_t) = 0 \qquad E(e_t e_s{}') = \begin{cases} R, & t = s \\ 0, & t \neq s \end{cases} \tag{10}$$

$$E(v_t) = 0 \qquad E(v_t v_s{}') = \begin{cases} Q, & t = s \\ 0, & t \neq s \end{cases} \tag{11}$$

The Kalman recursion is then given by

prediction step

(12) $$\hat{F}_{t|t-1} = A\hat{F}_{t-1|t-1}$$

(13) $$P_{t|t-1} = AP_{t-1|t-1}A^{'} + Q_t$$

update step

(14) $$\hat{F}_{t|t} = \hat{F}_{t|t-1} + K_t\left(z_t - \Lambda\hat{F}_{t|t-1}\right)$$

(15) $$P_{t|t} = P_{t|t-1} - K_t\Lambda P_{t|t-1}$$

where $\hat{F}_t \equiv E(F_t)$, $P_t \equiv Cov(F_t)$, and K_t is the "Kalman gain" matrix and is a function of the model parameters:[6]

(16) $$K_t = P_{t|t-1}\Lambda^{'}\left(\Lambda P_{t|t-1}\Lambda^{'} + R_t\right)^{-1}$$

A common application of the Kalman filter is to estimate the mean and covariance of the common factors over history — when observations on z_t are available — using the prediction and update steps, and then forecast the mean and covariance of the common factors by iterating only on the prediction step, as no observations on z_t are available to perform the update step. The forecast of the mean and covariance of the common factors can be used, along with expression (8), to forecast the mean and covariance of the observed data:

(17) $$E\left(z_{t|t-1}\right) = \Lambda\hat{F}_{t|t-1}$$

(18) $$Cov\left(z_{t|t-1}\right) = \Lambda P_{t|t-1}\Lambda^{'} + R$$

This would be one way to use the model to produce forecasts and standard-error bands for z_t.

This approach, however, abstracts from the complication that there typically is not a clean separation between historical months and forecast months. There is usually a small range at the end of a data set over which observations are available for some data series but not for others. This is commonly referred to as the "ragged edge" of the data. The traditional update step of the Kalman filter cannot be performed for a month in which only a subset of the data is available. Still, where data *are* available, we would like to use those observations to inform the values of the common factors and, in turn, to inform forecast values of the data that are not yet available.

To this end, we follow Giannone, Reichlin, and Small (2008) by further parameterizing the covariance matrix of the observation errors, *R*, as follows.[7] In a month in which some data are available and other data are not, we replace *R* with *R**, where

$$R^*_{ii} = \begin{cases} R_{ii}, & z_{i,t}\ available \\ \infty, & otherwise \end{cases}$$

That is, if $z_{i,t}$ is available at time t, we use the variance of $e_{i,t}$ in the i^{th} diagonal position of *R**, as usual. If $z_{i,t}$ is not available at time t, we replace the variance of $e_{i,t}$ with infinity. The intuition behind the effect of this parameterization can be seen by examination of expression (16), the Kalman gain matrix, and its role in the update step. The Kalman gain matrix is a measure of the relative uncertainty in the observation equations. The "larger" is *R*, the "smaller" is *K*, and the less the observed data updates the mean and covariance of the common factors. Setting the variance of a single observation equation to infinity ensures it plays no role in the updating of any of the common factors.

Because this approach handles missing data, it allows us to run the update step of the Kalman filter into the ragged edge of the data. Furthermore, it can be demonstrated that when all data are missing (beyond the ragged edge), the Kalman gain matrix, K_t, becomes zero and the update step of the Kalman filter reduces to the prediction step. In other words, this approach allows us to mechanically run the full Kalman filter (prediction and update) over history, into the ragged edge, and into the forecast. This also means that we can calculate the mean and covariance of z_t over history and in the forecast using *updated* values of the mean and covariance of the common factors:

[6] Good presentations of the Kalman filter can be found in chapter 13 of James D Hamilton, *Time Series Analysis*, Princeton University Press, 1994 and in chapter 3 of Andrew C Harvey, *Forecasting, Structural Time Series Models and the Kalman Filter*, Cambridge University Press, 1989.

[7] See citation above.

(19) $$E\left(z_{t|t}\right) = \Lambda \hat{F}_{t|t}$$

(20) $$Cov\left(z_{t|t}\right) = \Lambda P_{t|t} \Lambda' + R$$

Expressions (19) and (20) form the basis for how we integrate our DFM into our GDP tracking.

Integrating the Dynamic Factor Model into the GDP Tracking System

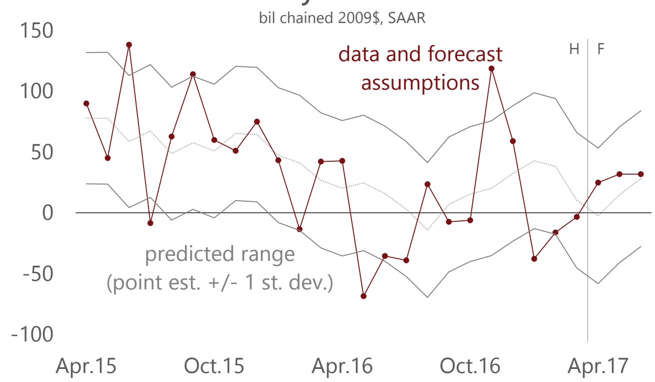
We integrate the DFM into our GDP tracking by "layering" the predictions of the DFM on top of our existing methodology. Our GDP tracking system comprises a few hundred monthly data series, most of which are aggregated into monthly indicator series and projected into the forecast. These indicator series then determine forecasted growth rates of several GDP components, which are then aggregated into a forecast of GDP. We have developed and employ a suite of small-scale statistical models that help determine the projections of several key monthly indicators series. We also employ judgment in projecting those and other key indicator series. In short, our current methodology employs a combination of small-scale models and judgment to predict GDP growth in the current quarter.

Expressions (19) and (20) provide alternative projections for several of the key monthly indicator series that are in our GDP tracking system. The DFM's projections for all of the series are internally consistent by design, and they bring to bear implications from measures that are generally not included in our existing suite of small-scale statistical models: business and consumer sentiment, financial indicators, and others. As alternative projections, we can use them as checks on the projections that result from our suite of small-scale statistical models and judgment; i.e., our long-standing methodology.

Figures 1 – 3 illustrate how we are using the DFM as a check on our existing models. Figure 1 shows historical values and forecast assumptions for monthly real nonfarm inventory investment excluding retail motor vehicles and parts. Also charted is the predicted range from our small-scale model. The drivers for that model are growth of monthly final sales excluding motor vehicles and parts (which we construct) and the recent trend in the inventory-to-sales ratio. Notice that in the

Figure 1

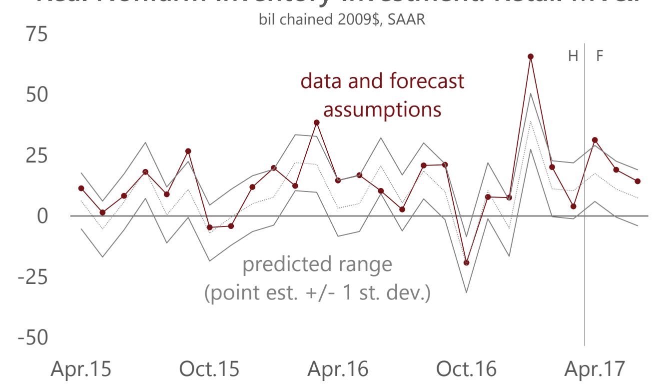
Real Nonfarm Inventory Investment ex. Retail MV&P



Sources: Macroeconomic Advisers, BEA.

Figure 2

Real Nonfarm Inventory Investment: Retail MV&P



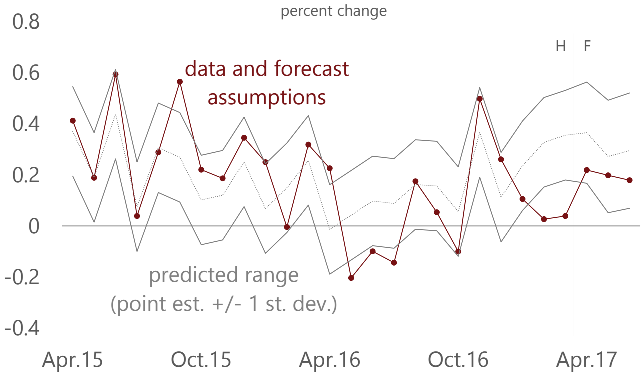
Sources: Macroeconomic Advisers, BEA.

three months of the second quarter, we are assuming values above the center of the predicted range.

Figure 2 shows historical values and our forecast assumptions for real retail motor vehicles and parts inventory investment along with the predicted range from our small-scale model. The drivers for that model are unit motor vehicle inventory change and wholesale and retail inventory change from Census. Notice again that our forecast assumptions for the second quarter are above the center of the predicted range.

The data charted in Figures 1 and 2 together make up total nonfarm inventory investment, whose one-month percent changes over history and in the forecast months are shown in Figure 3. Also shown in Figure 3 is the predicted range for this series from the DFM. It is evident that our forecast assumptions for this series represent a compromise between the predictions from the DFM and the predictions from our small-scale

Figure 3
Real Nonfarm Inventory Investment
percent change



Sources: Macroeconomic Advisers, BEA.

models. That is, the DFM argues for more nonfarm inventory investment in the second quarter, while the small-scale models argue for less. Raising our forecast assumptions for nonfarm inventory investment would resolve some upside risk from the perspective of the DFM, but would add to the downside risk already evident in the small-scale models. Our forecast assumptions balance these risks.

All throughout our GDP tracking system, we are now using the predictions from the DFM in this manner. We use our small-scale models and judgment to project key monthly indicators into the forecast. Those projections feed into our GDP tracking forecast. We subsequently run the DFM and use its projections as a check on our assumptions.[8] Where the DFM projections introduce risk to our monthly assumptions, we adjust those monthly assumptions to balance the risks from the perspective of the DFM, the small-scale models, and our judgment.

Conclusion

Our newly developed DFM is a state-space model in which a panel of 36 data series is modeled as a function of 10 latent common factors and idiosyncratic shocks. The latent common factors are extracted from the data using Kalman filter techniques and evolve as a vector autoregression with 3 lags. We project the model into the forecast and use the projections as a check on our monthly assumptions, which have long been produced with a mix of small-scale models and judgment. Where there are discrepancies between our forecast assumptions and the projections of the DFM, we adjust our forecast assumptions toward the projections of the DFM to resolve the risk. We believe the DFM introduces a new discipline to our monthly assumptions that will, over time, demonstrably improve our near-term GDP tracking accuracy.

[8] As of this writing, our practice is to run the DFM once per week, on Fridays.

Table 1: Data & Observation Equation Statistics

Stationarity Transformation: L=level, D=first difference, X=mo/mo % change

		R^2	DW
Business and Consumer Sentiment			
L	ISM Manufacturing PMI: Composite Index	0.74	1.1
L	Philly Fed Mfg Business Outlook: Current Activity Diffusion Index	0.61	1.1
X	University of Michigan Consumer Sentiment	0.48	2.2
Financial			
D	CBOE Market Volatility Index: VIX	0.75	1.9
D	Spread of Moody's Baa Corporate Bond Yield over 20-year Treasury	0.67	1.8
X	Stock Price Index: Standard & Poor's 500 Composite (month ave.)	0.76	1.8
Inflation			
X	PPI: All Commodities	0.84	2.3
X	PPI: Finished Goods	0.83	2.5
X	CPI-U: All Items	0.90	2.0
X	CPI-U: All Items Less Food and Energy	0.74	2.1
X	PCE Chain Price Index	0.90	2.2
X	PCE less Food & Energy Chain Price Index	0.78	2.2
Labor Market			
D	Nonfarm Payroll Employment	0.73	1.7
D	Unemployment Rate	0.70	2.3
D	Participation Rate	0.84	2.3
Consumption			
X	Retail Sales & Food Serv Excl Auto, Gas Stations & Building Materials	0.54	2.6
D	Light Vehicle Sales	0.73	2.4
X	Real Personal Consumption Expenditures	0.87	2.3
Equipment			
X	Mfrs' New Orders: Nondefense Capital Goods ex Aircraft	0.84	2.5
X	Mfrs' Shipments: Nondefense Capital Goods ex Aircraft	0.66	2.7
X	Mfrs' Unfilled Orders: Nondefense Capital Goods ex Aircraft	0.66	1.6
Housing and Construction			
X	1-Family Housing Units Authorized	0.65	2.3
X	1-Family Housing Starts	0.61	2.5
X	Construction Spending Less Res. Improvements and Federal	0.64	1.9
Inventories			
X	Manufacturers' Inventories: Durable Goods (Census)	0.71	1.6
X	Mfrs' Inventories: Nondurable Goods (Census)	0.63	1.9
X	Merchant Wholesalers Inventories (Census)	0.64	2.0
X	Retail Inventories (Census)	0.61	2.0
X	Real Nonfarm Inventories (BEA)	0.76	1.6
International Trade			
X	Nominal Goods Exports (Census)	0.51	2.6
X	Nominal Services Exports (BEA)	0.59	2.5
X	Nominal Goods Imports (Census)	0.50	2.6
X	Nominal Services Imports (BEA)	0.60	2.6
Production			
X	Industrial Production: Total Index	0.87	1.7
D	Capacity Utilization: Total Index	0.87	1.9
D	IP: Motor Vehicle Assemblies	0.54	2.3

Table 2: Transition Equation Statistics

Component	R^2	DW	p-values for test of joint null for all components of shown lag: 1st	2nd	3rd	p-values for test of joint null for all lags of shown component: 1	2	3	4	5	6	7	8	9	10
1	0.74	2.0	0.00	0.00	0.02	0.00	0.02	0.65	0.11	0.39	0.05	0.19	0.00	0.58	0.48
2	0.38	2.0	0.00	0.56	0.27	0.00	0.00	0.41	0.02	0.31	0.61	0.88	0.02	0.19	0.48
3	0.27	2.0	0.00	0.13	0.33	0.00	0.02	0.65	0.11	0.39	0.05	0.19	0.00	0.58	0.48
4	0.31	2.0	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.46	0.86	0.63	0.21	0.95	0.00
5	0.21	2.0	0.03	0.08	0.03	0.66	0.23	0.39	0.33	0.05	0.02	0.60	0.13	0.25	0.23
6	0.41	2.0	0.00	0.00	0.05	0.56	0.00	0.00	0.02	0.25	0.00	0.63	0.01	0.26	0.04
7	0.20	2.0	0.06	0.25	0.01	0.49	0.01	0.18	0.95	0.04	0.70	0.32	0.01	0.22	0.45
8	0.36	2.0	0.00	0.03	0.00	0.90	0.01	0.00	0.29	0.14	0.02	0.01	0.00	0.63	0.10
9	0.15	2.1	0.17	0.28	0.03	0.03	0.69	0.57	0.45	0.18	0.15	0.17	0.85	0.02	0.32
10	0.18	2.0	0.00	0.00	0.43	0.68	0.92	0.76	0.60	0.01	0.71	0.17	0.03	0.14	0.00

Notes: The p-values shown are for the F-test of the null hypothesis that all of the coefficients on the indicated terms are jointly equal to zero. P-values less than 0.01 are shaded dark gray. P-values between 0.01 and 0.05 are shaded light gray. P-values above 0.05 are not shaded

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