Stress Testing of Portfolios

In most practical applications of stress testing, portfolio managers have views on a small set of market variables. This means that, to estimate the portfolio return under the specified scenario, we need to estimate the relationships between these market variables and certain others that are relevant to the portfolio. One can utilize a historical covariance matrix of these market variables as an estimate for these relationships. Equal-weighted and time-weighted matrices are two commonly used alternatives. However, such matrices may not represent well the relationships under the specified scenario, especially if the scenario incorporates extreme movements in market variables. Under such conditions, correlations tend to breakdown, volatilities can jump, and there are observed asymmetries in the dependence structure. This paper describes an intuitive and robust methodology that produces a covariance matrix suited to the specified scenario and that incorporates such dynamics observed under distressed market conditions.

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1. Motivation

Scenario analysis helps portfolio managers in their decision-making process by providing a detailed analysis on the behavior of their portfolios under a specified scenario. These scenarios can take different forms. One could use a historical scenario, such as the recent credit crisis, and explore how a portfolio would perform if that historical episode were to reoccur. Alternatively, one can define a hypothetical scenario such as: What would be the profit/loss in a portfolio if the US equity market dropped by 10% over the next month?

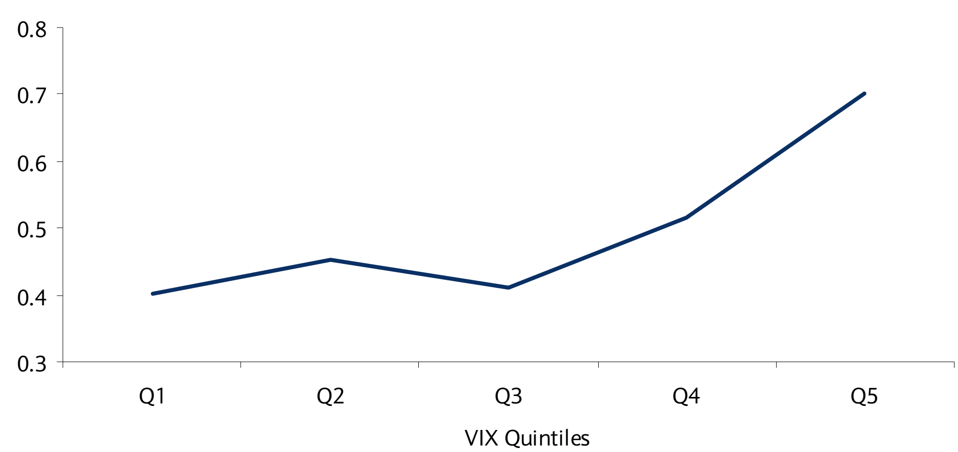
In most practical applications, users of scenario analysis have views on a small set of market variables. This means that, to estimate the portfolio return under such a scenario, we need to estimate the relationships between these market variables and certain others that are relevant to the portfolio (these market variables can be represented in different forms such as risk factors or indices). To illustrate this, let's assume that our portfolio is composed of a set of Treasury bonds. By estimating the relationship between the equity market and interest rates, we can estimate the return of the portfolio under the above equity market scenario.

To achieve this objective, we can use a covariance-based approach where the relationship between two market variables is modeled through their covariance. In this approach, we need to estimate the covariance matrix of a set of market variables in order to perform scenario analysis. Two commonly used covariance matrices in a variety of risk management applications are the equal-weighted and the time-weighted matrices. The former assigns equal weights to all historical observations whereas the latter assigns higher weights to recent observations in computing the covariances. These matrices are widely used in forecasting future relationships between market variables.

In the context of scenario analysis, we want to use a matrix that represents the potential relationships under the specified scenario. If the scenario is relatively moderate, time-weighted or equal-weighted matrices can be good estimates for the behavior of variables under such a scenario. However, under more extreme scenarios – which are more interesting for the purposes of stress testing – these matrices are not likely to represent the potential behavior of market variables. Under such conditions, correlations tend to breakdown, volatilities can jump, and there are observed asymmetries in the dependence structure: The use of a covariance matrix that does not explicitly account for such dynamics can have serious implications in the portfolio management process, typically leading to overestimated diversification benefits and ineffective tail hedges. In this paper, we propose a methodology to construct a covariance matrix that explicitly defines the relationship between variables as a function of the scenario under consideration.

To illustrate the issue under discussion, Figure 1 shows how the correlation between the S&P500 and the Barclays Capital US High Yield indices changes as a function of equity market volatility (here represented by the VIX Index). The correlation between the two indices increases from around 40% to 70% as we condition the observations to higher VIX levels. For a cross-asset class portfolio that has exposure to these two markets, the use of an average level of correlation may lead to an overstatement of diversification benefits for scenarios incorporating high volatility environments. When correlations change smoothly, the time-weighted matrix might capture these dynamics relatively well. However, when there are sudden large shocks to the market, as in most crises, the time-weighted matrix may not be able to capture potential correlation breakdowns.

Figure 1: Correlation between the Barclays Capital US High Yield Index and the S&P500 Index for Different VIX Quintiles (1990-2011)



Source: Barclays Capital

Another dynamic that cannot be captured by static matrices (e.g., equal/time weighted) is the asymmetries in the dependence structure across market variables. As an illustration, Figure 2 shows the 10 best and worst monthly returns for the S&P500 and the corresponding changes in the VIX index during the period from 1990-2011. We can see that during the worst episodes, the equity market volatility tends to jump significantly whereas during the best episodes, changes in market volatility are generally not that strong (in magnitude)[1]. This asymmetrical behavior cannot be captured using a static conditional

[1] This phenomenon is known in the literature as the leverage effect (see Bouchaud, Matacz, and Potters (2001)).

covariance matrix. By definition, this matrix is indifferent to the scenario specified and hence implies the same magnitude of movement in equity volatility for a negative or positive shock (of the same magnitude) in the underlying equity market.

Figure 2: Worst and the Best 10 Episodes for the US Equity Market (1990-2011)

Worst 10		Best 10	
Equity	VIX	Equity	VIX
-19.9%	52.0%	13.2%	-17.3%
-12.6%	78.5%	9.6%	3.2%
-10.9%	3.4%	9.3%	-9.0%
-9.9%	41.6%	8.6%	-4.7%
-9.6%	-7.7%	8.6%	-9.4%
-8.6%	34.3%	8.3%	-4.8%
-8.0%	26.1%	8.2%	7.8%
-7.9%	28.1%	8.1%	-1.6%
-7.5%	12.1%	7.8%	1.5%
-7.0%	47.1%	7.2%	-27.2%

Source: Barclays Capital

In the next section, we propose an intuitive and robust methodology that eliminates the drawbacks illustrated in the previous examples. It produces a covariance matrix consistent with the specified scenario and that has the flexibility to incorporate the above correlation and volatility dynamics (breakdowns in correlations, jumps in volatilities, asymmetrical dependence structure). As we describe the methodology, we use empirical analysis to illustrate its important features. Section 3 describes alternative approaches to this problem, while in section 4 we discuss the intervals of confidence around the estimates of the procedure. In section 5, we then present how it performs in the context of scenario analysis as compared to widely used static matrices such as the time-weighted matrix.

2. Methodology

In this section, we propose a methodology that produces a covariance matrix specifically designed to capture the relationships between market variables under a specified scenario. A straightforward application of this matrix is in the area of scenario analysis (see Shao and Ural (2010)). However, this covariance matrix can be utilized in various other portfolio management exercises such as designing optimal hedging strategies (e.g., tail risk hedging) and portfolio construction (e.g., constructing defensive allocations). Before we start describing this methodology in detail, we overview its highlights:

- Captures increasing correlations and volatilities under distressed conditions (e.g., flight to quality) while further allowing for flexible dynamics
- Captures asymmetries in the dependence structure
- Provides a generic solution for all types of scenarios
- Brings flexibility by modeling correlations and volatilities separately
- Is easy to interpret, as it involves only a reweighting of historical data

Our methodology can be divided into 3 steps, which are summarized in Figure 3.

Figure 3: Methodology Steps

Step 1	Estimate the correlations by dynamically weighting historical data
Step 2	Update volatilities as a function of the size of the shock
Step 3	Compute the covariance matrix from the above and use this matrix to perform scenario analysis or other exercises

Source: Barclays Capital

We now discuss each of these steps in detail. The major inputs to this methodology are the historical returns of the market variables and the scenario specified by the user. The output is the covariance matrix designed to reflect the relationships between these market variables under the specified scenario (i.e., this methodology produces a specific covariance matrix for each scenario). This matrix can then be used in a variety of portfolio management exercises including scenario analysis.

Step 1

In the first step, we estimate the correlations by dynamically weighting historical data. The main idea behind this technique is to assign larger weights to historical observations that are more similar to the scenario specified. To achieve this, we first define a function for the distance between the scenario and each historical observation and assign larger weights to the observations (e.g., months) that have smaller distance (i.e., are more similar) to the scenario under study. For instance, if the scenario is a 15% drop in the equity market and if we use monthly data, a month where the equity market dropped 10% will be assigned a larger weight than a month where the equity market dropped 5% in computing the weighted correlations. This methodology also applies to multivariate scenarios where there is more than one variable in the scenario definition (e.g., what happens if credit spreads widen by 50% and interest rates move down 100 bps?).

The following analysis can be done using asset classes, indices or risk factors as the unit of analysis ("market variables"). In what follows we focus primarily on risk factors, following the factor-based scenario analysis approach we typically use (see Shao and Ural (2010)). The risk factors used in our scenario analysis are high-level economic factors such as the level of the yield curve, credit spreads, and equity market factors that have straightforward interpretations.

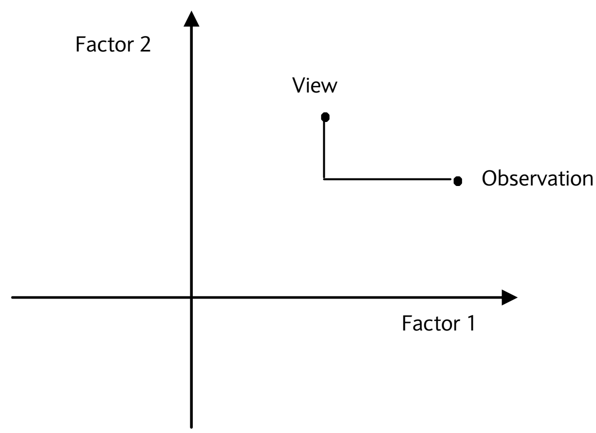
We define the distance function between the scenario and a historical observation *t* in the following way:

$$D_t = \sum_{i \in V} \left| \frac{F_t^i}{\sigma_t^i} - \frac{\theta^i}{\sigma_T^i} \right|$$

Where F_t^i is the realization of factor *i* during period *t*, θ^i is the scenario view on factor *i*, σ_t^i and σ_T^i are respectively the estimated volatility of factor i for time period *t* and *T* (current time), and V is the set of factors on which a scenario view is specified. We normalize each factor return and scenario view by the estimated volatility of the factor in the relevant time period, in order to make these quantities more comparable (invariant) across different factors and time periods. This distance function is the L1-norm[2] of the difference between the normalized scenario and historical observation vectors.

Figure 4 illustrates the above distance function in the case of two factors. The two axes in the figure represent the realizations of factor 1 and 2 in standard deviation terms (F/σ). Each historical observation and the scenario view can be represented by a point on this graph. The figure illustrates the scenario view and an observation, which correspond to certain specific values of the two factors and also the distance between the observation and the view, which corresponds to the length of the line (as shown) between the two.

Figure 4: The Distance Function in the Case of Two Factors



Source: Barclays Capital

As an example, if our scenario specifies -3 and -2 standard deviation moves in the US equity market and USD interest rates respectively, a historical observation where these markets both moved by -3 standard deviations would have the same distance to the scenario as another observation where both moved by -2 standard deviations (1+0=1).

Once we define the distance, we assign a weight to each historical observation based on its distance to the specified scenario:

$$w_t = 0.5^{D_t/(\lambda m)}$$

where D_t is the distance for historical observation t, λ is the half-life of exponential weighting (in units of standard deviations), and m is the number of factors on which scenario views are specified (this makes sure the weighting scheme is comparable between univariate and multivariate scenarios). This is an exponential weighting scheme where a larger weight is assigned to a historical observation that is more similar to the scenario (that has a smaller distance). Therefore, the procedure can be viewed as a weighted historical matrix, similar in spirit to the typical time-weighted covariance matrix. The difference is that the weights are not a function of time but instead a function of the distance between the

[2] An alternative would be to use the L2-norm where the distance is the sum of squared differences across different variables in the scenario definition as opposed to the absolute value of the differences in the L1-norm. If the scenario is specified using a single variable, L1-norm and L2-norm are equivalent. However in a multivariate scenario, a major difference between the two norms is that a significant difference between the scenario and a given historical observation in one of the variables would weight a lot more in the computation of distance in the L2-norm as it uses the squares of the differences as opposed to the absolute values.

historical observation and the scenario[3]. In what follows we set $\lambda = 1$. This means that an observation that is 1 standard deviation away from the scenario receives a 50% weight and one that is 2 standard deviations away receives only a 25% weight[4].

To demonstrate our weighting scheme, we focus on a particular hypothetical scenario: the US equity market drops by 3 standard deviations (which corresponds to 18.51%). Figure 5 shows the historical observations that are the closest to this scenario (the observations with the largest weights based on the above distance function). As we can see in the figure, these are months from different historical periods where the equity market experienced losses close to 3 standard deviations. For future reference, the table also shows the returns on two other markets – UK equities and US credit spreads – which are highly correlated with the US equity, especially under distressed conditions. Our historical data period in this exercise has around 250 observations; hence an equal-weighted matrix would assign 0.4% weight to each observation. In this case, our methodology assigns up to 7 times that weight to observations that are the closest to the scenario under consideration. By assigning significantly larger weights to observations that are more similar to the scenario, this dynamic weighting scheme allows us to capture changing correlations between market variables across different scenarios.

Figure 5: Historical Observations Most Similar to a -3 Stdev. US Equity Scenario

Month	Weight	US Equity (Stdev)	US Equity (Return)	UK Equity (Return)	US Credit Spread (% Change)
Jun-08	2.8%	-2.9	-8.6%	-8.4%	11.4%
Aug-98	2.1%	-3.6	-12.6%	-12.5%	47.7%
Sep-01	1.7%	-2.2	-7.9%	-12.2%	28.0%
Jul-02	1.7%	-2.1	-8.0%	-10.0%	21.5%
Aug-90	1.7%	-2.1	-9.9%	-7.0%	1.7%
Jul-07	1.5%	-2.0	-5.3%	-2.3%	38.0%
Sep-08	1.5%	-2.0	-6.9%	-13.0%	19.9%
Jul-96	1.3%	-1.8	-4.4%	-0.2%	-0.2%
Sep-02	1.3%	-1.7	-7.0%	-9.1%	8.7%
Feb-09	1.3%	-1.7	-10.9%	-4.8%	-0.9%

Source: Barclays Capital

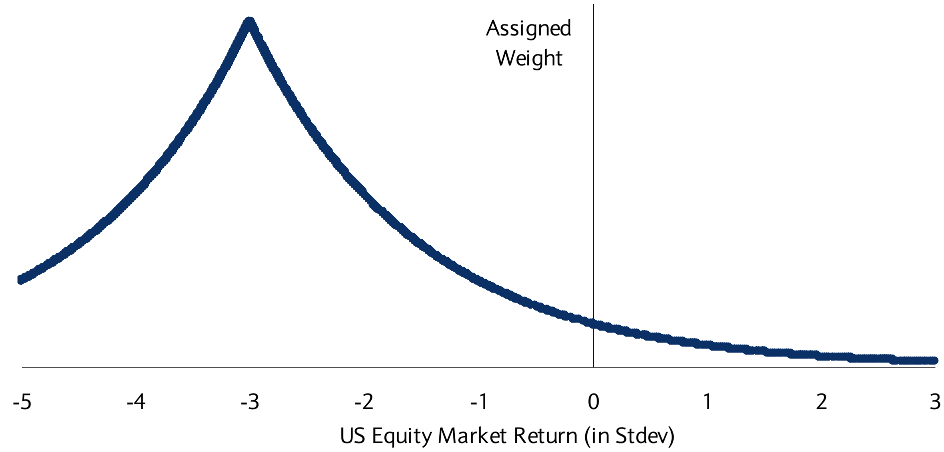
Figure 6 depicts the weighting function for all historical observations for the above scenario. We can see that the largest weights are assigned to observations that incorporate (close to) 3 standard deviation losses in the US equity market and the weights decrease in an exponential fashion as we move away from those observations[5].

[3] In particular, the weights in the typical time-weighted matrix for time period T are given as $w_t = 0.5^{D_t/\lambda}$, where $D_t = T - t$ and $\lambda = 12$ for a 12-month half-life.

[4] Weights are renormalized at the end to sum up to 1.

[5] The weighting function is symmetric around the scenario. For instance, for the above scenario, it assigns the same weight to an observation with a -2 or a -4 standard deviation move in the US equity market. Other weighting functions, namely asymmetric ones, can be used.

Figure 6: The Weighting Function in a -3 Standard Deviation US Equity Market Scenario



Source: Barclays Capital

Once we assign weights to each historical observation using the above algorithm, we compute the weighted sample correlation matrix across all market variables of interest. Figure 7 illustrates the correlation matrix coming out of this methodology for the above scenario for the set of markets described in Figure 5. From now on we call this matrix the scenario-weighted matrix. Figure 7 compares it with the equal weighted matrix, where we compute sample correlations assigning equal weights to each historical observation.

Figure 7: Scenario-Weighted vs. Equal-Weighted Correlations

Scenario Weighted	US Equity	UK Equity	US Credit Spread	Equal Weighted	US Equity	UK Equity	US Credit Spread
US Equity	1.00			US Equity	1.00		
UK Equity	0.82	1.00		UK Equity	0.75	1.00	
US Credit Spread	-0.64	-0.60	1.00	US Credit Spread	-0.53	-0.51	1.00

Source: Barclays Capital

As we discussed above, by assigning larger weights to the observations that are more similar to our stressed scenario, the scenario-weighted matrix captures increased correlations across different markets under such conditions.

As we mentioned earlier, when a large shock hits the market, correlations between market variables might breakdown leading to ineffective hedges and poor diversification. For example, a well diversified portfolio across asset classes may show poor diversification characteristics under stressed times, given flight-to-quality behaviors (correlations going to 1). Moreover, markets tend to react in a quite different way to large negative and positive shocks, resulting in asymmetries in the dependence structure. The correlation dynamics can be quite complex depending on variables under consideration. Can we capture such dynamics through the scenario-weighted correlation matrix coming out of our methodology?

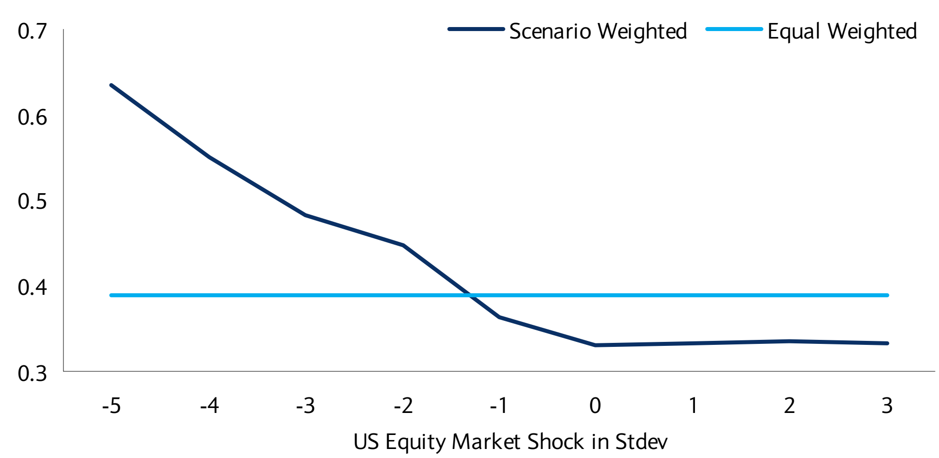
To answer the above question, we construct a sample portfolio and analyze a variety of its statistics across different scenarios. This is a multi-asset class US portfolio composed of equal weights in the following indices:

- Barclays Capital US Treasury Index
- Barclays Capital US Credit Index
- Barclays Capital US High Yield Caa Index
- S&P 500 Index
- Barclays Capital US Commodity Index

For the following illustrations, we use monthly data from 1990-2011. We construct 9 different scenarios on the US equity market, shocks ranging from -5 to +3 standard deviation movements[6]. Our methodology produces a custom correlation matrix (the scenario-weighted matrix) for each scenario; we analyze the dynamics of these correlation matrices across different scenarios.

As we mentioned earlier, we use a factor-based approach in scenario analysis, where a portfolio has exposure to a set of high-level economic risk factors from different asset classes that have straightforward interpretations. For the portfolio under analysis, for instance, the factors are the following: level and slope of the USD yield curve; US equity; US credit spreads; US commodity and US distressed debt (Appendix 1 illustrates the complete list of factors used in factor based scenario analysis in POINT). Figure 8 depicts the average absolute correlation between these risk factors for the 9 aforementioned US equity scenarios. Results are shown for two different matrices, scenario-weighted and equal-weighted.

Figure 8: Average Absolute Correlation across Different Scenarios



Source: Barclays Capital

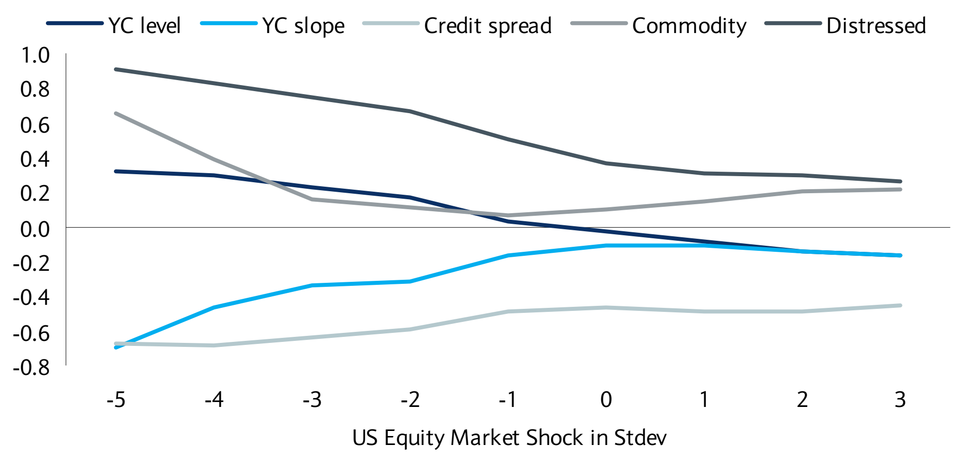
By construction, equal-weighted correlations are constant – at about 40% – across the range as this matrix is indifferent to the scenario specified. This would be the case for any matrix that is static, in the sense that it does not change with the nature of the scenario. We provide the results for this matrix as a baseline, for comparison purposes. On the other hand, average absolute correlation coming from the scenario-weighted matrix tends to increase significantly – from about 35% to 65% – as we move closer to the negative extreme scenarios. This result suggests that our methodology incorporates in a very intuitive way the well-known flight-to-

[6] These numbers are limited by historical data, they are the largest movements available in our data period.

quality phenomenon. Moreover, the scenario-weighted correlations are relatively flat across positive scenarios, illustrating the asymmetry between the reaction of markets to positive and negative shocks. Different asset classes such as equities, commodities, and credit that tend to show a moderate degree of comovement during normal/favorable times start moving much more in tandem under distressed conditions. As a result, our multi-asset class portfolio cannot realize the diversification benefits when we need it the most, under negative extreme scenarios, whereas diversification works relatively better when we do not want it, that is, under favorable scenarios. This is what we refer to as the downside concentration – upside diversification.

Figure 9 illustrates the asset class correlations in more detail. It depicts the individual correlations between the US equity market factor and each other factor coming out of the scenario-weighted matrix under the 9 different US equity scenarios.

Figure 9: Correlation between the US Equity Market Factor and Other Factors



Source: Barclays Capital

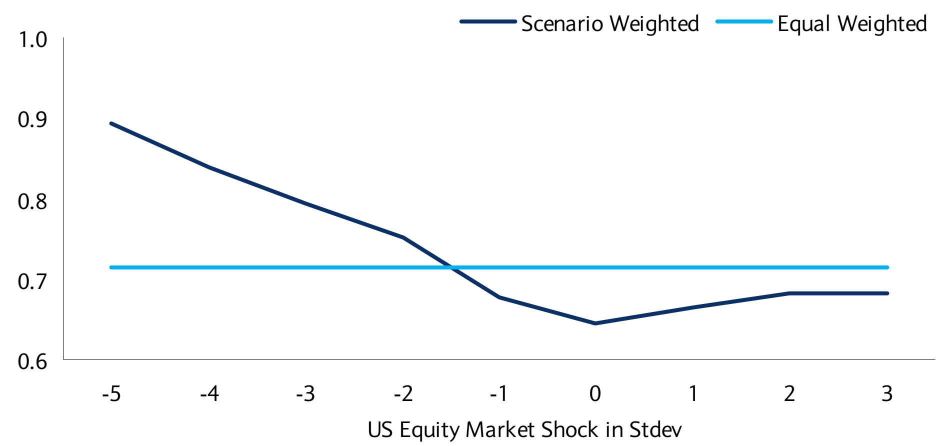
Results are line with the previous figure where scenario-weighted correlations generally rise in magnitude as we move to the negative extreme scenarios and they tend to be relatively flat across positive scenarios. What we want to emphasize in this figure is the varying behavior across different factors in terms of their reaction to the same set of shocks. Correlations generally increase under distressed conditions but not all correlations move to 1 and they do not increase at the same rate. Most practical applications of factor models would incorporate a larger number of factors where we would expect to see more pronounced variation across different factors in terms of their scenario behavior. Using empirical data in computing the scenario-weighted correlations allows us to capture the complex dynamics of correlations across different factors, under different scenarios.

We want to emphasize that our methodology is a generic approach in the sense that it can be applied to any type of scenario. For instance, as we mentioned before, it can be used with multivariate as well as univariate scenarios (e.g., what would be the P&L of my portfolio if credit spreads widen by 100 bps and at the same time interest rates go down by 50 bps?). Moreover, it can be used for scenarios that incorporate small to moderate shocks in the underlying market variables. In that case, the methodology would apply more weight to the historical observations that incorporate moderate movements in computing the correlations and thereby produce correlations that represent relationships under relatively "normal" market conditions. The framework is flexible enough to be a good and dynamic representation of historical behavior under various scenarios being analyzed.

Two typical approaches to diversify a global portfolio are to diversify across asset classes and across geographic regions. The recent illustrations in Figure 8 and Figure 9 focused on the former. We now look into the latter, by exploring the dynamics of correlations across different regions within the same asset class. In particular, we focus on equities and study these dynamics across the same set of scenarios analyzed before. We picked six regional equity market factors, namely US, UK, Continental Europe, Japan, Asia ex-Japan, and Emerging Markets.

Figure 10 depicts the average absolute correlation across all the equity market factors under the 9 US equity scenarios under consideration. Again, results are shown for both the scenario-weighted and equal-weighted matrices.

Figure 10: Average Absolute Correlation across Different Scenarios

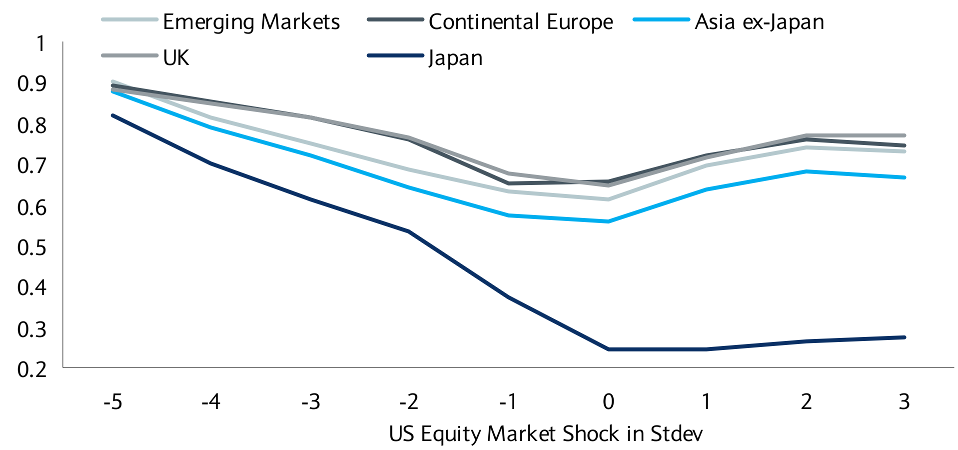


Source: Barclays Capital

The figure shows that average absolute correlation coming out of the scenario-weighted matrix tends to increase – from about 65% up to 90% – as we move closer to the negative extreme scenarios. This again illustrates a flight-to-quality phenomenon under stressed conditions (investors moving simultaneously out of relatively risky equity markets). Similar to the previous case, a portfolio that invests in these regional markets cannot realize the diversification benefits when we need it the most. Note that the baseline correlation is higher here as compared to the cross-asset class correlations as these regional equity markets tend to move relatively closely even under "normal" market conditions.

Figure 11 illustrates the cross-regional correlations in more detail. It depicts the individual correlation between the US equity market factor and each other regional factor coming out of the scenario-weighted matrix under the 9 different US equity scenarios.

Figure 11: Correlation between the US Equity Market Factor and Other Regional Factors

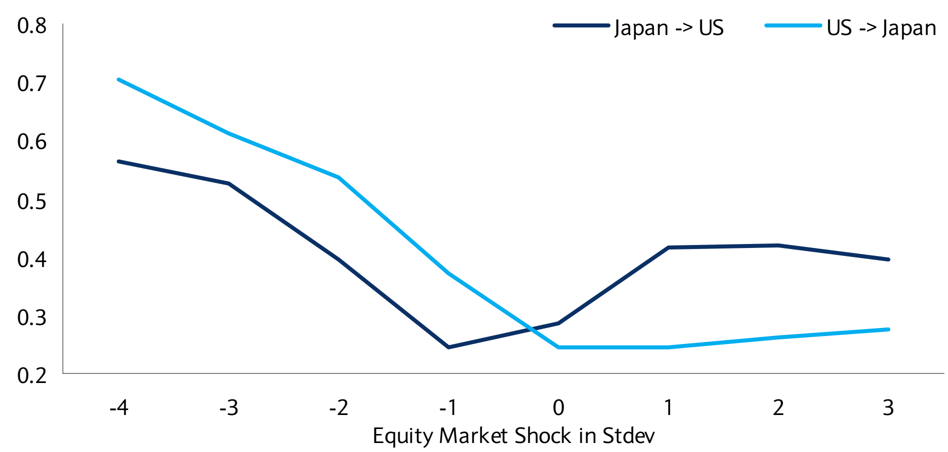


Source: Barclays Capital

It is interesting to note that there is still a certain degree of asymmetrical behavior between positive and negative shocks but it is less pronounced, when compared to the cross-asset class correlations analyzed before. The exception to this is the Japanese equity factor, which tends to move relatively independently of other markets under "normal" market conditions but which then starts moving much more in tandem with them under more extreme scenarios. We again want to emphasize that using empirical data in computing the scenario-weighted correlations allows us to capture these type of complex correlation dynamics.

So far we focused on the impact of a shock in the US equity market on other asset classes or regions. But what about shocks on the opposite direction? Would they have the same impact on the US equity market? A static correlation matrix such as equal-weighted or time-weighted would certainly imply the same impact in both directions. However this symmetry is not realistic for most pairs of market variables. To exemplify this, Figure 12 depicts the correlation between the US and Japanese equity markets under (i) shocks to the US equity market (US->Japan) and (ii) shocks to the Japanese equity market (Japan->US), using the scenario-weighted matrix. The former would represent the impact of a shock in the US equity market to Japanese equities and vice versa for the latter. We can see that the impact is typically different between the two directions: for example, a -4 standard deviation shock in the US equity market leads to a correlation of about 70% with the Japanese equities. However, when the same shock hits the Japanese market, the correlated effect on the US equity market is only about 55%. This intuitive result shows an asymmetric behavior across regions: shocks in the US market tend to propagate stronger to other regions than shocks originating from other regions. This is another type of asymmetry that our methodology can accommodate well, but that cannot be captured by static covariance matrices.

Figure 12: Correlation between the US Equity Market and the Japanese Equity Market[7]



Source: Barclays Capital

Further Conditioning on the Matrix

The major limitation to the scenario-weighted matrix methodology is the availability of historical observations similar to the scenario. As we assign weights to historical observations based on their similarity to the scenario specified, we need a certain degree of historical evidence for this scenario to be able to capture the relationships under such conditions. This evidence becomes harder to find in the historical data for multivariate scenarios as the number of variables in the scenario definition increases, because it becomes harder to find historical observations where all these variables move in a similar way to the specified scenario. If we don't have enough historical evidence, we might still be able to utilize our methodology with certain adjustments, for instance through further conditioning on the scenario-weighted matrix to better capture the specific nature of the scenario or the current market conditions. We exemplify such an approach in this section.

To illustrate how we can use this approach, we run scenario analysis on our US multi-asset class portfolio representing a turmoil in Middle East & North Africa (MENA). Let's assume that such a scenario results in a jump in oil prices and a general sell-off in emerging market (EMG) equities along with MENA equity markets. More specifically, we assume that EMG equities would lose 10% and oil prices would go up by 20%. Having defined our scenario, our goal is now to construct a covariance matrix that appropriately represents this scenario.

We can use our methodology to construct a multivariate scenario-weighted correlation matrix. This methodology would search for historical episodes similar to the above EMG equity-oil scenario and assign higher weights to them in computing empirical correlations. Figure 13 shows the 10 episodes (months) most similar to this scenario.

[7] The minimum is a -4 standard deviation shock in this illustration (as opposed to -5) as this is the most severe movement we observe in the Japanese equity market in our historical dataset.

Figure 13: Historical Episodes Most Similar to the EMG Equity – Oil Scenario

Month	Weight	EMG EQ	Oil Price
Dec-02	1.7%	-2.8%	16.2%
Jun-08	1.7%	-10.1%	9.7%
Sep-02	1.6%	-8.5%	4.7%
Jul-04	1.5%	-0.3%	17.9%
Feb-03	1.5%	-3.2%	12.5%
Aug-00	1.5%	0.6%	22.9%
May-08	1.4%	-1.0%	12.5%
Mar-05	1.4%	-4.7%	6.0%
Apr-04	1.4%	-4.8%	6.1%
Jun-02	1.3%	-5.5%	5.1%

Source: Barclays Capital

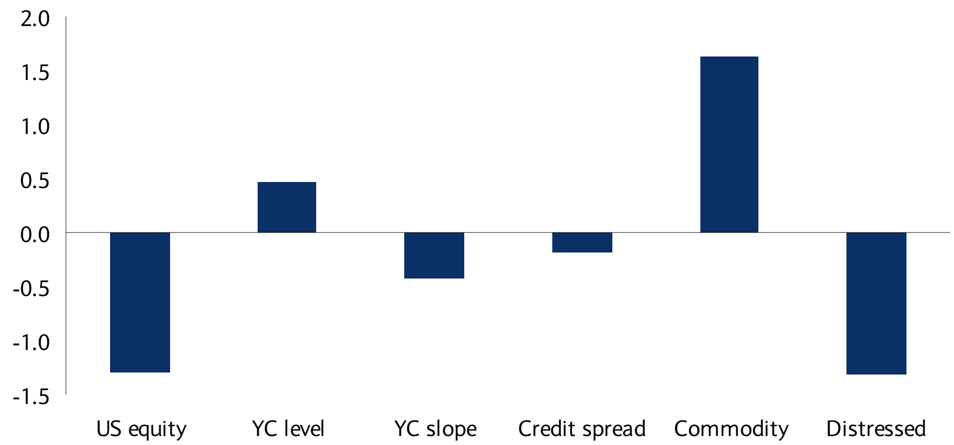
As we can see in the figure, there is not enough historical evidence for this scenario. Even the episodes that are closest to our scenario definition are generally not reasonably close to the shocks we specified in the EMG equity and oil markets. As a result, our methodology assigns weights to these observations that are not very different than equal weights[8] (0.8% in this case). Hence a scenario-weighted matrix here cannot capture well the specificity of this scenario.

To address this issue, we can use an alternative approach. First, we define a univariate scenario on EMG equities only (down 10%) and then construct our scenario-weighted matrix using this univariate shock. Historically when equity markets plummet, oil prices tend to follow. As a result, this matrix would imply a drop in oil prices, contrary to our view under the original scenario. Therefore once we construct the scenario-weighted matrix using the univariate EMG equity shock, we further condition this matrix to imply an appropriate rise in oil (and in relation to that also overall commodity) prices. There are different ways to achieve this, through individual correlations or eigenvalues of the correlation matrix. In this exercise, we specifically manipulate the correlations between oil prices/commodities and other factors[9].

Figure 14 shows the return contributions of different factors relevant to our US multi-asset class portfolio under this scenario using the manipulated matrix mentioned above in scenario analysis computations. As we can see in the figure, the commodity exposure of the portfolio acts now as a diversifier to other exposures, in line with oil prices moving up significantly as specified in this scenario.

[8] An interesting characteristic of our methodology is that the scenario-weighted matrix in the limit defaults to an equal-weighted matrix if there is not enough historical evidence for the scenario. This is due to all observations having very large and consequently similar distances to the scenario and therefore being assigned similar weights.
[9] In this example, we take a simple approach and multiply the correlations between oil prices/commodities and other factors by -1. This also ensures that the correlation matrix stays positive definite.

Figure 14: Return Contributions (%) Coming from Different Factors



Source: Barclays Capital

Step 2

In the first step we estimated the correlations between market variables under a specified scenario by assigning higher weights to historical observations that are more similar to the scenario. To construct the scenario-specific covariance matrix, we also need estimates for the volatilities. In the second step, given the correlations, we estimate the scenario volatility of each market variable. For simplicity, we use an exponentially weighted moving average (EWMA) model for that estimation. Many other models are available for this purpose. A straightforward alternative to this would be to estimate the volatilities via the scenario-weighted methodology as described in Step 1, by computing the standard deviation of historical returns using the scenario-specific weights. We think that the dynamics of the volatility process is not as complex as the correlations under a given scenario and therefore choose to impose a simple and intuitive structure as opposed to allowing empirical data to completely determine the scenario volatilities. This also allows the volatility forecast under the scenario to deviate from historical patterns.

In the EWMA model, the volatility of a given variable under the specified scenario is a weighted average of its current volatility and the magnitude of its movement under the scenario. This relationship can be formulized as follows:

$$\sigma_{i,s}^2 = (1-\alpha)\cdot\sigma_i^2 + \alpha\cdot f_i^2$$

where σ_i , $\sigma_{i,s}$, f_i are respectively the current volatility, scenario volatility, and the scenario return of variable *i*. As an example, if the current volatility of the US equity market is 5% (monthly) and this market experiences a drop of 15% in the specified scenario, this model assumes that its volatility will increase from 5% but will not move beyond 15%. The decay parameter (α) can be calibrated using historical data by optimizing the forecasting performance of the model. A time-weighted volatility model can be used as an estimate for the current volatility, σ_i .

If a view is specified on a market variable, we know the return of that variable under the given scenario and therefore we can use the above formula to estimate the scenario volatility. However, returns of all other market variables (ones with no view) are implied through an appropriate covariance matrix for the scenario. This matrix is composed of a correlation matrix coming from Step 1 and the volatilities, estimated in this step. So this means that to estimate the scenario volatility ($\sigma_{i,s}$) of a variable with no view, we need to know the scenario return (f_i) of this variable and to estimate this scenario return, we need to know its scenario volatility. To address this problem for variables with no views, we decompose f_i as :

$$f_i = z_{i,s} \cdot \sigma_{i,s}$$

where $\sigma_{i,s}$ is the scenario volatility of the variable and $z_{i,s}$ is the z-score of the variable under the given scenario (how many standard deviations a variable moves under the scenario). This z-score can be computed using all the information available before Step 2, hence is known in Step 2 (please see Appendix 2, Part 1 for details). Combining the previous two equations, we can write:

$$\sigma_{i,s}^2 = (1-\alpha) \cdot \sigma_i^2 + \alpha \cdot \left(z_{i,s}^2 \cdot \sigma_{i,s}^2 \right)$$

Now the only unknown in this equation in Step 2 is $\sigma_{i,s}^2$; we can solve for that.

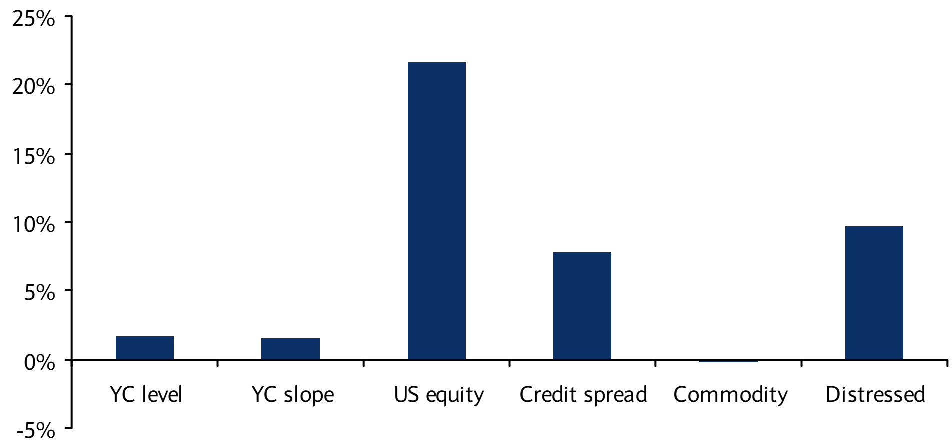
f_i is the expected return of variable *i* under the specified scenario. It is our best estimate as the scenario return but the actual realization of the variable (if the scenario realizes) will be a number around this expected return. Ignoring the volatility of this residual component (actual realization – expected return) may result in the underestimation of updated volatility (scenario volatility) in the above model. We modify our volatility estimation formula to incorporate this residual volatility (for details on how we estimate this residual volatility, please see Appendix 2, Part 2):

$$\sigma_{i,s}^2 = (1-\alpha) \cdot \sigma_i^2 + \alpha \cdot \left(z_{i,s}^2 \cdot \sigma_{i,s}^2 + \varepsilon_{i,s}^2 \right)$$

where $\varepsilon_{i,s}$ is the residual volatility of variable *i* under this scenario. We will be utilizing this residual component in Section 4 where we discuss confidence intervals for the expected returns of variables, f_i .

Figure 15 illustrates the percentage difference between the estimated scenario volatility and the current volatility for each factor relevant to our US multi-asset class portfolio for a -3 standard deviation scenario in US equities (e.g., volatility going up from 5% (current) to 6% (scenario) would be a 20% difference). We see that volatilities generally increase, with the largest changes in the US equity, credit spread, and the distressed factors. We want to note that US equity is the factor with the scenario view and the other two are the factors that exhibit the most significant correlations with the US equity under such a scenario.

Figure 15: Changes in Volatilities in a -3 Stdev. US Equity Market Scenario



Source: Barclays Capital

Given that we can produce the scenario-specific correlation matrix in Step 1, we are not limited by historical evidence for the scenario in Step 2. Because once we come to Step2, the only new information we need is the current volatilities of market variables, which are not specific to the given scenario. Therefore, this model can be applied to any type of scenario in producing the volatility forecasts.

Step 3

The major output of our methodology is a covariance matrix designed to capture the relationships between market variables under the specified scenario. In the third step, we compute this covariance matrix using the correlations from the first step and the volatilities from the second step. A straightforward application of this matrix is in the area of scenario analysis in estimating portfolio returns under the specified scenario. On the other hand, this covariance matrix can be utilized in various other portfolio management exercises such as designing optimal hedging strategies (e.g., tail risk hedging) and portfolio construction (e.g., constructing defensive allocations). In this section, we discuss the implications of using this matrix in the portfolio management process by illustrating such applications.

A major objective in the scenario analysis exercise is to estimate the P&L of a portfolio under a given scenario. Given a set of views on certain market variables, we can use the maximum likelihood estimation (MLE) procedure to estimate the returns of other market variables under this scenario. If we assume a multivariate normal distribution for the complete set of market variables, the following equation provides the MLE estimates:

$$f = \mu + \sum A^T (A \sum A^T)^{-1} (v - A\mu)$$

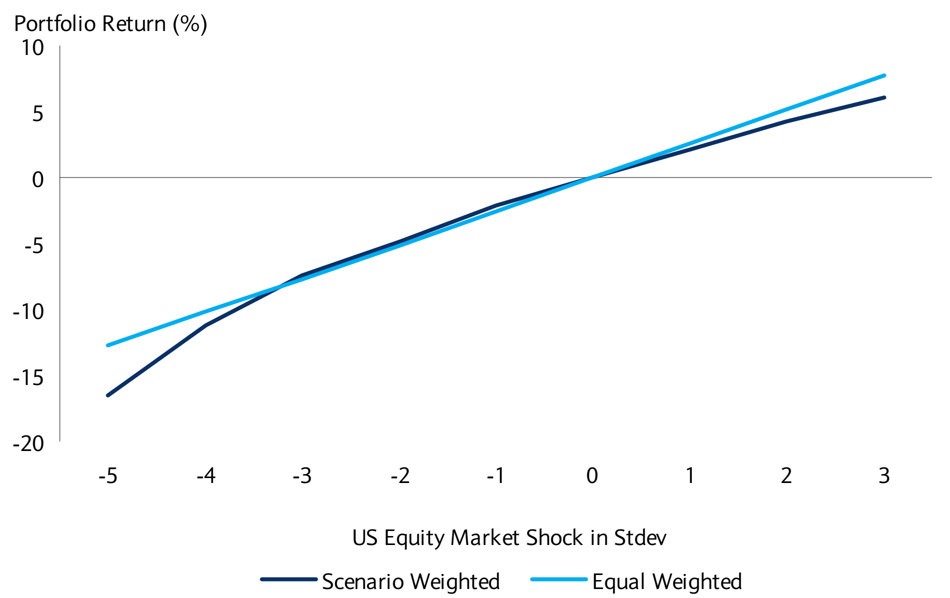
where μ and $\sum$ are respectively the mean and the covariance matrix of market variables, v and A represent the set of views, the scenario, specified as in $Af = v$. As we mentioned previously, in the context of factor models, these market variables are risk factors. Using a multi-factor model, we can write the systematic "scenario" return of a portfolio as

$$r = \sum L_i f_i$$

where f and L are respectively the factor returns and loadings (sensitivity) of the portfolio to these risk factors under the given scenario. Once we have the scenario returns of factors through the above MLE procedure, using this equation, we can estimate the scenario return of any portfolio (see Lazanas, Silva, Staal, and Ural (2011) and Lazanas, Silva, Gabudean, and Staal (2011) for detailed discussions on equity and fixed-income multi-factor risk models).

We use the aforementioned 9 scenarios on the US equity market (shocks ranging from -5 to +3 standard deviations) and estimate the systematic[10] return of our US multi-asset class portfolio under these scenarios using the above MLE. Figure 16 depicts the portfolio return using two different covariance matrices in this estimation, scenario-weighted and equal-weighted.

Figure 16: Estimated Return of the Portfolio under US Equity Scenarios



Source: Barclays Capital

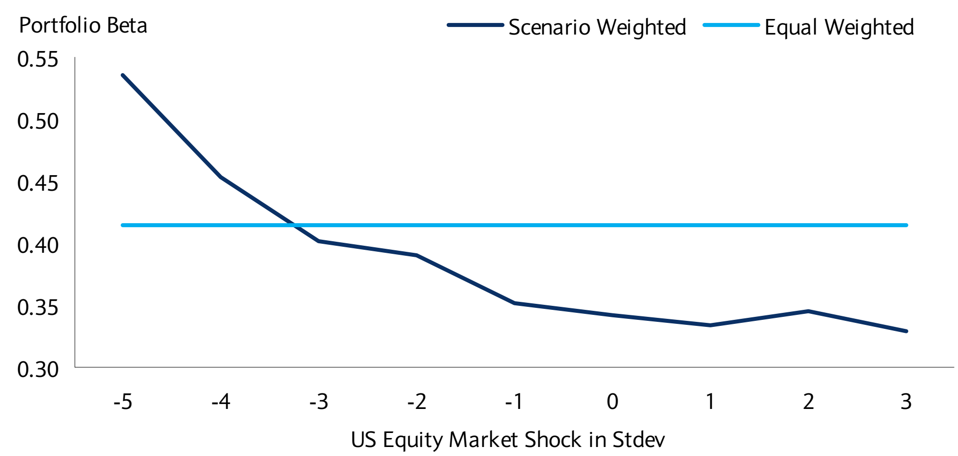
As discussed before, the equal weighted matrix is indifferent to the scenario specified. Therefore, the portfolio return is a linear function of the underlying shock. This is represented by the straight line in Figure 16. In contrast, our methodology (scenario-weighted) produces a specific matrix for each scenario. This provides a dynamic nature to the correlations and volatilities. This characteristic becomes especially useful in capturing the inherent negative convexity in the portfolio's returns, as a function of the equity shock. As a result, under large negative shocks, the scenario-weighted matrix implies reduced diversification benefits as compared to the equal weighted matrix, resulting in larger expected losses in the portfolio. The opposite happens for large positive shocks. The difference between the two matrices tends to be small for moderate shocks, but can be important for more extreme shocks, depending on the portfolio composition.

Reverse stress testing is a technique where the goal is to find the most likely scenario(s) that would result in a pre-specified amount of loss in the portfolio. For example, in the context of

[10] As idiosyncratic return tends to be small for market-level/diversified portfolios, we ignore that component for the purposes of this exercise.

our analysis, we can ask the following question: How large of an equity shock would result in a 12% loss in our multi-asset class portfolio? To answer this question using the above MLE procedure we need to find out the values on the X-axis that would correspond to a return of -12% on the two lines in Figure 16. We see that the scenario-weighted matrix produces a more conservative estimate: a shock of about -4 standard deviations would deliver that return using the scenario-weighted matrix. For the equal weighted matrix this shock has to be around -5 standard deviations.

Figure 17: Estimated Beta of the Portfolio under US Equity Scenarios



Source: Barclays Capital

Figure 17 depicts the sensitivity (beta) of the portfolio to the underlying US equity market movement. The equal-weighted beta is static, at around 0.42. However, using the scenario-weighted matrix, the portfolio beta increases significantly (from 0.33 to 0.54) as we move from positive to the negative extreme shocks. This result again comes from the dynamic nature of the scenario-weighted matrix, with correlations increasing significantly under distress (see Figure 8). The difference between the two approaches is significant. If we wanted to hedge our US multi-asset class portfolio using US equity futures, the optimal hedge ratio would depend on the particular scenario we are trying to hedge against. Hedges that are optimized under "normal" market conditions may not perform as expected when a large shock hits the market. This is typically when the hedge is needed the most. Note that using an average beta (e.g., the equal-weighted one) would result in a significant amount of unintended under/over exposure on both sides of the underlying shock spectrum.

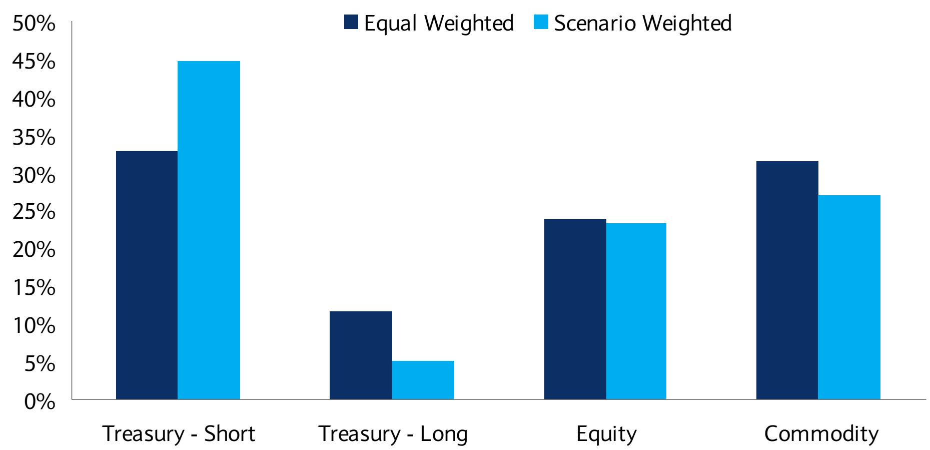
Defensive Asset Allocation

When constructing or rebalancing their portfolios, managers extensively rely on a risk model to understand the impact of their decisions in the risk of the portfolio. These models would typically involve a covariance matrix of risk factors. When these exercises are done with a "stressed" covariance matrix, the solution is typically referred to as a "defensive" one – the solution is optimal conditional on a distressed (not normal) environment.

In what follows, we illustrate how our methodology can be used in this context. For that, consider a portfolio of 4 asset class indices: the Barclays Capital US Treasury 1-3 Years (short Treasuries), US Treasury 10-20 Years (long Treasuries), US Commodity Index and the S&P 500 Index. We construct the minimum variance portfolio for these asset classes using two covariance matrices: scenario weighted and equal weighted[11]. We allow for only long positions in these indices and impose a minimum weight of 5% for each index. The scenario-weighted matrix in this exercise is coming from a -5 standard deviation US equity shock scenario.

Figure 18 illustrates the optimal allocations using the two matrices. Not surprisingly, the scenario-weighted methodology produces a more defensive allocation by using a stressed matrix. In particular, we can see that the optimal solution allocates significantly more weight to short Treasuries, which are prime safe-haven instruments under distressed conditions. That increase in weight comes mainly from a decrease in the allocation to the long Treasuries, as well as a smaller allocation to commodities. The results reflect the large negative correlation between short maturity Treasuries and equity/commodity asset classes under distressed conditions. The fact that short Treasuries exhibit a larger correlation (more negative) with the other two under such a market environment than long Treasuries justifies the concentration on the short index.

Figure 18: Composition of the Minimum Volatility Portfolio



Source: Barclays Capital

3. Alternative Approaches

In this section, we elaborate on alternative approaches in constructing a covariance matrix for the specified scenario. One such approach is to construct a custom covariance matrix for each scenario. This matrix would be primarily driven by market intuition and we would work on each individual correlation/volatility estimate to construct an appropriate matrix for the specified scenario. This approach provides a lot of flexibility and would produce a matrix that reflects experts' views on correlations and volatilities. However, as we work on each component of the matrix individually, we may end up with inconsistent correlations, resulting in a matrix that is not positive definite. This custom approach can be very time consuming and would produce matrices that may not be consistent across different scenarios. A more practical use case of this approach is to first construct the covariance matrix using the methodology described in the previous section and then further condition it locally to reflect particular strong views we have regarding certain factors.

[11] In fact, the optimal portfolios are constructed using the returns of the factors normalized by their volatility. It is as if we allocate weights using instead the correlation matrix and then re-normalize these weights using the individual factor volatilities. This optimal portfolio is referred sometimes in the literature as the most diversified portfolio.

An alternative approach would be to start with an initial matrix that captures current correlations/volatilities (e.g., time-weighted matrix) and move this matrix towards a target. An example is the latent factor approach where there is a small number of latent factors that drive correlations across various different market variables. We can capture changing correlations across these variables in a given scenario by manipulating the exposure of each individual variable to this latent factor. By imposing a certain structure in the change in correlations, this approach preserves the positive definiteness of the matrix. However, the methodology requires the estimation of the latent factor(s) and exposures of variables to the latent factor(s) under the given scenario. It can also be hard or impossible to capture the complex dynamics of correlations across different market variables by imposing a structure depending on the set of variables we are analyzing. This problem becomes more pronounced as the number of variables increases and as we include variables from different asset classes.

To capture increasing correlations and volatilities under a distressed scenario, one could use a covariance matrix from a historical crisis period. A limitation of this approach is that the results depend on the specific episode and may not hold under a different scenario. Moreover, this approach can only be applicable to a limited set of market variables due to data limitations – most historical crisis periods are relatively short, making it not so feasible to estimate in a robust way a large set of covariances.

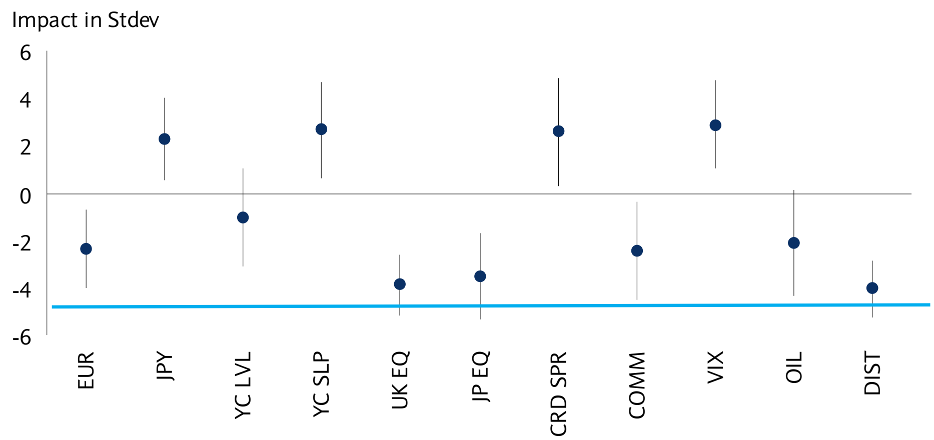
Using a mixture of distributions is another technique for capturing changing correlations under different market conditions, where we assume that the asset returns follow different multivariate distributions under different market regimes. However, it can be hard to capture complex correlation dynamics using a very small number of regimes. On the other hand, as we increase the number of regimes then it becomes harder to estimate such distributions for the underlying market variables.

4. Confidence Intervals

In the context of scenario analysis, Step 3 of our methodology provides the estimated returns of individual market variables and portfolios under the specified scenario. To understand these numbers better, we also need to analyze the confidence interval around these return estimates. For instance, when the equity market plummets, distressed bonds and commodities might be both experiencing significant losses on average but it could be the case that commodities exhibit a large variation around that average loss across different "equity market crash" episodes, whereas the reaction in the distressed bond market is more consistent across these episodes. If this is the case, when we apply large negative shocks to the equity market in the context of scenario analysis, we would get significantly negative return estimates for distressed bonds and commodities but our confidence in these numbers would be substantially different. To quantify this, we compute confidence intervals around our return estimates and compare them across different market variables and scenarios. These confidence intervals provide valuable information that can be incorporated into the portfolio management process.

Figure 19 depicts the return estimates for a selected set of risk factors and the (95%) confidence intervals around these estimates under a -5 standard deviation US equity market scenario. We see that the impact of such a shock is always significant for certain markets, such as UK equities and distressed bonds, where the confidence intervals are relatively small and far from zero. On the other hand, we see that distressed bonds and commodities both exhibit significantly negative estimated returns under this scenario but have quite different confidence intervals. The commodity market is expected on average to experience a significant loss under such a scenario (about -2 standard deviations), yet there is a large uncertainty around this estimate. This has serious implications, for instance in terms of efficacy of instruments in these markets as potential hedges to the US equity market. If we wanted to use short positions in commodity futures as a potential hedge to a US equity portfolio, there would be a large uncertainty about whether that hedge would work well when the equity market experiences drastic losses. Alternatively Figure 19 shows that a short position in UK equity futures can potentially be a more effective hedge under such market movements.

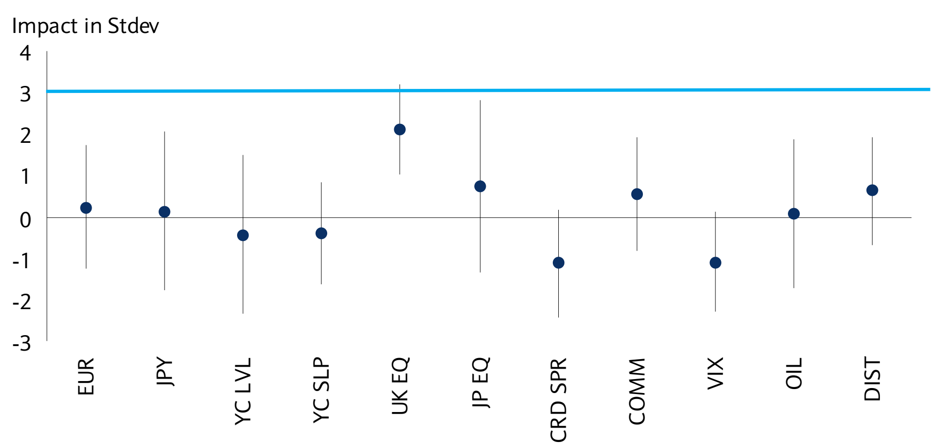
Figure 19: Confidence Intervals under a -5 Stdev. US Equity Market Scenario[12]



Source: Barclays Capital

Figure 20 depicts the return estimates for the same factors and the confidence intervals around these estimates this time under a +3 standard deviation US equity market scenario. We see that the impact of a positive shock in the US equity market is generally quite different when compared to the negative shock illustrated in Figure 19. Estimated returns of factors are closer to 0 in magnitude and except for the UK equities, they cannot be distinguished from zero (as confidence intervals include zero). Confidence intervals are generally larger as compared to the previous scenario. These are direct consequences of the aforementioned asymmetries in the dependence structure of asset classes.

Figure 20: Confidence Intervals under a +3 Stdev. US Equity Market Scenario



Source: Barclays Capital

[12] YC LVL: level of USD yield curve, YC SLP: slope of USD yield curve, UK EQ: UK equity market, JP EQ: Japanese equity market, CRD SPR: USD credit spreads, COMM: US commodities, DIST: global distressed bond prices

5. Testing the Methodology

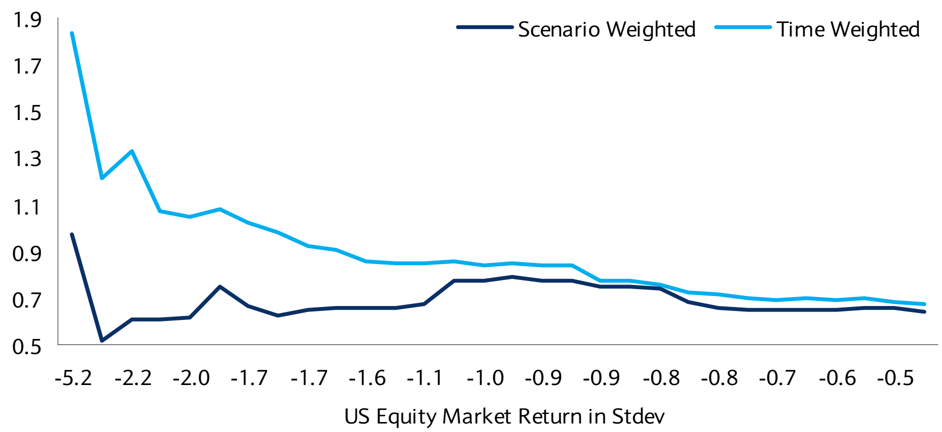
We perform a variety of tests to gauge the performance of our models either relative to certain well-established benchmarks or on an absolute basis. In this section, we present an out-of-sample test comparing the performance of our scenario-weighted covariance matrix to a widely used alternative, the time-weighted matrix. As the major use case of our methodology is in scenario analysis, we perform our testing in this context. We show that our methodology exhibits superior performance for scenarios incorporating more severe shocks and the two methodologies tend to exhibit similar performance as we move to more moderate scenarios.

For this exercise, we use monthly data starting from Jan-90 and a set of risk factors from various asset classes (FX, Yield Curve, Equity, Credit, Commodities, and Volatility). For a given month in the history, we assume that we have a perfect foresight of the US equity market return during that month and use two different covariance matrices (scenario-weighted vs. time-weighted[13]) via maximum likelihood estimation to forecast the return of all other factors during the same month. In the context of scenario analysis, the scenario definition is the actual return of the US equity market during the given month. For each month, we construct the matrices using only historical data prior to the given month (e.g., in constructing the matrix for Jun-05, we use monthly data only from Jan-90 to May-05). We perform this exercise for the worst 30 months of the US equity market since Jan-00[14]. Once we have the forecasts coming out of the two matrices, we compute the absolute error for each forecast as the absolute value of the difference between the forecasted and the actual factor return (normalized by the volatility of the factor).

Figure 21 illustrates the median absolute error for the forecasts coming from the scenario-weighted vs. the time-weighted matrices for different shocks to the US equity market. To explain the picture in more detail, let's take a specific point on the graph, -2.0 shock to the US equity market. This point shows us the median absolute error (across various factors) across all episodes where the US equity market experienced a loss larger than 2.0 standard deviations coming from the two matrices. A value of 1 on the graph means that on average, the forecasted return is 1 standard deviation away from the actual return of the factor. We see that the scenario-weighted matrix shows superior performance (smaller average error) for scenarios incorporating more severe shocks and the two methodologies tend to exhibit similar performance as we move to more moderate scenarios.

[13] For the time-weighted matrix, we use an exponential weighting scheme with a half-life of 12 months.
[14] The worst months are defined as the largest drops in standard deviation terms. Moreover, we use the period only after January 2000 because we need the first 10 years of data (120 observations) to construct the first set of covariance matrices.

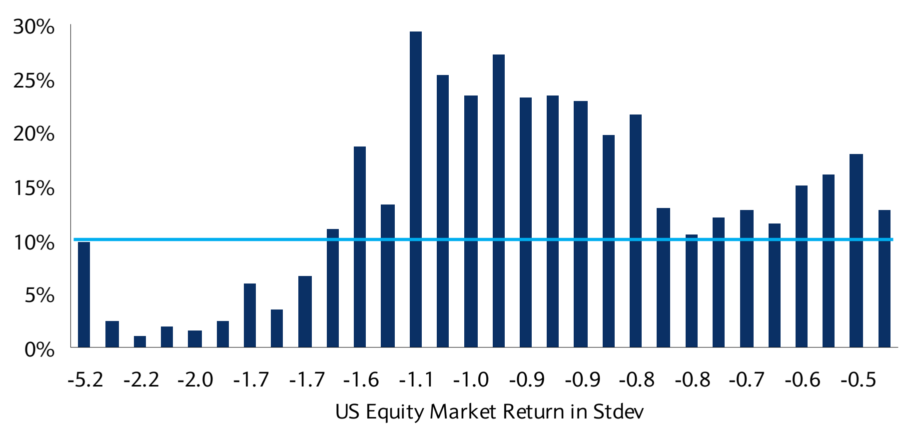
Figure 21: Median Absolute Error for Scenario-Weighted vs. Time-Weighted Matrices



Source: Barclays Capital

To understand the above picture better, we perform a paired t-test to find out if the performance difference between the scenario-weighted and the time-weighted matrices is statistically significant. Figure 22 shows the p-value for the corresponding t-statistic regarding the performance differential between the two matrices as shown in Figure 21. We see that the difference is statistically significant at α =10% level for the largest third of the scenarios. In Figure 21, we see that the scenario-weighted matrix always performs better but Figure 22 shows that we cannot distinguish between the performance of the two matrices as we move to more moderate scenarios. This means that the scenario-weighted matrix provides a significant improvement in the forecasting performance for more severe shocks but also performs as well as the time-weighted matrix for more moderate shocks, hence being a generic solution for different types of scenarios.

Figure 22: Statistical Significance of the Performance Difference (p-value)



Source: Barclays Capital

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APPENDIX 1: LIST OF FACTORS USED IN SCENARIO ANALYSIS

Figure 23: Factor Names and Types

Factor Type	Factor Name	Factor Type	Factor Name
Yield Curve	USD Yield Curve Level	Inflation Linked	USD Expected Inflation
Yield Curve	USD Yield Curve Slope	Inflation Linked	EUR Expected Inflation
Yield Curve	EUR Yield Curve Level	Inflation Linked	GBP Expected Inflation
Yield Curve	EUR Yield Curve Slope	Inflation Linked	JPY Expected Inflation
Yield Curve	JPY Yield Curve Level	Equity	US Equity
Yield Curve	JPY Yield Curve Slope	Equity	US Equity Volatility
Yield Curve	GBP Yield Curve Level	Equity	UK Equity
Yield Curve	GBP Yield Curve Slope	Equity	Continental Europe Equity
Yield Curve	CAD Swap Curve	Equity	Japan Equity
Yield Curve	SEK Swap Curve	Equity	Asia ex-JP Equity
Yield Curve	NOK Swap Curve	Equity	EMG Equity
Yield Curve	DKK Swap Curve	Commodity	USD Com. Energy
Yield Curve	CHF Swap Curve	Commodity	USD Com. Metals Industrial
Yield Curve	AUD Yield Curve	Commodity	USD Com. Metals Precious
Yield Curve	KRW Yield Curve	Commodity	USD Com. Agriculture
Yield Curve	TWD Yield Curve	Commodity	USD Com. Livestock
Yield Curve	SGD Yield Curve	Commodity	USD Com. Core
Spread	USD Swap Spread	Macro	VIX
Spread	EUR Swap Spread	Macro	Oil Price
Spread	GBP Swap Spread	Macro	Gold Price
Spread	JPY Swap Spread	Hedge Fund	USD Hedge Fund
Spread	USD Agency	Real Estate	USD RE Commercial Property
Spread	USD Agency Hybrid	FX	EUR
Spread	USD MBS	FX	JPY
Spread	USD ABS	FX	CHF
Spread	USD ABS HEL DTS	FX	GBP
Spread	USD CMBS DTS	FX	CAD
Spread	USD MUNI DTS	FX	AUD
Spread	USD Agency Strip IO	FX	KRW
Spread	USD Non-Agency OAS Drop	FX	TWD
Spread	USD Corporate DTS	FX	SEK
Spread	EUR Corporate DTS	FX	DKK
Spread	GBP Corporate DTS	FX	SGD
Spread	JPY Corporate DTS	FX	NOK
Spread	Global HY Distressed	FX	HKD
Spread	USD Mortgage Spread	FX	USD
Spread	EMG Fixed Income DTS	FX	CNY
Volatility	USD FI Volatility	FX	BRL
Volatility	EUR FI Volatility		
Volatility	GBP FI Volatility		

Source: Barclays Capital

APPENDIX 2: ESTIMATION OF SCENARIO VOLATILITIES

Part 1

As we mentioned in Step 3 of our methodology, if we assume a multivariate normal distribution for the complete set of market variables, the following equation provides the maximum likelihood estimates for the realization of market variables under the specified scenario:

$$f = \mu + \sum A^T (A \sum A^T)^{-1} (v - A\mu)$$

where μ and $\sum$ are respectively the mean and the covariance matrix (under the given scenario) of market variables, v and A represent the set of views in the scenario, specified as in $Af = v$. To simplify the above equation, let's assume that μ =0. This assumption does not change our conclusions and now we can rewrite f as

$$f = \phi_1 \phi_2^{-1} v$$

where $\phi_1 = \sum A^T$ and $\phi_2 = A \sum A^T$. v, set of views on certain variables, is known at the beginning of Step 2. To make the illustration simpler, let's also assume that out of a total of *n* variables, we specify views on the first *m* of them. Then we can rewrite $Af = v$ as

$$\begin{bmatrix} I & Z \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

where I is an identity matrix of size m x m and Z is a matrix of zeros of size m x (n-m). Now using A from the above equation, $\phi_2 = A \sum A^T$ simplifies to $\sum^*$, which is the subset of the full covariance matrix that corresponds to the covariances of variables (under the given scenario), on which views are specified. This part of the matrix is known in Step 2 as correlations are known at the end of Step 1 and the scenario volatilities for variables with views can be found in a straightforward way by using

$$\sigma_{i,s}^2 = (1-\alpha) \cdot \sigma_i^2 + \alpha \cdot f_i^2$$

where σ_i , $\sigma_{i,s}$, f_i are respectively the current volatility, scenario volatility, and the scenario return of variable *i*. For these variables $f_i = v_i$.

If we decompose the full covariance matrix $\sum$ into two components as $\left[\sum^1 \sum^2\right]$ where $\sum^1$ and $\sum^2$ are matrices of size n x m and n x (n-m) respectively. Then we can show that $\phi_1 = \sum A^T$ simplifies to $\sum^1$. If we go back to the original equation for the realizations of market variables $f = \phi_1 \phi_2^{-1} v$, we can re-write this as $f = \sum^1 \phi_2^{-1} v$. The (*i,j*)th element of $\sum^1$ can be written as $\rho_{i,j,s} \sigma_{i,s} \sigma_{j,s}$ where $\rho_{i,j,s}$ is the correlation between variables *i* and *j* under the specified scenario. Without changing the equality in $f = \sum^1 \phi_2^{-1} v$, we can divide each f_i and each row in $\sum^1$ by $\sigma_{i,s}$. Then what we get on the left hand side is the z-scores of the variables under the scenario, $z_{i,s}$. What is left inside $\sum^1$ on the right hand side is $\rho_{i,j,s} \sigma_{j,s}$ for *j*=1..m and let's denote this modified version of $\sum^1$ as $\sum^{1*}$. As we discussed previously, $\sigma_{j,s}$ can be found in a straightforward way from the equation above as *j* here is the index of variables with views. On the other hand, $\rho_{i,j,s}$ is known from Step 1. This means that

$\sum^{1*}$ can be computed using the known quantities. Looking into the equation $z = \sum^{1*} \phi_2^{-1} v$, we have shown that all three terms on the right hand side are known. Hence the z-scores for all variables (with and without views) are known and now we can solve for $\sigma_{i,s}$ also for variables without views in the below equation

$$\sigma_{i,s}^2 = (1-\alpha) \cdot \sigma_i^2 + \alpha \cdot \left(z_{i,s}^2 \cdot \sigma_{i,s}^2 \right)$$

Part 2

For a market variable *i* without a view, the maximum likelihood estimate of its realization under the scenario is its conditional expectation given the set of views in certain other m variables: $E(f_i \mid f_1 = v_1 ... f_m = v_m)$. Going back to the equations in Part 1, this expectation can be formulated as $f = \sum^1 \phi_2^{-1} v$. This formulation is equivalent to the expected value of f_i in the context of multiple linear regression where we regress variable f_i to variables $f_1 ... f_m$:

$$f_i = \beta_1 f_1 + ... + \beta_m f_m + \varepsilon_i$$

Here we can write the expected value of f_i as $E(f_i \mid f_1 = v_1 ... f_m = v_m) = \beta_1 v_1 + ... + \beta_m v_m$. Going back to the MLE estimates $f = \sum^1 \phi_2^{-1} v$, $\beta_1 ... \beta_m$ are individual elements in the *i*th row of $\sum^1 \phi_2^{-1}$. From Part 1, we know that the only unknown in $\sum^1 \phi_2^{-1}$ is the set of (scenario) volatilities for market variables without views. As this is what we are trying to estimate eventually, at this point, we need a prior for these volatilities, which are estimated by computing the weighted historical standard deviation of returns for each variable f_i using the scenario weights from Step 1. Once we extract $\beta_1 ... \beta_m$ through here, we go back to our regression equation to compute the volatility of ε_i. In the context of linear regression, as we assume that $f_1 ... f_m$ and ε_i are independent, we can estimate the volatility of ε_i under the given scenario *s* as

$$\varepsilon_{i,s}^2 = \sigma_{i,s}^2 - \beta^T \Sigma^* \beta$$

where $\sum^*$ is the covariance matrix of variables with views and β is the vector of $\beta_1 ... \beta_m$ and $\sigma_{i,s}$ is the scenario volatility of variable *i*. We again use the above prior for this volatility number (weighted historical standard deviation of returns for each variable f_i using the scenario weights from Step 1). Once we get this, then we can use it in

$$\sigma_{i,s}^2 = (1-\alpha) \cdot \sigma_i^2 + \alpha \cdot \left(z_{i,s}^2 \cdot \sigma_{i,s}^2 + \varepsilon_{i,s}^2 \right)$$

at the end of Step 2 to estimate the scenario volatility of each variable f_i.

APPENDIX 3: CHARACTERISTICS OF THE WEIGHTING FUNCTION

We would like to elaborate on certain characteristics of the weighting function that is used in estimating the scenario weighted correlations in Step 1 and discuss how it compares to some other functional forms. Here is a summary of its major features:

- Smooth function of the scenario definition: no jumps in weights
- Variables with extreme views play a significant role in determining weights
- Small weights in observations where one of the variables is very different than the specified view
- Converges to equal-weights when there is no historical evidence for the scenario
- All observations are utilized in the estimation

Now let's discuss each characteristic in more detail.

Smooth Function

As we change our view in one of the variables, weight of each observation changes in a smooth fashion (i.e., no discontinuities/jumps). A small change in the view does not result in a dramatic change in the weights assigned to observations. If we denote the volatility normalized factors in historical observations and views in the following distance function by $\widetilde{F}_t^i$ and $\widetilde{\theta}^i$ respectively

$$D_t = \sum_{i \in V} \left| \frac{F_t^i}{\sigma_t^i} - \frac{\theta^i}{\sigma_T^i} \right|$$

Then we can rewrite our weighting function as

$$w_t = 0.5^{\left|\widetilde{F}_t^1 - \widetilde{\theta}^1\right| + \ldots + \left|\widetilde{F}_t^m - \widetilde{\theta}^m\right| / (\lambda m)} = 0.5^{\left|\widetilde{F}_t^1 - \widetilde{\theta}^1\right| / \lambda m} * 0.5^{\left|\widetilde{F}_t^2 - \widetilde{\theta}^2\right| + \ldots + \left|\widetilde{F}_t^m - \widetilde{\theta}^m\right| / (\lambda m)}$$

As we change our view on the first variable (θ^1), the second component in the above product does not change and therefore acts as a constant. The first component changes as a smooth function of θ^1; its form was illustrated in Figure 6.

Variables with Extreme Views

Let's assume that scenario views are specified in two variables only. Then the exponent of the weighting function would be

$$\left|\widetilde{F}_t^1 - \widetilde{\theta}^1\right| + \left|\widetilde{F}_t^2 - \widetilde{\theta}^2\right| / 2\lambda$$

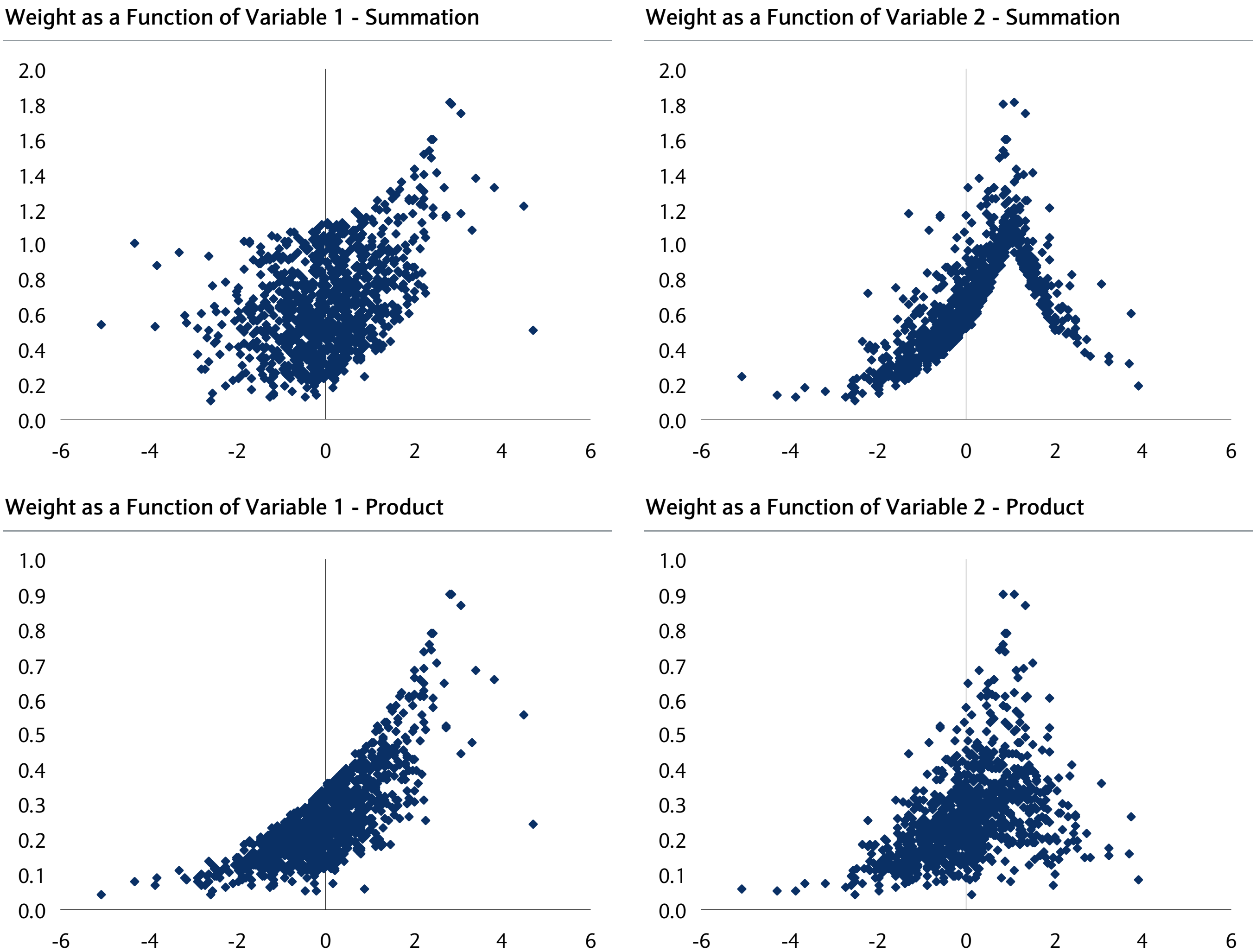
If one of the views is considerably more extreme compared to the other one, we would want the variable with that view to play a significant role in determining the weights in order to capture potentially stressed correlations. To illustrate this characteristic of our weighting function compared to other alternative forms, Figure 24 shows the weights assigned to a set of observations coming out of a simulation of two (uncorrelated) random variables with t-distribution with the same parameters, where df=10 (we use λ =1). We compare two different weighting schemes here: summation vs. product defined as follows:

Product: $$\left(0.5^{\left|\widetilde{F}_t^1-\widetilde{\theta}^1\right|} * 0.5^{\left|\widetilde{F}_t^2-\widetilde{\theta}^2\right|}\right)^{1/2}$$

Summation: $$\left(0.5^{\left|\widetilde{F}_t^1-\widetilde{\theta}^1\right|} + 0.5^{\left|\widetilde{F}_t^2-\widetilde{\theta}^2\right|}\right)\Big/2$$

Our weighting scheme corresponds to the former, the product. The overall weight assigned to each observation in this weighting scheme is the product of the weights corresponding to the first and the second variables. We plot weights coming from these two schemes as a function of the value of the first and the second variable in each observation. We assign a view of 3 and 1 (normalized) for the first and the second variables respectively. As these variables are uncorrelated, ideally we would expect the weighting function to exhibit a similar behaviour to what is shown in Figure 6, peaking around 3 when we plot against the values of the first variable and peaking around 1 when we plot against the second variable.

Figure 24: Comparing Summation vs. Product Weighting Schemes



Source: Barclays Capital

As we can see in the figure, for the first variable that incorporates the more extreme view, weights coming from the product function exhibit a behaviour that is closer to an exponential function peaking around 3 whereas for the summation function, the weights are more scattered. On the contrary, for the second variable, summation exhibits a clearer pattern with a peak around 1. Looking at the picture from another perspective, the summation function favours variables with more moderate views. If we look into the above equations, we can see that as the second variable incorporates a moderate view, the term corresponding to that variable in the summation function is on average significantly larger as compared to the first variable. This is due to observations being on average significantly further from the view for the first variable. As a result, the overall weight is driven more heavily by the variable with the moderate view. On the other hand, the product function treats all variables (with moderate vs. extreme views) in a more similar way. This can also be seen in Figure 24 where the behaviour of the weighting function is more comparable between the two bottom graphs, as a function of the first and the second variable. This characteristic helps in capturing potentially stressed correlations when one view is more on the extreme side and the other one is relatively moderate.

Other Characteristics

As we use a product function to compute the weights as illustrated in the previous section, when one of the variables in a given historical observation is very far from the view specified on that variable, the weight on that observation would be small as one of the components in the product is very close to 0. Going back to the comparison between different functional forms, this is not so much the case for a summation type of function where a very small weight with respect to one of the variables can still allow for a large weight for that observation.

As we mentioned in Step 1 of Section 2, the scenario-weighted matrix in the limit defaults to an equal-weighted matrix if there is not enough historical evidence for the scenario. This is due to all observations having very large and consequently similar distances to the scenario and therefore being assigned similar weights. This brings robustness into the procedure: without any additional useful information from the scenario, all historical observations are assigned equal weights in computing the correlations.

The last point we want to emphasize is that as opposed to certain methodologies that utilize subsets of all data available in computing conditional correlations (conditional on the given scenario), our methodology utilizes the full history but adjusts the weights across different historical observations as a function of the scenario specified. This allows us to use all data points, enhancing the accuracy of the estimation.

Analyst Certification(s)

We, Antonio Silva and Cenk Ural, hereby certify (1) that the views expressed in this research report accurately reflect our personal views about any or all of the subject securities or issuers referred to in this research report and (2) no part of our compensation was, is or will be directly or indirectly related to the specific recommendations or views expressed in this research report.

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