

Properties of the Most Diversified Portfolio

Yves Choueifaty[°]

President and CIO, TOBAM

Tristan Froidure[°]

Head of Research, TOBAM

Julien Reynier[°]

Senior Vice President, TOBAM

July 2011

Abstract

This article expands upon “Toward Maximum Diversification” by Choueifaty and Coignard [2008]. We present new mathematical properties of the Diversification Ratio and Most Diversified Portfolio (MDP), and investigate the optimality of the MDP in a mean-variance framework. We also introduce a set of “Portfolio Invariance Properties”, providing the basic rules an unbiased portfolio construction process should respect. The MDP is then compared in light of these rules to popular methodologies (equal weights, equal risk contribution, minimum variance), and their performance is investigated over the past decade, using the MSCI World as reference universe. We believe that the results obtained in this article show that the MDP is a strong candidate for being the un-diversifiable portfolio, and as such delivers investors with the full benefit of the equity premium.

[°] The authors can be reached at their firstname.lastname@tobam.fr

Ever since its introduction in the 1960's, the CAPM has come under intense scrutiny. In particular, the efficiency of the market capitalization weighted index has been questioned, with academics and practitioners offering numerous investment alternatives. In 1991, the seminal paper by Haugen and Baker, "The Efficient Market Inefficiency of Capitalization-Weighted Stock Portfolios," concisely proclaimed that "matching the market is an inefficient investment strategy." The authors argued that theory can still predict cap-weighted portfolios to be inefficient investments, even assuming that investors rationally optimize the relationship between risk and expected return in equilibrium, in an "informationally efficient" capital market. Putting theory into practice, Haugen and Baker presented one of the first empirical studies of the minimum variance portfolio. Over the 1972-1989 period, this portfolio delivered equal or greater returns compared to a broad market cap-weighted index of US stocks, while achieving consistently lower volatility, thus demonstrating the ex-post inefficiency of the market cap-weighted index. Nearly fifteen years later, Arnott, Hsu and Moore [2005] created indices with alternative measures of company size based on fundamental metrics. The authors showed such indices were "more mean-variance efficient" compared to market cap-weighted Indices, further challenging the CAPM. Subsequently, Choueifaty [2006] introduced the concept of maximum diversification, via a formal definition of portfolio diversification: the Diversification Ratio (DR). Choueifaty further went on to describe the portfolio which maximizes the DR – the *Most Diversified Portfolio* (MDP) – as an efficient alternative to the market cap-weighted index.

This article expands upon Choueifaty and Coignard [2008], which introduced the concepts of DR and MDP to a broad audience. First, we explore the mathematical properties of the DR. We also establish a new equivalent definition of the long-only MDP, and generalize the condition for the optimality of the MDP in a mean-variance framework. Next, we compare the MDP with three well-known long-only quantitative portfolio construction approaches: Equal Weighted; Equal Risk Contribution (Maillard, Roncalli and Teiletche [2010]); and minimum variance portfolios (Haugen and Baker [1991], Clarke, de Silva and Thorley [2006]). We introduce a set of basic invariance properties an unbiased portfolio construction process should respect, and then examine each approach in light of these properties, using synthetic examples. Finally, using one of the broadest equity universes available - the MSCI World - we study the four portfolios' empirical performance over the past decade.

Properties of the Diversification Ratio (DR)

Choueifaty [2006] proposed a measure of portfolio diversification, called the Diversification Ratio (DR), which he defined as the ratio of the portfolio's weighted average volatility to its overall volatility. This measure embodies the very nature of diversification whereby the volatility of a long-only portfolio of assets is less than or equal to the weighted sum of the assets' volatilities. As such, the DR of a long-only portfolio is greater than or equal to one, and equals unity for a single asset portfolio. Consider for example an equal-weighted portfolio of two independent assets with the same volatility: its DR is equal to $\sqrt{2}$, and to \sqrt{N} for N independent assets¹. In essence, the DR of a portfolio measures the diversification gained from holding assets that are not perfectly correlated. We formalize this intuition by introducing

a formal definition as well as establishing several properties of the DR. Note that all portfolios in this paper are constrained to be long-only, unless otherwise noted².

We consider a universe of N risky *single* assets $\{S_1, \dots, S_N\}$, with volatility $\boldsymbol{\sigma} = (\sigma_i)$, correlation matrix $C = (\rho_{i,j})$ and covariance matrix $\Sigma = (\rho_{i,j}\sigma_i\sigma_j)$, with $1 \leq i, j \leq N$. Let $\mathbf{w} = (w_i)$ be the weights of a long-only portfolio, $\sigma(\mathbf{w})$ its volatility, and $\langle \mathbf{w} | \boldsymbol{\sigma} \rangle = \sum_i w_i \sigma_i$ its average volatility. The Diversification Ratio $DR(\mathbf{w})$ of a portfolio is defined as the ratio of its weighted average volatility and its volatility:

$$DR(\mathbf{w}) = \frac{\langle \mathbf{w} | \boldsymbol{\sigma} \rangle}{\sigma(\mathbf{w})} \quad (1)$$

DR Decomposition

It is intuitive that portfolios with “concentrated” weights and/or highly correlated holdings would be poorly diversified, and hence be characterized by relatively low DRs. Here we formalize this intuition by decomposing the DR of a portfolio into its weighted-correlation and weighted-concentration measures. As shown in Appendix A, the DR decomposition is:

$$DR(\mathbf{w}) = [\rho(\mathbf{w})(1 - CR(\mathbf{w})) + CR(\mathbf{w})]^{-\frac{1}{2}} \quad (2)$$

Where $\rho(\mathbf{w})$ is the volatility-weighted average correlation of the assets in the portfolio,

$$\rho(\mathbf{w}) = \frac{\sum_{i \neq j} (w_i \sigma_i w_j \sigma_j) \rho_{i,j}}{\sum_{i \neq j} (w_i \sigma_i w_j \sigma_j)} \quad (3)$$

and $CR(\mathbf{w})$ is the volatility-weighted Concentration Ratio (CR) of the portfolio, with:

$$CR(\mathbf{w}) = \frac{\sum_i (w_i \sigma_i)^2}{(\sum_i w_i \sigma_i)^2} \quad (4)$$

A fully concentrated long-only portfolio has unit CR (a one asset portfolio), while an equal volatility weighted portfolio has the lowest CR, equal to the inverse of the number of assets it contains³. The CR introduces a generalization of the *Herfindahl-Hirschman* index (HHI in Rhoades [1993]), used for example, by US authorities as a sector concentration measure. In effect, the CR measures not only the concentration of weights, but also the concentration of risks; assets are weighted proportionally to their volatilities.

The above *DR decomposition* explicitly shows that the DR increases when the average correlation and/or the Concentration Ratio decrease. In the extreme, if correlations increase to unity, the DR is equal to one, regardless of the value of the Concentration Ratio, as portfolios of assets are no more diversified than a single asset. We note that when pair-wise asset correlations are equal, the DR varies only via the CR, and maximizing the Diversification Ratio is equivalent to minimizing the Concentration Ratio.

DR Composition

Determining the DR at the asset allocation level, for a multi-asset portfolio, is a potentially valuable tool for plan sponsors and their trustees. The *DR Composition* formula provides the overall DR of a portfolio, as a function of the DRs of its sub-portfolios. Consider S long-only sub-portfolios with weight vectors $(\theta_s)_{s=1\dots S}$. Each sub-portfolio s has overall non negative weight w_s , volatility σ_s and Diversification Ratio $DR_s = DR(\theta_s)$. The *DR Composition Formula* established in appendix A provides the overall DR of the portfolio:

$$DR(\mathbf{w}) = \frac{\langle \mathbf{w}_s \odot \sigma_s | \mathbf{DR}_s \rangle}{\sigma(\mathbf{w})} \quad (5)$$

where \odot is defined as the element-wise product of two vectors. The above DR Composition formula shows that the DR of a portfolio is the volatility-weighted average of its sub-portfolios' DRs, divided by its volatility⁴.

DR as a measure of degrees of freedom

We provide an intuitive *interpretation* of the DR, by first considering a universe of F independent risk factors, and a portfolio whose exposure to each risk factor is inversely proportional to the factor's volatility. Such a portfolio allocates its risk budget equally across all risk factors, and will have a DR squared (DR2) equal to F^5 . As such, its DR2 is equal to the number of independent risk factors, or degrees of freedom, represented in the portfolio. Therefore, the DR2 of *any* portfolio of assets can be *interpreted* as the number F of independent risk factors, necessary for a portfolio that allocates equal risk to independent risk factors, to achieve the same DR. As such, F can be interpreted as the *effective* number of independent risk factors, or degrees of freedom, represented in the portfolio.

For example, the DR of an index, such as the MSCI World, was 1.7 as of the end of 2010, implying that a passive MSCI World investor would have been effectively exposed to $1.7^2=2.9$ independent risk factors, in our interpretation. Taking this a step further, if one seeks to maximize the DR, the resulting DR would equal the square root of the effective number of independent risk factors available in the entire market. At the end of 2010, this resulted in a DR of 2.6, or 6.8 effective degrees of freedom. An interpretation of this result is that the market cap-weighted index misses out on the opportunity to effectively diversify across about four additional independent risk factors.

The Most Diversified Portfolio (MDP)

The MDP is defined as the long-only portfolio that maximizes the Diversification Ratio:

$$\mathbf{w}^{MDP} = \arg\max_{\mathbf{w} \in \Pi^+} DR(\mathbf{w})$$

where Π^+ is the set of long-only portfolios with weights summing to one⁶. As shown in Appendix B, the long-only MDP always exists and is unique when the covariance matrix Σ is definite. Furthermore, the portfolio's weights satisfy the first order condition:

$$\Sigma \mathbf{w}^{MDP} = \frac{\sigma(\mathbf{w}^{MDP})}{DR(\mathbf{w}^{MDP})} \boldsymbol{\sigma} + \boldsymbol{\lambda} \quad (6)$$

where the non negative (dual) variables $\boldsymbol{\lambda}$ are such that $\mathbf{Min}(\boldsymbol{\lambda}, \mathbf{w}^{MDP}) = \mathbf{0}$.

The Core Properties of the MDP

An equivalent definition⁷ of the MDP, which we call the Core Property of the MDP (1), provides a very intuitive understanding of its nature:

Any stock not held by the MDP is *more* correlated to the MDP than any of the stocks that belong to it. Furthermore, all stocks belonging to the MDP have the same correlation to it.

This property illustrates that all assets in the universe considered are *effectively* represented in the MDP, even if the portfolio does not *physically* hold them. For example, an MDP portfolio constructed using S&P500 stocks, may hold approximately 50 stocks. That does not mean however that this portfolio is not diversified, as the 450 stocks it does not hold are more correlated to the MDP compared the 50 stocks it actually holds. This is consistent with the notion that the Most Diversified portfolio is the un-diversifiable portfolio.

The Core Property of the MDP (1) is established in appendix B with the help of the above first order condition. It is also equivalent to the following alternative definition, which is more involved, and forms the basis of its proof. For this reason, we call it the Core Property of the MDP (2):

The long-only MDP is the long-only portfolio such that the correlation between any other long-only portfolio and itself is greater than or equal to the ratio of their DRs.

Equivalently, for any long-only portfolio with weights \mathbf{w} :

$$\rho_{\mathbf{w}, \mathbf{w}^{MDP}} \geq \frac{DR(\mathbf{w})}{DR(\mathbf{w}^{MDP})} \quad (7)$$

Accordingly, the more diversified a long-only portfolio is, the greater its correlation with the MDP. Note that when the covariance matrix Σ is not definite, all portfolios satisfying the Core Property (1) or (2), equivalently maximize the DR. As such, equation (7) also shows that all solutions are equivalent, as they have a correlation of one between themselves.

Optimality Properties of the MDP in a Mean-Variance Framework

In this section, we explore a mean-variance framework where the MDP is the optimal, equilibrium portfolio. This ideal setting is of course far from reality. Note, however, that the assumptions entertained here are not prerequisites for the MDP's outperformance in other contexts, in particular in the real world.

Consider a homogeneous investment universe of *single* assets where we have no reason to believe, ex-ante, that any *single* asset will reward risk more than another. In this universe, the ex-ante Sharpe ratios of *single* assets are identical, and thus each asset's expected excess

return (EER) is proportional to its volatility. Assume that a risk free asset is available, with rate r_f . Noting r_1, \dots, r_N the *single* assets' returns, and k a positive constant; *single* assets' EERs satisfy:

$$E(r_i) - r_f = k \sigma_i \quad (8)$$

As such, for any *portfolio* of single assets with weights \mathbf{w} , and return r_w :

$$E(r_w) - r_f = \sum_{i=1}^N w_i (E(r_i) - r_f) = k \langle \mathbf{w} | \boldsymbol{\sigma} \rangle$$

Using the definition of the Diversification Ratio, we finally obtain:

$$E(r_w) - r_f = k \sigma(\mathbf{w}) DR(\mathbf{w}) \quad (9)$$

Equation (9) shows that portfolios' EERs are proportional to their volatilities *multiplied* by their Diversification Ratios⁸. Dividing both sides of this equation by $\sigma(\mathbf{w})$ demonstrates that in this homogenous universe, maximizing the Diversification Ratio is equivalent to maximizing the Sharpe Ratio.

Going a step further, assume that all CAPM assumptions hold as in Sharpe [1990], whose Nobel lecture includes a very clear, self-contained, expose of the CAPM. One central assumption is that “all investors are in agreement concerning expected returns *and* (asset) covariances”. When equilibrium prices are attained, both expected returns and covariances are determined in such a way that markets clear. Let us explore further the case where all investors also agree that single assets' EERs are proportional to their volatilities. In this setting, assets' EERs depend on volatilities and on the proportionality constant k^9 (constant across assets). As such, assuming that equilibrium prices are attained, both asset covariances *and* the constant k are determined in equilibrium. Providing that markets have cleared, the Security Market Line relationship still obtains¹⁰. Also, as a risk free asset is available, the portfolio of risky assets held

by all investors maximizes the Sharpe Ratio¹¹, which in this particular situation also maximizes the DR, as EERs are proportional to volatilities. As a result, this portfolio is the MDP, and the Security Market Line relationship reads:

$$E(r_i) - r_f = \rho_{i,MDP} \frac{\sigma_i}{\sigma_{MDP}} (E(r_{MDP}) - r_f) \quad (10)$$

It is demonstrated in appendix B that the correlation of any asset to the unconstrained MDP is the same. Noting ρ_{MDP} this correlation, we finally obtain the pricing equation:

$$E(r_i) - r_f = \rho_{MDP} \frac{\sigma_i}{\sigma_{MDP}} (E(r_{MDP}) - r_f) \quad (11)$$

Naturally, this last result is consistent with the initial hypothesis that assets' EERs are proportional to volatility. It also shows that in equilibrium¹², the identical Sharpe Ratio of *single* assets is equal to the Sharpe Ratio of the equilibrium portfolio, the MDP, multiplied by the constant correlation of all assets to this portfolio. Importantly, it also demonstrates that we still have the original CAPM result that assets are rewarded in proportion to their systematic risk exposure, which in this setting corresponds to their exposure to the MDP.

Comparison of Quantitative Portfolios

Portfolio Invariance Properties

We propose in this section a set of basic properties that an unbiased, agnostic portfolio construction processes should respect, based on common sense and reasonable economic grounds. A starting point is the fact that portfolios resulting from these processes are highly dependent upon the structure of the universe of assets considered. As such, it may be reasonable to require an *unbiased* process to produce exactly the same portfolio when

considering a universe equivalent to the original one. We formalize this idea in the following three Portfolio Invariance Properties:

- (1) *Duplication Invariance*: Consider a universe where an asset is duplicated (for example, due to multiple listings of the same asset). An unbiased portfolio construction process should produce the same portfolio, regardless of whether the asset was duplicated.
- (2) *Leverage Invariance*: Imagine that a company chooses to deleverage (leverage). All else equal, the weights allocated by the portfolio to the company's underlying business(es) should not change, as its cash exposure is dealt with separately.
- (3) *Polico Invariance*: The addition of a positive linear combination of assets (for example, a leveraged long-only portfolio) already belonging to the universe (for example, the creation of a long-only leveraged ETF on a subset of the universe) should not impact the portfolio's weights to the original assets, as they were already available in the original universe. We abbreviate "positive linear combination" to read "Polico".

Comparison of well known quantitative approaches

Among the alternatives to cap-weighted indices that have been proposed, we compare the Equal Weighted (EW), Minimum Variance (MV), Equal Risk Contribution (ERC) and Most Diversified Portfolio (MDP). These portfolios are related to cap-weighted indices, insofar as they are all fully invested, unlevered, long-only, and provide comparable access to the equity risk premium. The MV portfolio, for example, minimizes volatility across all long-only portfolios, with weights summing to one. We examine each of these portfolios in the context of the aforementioned Portfolio Invariance Properties.

We consider a simple universe $\{A,B\}$ of two assets A and B , with volatilities $\sigma_A = 20\%$; $\sigma_B = 10\%$, respectively and pairwise correlation $\rho_{AB} = 50\%$. For each of the above four approaches, their portfolio weights and risk contributions¹³ are:

Portfolio	Weights		Risk contributions	
	A	B	A	B
EW	50%	50%	71%	29%
ERC	33%	67%	50%	50%
MV	-	100%	-	100%
MDP	33%	67%	50%	50%

By construction, the EW portfolio sees its largest risk contributions coming from the most volatile asset, whereas the MV invests 100% of its holdings in the low-risk asset¹⁴. Only the MDP and ERC portfolios provide a truly diversified risk allocation in this case, as seen from their risk contributions. In the next three sub-sections, we examine whether these portfolio construction methodologies respect the Portfolio Invariance Properties.

Duplication invariance

Consider a new universe derived from the first one, where asset A is duplicated: $\{A, A, B\}$.

Each of the four portfolios assigns weights as follows:

Portfolio	New weights			New weights in the original assets		Original weights		Compliant
	A	A	B	A	B	A	B	
EW	33%	33%	33%	67%	33%	50%	50%	no
ERC	23%	23%	54%	46%	54%	33%	67%	no
MV	-	-	100%	-	100%	-	100%	✓
MDP	17%	17%	67%	33%	67%	33%	67%	✓

Both the MV and MDP are duplication invariant, as their weights in the original assets are unchanged. The duplication invariance of the MV and MDP is true in general¹⁵. However, both the EW and ERC are not invariant, which shows that they are biased toward assets with multiple representations.

Leverage Invariance

Consider the new universe {LA, B} following the replacement of A with LA, a combination of $\frac{1}{4}$ A and $\frac{3}{4}$ cash. This leads to the following figures: $\sigma_{LA} = 5\%$, $\sigma_B = 10\%$ and $\rho_{LA,B} = 50\%$, and to the corresponding portfolio weights:

Portfolio	New weights		New weights in the original assets		Original weights		Compliant
	LA	B	A	B	A	B	
EW	50%	50%	20%	80%	50%	50%	no
ERC	67%	33%	33%	67%	33%	67%	✓
MV	100%	-	100%	-	-	100%	no
MDP	67%	33%	33%	67%	33%	67%	✓

The MDP and ERC are leverage invariant. This is true in general, and is shown for the MDP in Appendix C. On the contrary, the EW and MV portfolio are not leverage invariant, as the former invests a smaller weight in asset A and the latter is now fully concentrated in asset A, and not B. This shows that both the MV and EW are biased with respect to assets' leverage.

Polico Invariance

To illustrate Polico invariance, a Polico¹⁶ containing $\frac{1}{2}$ A, $\frac{1}{4}$ B and $\frac{1}{4}$ cash is added to the new universe {A, B, Polico}, leading to $\sigma_{Polico} = 11.46\%$, $\rho_{A,Polico} = 98.2\%$, $\rho_{B,Polico} = 65.5\%$ and to the following portfolio weights:

Portfolio	New weights			New weights in the original assets		Original weights		Compliant
	A	B	Polico	A	B	A	B	
EW	33%	33%	33%	55%	45%	50%	50%	no
ERC	21%	46%	34%	41%	59%	33%	67%	no
MV	-	69%	31%	17%	83%	-	100%	no
MDP	33%	67%	-	33%	67%	33%	67%	√

The MDP is Polico invariant, as it does not select the Polico, and has unchanged overall weights. This general fact, demonstrated in appendix C, shows that the MDP is robust to the misspecification of the nature of Policos. In effect, the Polico was treated in this example as a *single asset*, and not as a *portfolio* (its DR was assumed to equal one). On the contrary, the EW, ERC and MV portfolios are not Polico invariant, as the EW and ERC are biased toward assets with multiple representations, and the EW and MV are biased with respect to leverage. Note that in this situation, the MV has positive weights on *both* A and B, due to the selection of the Polico.

Summary of Results

To summarize, we present the following table describing the invariance properties respected by each portfolio:

Portfolio	Duplication	Leverage	Polico
EW	no	no	no
ERC	no	√	no
MV	√	no	no
MDP	√	√	√

The MDP's goal is to maximize diversification, and as such, to be unbiased. The fact that the MDP satisfies all three Portfolio Invariance Properties is consistent with this goal. The other

portfolios studied here make implicit bets. The EW and ERC portfolios reflect the belief that representativeness can only be achieved by investing in all stocks present in the universe. The EW and MV portfolios make implicit bets on companies' leverage.

Empirical Study

In this section, we compare the performance of five long-only portfolios: Market Capitalization-Weighted Index (MKT), Equal Risk Contribution (ERC), Equal Weighted (EW), Most Diversified Portfolio (MDP), and Minimum Variance (MV).

Our investment universe for backtesting is the MSCI World, which contains approximately 1500 stocks, spread across developed markets globally. MSCI creates the well-known MSCI World Minimum Volatility Index (MsMV). The MsMV would seem a natural candidate for our MV portfolio since it is also constructed using the MSCI World universe. Its construction methodology is well documented by the index provider, with data available since the creation of the Euro, i.e., December 31st, 1998. However, a careful reading of MSCI's Minimum Volatility Methodology reveals a complex set of minimum and maximum weight, country, sector and turnover constraints, and also minimum and maximum exposure to various risk factors. As a result, the MsMV may not be representative of a MV portfolio; for this reason we have implemented a simpler version of MV, in addition to the MsMV.

The ERC, EW, MV and MDP portfolios were rebalanced semi-annually¹⁷, and stocks belonging to the MSCI Index were selected at each rebalancing date. In order to avoid liquidity and price availability issues in such a broad universe, we only considered at each rebalancing date, the top half of stocks by market capitalization¹⁸ (793 stocks on average, representing 90%

of the index capitalization). To allow for a fair comparison between our portfolios and MKT, we also built a synthetic market cap-weighted index labeled MKT/2, comprised of the top half of stocks ranked by market capitalization. For an appropriate comparison with the MsMV portfolio, we simply added a maximum weight, a regional constraint, as well as a turnover penalty to the MV and MDP construction¹⁹.

In order to use data reflecting as much recent information as possible, we estimated the covariance matrices for the ERC, MV and MDP using a one-year window of past daily returns²⁰, at each rebalancing date. To account for the impact of time zone differences, we developed a “Plesiochronous²¹ Correlation Estimator,” which allows for the joint estimation of asset correlations, while taking into account the time delay between observations²². As having fewer observations than the number of assets results in a non-definite covariance matrix, we have also considered using a basic, yet robust, method consisting of shrinking half of the correlation matrix towards the identity matrix²³. Portfolios built using this method are labeled ERC_{PSD}, MDP_{PSD} and MV_{PSD}.

Finally, while it is straightforward to verify whether a portfolio has the ERC property, a direct implementation of the numerical optimization algorithms, as proposed in Maillard et Al. [2008], can require prohibitive computation time. For our purposes, we chose to implement the optimization problem (7) of their paper, which provides a unique, well-defined, long-only portfolio that respects the ERC property²⁴.

Performance

The portfolios’ empirical performance is summarized in Exhibit 1. All portfolios outperform MKT, which is consistent with the documented inefficiency of market cap-weighted indices,

even when assuming unrealistically high all-in trading costs of the order of two percent²⁵ to account for their higher turnover. The ERC, MV and MDP deliver significantly higher returns and lower volatility, whereas the EW outperforms the market cap-weighted index with comparable volatility. The ERC, in turn, functions as a risk-weighted version of the EW, with marginally higher returns and significantly lower risk. Among the portfolios with the lowest risk, the MsMV registers a modest performance advantage, with significantly less volatility than the cap-weighted index. Its MV counterpart, which has fewer constraints, has the lowest realized volatility, with returns similar in magnitude to the ERC portfolio.

Exhibit 2 provides performance for the ERC_{PSD} , MDP_{PSD} and MV_{PSD} portfolios. Overall returns and volatilities are mostly unchanged²⁶ compared to original versions of these portfolios. However, using the shrinkage method lessens turnover by 5 to 10%, with the MV and MDP portfolios holding 41 and 24 more stocks respectively. This can be expected, as shrunken correlation matrices are by design more stable over time, with the MV and MDP implicitly shrunk toward the equal-variance-weighted and equal-volatility-weighted portfolios.

Unsurprisingly, the Market Cap-Weighted Portfolio has the lowest Diversification Ratio, given its high concentration in large cap stocks and risk factors²⁷. The EW portfolio's diversified holdings result in slightly higher diversification, albeit less than the other portfolios, which use asset covariance information. The MDP both presents the highest DR - its primary objective- and also the highest Sharpe Ratio. As such, it is the closest candidate to being the tangency portfolio. Overall, both the MV and the MDP come close to delivering on their respective claims: to minimize ex-post volatility for the former, and to maximize ex-post Sharpe ratio for the latter.

Fama-French Regression

Exhibit 3 shows the results of a series of Fama and French [1993] 3-factor regressions for each portfolio construction methodology. The factors are labeled MKT for The MSCI World Gross USD index in excess of the 1 month LIBOR, HML for the performance difference between the MSCI World Value and Growth indexes, and SMB for the performance difference between the smallest 30% and the largest 30% of stocks by market capitalization. Month end data were used, with excess returns computed using US one-month LIBOR. Alphas are reported using annualized compounded returns.

All non-market capitalization strategies²⁸ have positive SMB factor exposure and are thus less biased toward large capitalizations stocks than the market cap-weighted index. Unsurprisingly, the EW has the largest exposure to SMB, both in terms of slope coefficient and statistical significance, as well as the largest market exposure of the strategies. For portfolios using a risk matrix (ERC, MMV, MV, and MDP), market exposures are substantially less than one, with the lowest being MV, followed by the MDP. All of the strategies load positively on HML, with the MV showing the largest exposure, as measured by both factor loading and statistical significance, consistent with its bias toward low volatility (value) stocks. The MV and MDP exhibit the lowest R^2 , revealing that the market cap-weighted index and the other two factors fail to explain a relatively large part of the performance of these two portfolios. Interestingly, the MsMV shows a negative Fama-French alpha, indicating that the numerous constraints placed on its construction may in fact be destructive of value. Finally, the MDP delivers the highest alpha of the five strategies tested, indicating that the performance of the MDP is significantly higher than what its Fama-French factor exposures would predict. This is

consistent with the MDP's goal of delivering maximum diversification, and thus a balanced exposure to the effective risk factors available in the universe.

Conclusion

In this paper, we have introduced additional properties of the Diversification Ratio and of the Most Diversified Portfolio (MDP), and proposed a basic set of rules an unbiased, agnostic portfolio construction process should respect: the Portfolio Invariance Properties. We find that the MDP adheres to these rules. Furthermore, using the MSCI World Index as a reference universe to compare the performance of the MDP with other approaches, we find that the MDP stands out, both in terms of relative performance and exposure to Fama-French factors.

Classical financial theory defines the equity risk premium as the return of the un-diversifiable portfolio. In developing the MDP, our goal was to articulate a theory and a consistent construction methodology that deliverer the full benefit of the equity risk premium to investors and their trustees, and we believe that our work shows that the MDP is a strong candidate for being the un-diversifiable portfolio.

Endnote

The authors would like to thank Robert Arnott, Robert Haugen and Jason Hsu for their very helpful feedback, remarks and encouragements. We would also like to thank our colleagues at TOBAM for their instrumental contributions and great support.

Appendix A

DR Decomposition

Noting $\tilde{\mathbf{w}} = \mathbf{w} \odot \boldsymbol{\sigma}$, where \odot is the element-wise product of two vectors, the variance of the portfolio with weights \mathbf{w} can be written as:

$$\sigma^2(\mathbf{w}) = \sum_i \tilde{w}_i^2 + \sum_{i \neq j} \tilde{w}_i \tilde{w}_j \rho_{i,j} = \sum_i \tilde{w}_i^2 + \rho(\mathbf{w}) \sum_{i \neq j} \tilde{w}_i \tilde{w}_j$$

Noticing that: $\sum_{i \neq j} \tilde{w}_i \tilde{w}_j = (\sum_i \tilde{w}_i)^2 - \sum_i \tilde{w}_i^2$ leads to:

$$\sigma^2(\mathbf{w}) = (1 - \rho(\tilde{\mathbf{w}})) \sum_i \tilde{w}_i^2 + \rho(\mathbf{w}) \left(\sum_i \tilde{w}_i \right)^2$$

Then, dividing this equality by $(\sum_i \tilde{w}_i)^2$ gives the decomposition:

$$\frac{1}{DR(\mathbf{w})^2} = (1 - \rho(\mathbf{w})) CR(\mathbf{w}) + \rho(\mathbf{w})$$

DR Composition Formula

With the overall holdings $= \sum_{s=1..S} w_s \boldsymbol{\theta}_s$, the overall average volatility reads:

$$\langle \mathbf{w} | \boldsymbol{\sigma} \rangle = \sum_{i=1..N} \left[\sum_{s=1..S} w_s \boldsymbol{\theta}_s \right] \sigma_i$$

By inverting the summations, factorizing w_s and using the definition of the DR of each of the S portfolios and replacing $\langle \boldsymbol{\theta}_s | \boldsymbol{\sigma} \rangle$ by $\sigma_s \mathbf{DR}_s$ gives: $\langle \mathbf{w} | \boldsymbol{\sigma} \rangle = \langle \mathbf{w}_s | \sigma_s \odot \mathbf{DR}_s \rangle$. Then, dividing the result by the overall portfolio volatility $\sigma(\mathbf{w})$ obtains the formula.

Appendix B

MDP's Existence and Uniqueness

The MDP optimization is a Quadratic Programming problem (QP) on a convex set: thanks to the fact that the DR is invariant by scalar multiplication, this is equivalent to: $\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}' \Sigma \mathbf{w}$, constrained by $w_i \geq 0$ and $\sum_i w_i \sigma_i = 1$, with weights rescaled to sum to one afterwards. The existence follows; uniqueness as well if the covariance matrix is definite (see Berkovitz [2001], pp. 210—215).

MDP's First Order Conditions

We first apply the KKT theorem: all admissible points qualify, according to the Linear Independence Constraint Qualification (equality and inequality are independent unless all the inequality constraints are active, which would mean that $\mathbf{w} = \mathbf{0}$). The log of our *positive* objective function is: $f(\mathbf{w}) = \ln(DR(\mathbf{w})) = \ln\langle \boldsymbol{\sigma} | \mathbf{w} \rangle - \frac{1}{2} \ln\langle \Sigma \mathbf{w} | \mathbf{w} \rangle$, with: $\nabla f_{\mathbf{w}} = \frac{1}{\langle \boldsymbol{\sigma} | \mathbf{w} \rangle} \boldsymbol{\sigma} - \frac{1}{2} \frac{1}{\langle \Sigma \mathbf{w} | \mathbf{w} \rangle} 2 \Sigma \mathbf{w}$. The KKT theorem states that at optimal points \mathbf{w} , there exists a vector $\boldsymbol{\nu} \in \mathbb{R}^N$ and a scalar μ such that:

$$\begin{cases} \frac{1}{\langle \boldsymbol{\sigma} | \mathbf{w} \rangle} \boldsymbol{\sigma} - \frac{1}{\sigma(\mathbf{w})^2} \Sigma \mathbf{w} + \mu \mathbf{1} + \boldsymbol{\nu} = \mathbf{0} \\ \mathbf{Min}(\boldsymbol{\nu}, \mathbf{w}) = \mathbf{0} \\ \langle \mathbf{w} | \mathbf{1} \rangle = 1 \end{cases}$$

Multiplying the first condition on the left by the transpose of \mathbf{w} , shows that μ must be 0, and that the first condition is independent of the constraint that weights sum to one. This does not come as a surprise, as the DR is invariant by scalar multiplication. Call $\boldsymbol{\lambda} = \sigma^2(\mathbf{w}) \boldsymbol{\nu}$; an optimal point \mathbf{w} is necessarily associated to a vector $\boldsymbol{\lambda} \in \mathbb{R}^N$ satisfying:

$$\begin{cases} \Sigma \mathbf{w} = \frac{\sigma(\mathbf{w})^2}{\langle \sigma | \mathbf{w} \rangle} \sigma + \lambda \\ \mathbf{Min}(\lambda, \mathbf{w}) = \mathbf{0} \\ \langle \mathbf{w} | \mathbf{1} \rangle = 1 \end{cases}$$

The Core Property of the MDP (2)

We first show that the MDP respects the core property (2). By definition, the correlation of the

MDP to any other portfolio reads: $\rho_{\mathbf{w}, \mathbf{w}^{MDP}} = \frac{\langle \Sigma \mathbf{w}^{MDP} | \mathbf{w} \rangle}{\sigma(\mathbf{w}^{MDP}) \sigma(\mathbf{w})}$. Given that $\Sigma \mathbf{w}^{MDP} = \delta \sigma + \lambda$, with

$= \frac{\sigma(\mathbf{w}^{MDP})}{DR(\mathbf{w}^{MDP})}$, we have: $\rho_{\mathbf{w}, \mathbf{w}^{MDP}} = \frac{1}{DR(\mathbf{w}^{MDP}) \sigma(\mathbf{w})} \left\langle \sigma + \frac{1}{\delta} \lambda | \mathbf{w} \right\rangle$. Keeping in mind that λ is non

negative, for all long-only portfolios with non negative weights \mathbf{w} ,

$\rho_{\mathbf{w}, \mathbf{w}^{MDP}} \geq \frac{1}{DR(\mathbf{w}^{MDP}) \sigma(\mathbf{w})} \langle \sigma | \mathbf{w} \rangle$, which leads us to the final result. We now prove that a

portfolio \mathbf{w}^* that respects the Core Property (2) necessarily maximizes the Diversification Ratio.

As \mathbf{w}^* respects the property (2), we have for all long-only portfolios: $\rho_{\mathbf{w}, \mathbf{w}^*} \geq \frac{DR(\mathbf{w})}{DR(\mathbf{w}^*)}$. Since

correlations are not greater than unity, for all long-only portfolios: $DR(\mathbf{w}^*) \geq DR(\mathbf{w})$, which

shows that \mathbf{w}^* maximizes the Diversification Ratio across all long-only portfolios.

The Core Property of the MDP (1)

Suppose that the MDP satisfies the Core Property (2). For any asset belonging to the MDP, the

inequality given by the Core Property (2) becomes an equality as $\mathbf{Min}(\lambda, \mathbf{w}^{MDP}) = \mathbf{0}$. Since the

DR of a single asset equals one, we have for any given asset *in* the MDP: $\rho_{in, \mathbf{w}^{MDP}} = \frac{1}{DR(\mathbf{w}^{MDP})}$.

Now, using the Core Property (2) for any stock *outside* of the MDP, we finally obtain:

$\rho_{out, \mathbf{w}^{MDP}} \geq \frac{1}{DR(\mathbf{w}^{MDP})} = \rho_{in, \mathbf{w}^{MDP}}$: the MDP satisfies the Core Property (1).

Conversely, suppose that a portfolio \mathbf{w}^* satisfies the Core Property (1), for a given correlation ρ_{in,\mathbf{w}^*} . Then for any long-only portfolio \mathbf{w} :

$$\begin{aligned}\rho_{\mathbf{w},\mathbf{w}^*} &= \frac{\mathbf{w}(\Sigma\mathbf{w}^*)}{\sigma(\mathbf{w})\sigma(\mathbf{w}^*)} = \frac{1}{\sigma(\mathbf{w})\sigma(\mathbf{w}^*)} \sum_{i=1..N} w_i \sigma_i \sigma(\mathbf{w}^*) \rho_{i,\mathbf{w}^*} \\ &\geq \rho_{in,\mathbf{w}^*} \frac{\sum_{i=1..N} w_i \sigma_i}{\sigma(\mathbf{w})} = \rho_{in,\mathbf{w}^*} DR(\mathbf{w})\end{aligned}$$

Applied with $\mathbf{w} = \mathbf{w}^*$, we have an equality. This shows that $\rho_{in,\mathbf{w}^*} DR(\mathbf{w}^*) = 1$, and :

$$DR(\mathbf{w}) \leq \rho_{\mathbf{w},\mathbf{w}^*} DR(\mathbf{w}^*) \leq DR(\mathbf{w}^*)$$

This demonstrates that \mathbf{w}^* is the MDP, as it has Maximum Diversification across all long-only portfolios. Overall, this shows that the Core Property of the MDP (1) is equivalent to the Core Property of the MDP (2).

Correlation of assets to the unconstrained MDP

When removing the long-only constraint, $\lambda = \mathbf{0}$, and for all portfolios, possibly long-short:

$$\rho_{\mathbf{w},\mathbf{w}^{MDP}} = \frac{DR(\mathbf{w})}{DR(\mathbf{w}^{MDP})}. \text{ In particular, the correlations of all assets to the MDP are constant, and}$$

equal to the inverse of the MDP's Diversification Ratio.

Appendix C

The MDP is Leverage Invariant

The first order condition for the MDP can be rewritten by splitting the covariance matrix into volatilities and correlations: $\sigma \odot C(\sigma \odot \mathbf{w}) = \delta \sigma + \lambda$. As volatilities are positive, this is equivalent to: $C(\sigma \odot \mathbf{w}) = \delta \mathbf{1} + \lambda'$ with $Min(\lambda', \mathbf{w}) = \mathbf{0}$ and $\lambda' = \lambda \oslash \sigma$. Now, applying a positive leverage vector $\mathbf{L} = (L_i)_{i=1..A}$, the leveraged assets have the same correlation matrix C and volatilities $\sigma^L = (L_i \sigma_i)$. The portfolio $\mathbf{w}^L = k\mathbf{w} \oslash \mathbf{L}$ is the MDP in the leveraged universe

(With k a positive normalization constant, such that $\langle \mathbf{w} | \mathbf{1} \rangle = 1$), as it verifies the first order condition: $C(\boldsymbol{\sigma}^L \odot \mathbf{w}^L) = k\delta\mathbf{1} + k\boldsymbol{\lambda}'$ with $\text{Min}(k\boldsymbol{\lambda}', \mathbf{w}^L) = \mathbf{0}$. This means that $\boldsymbol{\sigma}^L \odot \mathbf{w}^L = k\boldsymbol{\sigma} \odot \mathbf{w}$: the leverage invariance property is proved.

The MDP is Polico Invariant

The Core Property of the MDP (2) shows that the MDP is such that any asset not selected by the MDP has a correlation greater than $\frac{1}{DR(MDP)}$. This means the MDP is unchanged by adding to the universe any asset with a correlation strictly greater than $\frac{1}{DR(MDP)}$. Furthermore, if we consider any Polico Λ , we have: $\rho(\Lambda, MDP) \geq \frac{DR(\Lambda)}{DR(MDP)} > \frac{1}{DR(MDP)}$ as the DR of a Polico is greater than 1. This means that when the Polico is added to the universe, it is *never* selected, and the MDP remains unchanged (otherwise, we would have had $\rho(\Lambda, MDP) = \frac{1}{DR(MDP)}$ according to the Core Property (2)).

Bibliography

Arnott, Robert, Jason Hsu and Philip Moore. "Fundamental Indexation." *Financial Analysts Journal*, Vol. 61, No. 2 (2005), pp. 83–99.

Arnott, Robert, Vitali Kalesnik, Paul Moghtader and Craig Scholl. "Beyond Cap Weight." *Journal of Indexes* (January/February 2010).

Berkovitz Leonard D. "Convexity and Optimization in R^n ." Wiley-Interscience, New York, 2001.

Choueifaty, Yves. "Methods and systems for providing an anti-benchmark portfolio." USPTO No. 60/816,276 filed June 22nd, 2006.

Choueifaty, Yves, and Yves Coignard. "Toward Maximum Diversification." *Journal of Portfolio Management*, Vol. 34, No. 4 (2008), pp. 40–51.

Chow, Tzee-man, Jason Hsu, Vitali Kalesnik and Bryce Little (Research Affiliates). "A survey of Alternative Equity Index Strategies." *SSRN eLibrary Dec 2010*.

Clarke, Roger, Harindra de Silva and Steven Thorley. "Minimum-Variance Portfolios in the U.S. Equity Market." *Journal of Portfolio Management*, Vol. 33, No. 1 (2006), pp. 10–24.

Demey, Paul, Sébastien Maillard, and Thierry Roncalli. "Risk-Based Indexation." LYXOR white paper, 2010.

Fabozzi, Frank, Sergio Focardi and Petter Kolm, "Financial Modeling of the Equity Market, from CAPM to Cointegration," Wiley, 2006.

Fama Eugene F. and Kenneth R. French. "Common risk factors in the returns on stocks and bonds." *Journal of Financial Economics*, 33 (1993) pp. 3–56,.

Fernholz, Robert E. and Brian Shay. "Stochastic Portfolio Theory and Stock Market Equilibrium." *Journal of Finance*, Vol. 37, No. 2 (May 1982), pp. 615–624.

Haugen, Robert A. and James Heins. "On the Evidence Supporting the Existence of Risk Premiums in the Capital Market." *The Journal of Financial and Quantitative Analysis* (December 1975).

Haugen Robert and Nardin Baker, "The Efficient Market Inefficiency of Capitalization-Weighted Stock Portfolios." *The Journal of Portfolio Management* (Spring 1991), pp. 35–40.

Hayashi Takaki and Nakahiro Yoshida. "On covariance estimation of non-synchronously observed diffusion processes." *Bernoulli*, Vol. 11, No. 2 (2005), pp 359–379.

Hoffmann, Marc, Mathieu Rosenbaum and Nakahiro Yoshida. "Estimation of the lead-lag parameter from non-synchronous data". Working Paper, CREST, 2009.

Jagannathan, R. and Tongshu Ma. "Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps," *Journal of Finance*, vol. 58, No. 4 (2003).

Ledoit, Olivier, and Michael Wolf. "A Well-Conditioned Estimator for Large-Dimensional Covariance Matrices." *Journal of Multivariate Analysis*, 88 (2004), pp. 365–411.

Maillard, Sebastien, Thierry Roncalli and Jérôme Teiletche. "The properties of equally weighted risk contribution portfolio." *Journal of Portfolio Management*, Vol. 36, No. 4 (Summer 2010), pp. 60–70.

Markowitz, Harry "Portfolio Selection." *Journal of Finance*, Vol. 7, No. 1 (1952), pp. 77–91.

MSCI World and Minimum Volatility Indices are described at <http://www.msccibarra.com/>

Rhoades, Stephen A. "The Herfindahl-Hirschman index." *Federal Reserve Bulletin* (March 1993), pp. 188–189.

Sharpe, William. "Capital asset prices with and without negative holdings." *Nobel Lecture*, 1990.

EXHIBITS

Exhibit 1

Comparison of quantitative portfolios Performances, 1999-2010.

Statistic	MKT	MKT/2	MMV	EW	ERC	MV	MDP
Return	3.1%	2.9%	4.2%	5.8%	6.3%	6.7%	7.9%
Volatility (monthly)	16.6%	16.3%	11.7%	16.7%	13.1%	10.0%	11.4%
Volatility (daily)	17.2%	17.2%	12.3%	16.4%	12.9%	10.0%	11.2%
Turnover (one Way)	14%	11%	23%	29%	50%	76%	82%
Tracking Error (daily)	0.0%	0.8%	7.6%	3.6%	6.7%	10.4%	9.2%
DR (daily)	2.3	2.2	2.8	2.5	3.0	3.4	3.7
nbStocks (avg)	1,586	793	250	793	793	159	137
Sharpe (monthly)	-0.00	-0.01	0.05	0.16	0.24	0.36	0.42
Sharpe (daily)	-0.00	-0.01	0.06	0.16	0.24	0.36	0.43

Exhibit 2

Performances using a robust correlation matrix estimation method, 1999-2010.

Statistic	ERC _{PSD}	MV _{PSD}	MDP _{PSD}
Return	6.2%	6.7%	7.8%
Volatility (monthly)	13.4%	10.2%	11.5%
Volatility (daily)	13.2%	10.2%	11.3%
Turnover (one Way)	45%	66%	76%
Tracking Error (daily)	6.2%	10.1%	9.1%
DR (daily)	2.9	3.2	3.6
nbStocks (avg)	793	200	161
Sharpe (monthly)	0.23	0.35	0.41
Sharpe (daily)	0.24	0.35	0.41

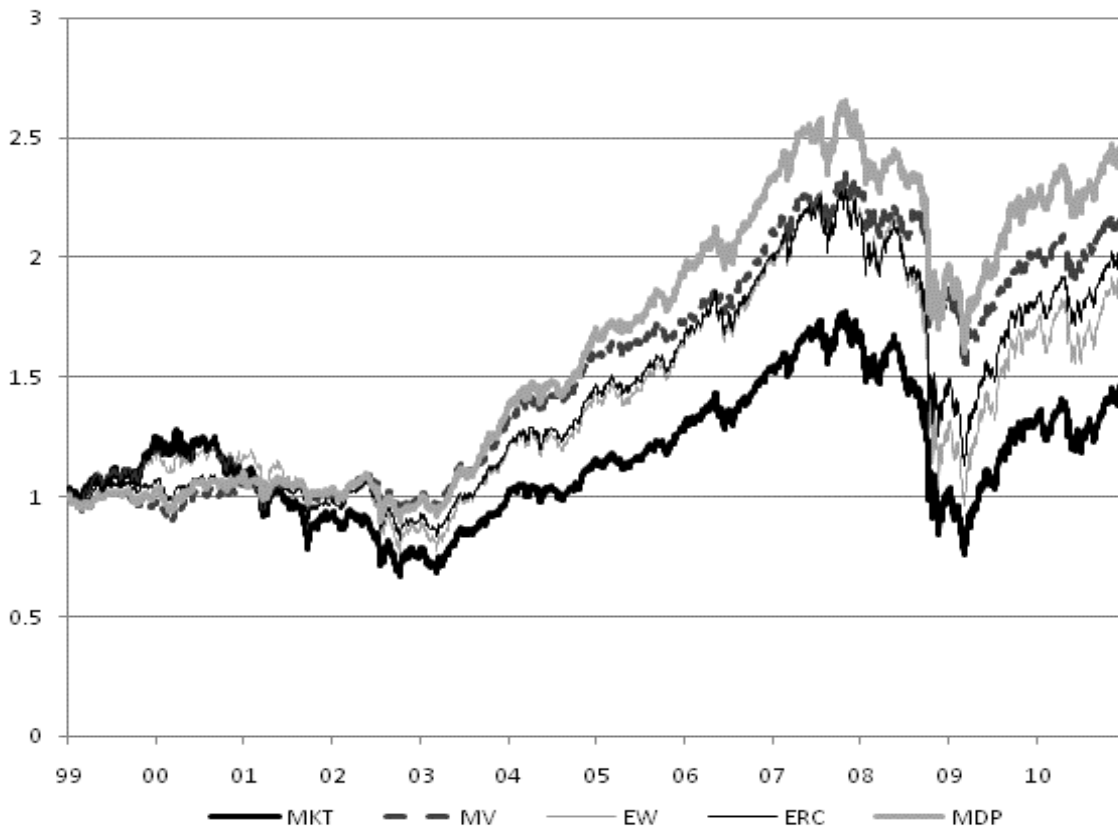
Exhibit 3

Fama-French Monthly Regression Coefficients, 1999-2010.

Portfolio	MKT	SMB	HML	Alpha	R2
EW	0.96	0.41	0.06	0.04%	99%
t-stat	132.28	21.63	4.33	0.09	
ERC	0.71	0.41	0.14	0.14%	93%
t-stat	40.60	8.99	3.99	0.20	
MMV	0.65	0.15	0.19	-0.56%	87%
t-stat	29.72	2.66	4.29	0.83	
MV	0.46	0.23	0.23	1.35%	70%
t-stat	16.32	3.15	4.02	0.83	
MDP	0.57	0.31	0.16	2.26%	80%
t-stat	21.29	4.35	2.85	1.46	

Exhibit 4

Comparison of quantitative portfolios Performances.



End Notes

¹In effect, the average volatility of the assets is equal to their common volatility, and the volatility of the portfolio equals their common volatility divided by the square root of the number of assets. We refer the reader to the “Definition of the Diversification Ratio” section of Choueifaty et al. (2008), for more examples.

²Definitions are provided accordingly.

³The *Herfindal* index attains its minimum value for an equal weighted portfolio. In our case, it suffices to rescale the portfolio weights by their associated volatilities to obtain this result.

⁴This reads identically to the original definition, except that the sub-portfolio volatilities in its numerator are multiplied by their respective DRs (they would be equal to one for portfolios of single assets).

⁵In effect, noting c the proportionality constant between the weights of the portfolio and the inverse of the volatilities, the numerator of the DR equals c times F , while its denominator equals c times the square root of F .

⁶Note that this section treats the long-only constrained MDP. We refer the reader to the “Properties” section of Choueifaty et al. (2008), for results addressing the unconstrained (long-short) case.

⁷We show in appendix D that when the covariance matrix is definite, the MDP is the only portfolio respecting this property, which uniquely *defines* the MDP. When this is not the case, all portfolios respecting this property have maximal diversification, and are fully correlated.

⁸Assuming that *single* assets Sharpe ratios are constant clearly does not mean that all *portfolios* also have constant Sharpe ratios, as their Sharpe ratios are proportional to their DR, which value varies across *portfolios*. As such, there is no internal inconsistency as noted in Chow et al. (2010), when assuming that *single* assets EERs are proportional to their volatilities and not those of *portfolios*.

⁹Assuming that single assets’ EERs are proportional to their volatilities does not mean that assets’ EERs are fixed prior to equilibrium, as they depend on the value of k which will be determined in equilibrium. In effect, equation (11) shows that in equilibrium, k is equal to the Sharpe Ratio of the MDP, multiplied by the constant correlation of all assets to the MDP.

¹⁰Using Sharpe’s notations, adding the assumption that EERs are proportional to volatility imposes the additional requirement that investors’ expectations are such that $E_i = k\sigma_i$. However, investors’ first order condition for portfolio optimality (2) in Sharpe’s lecture is unchanged, as is its aggregation over all investors (3), which form the basis for the CAPM’s pricing equation (5). Further assuming that a risk free asset is available leads to equation (8), which is the Security Market Line relationship we refer to in this paper. The requirement that $E_i = k\sigma_i$ naturally carries over to this last equation. It remains to be seen however, whether equilibrium can be reached with such additional requirement. See also note 12.

¹¹When a risk free asset is available with unlimited lending/borrowing, maximizing the mean variance utility function gives the same portfolio of risky assets, compared to directly maximizing the Sharpe ratio. The risk tolerance of the investor then determines the proportion of cash held.

¹²In this particular setting, any given market portfolio can be attained as the result of an equilibrium. It suffices for example that investors agree on zero expected correlations between assets, with expected volatilities being inversely proportional to the market portfolio’s weights. In such case, the Market Portfolio maximizes the Sharpe Ratio, as well as the DR.

¹³The risk contribution of an asset is defined here as the product of its portfolio weight and its marginal contribution to volatility, divided by the portfolio’s overall volatility.

¹⁴The fact that the Minimum Variance portfolio assigns a zero weight to asset A may come as a surprise, but there is no mistake here.

¹⁵Since the introduction of a redundant asset leads to a redundant equation in the first order conditions associated to the MV and MDP programs.

¹⁶As defined earlier, a Polico is a positive linear combination of assets (a leveraged long-only portfolio).

¹⁷Portfolios are rebalanced at the end of May and November, as is the MSCI Minimum Volatility Index.

¹⁸At each rebalancing date, we eliminated all stocks with less than six months price history, and selected the top half of the remaining stocks by market capitalization. Local currency total returns were converted to USD,

according to the MSCI methodology, and MSCI's official forex data used. Total Returns and Market Capitalization were obtained through Bloomberg.

¹⁹Having in mind MSCI's methodology, we added a 1.5% maximum weight constraint, and a maximum weights by time zone (America, Europe, Asia), to ensure allocation to the zones do not exceed those of the MSCI World (MKT) by more than 5%. We also added a turnover reduction penalty to the MDP and MV objective functions, such that the annualized tracking error to the un-penalized problem was no greater than 1.5%. We did not add those constraints to the ERC portfolio, as they were generally satisfied in its unconstrained version.

²⁰Having fewer observations than the number of assets results in a non definite covariance matrix. This was not an issue for the MDP and MV in the back tests presented here, as all portfolios contained fewer assets than observations (159 on average for the MV and 137 for the MDP), and were shown to be the unique solution of their optimization programs.

²¹Plesio means « near » in Greek, thus plesiochronous can be understood as “almost synchronous”. We chose this term to represent the fact that even if the Japanese and US stock markets for example never trade simultaneously, their time delay is mostly constant.

²²This estimator was developed in the spirit of the work done by Hayashi et al. [2005]. See also Hoffmann et al. [2009] for further references.

²³This method produces positive-definite matrices with eigenvalues greater than 0.5, and associated covariance matrices that are also positive-definite. We choose a high, constant, shrinkage intensity to ensure robustness. For references, see Ledoit et al. [2004], and Fabozzi et al. [2006], Chap. 9, p. 275.

²⁴The solution is unique, providing of course that the covariance matrix is definite. We found that with a standard PC (Intel Xeon @ 2.66 Ghz with 8Gb of Ram), those optimizations required less than a couple of seconds to converge, even when considering one thousand assets.

²⁵Unrealistic all-in trading costs of 3.4% (resp. 2.4%) would be needed for the MDP's higher turnover to be such that its outperformance relative to the market cap benchmark reduces to zero (resp. for both the EW and ERC).

²⁶This may come as a surprise to practitioners used to long-short portfolio optimizations, and to observe drastically different (and improved) results. However, the MV and MDP portfolios considered in this paper are long-only, and contain fewer assets than observations. As such, they are much less sensitive to the estimation errors of the covariance matrix. Also, the long-only constraint has already an effect similar to using a robust estimation technique. See Jagannathan et al. [2003], and Fabozzi et al. [2006], Chap. 9, p. 271.

²⁷For example, the financial sector weighted 24.4% of the MSCI EMU, representing 34.6% of its total risk, on average, over the year 2010.

²⁸The results obtained for the ERC_{PSD} , MDP_{PSD} and MV_{PSD} are not reported, as they are extremely close to their original counterparts. The only noticeable change concerns the R2s of the regressions, which increase by a few percents.