Risk Attribution with Custom-defined Risk Factors

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The standard approach to risk attribution breaks down volatility in terms of its contributions from a given set of factors, such as key rates. However, practitioners often need to monitor their exposures in terms of custom factors that are combinations of the original factors, such as the PCA shift/slope/butterfly movements of the curve. We propose a generalized approach to this risk-attribution exercise.

1. INTRODUCTION

Practitioners often rely on the factorization of the standard deviation to analyze the exposure of their portfolio to a given set of market risk factors (see Meucci (2005) for a comprehensive overview). In typical cases, the number of factors affecting the P&L is very large: practitioners need to organize their risk-attribution analysis according to a tree structure. Alternatively, they need the flexibility to define new factors as functions of the original ones and analyze risk in terms of these new factors. Here we propose a generalized solution to this type of problem.

In Section 2 we define the problem and its main statistics for the case where we are given a particular set of factors. In section 3 we extend the problem to arbitrary definitions of new risk factors as linear combinations of the original factors. Section 4 provides additional extensions of the general analysis and section 5 concludes.

2. RISK ATTRIBUTION WITH GIVEN FACTORS

Market practitioners closely monitor the risk in their portfolios. Monitoring risk means understanding the sources of risk in terms of a specified set of risk factors. This is the purpose of risk attribution. Denote by Π the P&L of a portfolio of securities. In what follows, we consider factor models for the portfolio P&L of the following form:

$$\Pi = \sum_{m=1}^{M} L_m F_m = L' F \qquad (1)$$

In this expression F_m is the m-th risk factor and L_m the corresponding portfolio loading, representing the portfolio manager's decision variable[2]. Formulation (1) is quite general. In particular, it includes as a special case standard APT-like linear pricing models. Indeed, it suffices to consider the last factor as a diversifiable idiosyncratic contribution that is independent of the remaining $M-1$ factors. Furthermore, it covers the non-linear theta-delta-gamma-vega P&L model routinely used on trading desks: in this case, the "gamma" factors are deterministic quadratic functions of the "delta" factors. Finally, it also encompasses the formulation $\Pi = \sum_{n=1}^{N} v_n R_n = V' R$, where R represents the returns on a set of securities or asset classes and V the market value of each of them. In this case, we can think of each security as being a specific risk factor.

[1] *We would like to thank Anthony Lazanas, Gary Wang, Bob Durie and Diego Pontoriero for their comments.*

[2] *Indeed,* L_m *is the weighted sum of the single-security loadings to the i-th risk factor* $L_m = \sum_{n=1}^{N} v_n L_{nm}$, *where* N *is the number of securities and* v_n *represents amount of security n in the portfolio. Note that without loss of generality, we can set the loadings to be net of a particular benchmark.*

The most widely used measure of risk in the industry is the standard deviation, also known as tracking error volatility (TEV) in the case of benchmark-driven allocations. In our case, it is a simple function of the market factors F and of the allocation L :

$$TEV = STD(\Pi) = \sqrt{L'\Omega L}$$

where Ω denotes the covariance matrix of the risk factors. The standard deviation above has homogeneity of degree one in the loadings. The following well known results are a direct consequence of that. If we define the marginal contribution to TEV from risk factor F_i as $MC_i = \partial TEV / \partial L_i$, then from Euler's homogenous function theorem:

$$TEV = \sum_{m=1}^{M} MC_m L_m$$

The total variance is not the weighted sum of the variance of the individual factors (except when the risk factors are uncorrelated – see example below). However, total volatility can now be naturally expressed as the sum of the contributions from each factor:

$$TEV = \sum_{m=1}^{M} C_m \qquad (2)$$

In this expression, the contribution from each factor is the product of the per-unit marginal contribution and the exposure of the portfolio to the factor, as represented by the respective loading:

$$C_m = MC_m L_m \qquad (3)$$

Furthermore, it is easy to check that the marginal contribution from factor *i* reads explicitly:

$$MC_m = \frac{\Omega_{m.} L}{TEV} \qquad (4)$$

where $\Omega_{m.}$ represents the *m*-th line of matrix Ω [3].

Example: Relating to POINT risk model report

In POINT[4], the risk of a particular portfolio is divided into three components: systematic (S), idiosyncratic (I) and default (D) risk. We can treat each of them as a factor F in the framework described above. By construction, these three components are uncorrelated to each other. Therefore the factor covariance matrix is diagonal:

$$\Omega = \begin{bmatrix} \sigma_S^2 & 0 & 0 \\ 0 & \sigma_I^2 & 0 \\ 0 & 0 & \sigma_D^2 \end{bmatrix}$$

Given the above definition, an intuitive normalization is to set the portfolio loadings to all three factors equal to 1, such that:

$$TEV = \sigma = \sqrt{L'\Omega L} = \sqrt{\sigma_S^2 + \sigma_I^2 + \sigma_D^2}$$

In this case, the marginal contributions and the contributions are equal and read:

[3] *More generally, we will represent $A_{i.}$ as the i-th line of matrix A.*

[4] *See Joneja, Dynkin et al (2005) for a detailed view of risk modeling in POINT.*

$$MC_i = C_i = \frac{\sigma_i^2}{TEV} \quad for \quad i = \{S, I, D\} \tag{5}$$

These concepts are illustrated in the following example, constructed in POINT. Consider the portfolio of the US issues in the USD investment grade credit index that have amounts outstanding larger than $1MM (as of March 27 2007). For simplicity, assume that we want to study its volatility (e.g. its benchmark is set to be cash) and that we hedge our portfolio from all sources of risk other than spreads. The following numbers are collected from the resulting POINT risk report:

Figure 1. Details of the TEV (bp/month)

Total TEV (bp/month)	49.6
Systematic TEV	48.4
Idiosyncratic TEV	10.4
Default TEV	3.0

Source: Lehman Brothers, POINT.

The numbers represent respectively $\sigma, \sigma_S, \sigma_I$, and σ_D. Using (5) we can now construct the contributions from these factors:

Figure 2. Contribution to TEV from Major Components (bp/month)

Contribution to TEV from:	C_i
Systematic Risk	47.24
Idiosyncratic Risk	2.18
Default Risk	0.19
Total	49.60

Source: Lehman Brothers, POINT.

Note that the contributions sum to the total volatility of the portfolio as in (2).

3. RISK ATTRIBUTION WITH NEW FACTORS

Often the portfolio manager needs to measure his exposure to a set of $K \le M$ new risk factors that are linear combinations of the original factors F. In formulas, we can represent the new factors in terms of a "pick" matrix P with K rows and M columns:

$$\widetilde{F} \equiv PF \tag{6}$$

Each row of P represents a linear combination of the original factors, and therefore it forms a specific "new" factor. How do we attribute risk to the new factors $\widetilde{F}$? In what follows we develop a general framework to think about this issue.

The new factors $\widetilde{F}$ drive the randomness in the P&L through some suitably defined factor loading vector $\widetilde{L}$. In general, if $K < M$, the new factors cannot account for all the risk in the portfolio. Therefore, there must exist some residual risk:

$$\Pi = \widetilde{L}'\widetilde{F} + \varepsilon \tag{7}$$

After defining P, we need to choose a suitable set of loadings $\widetilde{L}$. Is there any natural choice for $\widetilde{L}$? We propose one that satisfies two conditions. The first one is that the residual is minimized. The second is that the residual is uncorrelated with the new factors $\widetilde{F}$. Note that these two conditions are satisfied once we use regression analysis. In fact, if we regress the portfolio P&L on the new factors $\widetilde{F}$, the OLS estimate of $\widetilde{L}$ is:

$$\widetilde{L} = \left[\operatorname{cov}(\widetilde{F})\right]^{-1} \operatorname{cov}(\widetilde{F}, \Pi) \tag{8}$$

Or in terms of the primitives:

$$\widetilde{L} = (P\Omega P')^{-1} P\Omega L \tag{9}$$

By definition these loadings are such that the residual risk ε is not correlated with the new factors $\widetilde{F}$. Moreover, the residual is minimal, in a least-square sense. We therefore achieved the proposed goals. With this definition of $\widetilde{L}$, we can proceed in the same lines as (2) – (4). Specifically, we can define the contribution from each new factor $\widetilde{F}_k$ as:

$$\widetilde{C}_k = M\widetilde{C}_k \widetilde{L}_k \tag{10}$$

where it is easy to check that the per-unit contributions read:

$$M\widetilde{C}_k = \frac{\widetilde{\Omega}_{k.} \widetilde{L}}{TEV} \tag{11}$$

with $\widetilde{\Omega} = P\Omega P'$. In terms of the primitives the expression comes:

$$M\widetilde{C}_k = P_{k.} MC \tag{12}$$

Where MC is the vector of original marginal contributions with elements described in (4). Finally, note that contrary to (2), it may be the case that:

$$TEV \neq \sum_{k=1}^{K} \widetilde{C}_k$$

because of the residual ε in (7).

In what follows, we focus on several cases of particular interest. As we show, all of them can be analyzed using the framework outlined in (6) – (12). The cases allow us to convey some intuition on how to construct the P matrix as well as how to read the corresponding results and perform risk attribution.

Example (continued): Relating to the risk models in POINT

In Figure 1 we summarize the information regarding the systematic risk factors into a single number. However, this number comes, in our risk models, from a much richer framework[5]. If the portfolio described above in the example goes through POINT, we obtain a factor report that looks like Figure 3. In particular, note that the systematic spread volatility is explained by 34 risk factors. The contributions C_i from all 34 factors sum up to the contribution of the systematic risk, 47.24 as per Figure 2. In the following sections we illustrate various ways of reorganizing this information to better suit the risk-attribution process.

[5] *See Rosten and Silva (2007) for a description of the full credit risk model available through POINT.*

Figure 3. Details on the systematic risk factors

M	Factor name (F_i)	L_i	σ_{F_i}	MC_i	C_i
1	USD Ultra High Grade Industrials	2.702	6.25	3.648	9.86
2	USD Ultra High Grade Utilities	0.229	7.96	3.576	0.82
3	USD Ultra High Grade Financials	3.012	6.74	4.354	13.11
4	USD Ultra High Grade Non Corporate	0.223	4.85	2.303	0.51
5	USD IND Chemicals	0.025	6.96	3.318	0.08
6	USD IND Paper	0.048	8.68	4.862	0.23
7	USD IND Capital Goods	0.04	10.45	5.311	0.21
8	USD IND Div. Manufacturing	0.01	9.85	4.954	0.05
9	USD IND Auto	0.064	12.09	6.646	0.43
10	USD IND Consumer Cyclical	0.26	13.95	6.363	1.65
11	USD IND Retail	0.142	8.21	4.355	0.62
12	USD IND Cons. Non-cyclical	0.082	7.62	3.723	0.31
13	USD IND Pharmaceuticals	0.087	7.97	3.146	0.27
14	USD IND Energy	0.161	8.04	4.172	0.67
15	USD IND Technology	0.031	11.27	6	0.19
16	USD IND Media Cable	0.189	13.01	6.58	1.24
17	USD IND Media Non-cable	0.105	8.87	5.023	0.53
18	USD IND Wirelines	0.364	10.95	5.267	1.92
19	USD IND Wireless	0.331	11.96	6.095	2.02
20	USD UTI Electric	0.136	11.97	3.525	0.48
21	USD FIN Banking	0.491	7.68	3.945	1.94
22	USD FIN Brokerage	0.515	12.04	6.433	3.31
23	USD FIN Finance Companies	0.201	10.09	5.3	1.06
24	USD FIN Life and Health Insurance	0.058	10.64	5.282	0.31
25	USD FIN P&C Insurance	0.001	7.89	4.489	0.01
26	USD Non Corporate	0.015	14.09	5.53	0.08
27	USD Credit IG subordinated	0.642	3.62	0.346	0.22
28	USD IND Short Maturity	-0.639	1.65	0.178	-0.11
29	USD UTI Short Maturity	-0.087	2.23	0.23	-0.02
30	USD FIN Short Maturity	-1.016	2.95	-0.114	0.12
31	USD IND Long Maturity	12.017	1	0.253	3.05
32	USD UTI Long Maturity	1.249	1.32	0.223	0.28
33	USD FIN Long Maturity	9.844	1.12	0.154	1.52
34	USD NONCRP Long Maturity	1.697	1.04	0.165	0.28

Source: Lehman Brothers, POINT

Analysis with M new factors ($K = M$)

We start by considering the case where the investor defines a full set of $K = M$ non trivial new factors. In this case, the vector of new loadings $\widetilde{L}$ is M-dimensional. Moreover, if the new factors are non-trivial the "pick" matrix P is invertible. Using this fact and definition (9), we can write $\widetilde{L}$ as:

$$\begin{aligned}\tilde{L} &= (P\Omega P')^{-1} P\Omega L = \\ &= (P\Omega P')^{-1} P\Omega P' P'^{-1} L = \\ &P'^{-1} L\end{aligned} \tag{13}$$

Therefore the new factor loadings are unequivocally defined as $\tilde{L} = P'^{-1} L$. We can rewrite (1) as:

$$\Pi = L'F = L'P^{-1}PF = \tilde{L}'\tilde{F}$$

Substituting this expression into (7), we obtain:

$$\varepsilon = 0$$

This is not surprising: with a full set of $K = M$ new factors, the P&L is fully described by the new factors. We can represent the contributions and marginal contributions as in (10) and (11). In this case the residual is driven to zero, and so, as in (2):

$$TEV = \sum_{k=1}^{K} \tilde{C}_k$$

Therefore the new factors are able to fully describe the volatility of the portfolio.

Analysis with K new factors ($K < M$)

Now, suppose instead one is interested in computing the contributions from $K < M$ new factors. This is generally the case of interest. We can proceed using two distinct approaches. In the first, the new factors still fully describe the P&L of the report as we are able to eliminate the residual ε. As we show below, this is possible only with strong restrictions on the matrix P. In particular, the matrix is portfolio dependent.

In the second approach, we regain the control over P (namely, P does not depend on the portfolio), but at the expense of generating a residual. Both approaches are useful and deliver different intuitions regarding a particular portfolio or set of risk factors.

Full risk attribution

Suppose one is interested in computing the contributions from $K < M$ exhaustive sets of the original risk factors. For simplicity, let us assume for now that these K sets are also mutually exclusive. We can get this by constructing K buckets $\{M_1, M_2, ..., M_k, ..., M_K\}$ that represent a non-overlapping partition of the initial set of M risk factors.

We can think of each set k as representing a new risk factor. It seems natural to define the contribution to risk C_k from the generic *k*-th bucket as the sum of the individual contributions from each factor in the bucket:

$$C_k = \sum_{i \in M_k} C_i \tag{14}$$

That is re-write (2) as:

$$TEV = \sum_{k=1}^{K} \left(\sum_{i \in M_k} C_i \right) = \sum_{k=1}^{K} C_k \tag{15}$$

This result follows from a particular choice of the $K \times M$ matrix P [6]:

$$P_{ki} = \begin{cases} L_i & if \quad i \in M_k \\ 0 & otherwise \end{cases} \tag{16}$$

What is the corresponding set of loadings $\widetilde{L}$? Note that by definition of P , the initial vector of loadings can be represented by:

$$L' = 1_K ' P \tag{17}$$

Where 1_K as a K -dimensional vector of ones. Using (17) and (6) we can rewrite (1) as:

$$\Pi = L' F = 1_K ' PF = 1_K ' \widetilde{F}$$

So the natural choice for $\widetilde{L}$ is:

$$\widetilde{L} = 1_K$$

So that:

$$\Pi = \widetilde{L}' \widetilde{F} \tag{18}$$

The P&L, originally driven by M sources of risk, is now fully described by $K < M$ new factors (see (18)). Therefore we successfully drove the residual ε to zero. In particular, because $\widetilde{L}_k = 1, \forall_k$, the following is true:

$$\Pi = \sum_{k=1}^{K} \widetilde{L}_k \widetilde{F}_k = \sum_{k=1}^{K} \widetilde{F}_k$$

Moreover:

$$M\widetilde{C}_k = \widetilde{C}_k$$

whereas in (14) (using (12) and (16)):

$$\widetilde{C}_k = M\widetilde{C}_k = P_{k.} MC = \sum_{i \in M_k} L_i MC_i = \sum_{i \in M_k} L_i MC_i = \sum_{i \in M_k} C_i$$

Generalization

How can we set up this problem using the general framework developed in section 0? We do that by extending P with a set $M - K$ residual factors. Define that extended matrix as:

$$Q = \begin{bmatrix} P \\ S \end{bmatrix} \tag{19}$$

Where S is a $(M - K) \times M$ matrix of residual factors. We need only very mild conditions on S to be able to reconcile the general methodology laid down in section 0 with the analysis presented before in this section. Specifically, we need to make sure that the new matrix Q is invertible. If this is the case, the vector of new loadings is defined as (see (13)):

$$\widetilde{L}_Q = Q'^{-1} L \tag{20}$$

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[6] *We can generalize this approach by redefining* $P_{ki} = \alpha_{ki} L_i$ *where* α_{ki} *are any numbers such that* $\sum_{k=1}^{K} \alpha_{ki} = 1$ *for all i. This condition ensures that the new factors fully capture the risk in the portfolio, as in (15).*

As expected this vector has a very specific structure, given the definition of P in (16). In particular (see Appendix, proof #1):

$$\tilde{L}_Q' = \begin{bmatrix} 1_K' & 0_{M-K}' \end{bmatrix}, \tag{21}$$

where 0_{M-K} is a $M-K$ vector of zeros. Therefore, we are able to retain all results developed previously in this section. In particular,

$$TEV = \sum_{k=1}^{K} \tilde{C}_k = \sum_{m=1}^{M} \tilde{C}_m$$

As by (10) and (21) all loadings and therefore all contributions are zero for the $M-K$ residual factors:

$$\tilde{C}_m = 0 \quad \forall_{m>K}$$

Example (cont'd): Full risk attribution

Suppose we want to analyse the risk of our portfolio – as described in Figure 3 – using four new risk factors: industrials, utilities, financials and non-corporates. Specifically, suppose we define the new factors as:

Industrials	Factors 1, 5-19, 27, 28 and 31 in Figure 3
Utilities	Factors 2, 20, 29 and 32 in Figure 3
Financials	Factors 3, 21-25, 30 and 33 in Figure 3
Non-corporates	Factors 4, 26 and 34 in Figure 3

We can do that with the methodology described in this section. We start by using (16) and the loading information L_i in Figure 3 to define a 4×34 matrix P:

$$P = \begin{bmatrix} 2.702 & 0 & 0 & 0 & 0.025 & 0.048 & \ldots & 0 \\ 0 & 0.229 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 3.012 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0.223 & 0 & 0 & \ldots & 1.697 \end{bmatrix}$$

With this definition and setting $\tilde{L} = [1 \quad 1 \quad 1 \quad 1]'$, we can get $\tilde{\Omega}$ and $\tilde{C}_k$ as above. In our case:

$$C = \begin{bmatrix} 23.42 \\ 1.56 \\ 21.37 \\ 0.88 \end{bmatrix} \begin{matrix} \rightarrow & \text{Industrials} \\ \rightarrow & \text{Utilities} \\ \rightarrow & \text{Financials} \\ \rightarrow & \text{Non-corporates} \end{matrix}$$

So, we are able to attribute risk to the 4 new risk factors. Moreover, the sum of these contributions adds up to the contribution to the overall risk by the systematic component (see (15)). It is now much clearer than in Figure 3 that the bulk of risk comes from the exposure to industrials and financials. We could use the general approach to conduct this same analysis (see Appendix, Example #1).

Partial risk attribution

In the case above, the structure imposed on the P matrix allowed us fully to describe the P&L with the new set of factors and loadings (see (18)). The new factors were constructed as a specific weighted average of the initial factors using the portfolio loadings as weights. Hence the new factors are portfolio-specific. Often we need more flexibility in defining the new factors, that is, in forming the "pick" matrix P. In this case, the new factors do not generally explain all the risk of the portfolio. Therefore, there is some residual risk (see (7)):

$$\Pi = \widetilde{L}'\widetilde{F} + \varepsilon$$

The flexibility we gain in determining a matrix P that is suitable to our purposes comes with a cost, the residual risk ε. As discussed, we do have a natural choice for $\widetilde{L}$, provided by regression analysis (see (9)):

$$\widetilde{L} = (P\Omega P')^{-1} P\Omega L$$

By definition these loadings are such that the residual ε is not correlated with the new factors $\widetilde{F}$. Moreover, the r-square:

$$R^2 = 1 - \mathrm{var}(\varepsilon) / \mathrm{var}(\Pi)$$

attains a maximum with (9). Therefore, the regression coefficients split the risk into two separate components, namely the risk due to the new factors $\widetilde{F}$, which accounts for the r-square, and the risk due to the residual ε:

$$\mathrm{var}(\Pi) \quad = \quad [\underbrace{\mathrm{var}(\widetilde{L}'\widetilde{F})}_{R^2\,\mathrm{var}(\Pi)} \quad + \quad \underbrace{\mathrm{var}(\varepsilon)}_{(1-R^2)\,\mathrm{var}(\Pi)}\ ] \tag{22}$$

Now the sum of the contributions from the new factors $\widetilde{F}$ does not add up to the total volatility of the P&L (see (7)). Instead:

$$\sum_{k=1}^{K} \widetilde{C}_k = \sum_{k=1}^{K} M\widetilde{C}_k \widetilde{L}_k = \sum_{k=1}^{K} \frac{\widetilde{\Omega}_{k.}\widetilde{L}}{TEV} \widetilde{L}_k = \frac{\mathrm{var}(\widetilde{L}'\widetilde{F})}{TEV} = R^2 TEV$$

So, the (normalized) sum of the contributions from the new factors $\widetilde{F}$ assumes a very intuitive interpretation. Indeed their contributions to risk sum up to the r-square of the fit:

$$\frac{\sum_{k=1}^{K} \widetilde{C}_k}{TEV} = R^2 \tag{23}$$

Therefore, the risk-attribution formula (10) provides the contribution of each new factor to total risk and simultaneously provides the contribution to the r-square.

Generalization

We now focus on how to set up this problem using again the general framework developed in section 0. The solution comes again by extending P with a set $M - K$ residual factors. Once again we define the extended matrix as:

$$Q = \begin{bmatrix} P \\ S \end{bmatrix}$$

Where S is a $(M-K)\times M$ matrix of residual factors. Because we want to keep the freedom to define P, the new set of loadings $\widetilde{L}_Q = Q'^{-1} L$ will not obey (21):

$$\widetilde{L}_Q' \neq \begin{bmatrix} 1'_K & 0'_{M-K} \end{bmatrix}.$$

Therefore, the contributions from the residual factors are no longer zero. Instead:

$$TEV = \sum_{k=1}^{K} \widetilde{C}_k + \sum_{m=K+1}^{M} \widetilde{C}_m = \widetilde{C}^K + \widetilde{C}^{M-K}$$

To make things worse, different choices of S generally deliver different values for $\widetilde{C}^{M-K}$ and $\widetilde{C}^K$. This is a very undesirable result, as the residual factors have no specific interpretation. But let us remind ourselves of the principles we established on the natural choice for $\widetilde{L}$. In particular, we want (i) to maximize $\widetilde{C}^K$ (e.g. allow the maximum explanatory power to the factors we are interested on) and (ii) make $\widetilde{C}^K$ independent of the S chosen (e.g. the contributions from the factors we are interested in does not depend on the definition of the residual factors).

As we derived above, the first goal is accomplished by defining the loadings over the K factors of interest as in (9). The second is achieved by a judicious choice of S – a choice that imposes only a minor restriction in the selection of residual factors[7]. It can be expressed as an orthogonality constraint:

$$S = null(P\Omega)' \tag{24}$$

Where $null(A)$ represents the null space of matrix A. To see how these conditions helps us, let's start by defining:

$$\widetilde{L}_Q' = \begin{bmatrix} \widetilde{L}_K' & \widetilde{L}'_{M-K} \end{bmatrix}$$

$$\widetilde{F}_Q = QF = \begin{bmatrix} PF \\ SF \end{bmatrix} = \begin{bmatrix} \widetilde{F} \\ \widetilde{F}_{M-K} \end{bmatrix}$$

Note that the first set of loadings and factors ($\widetilde{L}_K$ and $\widetilde{F}$) are the ones we do care about. The others are residual factors and, as shown below, have no particular interpretation. With these definitions we can represent the variance of the P&L of the portfolio as:

$$\begin{aligned} \mathrm{var}(\Pi) &= \mathrm{var}(\widetilde{L}_Q' \widetilde{F}_Q) = \\ &= \mathrm{var}(\widetilde{L}'_K \widetilde{F}) + \mathrm{var}(\widetilde{L}'_{M-K} \widetilde{F}_{M-K}) + \mathrm{cov}(\widetilde{L}'_K \widetilde{F}, \widetilde{L}'_{M-K} \widetilde{F}_{M-K}) \end{aligned}$$

We show in the appendix (proof #2) that under (24), $\mathrm{cov}(\widetilde{L}'_K \widetilde{F}, \widetilde{L}'_{M-K} \widetilde{F}_{M-K}) = 0$, therefore:

$$\mathrm{var}(\Pi) = \mathrm{var}(\widetilde{L}'_K \widetilde{F}) + \mathrm{var}(\widetilde{L}'_{M-K} \widetilde{F}_{M-K})$$

Moreover, we show in the appendix (proof #3) that $\widetilde{L}_K = \widetilde{L}$, so:

$$\mathrm{var}(\Pi) = \mathrm{var}(\widetilde{L}' \widetilde{F}) + \mathrm{var}(\widetilde{L}'_{M-K} \widetilde{F}_{M-K})$$

[7] *This restriction is not important as we do not give any particular interpretation to the residual factors. Their only role is to help us generalize the results from the several cases we discuss in this paper into a single framework.*

Using (22), we get that:

$$\mathrm{var}(\varepsilon) = \mathrm{var}(\widetilde{L}'_{M-K}\widetilde{F}_{M-K})$$

So, the residual factors are the residual and are orthogonal to the factors of interest $\widetilde{F}$. With (24) in place, all the analysis in (22)-(23) follows, namely:

$$\widetilde{C}^K = \sum_{k=1}^{K}\widetilde{C}_k = \widetilde{L}\widetilde{\Omega}\widetilde{L}/TEV = R^2 TEV \tag{25}$$

$$\begin{aligned}\widetilde{C}^{M-K} &= \sum_{m=K+1}^{M}\widetilde{C}_m = \widetilde{L}_{M-K} S\Omega S'\widetilde{L}_{M-K}/TEV = \\ &= \mathrm{var}(\varepsilon)/TEV = (1-R^2)TEV\end{aligned} \tag{26}$$

Example (cont'd): Partial risk attribution

The partition imposed in (16) is rather restrictive. In particular, note that by using the portfolio loadings, the risk attribution is portfolio-specific. We may want instead to define generic factors that are not specific to a particular portfolio. For instance, suppose we want to aggregate the risk factors in Figure 3 by factor-type: Ultra-High-Grade (1 to 4), DTS Industry factors (5 to 26) and maturity (28 to 34). In particular, note that we are ignoring factor 27, e.g., we want it to be added to the residual. To that end, we can define generically P as:

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \tag{27}$$

Because we are not using any portfolio-specific information to define these new factors, we can use this set of new factors for any portfolio to be analyzed. However, by defining the matrix P this way, we are creating a residual into the risk attribution analysis. In our example, the procedure developed in this section with P defined as above delivers:

$$C = \begin{bmatrix} 16.74 \\ 20.63 \\ 1.30 \\ 8.58 \end{bmatrix} \begin{matrix} \rightarrow & \text{Ultra High Grade} \\ \rightarrow & \text{DTS factors} \\ \rightarrow & \text{Maturity} \\ \rightarrow & \text{Residual} \end{matrix} \tag{28}$$

And so the R^2 in our case is around 80%. Why is this number not close to 100%? Note that in (27) we are basically defining a new ultra high grade factor that is an equal sum of all the four ultra high grade factors. Our portfolio has almost no exposure to two of them (utilities and non-corporates, see Figure 3), so the equal sum will not be a faithful description of this particular portfolio. The same is true for the maturity factor. As one can see in Figure 3, the loadings to the short and long factors are very different. We may be imposing too much structure by joining them together under one single new factor "maturity" (more on this later on). This particular example shows us the trade-off we referred to above. To be able to define a generic partition that can be applied to different portfolios, we may lose some ability to attribute risk in a particular portfolio.

Note that we can use the extended matrix Q to achieve the same goal (see appendix, example #2).

4. EXTENSIONS

Defining the residual in terms of the original factors F

One question regards the interpretation of the risk due to the residual ε. If the choice of the user-defined new factors $\widetilde{F}$ is not the most suitable to describe the exposures of the portfolio, the residual can be large. In this situation, the portfolio manager needs to determine the nature of the residual, typically in terms of the original factors. Again, we can fully express the residual as a combination of the original factors:

$$\begin{aligned}\varepsilon &= \Pi - \widetilde{L}'\widetilde{F} = L'F - \widetilde{L}'PF = \\ &= (L' - \widetilde{L}'P)F = L^{\varepsilon}F = \\ &= \sum_{m=1}^{M} L_m^{\varepsilon} F_m\end{aligned}$$

where:

$$L^{\varepsilon} \equiv L - P'\widetilde{L}$$

The risk of the portfolio splits into two distinct components as in (22): a new-factor component and a residual component. The risk-attribution process further splits the new-factor component into separate terms due to each of the new factors:

$$\mathrm{var}(\Pi) \;=\; [\underbrace{\sum_{k=1}^{K} \widetilde{C}_k}_{R^2\,\mathrm{var}(\Pi)} \;+\; \underbrace{\sum_{m=1}^{M} C_m^{\varepsilon}}_{(1-R^2)\,\mathrm{var}(\Pi)}]$$

To compute in practice the contributions

$$\widetilde{C}_k = M\widetilde{C}_k \widetilde{L}_k$$

$$C_m^{\varepsilon} = MC_m^{\varepsilon} L_m^{\varepsilon}$$

With $M\widetilde{C}_k$ defined as in (11) and $MC_m^{\varepsilon} = \Omega_{m.} L^{\varepsilon} / TEV$

Example (cont'd): Checking the residual

If we calculate C^{ε} on the example we have been following, we can understand better where the residual 8.58 (see (28)) is coming from. In other words, we can understand in what dimensions the new factors mis-represent the old ones. Taking P as in (27), the following are the major elements in C^{ε}:

$$\begin{bmatrix} C_{30}^{\varepsilon} \\ C_{31}^{\varepsilon} \\ C_{33}^{\varepsilon} \end{bmatrix} = \begin{bmatrix} 1.25 \\ 3.13 \\ 2.33 \end{bmatrix}$$

Two conclusions follow: first, note that these three factors represent the bulk of the residual (6.71 out of the 8.58). If we could correct this situation, the residual would be drastically reduced. Second, all these three original factors are summarized by our new factor "maturity". This suggests that we only need to change the definition of this factor. A closer look at Figure 3 highlights the problem. The nature of the loadings on the short and long factors (#28 to #34) is quite different. By pooling them into a single new factor, we are blurring their individual contribution. With this diagnosis, we can proceed in various directions. For instance, we could separate the factors into a short and a long maturity. In this case, we would have 4 new factors, instead of the current 3. Suppose we want to keep the number of new factors to 3. Then, given the evidence in Figure 3, it seems more promising to drop the short factors from the new factor "maturity". This is because the extreme of the loadings is coming from the long maturity factors. If we do this, by redefining the third row of P in (27) accordingly (zero, except for factors 31 to 34), the contributions from the three factors plus residual come as:

$$C = \begin{bmatrix} 19.77 \\ 19.85 \\ 4.28 \\ 3.34 \end{bmatrix} \begin{matrix} \rightarrow & \text{Ultra High Grade} \\ \rightarrow & \text{DTS factors} \\ \rightarrow & \text{LONG Maturity} \\ \rightarrow & \text{Residual} \end{matrix} \tag{29}$$

Comparing with (28), we see that this redefinition contributed substantially to reduce the residual, now at 3.34. The R^2 of the current set of factors is now at 91%, a substantial increase from the previous 80%.

Adding idiosyncratic risk to the analysis

As referred above, the formulation in (1) is quite general. One specific practical application is to define the last "factor" as idiosyncratic:

$$\Pi = \sum_{i=1}^{M} L_i F_i = \sum_{i=1}^{M-1} L_i F_i + \nu$$

Here $\nu \equiv L_M F_M$ is independent of any of the other factors $\{F_1, F_2, ..., F_{M-1}\}$. In this case, all the previous analysis goes through, but we need to re-interpret some of the results.

Define I_M as the $M \times M$ identity matrix. Moreover, let the "pick" matrix P be a truncation of I_M. In particular let P be the first $M-1$ rows of I_M, $P^T = I_{1,2,...,M-1;1,2,...,M}$. Finally, define the (truncated (T)) loadings' vector as $L^T = L_{1,2,...,M-1}$ and $F^T = P^T F$, then:

$$\text{var}(\Pi) = \text{var}(L^T{}' F^T) + \text{var}(\nu)$$

Note that as previously:

$$\frac{\sum_{k=1}^{M-1} C_k^T}{TEV} = R^2$$

Where R^2 has the standard goodness-of-fit interpretation. If we define a new set of factors and loadings as in (6) and (9) with respect to the truncated series of factors F^T and loadings L^T, then:

$$\begin{aligned} \text{var}(\Pi) &= \sigma_{SYS}^2 + \sigma_{IDIO}^2 = \text{var}(\tilde{L}^T{}' \tilde{F}^T) + \text{var}(\varepsilon) + \text{var}(\nu) = \\ &= \text{var}(\tilde{L}^T{}' \tilde{F}^T) + \text{var}(\omega) \end{aligned}$$

Where $\omega = \varepsilon + \nu$. Then in the spirit of (22), if we define $r^2 = \text{var}(\tilde{L}^T{}' \tilde{F}^T) / \text{var}(\Pi)$:

$$\text{var}(\Pi) \quad = \quad [\underbrace{\text{var}(\tilde{L}^T{}' \tilde{F}^T)}_{r^2 \, \text{var}(\Pi)} \quad + \quad \underbrace{\text{var}(\omega)}_{(1-r^2)\,\text{var}(\Pi)} \;]$$

and the normalized sum of factor contributions is now:

$$\frac{\sum_{k=1}^{K} \widetilde{C}_k^T}{TEV} = r^2$$

So the r-square should now be interpretable against $\mathrm{var}(\Pi)$, not $\mathrm{var}(L^T{}'F^T)$ as before.

Example (cont'd): Adding idiosyncratic risk to the analysis

Recall that up to now we focus on the contribution to systematic risk only. To extend the approach to the full volatility of our portfolio, we need to add the volatility that is coming from idiosyncratic and default risk (see Figure 1 and Figure 2). We can do this using the definitions of R^2 and r^2. In particular, note that:

$$R^2 = \frac{\sum_{k=1}^{3} \widetilde{C}_k}{std(L^T F^T)} = \frac{43.9}{48.4} = 0.91$$

$$r^2 = \frac{\sum_{k=1}^{3} \widetilde{C}_k}{std(\Pi)} = \frac{43.9}{49.6} = 0.89$$

Where we use the numbers from (29) (the sum of the contributions from the 3 new factors) and Figure 1.

5. CONCLUSIONS

We propose a general framework to attribute risk to arbitrary user-defined combinations of factors in a given market. In this approach the portfolio P&L is represented as a function of the user-defined factors plus a residual. The loadings on the factors are defined as the standard regression coefficients. This way, the residual is minimal and uncorrelated with the user-defined factors. As intuition suggests, the sum of the contributions to total risk from the user-defined factors represents the r-square of the regression of the factors on the portfolio P&L. We also show that the residual can be broken down into contributions from the original factors allowing us to characterize the part of the portfolio risk (as described by the original factors) that the new user-defined factors fail to capture.

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APPENDIX

PROOF #1

Starting with (20):

$$\tilde{L} = Q'^{-1} L$$

We can rewrite it as:

$$Q'\tilde{L} = L$$

Using the definition of Q given by (19):

$$[P' \quad S']\tilde{L} = L$$

We also know from (17) that:

$$L = P'1_K$$

If we define $\tilde{L}' = [\tilde{L}'_K \quad \tilde{L}'_{M-K}]$, then:

$$P'\tilde{L}'_K + S'\tilde{L}'_{M-K} = P'1_K$$

or

$$P'(\tilde{L}'_K - 1_K) + S'\tilde{L}'_{M-K} = 0$$

We can use again (19) to get:

$$Q'\begin{bmatrix} \tilde{L}'_K - 1_K \\ \tilde{L}'_{M-K} \end{bmatrix} = 0$$

If Q is invertible then:

$$\begin{bmatrix} \tilde{L}'_K - 1 \\ \tilde{L}'_{M-K} \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} \tilde{L}'_K \\ \tilde{L}'_{M-K} \end{bmatrix} = \begin{bmatrix} 1_K \\ 0_{M-K} \end{bmatrix}$$

Or

$$\tilde{L}' = [1_K \quad 0_{M-K}]$$

As in (21). ■

PROOF #2

Start by defining:

$$\tilde{L}_Q' = [\tilde{L}_K' \quad \tilde{L}'_{M-K}]$$

$$\widetilde{F}_Q = QF = \begin{bmatrix} PF \\ SF \end{bmatrix} = \begin{bmatrix} \widetilde{F} \\ \widetilde{F}_{M-K} \end{bmatrix}$$

Therefore:

$$TEV^2 = \mathrm{var}(\widetilde{L}_Q{}'\widetilde{F}_Q) = \mathrm{var}(\widetilde{L}_K\widetilde{F}) + \mathrm{var}(\widetilde{L}_{M-K}'\widetilde{F}_{M-K}) + \\ + 2\,\mathrm{cov}(\widetilde{L}_K\widetilde{F}, \widetilde{L}_{M-K}'\widetilde{F}_{M-K})$$

Our goal is to make the two set of factors uncorrelated, so that we can interpret the second set as the OLS residual. To that end, we need that:

$$\mathrm{cov}(\widetilde{F}_K, \widetilde{F}_{M-K}) = 0 \tag{A.1}$$

or

$$\mathrm{cov}(PF, SF) = P\,\mathrm{cov}(F, F)S' = P\Omega S' = 0$$

By definition of the null space of a matrix, this happens if:

$$S = null(P\Omega)'$$

■

PROOF #3

Let's start with the definition given by (8):

$$\widetilde{L} = (\widetilde{F}\widetilde{F}')^{-1}\widetilde{F}\Pi$$

We can rewrite it as:

$$(\widetilde{F}\widetilde{F}')\widetilde{L} = \widetilde{F}\Pi$$

If we use (A.1), we can re-write the above expression as:

$$\begin{bmatrix} \widetilde{F}_K{}'\widetilde{F}_K & 0 \\ 0 & \widetilde{F}_{M-K}{}'\widetilde{F}_{M-K} \end{bmatrix} \begin{bmatrix} \widetilde{L}_K \\ \widetilde{L}_{M-K} \end{bmatrix} = \begin{bmatrix} \widetilde{F}_K\Pi \\ \widetilde{F}_{M-K}\Pi \end{bmatrix}$$

And in particular:

$$\left(\widetilde{F}_K{}'\widetilde{F}_K\right)\widetilde{L}_K = \widetilde{F}_K\Pi$$

Or:

$$\widetilde{L}_K = \left(\widetilde{F}_K{}'\widetilde{F}_K\right)^{-1}\widetilde{F}_K\Pi = \left(P\Omega P'\right)^{-1}P\Omega L$$

From (9) we can therefore establish that:

$$\widetilde{L}_K = \widetilde{L}$$

EXAMPLE #1

In particular, we can define Q as:

$$Q = \begin{bmatrix} 2.702 & 0 & 0 & 0 & 0.025 & 0.048 & ... & 0 \\ 0 & 0.229 & 0 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 3.012 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0.223 & 0 & 0 & ... & 1.697 \\ - & - & - & - & - & - & - & - \\ & & & & S_{30,34} & & & \end{bmatrix}$$

Where $S_{30,34}$ is any 30×34 matrix so that Q is invertible. In this case, $\widetilde{L} = Q'^{-1} L = \begin{bmatrix} 1'_4 & 0'_{30} \end{bmatrix}$ and:

$$C = \begin{bmatrix} 23.42 & 1.56 & 21.37 & 0.88 & 0 & ... & 0 \end{bmatrix}'$$

EXAMPLE #2

In particular, let's define $\widetilde{L}_Q = Q'^{-1} L$ and $\widetilde{F}_Q = QF$, where:

$$Q = \begin{bmatrix} P \\ null(P\Omega)' \end{bmatrix}$$

and P is defined as in (27). In this case, the vector of contributions C has a dimension of 34. In particular, the vector comes as:

$$C_Q = \begin{bmatrix} 16.74 & 20.63 & 1.30 & -0.11 & 0.18 & ... & 0.15 \end{bmatrix}'$$

As expected from (25) and (26), the first 3 elements of C_Q corresponds to the first 3 of C in (28). The other 31 elements of C_Q sum up to 8.58, the fourth element of C in (28). Remember these last 31 factors are artificially created. They do not convey individually any specific intuition or information. What is relevant is to understand that, as a whole, these residual factors do represent ε .

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