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Long-Term Bond Returns under Duration Targeting

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Although most bond portfolios maintain a relatively stable duration over time and are thus implicitly or explicitly "duration targeted," the distinctive nature of duration targeting (DT) is underappreciated. The authors' theoretical DT model demonstrates that over multi-year horizons, annualized DT returns converge back to the starting yield, regardless of the rate path. For example, for almost all six-year holding periods since 1985, Barclays bond index returns have converged to within 1% of the starting yield.

The standard analyses of bond behavior (and common intuitions about it) are based on either short-term returns or some form of a hold-to-maturity model. However, the vast majority of bond portfolios actually follow a process known as *duration targeting* (DT), which maintains a relatively stable duration over time. A DT approach is central to the rebalancing procedures of most active and passive institutional bond funds, bond mutual funds, and the fixed-income components of multi-asset funds. Even laddered bond portfolios with illiquid holdings implicitly function in a DT mode (Leibowitz and Bova 2012a).

Despite the widespread presence of DT, there remains an underappreciation of the very distinctive and rather surprising nature of long-term DT returns. This situation is particularly striking given the persistence of historically low interest rates and corresponding concerns about rising rates. In fact, most of the literature on DT investing has focused on either short-term price sensitivity or longer-term liability management through immunization or liability-driven investments (e.g., Reitano 1991; Ilmanen 1992; Leibowitz 1992; Rządkowski and Zaremna 2000; Siegel and Waring 2004; Fabozzi 2007; Moore and Nambia 2012). The literature also contains comprehensive treatments of various forms of advanced DT measures, including their application to term structure, forward rates, and various convexity environments (e.g., Winkelmann 1989; Coleman 2011; Howell 2011). In contrast, only a few studies have specifically analyzed the multi-year return behavior of DT funds (e.g., Langetieg, Leibowitz, and Kogelman 1990; Leibowitz and Bova 2012b).

Langetieg et al. (1990) developed a DT simulation model that showed that (1) total return volatility is minimized when the holding period is close to twice the duration target and (2) DT is less volatile than immunization in early years but slightly more volatile in later years.

In our study, in order to gain deeper insight into the DT process, we began with a simplified model of a flat yield curve, no default risk, annual rebalancing, and an investment in a zero-coupon bond. We tested our results against 25–30 years of bond index return data.

In our simplified zero-coupon bond model, the duration target is equal to the bond maturity (i.e., the Macaulay duration) and the year-end price factor is equal to the initial duration reduced by one year (see Appendix A). This year-end effect stands in contrast to the instantaneous price sensitivity, which is measured by the modified duration.

Discussion of findings. We found that along a trendline path from the initial yield to the terminal yield, the average return depends on only the annual yield change and the duration target. For an investment horizon that is one year less than twice the duration target, the annualized return across all trendline paths is equal to the starting yield.

For the more general case of random rate paths, we partitioned the total return volatility into two distinct, independent components: trendline-based volatility and tracking error relative to a given trendline path. In contrast to trendline volatility, tracking error increases gradually with the investment horizon and tends to stabilize within three years of the *convergence horizon*. Within this three-year band, the annualized return across all paths tends to converge to within 100 bps of the starting yield.

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Our model's results are confirmed by both a random walk simulation of interest rates and the actual multi-year returns of the Barclays U.S. Aggregate Government/Credit Index and the Barclays U.S. Credit Index. These indices have had fairly stable durations (5.4 years since 1985 and 6.1 years since 1981, respectively) and can be viewed as implicitly following a DT approach. In both cases, the average returns over almost all six-year holding periods have been within about 100 bps of the initial yield.

We found that the key to understanding DT lies in recognizing the offsetting relationship between price effects and incremental accruals. A simple example of this effect can be seen in a trendline path with constant annual yield increases. With a fixed zero-coupon bond duration, each year's yield increase leads to roughly the same capital loss. However, this price loss is offset by a greater accrual from the higher yield.

At the outset, annual price losses dominate the incremental accruals. But because the annual price losses remain constant along a trendline whereas the annual accrual rate rises with each year's higher level of yields, the cumulative accrual is sufficiently above the starting yield to fully offset the cumulative price loss. The incremental return is then reduced to zero, and the annualized return just equals the starting yield.

Of course, the trendline is only one of the myriad paths that rates can follow over time. Each nontrendline path to a particular terminal yield can be framed in terms of positive or negative yield deviations from the trendline path leading to that terminal yield. Under a DT regimen, a nontrendline path and a trendline path with the same overall yield change incur about the same capital gain or loss. In contrast, the accruals are highly sensitive to the actual rate path, and deviations from the nontrendline path may lead to significantly different accruals.

When we went beyond any single nontrendline path to the universe of "all" conceivable paths, we found that for any given nontrendline path to a terminal yield point, one can generally identify a second nontrendline path whose deviations are the exact opposites of those of the associated trendline. The yield deviations from this "mirror image" offset those from the first nontrendline path, and the average of the accruals from the two nontrendline paths matches the trendline path accrual. With both the price effect and the average accrual matching the trendline path values, the two paired nontrendline paths have the same expected return as the trendline path—so long as the deviations have symmetric probabilities. By extension, the set of all nontrendline paths to a specified terminal yield can be viewed as essentially comprising mirror pairs that all have the same expected return as the trendline path. Thus, even though the earlier returns along a trendline may vary widely, the accumulation of offsets exerts a kind of "gravitational force" that drives DT returns back toward the starting yield.

Because all terminal yields have an associated trendline path along which returns converge toward the starting yield, this same convergence property must hold for all the nontrendline paths that can be configured as symmetric-probability mirror pairs around some trendline path. This reasoning leads to the striking conclusion that the expected return across virtually *all* paths leading to *any* terminal yield should also converge toward the starting yield. (One exception is a mirror pair in which the mirror image involves implied negative rates.) For any assumed random walk of yields, however, total returns disperse around the trendline mean.

In our study, we developed an analytic model for estimating the standard deviation of the dispersion of returns relative to the starting yield. For a typical DT fund with a five-year duration, this standard deviation declines over time until flattening out around 1% by the sixth year. Taken together, these findings suggest that a DT fund with a five-year target should have returns that converge, by the sixth year, to within 1% of the starting yield—regardless of the intervening rate paths.

We first tested these theoretical DT models by using the 1977–2011 history of constant-maturity US Treasury yields to simulate returns for both point and ladder portfolios with five-year duration targets. More-market-related tests are based on several bond indices that have had relatively stable durations over a span of years and, in effect, have functioned as DT funds. In all cases, these tests confirmed that the starting yield does indeed approximate average six-year returns, with standard deviations close to the theoretical level of 1%.

These findings have a number of investment implications for bond volatility, the expected returns associated with scenario analysis, and the volatility of the bond components of a multi-asset framework:

1. The DT process is far more widespread than generally recognized.
2. In scenario analysis, the trendline formula can provide a reasonable estimate of the expected return to a given terminal yield.
3. Over an appropriate horizon that depends on only the duration target, the annualized return converges toward the starting yield, subject to a modest standard deviation (for a five-year duration, the convergence to within ±1% occurs over six years).

4. On the one hand, for DT investors content with the current yield levels, this convergence suggests that they need not be unduly worried about the impact of higher rates on their multiyear returns.
5. On the other hand, DT investors committed to a fixed duration target should not count on higher—or lower—yields to augment their multiyear returns above the current levels of yields.
6. To avoid being trapped in the gravitational pull back to the starting yield, a DT fund must make a significant shift away from its current duration target.
7. A DT bond portfolio has a much lower multiyear volatility than generally believed.
8. In an asset allocation framework, the overall fund-level volatility contributed by the bond component is generally far less than that of a standard mean–variance model.

A Simple DT Model vs. a Buy-and-Hold Model

Most bond portfolio management can be characterized as either buy and hold (B&H) or duration targeting (DT). B&H is the ultimate passive approach. For example, if a five-year zero-coupon bond is held to maturity, the duration simply "ages down" year by year until maturity and the annualized return equals the initial yield.

In contrast to B&H, the constant duration of DT is maintained over the entire investment horizon by periodic rebalancing. At the end of the first year, the five-year bond ages to a four-year bond. The four-year bond is then sold and replaced with a new five-year bond. **Figure 1** contrasts the time evolutions of the durations of B&H and DT for a five-year zero-coupon bond.

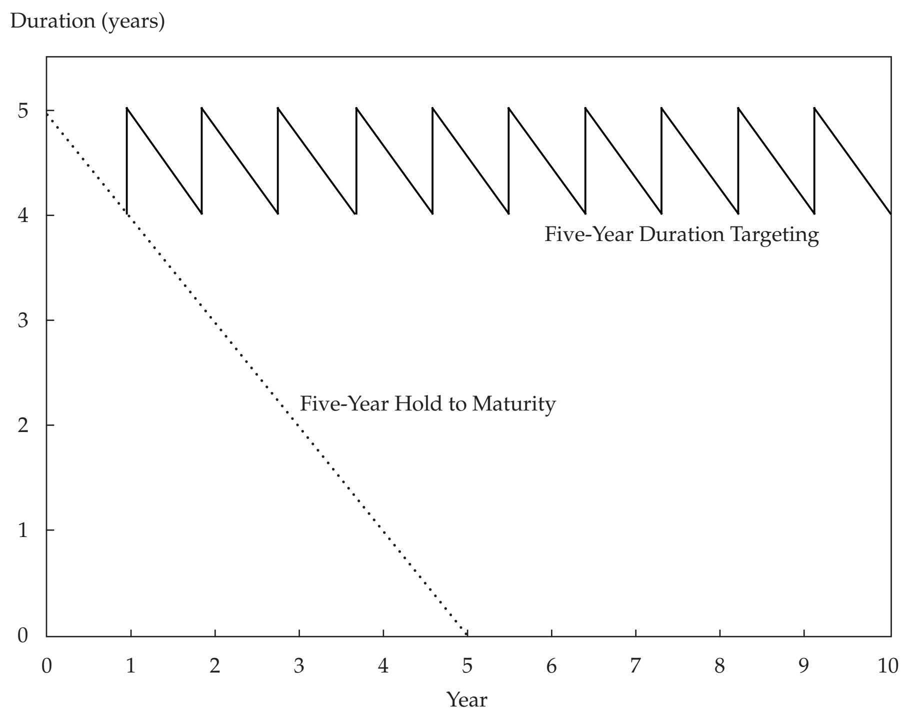
There are important parallels between single-bond B&H returns and DT portfolio returns. For example, the aging duration of B&H results in decreasing volatility of the mark-to-market return as the holding period approaches the bond maturity. At its stated maturity, the B&H bond's return equals the initial yield.

With DT portfolios, the duration does not age down but, rather, is sustained at the targeted level. Thus, DT portfolios are not generally thought to share the B&H property of convergence toward the starting yield. But the annualized return volatility for DT structures also decreases with longer holding periods (discussed later in the article).

DT Portfolio Rebalancing

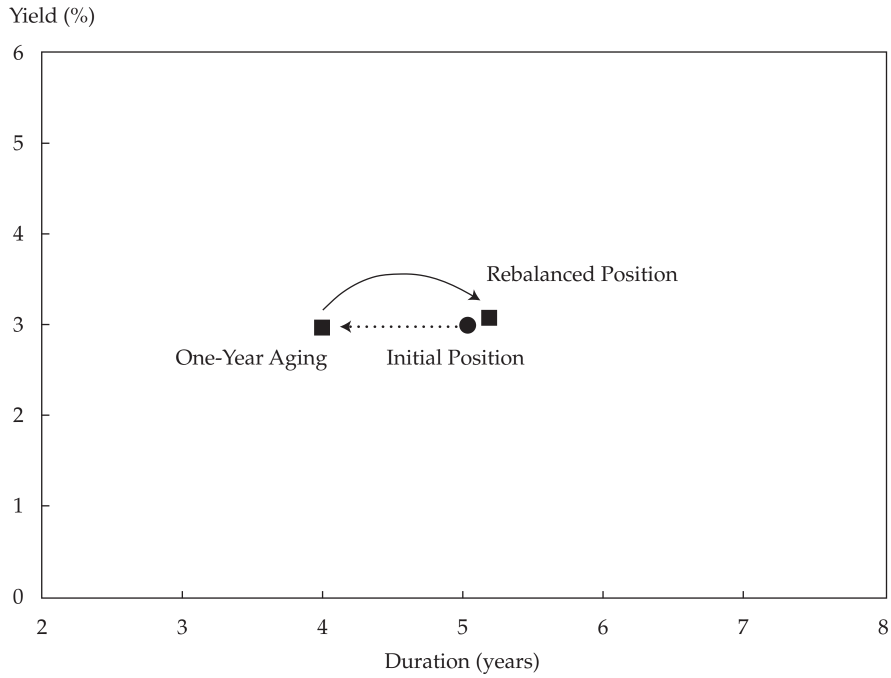
The DT process can be achieved in a variety of ways. **Figure 2** illustrates the simplest case: a single bond with an initial duration of five years. Over the first year, the bond ages down to a four-year duration. The bond is sold, and the proceeds are used

Figure 1. Duration Targeting vs. Buy and Hold



Source: Morgan Stanley Research.

Figure 2. Sale-and-Repurchase Rebalancing



Source: Morgan Stanley Research.

to purchase a new five-year bond. This simplest of all DT processes relies on having liquid bonds with minimal transaction costs.

However, a DT process can be at work in a more complex bond portfolio with illiquid holdings. For example, **Figure 3** depicts an equal-weighted bond "ladder" with maturities ranging from one to five years. The average maturity/duration of this portfolio is three years. After the first year, each bond (and the portfolio) ages by one year. The one-year bond matures, and the five-year bond position is vacant. The portfolio can be "rebalanced" without selling any bonds by simply using the proceeds of the maturing bond to buy a new five-year bond. The bond durations again range from one to five years, and the initial three-year average duration has been restored.

This simple example can be generalized to more complex coupon bond portfolios with unequal bond weights. Even for such portfolios, rebalancing can often be accomplished by using the proceeds of maturing bonds and interim coupon flows. When few forced bond sales are required, many bonds can be held to maturity, and such DT portfolios may have a substantial B&H component.

In the case of bond indices, bonds typically must have a maturity greater than one year. Bonds that age to the one-year point must be sold and the proceeds used to purchase newly available bonds of usually longer maturities. Absent any explicit commitment to a DT framework, this sell-and-buy process in bond indices tends to lead to relatively stable durations.

DT Returns along Idealized Trendline Paths

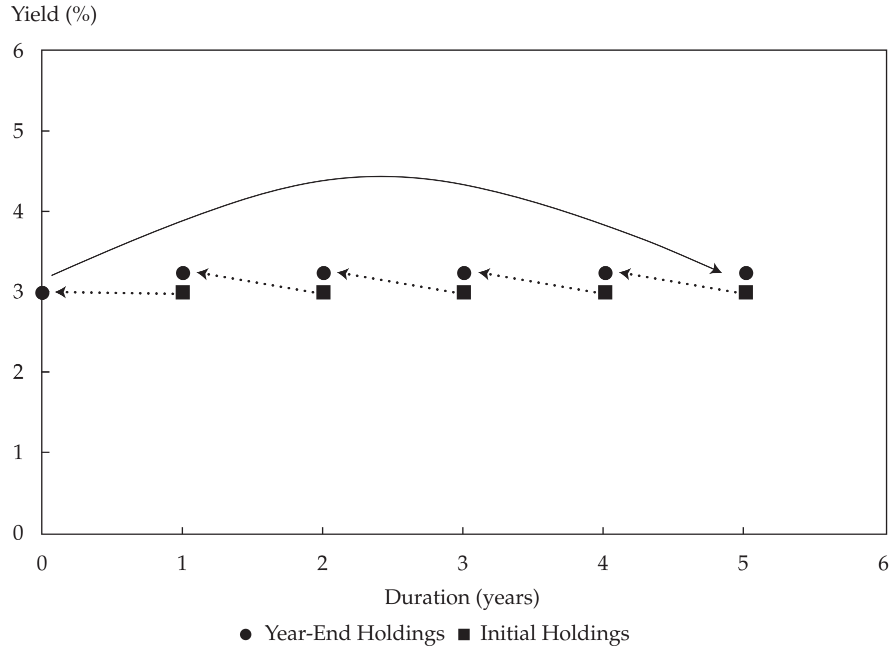
To illustrate the DT process, we can use the most basic model of a zero-coupon bond with a five-year duration. At the end of each one-year holding period, the aged bond is sold and the proceeds are invested in a new five-year bond.

All returns from zero-coupon bonds take the form of price movements. In this analysis, however, we can differentiate between two types of price movements: (1) price effects derived from changes in yield and (2) "accruals" that result from the passage of time without any change in yield.

To illustrate how yield changes affect returns, we begin with a simple nine-year-yield trendline path that proceeds from an initial yield of 3% to a final yield of 7%. **Table 1** is based on yields that rise at a constant 50 bps a year.

The first-year-yield accrual is derived from the 3% initial yield, and so the excess yield accrual over the 3% initial yield is just 0%. Over the year, the bond ages and the initial five-year bond duration is reduced to four. Because the year-end trendline yield is 0.5% higher than the initial yield, the rebalancing bond sale results in a price loss of approximately $-4 \times 0.5\% = -2\%$. (For clarity of illustration,

Figure 3. Rebalancing of Illiquid Ladders



Source: Morgan Stanley Research.

Table 1. Trendline Returns with a 50 bp Drift

Year	Yield at Beginning of Year	Excess Yield Accrual over Year	Cumulative Excess Accrual	Price Loss over Year	Cumulative Price Loss	Cumulative Excess Return	Annualized Excess Return
1	3.0%	0.0%	0.0%	–2.0%	–2.0%	–2.0%	–2.00%
2	3.5	0.5	0.5	–2.0	–4.0	–3.5	–1.75
3	4.0	1.0	1.5	–2.0	–6.0	–4.5	–1.50
4	4.5	1.5	3.0	–2.0	–8.0	–5.0	–1.25
5	5.0	2.0	5.0	–2.0	–10.0	–5.0	–1.00
6	5.5	2.5	7.5	–2.0	–12.0	–4.5	–0.75
7	6.0	3.0	10.5	–2.0	–14.0	–3.5	–0.50
8	6.5	3.5	14.0	–2.0	–16.0	–2.0	–0.25
9	7.0	4.0	18.0	–2.0	–18.0	0.0	0.00

Source: Morgan Stanley Research.

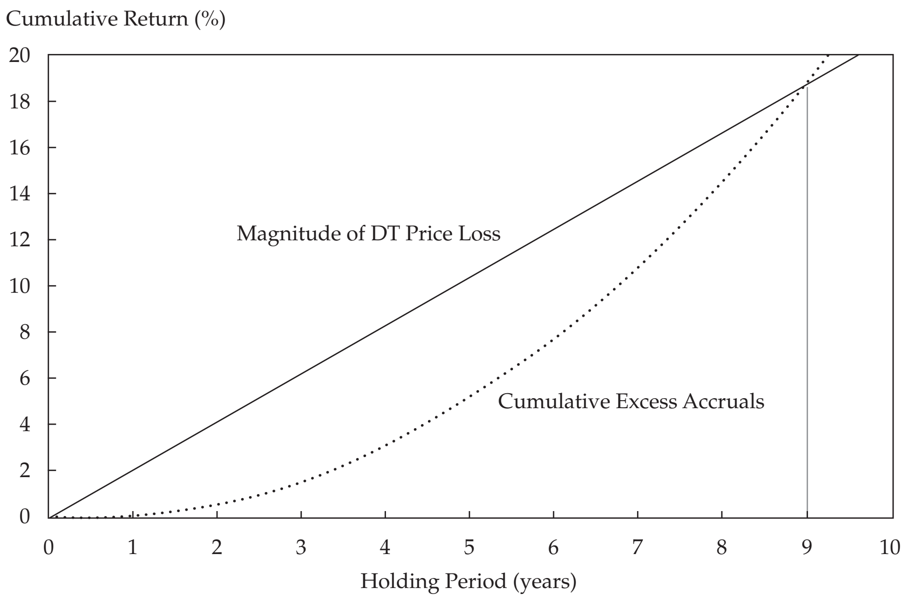
all returns are in a simple-interest format that ignores the price effects of compounding and multi-year compounding.)

The sale proceeds are then reinvested in a new five-year bond with a 3.5% yield, which becomes the basis for the second-year accrual. The trendline yield increases by another 0.5% at the end of the second year, resulting in another –2% price loss. This –2% loss combines with the second-year excess yield accrual of 0.5% to provide an excess return of –1.5%, a cumulative excess return of –3.5%, and an annualized excess return of –1.75% after two years. With each subsequent year, the excess accrual rate increases by another 50 bps, while the price loss remains constant at –2%. At the end of four years, the cumulative excess return is –5%.

By the fifth year, yields and accruals have risen by 2% (from 3% to 5%). The excess accrual of 2% precisely offsets the –2% price loss, and so the cumulative excess return remains unchanged at –5%. In later years, the excess accruals continue to grow such that, after nine years, the cumulative excess yield accrual of 18% completely offsets the cumulative price loss of –18%. At this nine-year "effective maturity," the cumulative excess return is 0% and the annualized return equals the 3% starting yield. The annualized excess returns are linear over time, eventually reaching 0% by the ninth year.

To better illustrate how the relationship between accruals and price changes develops over time, **Figure 4** focuses on a hypothetical framework of continuous compounding in which the

Figure 4. Continuous Excess Yield Accruals and Price Losses



Source: Morgan Stanley Research.

cumulative price loss appears to be linear over time (although there is actually a slight upward curvature). At the outset, the price loss far exceeds the nominal excess yield accrual. However, the new accruals build on the preceding yields, and the net effect is that the cumulative accrual grows rapidly and ultimately overcomes the cumulative price loss by the ninth year.

DT Effective Maturity

The annualized excess return in the previous trendline example can be calculated directly from a formula (derived in Appendix A). This formula shows that the effective maturity for a trendline-based DT path is equal to twice the duration target minus 1. Thus, the effective maturity depends on only the duration and is independent of the size and direction of yield changes. For a five-year bond, the effective maturity is nine years (2 × 5 – 1).

Figure 5 shows that annualized excess returns approach zero as the holding period approaches the nine-year effective maturity for a five-year duration target. Each line represents a different yield path. When yields decline at –50 bps a year (the uppermost line), price gains in the early years dominate accruals, resulting in relatively high returns. Over time, ever-lower yields lead to declining accruals that eventually offset all price gains completely.

At the other extreme of 50 bp annual yield increases, initial price losses lead to negative excess returns in the early years. But those early losses are gradually mitigated by accrual gains, and the excess return again zeroes out at the same nine-year horizon. More generally, Figure 5 illustrates that a given duration target has the same effective maturity for any yield drift so long as yields progress in a trendline fashion.

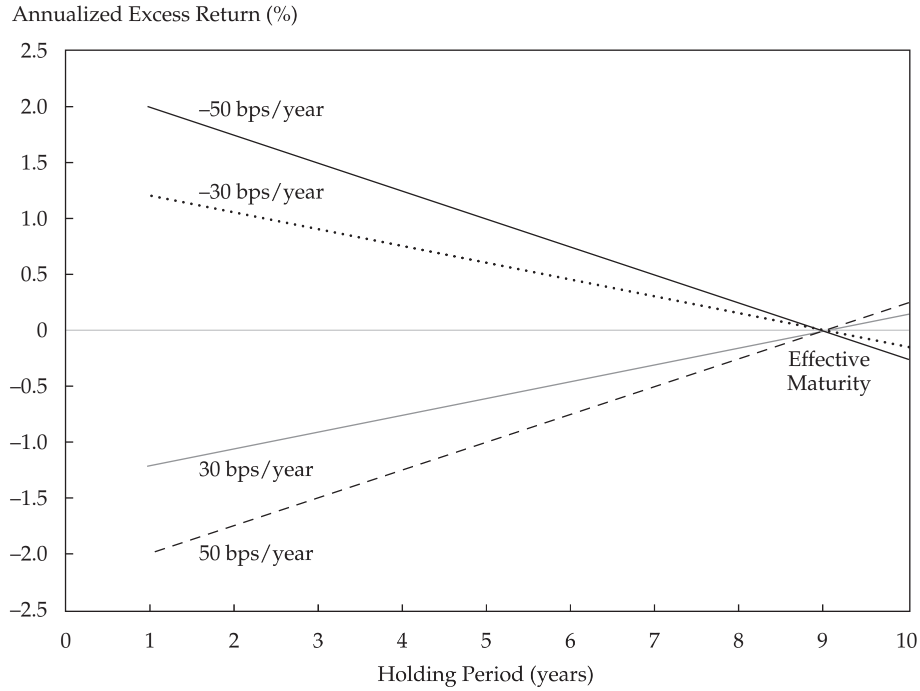
DT Return Sensitivity at a Fixed Horizon

At a specified time horizon, each trendline path leads to a certain total yield change. Conversely, a given total yield change at a specified horizon can always be reached by some trendline path with the appropriate annual yield change. The corresponding excess DT return can then be found by applying the trendline formula in Appendix A.

Figure 6 focuses on this relationship between excess DT returns and total yield changes over two-, five-, seven-, and nine-year holding periods. At the shorter horizons, the price effects dominate and the excess return is highly sensitive to the yield change. As the holding period increases, the sensitivity to total yield change decreases and the return line rotates counterclockwise.

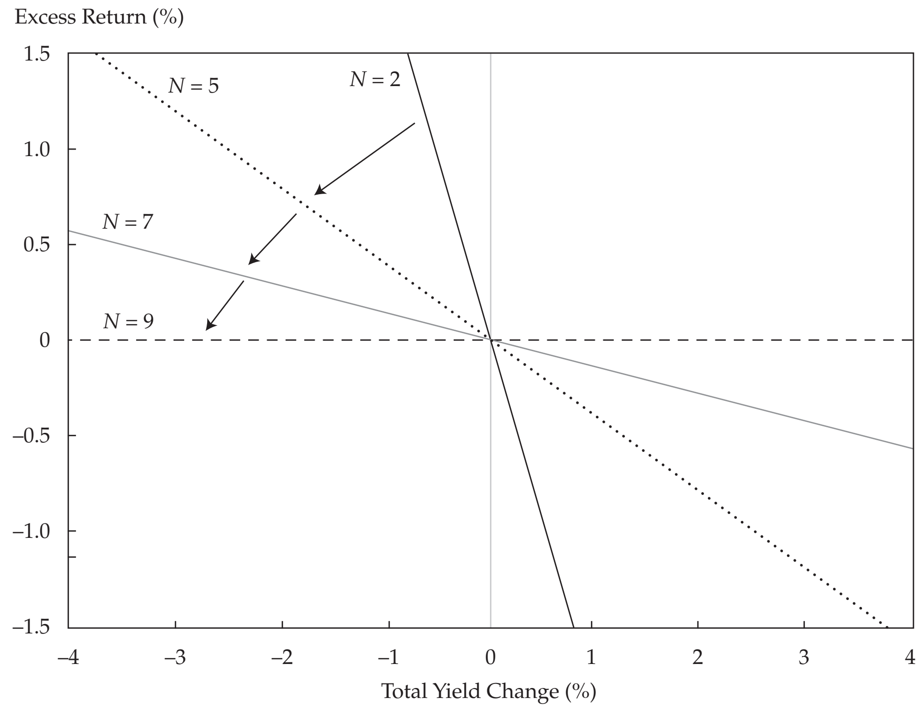
Over the nine-year holding period, the excess accruals match the price losses and the excess return is zero, regardless of the magnitude of the yield change. Thus, this zero excess return implies that the annualized return comes back to the starting yield along the trendline path associated with *any* terminal yield, regardless of the pace of yield changes.

Figure 5. Excess Returns for Various Trendline Drift Rates



Note: Duration target = five years.
Source: Morgan Stanley Research.

Figure 6. Excess Returns vs. Total Yield Change for Various Time Horizons



Note: Duration target = five years.
Source: Morgan Stanley Research.

Terminal Yield and Return Distributions

Until this point, we have focused on trendline paths that lead to a specific terminal yield. Let us now turn our attention to probability distributions of terminal yields as generated by a random walk.

In the following example, let us assume that yield changes are subject to an annual volatility of 1% around a 50 bp upward drift from the 3% starting yield. **Figure 7** shows the progression of yield distributions over time. At the outset, the investment horizon, N, equals zero, and there is no volatility around the 3% initial yield. As the horizon increases, the mean of the yield distribution increases at the drift rate of +50 bps a year while the "volatility"—the standard deviation of the yield change distribution—increases by a factor of $\sqrt{N}$ times the 1% yield volatility.

At the five-year horizon, the yield distribution has a mean of 5.5% (= 3.0% + 5 × 0.5%) and a volatility of 2.24% ($= \sqrt{5} \times 1\%$). A unique trendline leads from the initial 3% to each terminal yield in the distribution. The annualized return for each trendline path can be determined by applying the return formula in Appendix A. Thus, the total annualized return to the five-year yield of 5.5% is 2.0% (the 3% initial yield minus the 1% annualized excess return, as in Table 1). The probability of terminal yields of 5.5% ± 0.5% is 18%. Similarly, a terminal yield of 2.5% ± 0.5% has a 7% probability, and the trendline path to this 2.5% yield would produce a return of 3.2%.

Table 2 shows how associating the probability of the terminal yields with the corresponding returns can translate into a distribution of returns. However, we can see that the trendline returns for any terminal yield are compressed over longer horizons, eventually converging back to the starting yield. Thus, even though the yield distributions have an underlying volatility that increases over time, this return compression drives the return distribution to follow the strikingly different sequence shown in **Figure 8**.

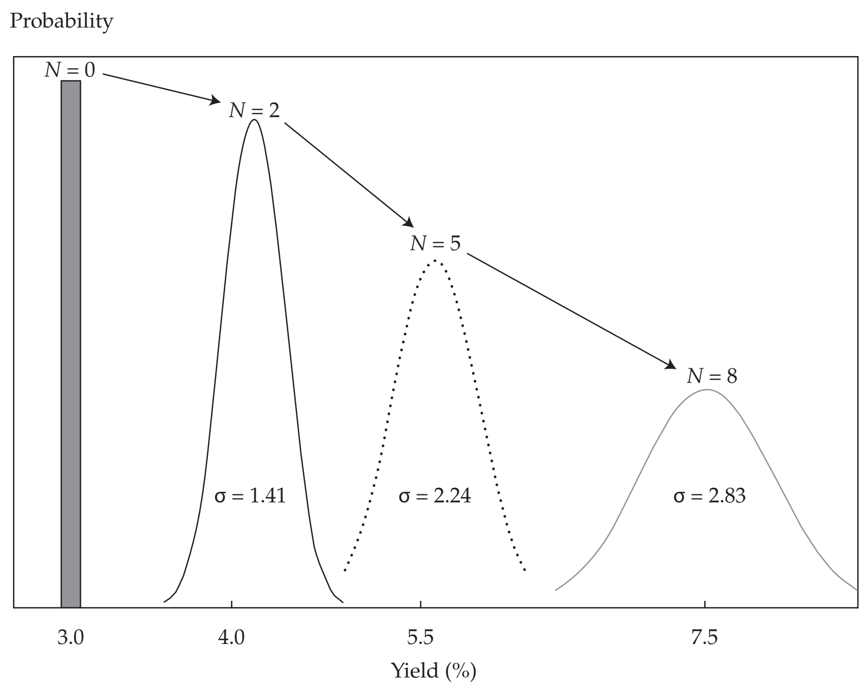
Table 2. Probability Distributions of Five-Year Trendline Returns

Five-Year Yield	Probability	Annualized Return
2.5%	7%	3.2%
3.5	12	2.8
4.5	16	2.4
5.5	18	2.0
6.5	16	1.6
7.5	12	1.2
8.5	7	0.8

Note: The average yield is 5.5%, the average return is 2%, the yield volatility is 2.24%, and the return volatility is 0.9%.

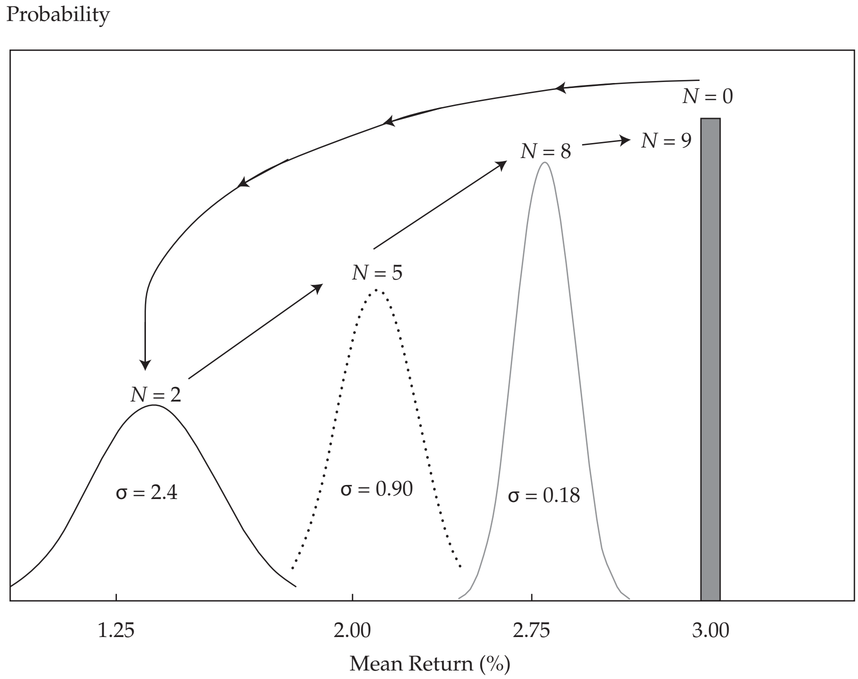
Source: Morgan Stanley Research.

Figure 7. The Progression of Terminal Yield Distributions over Time



Source: Morgan Stanley Research.

Figure 8. The Progression of Annualized Returns over Time



Source: Morgan Stanley Research.

For shorter horizons, the rising drift in yields leads to price losses that result in lower mean returns; for example, for $N = 2$, the mean return is 1.25%. The relatively wide volatility of returns reflects the high sensitivity to the yield changes shown in Figure 6. As the investment horizon lengthens, however, the returns are compressed for all terminal yields, resulting in a mean return that converges back to 3% with an ever-decreasing volatility.

In Figure 7, the mean of the yield distributions drifts further and the dispersions continue to widen over time. In contrast, as shown in Figure 8, the return distributions follow a dramatically different progression, ultimately cycling back to having all probability weight concentrated at the starting yield.

Mirror Image Paths

To this point, we have focused on a trendline model in which yields move in equal annual increments from the initial yield to the terminal yield. Let us now turn to nontrendline paths with yields that move intermittently above and below the trendline. One such random path is shown in **Figure 9**.

The second path in Figure 9 is a "mirror image" of the first random path relative to the trendline. The two mirror paths have equal vertical deviations from the trendline—but with opposite signs. Consequently, at the end of each year, the accrual deviations for the mirror paths cancel each other.

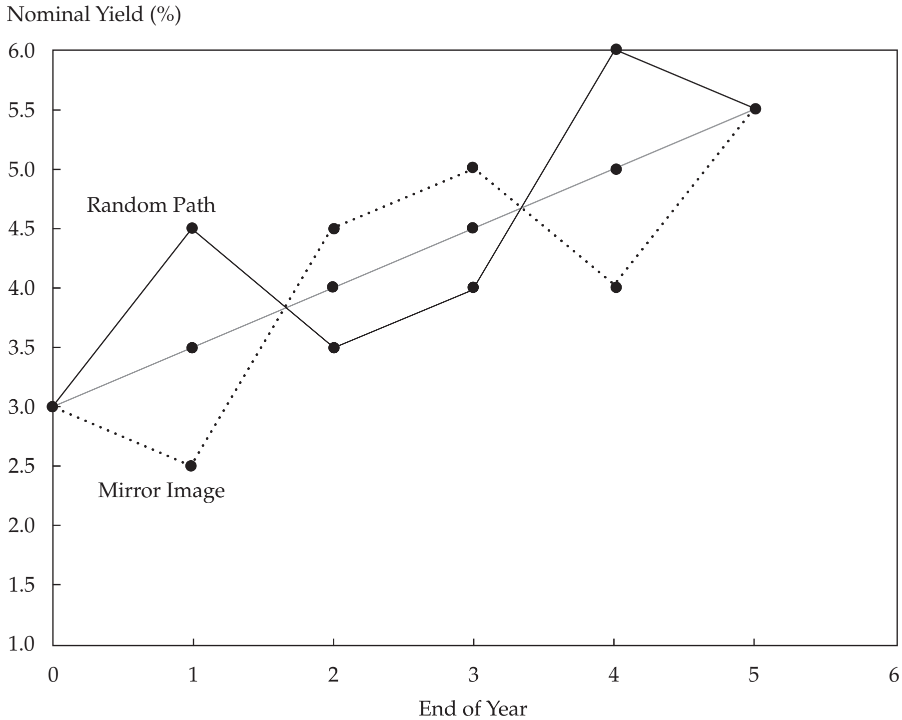
As shown in **Table 3**, if the two nontrendline paths have an equal probability of occurring, the expected accrual of the mirror pair equals the trendline accrual. The total capital gain/loss for the random path (and mirror path) is also the same for the trendline because price effects depend on only the initial and terminal yields. Therefore, the expected return across any such mirror pairs will just equal the trendline return.

The averaging reported in Table 3 can be extended to the full range of terminal yields. Each terminal yield has a unique trendline with a specific trendline return. If both nontrendline paths have comparable probabilities, the mirror paths will have the same expected return as the trendline. By extension, each trendline return to a given terminal yield can be taken as a reasonable estimate of the average return across all the many nontrendline paths leading to that same terminal yield. Moreover, the overall expected trendline return across all terminal yields would reflect the expected return across all paths derived from the random walk.

Tracking Error around the Trendline Return

From Table 3, we can see that even though the mirror pairs may have returns that average to the trendline return, each nontrendline path has a return that differs from the trendline return. This "scattering"

Figure 9. Mirror Image Paths Relative to a Trendline



Source: Morgan Stanley Research.

Table 3. Trendline and Mirror Pair Returns

	Accruals			
	Trendline Path	Random Path	Mirror Image	Mirror Pair Average
Year 1	3.0%	3.0%	3.0%	3.0%
Year 2	3.5	4.5	2.5	3.5
Year 3	4.0	3.5	4.5	4.0
Year 4	4.5	4.0	5.0	4.5
Year 5	5.0	6.0	4.0	5.0
Annualized accrual	4.0%	4.2%	3.8%	4.0%
Annualized price effect	–2.0	–2.0	–2.0	–2.0
Excess return	2.0%	2.2%	1.8%	2.0%

Source: Morgan Stanley Research.

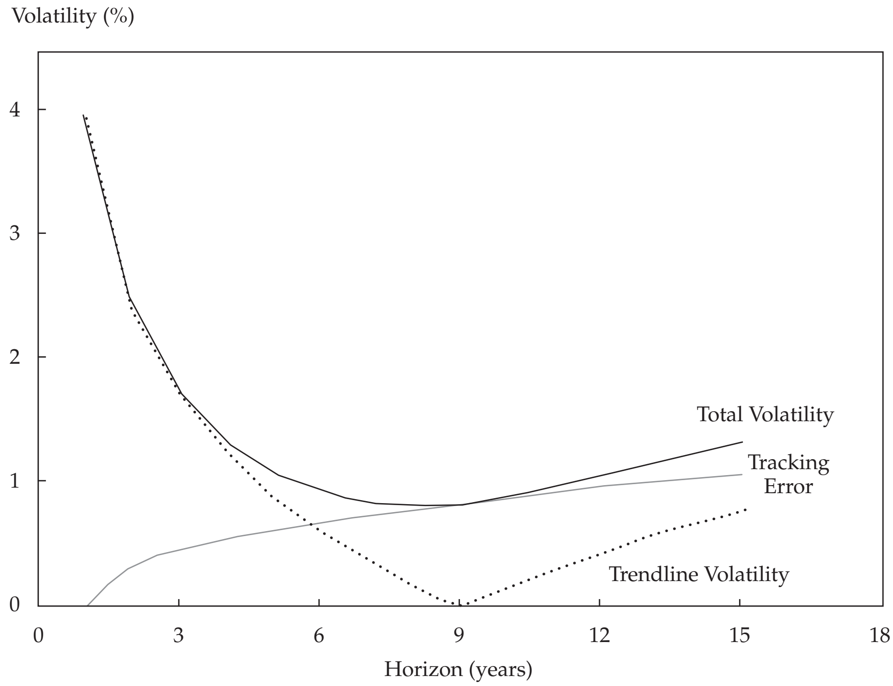
from the nontrendline paths is a source of tracking error (TE) relative to the trendline returns and has the effect of widening the distribution of returns around the mean return.

For a typical random walk process, we developed an approximation formula for the TE and the "total volatility" of the all-path return distribution (see Appendix A).

Figure 10 shows the three volatility measurements as a function of the horizon period. The trendline volatility decreases over time and reaches zero at the nine-year effective maturity. The TE volatility for nontrendline paths increases over time as the $\sqrt{N}$. The total all-path volatility is the sum of the squares of the first two volatilities. This total volatility curve declines sharply at the outset and then becomes relatively flat at around 1% for horizons in the range of six to nine years.

This flattening is important because it implies that the uncertainty around the trendline mean return is 1% whether the horizon is six or nine years. In a sense, the six-year horizon could be interpreted as a much shorter "statistical effective maturity" that provides roughly the same all-path total volatility as the nine-year effective maturity.

Figure 10. Total Return Volatility



Source: Morgan Stanley Research.

This flattening of the total volatility curve allows the six-year horizon to serve as a reasonable time frame for returns to fall within ±1% of the starting yield.

Model Testing

We tested and confirmed the results of our theoretical models with both a Monte Carlo simulation and actual return data from the Barclays U.S. Government/Credit Index.

Simulation. As one way of testing these theoretical models, we performed a Monte Carlo simulation based on a five-year random walk with a mean yield change of 0% and a standard deviation of 1%. In **Figure 11**, each scatterpoint represents a five-year simulation path for a DT strategy with a five-year duration.

The vertical axis represents the excess return for each path—the annualized return above or below the initial yield—and the horizontal axis corresponds to the total yield change by the end of the fifth year. The solid line represents the trendline return for each ending yield change. This trendline has a slope of –0.4 (applying the trendline formula derived in Appendix A).

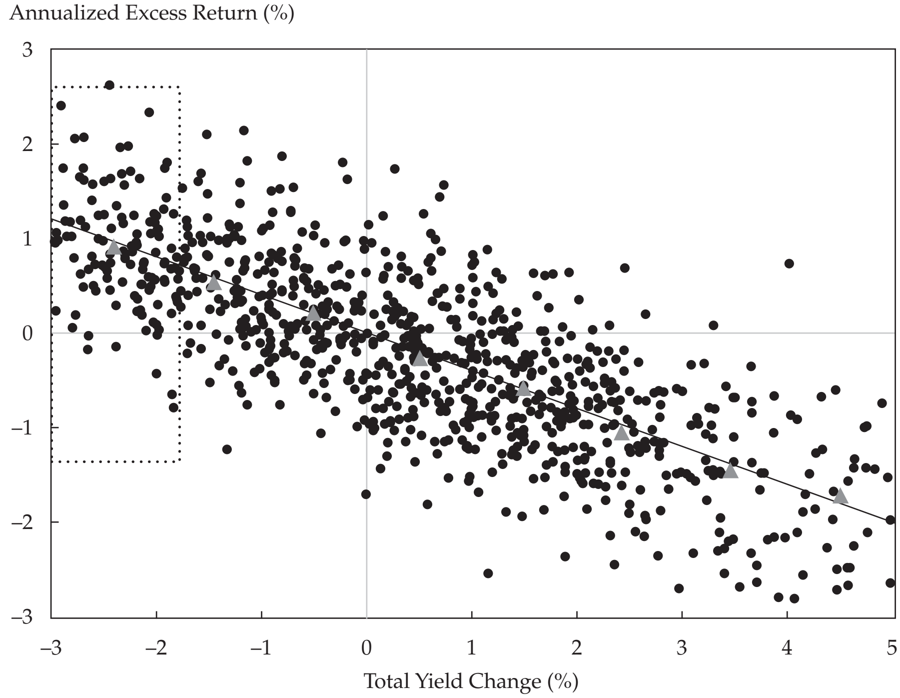
The triangles represent the average of the simulated path returns within each yield change interval. We can see that these averages fall very close to the trendline return. Moreover, the TE and total volatility from the simulation are extraordinarily close to the model estimates of 0.6% and 1.1%, respectively.

Each point in Figure 11 represents a specific path. To show how these paths evolve over time, **Figure 12** focuses on a sampling of the simulated paths in the boxed area within Figure 11. Taking some artistic license for sake of clarity, we chose to depict these paths as smooth curves over time. Although the returns vary considerably in the early years, they converge rapidly toward the expected 1% trendline return in years 4–5.

Historical Data from Barclays U.S. Government/Credit Index. Because duration stability is the key characteristic of DT portfolios, our starting point for testing the theoretical DT model against actual market returns was verifying duration stability. **Figure 13** shows that over 1973–2011, the Barclays U.S. Aggregate Government/Credit Index duration ranged between 4 and 6 years but that after 1985, it remained quite stable at around 5.4 years. (The average duration over nonoverlapping six-year periods is represented by black squares.) From 1985 to 2011, we can view this index as an implicit representation of a DT portfolio.

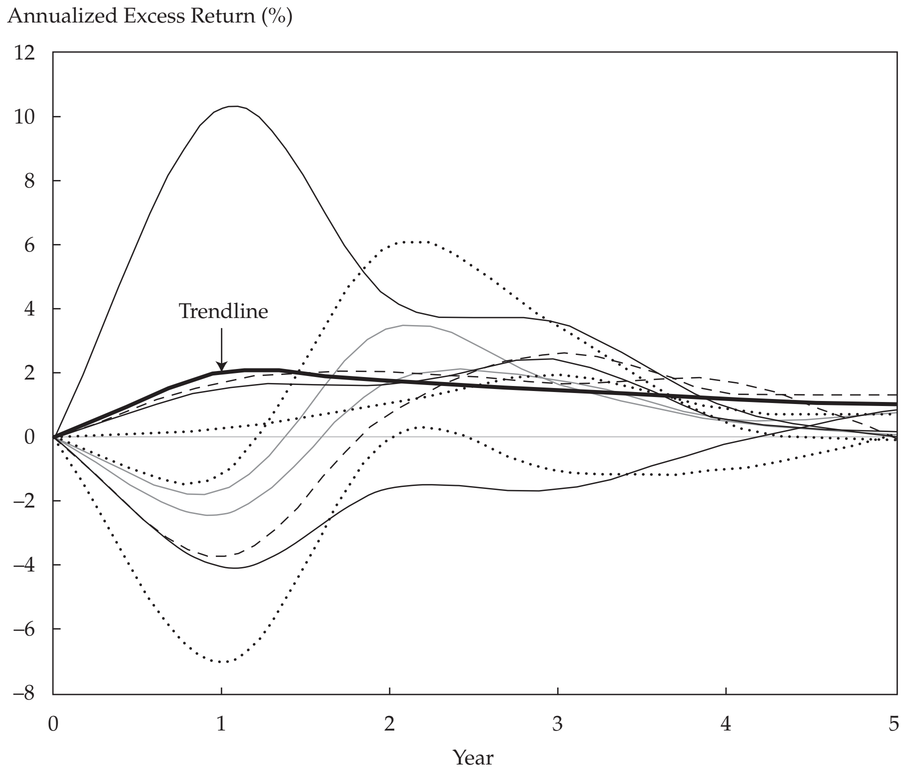
Figure 14 plots both the reported monthly yield of the index over 1973–2011 and the annualized compound returns for six-year holding periods that start each month beginning in 1985. These returns are aligned with the yields at the starting date of the six-year investment horizon. Thus, the first

Figure 11. Simulation Returns for D = 5, N = 5



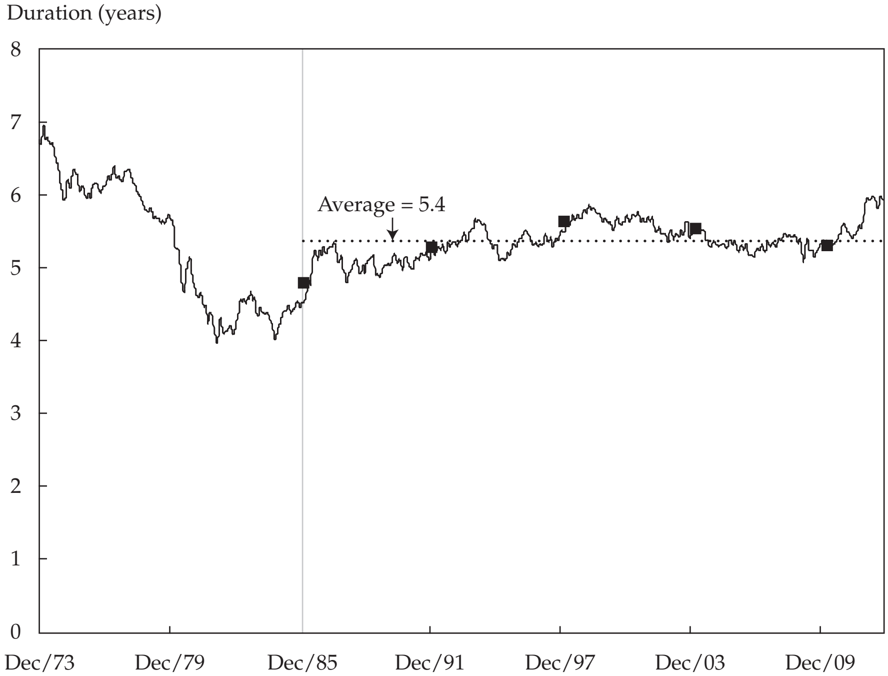
Note: Yield volatility = 1.0%; overall TE = 0.6%.
Source: Morgan Stanley Research.

Figure 12. Simulated Return Paths with Yield Declines of 2%–3%



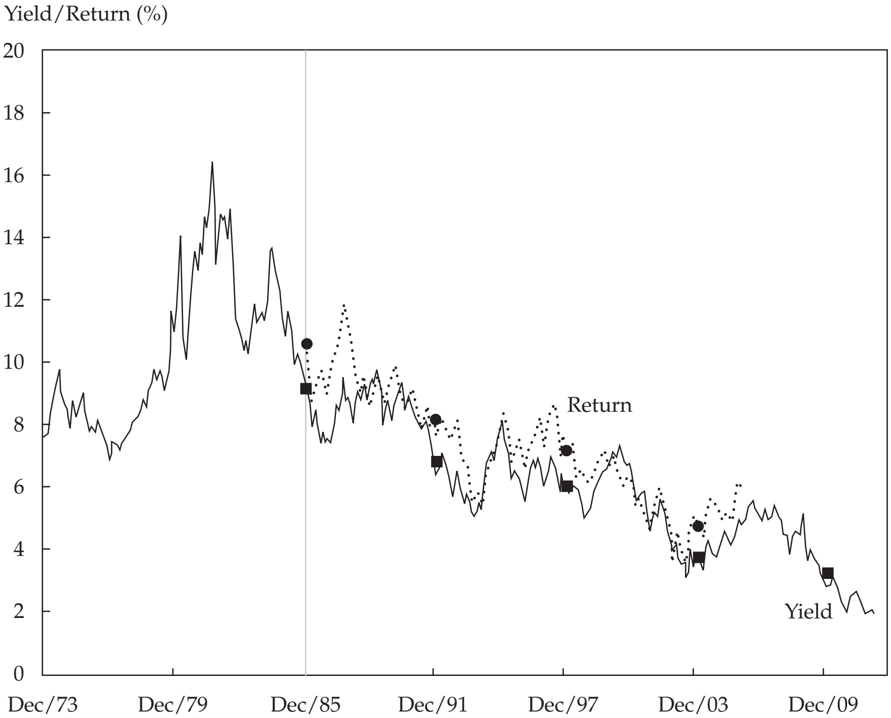
Source: Morgan Stanley Research.

Figure 13. Barclays U.S. Government/Credit Index Duration, 31 December 1973 to 31 December 2011



Sources: Morgan Stanley Research; DataStream.

Figure 14. Barclays U.S. Government/Credit Index Yields and Shifted Returns over Six-Year Holding Periods



Sources: Morgan Stanley Research; DataStream.

return point in Figure 14 represents the annualized return for the holding period of 31 December 1985 to 31 December 1991. The four nonoverlapping six-year returns are also shown (as black squares).

As depicted in Figure 14, the six-year holding period returns have followed the same general declining pattern as the yields. However, far beyond simply sharing a general downward trend, these six-year returns evidently correlate very tightly with the starting yield.

Index Returns vs. Starting Yields

To more directly test the starting yield as a return estimate, **Figure 15** plots the six-year Barclays U.S. Aggregate Government/Credit Index returns against the starting yields. The rolling (and overlapping) six-year returns are plotted as black squares, and the nonoverlapping returns are shown as gray circles. The 45-degree line represents a "forced" yield-matching relationship. We can see that the starting yields seem to provide a remarkably good estimate of the subsequent six-year returns, especially given the historic decline in yields over 1985–2011.

Figure 16 goes beyond Figure 15 to illustrate the progression of convergence over a range of holding periods. The average deviation from the initial yield exhibits a clear declining path as the holding period increases from one to nine years. In contrast, the tracking error (relative to the initial yield) steadily decreases only in years 1–5. From years 5 to 9, the tracking error—and the total volatility—stabilizes at just under 1%.

The Barclays U.S. Credit Index

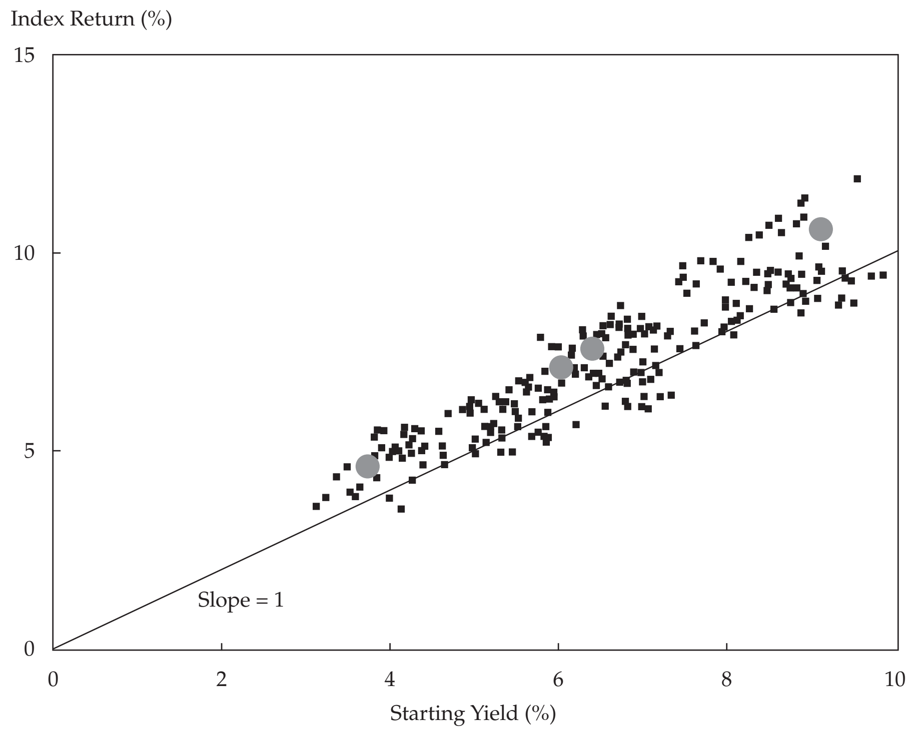
Let us now turn to the Barclays U.S. Credit Index. As shown in **Figure 17**, this index's duration fell from more than nine years to less than six years over 1973–1981. From 1981 to 2011, however, the duration remained relatively stable, with an average of 6.1 years. The squares represent the trailing average duration over nonoverlapping six-year periods.

Figure 18 analyzes Barclays U.S. Credit Index returns versus the yield-matching line. Once again, we can see that both the overlapping (black squares) and the nonoverlapping (gray circles) six-year returns are very close to their respective starting yields.

Conclusion

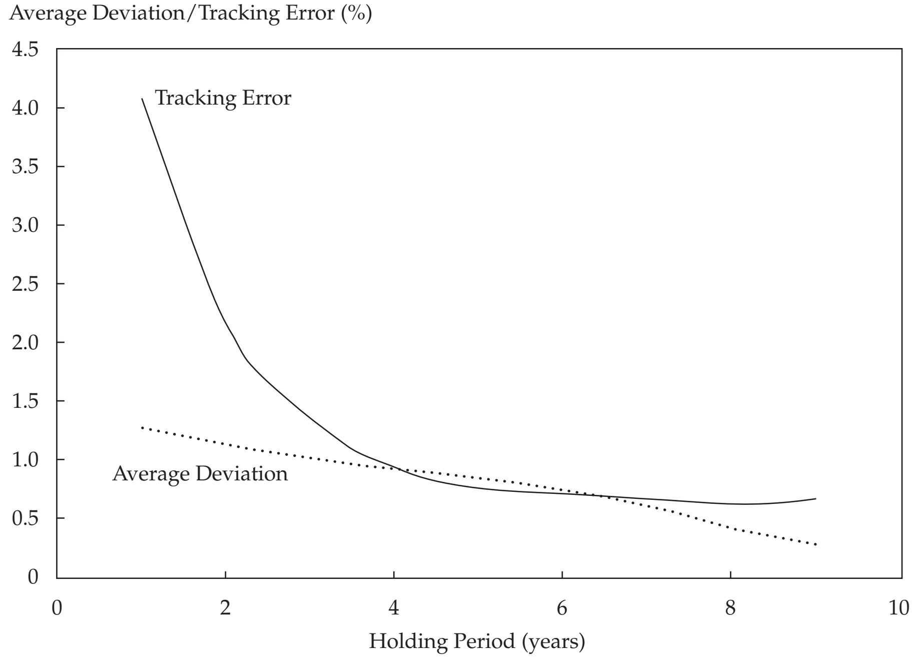
Duration targeting keeps the end-of-year duration relatively constant while incorporating new levels of accruals from rising or falling rates. These accruals act to offset the duration-based price effects. As

Figure 15. Barclays U.S. Government/Credit Index Returns over Six-Year Holding Periods (1985–2006) vs. Yield-Matching Line



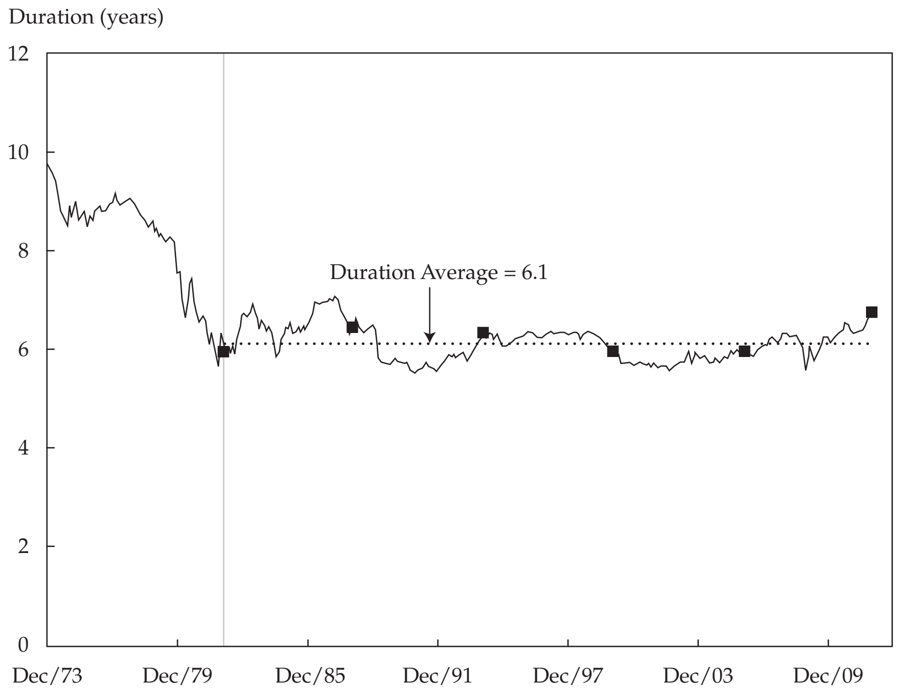
Sources: Morgan Stanley Research; DataStream.

Figure 16. Deviations of Returns from Starting Yields over One- to Nine-Year Holding Periods: Barclays U.S. Government/Credit Index, 1985–2006



Sources: Morgan Stanley Research; DataStream.

Figure 17. Barclays U.S. Credit Index Duration, 1973–2011

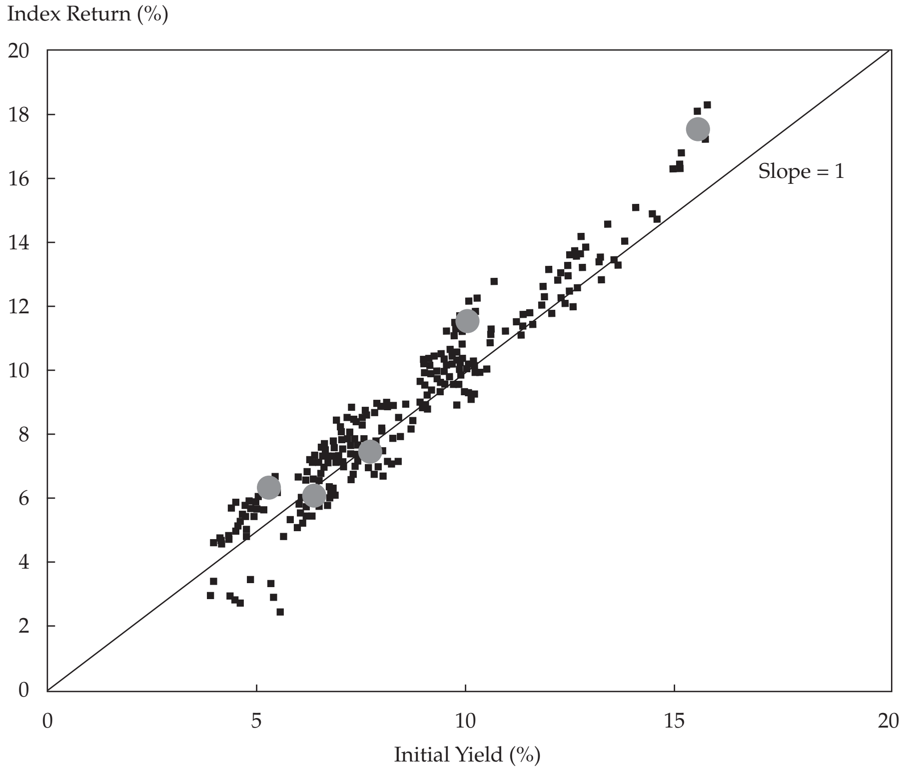


Sources: Morgan Stanley Research; DataStream.

the investment horizon lengthens, the role of accruals grows, leading to multi-year bond returns that converge in both mean and volatility around the starting yield.

The theoretical model that we developed to represent this convergence effect generally stood up well in a series of tests involving simulation-based yield paths derived from constant-maturity

Figure 18. Barclays U.S. Credit Index Returns over Six-Year Holding Periods (1982–2006) vs. Yield-Matching Line



Sources: Morgan Stanley Research; DataStream.

Treasury data and from Monte Carlo random walks. We conducted more stringent tests (based on Barclays indices), which also proved supportive.

Our key findings are as follows:

1. In DT funds, higher going-forward yields provide some offset to price losses.
2. In DT funds, lower going-forward yields provide some offset to price gains.
3. Along trendline rate paths, price effects dominate accruals at the outset.
4. But yield accruals grow at an increasing pace.
5. With intermediate DT durations, accruals can fully offset price effects over time.
6. As the incremental return is reduced, total return converges back toward starting yield.
7. This convergence can occur even in the face of strongly rising or falling rates.
8. Standard models overstate the multi-year volatility of DT funds.
9. DT components contribute less than expected to the overall volatility of multi-asset funds.

This article qualifies for 1 CE credit.

Appendix A. Derivations of the Formulas

In this appendix, we provide derivations of the various formulas and approximations referred to in the article. Specifically, we focus on trendline total return and volatility, total volatility along nontrendline paths, and trendline tracking error. We also develop a simple approximation for zero-coupon bond returns, and we extend our trendline-based results to the case of continuous rebalancing.

Path Return and Volatility

Under the assumption of a random walk model of interest rates, an initial investment is made in a D-year zero-coupon bond with nominal yield Y_0. Note that the model can easily be adjusted to accommodate rate drift (though we do not do so).

We assume a DT strategy in which the same duration is maintained throughout the holding period by repricing the bond at the end of each year and "rolling" the proceeds into a new D-year bond.

For zero-coupon bonds, the time to maturity (D) is the same as the Macaulay duration. After one year, the aged-bond duration is D – 1. The return over any year is approximately the sum of the accrual (the yield at the beginning of the year) and –(D – 1) multiplied by the simulated yield change. Later in the appendix, we show that this approximation is quite accurate for modest yield changes and moderate investment horizons.

Returns are calculated over N-year yield paths. Each path, j, originates at the same initial yield, Y_0. The yield at time i along path j is $Y_{j,i}$. The change in yield from time i – 1 to time i is $\Delta Y_{j,i}$. The terminal yield for path j, $Y_{j,N}$, is Y_0 plus the sum of all the yield changes along path j.

$$Y_{j,0} = Y_0. \tag{A1}$$

$$Y_{j,1} = Y_0 + \Delta Y_{j,1}. \tag{A2}$$

$$Y_{j,2} = Y_0 + \Delta Y_{j,1} + \Delta Y_{j,2}. \tag{A3}$$

$$\begin{aligned} Y_{j,N} &= Y_0 + \Delta Y_{j,1} + \Delta Y_{j,2} + \cdots + \Delta Y_{j,N} \\ &= Y_0 + \sum_{i=1}^{N} \Delta Y_{j,i}. \end{aligned} \tag{A4}$$

The return over period i for path j, $R_{j,i}$, equals the accrual, $A_{j,i}$, minus the year-end duration, D – 1, multiplied by the yield change, $\Delta Y_{j,i}$. $A_{j,i}$ is the yield at the beginning of the period, $Y_{j,i-1}$.

$$A_{j,i} = Y_{j,i-1}.$$

$$R_{j,i} = A_{j,i} - (D-1)\Delta Y_{j,i}.$$

The average N-year return associated with path j, denoted $\overline{R_j}$, is equal to the average accrual, $\overline{A_j}$, minus ($D-1$) multiplied by the average annual yield change, $\overline{\Delta Y_j}$.

$$\overline{R_j} = \overline{A_j} - (D-1)\overline{\Delta Y_j}. \tag{A5}$$

$$\begin{aligned} \overline{A_j} &= \frac{1}{N}\sum_{i=1}^{N} A_{j,i} \\ &= \frac{1}{N}\sum_{i=1}^{N} Y_{j,i-1}. \end{aligned} \tag{A6}$$

$$\overline{\Delta Y_j} = \frac{1}{N}\sum_{i=1}^{N} \Delta Y_{j,i}. \tag{A7}$$

The summation on the right side of Equation A6 is "calculated" by substituting the expressions from Equations A1–A4.

$$\begin{aligned} \overline{A_j} &= \frac{1}{N}\begin{bmatrix} NY_0 + (N-1)\Delta Y_{j,1} \\ +(N-2)\Delta Y_{j,2} + \cdots + \Delta Y_{j,N-1} \end{bmatrix} \\ &= Y_0 + \frac{1}{N}\begin{bmatrix} (N-1)\Delta Y_{j,1} + (N-2)\Delta Y_{j,2} \\ + \cdots + \Delta Y_{j,N-1} \end{bmatrix}. \end{aligned} \tag{A8}$$

The average path return is found by substituting Equation A8 in Equation A5.

$$\overline{R_j} = Y_0 + \frac{1}{N}\begin{bmatrix} (N-1)\Delta Y_{j,1} \\ +(N-2)\Delta Y_{j,2} \\ + \cdots + \Delta Y_{j,N-1} \end{bmatrix} - (D-1)\overline{\Delta Y_j}. \tag{A9}$$

Trendline Returns

When the path from Y_0 to any final yield $Y_{j,N}$ is a trendline, yields change in N equal increments of the average yield change, $\overline{\Delta Y_j}$. In Equation A8, when we replace $\Delta Y_{j,i}$ with $\overline{\Delta Y_j}$ for all i, Equation A8 takes a much simpler form:

$$\begin{aligned} \overline{A_{TL,j}} &= Y_0 + \frac{1}{N}\begin{bmatrix} (N-1)\overline{\Delta Y_j} \\ +(N-2)\overline{\Delta Y_j} + \cdots + \overline{\Delta Y_j} \end{bmatrix} \\ &= Y_0 + \frac{1}{N}\left[(N-1) + (N-2) + \cdots + 1\right]\overline{\Delta Y_j} \\ &= Y_0 + \frac{1}{N}\frac{(N-1)N}{2}\overline{\Delta Y_j} \\ &= Y_0 + \frac{(N-1)}{2}\overline{\Delta Y_j}. \\ \overline{R_{TL,j}} &= \overline{A_{TL,j}} - (D-1)\overline{\Delta Y_j} \\ &= Y_0 + \frac{(N-1)}{2}\overline{\Delta Y_j} - (D-1)\overline{\Delta Y_j}. \end{aligned}$$

This expression can be written more compactly so that the average return can be expressed in terms of the total yield change, $N\overline{\Delta Y_j}$.

$$\begin{aligned} \overline{R_{TL,j}} &= Y_0 - \left[(D-1) - \frac{(N-1)}{2}\right]\overline{\Delta Y_j} \\ &= Y_0 - \left(\frac{D-1}{N} - \frac{N-1}{2N}\right)N\overline{\Delta Y_j}. \end{aligned} \tag{A10}$$

The quantity in parentheses in Equation A10 can be interpreted as a "trendline duration" that incorporates both accrual and price effects. D_{TL} is a measure of the sensitivity of trendline returns to the total yield change, $N\overline{\Delta Y_j}$. The average annual return can, in turn, be expressed more compactly in terms of the initial yield, total yield change, and trendline duration:

$$D_{TL} = \frac{D-1}{N} - \frac{N-1}{2N}. \tag{A11}$$

$$\overline{R_{TL,j}} = Y_0 - D_{TL}N\overline{\Delta Y_j}. \tag{A12}$$

We also note that D_{TL} can be written more compactly:

$$D_{TL} = \frac{D}{N} - \frac{(N+1)}{2N}. \tag{A13}$$

From Equation A13, it is clear that $D_{TL} = 0$ when $N = 2D - 1$. We refer to $2D - 1$ as the "effective maturity" of the DT process. Over that holding period, the average trendline return equals the initial yield, regardless of the interim yield changes, and the volatility of trendline returns is zero.

Looking again at Equation A11, we observe that the trendline duration incorporates both a duration factor and an accrual factor that reflect the separate duration and accrual impacts of the total yield change.

$$\frac{D-1}{N} = \text{Duration factor.} \tag{A14}$$

$$\frac{N-1}{2N} = \text{Accrual factor.} \tag{A15}$$

Duration effect = Duration factor × Total yield change.

Accrual effect = Accrual factor × Total yield change.

Note that as N increases, the duration factor decreases and ultimately approaches zero as $N \to \infty$. In contrast, the accrual factor approaches 1/2 as $N \to \infty$. Therefore, the trendline duration also approaches 1/2 as $N \to \infty$.

Trendline Volatility

Yield changes $\Delta Y_{j,i}$, for $i = 1, 2, \ldots, N$, are assumed to be independent, and their distributions have a mean of zero and a standard deviation of $\sigma_{\Delta Y}$. Consequently, the mean of the average yield changes $\overline{\Delta Y_j}$ across all paths is also zero. From Equation A12, it follows that μ_{TL}, the mean of annualized N-period trendline returns across all paths, equals Y_0. The standard deviation of annualized trendline returns, σ_{TL}, turns out to be proportional to the underlying yield change volatility, $\sigma_{\Delta Y}$, multiplied by $\sqrt{N}$. The constant of proportionality is $|D_{TL}|$.

$$\mu_{TL} = Y_0.$$

To calculate σ_{TL}, we first compute the variance (i.e., the expected value of the square of $\left[\overline{R_{TL,j}} - Y_0\right]$):

$$\begin{aligned}
(\sigma_{TL})^2 &= \mathrm{E}\left[\left(D_{TL} N \overline{\Delta Y_j}\right)^2\right] \\
&= (D_{TL})^2 \bullet \mathrm{E}\left[\left(N \overline{\Delta Y_j}\right)^2\right] \\
&= (D_{TL})^2 \bullet \mathrm{E}\left[\left(\sum_{i=1}^{N} \Delta Y_{j,i}\right)^2\right] \\
&= (D_{TL})^2 \bullet \mathrm{E}\left[\begin{array}{l} \sum_{i=1}^{N} \left(\Delta Y_{j,i}\right)^2 \\ + \text{Cross terms} \end{array}\right].
\end{aligned}$$

The cross terms are similar to $\Delta Y_{j,1} \bullet \Delta Y_{j,2}$. Because yield changes are assumed to be independent, the expected value of such terms is zero:

$$\begin{aligned}
(\sigma_{TL})^2 &= (D_{TL})^2 \bullet \left[\begin{array}{l} N(\sigma_{\Delta Y})^2 \\ + \mathrm{E}(\text{Cross terms}) \end{array}\right] \\
&= (D_{TL})^2 \bullet N(\sigma_{\Delta Y})^2. \\
\sigma_{TL} &= |D_{TL}| \bullet \sqrt{N}\sigma_{\Delta Y}.
\end{aligned} \tag{A16}$$

Total Volatility

Returning to Equation A9, we focus on the total volatility of returns, $\overline{R_j}$, across all paths derived from a random walk model of yield changes over an N-year horizon. As with trendlines, the mean $\mu_{TOT} = Y_0$ because the distributions of yield changes $\Delta Y_{j,i}$ are assumed to be independent with a zero mean. There is no loss of generality in this zero-mean assumption because the analysis can easily be extended to include rate drift. The calculation of the total volatility, σ_{TOT}, is similar to the calculation of trendline volatility but is a bit more algebraically complicated:

$$\mu_{TOT} = Y_0.$$

$$\begin{aligned}
(\sigma_{TOT})^2 &= \mathrm{E}\left\langle \left\{ \begin{array}{l} \frac{1}{N}\left[\begin{array}{l} (N-1)\Delta Y_{j,1} \\ +(N-2)\Delta Y_{j,2} \\ +\cdots+\Delta Y_{j,N-1} \end{array}\right] \\ -(D-1)\frac{1}{N}\sum_{i=1}^{N}\Delta Y_{j,i} \end{array} \right\}^2 \right\rangle \\
&= \mathrm{E}\left\{ \begin{array}{l} \frac{1}{N^2}\left[\sum_{i=1}^{N-1}(N-i)^2\left(\Delta Y_{j,i}\right)^2\right] \\ -\frac{2(D-1)}{N}\frac{1}{N}\sum_{i=1}^{N}(N-i)\left(\Delta Y_{j,i}\right)^2 \\ +\left[(D-1)\overline{\Delta Y_j}\right]^2 + \text{Cross terms} \end{array} \right\} \\
&= \frac{1}{N^2}\left\{ \begin{array}{l} \sum_{i=1}^{N-1}(N-i)^2\,\mathrm{E}\left[\left(\Delta Y_{j,i}\right)^2\right] \\ -2(D-1)\sum_{i=1}^{N-1}(N-i)\,\mathrm{E}\left[\left(\Delta Y_{j,i}\right)^2\right] \\ +(D-1)^2\,\mathrm{E}\left[\left(\overline{\Delta Y_j}\right)^2\right] \\ +\mathrm{E}(\text{Cross terms}) \end{array} \right\}.
\end{aligned}$$

The assumed independence of yield changes implies that the expected value of cross terms is zero. Also, we noted earlier that $\mathrm{E}\left[\left(\Delta Y_{j,i}\right)^2\right] = (\sigma_{\Delta Y})^2$ for all i

and $\mathrm{E}\left[\left(\overline{\Delta Y_j}\right)^2\right] = N\left(\sigma_{\Delta Y}\right)^2$. The previous equation can thus be simplified as follows:

$$\left(\sigma_{TOT}\right)^2 = \frac{\left(\sigma_{\Delta Y}\right)^2}{N^2}\begin{bmatrix} \sum_{i=1}^{N-1}\left(N-i\right)^2 \\ -2\left(D-1\right)\sum_{i=1}^{N-1}\left(N-i\right) \\ +\left(D-1\right)^2 N \end{bmatrix}$$

$$= \frac{\left(\sigma_{\Delta Y}\right)^2}{N^2}\begin{bmatrix} \frac{\left(N-1\right)\left(N\right)\left(2N-1\right)}{6} \\ -2\left(D-1\right)\bullet\frac{\left(N-1\right)N}{2} \\ +\left(D-1\right)^2 N \end{bmatrix}$$

$$= \frac{\left(\sigma_{\Delta Y}\right)^2}{N}\begin{Bmatrix} \frac{\left(N-1\right)\left(2N-1\right)}{6} \\ +\left(D-1\right)^2 \\ -2\left(D-1\right)\bullet\frac{\left(N-1\right)}{2} \\ +\left[\frac{\left(N-1\right)}{2}\right]^2-\left[\frac{\left(N-1\right)}{2}\right]^2 \end{Bmatrix}$$

$$= \frac{\left(\sigma_{\Delta Y}\right)^2}{N}\begin{Bmatrix} \frac{\left(N-1\right)\left(2N-1\right)}{6} \\ +\left[\left(D-1\right)-\frac{\left(N-1\right)}{2}\right]^2 \\ -\left[\frac{\left(N-1\right)}{2}\right]^2 \end{Bmatrix}$$

$$= \left(\sigma_{\Delta Y}\right)^2\begin{bmatrix} \frac{\left(N-1\right)}{2N}\left(\frac{2N-1}{3}-\frac{N-1}{2}\right) \\ +N\left(\frac{D-1}{N}-\frac{N-1}{2N}\right)^2 \end{bmatrix}.$$

The last term in the brackets incorporates the trendline duration, D_{TL}. After further simplification of the first term in the brackets, the equation becomes

$$\begin{aligned}\left(\sigma_{TOT}\right)^2 &= \left(\sigma_{\Delta Y}\right)^2\left[\frac{\left(N^2-1\right)}{12N}+N\left(D_{TL}\right)^2\right] \\ &= \left(\sigma_{\Delta Y}\right)^2 N\left(D_{TL}\right)^2 \\ &\quad +\left(\sigma_{\Delta Y}\right)^2 N\frac{\left(N^2-1\right)}{12N^2}.\end{aligned} \tag{A17}$$

$$\sigma_{TOT} = \sigma_{\Delta Y}\sqrt{N}\sqrt{\left(D_{TL}\right)^2+\frac{\left(N^2-1\right)}{12N^2}}.$$

The first term on the right side of Equation A17 is the trendline variance derived in the previous section. The second variance term reflects the additional volatility associated with deviations of individual paths from their associated trendline. These additional deviations can be interpreted as tracking error.

Tracking Error

We can view $\overline{R_j}$, the annualized return across any path j, as comprising the trendline return, $\overline{R_{TL,j}}$, plus an uncorrelated residual return, $\overline{r_j}$. The residual return is simply the difference between the annualized path return and the annualized trendline return. The volatility of residual (or excess) returns across all random paths is the tracking error (TE).

Specifically, we define TE as the square root of the average of the squared residuals. Because the means of both $\overline{R_j}$ and $\overline{R_{TL,j}}$ equal Y_0, the mean residual must be zero. In this case, TE is the same as the standard deviation of the residuals. Because the residuals are assumed to be uncorrelated, the total path volatility can be decomposed into the sum of the squares of the trendline volatility and the residual volatility. This relationship can, in turn, be used to derive a formula for TE.

$$\overline{R_j} = \overline{R_{TL,j}}+\overline{r_j}.$$

$$\left(\sigma_{TOT}\right)^2 = \left(\sigma_{TL}\right)^2+\left(TE\right)^2.$$

$$\left(TE\right)^2 = \left(\sigma_{TOT}\right)^2-\left(\sigma_{TL}\right)^2.$$

From Equation A17, it follows that

$$\left(TE\right)^2 = \left(\sigma_{\Delta Y}\right)^2 N\frac{\left(N^2-1\right)}{12N^2}.$$

$$\begin{aligned}TE &= \frac{1}{\sqrt{12}}\sqrt{N}\sigma_{\Delta Y}\sqrt{1-\frac{1}{N^2}} \\ &= 0.29\sqrt{N}\sigma_{\Delta Y}\sqrt{1-\frac{1}{N^2}}.\end{aligned} \tag{A18}$$

$$\begin{aligned}TE &\approx 0.29\sqrt{N}\sigma_{\Delta Y}\left(1-\frac{1}{2N^2}\right) \\ &\approx 0.29\sqrt{N}\sigma_{\Delta Y} \text{ for } N>3.\end{aligned} \tag{A19}$$

This simple approximation implies that TE is proportional to the rate change volatility and increases with $\sqrt{N}$.

A Simple Approximation for Zero-Coupon Bond Returns

To develop simple formulas for DT bond returns, we use a linear model of annual bond returns. Specifically, we show that for zero-coupon bonds, modest yield changes, and moderate investment horizons, the one-year return is approximately the initial yield reduced by (D – 1) times the actual yield change.

At the outset, the price (per dollar of maturity value), P_0, of a zero-coupon bond with duration D and yield Y_0 is found by taking the present value of 1:

$$P_0 = \left(1+Y_0\right)^{-D}.$$

At the end of one year, the time to maturity decreases by 1. If the yield increases by ΔY, the year-end price is

$$P_1 = \left[1+\left(Y_0+\Delta Y\right)\right]^{-(D-1)}.$$

The return for the first year is 1 less than the ratio of the final price to the initial price:

$$\begin{aligned} R_1 &= \frac{P_1}{P_0} - 1 \\ &= \frac{\left(1+Y_0+\Delta Y\right)^{-(D-1)}}{\left(1+Y_0\right)^{-D}} - 1 \\ &= \left(1+Y_0+\Delta Y\right)\left(\frac{1+Y_0+\Delta Y}{1+Y_0}\right)^{-D} - 1 \\ &= \left(1+Y_0+\Delta Y\right)\left(1+\frac{\Delta Y}{1+Y_0}\right)^{-D} - 1. \end{aligned}$$

The second quantity in parentheses can be replaced with the first two terms in a Taylor series expansion:

$$\begin{aligned} R_1 &= \left(1+Y_0+\Delta Y\right)\left[1 - D\frac{\Delta Y}{1+Y_0} + \frac{D(D+1)}{2}\left(\frac{\Delta Y}{1+Y_0}\right)^2 + \cdots\right] - 1 \\ &= Y_0 + \Delta Y - D\Delta Y - D\frac{\left(\Delta Y\right)^2}{1+Y_0} \\ &\quad + \frac{D(D+1)}{2}\frac{\left(\Delta Y\right)^2}{1+Y_0} \qquad \text{(A20)} \\ &\quad + \text{Higher-order terms in } \Delta Y \\ &= Y_0 - (D-1)\Delta Y + \frac{D(D-1)}{2}\frac{\left(\Delta Y\right)^2}{1+Y_0} + \cdots. \end{aligned}$$

The second term on the right represents the percentage price change that is directly attributable to a "duration" effect when the bond ages by one year and the yield changes by ΔY. The third term on the right represents a "convexity" effect. That term is always positive and thus provides an offset to the pure duration effect.

The convexity is relatively small within a reasonable range of durations and yield changes. For example, if D = 5, ΔY = 0.5%, and Y_0 = 3%, the convexity correction amounts to only 1% of the duration effect. If we ignore the convexity term, the relationship between return and yield change becomes linear:

$$R_1 \approx Y_0 - (D-1)\Delta Y. \qquad \text{(A21)}$$

Continuous Rebalancing

In the first section of this appendix, we showed that the accrual factor approaches 1/2 as N increases. Alternatively, we can view N as the rebalancing frequency. As N approaches infinity, rebalancing is continuous. We can also directly derive the accrual factor for continuous rebalancing:

A = Yield change per unit time.

$C(t)$ = Cumulative yield change to time t

$= At$.

$Y(t) = Y_0 + at$.

$A(t)$ = Cumulative accrual to time t

$$= \int_0^t \left(Y_0 + au\right) du$$

$= Y_0 t + ½at^2$.

Cumulative return(t) = $A(t) - D[Y(t) - Y_0]$

$= Y_0 t + ½at^2 - Dat$.

$R(t)$ = Annualized return to time t

= Cumulative return(t)/t.

$R(t) = Y_0 + ½at - (D/t)at$.

Net excess return(t) = $R(t) - Y_0$

$= ½at - (D/t)at$.

The term $½at$ is the average excess accrual resulting from a total yield change (at).

The accrual factor is the average excess accrual divided by the total yield change. So, for continuous rebalancing,

Accrual factor = $(½at)/(at)$

= ½.

The second term in the excess return equation is the average price return resulting from the change at. The factor D/t is the associated duration factor. As t increases, D/t approaches zero.

We can combine the duration and accrual factors into a "trendline" duration. The excess return that results from a yield change is the negative of the trendline duration times total yield change.

Duration factor = D/t.

Trendline duration = $D/t - ½$.

Net excess return(t) = −(Trendline duration)(at).

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