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Axioma Research Paper
No. 007

July 15, 2008

Real World Case Studies in Portfolio Construction Using Robust Optimization

Anthony Renshaw, PhD
Director, Applied Research

Robust portfolio optimization has been a core feature of Axioma's portfolio construction tools for years, and many of our clients use robust optimization in their portfolio construction processes to deliver higher value added. This study reports a series of real world portfolio construction case studies documenting different approaches for implementing robust portfolio optimization and their benefits. The results provide guidance for designing robust portfolio construction strategies.

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Executive Summary

All active portfolio managers are forecasters. Their various levels of skill are reflected in their respective Information Ratio (IR). The IR combines into one statistic both forecasting skill and skill at constructing efficient portfolios with the forecasted data. Grinold (1989) developed the notion of the Information Coefficient (IC) to differentiate between managers with respect to their forecasting skills. More recently, Clarke, Silva, Thorely (2002) introduced the notion of Transfer Coefficient in order to differentiate between managers with respect to their efficiency in constructing portfolios with the forecasted data. While robust optimization aims to deliver improvements along both dimensions, this paper deals with ways in which this technique can be used to improve the realized portfolio return in the presence of noisy forecasts.

Robust portfolio optimization (RPO) improves performance by mitigating the deficiencies associated with classical portfolio construction methods.[1] This is achieved by penalizing large asset bets that are likely to be based on error-prone expected return estimates (alphas). Robust portfolio optimization has been a core feature of Axioma's portfolio construction tools for years,

[1] These deficiencies include insufficient diversification, over-weighting of assets with large, unreliable expected returns, and excessive sensitivity of the optimized portfolio holdings to small changes in the portfolio construction parameters.

and many of our clients use robust optimization in their portfolio construction processes to deliver higher value added. This study reports a series of real world portfolio construction case studies documenting different approaches for implementing robust portfolio optimization and their benefits.[2] The results provide guidance for designing robust portfolio construction strategies.

High level conclusions from these case studies include:

- Given the (real-world) sample set of investment strategies examined, RPO has a demonstrable effect in improving the portfolios of those managers with a lower IR than the top quartile (i.e. IR < 0.5)
- Portfolio diversification increases as estimation error aversion is increased, and proper RPO calibration must consider both the number of names held as well as other metrics of portfolio performance.
- Since RPO reduces turnover, turnover constraints may be able to be loosened when RPO is employed. Other constraints could also potentially be loosened when RPO is used.
- RPO calibration should include a cost-to-benefit analysis using the gains achieved in the information ratio, plus the lower turnover costs over time, minus the cost of holding more names.

Intuitively the only "help" a top quartile forecaster (i.e., IR > 0.5) would need is avoiding excessive sensitivity of the optimized portfolio holdings to ongoing small changes in the forecasts. For these managers, the marginal value of robust optimization is the value of IR gain and reduced ongoing turnover, compared to the costs associated with initially holding and trading more names. For managers with an IR lower than 0.5, robust optimization not only improves by this marginal value but also helps reverse the detractions to returns from insufficient diversification and the over-weighting of assets with large, unreliable forecasted returns.

[2] Other studies have described the motivation and formulation of robust optimization as well as documenting improved portfolio performance. See, for example, Ceria and Stubbs (2006) or Longerstaey and Weed (2007).

Robust Optimization

Consider a portfolio construction strategy whose objective function maximizes the expected return of the portfolio. Transaction costs, market impact, the cost of shorting, and other quantities may also be included in the strategy's objective function. The strategy could also maximize utility by subtracting a risk aversion value times the portfolio variance from the expected return.

In robust portfolio optimization, the objective function is modified by *subtracting* the risk

$$\kappa\sqrt{w^T Q w} \qquad (1)$$

where:

- w is a vector of portfolio holdings or weights, either managed or active (managed minus benchmark);
- Q is a covariance matrix measuring alpha uncertainty, the magnitude of alpha estimation error; and
- κ is a non-negative, scalar constant.[3]

Of these three terms, the most challenging term to estimate is Q, the alpha uncertainty covariance matrix. Q is not the *return* covariance matrix, normally specified by a risk model, which measures the variance in expected returns (alphas). Instead, Q measures variance of the *error* in the alphas, not the variance of the alphas themselves.

For some alpha construction techniques, theoretical confidence regions for the alpha estimates can be determined (for example, Stubbs and Vance (2005)). For most alpha construction techniques, such estimates are difficult to obtain. Many approaches construct Q using historical alpha estimates. Unfortunately, historical alphas are often not available, and even when they are, there is frequently insufficient history to make statistically meaningful estimates of all elements of Q. Consequently, simplifying assumptions are normally made, the most common of which is that Q is diagonal.

In this study, we consider some simple and easily implementable alpha uncertainty models, all of which are diagonal:

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[3] In Axioma Portfolio™, a "maximize return" strategy that *subtracts* the robust risk correction in equation (1) can be implemented in several different ways. See Appendix A for details. "Axioma Portfolio" is a trademark of Axioma, Inc.

(1) **Constant *Q***. Q is the identity matrix; that is, the diagonal elements of Q are all one. This is the simplest possible Q.

(2) **Cross-Sectional *Q*.** The diagonal elements of Q are cross-sectional statistics of the alphas for the current time period. Consider a universe with M assets, each of whose current expected return is $\alpha_i,\ i = 1,\ldots,M$. Let $\sigma^2 = Var\{\alpha_1, \alpha_2, \ldots, \alpha_M\}$ be the variance of the current time period's alphas. Then,

$$Q_{ii} = \sigma^2 \text{ for each and every asset } i$$

All M assets have the same magnitude of alpha uncertainty for each time period, but the magnitude varies through time.[4] This approach captures the uncertainty of the alpha generation process rather than uncertainty of each asset's alpha since time periods with a wide dispersion of alpha views across assets are penalized more than time periods with less alpha dispersion. No historical data is required to estimate Q in this method.

(3) **Time Series *Q*.** The diagonal elements of Q are time series statistics of the current and historical expected asset returns and realized asset returns. Let $\alpha_i(t_j)$ be the expected returns for a universe of M assets ($i = 1,\ldots,M$) that includes the current time period ($j = 0$) as well as P historical time periods ($j = -1, -2, \ldots, -P$). In addition, let $r_i(t_j)$ be the realized asset returns for the same universe and time history.[5] The diagonal elements of Q are:

$$Q_{ii} = Var\{\alpha_i(t_{-1}) - r_i(t_{-1}),\ \alpha_i(t_{-2}) - r_i(t_{-2}), \ldots, \alpha_i(t_{-P}) - r_i(t_{-P})\}$$

The variance is computed over index j (the historical time periods), with index i fixed. This produces a different variance for each asset that captures the variance of the difference between the alpha estimates and the realized returns.

Appendix A gives a detailed, numerical example constructing Q for each of the above formulas.

Each of these estimation methods for Q can be extended to incorporate more granularity or additional knowledge of an alpha generation process. For example, assets can be classified by industry sectors, and individual cross-sectional or time series estimates can be formed for each sector and, if desired, sector-sector correlations can be included in the off-diagonal of Q.

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[4] In Axioma Portfolio, diagonal alpha uncertainty matrices are specified in terms of risk, i.e., $\sqrt{Q_{ii}}$, not variance, Q_{ii} . See Appendix A.

[5] The realized returns should be in the same units as the expected returns, e.g., monthly, annual, etc.

Knowledge of the alpha generation process can provide insight into which alphas are most accurate, and the corresponding elements of Q can be adjusted accordingly. Of course, additional granularity in Q requires additional data processing effort.

Once Q has been specified, w and κ must be set. Both managed and active w can be considered and used in robust optimization, regardless of whether a portfolio construction strategy is benchmark relative. The choice of weight type for the robust risk correction term in equation (1) does not restrict the other terms in the objective function. In the case studies that follow, both active and managed weights are tested for the robust correction even though all strategies constrain active risk (i.e., are benchmark relative).

When managed weights are used, we denote this as *Absolute robust portfolio optimization*, or *Absolute RPO*. When active weights are used, we denote this as *Benchmark Relative* or *Relative RPO.*

When $\kappa = 0$, robust optimization is not incorporated in the optimization strategy. [6] As κ increases, the alpha estimation error increases, and less aggressive portfolios are produced since the alphas are considered less reliable. For any portfolio construction process, κ must be calibrated, typically by performing a series of backtests. Reasonable values of κ can normally be found with a handful of backtests. In the tests performed here, we calibrate κ by maximizing the Information Ratio. Other criteria such as risk-adjusted return (Sharpe Ratio), returns adjusted for transaction costs, and Transfer Coefficient can also be used.[7]

Case Studies

Axioma obtained real world backtest data including historical alphas and portfolio construction parameters (e.g., asset bounds, tracking errors, etc.) from nine portfolio management teams. The backtests all use monthly rebalancing for time windows between 1995 and 2006. The backtests

[6] Although mathematically identical, the backtest results for the two cases (a), $\kappa = 0$ with the robust correction term included in the objective function; and (b), no robust correction term in the objective function may not be identical. This occurs if any of the rebalancings possess non-unique solutions. For example, when the risk constraint is not binding and the alphas of two assets are identical, the optimal solution may be unable to distinguish these two assets. In this case, the optimizer may return different backtest solutions for the two objective functions, leading to slightly different backtest results.

[7] Transfer Coefficient statistics will be available in the next release of Axioma Portfolio.

range from a minimum of 58 monthly rebalancings to a maximum of 142 monthly rebalancings. Table 1 provides a high-level description of each of these data sets.

Case	Bench-mark	Target Tracking Error	Active Asset Bound (+/- %)	Active Sector Bound (+/- %)	Active Industry Bound (+/- %)	Two-Sided Turnover (%)
A	R1000V	5%	2.5%	4%	4%	10%
B	R1000	4%	Per Asset	3%	2%	16%
C	SP500	4%	Per Asset	3%	2%	16%
D	SP500	5%	2.5%	4%	4%	10%
E	R2000V	5%	1.5%	6%	3%	15%
F	SP500	2%	2%			
G	R1000	8.4%	Per Asset	5%		33%
H	R1000	8%	1%		10%	40%
I	SP500	2.75%	1.75%			30%

Table 1. Summary description of the case study data sets.

All test cases use benchmark relative portfolio construction strategies, with a targeted level of active risk (tracking error) set as a constraint. All but Case G are long-only portfolios. Case G is a 120/20 strategy. The Two-Sided Turnover constraints are monthly turnover limits. Table 1 does not list the complete portfolio construction strategy. Several cases impose additional constraints on active style bets, portfolio beta, and asset level trade limits. In addition, all constraints except the tracking error and budget constraints were placed in Axioma Portfolio's Constraint Hierarchy[8]. This ensures that the best possible solution to the portfolio construction problem was used during the backtest even if no solution exists that satisfies all constraints.

The baseline performance of these data sets *without* robust optimization is shown in Table 2.

[8] Axioma Portfolio's Constraint Hierarchy is an automated approach to softening portfolio construction constraints to obtain a solution when the base strategy is infeasible. See the White Paper "Using Soft Constraints in PortfolioPrecision."

Case	Ann. Port. Return	Ann. Port. Real. Risk	Sharpe Ratio	Ann. Active Return	Ann. Active Real. Risk	Info. Ratio	Ave. Univ. Size	Ave. Names Held	Ave. Turn-over
A	7.5%	13.5%	0.55	0.6%	4.1%	0.15	1061	139	11%
B	0.0%	16.8%	0.00	3.3%	4.7%	0.70	2768	573	16%
C	-0.5%	16.6%	-0.03	3.2%	4.5%	0.72	2694	119	16%
D	6.0%	13.8%	0.43	3.1%	4.3%	0.72	1061	136	12%
E	22.4%	14.1%	1.59	7.2%	4.7%	1.53	1256	105	15%
F	15.0%	14.7%	1.02	3.9%	2.5%	1.61	500	172	56%
G	24.3%	22.8%	1.07	15.3%	9.3%	1.65	981	117	30%
H	25.7%	19.4%	1.32	14.1%	7.9%	1.79	1129	82	40%
I	20.0%	15.7%	1.27	9.1%	3.1%	2.97	499	124	30%

Table 2. Baseline case study performance without RPO.

The Sharpe Ratio shown is the ratio of the annualized portfolio return divided by the annualized, realized risk. The Information Ratio is annualized active return (the difference of the annualized portfolio and benchmark returns) divided by annualized, realized active risk. For Case B, the average number of names held was 573. This was driven by tight asset-level holding limits.

The cases have been ordered from the lowest to highest Information Ratio. In order to preserve the real-world character of these backtests, no attempt was made to modify the baseline portfolio construction strategy or the time window of the backtest. The only change that was made was the inclusion of the robust portfolio correction term in the objective function.

Robust Optimization Results

For each case study, we use robust optimization to maximize the Information Ratio. We consider six different robust formulations:

(1) *Absolute RPO* using a *constant Q* (plotted with solid blue lines in Figures).
(2) *Absolute RPO* using a *cross-sectional Q* (dashed blue lines).
(3) *Absolute RPO* using a *time series Q* with P = 12 (dash-dot blue lines).
(4) *Benchmark Relative RPO* using a *constant Q* (solid brown lines).
(5) *Benchmark Relative RPO* using a *cross-sectional Q* (dashed brown lines).
(6) *Benchmark Relative RPO* using a *time series Q* with P = 12 (dash-dot brown lines).

Figures 1 to 4 show calibration results for four of the nine cases, starting with the lowest baseline Information Ratio. Figure 1 shows the results for Case A with a baseline Information Ratio of 0.15.

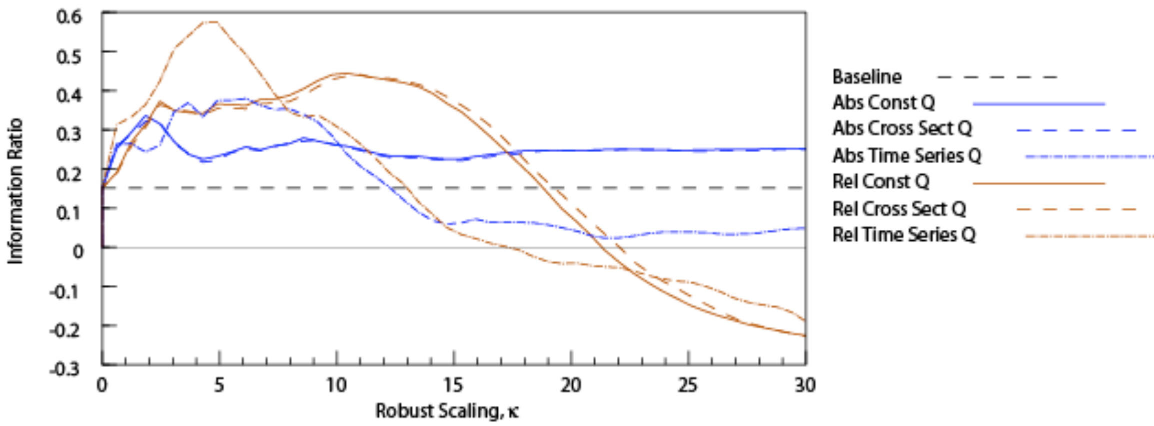

Figure 1. Robust portfolio optimization results for **Case A** with a baseline Information Ratio of 0.15 (dashed, black line). Blue = Absolute; Brown = Relative. Solid lines = constant Q; dashed lines = cross-sectional Q; dash-dot line = time series Q.

For Case A, all six methods improve the baseline Information Ratio when properly calibrated. The maximum increase is achieved using Relative RPO and the time series Q, giving an Information Ratio of 0.58 when $\kappa = 4.9$. The other methods increase Information Ratio to values between 0.32 and 0.44. The results using Absolute RPO and constant and cross-sectional Q are almost indistinguishable, and reach an asymptotic value of 0.25 for large κ.

In Figure 2, we show Case B with an initial Information Ratio of 0.70. For this case, Absolute RPO consistently works well.

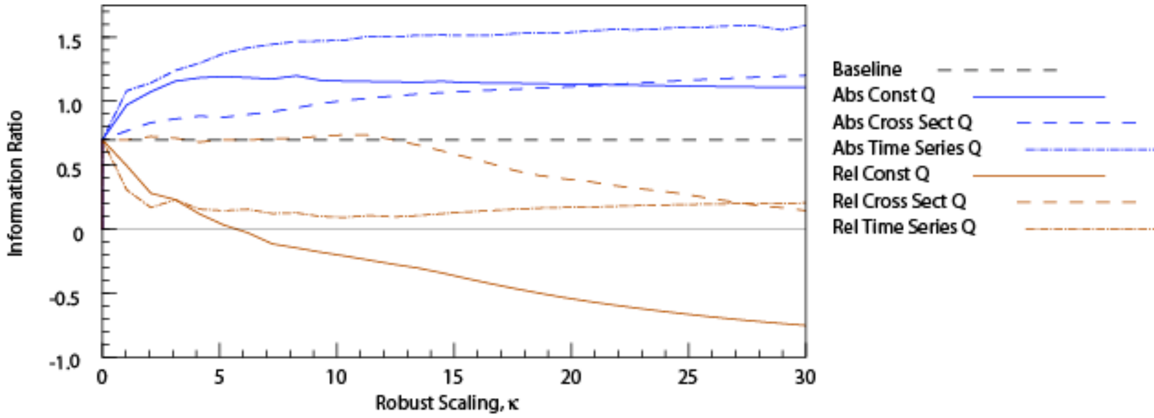

Figure 2. Robust portfolio optimization results for **Case B** with a baseline Information Ratio of 0.70.

In Figure 3, we show Case E with an initial Information Ratio of 1.53. For this case, there are regions in which Absolute RPO with constant and cross-sectional Q modestly increases the Information Ratio, but these regions are not unique. Cross-sectional Relative RPO also has a region of κ in which the Information Ratio is modestly improved.

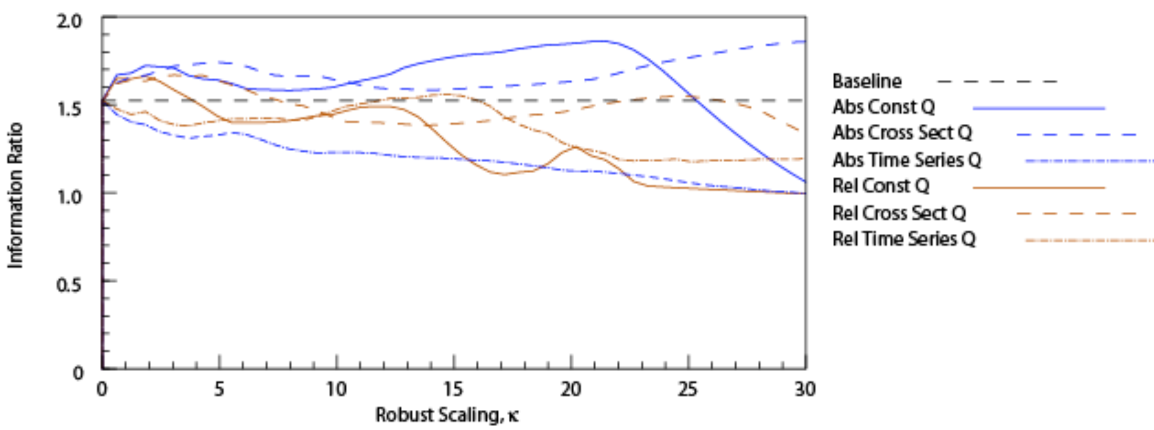

Figure 3. Robust portfolio optimization results for **Case E** with a baseline Information Ratio of 1.53.

Finally, in Figure 4, we show results for Case I, the case with the largest baseline Information Ratio of 2.97. Two of the Relative RPO cases (constant and cross-sectional) have narrow regions of κ in which the Information Ratio increases slightly, but both of these are followed by a steep decline in Information Ratio as κ increases. All of the methods using Absolute RPO reduce Information Ratio.

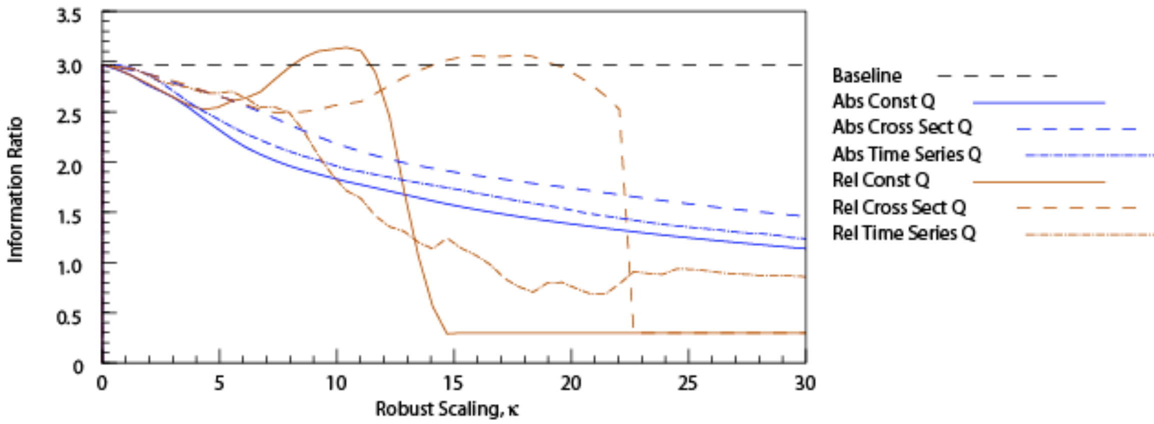

Figure 4. Robust portfolio optimization results for **Case I** with a baseline Information Ratio of 2.97.

Figures 5 and 6 show the number of assets held as a function of the κ for Cases A and I.

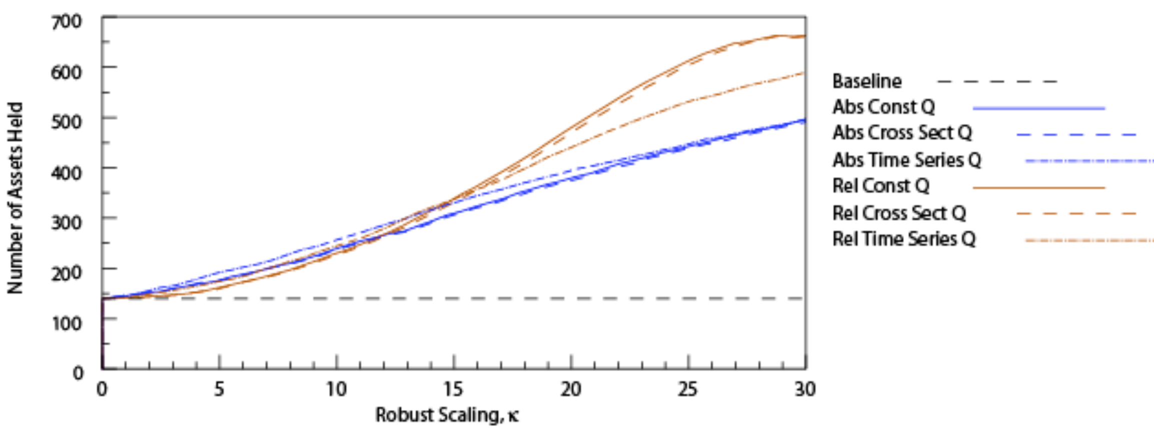

Figure 5. Average number of assets held as a function of κ for **Case A**. The benchmark is the Russell 1000 Value.

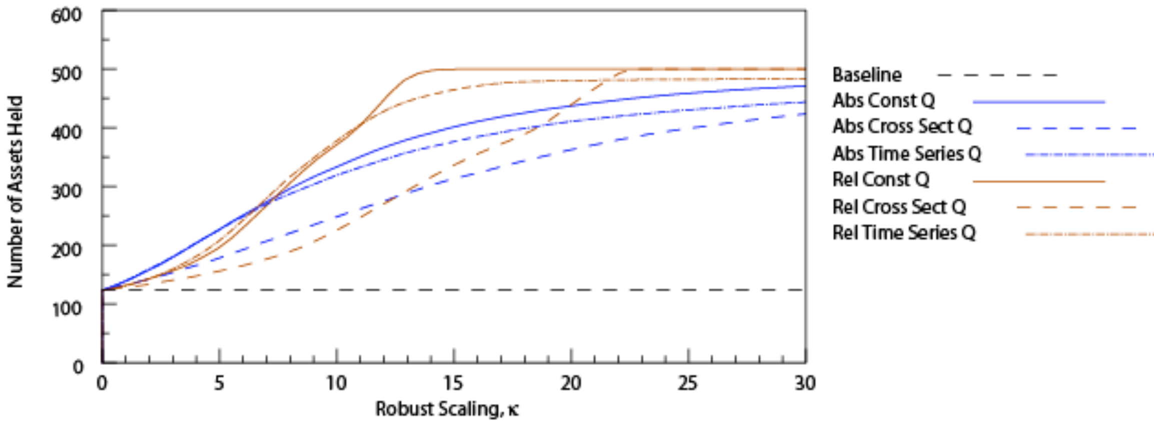

Figure 6. Average number of assets held as a function of κ for **Case I**.
The benchmark is the S & P 500.

The graphs show that robust portfolio optimization increases the number of assets held. It is important to properly calibrate κ in order to keep the number of names held within the desired range or add a constraint to the portfolio construction strategy to limit the maximum number of names held.

Figure 7 shows the average two-sided, per-period turnover for Case F, the only case with no constraint on turnover in the portfolio construction strategy.

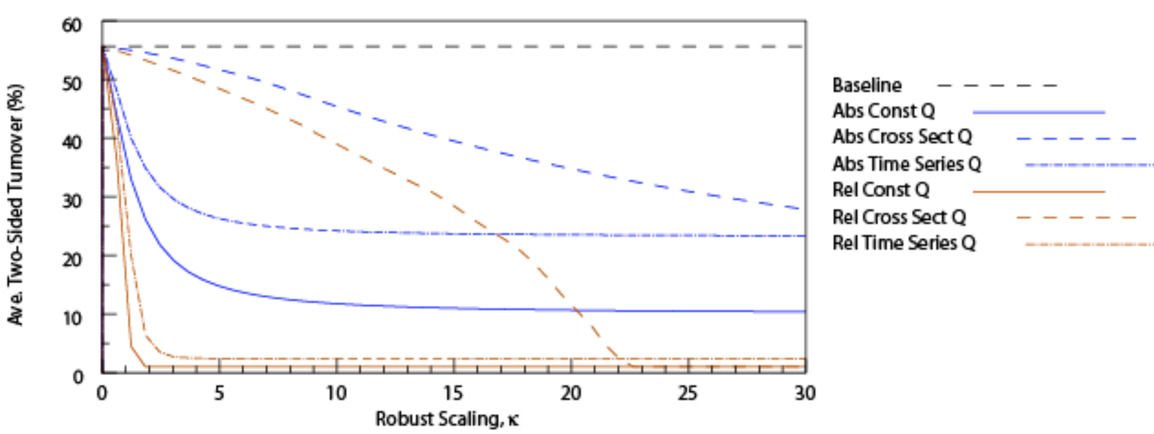

Figure 7. Average, two-sided, per-period turnover as a function of κ for **Case F**, the only case with no turnover constraint in the portfolio construction strategy.

Robust optimization decreases turnover, but different methods decrease turnover at different rates. In the other cases, the turnover constraint is binding in all periods both with and without RPO.

Table 3 summarizes the maximum increase in Information Ratio for each case and RPO method.

		Absolute RPO			Bench. Relative RPO		
Case	Base. Info. Ratio	Const. Q	Cross-Sect. Q	Time Series Q	Const. Q	Cross-Sect. Q	Time Series Q
A	0.15	121%	111%	150%	192%	190%	278%
B	0.70	72%	72%	128%	0%	6%	0%
C	0.72	42%	43%	53%	0%	2%	0%
D	0.72	86%	86%	58%	71%	71%	57%
E	1.53	22%	22%	0%	9%	10%	2%
F	1.61	0%	10%	0%	0%	3%	0%
G	1.65	0%	12%	0%	0%	9%	0%
H	1.79	0%	10%	0%	0%	12%	0%
I	2.97	0%	0%	0%	6%	3%	0%
Cases > 10%		5	8	4	2	4	2

Table 3. Maximum increase in Information Ratio for each case study and RPO method. The summary line at the bottom indicates the total number of cases in which the Information Ratio increased by at least 10%.

The most success was had by the Absolute RPO using Cross-Sectional Q, which increased the Information Ratio by at least 10% in eight of the nine test cases. These tests suggest that Absolute RPO using Cross-Sectional Q should be included in the methods considered when first experimenting with RPO.

Information Ratio increases can be driven by changes in both the active return and the active risk. Table 4 shows the relative changes in annual, realized active return while Table 5 shows the relative changes in annual, realized active risk for the increased Information Ratios shown in Table 3.

	Absolute RPO			Bench. Relative RPO		
Case	Const. Q	Cross-Sect. Q	Time Series Q	Const. Q	Cross-Sect. Q	Time Series Q
A	119%	110%	119%	134%	128%	219%
B	79%	84%	100%	0%	-6%	0%
C	63%	67%	41%	0%	9%	0%
D	28%	28%	18%	10%	5%	44%
E	-30%	-13%	0%	7%	8%	-45%
F	0%	11%	0%	0%	7%	0%
G	0%	12%	0%	0%	5%	0%
H	0%	14%	0%	0%	14%	0%
I	0%	0%	1%	-61%	-62%	0%

Table 4. Relative change in annual, realized active return corresponding to the increased Information Ratios shown in Table 3.

	Absolute RPO			Bench. Relative RPO		
Case	Const. Q	Cross-Sect. Q	Time Series Q	Const. Q	Cross-Sect. Q	Time Series Q
A	-1%	-1%	-12%	-20%	-21%	-16%
B	4%	7%	-12%	0%	-11%	0%
C	15%	16%	-8%	0%	7%	0%
D	-31%	-31%	-25%	-36%	-38%	-8%
E	-42%	-29%	0%	-2%	-1%	-46%
F	0%	1%	0%	0%	3%	0%
G	0%	-1%	0%	0%	-4%	0%
H	0%	4%	0%	0%	2%	0%
I	0%	0%	0%	-63%	-63%	0%

Table 5. Relative change in annual, realized active risk corresponding to the increased Information Ratio shown in Table 3.

Tables 4 and 5 show that RPO generally increases active realized return and decreases active realized risk. There are exceptions, of course, as indicated in the Tables, but RPO often improves performance in both risk and return.

As indicated in the Figures, the optimal value of κ varies from one method to another, and across different cases. Table 6 shows the relative increases in Information Ratio and names held for the case of Absolute RPO with cross-sectional Q for three values of κ: $\kappa = 1$, $\kappa = 5$, and the value of κ corresponding to the Information Ratio shown in Table 3. The maximum value of κ tested was 30, which is reported as optimal in three cases.

				Info. Ratio Increase			Names Held Increase		
Case	Base. Info. Ratio	Base. Names Held	Opt. Value of κ	κ = 1	κ = 5	Opt. κ	κ = 1	κ = 5	Opt. κ
A	0.15	139.5	1.8	91%	48%	111%	6%	25%	7%
B	0.70	573.0	30.0	10%	25%	72%	3%	13%	59%
C	0.72	119.4	15.0	9%	25%	43%	1%	79%	233%
D	0.72	135.7	30.0	22%	49%	86%	4%	23%	225%
E	1.53	105.0	30.0	8%	14%	22%	9%	62%	668%
F	1.61	172.0	4.3	5%	10%	10%	6%	30%	25%
G	1.65	83.7	3.1	4%	7%	12%	31%	283%	140%
H	1.79	67.3	5.0	6%	10%	10%	18%	96%	96%
I	2.97	123.8	0.0	-2%	-10%	0%	9%	43%	0%

Table 6. Comparison of Information Ratio and number of names held for the case of Absolute RPO and Cross-Sectional Q.

Table 6 illustrates the dependence of the maximum increase in Information Ratio and the number of names held. In some cases, the value of κ that produces the largest Information Ratio produces increases in the number of names held greater than 200%. In most cases, reasonable increases in the number of names held (5 – 30%) can be achieved with significant increases in Information Ratio by properly calibrating κ.[9]

When a reliable transaction cost model is available, the marginal value of robust optimization can be computed by comparing the benefit of RPO (in terms of IR profit and reduced turnover) to the costs associated with holding and trading more names. This will give the aggregate P&L with and without RPO.

Conclusions

A number of observations can be drawn from these results, bearing in mind the limited number of test cases available.

[9] The maximum number of names held can also be constrained by the portfolio construction strategy. However, since this is a combinatorial constraint, the time required to determine a rebalanced portfolio may increase.

- Absolute RPO with cross-sectional Q was the most successful method in the limited number of case studies examined here. All other things being equal, this may be a good method to test when beginning to design an RPO strategy.
- The number of assets held increases with κ, and proper RPO calibration must consider both the number of names held as well as other metrics of portfolio performance.
- Since RPO reduces turnover, turnover constraints may be able to be loosened when RPO is employed. Other constraints could also potentially be loosened when RPO is used.
- RPO calibration should include a cost-to-benefit analysis using the gains achieved in the information ratio, plus the lower turnover costs over time, minus the cost of holding more names.

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Appendix A – A Detailed Numerical Example

Constructing *Q*. We construct Q for a universe of eight assets for 3/30/2001 for each of the three methods described in this paper. Tables 7 and 8 show the alphas (expected returns) and realized, forward monthly returns for the universe over the previous three months. Table 7 also gives the cross-sectional standard deviation of the alphas for 3/30/2001 (0.725%).

Alphas -- Monthly Return, %				
Asset ID	**3/30/2001**	**2/28/2001**	**1/31/2001**	**12/29/2000**
Asset01	-0.35	-0.26	-0.24	-0.23
Asset02	1.03	0.33	-0.14	-0.1
Asset03	-0.12	-0.22	0.1	-0.07
Asset04	1.2	-2.27	0.1	0.09
Asset05	1.65	1.42	-2.55	-0.34
Asset06	0.79	1.88	1.64	0.92
Asset07	0.87	0.31	1.33	1.09
Asset08	-0.11	-0.86	0.08	0.03
Cross Sectional Stdev	0.725			

Table 7. Alphas (expected returns) for an eight asset universe for four months. The cross-sectional, standard deviation of the alphas for 3/30/2001 is shown at the bottom.

Realized, Forward, Monthly Returns, %				
Asset ID	**3/30/2001**	**2/28/2001**	**1/31/2001**	**12/29/2000**
Asset01		1.715	1.751	-1.883
Asset02		0.000	-0.382	1.723
Asset03		-0.465	0.651	-4.341
Asset04		-1.920	-0.408	-0.451
Asset05		-1.292	-1.637	2.784
Asset06		-0.755	-3.003	0.320
Asset07		-1.920	-1.424	1.478
Asset08		-0.446	1.233	-1.031

Table 8. Realized, forward monthly asset returns.

Case 1: Constant *Q*. Q is the identity matrix. So, for the eight asset universe:

$$Q = \sqrt{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Case 2: Cross-Sectional *Q*. In this case, $\sqrt{Q_{ii}} = StDev\{\alpha_1, \alpha_2, ..., \alpha_8\}$. As shown in Table 7 for 3/30/2001, the cross-sectional standard deviation is $\sqrt{Q_{ii}} = 0.725$. In Axioma Portfolio, we specify diagonal covariance matrices in terms of risk instead of variance. Hence, for this case, we have

$$\sqrt{Q} = \begin{bmatrix} 0.725 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.725 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.725 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.725 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.725 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.725 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.725 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.725 \end{bmatrix}$$

In Axioma Portfolio, both the alpha and the diagonal of $\sqrt{Q}$ are given in the same units. In this example, the units are monthly percent return.

Case 3: Time Series *Q*. In this case, $\sqrt{Q_{ii}} = StDev\{\alpha_i(t_{-1}) - r_i(t_{-1}), \alpha_i(t_{-2}) - r_i(t_{-2}), \alpha_i(t_{-3}) - r_i(t_{-3})\}$. We only have three historical sets of alphas and realized returns. Usually, more historical time periods are used (if available) to improve the estimate of Q. First, we construct the differences between alpha and the realized returns. This is shown in Table 9. Table 10 shows the Time Series standard deviation for each asset.

Difference: Alpha Minus Realized Return (Monthly, %)				
Asset ID	**3/30/2001**	**2/28/2001**	**1/31/2001**	**12/29/2000**
Asset01		-1.975	-1.991	1.653
Asset02		0.330	0.242	-1.823
Asset03		0.245	-0.551	4.271
Asset04		-0.350	0.508	0.541
Asset05		2.712	-0.913	-3.124
Asset06		2.635	4.643	0.600
Asset07		2.230	2.754	-0.388
Asset08		-0.414	-1.153	1.061

Table 9. Historical differences between alpha and realized returns for each asset.

Time Series Stdev of (Alpha Minus Realized Return)	
Asset ID	**3/30/2001**
Asset01	2.099
Asset02	1.218
Asset03	2.585
Asset04	0.505
Asset05	2.947
Asset06	2.021
Asset07	1.683
Asset08	1.127

Table 10. Standard deviation of historical differences between alpha and realized returns

The results in Table 10 imply that

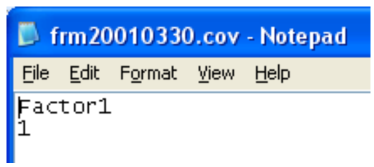
$$\sqrt{Q} = \begin{bmatrix} 2.099 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.218 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.585 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.505 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.947 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.021 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.683 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.127 \end{bmatrix}$$

The order in which the assets are listed in Q is the same as in the table.

Importing *Q* into Axioma Portfolio. There are several ways to import Q into Axioma Portfolio for use with Robust Portfolio construction. Here, we detail two methods: (1), importing Q as a risk model using the "Delimited, Factor Risk Model" tool (FRM); and (2), using the Alpha Uncertainty Model tool (AUM).

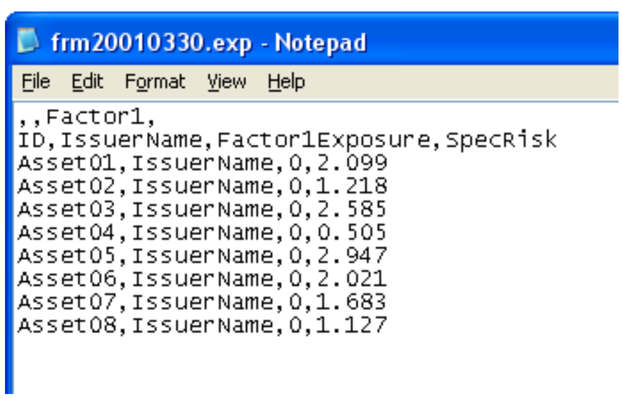
First, we describe the FRM approach. We set up two input text files to specify the risk model. The risk model has one "dummy" factor. The exposure to this factor is zero for all assets, so this factor does not contribute any risk. We then specify $\sqrt{Q}$ as specific risk.

The first input file is the covariance matrix, named frm20010330.cov in this case. This file only has two lines, the first giving the name of the factor (Factor1), the second giving its covariance (this number must be positive; see the Axioma Portfolio Reference Manual for further details.)




```
Factor1
1
```

The second file is the exposure file, frm20010330.exp, which shows Case 3, Time Series Q. The file contains two, comma separated header lines. It lists the eight assets, the string "Issuer Name" (which could be changed to something more meaningful), the exposure to Factor 1 (which must be zero), and finally, the specific risk, given in the same units as the alphas.




```
,,Factor1,
ID,IssuerName,Factor1Exposure,SpecRisk
Asset01,IssuerName,0,2.099
Asset02,IssuerName,0,1.218
Asset03,IssuerName,0,2.585
Asset04,IssuerName,0,0.505
Asset05,IssuerName,0,2.947
Asset06,IssuerName,0,2.021
Asset07,IssuerName,0,1.683
Asset08,IssuerName,0,1.127
```

To input this risk model into Axioma Portfolio, first create a workspace with these eight assets. Then, click on

Tools → Import → Risk Model → Delimited Factor Model

fill out the importer worksheet to point to the covariance and exposure files, indicate which asset maps to use, and give the risk model a name (for example, RobustFRM). This imports the risk model.

Next, we alter the strategy to incorporate the robust correction factor, $\kappa\sqrt{w^T Q w}$. We assume that a strategy already exists that maximizes expected return. That is, the strategy maximizes the objective function with the term "expectedreturn" with a coefficient or "Weight" of +1.0.

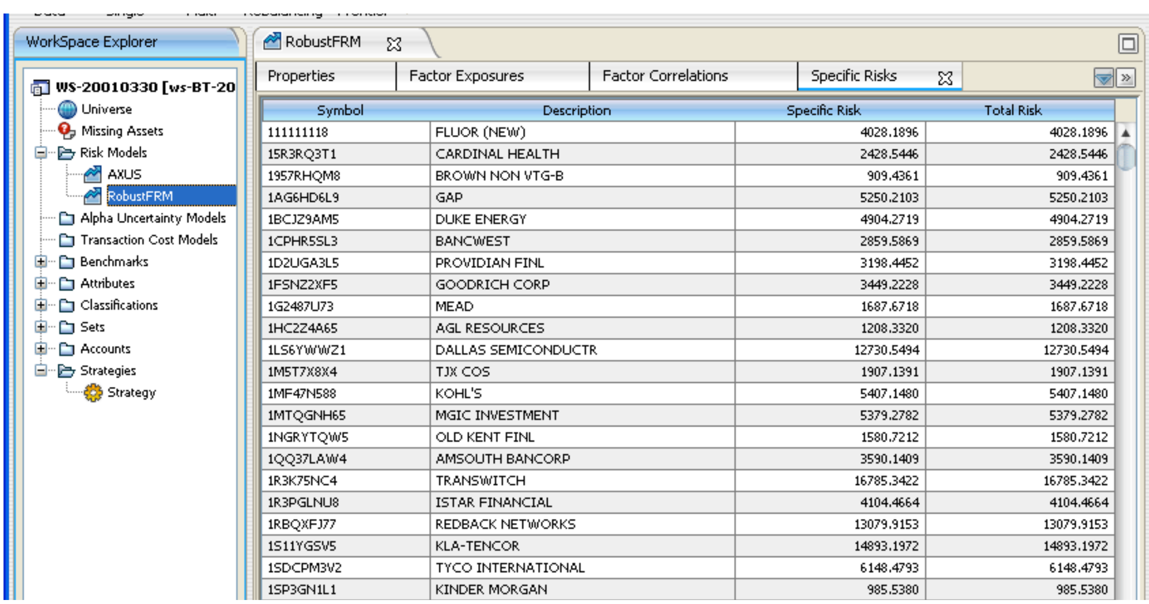
We create a new objective term, called Robust Risk, in the Objective Term editor, by clicking

New Objective Term → Risk → Standard Deviation (Risk)

For this term, we use the robust risk model (RobustFRM). If we want the robust correction to be absolute, the Benchmark should be "No Benchmark" (i.e. selecting "No Benchmark " will use Absolute RPO, meaning managed weights will be used instead of active weights). If we wish to use benchmark relative RPO, change the benchmark to the appropriate benchmark (or use REBALANCING.BENCHMARK to use the default benchmark).

Finally, add this term to the objective function. When it is added to an objective function to be maximized, the default weight should be -1.0. For maximized objective functions, this weight equals $-\kappa$ and should always be negative. To increase the magnitude of the robust correction, make this more negative. If the original objective function is set to "minimize" as in "minimize risk", then the weight for robustFRM term value should be set to $+\kappa$

The simple example above only has eight assets. In a more realistic case, the universe of assets will be larger. Figure 8 shows a screenshot taken from a real world workspace in which a factor risk model has been imported using the Time Series method for Q. If the Cross-Sectional method for Q were used, the Specific Risk and Total Risk Columns in the application would list the same value for every asset.



Symbol	Description	Specific Risk	Total Risk
111111118	FLUOR (NEW)	4028.1896	4028.1896
15R3RQ3T1	CARDINAL HEALTH	2428.5446	2428.5446
1957RHQM8	BROWN NON VTG-B	909.4361	909.4361
1AG6HD6L9	GAP	5250.2103	5250.2103
1BCJZ9AM5	DUKE ENERGY	4904.2719	4904.2719
1CPHR5SL3	BANCWEST	2859.5869	2859.5869
1D2UGA3L5	PROVIDIAN FINL	3198.4452	3198.4452
1FSNZ2XF5	GOODRICH CORP	3449.2228	3449.2228
1G2487U73	MEAD	1687.6718	1687.6718
1HC2Z4A65	AGL RESOURCES	1208.3320	1208.3320
1LS6YWWZ1	DALLAS SEMICONDUCTR	12730.5494	12730.5494
1M5T7X8X4	TJX COS	1907.1391	1907.1391
1MF47N588	KOHL'S	5407.1480	5407.1480
1MTQGNH65	MGIC INVESTMENT	5379.2782	5379.2782
1NGRYTQW5	OLD KENT FINL	1580.7212	1580.7212
1QQ37LAW4	AMSOUTH BANCORP	3590.1409	3590.1409
1R3K75NC4	TRANSWITCH	16785.3422	16785.3422
1R3PGLNU8	ISTAR FINANCIAL	4104.4664	4104.4664
1RBQXFJ77	REDBACK NETWORKS	13079.9153	13079.9153
1S11YGSV5	KLA-TENCOR	14893.1972	14893.1972
1SDCPM3V2	TYCO INTERNATIONAL	6148.4793	6148.4793
1SP3GN1L1	KINDER MORGAN	985.5380	985.5380

Figure 8. Screen shot showing a Time Series Q as the factor risk model (FRM) named RobustFRM.

In Figure 9 we show an objective function that incorporates the Robust risk term.

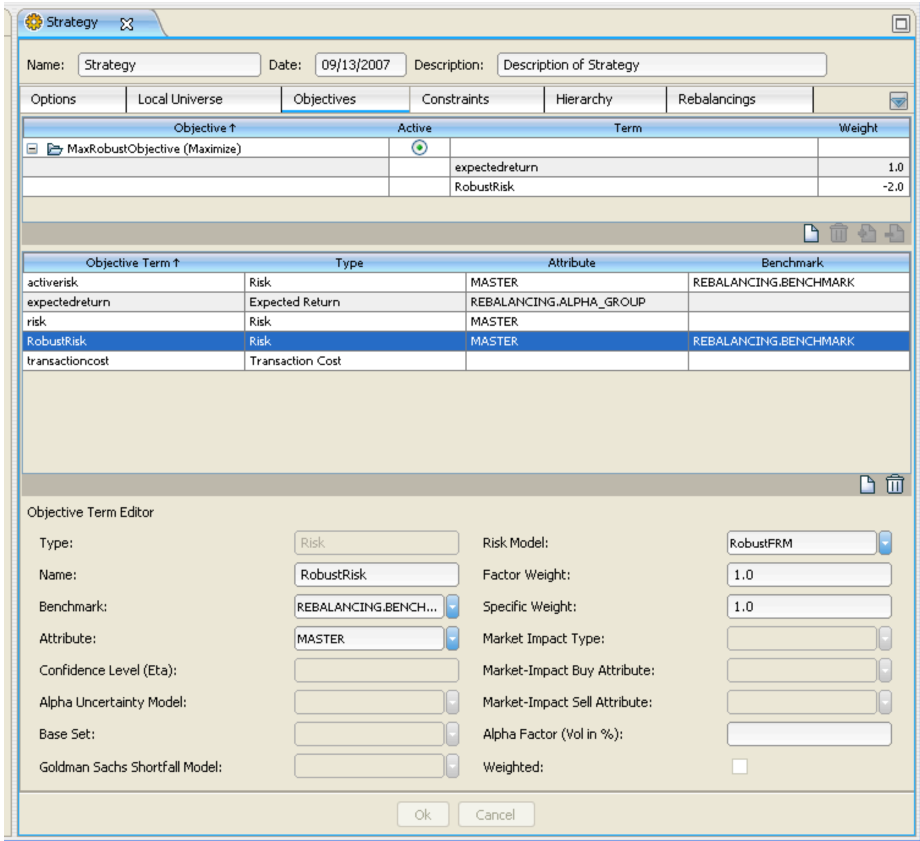


Figure 9. Screen shot for the strategy with the robust risk correction of a factor risk model (FRM). $\kappa = 2.0$ (i.e., the Weight of the RobustRisk term is -2.0).

In Axioma Portfolio, users can also import Q into the application using the Alpha Uncertainty Model (AUM) tool instead of as a factor risk model (FRM). The AUM tool allows users to specify the AUM risk $\sqrt{Q}$ at the Portfolio, Attribute and Asset levels, which can be used to construct Constant, Cross-Sectional, and Time Series Q's. In many instances, this is the quickest and easiest way to test RPO. AUM's are created by right-clicking on the Alpha Uncertainty Models node of the Workspace Explorer in the Data Perspective. Figure 10 shows an AUM model in which the uncertainty of every asset is 0.725.

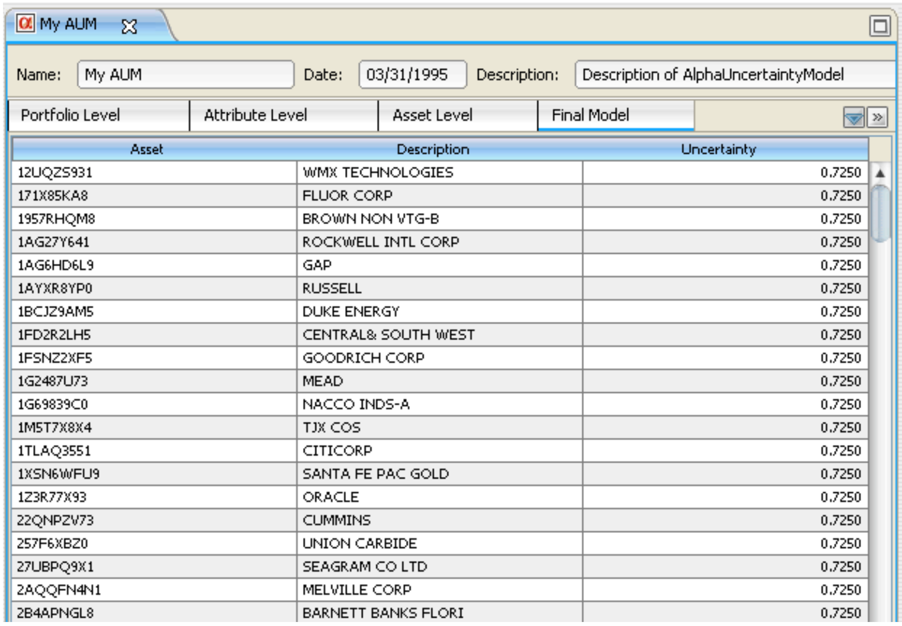


Figure 10. Screen shot showing an Alpha Uncertainty Model named "My AUM" in which the risk $\sqrt{Q}$ for every asset is 0.725.

When creating an RPO strategy using the AUM tool, the user specifies a "Confidence Level (Eta)" (η), which is a probability between 0 and 1 measuring the magnitude of the alpha uncertainty. The portfolio construction strategy maximizes the Robust Objective term:

$$\alpha^T w - \sqrt{\left(\chi_M^2\right)^{-1}(\eta)}\sqrt{w^T Q\, w} \tag{2}$$

where:

- $\alpha^T w$ is the expected return of the portfolio
- $\left(\chi_M^2\right)^{-1}(\eta)$ is the inverse, cumulative chi-squared distribution with M degrees of freedom for probability η

Comparing equations (1) and (2), the FRM and AUM formulations are identical when

$$\kappa = \sqrt{\left(\chi_M^2\right)^{-1}(\eta)} \tag{3}$$

MVO results are produced when $\eta = 0$. Table 11 gives equivalent RPO values of κ for different values of η for universe sizes of M = 50, 100, 500, 1000, 1500, and 3000.

	Universe Size (Degrees of Freedom)					
η	50	100	500	1000	1500	3000
0.0001	4.58	7.46	19.77	29.02	36.12	52.16
0.001	4.97	7.87	20.20	29.45	36.56	52.60
0.01	5.45	8.37	20.72	29.98	37.09	53.13
0.1	6.14	9.08	21.45	30.71	37.82	53.86
0.5	7.02	9.97	22.35	31.61	38.72	54.77
0.9	7.95	10.89	23.26	32.52	39.63	55.67
0.95	8.22	11.15	23.52	32.78	39.89	55.93
0.99	8.73	11.65	24.01	33.27	40.38	56.42
0.999	9.31	12.22	24.57	33.82	40.93	56.97
0.9999	9.80	12.70	25.02	34.28	41.38	57.42

Table 11. Equivalent RPO values of κ for different combinations of Eta (η) and universe size (degrees of freedom M).

To create the RPO objective term described by (2), first create an Alpha Uncertainty Model, and then, in the Objective Term editor, click

New Objective Term → Expected Return → Robust

In the Objective Term Editor, set the Attribute equal to the expected return (alpha), give it a Confidence Level (Eta), and select an Alpha Uncertainty Model to use. Then, include this term in the objective to be maximized, typically with a weight of 1.0. Figure 11 shows a sample RPO objective constructed using an AUM model.

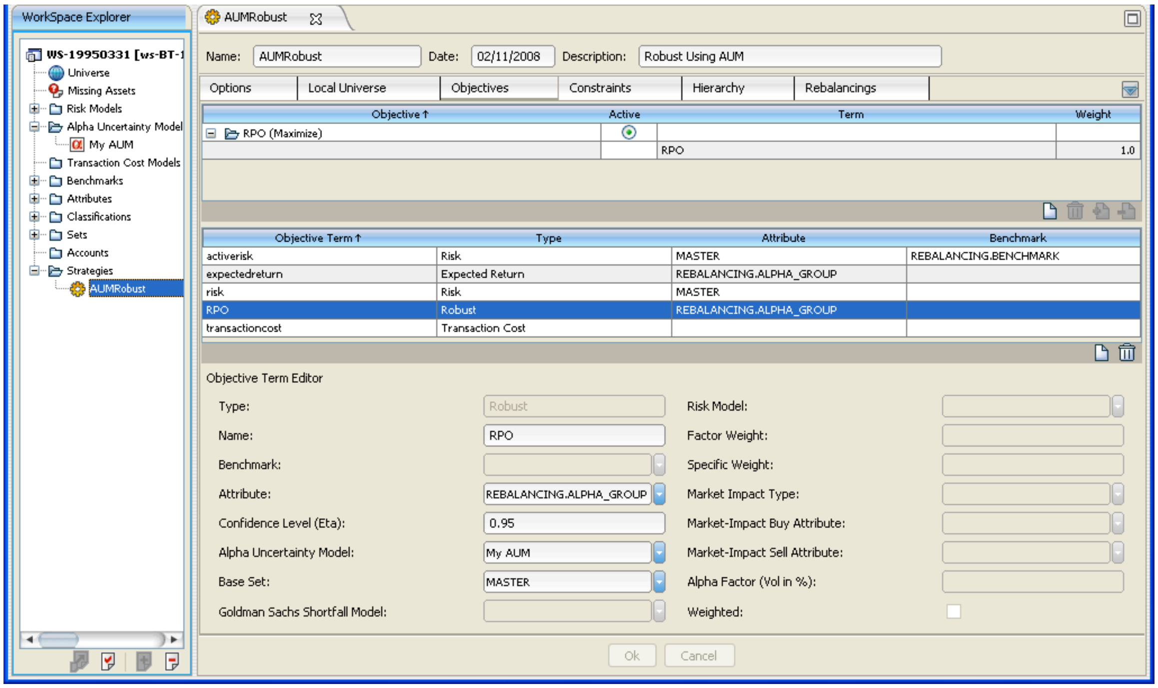


Figure 11. Screen shot showing an RPO strategy using the Alpha Uncertainty Model named "My AUM" and a Confidence Level (Eta) = 0.95.

Flexible is better.™

United States and Canada: 212-991-4500
Europe: +44 20 3170 8745
Asia: +852-8203-2790

New York Office
Axioma, Inc.
17 State Street
Suite 800
New York, NY 10004

Phone: **212-991-4500**
Fax: **212-991-4539**

Atlanta Office
Axioma, Inc.
8800 Roswell Road
Building B, Suite 295
Atlanta, GA 30350

Phone: **678-672-5400**
Fax: **678-672-5401**

San Francisco Office
Axioma, Inc.
201 Mission Street
Suite #2230
San Francisco, CA 94105

Phone: **415-614-4170**
Fax: **415-614-4169**

London Office
Axioma, (UK) Ltd.
288 Bishopsgate
London EC2M 4QP

Phone: **+44 20 3170 8745**

Hong Kong Office
Axioma, (HK) Ltd.
Unit C, 17/F
30 Queen's Road Central
Hong Kong

Phone: **+852-8203-2790**
Fax: **+852-8203-2774**

Singapore Office
Axioma, (Asia) Pte Ltd.
30 Raffles Place
#23-00 Chevron House
Singapore 048622

Phone: **+65 6233 6835**
Fax: **+65 6233 6891**

Sales: sales@axiomainc.com
Client Support: support@axiomainc.com
Careers: careers@axiomainc.com