A Quick Reviw of Portfolio Optimization

Iman Moazzen, Institute for Integrated Energy Systems, University of Victoria

Imanmoaz@uvic.ca, www.ece.uvic.ca/~imanmoaz

Concept

Portfolio optimization is the process of choosing the proportions of various assets to be held in a portfolio, in such a way as to make the portfolio better than any other according to some criterion. The criterion will combine, directly or indirectly, **considerations of the expected value** of the portfolio's rate of return as well as of the return's **dispersion** [1]. The dispersion in many contexts is translated to risk.

Graphical Representation of the Concept

Let's assume the probability distribution of the cost is shown in Fig.1. To minimize the expected value, the optimization algorithm tries to shift the red dot on the curve to the left as much as possible. To minimize the dispersion (risk), the optimization algorithm tries to move the black bars toward each other as much as possible.

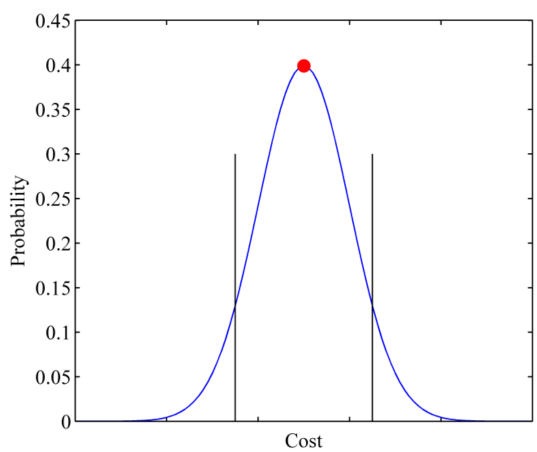


Figure 1. Probability distribution function

Three common ways to formulate the portfolio optimization are as follows:

(a)	(b)	(c)
Min (ω_1Expected Value + ω_2Risk)	**Min** (Expected Value) Risk $\leq$ *threshold*	**Min** (Risk) Expected Value $\leq$ *threshold*

Form (a) basically formulates the cost function as a weighted combination of both terms. Form (b) is useful when the maximum risk can be quantified in the system. Form (c) is useful when the budget limitation is defined. In this case we are willing to pay additional cost to control the risk. It is important to note that the three formulations are equivalent with the appropriate chosen parameters.

Example

Let's assume the demand for the next hour is one kW. The generation portfolio includes nuclear, natural gas and coal. The generation cost for three technologies is a random variable (c_i) with normal distribution as summarized in Table 1. $\bar{c}_i$ and σ_i are the expected value and standard deviation, respectively.

Table 1. Generation cost for different technologies [2]

Variable	Technology	$\bar{c}_i$ [ct/kWh]	σ_i [ct/kWh]
x	Nuclear Power Plant	4.5	1.0
y	Natural Gas Power Plant	3.5	1.5
z	Coal Power Plant	4.0	1.25

The optimization problem can be formulated as follows:

$$\left\{\begin{array}{ll} \text{Minimize } c_x x + c_y y + c_z z & (1) \\ \text{subject to:} & \\ x \geq 0, y \geq 0, z \geq 0 \quad \text{(physical limitation)} & \\ x + y + z = 1 \qquad \text{(demand must be met)} & \end{array}\right.$$

This formulation can be easily transformed to:

$$\left\{\begin{array}{ll} \text{Minimize } (c_x - c_z)x + (c_y - c_z)y + c_z & (2) \\ \text{subject to:} & \\ x \geq 0, y \geq 0, x + y \leq 1 & \\ z = 1 - x - y & \end{array}\right.$$

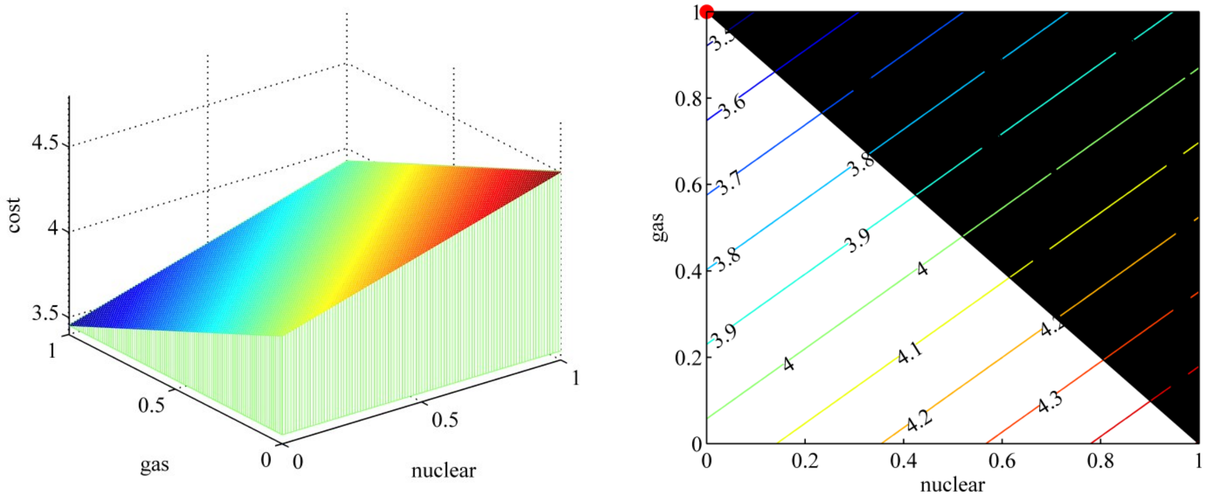
The expected cost for different combination of nuclear and gas is shown in Fig. 2 (a). The coal variable is not explicitly shown because its contribution corresponds to the gap between the other two technologies' supplies and the demand $(1 - x - y)$. The lines in Fig. 2 (b) show the systems with identical cost. The black area is not feasible. To minimize just the expected cost, the objective function of the optimization problem in (2) can be reformulated as follows (constraints remain the same):

Minimize $\quad E\{\ (c_x - c_z)x + (c_y - c_z)y + c_z\}$ =>

Minimize $\quad (\bar{c}_x - \bar{c}_z)x + (\bar{c}_y - \bar{c}_z)y + \bar{c}_z$ =>

Minimize $\quad x - 0.5\, y + 4$

It is clear that the minimum value (shown by a red dot in Fig. 2(b)) is for the case when x, z = 0 and y =1.



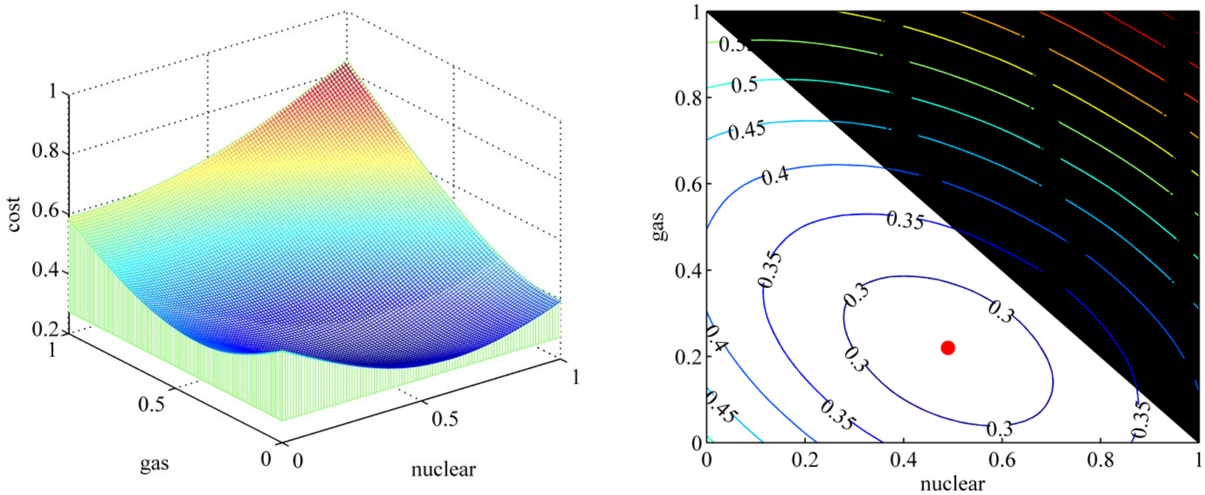
(a) (b)

Figure 2. Expected cost function versus nuclear and gas generation

Now, let's consider the case when we just want to minimize the risk with the following definition (constraints are similar to (2)):

$$R = \frac{1}{N}\sum_{w=1}^{N} max\{0, [c_x(w) - \bar{c}_x]x + [c_y(w) - \bar{c}_y]y + [c_z(w) - \bar{c}_z]z\} \qquad (3)$$

where $c_i(w)$ is one realization of the random variable. The risk function for 2000 realizations of random variables (Monte Carlo experiments) is calculated and plotted in Fig.3 (a). The ellipses in Fig.3 (b) denote portfolios with the same risk. The generation portfolio with the minimum risk is shown by a red dot in Fig. 3 (b).



(a) (b)

Figure 3. Risk function versus nuclear and gas generation

From Fig.2 (b), it is clear that as we move from the red dot (top left) to the bottom right the expected cost increases. Also, from Fig.3 (b), it is clear that as we move away from the red dot (in all directions), the risk increases. To provide a better understanding of the expected value and risk, Fig.2 (b) and Fig.3 (b) are plotted on top of each other in Fig.4 and three arbitrary points are selected. The characteristics of these points are summarized in Table 2. As we move from the first point to the third point the expected cost increases but the associated risk decreases. The probability distribution function of the observed cost for choosing each of these points is shown in Fig.5. The expected value is also shown by a big dot in each case. It is clear that as we move from the first point to the third point the expected cost goes up (big dot moves to the right) but the risk decreases (less deviation).

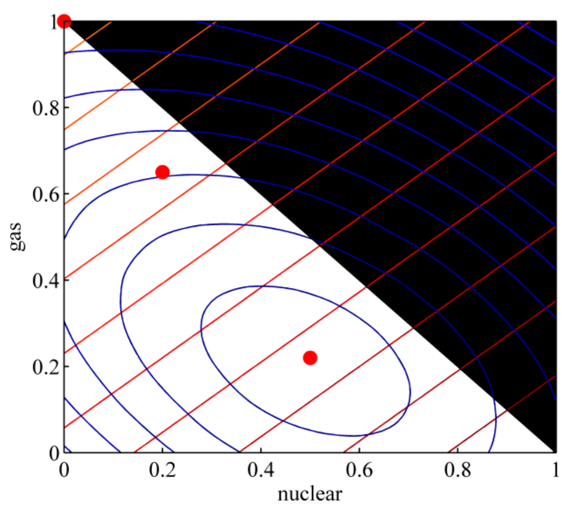


Figure 4. Contours for expected value and risk functions

Table 2. Three selected points in Fig.4

	First Point	**Second Point**	**Third Point**
x (nuclear)	0	0.2	0.5
y (gas)	1	0.65	0.22
z (coal)	0	0.15	0.28
Expected Cost	3.5	3.87	4.39
Associated Risk	0.59	0.40	0.27

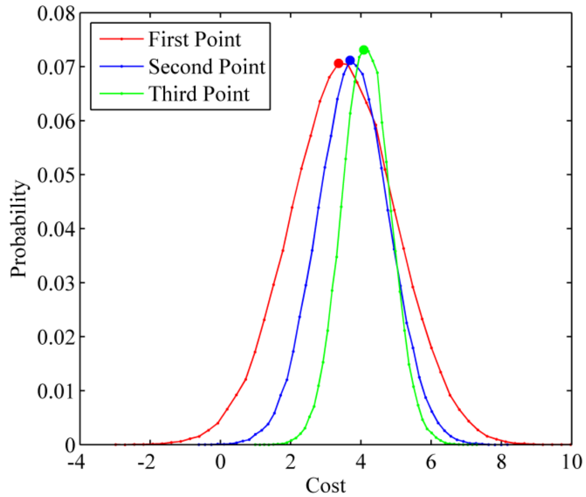


Figure 5. Probability distribution function of the observed cost for the three selected points in Fig.4

So far, it was assumed that all the random variables are not correlated. The correlation between the variables can change the risk formulation drastically. For simplicity, let's replace the risk formulation in (3) with the basic definition of variance and let's assume the expected vale ($\bar{c_i}$) for all the variables is zero. So, risk becomes:

$$R = E\left\{\left[c_x x + c_y y + c_z z\right]^2\right\} \qquad (4)$$

For the case that c_x, c_y and c_z are not correlated, R can be simply replaced by:

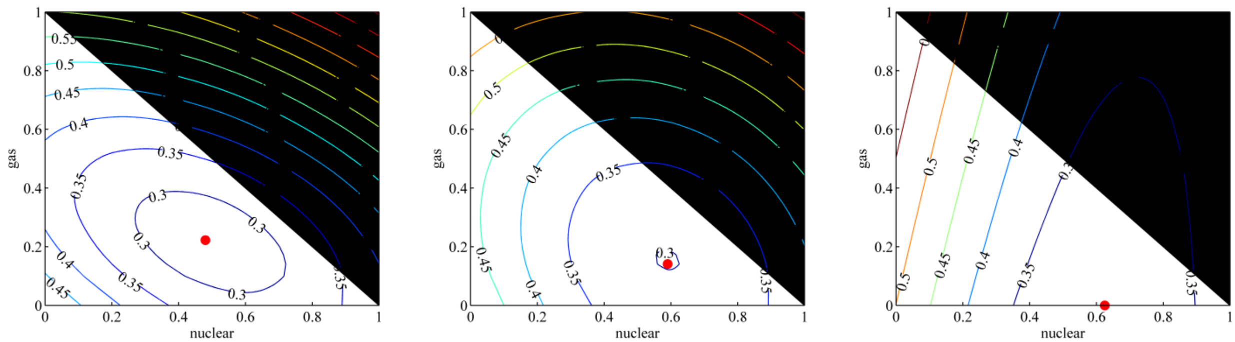
$$R = E\{[c_x]^2\}x + E\left\{\left[c_y\right]^2\right\}y + E\{[c_z]^2\}z = \sigma_x^2 x + \sigma_y^2 y + \sigma_z^2 z \qquad (5)$$

where σ_i^2 is the variance of each random variable. Now, let's assume the correlation between y and z is ρ. In this case, it can be easily shown that:

$$R = \sigma_x^2 x + \sigma_y^2 y + \sigma_z^2 z + \rho yz \qquad (6)$$

It is obvious that a simple correlation makes the optimization non-linear!

To graphically illustrate the impact of correlation on the risk, ρ (correlation between gas and coal generation cost) is set to 0, 0.5 and 1. The risk contours along with the minimum risk (red dot) for each case are shown in Fig.6.



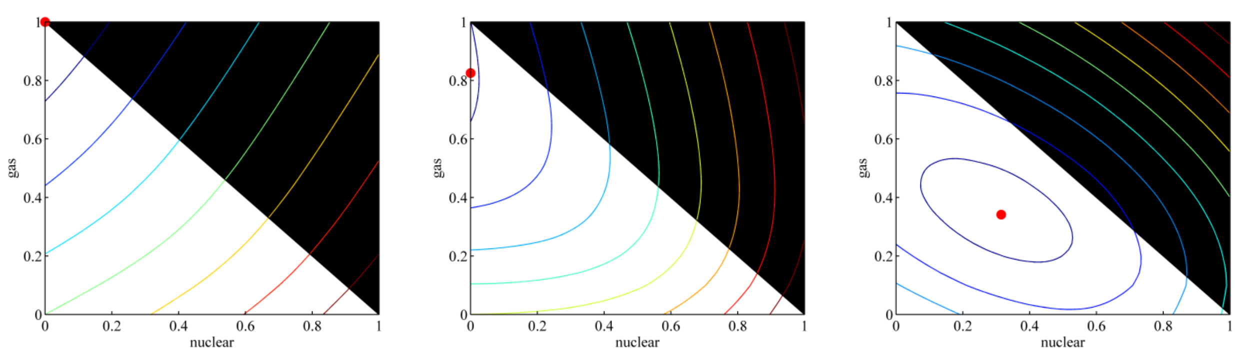
(a) $\rho = 0$ (b) $\rho = 0.5$ (c) $\rho = 1$

Figure 6. Impact of correlation between random variables on the risk function

So far, the minimization of either expected value or risk is discussed. Now, let's look at the three common forms of the portfolio optimization as discussed in Page 1. For simplicity, all the random variables are considered not correlated.

Form a: Minimize (ω_1Expected Value + ω_2Risk)

The cost function in this form is a weighted combination of expected value and risk. The objective function contours along with the minimum point (red dot) for three different weighting systems is shown in Fig.7.



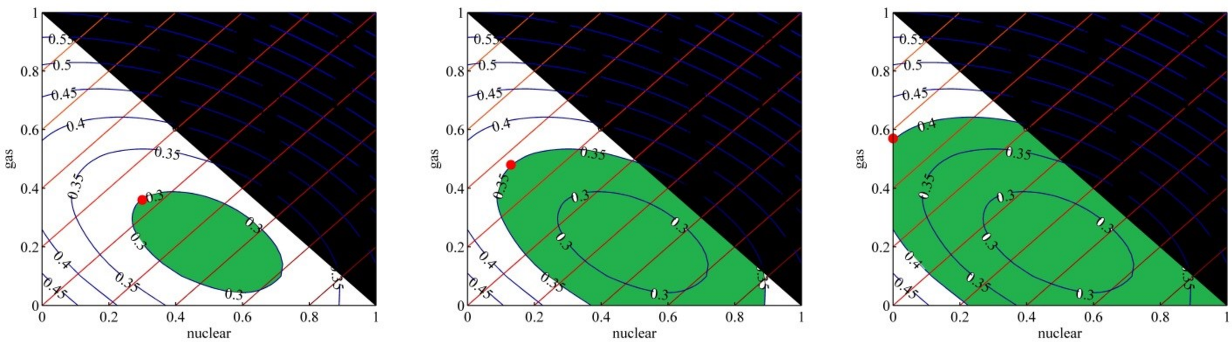
$\omega_1 = 0.8$ and $\omega_2 = 0.2$ $\omega_1 = 0.5$ and $\omega_2 = 0.5$ $\omega_1 = 0.2$ and $\omega_2 = 0.8$

Figure 7. Portfolio optimization (form a)

Form b:

$$\begin{cases} \text{Minimize (Expected Value)} \\ \text{Risk} \leq \textit{threshold (maximum allowed risk)} \end{cases}$$

This form is useful when the maximum risk can be quantified in the system. The objective function along with the minimum point (red dot) for three different thresholds is shown in Fig.8. The feasible area is shown by green color.



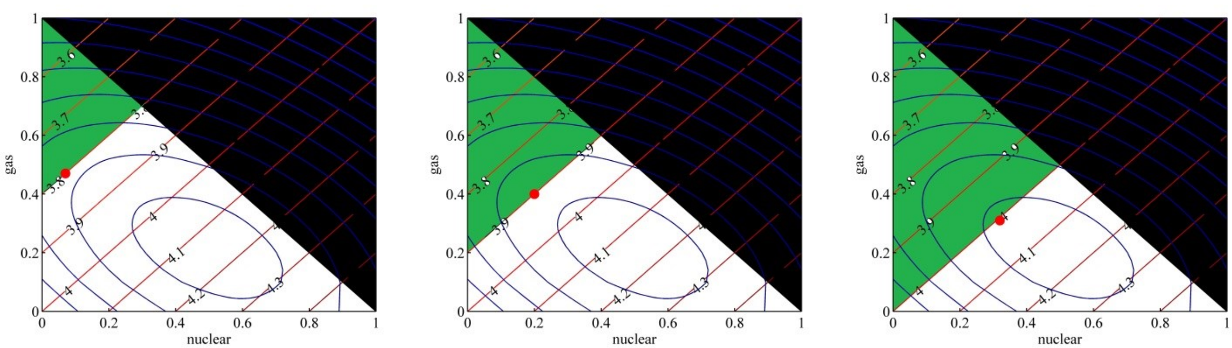
threshold = 0.3 *threshold* = 0.35 *threshold* = 0.4

Figure 8. Portfolio optimization (form b)

Form c:

$$\begin{cases} \text{Minimize (Risk)} \\ \text{Expected Value} \leq \textit{threshold (maximum allowed budget)} \end{cases}$$

This form is useful when the budget limitation is defined. In this case, we are willing to pay additional cost to control the risk. The objective function along with the minimum point (red dot) for three different thresholds is shown in Fig.9. The feasible area is shown by green color.



threshold = 3.8 *threshold* = 3.9 *threshold* = 4

Figure 9. Portfolio optimization (form c)

In summary, a trade-off between cost and risk can be provided using portfolio optimization.

References:

[1] Wikipedia, Portfolio Optimization, accessed at https://en.wikipedia.org/wiki/Portfolio_optimization

[2] Volker Krey, Keywan Riahi, “Risk Hedging Strategies Under Energy System and Climate Policy Uncertainties”, *Handbook of Risk Management in Energy Production and Trading*, pp 435-474, November 2013.

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