A Journey through the "Mathematical Underworld" of Portfolio Optimization

Marcos López de Prado

Lawrence Berkeley National Laboratory
Computational Research Division

BERKELEY LAB
LAWRENCE BERKELEY NATIONAL LABORATORY
U.S. DEPARTMENT OF ENERGY

Key Points

- Portfolio optimization is one of the problems most frequently encountered by financial practitioners. It appears in various forms in the context of Trading, Risk Management and Capital Allocation.
- The *Critical Line Algorithm* (CLA) is the only algorithm specifically designed for inequality-constrained portfolio optimization problems, which guarantees that the exact solution is found after a predefined number of iterations.
- Surprisingly, open-source implementations of CLA in a scientific language appear to be inexistent or unavailable.
- **Most financial practitioners are relying on generic-purpose optimizers which often deliver suboptimal solutions**.

SECTION I
The Problem

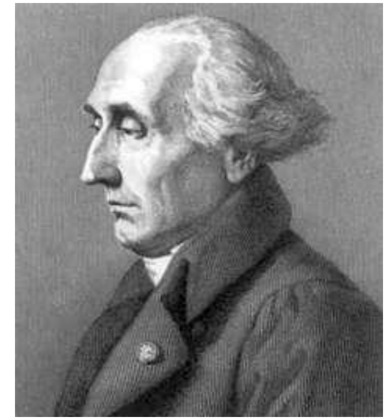
Quadratic Program s.t. Equality Constraints (1/2)

- **Problem #1**: Consider an investment universe of *n* assets with
  - *(nx1)* vector of means μ.
  - *(nxn)* positive definite covariance matrix Σ.
  - ω is the *(nx1)* vector of asset weights, which is our optimization variable.
- There are analytical solutions to the "unconstrained" problem (i.e., subject to equality constraints only).
  - Minimizing the Lagrange function with respect to the vector of weights ω and the multipliers γ and λ:

$$L[\omega, \gamma, \lambda] = \frac{1}{2}\omega'\Sigma\omega - \gamma(\omega' 1_n - 1) - \lambda(\omega'\mu - \mu_p)$$

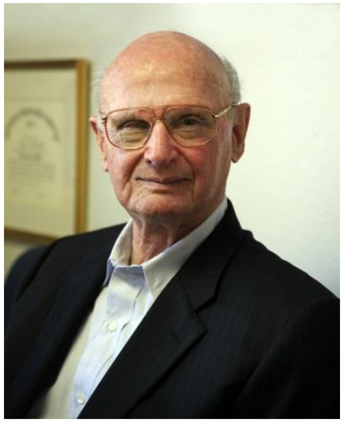
where 1_n is the *(nx1)* vector of ones and μ_p is the targeted excess return.

Quadratic Program s.t. Equality Constraints (2/2)



- **Solution #1**: The method of *Lagrange multipliers* applies first order necessary conditions on each weight and Lagrange multiplier, leading to a linear system of *n+2* conditions.
- An analytical solution exists… So what is the problem?
- Unfortunately, this solution is useless in most applications:
  - Weights are highly unstable.
  - Weights can take any value in the real domain.
  - In particular, weights can be negative. This is an unacceptable outcome in capital allocation and many other applications.
  - Very often, we have *a priori* information regarding sensible values for ω, and we would like to incorporate that information into the optimization problem, in the form of *inequality constraints*.

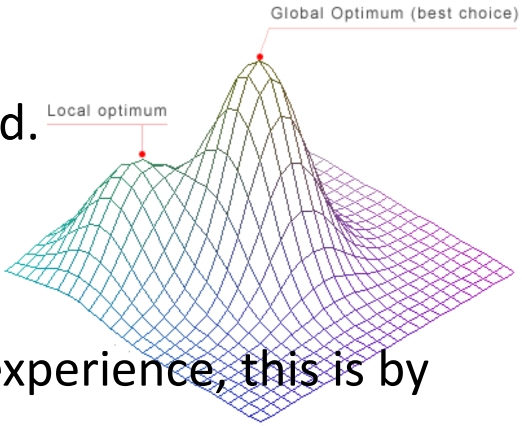
Quadratic Program s.t. Inequality Constraints (1/3)



- **Problem #2**: The method of Lagrange multipliers cannot be applied in the "constrained" problem (i.e., when inequality constraints are present).
- **Solution #2**: Markowitz's CLA provides the exact solution to quadratic programming problems subject to inequality constraints.
- The only disadvantage of CLA is that its implementation is relatively complex.
- However, the advantages of CLA clearly pay off:
  – Accuracy: It provides exact solutions.
  – Robustness: It is numerically stable.
  – Speed: It requires a low and predefined number of iterations.
  – Completeness: The output is the entire efficient frontier, not only a particular point.

Quadratic Program s.t. Inequality Constraints (2/3)

- **Question #1: Why are there no well-documented, open-source implementations of CLA in a scientific language?**
- **Answer #1**: Most firms rely on generic-purpose optimization algorithms or packages (AMPL, NAG, TOMLAB). Instead of CLA, they use:
  - BFGS, TNS: Gradient-based, they fail when a boundary is reached.
  - COBYLA: A quadratic problem should not be approximated linearly.
  - NNLS: Does not accept inequality constraints.
  - SLSQP: Converges to a local optimum near the seed.



- The reasons may be that:
  - CLA is non-trivial to implement.
  - They are simply unaware of CLA! (in my personal experience, this is by far the most prevalent reason)

Quadratic Program s.t. Inequality Constraints (3/3)

- In the context of this particular problem, these optimizers are:
  - Inaccurate: The solution is simply wrong. This is quite a problem, when trillions of dollars are on the line, and a small deviation means billions.
  - Numerically expensive: The calculations needed to guarantee a certain degree of accuracy can take forever (literally!)
  - Unreliable: You may get as many different solutions as runs.
  - Unstable: Move a constraint, and the new “optimum” may be far off.

In my experience, even large financial firms are using **heuristic** approaches for portfolio optimization… Heuristics are OK when we have very limited understanding of a problem. But portfolio optimization is well defined (no need for heuristics).

The *“mathematical underworld”* of investments: Heuristics can be misused to sell as quant-driven an investment product that, in reality, is gut-driven.

SECTION II
The Solution

The Critical Line Algorithm (1/4)

- **Key Concept #1:** **Free assets** are the set of weights that do not lie on their respective boundaries.
- Let us define:
  - $N = \{1,2, \ldots, n\}$ is a set of indices that number the investment universe.
  - *l* is the *(nx1)* vector of lower bounds, with $\omega_i \geq l_i, \forall i \in N$.
  - *u* is the *(nx1)* vector of upper bounds, with $\omega_i \leq u_i, \forall i \in N$.
  - $F \subseteq N$ is the subset of *free* assets, where $l_i < \omega_i < u_i$. In words, free assets are those that do not lie on their respective boundaries. *F* has length $1 \leq k \leq n$.
  - $B \subset N$ is the subset of weights that lie on one of the bounds. By definition, $B \cup F = N$.

The Critical Line Algorithm (2/4)

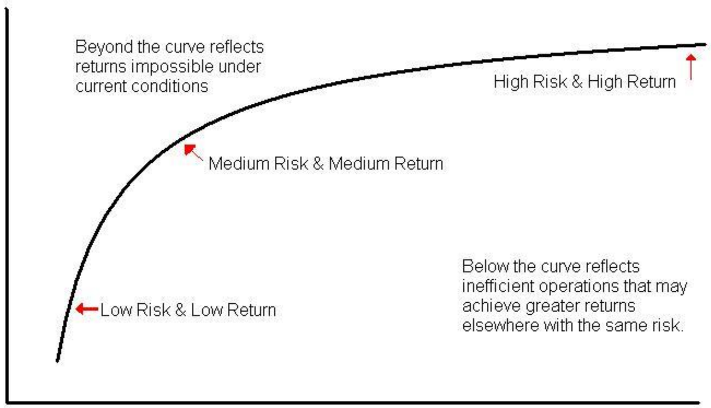
- **Key Concept #2:** A solution vector ω^* is a **turning point** if in its vicinity there is another solution vector with different free assets.
- **Key Insight #1**: Harry Markowitz (Nobel Prize, 1990) realized that between any two consecutive turning points the inequality constraints are irrelevant. The constrained solution reduces to solving the following unconstrained problem on the free assets (one solution per turning point):

$$L[\omega,\gamma,\lambda] = \frac{1}{2}\omega_F{}'\Sigma_F\omega_F + \frac{1}{2}\omega_F{}'\Sigma_{FB}\omega_B + \frac{1}{2}\omega_B{}'\Sigma_{BF}\omega_F + \frac{1}{2}\omega_B{}'\Sigma_B\omega_B - \gamma(\omega_F{}'1_k + \omega_B{}'1_{n-k} - 1) - \lambda\left(\omega_F{}'\mu_F + \omega_B{}'\mu_B - \mu_p\right)$$

where ω_B is known and does not change between turning points.

The Critical Line Algorithm (3/4)

- **Key Concept #3 (the “two-funds theorem”)**: The unconstrained efficient frontier can be simply derived as a convex combination between any two optimal portfolios.

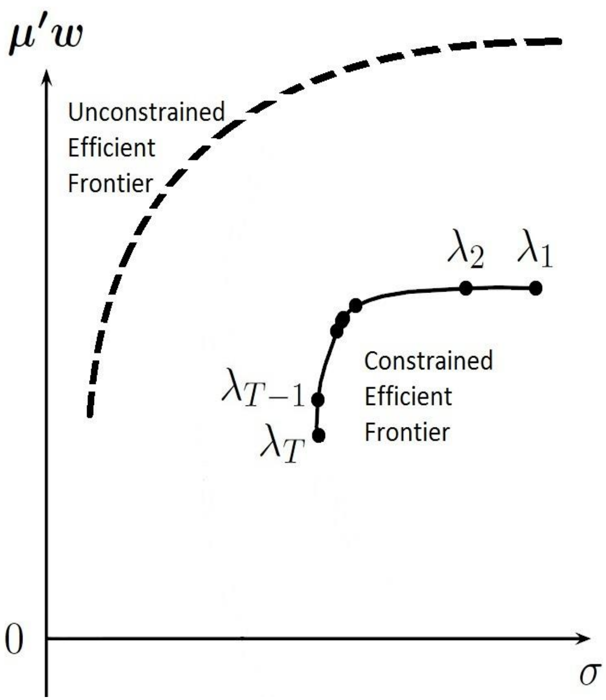


In the unconstrained case, any two optimal solutions suffice to generate the entire efficient frontier. The two-funds theorem does not hold in the presence of inequality constraints.

- **Key Insight #2**: Markowitz realized that, since the constrained efficient frontier between two neighbor turning points is the same as in the unconstrained case, he only needed to compute *the sequence of turning points* to derive the entire constrained efficient frontier!

The Critical Line Algorithm (4/4)

- **Key Insight #3**: Find first the turning point associated with the highest expected return, and then compute the sequence of turning points, each with a lower expected return than the previous.



- That first turning point consists in the smallest subset of assets with highest return such that the sum of their upper boundaries equals or exceeds one.
- The transition from one turning point to the next requires that one element is either added or removed from the subset of free assets, *F*.
- Because λ and $\omega'\mu$ are linearly and positively related, this means that **each subsequent turning point will lead to a lower value for $\boldsymbol{\lambda}$**: $\lambda_i < \lambda_j, \forall i > j$.
- This recursion of adding or removing one asset from *F* continues until the algorithm determines that the optimal expected return cannot be further reduced.

SECTION III
The CLA Open-Source Python Class

Why Open-Source?

- It has been estimated that the current size of the asset management industry is approximately US$58 trillion.
- The lack of publicly available CLA software, commercially or open-source, means that trillions of dollars are likely to be suboptimally allocated as a result of practitioners using general-purpose quadratic optimizers.

The *Centre for Innovative Financial Technology* (www.lbl.gov/CS/CIFT.html) is part of Berkeley Lab's Computational Research Division. Its mission is to develop mathematical solutions aimed at more stable and efficient financial markets.

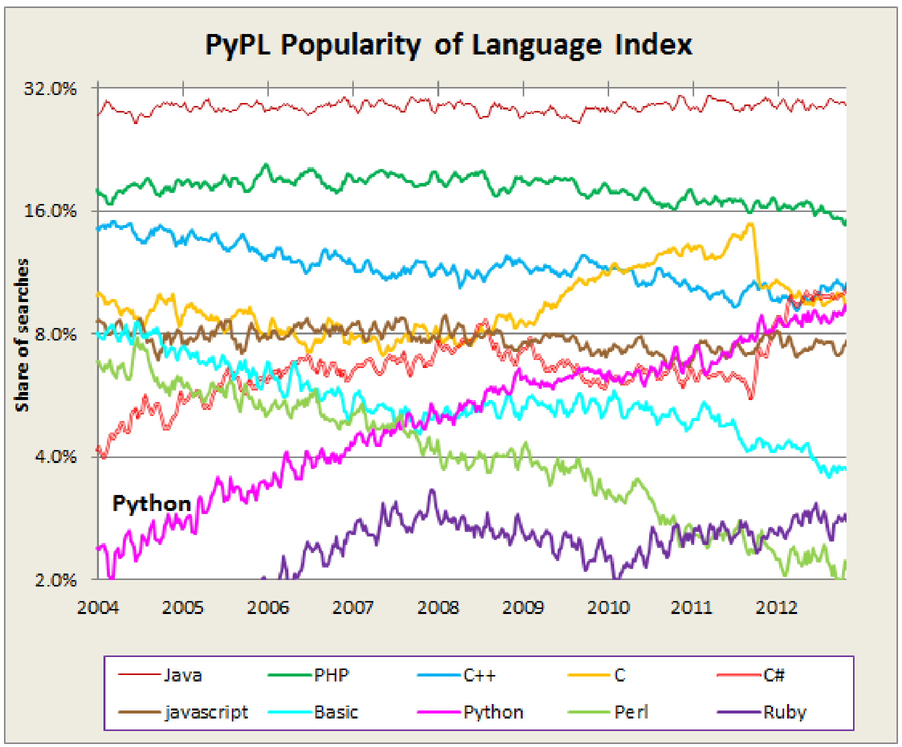
Why Python? (1/2)

- VBA-Excel is the most widely used platform in Finance.
- Excerpts from the JP Morgan internal investigation into the US$6 billion "London Whale" losses of 2012 (pp. 124, 128):

> During the review process, additional operational issues became apparent. For example, the model operated through a series of Excel spreadsheets, which had to be completed manually, by a process of copying and pasting data from one spreadsheet to another. In addition, many of the tranches were less liquid, and therefore, the same price was given for those tranches on multiple consecutive days, leading the model to convey a lack of volatility. While there was some effort to map less liquid instruments to more liquid ones (*i.e.*, calculate price changes in the less liquid instruments derived from price changes in more liquid ones), this effort was not organized or consistent.
>
> After this re-review, a decision was made to stop using the Basel II.5 model and not to rely on it for purposes of reporting CIO VaR in the Firm's first-quarter Form 10-Q. Following that decision, further errors were discovered in the Basel II.5 model, including, most significantly, an operational error in the calculation of the relative changes in hazard rates and correlation estimates. Specifically, after subtracting the old rate from the new rate, the spreadsheet divided by their sum instead of their average, as the modeler had intended. This error likely had the effect of muting volatility by a factor of two and of lowering the VaR,
>
> 128

Why Python? (2/2)

- We believe that a large amount of financial firms and practitioners will benefit from our robust implementation of CLA in a scientific language: http://ssrn.com/abstract=2197616



We have chosen Python because:

- It is widely used among National Laboratory researchers and computer scientists in general (example here).
- It has more than 26,000 extension modules currently available.
- It stresses readability. Its "pseudocode" appearance makes it a good choice for discussion in an academic paper.
- The EPD 7.3 product (Enthought Python Distribution) efficiently integrates all the necessary scientific libraries.

SECTION IV
Conclusions

Conclusions

- Portfolio optimization subject to equality constraints (also known as “unconstrained”) has analytical solution. However, it is useless in most practical applications.
- Portfolio optimization subject to inequality constraints (also known as “constrained”) has no analytical solution. However, CLA provides an exact numerical solution.
- The lack of open-source, well-documented implementations of CLA in scientific languages implies that most firms rely on generic-purpose optimization algorithms, optimization packages or heuristics.
- We expect that many financial firms and practitioners will benefit from our robust implementation of CLA in Python.

THANKS FOR YOUR ATTENTION!

SECTION V
The stuff nobody reads

Bibliography (1/2)

- Avriel, M. and D. Wilde (1966): "Optimality proof for the symmetric Fibonacci search technique", *Fibonacci Quarterly*, 4, 265-269.
- Bailey, D. and M. López de Prado (2013): "An Open-Source Implementation of the Critical-Line Algorithm for Portfolio Optimization". Research Paper, Lawrence Berkeley National Laboratory. Available at http://ssrn.com/abstract=2197616. Code can be downloaded at http://www.quantresearch.info/Software.htm
- Beardsley, B., H. Donnadieu, K. Kramer, M. Kumar, A. Maguire, P. Morel, T. Tang (2012): "Capturing Growth in Adverse Times: Global Asset Management 2012". Research Paper, The Boston Consulting Group.
- Dalton, S. (2007): "Financial applications using Excel Add-in development in C/C++", Wiley, 2nd Edition, pp. 13-14.
- Hirschberger, M., Y. Qi, and R. E. Steuer (2004): "Quadratic Parametric Programming for Portfolio Selection with Random Problem Generation and Computational Experience". Working paper, Terry College of Business, University of Georgia.
- Kiefer, J. (1953): "Sequential minimax search for a maximum". *Proceedings of the American Mathematical Society*, 4(3), 502-506.
- Kopman, L. and S. Liu (2009): "Maximizing the Sharpe Ratio". MSCI Barra Research Paper No. 2009-22.

Bibliography (2/2)

- Markowitz, H.M. (1952): “Portfolio Selection”. *Journal of Finance* 7(1), pp. 77–91.
- Markowitz, H.M. (1956): “The Optimization of a Quadratic Function Subject to Linear Constraints”. *Naval Research Logistics Quarterly*, III, 111–133.
- Markowitz, H.M. (1959): “Portfolio Selection: Efficient Diversification of Investments”. John Wiley and Sons.
- Markowitz, H.M., A. Malhotra and D.P. Pazel (1984): “The EAS-E application development system: principles and language summary”, Communications of the ACM, 27(8), August, 785-799.
- Markowitz, H.M., G. P. Todd (2000): “Mean variance analysis in portfolio choice and capital markets”, Wiley.
- Meucci, A. (2005): “Risk and Asset Allocation”. Springer.
- Niedermayer, A. and D. Niedermayer (2007): “Applying Markowitz’s Critical Line Algorithm”, Research Paper Series, Department of Economics, University of Bern.
- Steuer, R.E., Y. Qi, and M. Hirschberger (2006): “Portfolio Optimization: New Capabilities and Future Methods”. *Zeitschrift für Betriebswirtschaft*, 76(2), 199-219.
- Wolfe, P. (1959): “The Simplex Method for Quadratic Programming”. *Econometrica*, 27(3), 382–398.

Bio

Marcos López de Prado is Senior Managing Director at Guggenheim Partners. He is also a Research Affiliate at Lawrence Berkeley National Laboratory's Computational Research Division (U.S. Department of Energy's Office of Science).

Before that, Marcos was Head of Quantitative Trading & Research at Hess Energy Trading Company (the trading arm of Hess Corporation, a Fortune 100 company) and Head of Global Quantitative Research at Tudor Investment Corporation. In addition to his 15+ years of trading and investment management experience at some of the largest corporations, he has received several academic appointments, including Postdoctoral Research Fellow of RCC at Harvard University and Visiting Scholar at Cornell University. Marcos earned a Ph.D. in Financial Economics (2003), a second Ph.D. in Mathematical Finance (2011) from Complutense University, is a recipient of the National Award for Excellence in Academic Performance by the Government of Spain (National Valedictorian, 1998) among other awards, and was admitted into American Mensa with a perfect test score.

Marcos is the co-inventor of four international patent applications on High Frequency Trading. He has collaborated with ~30 leading academics, resulting in some of the most read papers in Finance (SSRN), three textbooks, publications in the top Mathematical Finance journals, etc. Marcos has an Erdös #3 and an Einstein #4 according to the American Mathematical Society.

Notice:

The research contained in this presentation is the result of a continuing collaboration with

Prof. David H. Bailey, LBNL

The full paper is available at:
http://ssrn.com/abstract=2197616

For additional details, please visit:
http://ssrn.com/author=434076
www.QuantResearch.info

Disclaimer

- The views expressed in this document are the authors' and do not necessarily reflect those of the organizations he is affiliated with.
- No investment decision or particular course of action is recommended by this presentation.
- All Rights Reserved.