Multi-Period Integer Portfolio Optimization Using a Quantum Annealer

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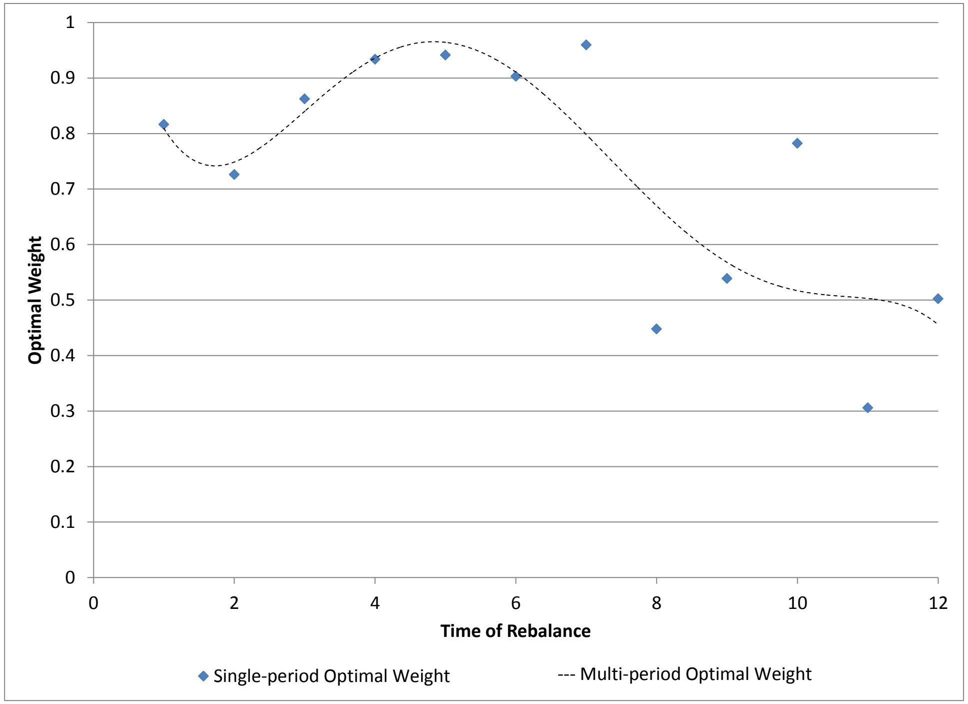
Key Points

- The problem: For large portfolio managers, a sequence of single-period optimal positions is rarely multi-period optimal.
- A major cause: Transaction costs can prevent large portfolio managers from monetizing most of their forecasting power.
- The solution: Compute the trading trajectory that comes sufficiently close to the single-period optima while minimizing turnover costs.
- A few solutions exist that compute trading trajectories, under rather idealistic/simplistic scenarios.
- Computing a trading trajectory in general terms is a NP-Complete problem.
- In this presentation we will illustrate how quantum computers can solve this problem in the most general terms.

SECTION I
The Trading Trajectory Problem

The Trading Trajectory Problem

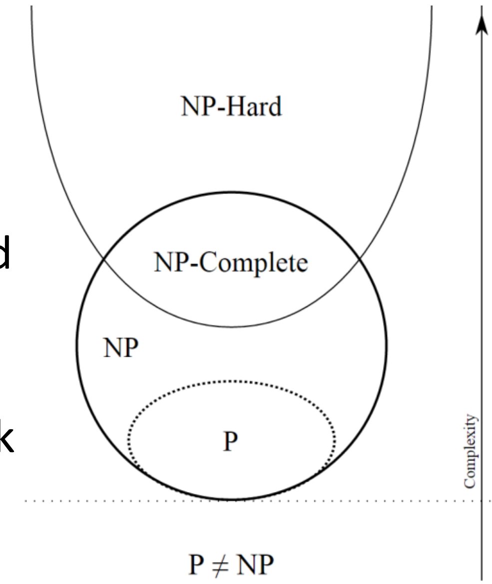
- Large portfolio managers need to optimize their portfolio over multiple forecasted horizons, in order to properly account for significant market impact (temporary & permanent).
- Some positions can only be traded in blocks (e.g., ETF units, real estate, private offerings, etc.), thus requiring integer solutions.



- Rebalancing the portfolio to align it with each single-step optimum (blue dots) is often prohibitively expensive.
- We could instead compute a trajectory that maximizes overall risk-adjusted performance after all transaction costs (black line).

How “hard” is this problem?

- **Single-period**:
  - Jobst et al. [2001] showed that the efficient frontier of the *integer* problem is discontinuous.
  - Bonami and Lejeune [2009] have solved the single-period problem with *integer* holdings without transaction costs.
  - The single-period *integer* portfolio optimization problem has been shown to be NP-complete, regardless of the risk measure used (Kellerer, Mansini, Speranza [2000]).



- **Multi-period**:
  - Garleanu and Pedersen [2013] solved a *continuous* multi-period problem for a Brownian motion via dynamic programming (Bellman equations), deriving a closed-form solution when the covariance matrix is positive definite and transaction costs are proportional to market risk.
  - Lopez de Prado [2014] showed that the *integer* problem is, in its most general formulations, amenable to quantum computing.

Integer Optimization Formulation

- This multi-period *integer* optimization problem may be written as

$$w = \text{argmax}_w \left\{ \sum_{t=1}^{T} \left(\underbrace{\mu_t^T w_t}_{\text{returns}} - \frac{\gamma}{2} \underbrace{w_t^T \Sigma_t w_t}_{\text{risk}} - \underbrace{\Delta w_t^T \Lambda_t \Delta w_t}_{\substack{\text{direct costs,} \\ \text{temp.impact}}} - \underbrace{\Delta w_t^T \Lambda_t' \Delta w_t}_{\text{perm.impact}} \right) \right\}$$

s.t.:

$$\forall t: \sum_{n=1}^{N} w_{n,t} \leq K \; ; \forall t, \forall n: w_{n,t} \leq K'$$

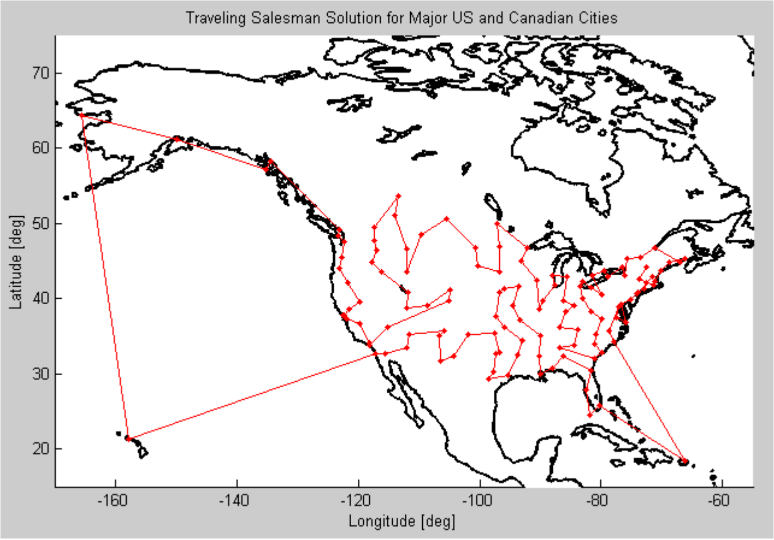
Symbol	Type	Description
K	$\mathbb{N}_1$	Number of units to be allocated at each time step
K'	$\mathbb{N}_1$	Largest allowed holding for any asset
N	$\mathbb{N}_1$	Number of assets
T	$\mathbb{N}_1$	Number of time steps
μ	$\mathbb{R}^{N\times T}$	Forecast mean returns of each asset at each time step
γ	$\mathbb{R}$	Risk aversion
Σ	$\mathbb{R}^{T\times N\times N}$	Forecast covariance matrix for each time step
c'	$\mathbb{R}^{N\times T}$	Permanent market impact coefficients for each asset at each time step
c	$\mathbb{R}^{N\times T}$	Transaction cost coefficients for each asset at each time step
w_0	$\mathbb{N}_0^N$	Initial holdings for each asset
w_{T+1}	$\mathbb{N}_0^N$	Final holdings for each asset
w	$\mathbb{N}_0^{N\times T}$	Holdings for each asset at each time step (the trading trajectory)

Temporary t-costs *c* are represented using the tensor Λ, where $\Lambda_{t,n,n'} = c_{n,t}\delta_{n,n'}$, and similarly for the permanent price impact c' and Λ'.

SECTION II
How does a Quantum Annealer solve this problem?

What is combinatorial optimization?

- Combinatorial optimization problems can be described as problems where there is a finite number of feasible solutions, which result from combining the discrete values of a finite number of variables.
- As the number of feasible combinations grows, an exhaustive search becomes impractical.
- The traveling salesman problem is an example of a combinatorial optimization problem that is known to be NP-hard.



- Digital computers evaluate and store feasible solutions *sequentially*.
  - The bits of a standard computer can only adopt one of two possible states ({0,1}) at once.
- This is a major disadvantage for finding path-dependent solutions.

How do Quantum Computers work? (1/2)

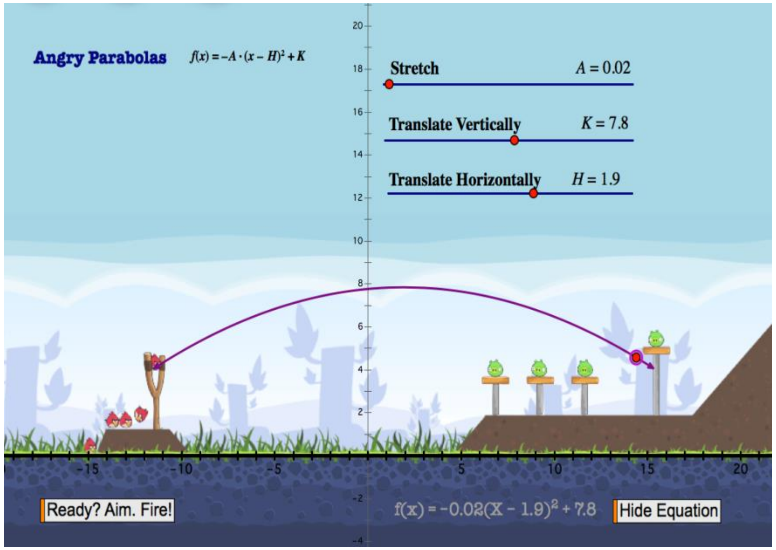
- Quantum computers rely on **qubits**, which are memory elements that may hold a linear superposition of both states ({0,1}).
  - This allows them to evaluate and store all feasible solutions simultaneously.

Superposition	States
δ	$\|(1,1) >$
γ	$\|(0,1) >$
β	$\|(1,0) >$
α	$\|(0,0) >$

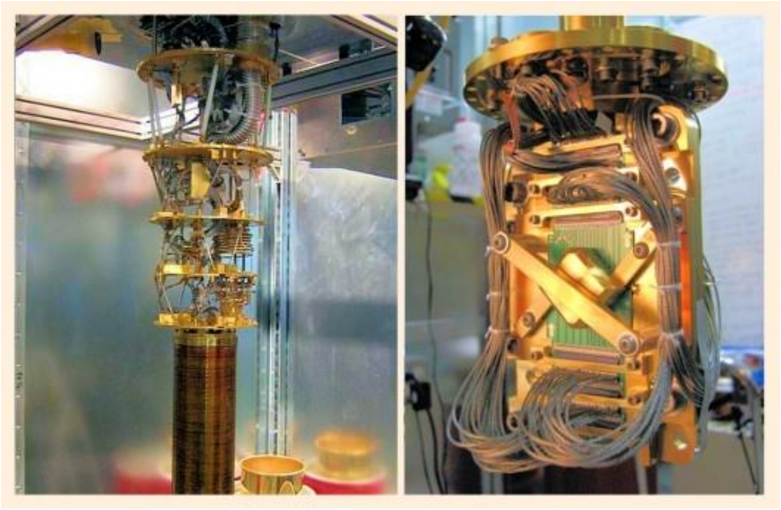
- The state of a 2-bit digital system is determined by 2 binary digits.
- The state of a 2-bit quantum system is determined by 4 coefficients: $(\alpha, \beta, \gamma, \delta)$, normalized.
- N-qubits contain 2^N units of classical information.

- Quantum computers may require an exponentially smaller number of operations to reach a solution.
- Quantum effects such as *tunneling* and *entanglement*, may help speed up the convergence of the procedure.

How do Quantum Computers work? (2/2)



- The digital-computer game "Angry Birds" uses mathematics to solve a physics problem.
  - A PDE library solves classical mechanics problems as demanded by the user.
- Conversely, Quantum Computers use physics to solve a mathematical problem.
- The Quantum Computing analogue of "Angry Birds" is to program the chip to behave like gravity, toss the bird, measure how it bounces.



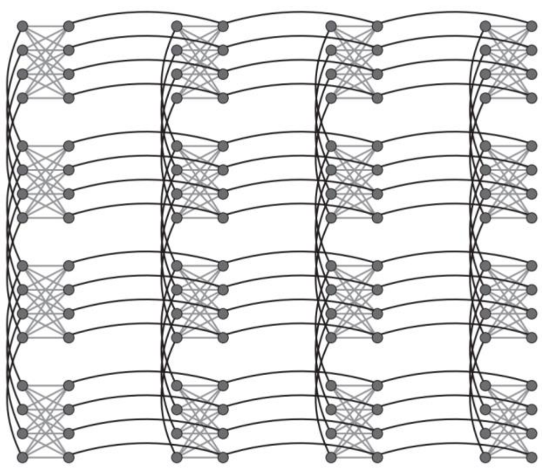
- In order to keep external disturbances to a minimum, the D-Wave machine is cooled to 15 mK (about 180 times colder than interstellar space), and operates in a near vacuum shielded from electromagnetic fields.
- This makes the D-Wave machine one of the coolest things in the Galaxy… quite literally!

How does a Quantum Annealer work? (1/2)

- A Quantum Annealer is a quantum computer that solves optimization problems that can be encoded as a *Hamiltonian*.
- D-Wave Systems has developed a device which minimizes unconstrained binary quadratic functions of the form

$$\min_x x^T Q x$$
$$s.t.: x \in \{0,1\}^N$$

where $Q \in \mathbb{R}^{N\text{x}N}$.



- The connectivity of the qubits in D-Wave's quantum annealer is currently described by a squared *Chimera graph* (left figure, a 128-Qubit architecture).
- The hardware graph is sparse and in general does not match the problem graph, which is defined by the adjacency matrix of *Q*.
- Minor embedding: Several physical qubits can be treated as a single logical qubit.

How does a Quantum Annealer work? (2/2)

- Superposition is useful to reach a solution quickly, however the final result must be representable in terms of the basis states.
- Encoding: Integer variables are recast as binary variables.

TABLE II: Comparison of the four encodings described in Section II-B. The column "Variables" refers to the number of binary variables required to represent a particular problem. The column "Largest integer" refers to a worst-case estimate of the largest integer that could be represented based on the limitation imposed by the noise level ϵ and the ratio between the largest and smallest problem coefficients δ and $n \equiv 1/\sqrt{\epsilon\delta}$.

Encoding	Variables	Largest integer	Notes
Binary	$TN \log_2(K' + 1)$	$\lfloor 2n \rfloor - 1$	Most efficient in number of variables; allows representing of the second-lowest integer.
Unary	TNK'	No limit	Biases the quantum annealer due to differing redundancy of code words for each value; encoding coefficients are even, giving no dependence on noise, so it allows representing of the largest integer.
Sequential	$\frac{1}{2}TN\left(\sqrt{1+8K'} - 1\right)$	$\frac{1}{2}\lfloor n \rfloor \left(\lfloor n \rfloor + 1\right)$	Biases the quantum annealer (but less than unary encoding); second-most-efficient in number of variables; allows representing of the second-largest integer.
Partition	$\leq T\binom{K+N-1}{N-1}$	$\lfloor n \rfloor$	Can incorporate complicated constraints easily; least efficient in number of variables; only applicable for problems in which groups of variables are required to sum to a constant; allows representing the lowest integer.

SECTION III
Experimental Results

Experiment Design

1. Generate a random multi-period integer optimization problem.
2. Using a digital computer:
   a) Compute the exact solution by brute force.
   b) Compute the variance of that solution by perturbating the covariance matrix.
3. Solve the problem using the Quantum Annealer.
4. Evaluate whether the Quantum solution falls within the exact solution's margin of error.

Why do we need to repeat the procedure multiple times?:

- Remember, **Quantum Computers will not return the exact solution**, because of hardware limitations.
- They are non-deterministic devices that search for the optimal linear combinations of basis states!
- In addition, they are physical devices, and the solution is recovered with a measurement error.

Accuracy

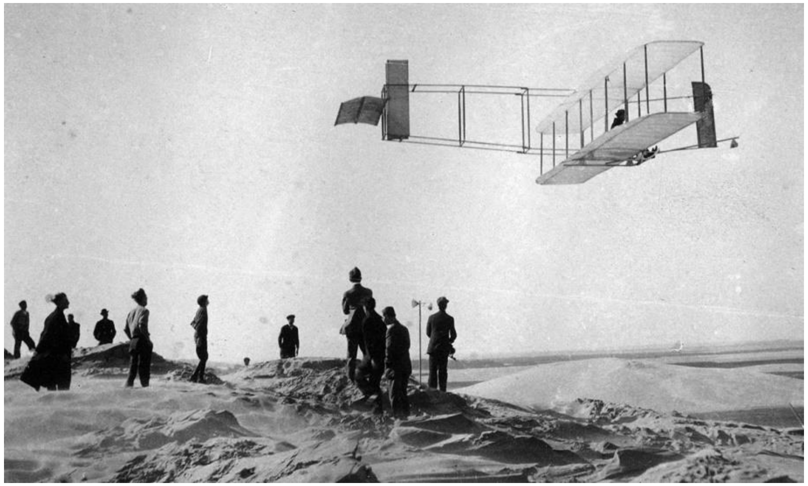
TABLE IV: Results using D-Wave's quantum annealer (with 200 instances per problem): N is the number of assets, T is the number of time steps, K is the number of units to be allocated at each time step and the maximum allowed holding (with $K' = K$), "encoding" refers to the method of encoding the integer problem into binary variables, "vars" is the number of binary variables required to encode the given problem, "density" is the density of the quadratic couplers, "qubits" is the number of physical qubits that were used, "chain" is the maximum number of physical qubits identified with a single binary variable, and $S(\alpha)$ refers to the success rate given a perturbation magnitude $\alpha\%$ (explained in the text).

N	T	K	encoding	vars	density	qubits	chain	$S(0)$	$S(1)$	$S(2)$
2	3	3	binary	12	0.52	31	3	100.00	100.00	100.00
2	2	3	unary	12	0.73	45	4	97.00	100.00	100.00
2	4	3	binary	16	0.40	52	4	96.00	100.00	100.00
2	3	3	unary	18	0.53	76	5	94.50	100.00	100.00
2	2	7	binary	12	0.73	38	4	90.50	100.00	100.00
2	5	3	binary	20	0.33	63	4	89.00	100.00	100.00
2	6	3	binary	24	0.28	74	4	50.00	100.00	100.00
3	2	3	unary	18	0.65	91	6	38.50	80.50	95.50
3	3	3	binary	18	0.45	84	5	35.50	80.50	96.50
3	4	3	binary	24	0.35	106	6	9.50	89.50	100.00

- **The Quantum Annealer's solution was within the margin of error in a large majority of experiments.**

Why is this relevant?

- Our approach is general and scalable:
  - It computes integer (discrete) solutions (a NP-complete problem) that are globally optimal over multiple-periods.
  - We don't impose a particular returns process.
  - The procedure accepts very general transaction cost functions, and there is no need to impose simplifying assumptions such as costs proportional to risk.
  - The covariance matrix does not need to be positive definite. The solution does not even require covariance invertibility!



The size of problems currently solvable by Quantum computers is relatively small.

Numerous challenges need to be addressed.

The purpose of our work is to show that, in the near future, Quantum computers will solve many currently intractable problems, and render obsolete some existing approaches.

THANKS FOR YOUR ATTENTION!

SECTION IV
The stuff nobody reads

Notice:

The research contained in this presentation is the result of a continuing collaboration with

Peter Carr (Morgan Stanley, NYU), Phil Goddard (*1QBit),*
Poya Haghnegahdar (1QBit), Gili Rosenberg (1QBit),
Kesheng Wu (Berkeley Lab)

The full paper is available at:
http://ssrn.com/abstract=2649376

For additional details, please visit:
http://ssrn.com/author=434076
www.QuantResearch.info

Bio

Marcos López de Prado is Senior Managing Director at *Guggenheim Partners*. He is also a Research Fellow at *Lawrence Berkeley National Laboratory*'s Computational Research Division (U.S. Department of Energy's Office of Science), where he conducts unclassified research in the mathematics of large-scale financial problems and supercomputing.

Before that, Marcos was Head of Quantitative Trading & Research at Hess Energy Trading Company (the trading arm of *Hess Corporation*, a Fortune 100 company) and Head of Global Quantitative Research at *Tudor Investment Corporation*. In addition to his 17 years of trading and investment management experience at some of the largest corporations, he has received several academic appointments, including Postdoctoral Research Fellow of *RCC at Harvard University* and Visiting Scholar at *Cornell University*. Marcos earned a Ph.D. in Financial Economics (2003), a second Ph.D. in Mathematical Finance (2011) from *Complutense University*, is a recipient of the National Award for Excellence in Academic Performance by the Government of Spain (National Valedictorian, 1998) among other awards, and was admitted into *American Mensa* with a perfect test score.

Marcos serves on the Editorial Board of the *Journal of Portfolio Management* (IIJ) and the *Journal of Investment Strategies* (Risk). He has collaborated with ~30 leading academics, resulting in some of the most read papers in Finance (SSRN), four international patent applications on High Frequency Trading, three textbooks, numerous publications in the top Mathematical Finance journals, etc. Marcos has an Erdös #2 and an Einstein #4 according to the *American Mathematical Society*.

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