# Risk-Based Indexation*

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**Abstract**

A capitalization-weighted index is the most common way to gain access to broad equity market performance. These portfolios are generally concentrated in a few stocks and present some lack of diversification. In order to avoid this drawback or to simply diversify market exposure, alternative indexation methods have recently prompted great interest, both from academic researchers and market practitioners. Fundamental indexation computes weights with regard to economic measures, while risk-based indexation focuses on risk and diversification criteria. This paper describes risk-based indexation methodologies, highlights potential practical issues when implemented, and illustrates these issues as it applies to the Euro Stoxx 50 universe.

**Keywords:** Risk-based indexation, fundamental indexation, market capitalization, equity indexes, diversification, portfolio optimization, robust estimation.

**JEL classification:** G11, C60.

## 1 Introduction

For the past forty years, asset allocation has relied on the Capital Asset Pricing Model (CAPM) theory, originated by William F. Sharpe (1964). CAPM theory concludes that under some assumptions, market-capitalization weighting is efficient for asset allocation, in the sense that no other portfolio with the same risk (i.e. volatility) will have a higher expected return. Since then, capitalization-weighted indexes (hereafter CW) have played a central role in the investment industry. First, they provide convenient access to broad equity markets and serve as a natural investment vehicle in financial markets (index funds, electronic-traded funds, derivatives). Second, they represent a reference and benchmark for active management.

Realistically, the assumptions of CAPM do not hold (investors do not all have the same expectations, they cannot sell short without a penalty), and CAPM appears to be inefficient

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(see, for example, Haugen *et al.* (1991), Amenc *et al.* (2006), Hsu (2006)). In this context, investors have recently shown great interest in alternative-weighted indexes (hereafter AW). An alternative-weighted index is defined as an index in which assets are weighted in a different way than those based on market capitalization. Alternative-weighted indexes can be split into two families: fundamental indexes and risk-based indexes. Fundamental indexation defines the weights as a function of economic metrics like dividends or earnings. This indexation has been studied in numerous articles (for example Arnott *et al.* (2005), Estrada (2008) or Haugen *et al.* (2010)) and aims to provide higher returns and lower risk than capitalization-weighted indexes.

On the other hand, risk-based indexes are meant to diversify the risk of the portfolio. Two well-known examples are the minimum-variance portfolio (or MV portfolio) and the equally-weighted portfolio (or 1/n portfolio). The MV portfolio is located on the mean-variance efficient frontier with the lowest risk. Many equity funds have recently been launched using this concept as it is both easy to compute, due to its unique solution, and recognized as robust, since it is the only one among mean-variance efficient portfolios that does not incorporate any information on expected returns. However, minimum-variance portfolios are generally suffering from – and even deepening – the drawback of portfolio concentration. A natural and simple way to deal with this last issue is to attribute the same weight to all the assets of the portfolio. Equally-weighted portfolios are widely used in practice (Bernartzi and Thaler (2001), Windcliff and Boyle (2004)) and have been shown to be efficient in out-of-sample exercises (DeMiguel *et al.*, 2007). Recently, Choueifaty and Coignard (2008) introduced the concept of the most diversified portfolio (or MDP portfolio). As in the case of minimum-variance portfolios (Clarke *et al.*, 2006), the weights of the portfolio depend only on the covariance matrix. Later, Maillard *et al.* (2008) studied the properties of the equally-weighted risk contributions portfolio (or ERC portfolio) as a new methodology for building a diversified portfolio. All of these methods have contributed to the emergence of the concept of risk-based indexation. The main difference between fundamental and risk-based indexes is that the former promises alpha, whereas the latter promises diversification.

This paper, which aims at comparing the different risk-based indexes, is organized in the following structure: Section 2 analyzes the properties of capitalization-weighted and fundamental indexes; Section 3 details the various risk-based indexes (MV, 1/n, MDP/MSR and ERC portfolios) by comparing them in terms of mathematical properties; Section 4 presents empirical results based on the DJ Euro Stoxx 50 universe over the period 1992-2009 and Section 5 draws conclusions.

## 2 Beyond capitalization-weighted indexes

Let us consider an index composed of $n$ stocks. Let $P_i(t)$ be the price of the $i$-th stock and $R_i(t)$ be the corresponding return between time $t-1$ and $t$:

$$R_i(t) = \frac{P_i(t)}{P_i(t-1)} - 1$$

The value of the index (or benchmark) $B(t)$ at time $t$ is defined by:

$$B(t) = B(t-1) \sum_{i=1}^{n} w_i(t) \left(1 + R_i(t)\right)$$

where $w_i(t)$ is the weight of the $i$-th stock in the index satisfying $\sum_{i=1}^{n} w_i(t) = 1$. The computation of the value of the index $B(t)$ is generally calculated at the closing time $t$.

However, this computation is purely theoretical. In order to replicate this index, we have to build a hedging strategy that consists of investing in stocks. Let $S(t)$ be the value of the strategy (or the index fund). We have:

$$S(t) = \sum_{i=1}^{n} n_i(t) P_i(t)$$

where $n_i(t)$ is the number of stock $i$ held between $t-1$ and $t$. We define the tracking error as the difference between the return of the strategy and the return of the index:

$$e_{S|B}(t) = R_S(t) - R_B(t)$$

The quality of the replication process is generally measured by the volatility $\sigma\left(e_{S|B}(t)\right)$ of the tracking error[1]. We may distinguish several cases:

1. We may have an index fund with a low tracking error volatility (less than 10 bps). It can be achieved by a pure physical replication (by buying all of the components with the appropriate weights each time) or by a synthetic replication (i.e. entering into a swap agreement with an investment bank).

2. We may have an index fund with moderate tracking error volatility (between 10 bps and 30 bps). For example, this is the case with an index fund based on sampling techniques.

3. An index fund with a higher tracking error volatility (between 30 bps and 1%) exists and corresponds either to some universes presenting liquidity problems or to enhanced index funds as a part of active management.

It is also important to note the difference between an investable index and a non-investable index. The frontier between these two categories is not precise. From a theoretical point of view, an investable index may be replicated with a tracking error volatility close to zero. For a non-investable index, it is impossible to replicate it perfectly. For example, stock indexes of major market places are investable. This is the case with the S&P 500, DAX, CAC and/or Nikkei indexes. This is not the case with private equity indexes, some small cap indexes or for certain market places where it is difficult to invest (i.e. the Middle East). Interesting examples are global stock indexes, like the MSCI World Index or the DJ Islamic Market Index. They contain many stocks (more than 2000 for the two cited indexes) and cover a variety of countries.

## 2.1 Capitalization-weighted indexes

By definition, the weights are given by:

$$w_i(t) = \frac{N_i(t) P_i(t)}{\sum_{j=1}^{n} N_j(t) P_j(t)} \tag{1}$$

where $N_i(t)$ is the number of shares outstanding for the $i$-th stock. We notice that $C_i(t) = N_i(t) P_i(t)$ is the market capitalization of the $i$-th stock. The weight $w_i(t)$ then corresponds to the ratio of the market capitalization $C_i(t) = N_i(t) P_i(t)$ of the $i$-th stock with respect to the market capitalization of the index. Generally, the number of shares is constant

[1] People often confuse the notions of tracking error and volatility of the tracking error.

$N_i(t) = N_i(t-1)$ or changes at a low frequency. We also have:

$$\begin{aligned} w_i(t) &= \frac{N_i(t) P_i(t)}{\sum_{j=1}^n N_j(t) P_j(t)} \\ &= \frac{N_i(t-1) P_i(t)}{\sum_{j=1}^n N_j(t-1) P_j(t)} \\ &\neq w_i(t-1) \end{aligned}$$

Regardless of whether the number of shares is constant, the weights of CW indexes move every day because of the price effect, giving us:

$$\begin{aligned} w_i(t) \geq w_i(t-1) &\Leftrightarrow \frac{C_i(t)}{\sum_{j=1}^n C_j(t)} \geq \frac{C_i(t-1)}{\sum_{j=1}^n C_j(t-1)} \\ &\Leftrightarrow \frac{C_i(t)}{C_i(t-1)} \geq \frac{\sum_{j=1}^n C_j(t)}{\sum_{j=1}^n C_j(t-1)} \\ &\Leftrightarrow R_i(t) \geq R_B(t) \end{aligned} \tag{2}$$

Another interesting result of CW indexes is that the portfolio of the hedging strategy does not change if the structure of the market remains the same (or $N_i(t) = N_i(t-1)$). We verify that:

$$n_i(t) = n_i(t-1) \tag{3}$$

We do not need to rebalance the portfolio of the hedging portfolio because of the relationship:

$$n_i(t) P_i(t) \propto w_i(t) P_i(t)$$

This property is one of the main benefits of CW indexes and implies **low trading costs**.

Another important advantage of CW indexes is that they are considered to be a good proxy of the market portfolio defined in the CAPM-Sharpe model. To define the market portfolio, we proceed with two steps (see Figure 1):

1. First, we build the efficient frontier by computing the convex hull of the risk/return ratio for every possible portfolio. This is equivalent to finding all of the portfolios $w^\star$ defined by:

$$\begin{aligned} w^\star &= \arg\max \mu^\top w \\ \text{u.c.} &\quad \sqrt{w^\top \Sigma w} \leq \sigma^\star,\ \mathbf{1}^\top w = 1 \text{ and } \mathbf{0} \leq w \leq \mathbf{1} \end{aligned}$$

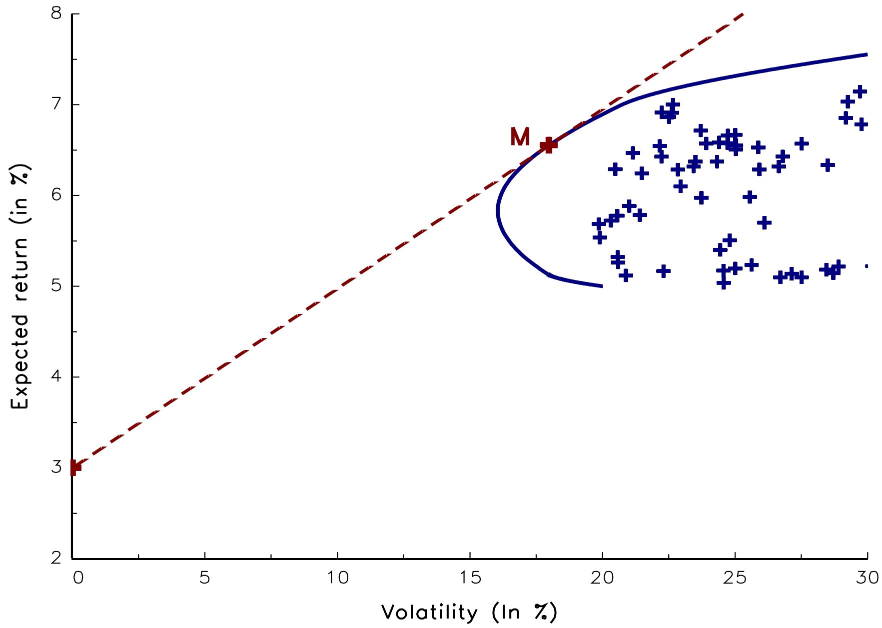
   where $\mu$ is the vector of expected returns, $\Sigma$ is the covariance matrix and $\sigma^\star$ is the desired level of volatility.

2. Second, we determine the capital market line, which is graphically, the tangent line connecting the return of a risk-free-asset with the efficient market frontier. The tangency portfolio belonging to both the efficient frontier and the market line is the market portfolio.

Under certain assumptions, such as the efficient market hypothesis (EMH), the theory states that the tangency portfolio is the unique risky portfolio owned by investors. In a manner of speaking, this means that a CW index defines a tangency portfolio.

The main criticisms we may address to CW indexes are the following:

Figure 1: The Market Portfolio




- By construction, a CW index is trend-following, meaning that it incorporates momentum bias which leads to bubble exposure risk as the weight of the best performers increases and the weight of the worst performers decreases (see Equation 2).
- A CW index generally contains a growth bias, because high-valuation multiple stocks weigh more than low valuation multiple stocks with equivalent realized earnings.
- The index may suffer from a high drawdown risk and a lack of risk diversification.

Another important issue is the concentration of underlying portfolios. In Table 1 and Figure 2, we presented the Gini coefficient and the Lorenz curve for several equity indexes. These two concepts are presented in Appendix A. The Lorenz curve $\mathcal{L}(x)$ is a graphical representation of the concentration. It represents the cumulative weight of the first $x\%$ most representative stocks. If the index is not concentrated, the Lorenz curve is a straight 45° line. The more concentrated the index, the steeper the Lorenz curve. For example, we note in Figure 2 that the DJ Stoxx 50 Index is less concentrated than the OMX Index. Generally, we summarize the information contained in the Lorenz curve with the Gini coefficient, which is a statistical measure of concentration or dispersion. When the Gini coefficient is equal to 0, it corresponds to perfect equality between weights. Total concentration corresponds to a Gini coefficient equal to 1. The results are given in Table 1 where we have ranked several stock indexes using the Gini coefficient from the least concentrated index to the most concentrated index. For each index, we have also indicated the values taken by $\mathcal{L}(x)$ and $\mathcal{L}^{-1}(x)$ for different values of $x$ (expressed as a %). For example, in the case of the NASDAQ 100 Stock Index (NDX), $\mathcal{L}(10\%)$ is equal to 47%, meaning that 10% of the stocks represent almost 50% of the weight, and $\mathcal{L}^{-1}(25\%)$ is equal to 3%, meaning that 25% of the weight is concentrated in 3% of the stocks. We can also note that some non-large indexes (for example SMI or IBEX indexes) are highly concentrated.

Table 1: Gini coefficient of several equity indexes (December 31, 2009)

| | | $\mathcal{L}(x)$ | | | | | | $\mathcal{L}^{-1}(x)$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Index | Gini | 10 | 25 | 50 | 75 | 90 | 95 | 10 | 25 | 50 | 75 | 90 | 95 |
| SX5P | 0.27 | 23 | 45 | 68 | 86 | 95 | 98 | 4 | 11 | 30 | 59 | 81 | 90 |
| INDU | 0.29 | 21 | 42 | 71 | 89 | 97 | 98 | 4 | 13 | 31 | 55 | 76 | 86 |
| SX5E | 0.31 | 24 | 45 | 71 | 90 | 97 | 99 | 3 | 11 | 29 | 55 | 76 | 84 |
| BEL20 | 0.41 | 28 | 51 | 79 | 93 | 98 | 99 | 2 | 8 | 24 | 45 | 67 | 80 |
| OMX | 0.44 | 33 | 57 | 79 | 93 | 98 | 99 | 3 | 7 | 19 | 44 | 69 | 81 |
| CAC | 0.47 | 34 | 58 | 82 | 94 | 98 | 99 | 2 | 6 | 19 | 41 | 65 | 79 |
| DAX | 0.47 | 29 | 58 | 84 | 94 | 98 | 99 | 3 | 9 | 20 | 36 | 63 | 78 |
| HSI | 0.51 | 39 | 63 | 83 | 95 | 99 | 99 | 2 | 6 | 15 | 38 | 64 | 76 |
| AEX | 0.51 | 34 | 62 | 85 | 96 | 99 | 100 | 3 | 7 | 18 | 37 | 58 | 70 |
| NDX | 0.53 | 47 | 66 | 82 | 93 | 98 | 99 | 1 | 3 | 12 | 38 | 66 | 81 |
| NKY | 0.59 | 47 | 69 | 87 | 96 | 99 | 100 | 1 | 4 | 12 | 31 | 56 | 70 |
| MEXBOL | 0.59 | 44 | 68 | 89 | 97 | 99 | 100 | 1 | 4 | 13 | 31 | 52 | 67 |
| SMI | 0.60 | 41 | 71 | 90 | 96 | 99 | 100 | 2 | 5 | 13 | 28 | 50 | 68 |
| SPX | 0.63 | 52 | 73 | 89 | 96 | 99 | 100 | 1 | 3 | 9 | 28 | 54 | 69 |
| UKX | 0.63 | 49 | 76 | 89 | 96 | 99 | 99 | 1 | 3 | 10 | 24 | 52 | 71 |
| SXXE | 0.64 | 52 | 76 | 90 | 96 | 99 | 99 | 1 | 3 | 9 | 24 | 50 | 68 |
| HSCEI | 0.64 | 53 | 77 | 90 | 97 | 99 | 100 | 1 | 4 | 9 | 23 | 49 | 67 |
| SPTSX | 0.66 | 55 | 77 | 90 | 97 | 99 | 100 | 1 | 3 | 8 | 23 | 49 | 67 |
| SXXP | 0.67 | 57 | 78 | 90 | 97 | 99 | 100 | 1 | 2 | 8 | 22 | 49 | 67 |
| IBEX | 0.69 | 61 | 81 | 91 | 97 | 99 | 100 | 1 | 3 | 7 | 17 | 47 | 66 |
| TWSE | 0.78 | 71 | 85 | 94 | 98 | 100 | 100 | 0 | 1 | 3 | 13 | 36 | 53 |
| TPX | 0.82 | 74 | 90 | 97 | 99 | 100 | 100 | 0 | 1 | 3 | 11 | 26 | 41 |
| KOSPI | 0.86 | 81 | 94 | 98 | 100 | 100 | 100 | 0 | 1 | 3 | 8 | 17 | 30 |
| 1/n | 0.00 | 10 | 25 | 50 | 75 | 90 | 95 | 10 | 25 | 50 | 75 | 90 | 95 |

Notes on the indexes in Table 1.

AEX (AEX Index) is an index of 25 leading Dutch stocks traded on the Amsterdam Exchange. BEL20 (BEL 20 Index) is an index of 20 leading Belgian stocks traded on the Brussels Stock Exchange. CAC (CAC 40 Index) is an index of 40 leading French stocks traded on the Paris Bourse. DAX (DAX Index) is an index of 30 leading German stocks traded on the Frankfurt Stock Exchange. HSCEI (Hang Seng China Enterprises Index) is an index of H-Shares listed on the Hong Kong Stock Exchange and included in the Hang Seng Mainland Composite Index. HSI (Hang Seng Index) is an index of leading stocks traded on the Hong Kong Stock Exchange. IBEX (IBEX 35 Index) is an index of 35 leading Spanish stocks traded on the Spanish Continuous Market. INDU (Dow Jones Industrial Average) is an index of 30 blue-chip stocks. KOSPI (KOSPI Index) is an index of all common shares on the Korean Stock Exchanges. MEXBOL (Mexico Bolza Index) is an index of leading Mexican stocks traded on the Mexican Stock Exchange. NDX (NASDAQ 100 Stock Index) is an index of the 100 largest and most active non-financial domestic and international issues listed on the NASDAQ. NKY (NIKKEI 225 Index) is an index of 225 top-rated Japanese companies listed on the First Section of the Tokyo Stock Exchange. OMX (OMX Stockholm 30 Index) is an index of 30 Swedish stocks traded on the Stockholm Stock Exchange. SMI (Swiss Market Index) is an index of the 20 leading Swiss stocks of the SPI universe. SPTSX (S&P/Toronto Stock Exchange Composite Index) is an index of leading stocks listed on the TSX. SPX (S&P 500 Index) is an index of 500 leading US stocks. SX5E (Dow Jones Euro Stoxx 50) is an index of 50 European blue-chip stocks from countries participating in the EMU. SX5P (Dow Jones Stoxx 50) is an index of 50 European blue-chip stocks. SXXE (Dow Jones Euro Stoxx)is an index that includes stocks of the DJ Stoxx 600 Index from countries participating in the EMU. SXXP (Dow Jones Stoxx 600) is an index of 600 leading European stocks. TPX (Tokyo Stock Price Index) is an index of all companies listed on the First Section of the Tokyo Stock Exchange. TWSE (Taiwan Taiex Index) is an index of all listed common shares traded on the Taiwan Stock Exchange. UKX (FTSE 100 Index) is an index of 100 leading stocks traded on the London Stock Exchange.

Figure 2: Lorenz curve of several equity indexes (December 31, 2009)

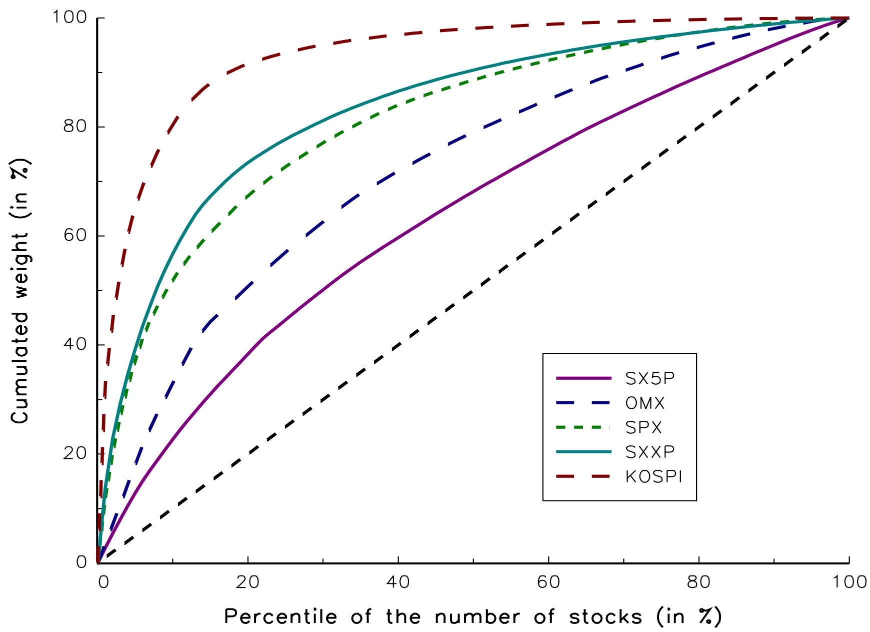



Figure 3: Change in the Gini coefficient for the DJ Euro Stoxx 50 Index

**Remark 1** *Table 1 is a snapshot of the Gini coefficient at one trading date (December 13, 2009) but it changes over time. In Figure 3, we indicate the coefficient from 1992 to 2009 on a monthly basis for the DJ Euro Stoxx 50 Index, which is the universe examined in Section 4 presenting the empirical results. We see that the Gini coefficient is higher in the dot-com period. We have also indicated the Gini coefficient computed with risk contributions. We will comment on these results in Section 4.*

### 2.2 Price-weighted indexes

In the previous paragraph, we saw that the dynamic of a capitalized-weighted index has two components: a **share component** and a **price component**. Generally, the share component is very stable over time, whereas the price component changes every day, meaning that the high-frequency dynamic of the CW index is driven entirely by the price dynamics. Price-weighted (PW) indexation utilizes this idea. We have:

$$B(t) = B(t-1) \frac{\sum_{i=1}^{n} P_i(t)}{\sum_{i=1}^{n} P_i(t-1)}$$

Two well-known PW indexes are the Nikkei Index and the Dow Jones Industrial Average Index. Price-weighted indexation has been criticized extensively, but from our point of view, price-weighted indexation and capitalized-weighted indexation are very close.

### 2.3 Fundamental indexes

Fundamental indexation has been extensively studied by Arnott *et al.* (2005). The idea is to define the weights as a function of fundamental statistics. A basic example of fundamental statistics is the dividend yield. In November 2003, Dow Jones launched the DJ US Select Dividend Index followed by other dividend indexes by country and region. Now, there are other dividend index providers like WisdomTree, MarketGrader, etc. In November 2005, FTSE launched the RAFI® Index (Research Affiliates Fundamental Index) with Research Affiliates, LLC, formed by Robert D. Arnott. The RAFI method uses four fundamental measures: Cash Flow, Sales, Book Value and Dividends. In Figure 4, we represent the performance of these two indexes (with base 100 in January 1992). It is Interesting to note that the two fundamental indexes outperformed the S&P 500 Index.

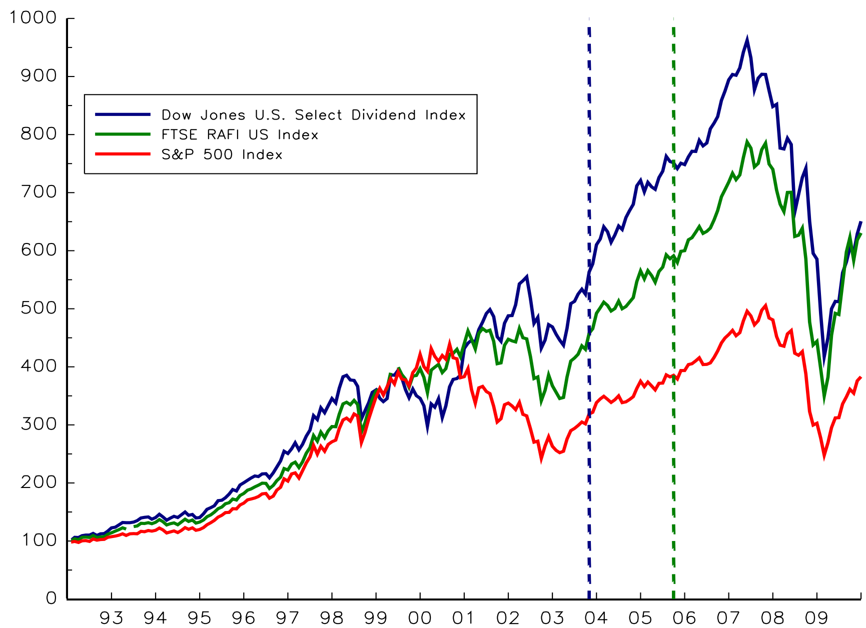
One of the reasons for the success of fundamental indexes is the promise of potentially superior returns:

> "*We show that the fundamentals-weighted, noncapitalization-based indexes consistently provide higher returns and lower risks than the traditional cap-weighted equity market indexes while retaining many of the benefits of traditional indexing*" [Arnott *et al.* (2005), page 83].

Figure 4 shows the impressiveness of backtestings. Despite the lack of any significant overperformance in 2007 and 2008, these indexes performed solidly in 2009. Still, the construction of these indexes is not obvious and some practitioners associate them with a stock-picking strategy packaged in a passive strategy[2].

[2] In asset management, many quantitative stock screening models and value investment strategies are based on these fundamental or economic variables.

Figure 4: Performance since January 1992




## 2.4 Construction of alternative-weighted indexes

There are several arguments supporting capitalization-weighted indexation. First, a CW index is comprehensible and data-independent; closing prices and the number of shares are available without any measurement errors. Second, trading costs are low, because it corresponds to a buy-and-hold strategy if the number of shares remains unchanged. Lastly, a CW index may be easily hedged and replicated because liquidity is highly correlated with market capitalizations.

We will now define what are the necessary properties of an appropriate alternative-weighted index. In what follows, the CW index is considered as the reference index.

- The universe of the AW index is included within the universe of the CW index:

$$\mathcal{U}_{\mathrm{AW}} \subseteq \mathcal{U}_{\mathrm{CW}}$$

  This conveys that the AW index contains only assets belonging to the corresponding reference index. For us, it is a key property in order to qualify the AW index as a passive strategy and to minimize some style biases between the AW and CW indexes.

- In the long-term, the AW index must perform better than the CW index, and/or the volatility of the AW index must be lower than that of the CW index. With respect to the CW index, the AW index must obviously be characterized as an alpha index (like fundamental indexation) or a beta index (like risk-based indexation).

- The correlation between the performance of the AW index and the performance of the CW index is strictly different from one:

$$\rho\left(R_{\mathrm{AW}}\left(t;t+h\right),R_{\mathrm{CW}}\left(t;t+h\right)\right)=\frac{w_{\mathrm{AW}}^{\top}\Sigma w_{\mathrm{CW}}}{\sqrt{w_{\mathrm{AW}}^{\top}\Sigma w_{\mathrm{AW}}}\sqrt{w_{\mathrm{CW}}^{\top}\Sigma w_{\mathrm{CW}}}}<1$$

The lower the correlation, the higher the interest of the AW index. This third point is the main rationale for the AW index. An investor who prefers to invest in both AW and CW indexes does not necessarily seek better performance, but wants to diversify the risk in order to achieve a better risk-return profile.

- The rules for constructing the AW index are clearly defined, and the computation of the index values may be performed by several third-parties.

# 3 Risk-based indexation

We have seen that there are two methods for building alternative-weighted indexes:

1. Fundamental indexes (promising alpha)
2. Risk-based indexes (promising diversification)

In some sense, the difference between the two methods comes from the opinion of modifying the risk-adjusted return ratio. In other words, in the case of fundamental indexes, one expects to have superior returns, and one expects to create alpha with respect to the CW index. In the case of risk-based indexes, one expects to decrease the risk of the portfolio in either absolute or relative value. The AW index may have a smaller risk than the CW index or the combination of the AW index and the CW index may produce lower risk because they are not perfectly correlated. Consider a portfolio with $(1-x)\,\%$ of the CW index and $x\%$ of the AW index. We denote the volatility of the returns of the two indexes as $\sigma_{\mathrm{CW}}$ and $\sigma_{\mathrm{AW}}$ and the correlation between them as $\rho$. If we assume that the two indexes have the same expected return, then we can compute the optimal portfolio. It appears that $x$ is equal to 0 if and only if we have $\sigma_{\mathrm{CW}} < \rho\sigma_{\mathrm{AW}}$. As such, financial theory explains that if the two indexes have the same expected return, it is better to invest in the two indexes rather than only in the CW index, except if the volatility of the CW index is particularly low with respect to the volatility of the AW index. In Figure 5, we have provided some examples with $\sigma_{\mathrm{CW}} = 18\%$. In the first case ($\sigma_{\mathrm{AW}} = 20\%$ and $\rho = 95\%$), there is no interest in diversifying the portfolio, because the inequality $\sigma_{\mathrm{CW}} > \rho\sigma_{\mathrm{AW}}$ does not hold. In other illustrative cases, the investor is particularly interested in diversification if the correlation is low.

In this section, we focus on four risk-based indexation methods: 1/n, MV, MDP/MSR and ERC portfolios. We have chosen these because they are already used by the investment industry, but other solutions, which are more complex, also exist (see for example Bera *et al.*, 2008). The main advantage of the four methods is that they have been studied both by academics and professionals.

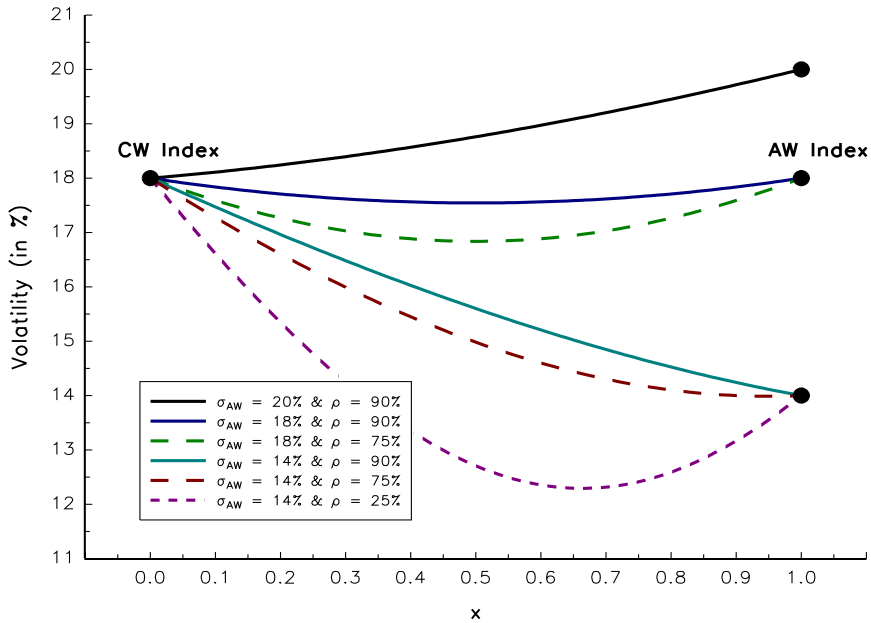
## 3.1 The 1/n portfolio

The idea of the 1/n portfolio is to define a portfolio independently from estimated statistics and properties of stocks (Windcliff *et al.*, 2004). If we assume that it is impossible to predict return and risk, then attributing an equal weight to all of the portfolio components constitutes a natural choice. The structure of the portfolio depends only on the number $n$ of stocks because the weights are equal and uniform:

$$w_i = \frac{1}{n}$$

This type of indexation is easy to understand, thanks to the uncomplicated rules of construction. It corresponds to a contrarian strategy with a take-profit scheme, because if one stock has a substantial return between two rebalancing dates, its weight will reset to $\frac{1}{n}$ at

Figure 5: Illustration of the diversification effect




the next rebalancing date. However, in a CW index, the opposite is true: the larger the return of the stock, the greater its weight. An appealing property is that the 1/n index is the least concentrated portfolio in terms of weights[3]. The main weakness is that it does not consider individual risks and correlations between these risks, which implies that it is difficult to locate this portfolio in a mean-variance framework. From a theoretical point of view, the 1/n portfolio coincides with the efficient portfolio if the expected returns and volatilities of stocks are assumed to be equal and correlation is uniform. If we are not far from these assumptions, we may consider the 1/n portfolio to be similar to the efficient portfolio

## 3.2 The minimum-variance portfolio

The minimum-variance (MV) portfolio is defined by the following mathematical problem:

$$\begin{aligned} w^{\star} &= \arg\min w^{\top}\Sigma w \\ &\text{u.c.} \quad \mathbf{1}^{\top}w = 1 \text{ and } \mathbf{0} \leq w \leq \mathbf{1} \end{aligned}$$

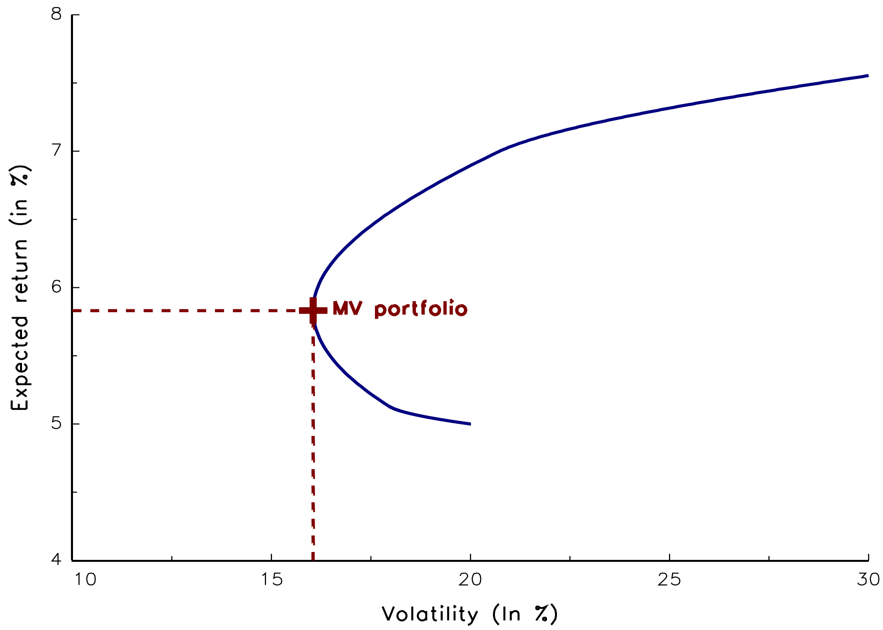
The MV portfolio is the only portfolio located on the efficient frontier that is not dependent on the expected returns hypothesis. The main advantages of the MV portfolio are:

- It is easy to understand (like the 1/n portfolio).
- By construction, its volatility is low, at least on an ex-ante basis.
- Academic literature generally reports that it presents some good out-of-sample performance in a complete economic cycle (Clarke *et al.*, 2006).

[3]For the 1/n portfolio, the Gini coefficient of weights is equal to zero.

This portfolio is the tangency portfolio if and only if expected returns are equal for all stocks[4]. Notwithstanding these advantages, the MV portfolio poses a serious drawback: diversification of volatility, but not weight. This implies that the portfolio is concentrated in relatively few stocks. All portfolios that are computed through optimization based on the covariance matrix $\Sigma$ share this drawback.

Figure 6: The MV portfolio and the efficient frontier




## 3.3 The MDP/MSR portfolio

Academic literature defines the MDP portfolio as the most-diversified portfolio (Choueifaty and Coignard, 2008) and is synonymous with the MSR portfolio (maximum Sharpe ratio) presented by Martellini (2008). It corresponds to the tangency portfolio in which it is assumed that the risk premium is proportional to the volatility or, equivalently, when all of the assets have the same Sharpe ratio. The traditional optimization program is defined as follows:

$$\begin{aligned} w^{\star} &= \arg\max \operatorname{sh}(w) \\ &\text{u.c.} \quad \mathbf{1}^{\top} w = 1 \text{ and } \mathbf{0} \leq w \leq \mathbf{1} \end{aligned}$$

where:[5]:

$$\operatorname{sh}(w) = \frac{\mu^{\top} w - r}{\sqrt{w^{\top} \Sigma w}}$$

[4]In the case of no risk-free asset and no short-selling constraint, we may show that every optimal portfolio located in the efficient frontier contains a proportion of the MV portfolio. This result derives from Black's two-fund separation theorem. However, it does not hold when there is a risk-free asset or when we impose a no-short-selling constraint.

[5]We remind that $\mu$ is the vector of expected returns and $r$ is the risk free rate.

In the case of the MDP/MSR portfolio, we assume that $\mu_i = r + s\sigma_i$, and the objective function becomes:

$$\mathrm{sh}\left(w\right) = s\frac{\sigma^\top w}{\sqrt{w^\top \Sigma w}}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$ is the vector of volatilities. This portfolio presents appealing properties, such as better diversification and less sensitivity to inputs (than the MV portfolio) and is not dependent upon any expected return hypothesis. Moreover, it is the optimal portfolio when it is assumed that all Sharpe ratios are equal. Similar to the MV portfolio, it may still be concentrated in a few stocks.

## 3.4 The ERC portfolio

The last construction is the ERC portfolio, which is derived from the techniques of risk-budgeting (Scherer, 2007). The ERC portfolio corresponds to the portfolio in which the risk contribution from each stock is made equal. It is the simplest risk budgeting rule. If we assume that risk and correlation can be reasonably forecast but that it is impossible to predict return, then attributing an equal budget of risk to all of the portfolio components seems natural.

Let $\sigma\left(w\right) = \sqrt{w^\top \Sigma w}$ be the volatility of the portfolio with weights $w$. We can show that we have the following decomposition:

$$\sigma\left(w\right) = \sum_{i=1}^{n} \mathrm{RC}_i = \sum_{i=1}^{n} w_i \frac{\partial\, \sigma\left(w\right)}{\partial\, w_i}$$

The term $\mathrm{RC}_i = w_i \frac{\partial\, \sigma(w)}{\partial\, w_i}$ is called the risk contribution of the $i^{\mathrm{th}}$ asset to the volatility of the portfolio, which is the product of the weight times the marginal risk. The ERC portfolio then corresponds to the portfolio in which the risk contribution from each stock is equal[6] – the most straightforward risk budgeting rule. The main advantages of this method are the following:

1. It defines a portfolio that is well diversified in terms of risk and weights.
2. Like the MV and 1/n portfolios, it does not depend on any expected returns hypothesis.
3. It is less sensitive to small changes in the covariance matrix than the MV and MDP/MSR portfolios (Maillard *et al.*, 2008).

Similar to the 1/n portfolio, it is difficult to locate on the mean-variance framework, but it corresponds to the optimal portfolio when the correlation is uniform and the assets have the same Sharpe ratio. The ERC portfolio coincides with the MDP/MSR portfolio when the correlation is uniform.

[6]The ERC portfolio is also the least concentrated portfolio in terms of risk contributions, because the corresponding Gini coefficient is zero.

## 3.5 Comparison of the four passive indexation methods

### 3.5.1 Some properties

Although the four methods are based on different approaches, they present similarities. First, we can compare them in terms of weights and risk contributions:

$$w_i = w_j \tag{1/n}$$
$$\frac{\partial\,\sigma(w)}{\partial\,w_i} = \frac{\partial\,\sigma(w)}{\partial\,w_j} \tag{MV}$$
$$w_i \times \frac{\partial\,\sigma(w)}{\partial\,w_i} = w_j \times \frac{\partial\,\sigma(w)}{\partial\,w_j} \tag{ERC}$$
$$\frac{1}{\sigma_i} \times \frac{\partial\,\sigma(w)}{\partial\,w_i} = \frac{1}{\sigma_j} \times \frac{\partial\,\sigma(w)}{\partial\,w_j} \tag{MDP}$$

The weights are equal in the 1/n portfolio whereas the marginal risk is equal in the MV portfolio. In the case of the ERC portfolio, this is the product of the weight times the marginal which is equal. For the MDP/MSR portfolio, the equality is on the marginal risk divided by the volatility (this measure may be interpreted as relative or scaled marginal risk). We notice that the equalities are verified in the case of the MV or the MDP/MSR portfolio only for the assets with a non-zero weight.

Another important result is that the volatility of the MV, ERC and 1/n portfolios may be ranked in the following order (Maillard *et al.*, 2008):

$$\sigma_{\mathrm{MV}} \leq \sigma_{\mathrm{ERC}} \leq \sigma_{1/\mathrm{n}}$$

The ERC portfolio may then be viewed as a portfolio between the MV and the 1/n portfolios. Of course, in the case of the MDP/MSR portfolio, we also have $\sigma_{\mathrm{MV}} \leq \sigma_{\mathrm{MDP}}$, but a comparison with the ERC and 1/n portfolios is not possible. The volatility of the MDP/MSR portfolio may be either greater or lower than the volatility of the ERC and 1/n portfolios.

It is important to mention that the ERC and MDP/MSR portfolios coincide when the correlation is uniform across assets returns. In this case, the weight $w_i$ of the $i^{\mathrm{th}}$ stock is inversely proportional to its volatility $\sigma_i$. The MDP/MSR portfolio corresponds to the MV portfolio when the individual volatilities $\sigma_i$ are equal. Curiously, the ERC and MV portfolios are the same when the correlation is uniform and is equal to the lower bound $\rho = -1/(n-1)$ (i.e. when diversification from correlation is maximum).

### 3.5.2 Some examples

We illustrate here the properties of these four portfolios by looking at six theoretical examples, for a better understanding of the characteristics of these four risk-based portfolios and the issues faced in practice when they are implemented.

**Example 1** *We consider an example with four assets. We first assume that the volatility $\sigma_i$ is the same and equal to 20% for all four assets. The correlation matrix $C$ is equal to:*

$$C = \begin{pmatrix} 100\% & & & \\ 80\% & 100\% & & \\ 0\% & 0\% & 100\% & \\ 0\% & 0\% & -50\% & 100\% \end{pmatrix}$$

*The results (expressed as a %) are reported in Table 2. We verify that because the volatility of the assets is the same, the MDP/MSR portfolio is equal to the MV portfolio. We also check* *that the marginal risks* $\mathrm{MR}_i$ *are equal for the MV portfolio whereas it is the risk contributions* $\mathrm{RC}_i$ *that are equal for the ERC portfolio. We notice that the MV and the ERC portfolios are similar in terms of weights.*

Table 2: Weights and risk contributions (Example 1)

| | MV | | | ERC | | | MDP/MSR | | | 1/n | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Asset | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ |
| 1 | 10.9 | 8.8 | 1.0 | 17.3 | 13.4 | 2.3 | 10.9 | 8.8 | 1.0 | 25.0 | 16.8 | 4.2 |
| 2 | 10.9 | 8.8 | 1.0 | 17.3 | 13.4 | 2.3 | 10.9 | 8.8 | 1.0 | 25.0 | 16.8 | 4.2 |
| 3 | 39.1 | 8.8 | 3.5 | 32.7 | 7.1 | 2.3 | 39.1 | 8.8 | 3.5 | 25.0 | 4.7 | 1.2 |
| 4 | 39.1 | 8.8 | 3.5 | 32.7 | 7.1 | 2.3 | 39.1 | 8.8 | 3.5 | 25.0 | 4.7 | 1.2 |
| $\sigma(w)$ | | | 8.8 | | | 9.3 | | | 8.8 | | | 10.7 |

**Example 2** *We modify Example 1 by introducing differences in volatilities. They are* 10%, 20%, 30% *and* 40% *respectively. We consider the same correlation matrix as in Example 1. We notice that the MV portfolio is concentrated in the first asset because of its low level of volatility. The weight of the second asset is 0, because although its volatility is smaller than that of the third and fourth asset, its correlation with the first asset is high. We verify that the values of the marginal risk* $\mathrm{MR}_i$ *in the MV portfolio are equal for assets with non-zero weights (they are equal to* 8.6%*). The ERC and MDP/MSR portfolios produce more balanced portfolios in terms of weights. In this example, we check the inequalities* $\sigma_{\mathrm{MV}} \leq \sigma_{\mathrm{ERC}} \leq \sigma_{1/\mathrm{n}}$. *Unlike the first example, however, the volatility of the MDP/MSR portfolio is now higher than the volatility of the ERC portfolio.*

Table 3: Weights and risk contributions (Example 2)

| | MV | | | ERC | | | MDP/MSR | | | 1/n | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Asset | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ |
| 1 | 74.5 | 8.6 | 6.4 | 38.4 | 6.7 | 2.6 | 27.8 | 4.4 | 1.2 | 25.0 | 5.6 | 1.4 |
| 2 | 0.0 | 13.8 | 0.0 | 19.2 | 13.4 | 2.6 | 13.9 | 8.8 | 1.2 | 25.0 | 12.2 | 3.0 |
| 3 | 15.2 | 8.6 | 1.3 | 24.3 | 10.6 | 2.6 | 33.3 | 13.3 | 4.4 | 25.0 | 6.5 | 1.6 |
| 4 | 10.3 | 8.6 | 0.9 | 18.2 | 14.1 | 2.6 | 25.0 | 17.7 | 4.4 | 25.0 | 21.7 | 5.4 |
| $\sigma(w)$ | | | 8.6 | | | 10.3 | | | 11.3 | | | 11.5 |

**Example 3** *We now reverse the volatilities of Example 2. They are now equal to* 40%, 30%, 20% *and* 10%. *Weights of the MV, ERC and MDP/MSR portfolios are similar. The volatilities of the corresponding portfolios are comparable, but the 1/n portfolio has a high volatility with respect to the three other portfolios. It can be explained by the fact that the first and second assets are more volatile and highly correlated, meaning that diversification effects are low.*

Table 4: Weights and risk contributions (Example 3)

| | MV | | | ERC | | | MDP/MSR | | | 1/n | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Asset | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ |
| 1 | 0.0 | 6.8 | 0.0 | 7.3 | 26.8 | 2.0 | 4.2 | 17.7 | 0.7 | 25.0 | 37.3 | 9.3 |
| 2 | 4.5 | 6.4 | 0.3 | 9.7 | 20.1 | 2.0 | 5.6 | 13.3 | 0.7 | 25.0 | 27.1 | 6.8 |
| 3 | 27.3 | 6.4 | 1.7 | 27.7 | 7.1 | 2.0 | 30.1 | 8.8 | 2.7 | 25.0 | 4.4 | 1.1 |
| 4 | 68.2 | 6.4 | 4.4 | 55.3 | 3.5 | 2.0 | 60.2 | 4.4 | 2.7 | 25.0 | 0.0 | 0.0 |
| $\sigma(w)$ | | | 6.4 | | | 7.8 | | | 6.8 | | | 17.2 |

**Example 4** *Now we consider an example with six assets. The volatilities are* 25%, 22%, 14%, 30%, 40% *and* 30% *respectively. We use the following correlation matrix:*

$$C = \begin{pmatrix} 100\% & & & & & \\ 60\% & 100\% & & & & \\ 60\% & 60\% & 100\% & & & \\ 60\% & 60\% & 60\% & 100\% & & \\ 60\% & 60\% & 60\% & 60\% & 100\% & \\ 60\% & 60\% & 60\% & 60\% & 20\% & 100\% \end{pmatrix}$$

*The correlation matrix is specific, because the correlation is uniform and equal to* 60% *for all assets except for the correlation between the fifth and sixth assets which is equal to* 20%. *The results are surprising. Whereas the MV portfolio concentrates the weights in the second and third assets, the MDP/MSR portfolio concentrates the weights in the fifth and sixth assets. The ERC portfolio overweight the third asset, but the weights are close to the 1/n portfolio. In this example, it is the MDP/MSR portfolio that has the largest volatility. It is interesting to note that in this example, the MV portfolio is sensitive to the* ***specific volatility risk*** *whereas the MDP/MSR portfolio is sensitive to the* ***specific correlation risk****.*

Table 5: Weights and risk contributions (Example 4)

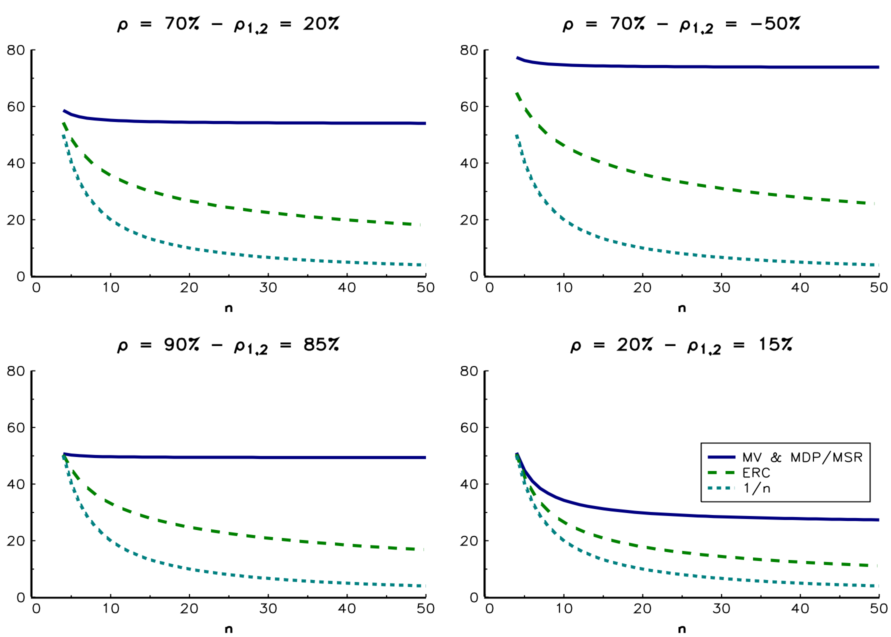
| | MV | | | ERC | | | MDP/MSR | | | 1/n | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Asset | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ | $w_i$ | $\mathrm{MR}_i$ | $\mathrm{RC}_i$ |
| 1 | 0.0 | 15.3 | 0.0 | 15.7 | 20.7 | 3.3 | 0.0 | 19.4 | 0.0 | 16.7 | 20.8 | 3.5 |
| 2 | 3.6 | 14.0 | 0.5 | 17.8 | 18.2 | 3.3 | 0.0 | 17.0 | 0.0 | 16.7 | 18.1 | 3.0 |
| 3 | 96.4 | 14.0 | 13.5 | 28.0 | 11.6 | 3.3 | 0.0 | 10.8 | 0.0 | 16.7 | 11.1 | 1.9 |
| 4 | 0.0 | 18.4 | 0.0 | 13.1 | 24.9 | 3.3 | 0.0 | 23.2 | 0.0 | 16.7 | 25.4 | 4.2 |
| 5 | 0.0 | 24.5 | 0.0 | 10.9 | 30.0 | 3.3 | 42.9 | 31.0 | 13.3 | 16.7 | 31.4 | 5.2 |
| 6 | 0.0 | 18.4 | 0.0 | 14.5 | 22.5 | 3.3 | 57.1 | 23.2 | 13.3 | 16.7 | 21.6 | 3.6 |
| $\sigma(w)$ | | | 14.0 | | | 19.5 | | | 26.6 | | | 21.4 |

**Example 5** *To illustrate how the MV and MDP/MSR portfolios are sensitive to specific risks, we consider a universe of $n$ assets with volatility equal to* 20%. *The structure of the correlation matrix is the following:*

$$C = \begin{pmatrix} 100\% & & & & \\ \rho_{1,2} & 100\% & & & \\ 0 & \rho & 100\% & & \\ \vdots & \vdots & \ddots & 100\% & \\ 0 & \rho & \cdots & \rho & 100\% \end{pmatrix}$$

*The first asset is not correlated with the other assets, except with the second asset. The correlation of the second asset to n is uniform and equal to* $\rho$*. The correlation of the first two assets is set to* $\rho_{1,2}$*. The correlation matrix is more specific, because it is similar to a constant-correlation matrix except for one asset. In Figure 7, we report the sum* $w_1 + w_2$ *with respect to the number* $n$ *of assets by considering several values of* $\rho$ *and* $\rho_{1,2}$*. It is interesting to note the significant difference between the MV and MDP/MSR portfolios*[7] *on one side and the ERC and 1/n portfolios on the other side. The sum* $w_1 + w_2$ *decreases faster for the ERC and 1/n portfolio whereas the decrease is low for the MV and MDP/MSR portfolios. For example, if* $\rho = 70\%$ *and* $\rho_{1,2} = 20\%$*, the sum* $w_1 + w_2$ *is equal to* 58.6% *(MV and MDP/MSR),* 54.3% *(ERC) and* 50% *(1/n) respectively if n is equal to 4. If n is now set to 50, the sum* $w_1 + w_2$ *becomes* 54.1% *(MV and MDP/MSR),* 18.1% *(ERC) and* 4% *(1/n) respectively. The number of assets only marginally impacts MV and MDP/MSR weights, because they rely solely on the covariance matrix. They concentrate their weights in the relatively least volatile or least correlated assets. The ERC portfolio with its implicit diversification constraint, naturally dilutes weights among components as the number of assets increases.*

Figure 7: Weights (in %) of the first two assets (Example 5)




# 4 Empirical results

In order to illustrate the preceding methods, we consider the universe of the DJ Euro Stoxx 50 Index from December 31, 1991, to December 31, 2009. Results are purely indicative. If we use another universe, we may find other empirical results. However, we believe that we can draw some interesting conclusions from this specific example.

We build risk-based indexes by using the following characteristics:

[7] They are equal because the volatilities are the same.

- Every month, we consider only the stocks which belong to the DJ Euro Stoxx 50 Index.
- We compute the empirical covariance matrix using a 1 year (or 260 trading days) window lag at the last trading day of the month.
- The portfolio is rebalanced on the first trading day of the next month.
- The risk-based index is computed daily as a price index.

## 4.1 Performance and risk

The results are presented in Table 6. We first notice that the four risk-based indexes outperform the DJ Euro Stoxx 50 Index. We have to be cautious that we do not take trading costs into account, an issue discussed in the next paragraph. Comparing the risk-based indexes with the CW index in terms of risk, we observe that they present lower risk if we consider the volatility or drawdown measures except for the 1/n portfolio. It is not true for the kurtosis measure. Finally, we report the correlation of daily returns with the DJ Euro Stoxx 50 Index. The least correlated is the MV portfolio followed by the MDP/MSR portfolio. Results by calendar year are reported in Table 7.

Table 6: Statistics of performance and risk

| | CW | MV | ERC | MDP | 1/n |
|---|---|---|---|---|---|
| Performance | 6.39 | 8.08 | 10.30 | 12.63 | 9.22 |
| Volatility | 22.41 | 17.65 | 20.66 | 20.00 | 22.43 |
| Sharpe | 0.29 | 0.46 | 0.50 | 0.62 | 0.41 |
| Volatility of TE | | 14.85 | 5.98 | 13.19 | 4.37 |
| IR | | 0.11 | 0.65 | 0.47 | 0.65 |
| Drawdown | 66.88 | 55.89 | 56.84 | 49.95 | 61.79 |
| Skewness (monthly) | −0.50 | −1.06 | −0.55 | −0.58 | −0.45 |
| Kurtosis (monthly) | 3.87 | 5.31 | 4.42 | 4.25 | 4.70 |
| Skewness | 0.06 | 2.12 | 0.24 | 3.44 | 0.08 |
| Kurtosis | 8.63 | 59.59 | 11.05 | 90.58 | 9.71 |
| Correlation | 100.00 | 75.00 | 94.66 | 81.24 | 98.10 |

Notes about the statistics in Table 6.
All the statistics are computed on a daily basis if not mentioned (otherwise only the skewness and the kurtosis are computed on a monthly basis). They are expressed in % except for the Sharpe and IR ratios, the skewness and kurtosis coefficients which are measured in decimals. The performance corresponds to the annualized return in %. The Sharpe ratio is the annualized performance divided by the volatility without taking into account the risk-free rate. The volatility of the index and the volatility of the tracking error are equal to the standard deviation of the daily return or difference of returns multiplied by the square root of 260. The information ratio (IR) is the annualized excess performance divided by the volatility of the tracking error.

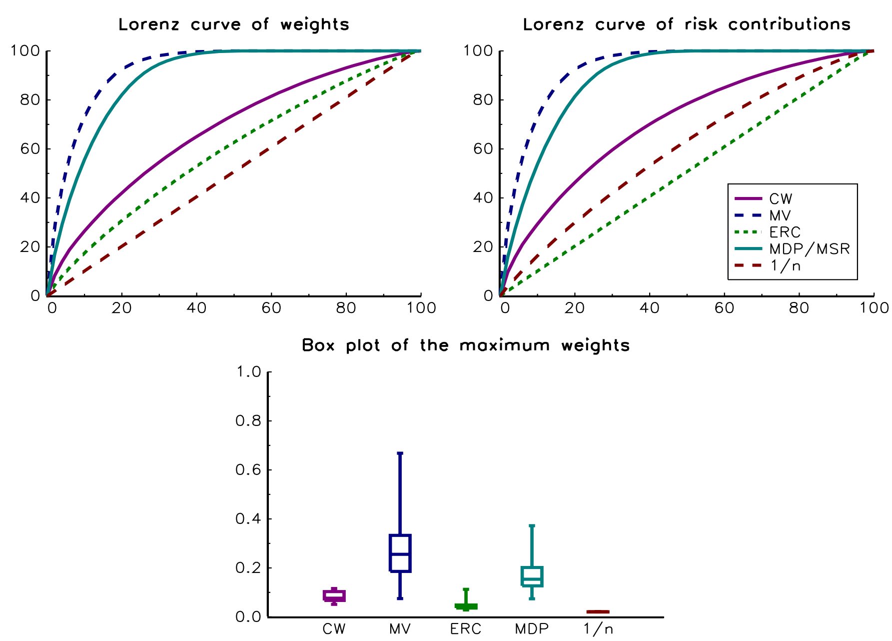
## 4.2 On the importance of constraints

Figure 8 represents some statistics regarding concentration. The top/left graph corresponds to the average of all the Lorenz curves of weights. The MV and MDP portfolios are more concentrated than the CW index, but it is not the case for the ERC portfolio. Of course, the 1/n portfolio appears the least concentrated. If we build the Lorenz curve on risk contributions (top/right graph), we obtain the same conclusions: the MV and MDP portfolios are the most concentrated indexes. Another way to illustrate this concentration is to consider

Table 7: Statistics of performance and risk by year

| | Performance (in %) | | | | | Volatility (in %) | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Year | CW | MV | ERC | MDP | 1/n | CW | MV | ERC | MDP | 1/n |
| 1993 | 38.7 | 38.1 | 43.5 | 45.5 | 44.5 | 10.8 | 7.9 | 8.7 | 8.2 | 10.0 |
| 1994 | −7.9 | −4.0 | −2.1 | 6.6 | −2.7 | 13.2 | 9.5 | 11.5 | 10.4 | 12.5 |
| 1995 | 14.5 | 16.0 | 14.3 | 18.1 | 13.0 | 10.2 | 8.0 | 9.5 | 9.0 | 10.3 |
| 1996 | 22.8 | 26.4 | 29.5 | 33.2 | 30.0 | 10.2 | 10.2 | 9.9 | 10.4 | 10.2 |
| 1997 | 36.8 | 38.8 | 45.0 | 46.7 | 44.6 | 20.0 | 12.9 | 18.4 | 17.2 | 19.4 |
| 1998 | 32.0 | 47.6 | 34.4 | 52.9 | 33.6 | 26.6 | 19.2 | 24.0 | 20.7 | 25.5 |
| 1999 | 46.7 | 20.8 | 36.6 | 25.6 | 41.9 | 20.1 | 17.2 | 17.5 | 18.0 | 18.3 |
| 2000 | −2.7 | 4.6 | 5.8 | 3.8 | 2.5 | 23.1 | 15.4 | 15.7 | 16.3 | 17.5 |
| 2001 | −20.2 | −11.2 | −13.8 | −10.3 | −17.7 | 27.2 | 18.7 | 22.9 | 19.7 | 26.2 |
| 2002 | −37.3 | −34.7 | −32.9 | −29.2 | −34.8 | 37.2 | 24.6 | 35.0 | 29.9 | 38.4 |
| 2003 | 15.7 | 4.3 | 18.8 | 24.9 | 23.3 | 27.1 | 18.5 | 25.9 | 24.9 | 28.6 |
| 2004 | 6.9 | 15.6 | 10.0 | 8.3 | 8.0 | 14.0 | 9.6 | 13.0 | 12.2 | 14.3 |
| 2005 | 21.3 | 16.9 | 20.0 | 16.1 | 20.4 | 11.0 | 9.7 | 10.8 | 10.4 | 11.2 |
| 2006 | 14.3 | 16.7 | 17.9 | 15.5 | 17.5 | 14.7 | 11.7 | 14.1 | 12.9 | 14.6 |
| 2007 | 6.8 | −2.3 | 5.0 | −2.5 | 5.2 | 15.9 | 11.9 | 14.6 | 12.4 | 15.5 |
| 2008 | −44.4 | −15.7 | −36.1 | −20.1 | −44.2 | 39.4 | 40.1 | 38.6 | 44.0 | 40.6 |
| 2009 | 21.1 | −5.2 | 25.5 | 16.9 | 29.4 | 28.0 | 23.0 | 27.8 | 26.5 | 31.7 |

Figure 8: Lorenz curve of weights



the maximum weights of the portfolio. We have represented their box plots[8] for the 216 monthly rebalanced dates (below graph).

These observations imply that we have to introduce some constraints in the MV and MDP/MSR indexation if we want to obtain an investment strategy that makes sense. We report in Table 8 the composition of the different portfolios as of December 31, 2009. The MV index is invested in exactly 11 stocks, whereas the MDP/MSR portfolio is composed of 17 stocks. By construction, the ERC and 1/n portfolios are invested in all the stocks of the universe. So, the MV and MDP/MSR portfolios take big bets on some stocks contrary to the ERC and 1/n portfolios. We think that it is certainly the main difference between these risk-based indexes. We have also reported the composition of the MV and MDP/MSR portfolios when we impose an upper bound (10% or 5%).

In Table 9, we have computed the annualized turnover[9] based on the monthly weights. In the case of the 1Y estimated covariance matrix (which is our default case), the turnover is equal respectively to 327%, 65%, 340% and 20% for the MV, ERC, MDP/MSR and 1/n indexes. We notice that the turnover is not equal to zero for the 1/n portfolio because of the entry/exit in the universe. The turnover of the ERC portfolio is relatively small whereas it is high for the MV and MDP/MSR portfolios. To reduce these turnovers, we may impose some constraints (like an upper bound) or use a covariance matrix estimated with a longer window lag. For example, turnover is reduced on average by a factor of two with a 3 year window lag. Moreover, if we impose a 5% upper bound on the weights, the turnover becomes 74% and 95% for the MV and MDP/MSR portfolios.

In practice, we can not build a MV index or a MDP/MSR index without imposing some constraints[10] because these portfolios are too concentrated (more than the capitalized-weighted index). We report the Gini coefficient for the different portfolios at the end of year 2009 in Table 10. We compute this coefficient for both the weights ($G_{\mathrm{W}}$) and the risk contributions ($G_{\mathrm{RC}}$). For the CW index, we retrieve the value of 0.31 for the weights (computed in Table 1). In terms of risk contributions, this index is more concentrated ($G_{\mathrm{RC}} = 0.38$). We also verify that MV and MDP/MSR indexes are more concentrated. Even if we impose an upper bound of 5% for the weights, we have $G_{\mathrm{W}} = 0.60$ and $G_{\mathrm{RC}} = 0.64$. We do not encounter this problem with the 1/n index because it is the least concentrated portfolio in terms of weight by definition. For the ERC index, concentration is not a concern. Indeed, Maillard *et al.* (2008) show that if $w^{\star}$ is the solution of the ERC problem, then there exists a constant $c$ such that it is also the solution of the following optimization problem :

$$\begin{aligned} w^{\star} &= \arg\min w^{\top}\Sigma w \\ &\text{u.c.} \quad \begin{cases} \sum_{i=1}^{n} \ln w_i \geq c \\ \mathbf{1}^{\top} w = 1 \\ \mathbf{0} \leq w \leq \mathbf{1} \end{cases} \end{aligned}$$

We may also interpret the ERC method as a minimum-variance optimization problem under a constraint of sufficient diversification ($\sum_{i=1}^{n} \ln w_i \geq c$).

**Remark 2** *It is important to reduce the turnover not only because of trading costs, but above all because of possible market price impact. Moreover, two portfolios with same turnover may produce different market price impacts. Let us consider the two portfolios with 10 assets in*

[8] In the box plot, we indicate the statistics of maximum, minimum, median and $25^{\text{th}}$ and $75^{\text{th}}$ percentiles.

[9] Let $w_i(t)$ be the weight of asset $i$ at time $t$. The turnover is defined by $T_{\mathrm{W}} = \sum_{i=1}^{n} |w_i(t) - w_i(t-1)|$.

[10] Moreover, without constraints, the MV and MDP/MSR indexes do not satisfy the 5/10/40 UCITS III rule.

Table 8: Weights for the Euro Stoxx 50 universe (31/12/2009)

| Name | CW | MV | ERC | MDP | 1/n | MV 10% | MDP 10% | MV 5% | MDP 5% |
|---|---|---|---|---|---|---|---|---|---|
| TOTAL | 6.1 | | 2.1 | | 2.0 | | | 5.0 | |
| BANCO SANTANDER | 5.8 | | 1.3 | | 2.0 | | | | |
| TELEFONICA SA | 5.0 | 31.2 | 3.5 | | 2.0 | 10.0 | | 5.0 | 5.0 |
| SANOFI-AVENTIS | 3.6 | 12.1 | 4.5 | 15.5 | 2.0 | 10.0 | 10.0 | 5.0 | 5.0 |
| E.ON AG | 3.6 | | 2.1 | | 2.0 | | | | 1.4 |
| BNP PARIBAS | 3.4 | | 1.1 | | 2.0 | | | | |
| SIEMENS AG | 3.2 | | 1.5 | | 2.0 | | | | |
| BBVA(BILB-VIZ-ARG) | 2.9 | | 1.4 | | 2.0 | | | | |
| BAYER AG | 2.9 | | 2.6 | 3.7 | 2.0 | 2.2 | 5.0 | 5.0 | 5.0 |
| ENI | 2.7 | | 2.1 | | 2.0 | | | | |
| GDF SUEZ | 2.5 | | 2.6 | 4.5 | 2.0 | | 5.4 | 5.0 | 5.0 |
| BASF SE | 2.5 | | 1.5 | | 2.0 | | | | |
| ALLIANZ SE | 2.4 | | 1.4 | | 2.0 | | | | |
| UNICREDIT SPA | 2.3 | | 1.1 | | 2.0 | | | | |
| SOC GENERALE | 2.2 | | 1.2 | 3.9 | 2.0 | | 3.7 | | 5.0 |
| UNILEVER NV | 2.2 | 11.4 | 3.7 | 10.8 | 2.0 | 10.0 | 10.0 | 5.0 | 5.0 |
| FRANCE TELECOM | 2.1 | 14.9 | 4.1 | 10.2 | 2.0 | 10.0 | 10.0 | 5.0 | 5.0 |
| NOKIA OYJ | 2.1 | | 1.8 | 4.5 | 2.0 | | 4.8 | | 5.0 |
| DAIMLER AG | 2.1 | | 1.3 | | 2.0 | | | | |
| DEUTSCHE BANK AG | 1.9 | | 1.0 | | 2.0 | | | | |
| DEUTSCHE TELEKOM | 1.9 | | 3.2 | 2.6 | 2.0 | 5.7 | 3.7 | 5.0 | 5.0 |
| INTESA SANPAOLO | 1.9 | | 1.3 | | 2.0 | | | | |
| AXA | 1.8 | | 1.0 | | 2.0 | | | | |
| ARCELORMITTAL | 1.8 | | 1.0 | | 2.0 | | | | |
| SAP AG | 1.8 | 21.0 | 3.4 | 11.2 | 2.0 | 10.0 | 10.0 | 5.0 | 5.0 |
| RWE AG (NEU) | 1.7 | 2.7 | 2.7 | | 2.0 | 7.0 | | 5.0 | |
| ING GROEP NV | 1.6 | | 0.8 | 0.4 | 2.0 | | | | |
| DANONE | 1.6 | 1.9 | 3.4 | 1.8 | 2.0 | 8.7 | 3.3 | 5.0 | 5.0 |
| IBERDROLA SA | 1.6 | | 2.5 | | 2.0 | 5.1 | | 5.0 | 1.2 |
| ENEL | 1.6 | | 2.1 | | 2.0 | | | 5.0 | 2.9 |
| VIVENDI SA | 1.6 | 2.8 | 3.1 | 4.5 | 2.0 | 10.0 | 5.9 | 5.0 | 5.0 |
| ANHEUSER-BUSCH INB | 1.6 | 0.2 | 2.7 | 10.9 | 2.0 | 2.1 | 10.0 | 5.0 | 5.0 |
| ASSIC GENERALI SPA | 1.6 | | 1.8 | | 2.0 | | | | |
| AIR LIQUIDE(L') | 1.4 | | 2.1 | | 2.0 | | | 5.0 | |
| MUENCHENER RUECKVE | 1.3 | | 2.1 | 2.1 | 2.0 | | 3.1 | 5.0 | 5.0 |
| SCHNEIDER ELECTRIC | 1.3 | | 1.5 | | 2.0 | | | | |
| CARREFOUR | 1.3 | 1.0 | 2.7 | 1.3 | 2.0 | 3.7 | 2.5 | 5.0 | 5.0 |
| VINCI | 1.3 | | 1.6 | | 2.0 | | | | |
| LVMH MOET HENNESSY | 1.2 | | 1.8 | | 2.0 | | | | |
| PHILIPS ELEC(KON) | 1.2 | | 1.4 | | 2.0 | | | | |
| L'OREAL | 1.1 | 0.8 | 2.8 | | 2.0 | 5.5 | | 5.0 | 5.0 |
| CIE DE ST-GOBAIN | 1.0 | | 1.1 | | 2.0 | | | | |
| REPSOL YPF SA | 0.9 | | 2.0 | | 2.0 | | | 5.0 | |
| CRH | 0.8 | | 1.7 | 5.1 | 2.0 | | 5.2 | | 5.0 |
| CREDIT AGRICOLE SA | 0.8 | | 1.1 | | 2.0 | | | | |
| DEUTSCHE BOERSE AG | 0.7 | | 1.5 | | 2.0 | | | | 1.9 |
| TELECOM ITALIA SPA | 0.7 | | 2.0 | | 2.0 | | | | 2.5 |
| ALSTOM | 0.6 | | 1.5 | | 2.0 | | | | |
| AEGON NV | 0.4 | | 0.7 | | 2.0 | | | | |
| VOLKSWAGEN AG | 0.2 | | 1.8 | 7.1 | 2.0 | | 7.4 | | 5.0 |
| Total of components | 50 | 11 | 50 | 17 | 50 | 14 | 16 | 20 | 23 |

Table 9: Annualized monthly turnover (in %)

| **Lag** | **MV** | **ERC** | **MDP** | **1/n** | **MV 10%** | **MDP 10%** | **MV 5%** | **MDP 5%** |
|---|---|---|---|---|---|---|---|---|
| 1M | 1791 | 578 | 1932 | 20 | 1444 | 1597 | 991 | 1113 |
| 2M | 1248 | 304 | 1321 | 20 | 939 | 1064 | 636 | 727 |
| 3M | 913 | 205 | 984 | 20 | 705 | 818 | 472 | 548 |
| 1Y | 327 | 65 | 340 | 20 | 249 | 292 | 162 | 202 |
| 2Y | 194 | 43 | 212 | 20 | 149 | 190 | 99 | 129 |
| 3Y | 145 | 36 | 157 | 20 | 112 | 144 | 74 | 95 |

Table 10: Gini coefficient of weights and risk contribution (31/12/2009)

| Index | | $G_{\mathrm{W}}$ | $G_{\mathrm{RC}}$ |
|---|---|---|---|
| **CW** | | 0.31 | 0.38 |
| **MV** | | 0.90 | 0.90 |
| **ERC** | | 0.25 | 0.00 |
| **MDP** | | 0.79 | 0.79 |
| **1/n** | | 0.00 | 0.26 |
| **MV** | **10%** | 0.78 | 0.78 |
| **MDP** | **10%** | 0.76 | 0.76 |
| **MV** | **5%** | 0.60 | 0.64 |
| **MDP** | **5%** | 0.60 | 0.64 |

*Table 11. These two portfolios have the same turnover (it is equal to* 100%*). However, because the portfolio B is more concentrated than the portfolio A, we may think that the market price impact will be higher for the portfolio B.*

Table 11: Weights (in %) of two portfolios with same turnover of 100%

| | Portfolio A | | | | | | | | | | $T_{\mathrm{W}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_i\,(t-1)$ | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | |
| $w_i\,(t)$ | 20 | 0 | 20 | 0 | 20 | 0 | 20 | 0 | 20 | 0 | 100 |
| | Portfolio B | | | | | | | | | | $T_{\mathrm{W}}$ |
| $w_i\,(t-1)$ | 25 | 25 | 0 | 0 | 0 | 0 | 0 | 0 | 25 | 25 | |
| $w_i\,(t)$ | 25 | 0 | 50 | 0 | 0 | 0 | 0 | 0 | 25 | 0 | 100 |

## 4.3 Robust estimation of the covariance matrix

Imposing constraints is not the only way to improve the MV and MDP/MSR methods in terms of turnover. Another important issue is the statistical method to estimate the covariance matrix. Using a more robust method may have two effects:

1. Obtaining a more weight-balanced portfolio and reducing the turnover.

2. Defining a more forward looking covariance matrix.

The first point is related to the MV and MDP/MSR methods whereas the second point concerns the ERC method too. Throughout this paper, we have used the empirical covariance matrix. In the next paragraph, we review general methods to improve the estimated covariance matrix.

### 4.3.1 The maximum likelihood estimator

Let $X$ be a $n \times 1$ random vector such that $X \sim \mathcal{N}(\mu, \Sigma)$. We denote $x_t$ the observed value of $X$ at time $t$. The log-likelihood of the sample $x = \{x_t, t = 1, \ldots, T\}$ is:

$$\ell(x) = -\frac{nT}{2} \ln 2\pi - \frac{T}{2} \ln(\det \Sigma) - \frac{1}{2} \sum_{t=1}^{T} (x_t - \mu)^\top \Sigma^{-1} (x_t - \mu) \tag{4}$$

We may show that the ML estimator is:

$$\hat{\Sigma} = \frac{1}{T} \sum_{t=1}^{T} (x_t - \bar{x})^\top (x_t - \bar{x})$$

The ML estimate $\hat{\Sigma}$ corresponds to the empirical covariance matrix. Using ML theory, it converges to the true value $\Sigma$ as $T \to \infty$ and $n$ is fixed.

If the sample size $T$ is small and the number $n$ of variables is large, the ML estimator $\hat{\Sigma}$ is not stable. Moreover, if $n \geq T$, $\hat{\Sigma}$ is a singular matrix. For example, we encounter this type of problem when we estimate the covariance matrix of large stock market indexes (S&P 500, etc.) with a one-year lag window – here $T = 260$ and $n = 500$. In this case, it may be valuable to estimate $\Sigma$ in another way. A first way consists in defining a factor model to explain the stock returns whereas the second case is based on Bayesian methods and the concept of shrinkage.

### 4.3.2 Factor models

Let $R_t$ be the return of the $n$ stocks at time $t$. We assume that it is related to $m$ exogenous factors $F_t$:

$$R_t = AF_t + u_t$$

with $A$ a $n \times m$ matrix and $\operatorname{cov}[F_t] = \Omega$. $AF_t$ represents the factor or common component whereas $u_t$ is the idiosyncratic component of the returns. Moreover, we assume that $\mathbb{E}[u_t] = 0$ and $\operatorname{cov}[u_t] = D$. By assuming that returns are gaussian, we may write the concentrated log-likelihood function in the following way:

$$\ell \propto -\frac{T}{2} \left( \ln |\Phi| + \operatorname{tr}\left(\Phi^{-1} \hat{\Sigma}\right) \right) \tag{5}$$

with $\hat{\Sigma}$ the empirical covariance matrix and $\Phi = A \Omega A^\top + D$. The idea of factor models (FM) is to reduce the number of estimated parameters. That's why we generally use the following hypothesis: $D$ is a diagonal matrix. Moreover, if we want to identify the model, we have to assume that $\Omega$ is given[11]. In this case, the number of estimated parameters associated to the log-likelihood (5) is $p_2 = n \times (m + 1)$ whereas it is $p_1 = n \times (n + 1)/2$ for the log-likelihood (4). In the case of the DJ Euro Stoxx 50 universe, we have $n = 50$ and $p_1 = 1\,275$ parameters. If we use a ten factors model, the number of parameters becomes $p_2 = 550$, which is lower. For the S&P 500 universe, we have $p_1 = 125\,250$ and $p_2 = 5\,500$ (22 times less parameters than for the estimated covariance matrix).

[11] In the case when one uses exogenous economic factors, $\Omega$ is set to the empirical covariance matrix of the factors. If we don't want to define the factors, we use the identity matrix.

The simplest factor model is to consider that the correlation between returns is uniform. This model is known as the constant correlation (CC) model and is used in finance for a long time (Elton and Gruber, 1973). It corresponds to a special case of the 1F model when the vector  of loadings is proportional to the volatilities of the stock returns. Let $\rho$ be the uniform correlation. One generally estimates it as the average of the empirical correlations:

$$\hat{\rho} = \frac{2}{n(n-1)} \sum_{i>j} \frac{\hat{\Sigma}_{i,j}}{\sqrt{\hat{\Sigma}_{i,i}\hat{\Sigma}_{j,j}}}$$

However, this estimator is biased and we prefer to estimate the parameter $\rho$ by maximum likelihood.

Naturally, the number of parameters in factor models is smaller implying that the instability of the estimated covariance matrix is reduced. Previously, we have assumed that factors are exogenous. Another way to improve the estimation of $\Sigma$ is to use principal components analysis (PCA). By construction, the ML estimate $\hat{\Sigma}$ is a symmetric, positive semi-definite matrix. We may also apply eigenvalues decomposition:

$$\hat{\Sigma} = U \Lambda U^{\top} \tag{6}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix[12] and $U$ is an orthogonal matrix. In PCA, the endogenous factors take the form $F_t = \Lambda^{-1/2} U^{\top} R_t$. By considering only the first $m$ components or factors, we may build a more robust estimation of $\Sigma$. The main question is to determine the number $m$ of components. One solution is to consider only the factors which explains more than $1/n$ of the variance. In this case, we have $m = \max\{i : \lambda_i \geq (\lambda_1 + \ldots + \lambda_n)/n\}$. Another solution is to exploit the algorithm of Laloux *et al.* (1999). Using results on random matrix theory (RMT), they show that the eigenvalues of a $T \times n$ random matrix $X$ has asymptotically a maximum equal to $\lambda_{\max} = \sigma^2 \left(1 + n/T + 2\sqrt{n/T}\right)$ with $\sigma^2$ the variance of $X$. In this case, $m = \max\{i : \lambda_i > \lambda_{\max}\}$.

### 4.3.3 The shrinkage estimator

On one hand, we know that the empirical covariance matrix estimator is unbiased but its convergence rate is very low if the number of variables is high. On the other hand, we know that covariance matrix estimators based on factor models are faster to converge but are biased. The idea of shrinkage estimation consists in combining a biased estimator with other information (for example an unbiased estimator) to produce another estimator with a smaller mean squared error. Some examples are the Bayes-Stein estimator (Jorion, 1996) or the shrinkage method presented by Ledoit and Wolf (2003).

Let $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ be two estimators of the covariance matrix $\Sigma$. Let us build the estimator $\hat{\Sigma}_\alpha$ as a linear combination of the previous estimators :

$$\hat{\Sigma}_\alpha = \alpha \hat{\Sigma}_1 + (1-\alpha) \hat{\Sigma}_2 \tag{7}$$

Suppose that we want to minimize the expected value of the quadratic loss function:

$$\alpha^\star = \arg\max \mathbb{E}\left(\left\| \alpha \hat{\Sigma}_1 + (1-\alpha) \hat{\Sigma}_2 - \Sigma \right\|^2\right)$$

Ledoit and Wolf (2003) derive the analytical expression of the optimal value $\alpha^\star$ when $\hat{\Sigma}_1$ is the estimated covariance matrix based on a factor model and when $\hat{\Sigma}_2$ is the empirical covariance matrix. The case when $\hat{\Sigma}_1$ is built with a CC matrix is treated in Ledoit and Wolf (2004).

[12] The eigenvalues are ordered in descending order: $\lambda_1 > \lambda_2 > \ldots > \lambda_n$.

Table 12: Gini coefficient of weights and risk contribution (31/12/2009)

| **Stats** | **MV** | **ERC** | **MDP** | **1/n** | **MV 10%** | **MDP 10%** | **MV 5%** | **MDP 5%** |
|---|---|---|---|---|---|---|---|---|
| | Empirical covariance matrix (EMP) | | | | | | | |
| $T_{\mathrm{W}}$ | 327 | 65 | 340 | 20 | 249 | 292 | 162 | 202 |
| $\bar{G}_{\mathrm{W}}$ | 0.85 | 0.18 | 0.78 | 0.00 | 0.77 | 0.74 | 0.59 | 0.58 |
| $\bar{G}_{\mathrm{RC}}$ | 0.85 | 0.00 | 0.77 | 0.19 | 0.76 | 0.74 | 0.60 | 0.62 |
| IR | 0.11 | 0.65 | 0.47 | 0.65 | 0.29 | 0.54 | 0.45 | 0.67 |
| | Constant correlation matrix (CC) | | | | | | | |
| $T_{\mathrm{W}}$ | 280 | 47 | 47 | 20 | 213 | 47 | 136 | 47 |
| $\bar{G}_{\mathrm{W}}$ | 0.86 | 0.14 | 0.14 | 0.00 | 0.76 | 0.14 | 0.59 | 0.14 |
| $\bar{G}_{\mathrm{RC}}$ | 0.86 | 0.00 | 0.01 | 0.16 | 0.76 | 0.01 | 0.60 | 0.01 |
| IR | 0.02 | 0.62 | 0.63 | 0.65 | 0.19 | 0.62 | 0.29 | 0.62 |
| | Principal component analysis (PCA) | | | | | | | |
| $T_{\mathrm{W}}$ | 264 | 65 | 259 | 20 | 206 | 230 | 144 | 173 |
| $\bar{G}_{\mathrm{W}}$ | 0.85 | 0.19 | 0.76 | 0.00 | 0.77 | 0.73 | 0.59 | 0.58 |
| $\bar{G}_{\mathrm{RC}}$ | 0.85 | 0.00 | 0.75 | 0.19 | 0.77 | 0.73 | 0.61 | 0.61 |
| IR | 0.14 | 0.68 | 0.55 | 0.65 | 0.25 | 0.54 | 0.48 | 0.69 |
| | Shrinkage estimator with CC (CC-SHRINK) | | | | | | | |
| $T_{\mathrm{W}}$ | 310 | 59 | 306 | 20 | 236 | 272 | 151 | 198 |
| $\bar{G}_{\mathrm{W}}$ | 0.86 | 0.17 | 0.73 | 0.00 | 0.76 | 0.70 | 0.59 | 0.57 |
| $\bar{G}_{\mathrm{RC}}$ | 0.86 | 0.00 | 0.72 | 0.18 | 0.76 | 0.70 | 0.60 | 0.59 |
| IR | 0.10 | 0.72 | 0.52 | 0.65 | 0.29 | 0.55 | 0.37 | 0.67 |
| | Shrinkage estimator with PCA (PCA-SHRINK) | | | | | | | |
| $T_{\mathrm{W}}$ | 303 | 65 | 307 | 20 | 233 | 271 | 155 | 192 |
| $\bar{G}_{\mathrm{W}}$ | 0.85 | 0.19 | 0.76 | 0.00 | 0.76 | 0.73 | 0.59 | 0.58 |
| $\bar{G}_{\mathrm{RC}}$ | 0.85 | 0.00 | 0.75 | 0.19 | 0.76 | 0.73 | 0.61 | 0.61 |
| IR | 0.13 | 0.66 | 0.54 | 0.65 | 0.28 | 0.54 | 0.44 | 0.67 |

### 4.3.4 Application to our example

Using different estimators for the covariance matrix, we obtain results in Table 12. For each estimator, we report the yearly turnover $T_{\mathrm{W}}$ (expressed in %), the average Gini coefficients $\bar{G}_{\mathrm{W}}$ and $\bar{G}_{\mathrm{RC}}$, and the information ratio IR. All estimators use a window lag of one year. The first estimator EMP is empirical (or ML) estimate of the covariance matrix. The second estimator CC is the estimated covariance matrix built with empirical volatilities and a uniform correlation. PCA corresponds to the covariance matrix deduced from principal component analysis with a number of factors computed with the formula of Laloux *et al.* (1999). CC-SHRINK and PCA-SHRINK are the shrinkage estimators deduced from the previous CC and PCA estimators. We notice that using a more robust covariance matrix estimate allows generally to reduce the turnover. But except for the CC estimator applied to the MDP/MSR method, they are not sufficient to solve the concentration issue. Imposing bounds (and using longer window lag) remains the main method to reduce this problem.

## 5 Conclusion

The failure of many active managers to generate value has caused investors to shift their focus from active management to passive exposure to the market. This encourages investors to study the alternative ways in which they can gain equity market exposure. The classical and natural way consists of a capitalization-weighted indexation, which under CAPM framework and tedious assumptions, can be presented as the efficient portfolio. But this does not seem to be efficient. As illustrated in this paper a lot of capitalization-weighted equity indexes are concentrated in their biggest members. They suffer from a lack of diversification and expose investors to high shortfalls of risk. Another issue is that they adopt a trend following behavior and contain a growth bias.

Alternative methods have thus emerged and have been promoted. There are two distinct methods that offer more efficient exposure to the market, or get a better beta. The first one uses fundamental statistics such as earnings or dividend yields to compute weights. These fundamental indexes are thus value-tilted and promote this value bias as a potential return enhancement or at least as diversifying to growth oriented capitalization weighted indexes. The second kind is risk-based indexation. By looking for an efficient way to split the risk within the portfolio they promise lower risk and diversification regarding their cap-weighted counterpart. We described four risk-based indexations. The simplest way is to equalize the weights between stocks (the 1/n portfolio), but this does not necessarily guarantee that the portfolio will be balanced in terms of risk. This naturally leads to a portfolio where the risks, no longer the weights, is split equally among components: every component has the same contribution to the portfolio total risk in an ERC portfolio. The third and fourth portfolios are the result of an optimization program. The former is the minimum-variance portfolio, i.e. the one with the minimum volatility on an ex-ante basis. The latter is the most-diversified portfolio, the portfolio maximizing a diversification measure.

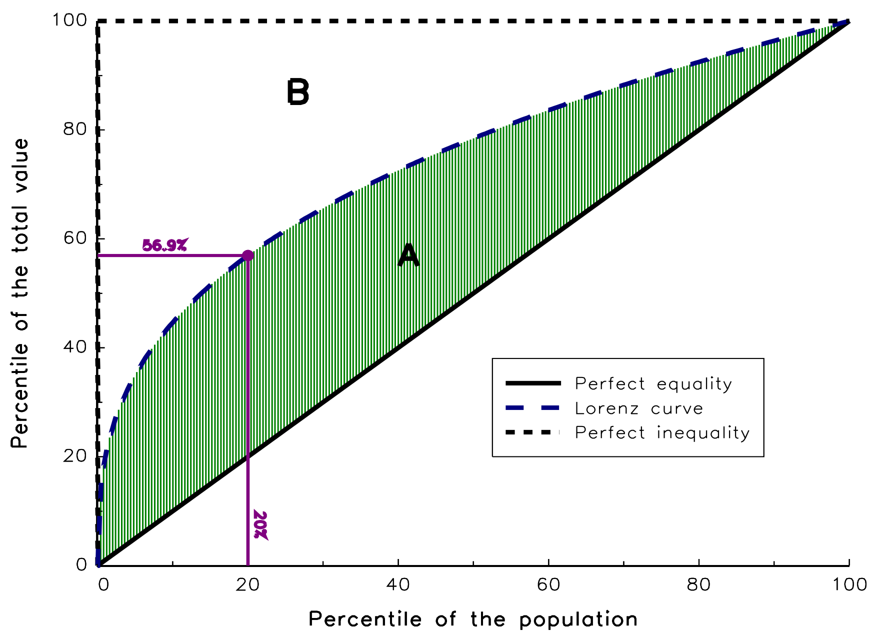
This paper then illustrated these methods with an application using the DJ Euro Stoxx 50 universe. It showed that restrictive constraints on weights should be imposed with MV or MDP/MSR methods to avoid extreme concentration in the less volatile or less correlated assets. Finally non-constrained and constrained portfolios may be different and constrained thresholds show importance in the design of the portfolio. The definition of constraints is however an open question, because it introduces a discretionary part in the strategy and they are not generally rigourously justified. The ERC portfolio is balanced in weights such that there is no need for constraints as does the 1/n portfolio. Regarding correlation with CW index, MDP/MSR and MV portfolios are the least correlated. Regarding index volatility, as expected, the MV portfolio is the least volatile. The volatility of the 1/n portfolio is similar to those of the CW index while the out-of-sample volatility of the ERC and MDP/MSR portfolios stand middle grounded between the volatilities of the MV and 1/n portfolios. In terms of risk adjusted performance, the four alternative indexation post higher figures than cap weighted performance. Lastly, risk-based indexation methods all experience smaller drawdown compared to the CW index. Avoiding portfolio concentration is critical to capital preservation. With alternative indexes, investors can thus access to a well diversified and diversifying exposure to broad equity markets.

# Appendix

## A The Lorenz curve and the Gini coefficient

In economics, the Lorenz curve is a graphical representation of the cumulative distribution function of the empirical probability distribution of wealth. The $x$ value of the curve corresponds to a percentile of the population ordered according to the characteristic in question. The $y$ value of the curve represents that portion of the total value of the characteristic in question. If we apply this concept to the composition of an index, the Lorenz curve represents the cumulated weights of the first $x\%$ most important stocks. In Figure 9, we have represented an example of a Lorenz curve. In this case, we have $\mathcal{L}(20\%) = 56.9\%$ meaning that the 20% most weighted stocks represent 56.9% of the weight in the index. In the case of no concentration, the Lorenz curve is the straight 45° line which corresponds to the perfect equality of the weights and we have $\mathcal{L}(x) = x$. Perfect concentration corresponds to the case when one stock has a weight of 100% and we have $\mathcal{L}(x) = 0$ if $x < 1/n$ and $\mathcal{L}(x) = 1$ if $x \geq 1/n$.

Figure 9: The concept of Lorenz curve




The Gini Coefficient $G$ is a measure of the dispersion using the Lorenz curve. Graphically, we have $G = A/(A+B)$ with $A$ and $B$ the two areas defined in Figure 9. Another expression is:

$$G = 2\int_0^1 \mathcal{L}(x)\;\mathrm{d}x - 1$$

The statistical measure $G$ takes the value 1 for a perfectly concentrated portfolio and 0 for the portfolio with uniform weights.

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