The Lehman Brothers Multi-factor Risk Model

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SUMMARY

- We describe the proprietary Lehman Brothers Risk Model for dollar-denominated government, corporate, and mortgage-backed securities. The model quantifies expected deviation in performance ("tracking error") between a portfolio of fixed-income securities and an index representing the market, such as the Lehman Brothers Aggregate, Corporate, or High Yield Index.

- The forecast of the return deviation is based on specific mismatches between the sensitivities of the portfolio and the benchmark to major market forces ("risk factors") that drive security returns. The model uses historical variances and correlations of the risk factors to translate the structural differences of the portfolio and the index into an expected tracking error. The model quantifies not only this systematic market risk, but security-specific (non-systematic) risk as well.

- Using an illustrative portfolio, we discuss the implementation of the model. We show how each component of tracking error can be traced back to the corresponding difference between the portfolio and benchmark risk exposures. We describe our methodology for the minimization of tracking error and discuss a variety of portfolio management applications based on our experience with investors.

- The Appendix presents a theoretical discussion of our approach to modeling risk and a detailed mathematical description of the model. It contains detailed risk factor definitions, formulation of our model for security-specific risk, an overview of the risk optimization algorithm and a description of the procedure for deriving historical factor realizations.

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TABLE OF CONTENTS

1. INTRODUCTION

CONCEPT OF RISK IN HISTORICAL CONTEXT

Risk, an ancient subject predating civilization, has taken on a new urgency in modern capital markets. In a pre-capitalist agrarian society, surplus harvest in excess of seed represents the main productive risk. For a medieval merchant in the Low Countries, risk could range from an inability to barter to a complete loss of product through theft, fire, or general economic hardship. In the wake of Great Global Depression of the 1930s and up until the conversion of bond managers from absolute return maximizers to competitive relative-value operators, bond market risk typically centered on the possibility of complete loss of the coupon payment stream and even principal. Regrettably, the 19th and early 20th Centuries contain multiple corporate and sovereign reminders of this absolute risk.

The concept of risk has undergone a metamorphosis during the past three decades. Commercial bankers, chastened by hardships in commercial real estate and emerging market lending, still view loan risks in these classic absolute terms. But at the command of accounting authorities and in acknowledgement of demanding investors in both equity and bond products, risk has moved toward a daily, mark-to-market framework. In the wake of a decade like the 1990s, which sequentially featured major asset-quality problems in certain U.S., European, and Asian financial institutions, this enhanced emphasis on sturdier risk control has found broad regulatory, investor, and management support.

In this context, our risk model mainly covers risk of the return deviation between a bond portfolio and a benchmark. Our model does not project the likelihood of currency, yield curve, or spread movements. Our methodology does not make any predictions about the absolute risk of default by any issue.

DEFINITION OF RISK

The classical definition of investment risk is uncertainty of returns, measured by their volatility. Investments with greater risk are expected to earn greater returns than less risky alternatives. Asset allocation models help investors choose the asset mix with the highest expected return given their risk constraints (for example, avoid a loss of more than 2% per year in a given portfolio).

Once investors have selected a desired asset mix, they often enlist specialized asset managers to implement their investment goals. The performance of the portfolio is usually compared with a benchmark that reflects the investor's asset selection decision. From the perspective of most asset managers, risk is defined by performance relative to the benchmark rather than by absolute return. In this sense, the least-risky investment portfolio is one that replicates the benchmark. Any portfolio deviation from the benchmark entails some risk. For example, to the manager of a bond fund benchmarked against the High Yield Index, investing 100% in U.S. Treasuries would involve a much greater long-term risk than investing 100% in high yield corporate bonds. In other words, benchmark risk belongs to the plan sponsor, while the asset manager bears the risk of deviating from the benchmark.

As the leading global provider of fixed-income indices, Lehman Brothers provides quantitative models to aid the management of portfolios relative to its indices. The Lehman Brothers Risk Model, created in the early 1990s, focuses on the latter type of risk—portfolio risk relative to a benchmark. Our risk model is designed for use by fixed-income portfolio managers benchmarked against broad market indices.

QUANTIFYING RISK

Given our premise that the least-risky portfolio is the one that exactly replicates the benchmark, we proceed to compare the composition of a fixed-income portfolio to that of its benchmark. Are they similar in exposures to changes in the term structure of interest rates, in allocations to different asset classes within the benchmark, and in allocations to different quality ratings? Such portfolio versus benchmark comparisons form the foundation for modern fixed-income portfolio management. Techniques such as "stratified sampling" or "cell-matching" have been used to construct portfolios that are similar to their benchmarks in many components (i.e., duration, quality etc.). However, these techniques can not answer quantitative questions concerning portfolio risk. How much risk is there? Is portfolio A more or less risky than portfolio B? Will a given transaction increase or decrease risk? To best decrease risk relative to the benchmark, should the focus be on better aligning term structure exposures or sector allocations? How do we weigh these different types of risk against each other? What actions can be taken to mitigate the overall risk exposure? Any quantitative model of risk must account for the magnitude of a particular event as well as its likelihood. When multiple risks are modeled simultaneously, the issue of correlation also must be addressed.

The Lehman Brothers Risk Model provides quantitative answers to such questions. This multi-factor model compares portfolio and benchmark exposures along all dimensions of risk, such as yield curve movement, changes in sector spreads, and changes in implied volatility. Exposures to each *risk factor* are calculated on a bond-by-bond basis and aggregated to obtain the exposures of the portfolio and the benchmark.

Tracking error, which quantifies the risk of performance difference (projected standard deviation of the return difference) between the portfolio and the benchmark, is projected based on the differences in risk factor exposures. This calculation of overall risk incorporates historical information about the volatility of each risk factor and the correlations among them. The volatilities and correlations of all the risk factors are stored in a *covariance matrix*, which is calibrated based on monthly returns of individual bonds in the Lehman Brothers Aggregate Index dating back to 1987. The model is updated monthly with historical information. The choice of risk factors has been reviewed periodically since the model's introduction in 1990. The model covers U.S. dollar-denominated securities in most Lehman Brothers domestic fixed-rate bond indices (Aggregate, High Yield, Eurobond). The effect of non-index securities on portfolio risk is measured by mapping onto index risk categories. The net effect of all risk factors is known as *systematic risk*.

The model is based on historical returns of individual securities and its risk projections are a function of portfolio and benchmark positions in individual securities. Instead of deriving risk factor realizations from changes in market averages (such as a Treasury curve spline, sector spread changes, etc) the model derives them from historical returns of securities in Lehman Indices. While this approach is much more data and labor intensive, it allows us to quantify residual return volatility of each security after all systematic risk factors have been applied. As a result, we can measure *non-systematic risk* of a portfolio relative to the benchmark based on differences in their diversification. This form of risk, also known as concentration risk or *security-specific risk*, is the result of a portfolio's exposure to individual bonds or issuers. Non-systematic risk can represent a significant portion of the overall risk, particularly for portfolios containing relatively few securities, even for assets without any credit risk.

PORTFOLIO MANAGEMENT WITH THE LEHMAN BROTHERS RISK MODEL

Lehman Brothers developed its risk model for the benefit of portfolio managers benchmarked to Lehman indices and implemented it as part of its portfolio analytics platform (PC Product/Sunbond). The model has been used with much success by investors with diverse portfolio management styles. Passive portfolio managers, or "indexers," seek to replicate the returns of a broad market index. They use the risk model to help keep the portfolio closely aligned with the index along all risk dimensions. Active portfolio managers attempt to outperform the benchmark by positioning the portfolio to capitalize on market views. They use the risk model to quantify the risk entailed in a particular portfolio position relative to the market. This information is often incorporated into the performance review process, where returns achieved by a particular strategy are weighed against the risk taken. Enhanced indexers express views against the index, but limit the amount of risk assumed. They use the model to keep risk within acceptable limits and to highlight unanticipated market exposures that might arise as the portfolio and index change over time. These management styles can be associated with approximate ranges of tracking errors. Passive managers typically seek tracking errors of 5 to 25 basis points per year. Tracking errors for enhanced indexers range from 25 to 75 bp, and those of active managers are even higher.

WHY A MULTI-FACTOR MODEL?

With the abundance of data available in today's marketplace, an asset manager might be tempted to build a risk model directly from the historical return characteristics of individual securities. The standard deviation of a security's return in the upcoming period can be projected to match its past volatility; the correlation between any two securities can be determined from their historical performance. Despite the simplicity of this scheme, the multi-factor approach has several important advantages. First of all, the number of risk factors in the model is much smaller than the number of securities in a typical investment universe. This greatly reduces the matrix operations needed to calculate portfolio risk. This increases the speed of computation (which is becoming less important with gains in processing power) and, more importantly, improves the numerical stability of the calculations. A large covariance matrix of individual security volatilities and correlations is likely to cause numerical instability. This is especially true in the fixed income world, where returns of many securities are very highly correlated. Risk factors may also exhibit moderately high correlations with each other, but much less so than for individual securities.[1]

A more fundamental problem with relying on individual security data is that not all securities can be modeled adequately in this way. For illiquid securities, pricing histories are either unavailable or unreliable; for new securities, histories do not exist. For still other securities, there may be plenty of reliable historical data, but changes in security characteristics make this data irrelevant to future results. For instance, a ratings upgrade of an issuer would make future returns less volatile than those of the past. A change in interest rates can significantly alter the effective duration of a callable bond. As any bond ages, its duration shortens, making its price less sensitive to interest rates. A multi-factor model estimates the risk from owning a particular bond based not on the historical performance of that bond, but on historical returns of all bonds with characteristics similar to those currently pertaining to the bond.

[1] Some practitioners insist on a set of risk factors that are uncorrelated to each other. We have found it more useful to select risk factors that are intuitively clear to investors, even at the expense of allowing positive correlations among the factors.

In this report, we present the risk model by way of example. In each of the following sections, a numerical example of the model's application motivates the discussion of a particular feature. In Section 2, we take a detailed tour of the risk report and discuss the various sources of tracking error and how they combine. In Section 3, we explore several applications of the model to portfolio management, including portfolio optimization and the creation of proxy portfolios. In Section 4, we discuss the modeling of risk for non-index securities. Section 5 presents advanced features of the model, while Section 6 describes historical tests that were carried out to validate the model. In Section 7, we discuss relationships between the risk model and other analytical tools for fixed income portfolio management. The Appendix contains a complete mathematical presentation of the model, as well as a glossary that gives simple definitions for many of the terms used in this paper.

2. THE RISK REPORT

For illustration, we apply the risk model to a sample portfolio of 57 bonds benchmarked against the Lehman Brothers Aggregate Index. The model produces two important outputs: a tracking error summary report and a set of risk sensitivities reports that compare the portfolio composition to that of the benchmark. These various comparative reports form the basis of our risk analysis, by identifying structural differences between the two. Of themselves, however, they fail to quantify the risk due to these mismatches. The model's anchor is therefore the tracking error report, which quantifies the risks associated with each cross-sectional comparison. Taken together, the various reports produced by the model provide a complete understanding of the risk of this portfolio versus its benchmark.

From the overall statistical summary shown in Figure 1, it is obvious that the portfolio has a significant term structure exposure, as its duration (4.82 years) is longer than that of the benchmark (4.29 years). In addition, the portfolio is over-exposed to corporate bonds and under-exposed to Treasuries. We will see this explicitly in the sector report (Figure 4); it is reflected in the statistics in Figure 1 by a higher average yield and coupon. The overall annualized tracking error, shown at the bottom of the statistics report, is 52 bp. *Tracking error* is defined as one standard deviation of the difference between the portfolio and benchmark annualized returns. In

Figure 1. **Top-level Statistics Comparison**
Sample Portfolio vs. Aggregate Index, 9/30/98

	Portfolio	Benchmark
Number of Issues	57	6932
Average Maturity/Average Life (years)	9.57	8.47
Internal Rate of Return (%)	5.76	5.54
Average Yield to Maturity (%)	5.59	5.46
Average Yield to Worst (%)	5.53	5.37
Average Option-adjusted Convexity	0.04	-0.22
Average OAS to Maturity (bp)	74	61
Average OAS to Worst (bp)	74	61
Portfolio Mod. Adjust Duration (years)	4.82	4.29
Portfolio Average Price	108.45	107.70
Portfolio Average Coupon (%)	7.33	6.98
Risk Characteristics		
Estimated Total Tracking Error (bp/year)	52	
Portfolio Beta	1.05	

simple terms, this means that with a probability of about 68% the portfolio return over the next year will be within +/- 52 bp of the benchmark return.[2]

SOURCES OF SYSTEMATIC TRACKING ERROR

What are the main sources of this tracking error? The model identifies market forces influencing *all* securities in a certain category as *systematic risk factors*. Figure 2 divides the tracking error into components corresponding to different categories of risk. Looking down the first column, we see that the largest sources of systematic tracking error between this portfolio and its benchmark are the differences in sensitivity to term structure movements (36.3 bp) and to changes in credit spreads by sector (32 bp) and quality (14.7 bp). The components of systematic tracking error

[2] This interpretation requires several simplifying assumptions. The 68% confidence interval assumes that returns are normally distributed, which may not be the case. Second, this presentation ignores differences in the expected returns of portfolio and benchmark (due, for example, to a higher portfolio yield). Strictly speaking, the confidence interval should be drawn around the expected outperformance.

Figure 2. **Tracking Error Breakdown for Sample Portfolio**
Sample Portfolio vs. Aggregate Index, 9/30/98

	Tracking Error (bp/year)		
	Isolated	**Cumulative**	**Change in Cumulative***
Tracking Error Term Structure	36.3	36.3	36.3
Non-Term Structure	39.5		
Tracking Error Sector	32.0	38.3	2.0
Tracking Error Quality	14.7	44.1	5.8
Tracking Error Optionality	1.6	44.0	-0.1
Tracking Error Coupon	3.2	45.5	1.5
Tracking Error MBS Sector	4.9	43.8	-1.7
Tracking Error MBS Volatility	7.2	44.5	0.7
Tracking Error MBS Prepayment	2.5	45.0	0.4
Total Systematic Tracking Error			45.0
Non-Systematic Tracking Error			
Issuer-specific	25.9		
Issue-specific	26.4		
Total	26.1		
Total Tracking Error			52
	Systematic	**Non-systematic**	**Total**
Benchmark Return Standard Deviation	417	4	417
Portfolio Return Standard Deviation	440	27	440

* Isolated Tracking Error is the projected deviation between the portfolio and benchmark return due to a single category of systematic risk. Cumulative Tracking Error shows the combined effect of all risk categories from the first one in the table to current. See glossary of terms and discussion of tracking error components further in this section.

correspond directly to the groups of risk factors. A detailed report of the differences in portfolio and benchmark exposures (sensitivities) to the relevant set of risk factors illustrates the origin of each component of systematic risk.

Sensitivities to risk factors are called *factor loadings*. They are expressed in units that depend on the definition of each particular risk factor. For example, for risk factors representing volatility of corporate spreads factor loadings are given by spread durations, for risk factors measuring volatility of prepayment speed (in units of PSA) factor loadings are given by "PSA Duration". The factor loadings of a portfolio or an index are calculated as a market-value weighted average over all constituent securities. Differences between portfolio and benchmark factor loadings form a vector of *active portfolio exposures*. A quick comparison of the magnitudes of the different components of tracking error highlights the most significant mismatches.

Because the largest component of tracking error is due to term structure, let us examine the term structure risk in our example. Risk factors associated with term structure movements are represented by the fixed set of points on the theoretical Treasury spot curve shown in Figure 3. Each of these risk factors exhibits a certain historical return volatility. The extent to which the portfolio and the benchmark returns are affected by this volatility is measured

Figure 3. **Term Structure Report**
Sample Portfolio vs. Aggregate Index, 9/30/98

	Cash Flows		
Year	**Portfolio**	**Benchmark**	**Difference**
0.00	1.45%	1.85%	-0.40%
0.25	3.89	4.25	-0.36
0.50	4.69	4.25	0.45
0.75	4.34	3.76	0.58
1.00	8.90	7.37	1.53
1.50	7.47	10.29	-2.82
2.00	10.43	8.09	2.34
2.50	8.63	6.42	2.20
3.00	4.28	5.50	-1.23
3.50	3.90	4.81	-0.92
4.00	6.74	7.19	-0.46
5.00	6.13	6.96	-0.83
6.00	3.63	4.67	-1.04
7.00	5.77	7.84	-2.07
10.00	7.16	7.37	-0.21
15.00	4.63	3.88	0.75
20.00	3.52	3.04	0.48
25.00	3.18	1.73	1.45
30.00	1.22	0.68	0.54
40.00	0.08	0.07	0.01

by factor loadings (exposures). These exposures are computed as percentages of the total present value of the portfolio and benchmark cashflows allocated to each point on the curve.[3] The risk of the portfolio performing differently from the benchmark due to term structure movements is due to the differences in the portfolio and benchmark exposures to these risk factors and to their volatilities and correlations. Figure 3 compares the term structure exposures of the portfolio and benchmark for our example. The Difference column shows the portfolio to be overweighted in the 2-year section of the curve, underweighted in the 3- to 10-year range, and overweighted at the long end. This makes the portfolio longer than the benchmark and more barbelled.

The tracking error is calculated from this vector of differences between portfolio and benchmark exposures. However, mismatches at different points are not treated equally. Exposures to factors with higher volatilities have a larger effect on tracking error. In this example, the risk exposure with the largest contribution to tracking error is the overweight of 1.45% to the 25-year point on the curve. While other vertices have larger mismatches (e.g., –2.07% at 7 years), their overall effect on risk is not as strong because the longer duration of a 25-year zero causes it to have a higher return volatility. It should also be noted that the risk caused by overweighting one segment of the yield curve can sometimes be offset by underweighting another. Figure 3 shows the portfolio to be underexposed to the 1.50-year point on the yield curve by –2.82% and overexposed to the 2.00-year point on the curve by +2.34%. Those are largely offsetting positions in terms of risk because these two adjacent points on the curve are highly correlated and almost always move together. To eliminate completely the tracking error due to term structure, differences in exposures to each term structure risk factor need to be reduced to zero. To lower term structure risk, it is most important to focus first on reducing exposures at the long end of the curve, particularly those that are not offset by opposing positions in nearby points.

The tracking error due to sector exposures is explained by the detailed sector report shown in Figure 4. This report shows the sector allocations of the portfolio and the benchmark in two ways. In addition to reporting the percentage of market value allocated to each sector, it shows the contribution of each sector to the overall spread duration.[4] These contributions are computed as the product of the percentage allocations to a sector and the

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[3] The details of this calculation may be found in Appendix 2, which offers a full description of all risk factors and factor loadings in the model.

[4] Just as traditional duration can be defined as the sensitivity of bond price to a change in yield, spread duration is defined as the sensitivity of bond price to a change in spread. While this distinction is largely academic for bullet bonds, it can be significant for other securities, such as bonds with embedded options and floating-rate securities. The sensitivity to spread change is the correct measure of sector risk.

Figure 4. **Detailed Sector Report**

Sample Portfolio vs. Aggregate Index, 9/30/98

	Portfolio			Benchmark			Difference	
Detailed Sector	**% of Portf.**	**Adj. Dur.**	**Contrib. to Adj. Dur.**	**% of Portf.**	**Adj. Dur.**	**Contrib. to Adj. Dur.**	**% of Portf.**	**Contrib. to Adj. Dur.**
Treasury								
Coupon	27.09	5.37	1.45	39.82	5.58	2.22	-12.73	-0.77
Strip	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Agencies								
FNMA	4.13	3.40	0.14	3.56	3.44	0.12	0.57	0.02
FHLB	0.00	0.00	0.00	1.21	2.32	0.03	-1.21	-0.03
FHLMC	0.00	0.00	0.00	0.91	3.24	0.03	-0.91	-0.03
REFCORP	3.51	11.22	0.39	0.83	12.18	0.10	2.68	0.29
Other Agencies	0.00	0.00	0.00	1.31	5.58	0.07	-1.31	-0.07
Financial Inst.								
Banking	1.91	5.31	0.10	2.02	5.55	0.11	-0.11	-0.01
Brokerage	1.35	3.52	0.05	0.81	4.14	0.03	0.53	0.01
Financial Cos.	1.88	2.92	0.06	2.11	3.78	0.08	-0.23	-0.02
Insurance	0.00	0.00	0.00	0.52	7.47	0.04	-0.52	-0.04
Other	0.00	0.00	0.00	0.28	5.76	0.02	-0.28	-0.02
Industrials								
Basic	0.63	6.68	0.04	0.89	6.39	0.06	-0.26	-0.01
Capital Goods	4.43	5.35	0.24	1.16	6.94	0.08	3.26	0.16
Consumer Cycl.	2.01	8.37	0.17	2.28	7.10	0.16	-0.27	0.01
Consum. Non-cycl.	8.88	12.54	1.11	1.66	6.84	0.11	7.22	1.00
Energy	1.50	6.82	0.10	0.69	6.89	0.05	0.81	0.05
Technology	1.55	1.58	0.02	0.42	7.39	0.03	1.13	-0.01
Transportation	0.71	12.22	0.09	0.57	7.41	0.04	0.14	0.04
Utilities								
Electric	0.47	3.36	0.02	1.39	5.02	0.07	-0.93	-0.05
Telephone	9.18	2.08	0.19	1.54	6.58	0.10	7.64	0.09
Natural Gas	0.80	5.53	0.04	0.49	6.50	0.03	0.31	0.01
Water	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Yankee								
Canadians	1.45	7.87	0.11	1.06	6.67	0.07	0.38	0.04
Corporates	0.49	3.34	0.02	1.79	6.06	0.11	-1.30	-0.09
Supranational	1.00	6.76	0.07	0.38	6.33	0.02	0.62	0.04
Sovereigns	0.00	0.00	0.00	0.66	5.95	0.04	-0.66	-0.04
Hypothetical	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Cash	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Mortgage								
Conventnl. 30-yr.	12.96	1.52	0.20	16.60	1.42	0.24	-3.64	-0.04
GNMA 30-yr.	7.53	1.23	0.09	7.70	1.12	0.09	-0.16	0.01
MBS 15-yr.	3.52	1.95	0.07	5.59	1.63	0.09	-2.06	-0.02
Balloons	3.03	1.69	0.05	0.78	1.02	0.01	2.25	0.04
OTM	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
European & International								
Eurobonds	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
International	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Asset Backed	0.00	0.00	0.00	0.96	3.14	0.03	-0.96	-0.03
CMO	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Other	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Totals	100.00		4.82	100.00		4.29	0.00	0.54

market-weighted average spread duration of the holdings in that sector. Contributions to spread duration (factor loadings) measure the sensitivity of return to systematic changes in particular sector spreads (risk factors) and are a better measure of risk than simple market allocations. The rightmost column in this report, the difference between portfolio and benchmark contributions to spread duration in each sector, is the exposure vector that is used to compute tracking error due to sector. A quick look down this column shows that the largest exposures in our example are an underweight of 0.77 years to Treasuries and an overweight of 1.00 years to consumer non-cyclicals in the industrial sector. (The fine-grained breakdown of the corporate market into industry groups corresponds to the second tier of the Lehman Brothers hierarchical industry classification scheme.) Note that the units of risk factors and factor loadings for sector risk differ from those used to model the term structure risk.

The analysis of credit quality risk shown in Figure 5 follows the same approach. Portfolio and benchmark allocations to different credit rating levels are compared in terms of contributions to spread duration. Once again we see the effect of the overweighting of corporates: there is an overweight of 0.80 years to single As and an underweight of –0.57 years in AAAs (U.S. Government debt). The risk represented by tracking error due to quality corresponds to a systematic widening or tightening of spreads for a particular credit rating, uniformly across all industry groups.

Figure 5. **Quality Report**
Sample Portfolio vs. Aggregate Index, 9/30/98

	Portfolio			Benchmark			Difference	
Quality	% of Portf.	Adj. Dur.	Cntrb. to Adj. Dur.	% of Portf.	Adj. Dur.	Cntrb. to Adj. Dur.	% of Portf.	Cntrb. to Adj. Dur.
Aaa+	34.72	5.72	1.99	47.32	5.41	2.56	-12.60	-0.57
MBS	27.04	1.51	0.41	30.67	1.37	0.42	-3.62	-0.01
Aaa	1.00	6.76	0.07	2.33	4.84	0.11	-1.33	-0.05
Aa	5.54	5.67	0.31	4.19	5.32	0.22	1.35	0.09
A	17.82	7.65	1.36	9.09	6.23	0.57	8.73	0.80
Baa	13.89	4.92	0.68	6.42	6.28	0.40	7.47	0.28
Ba	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Caa	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Ca or lower	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
NR	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Totals	100.00		4.82	100.00		4.29	0.00	0.54

As we saw in Figure 2, the largest sources of systematic risk in our sample portfolio are term structure, sector, and quality. We have therefore directed our attention first to the reports that address these risk components; we will return to them later. Next we examine the reports explaining optionality risk and mortgage risk, even though these risks do not contribute significantly to the risk of this particular portfolio.

Figure 6 shows the optionality report. Several different measures are used to analyze portfolio and benchmark exposures to changes in the value of embedded options. For callable and putable bonds, the difference between a bond's static duration[5] and its option-adjusted duration, known as "reduction due to call," gives one measure of the effect of optionality on pricing. This "reduction" is positive for bonds trading to maturity and negative for bonds trading to a call. These two categories of bonds are represented by separate risk factors. The exposures of the portfolio and benchmark to this "reduction," divided into option categories, constitute one set of factor loadings due to optionality. The model also looks at option delta and gamma, the first and second derivatives of option price with respect to security price.

The risks particular to mortgage-backed securities consist of spread risk, prepayment risk, and convexity risk. The underpinnings for MBS sector spread risk, like those for corporate sectors, are found in the detailed sector report shown in Figure 4. Mortgage-backed securities are divided into four broad sectors based on a combination of originating agency and product: conventional 30-year; GNMA 30-year; all 15-year; and all balloons. The contributions of these four sectors to the portfolio and benchmark spread durations form the factor loadings for mortgage sector risk. Exposures to prepayments are shown in Figure 7. This group of risk factors corresponds to systematic changes in prepayment speeds by sector. Thus, the factor loadings represent the sensitivities of mortgage prices to changes in prepayment speeds (PSA durations). Premium mortgages will show negative prepayment sensitivities (i.e., prices will decrease with increasing prepayment speed), while those of discount mortgages will be positive. To curtail the exposure to sudden changes in prepayment rates, the portfolio should match the benchmark contributions to prepayment sensitivity in each mortgage sector. The third mortgage-specific component of tracking error is due to MBS volatility. Convexity is used as a measure of volatility sensitivity because volatility shocks will have the strongest impact on prices of those mortgages whose prepayment options are at the money

[5] "Static duration" refers to the traditional duration of the bond assuming a fixed set of cashflows. Depending on how the bond is trading, these will be the bond's natural cashflows either to maturity or to the most likely option redemption date.

Figure 6. **Optionality Report**
Sample Portfolio vs. Aggregate Index, 9/30/98

Optionality	% of Portfolio	Duration	Contrib. to Duration	Adjusted Duration	Contrib. to Adj. Dur.	Reduction Due to Call
PORTFOLIO						
Bullet	63.95	5.76	3.68	5.76	3.68	0.00
Callable Traded to Matur.	4.74	10.96	0.52	10.96	0.52	0.00
Callable Traded to Call	4.26	8.43	0.36	4.97	0.21	0.15
Putable Traded to Matur.	0.00	0.00	0.00	0.00	0.00	0.00
Putable Traded to Put	0.00	0.00	0.00	0.00	0.00	0.00
MBS	27.04	3.28	0.89	1.51	0.41	0.48
ABS	0.00	0.00	0.00	0.00	0.00	0.00
CMO	0.00	0.00	0.00	0.00	0.00	0.00
Others	0.00	0.00	0.00	0.00	0.00	0.00
Totals	100.00		5.45		4.82	0.63
BENCHMARK						
Bullet	57.53	5.70	3.28	5.70	3.28	0.00
Callable Traded to Matur.	2.66	9.06	0.24	8.50	0.23	0.01
Callable Traded to Call	7.06	6.93	0.49	3.56	0.25	0.24
Putable Traded to Matur.	0.35	11.27	0.04	9.64	0.03	0.01
Putable Traded to Put	0.78	11.59	0.09	5.77	0.04	0.05
MBS	30.67	3.25	1.00	1.37	0.42	0.58
ABS	0.96	3.14	0.03	3.14	0.03	0.00
CMO	0.00	0.00	0.00	0.00	0.00	0.00
Others	0.00	0.00	0.00	0.00	0.00	0.00
Totals	100.00		5.17		4.29	0.88

Option Delta Analysis

	Portfolio			Benchmark			Difference	
Option Delta	**% of Portf.**	**Delta**	**Cntrb. to Delta**	**% of Portf.**	**Delta**	**Cntrb.to Delta**	**% of Portf.**	**Cntrb. to Delta**
Bullet	63.95	0.000	0.000	57.53	0.000	0.000	6.43	0.000
Callable Traded to Matur.	4.74	0.000	0.000	2.66	0.057	0.002	2.08	-0.002
Callable Traded to Call	4.26	0.474	0.020	7.06	0.584	0.041	-2.80	-0.021
Putable Traded to Matur.	0.00	0.000	0.000	0.35	0.129	0.001	-0.35	-0.001
Putable Traded to Put	0.00	0.000	0.000	0.78	0.507	0.004	-0.78	-0.004
Totals	72.96		0.020	68.38		0.047	4.58	-0.027

Option Gamma Analysis

	Portfolio			Benchmark			Difference	
Option Gamma	**% of Portf.**	**Gamma**	**Cntrb. to Gamma**	**% of Portf.**	**Gamma**	**Cntrb.to Gamma**	**% of Portf.**	**Cntrb. to Gamma**
Bullet	63.95	0.0000	0.0000	57.53	0.0000	0.0000	6.43	0.0000
Callable Traded to Matur.	4.74	0.0000	0.0000	2.66	0.0024	0.0001	2.08	-0.0001
Callable Traded to Call	4.26	0.0059	0.0002	7.06	0.0125	0.0009	-2.80	-0.0006
Putable Traded to Matur.	0.00	0.0000	0.0000	0.35	-0.0029	-0.0000	-0.35	0.0000
Putable Traded to Put	0.00	0.0000	0.0000	0.78	-0.0008	-0.0000	-0.78	0.0000
Totals	72.96		0.0002	68.38		0.0009	4.58	-0.0007

(current coupons). These securities tend to have the most negative convexity. Figure 8 shows the comparison of portfolio and benchmark contributions to convexity in each mortgage sector, which forms the basis for this component of tracking error.

SOURCES OF NON-SYSTEMATIC TRACKING ERROR

In addition to the various sources of systematic risk, Figure 2 indicates that the sample portfolio has 26 bp of non-systematic tracking error, or special risk. This risk stems from portfolio concentrations in individual securities

Figure 7. **MBS Prepayment Sensitivity Report**
Sample Portfolio vs. Aggregate Index, 9/30/98

	Portfolio			Benchmark			Difference	
MBS Sector	**% of Portfolio**	**PSA Sens.**	**Cntrb. to PSA Sens.**	**% of Portfolio**	**PSA Sens.**	**Cntrb. to PSA Sens.**	**% of Portfolio**	**Cntrb. to PSA Sens.**
COUPON < 6.0%								
Conventional	0.00	0.00	0.00	0.00	1.28	0.00	0.00	0.00
GNMA 30-yr.	0.00	0.00	0.00	0.00	1.03	0.00	0.00	0.00
15-year MBS	0.00	0.00	0.00	0.14	0.01	0.00	-0.14	0.00
Balloon	0.00	0.00	0.00	0.05	-0.08	0.00	-0.05	0.00
6.0 % <= COUPON < 7.0%								
Conventional	2.90	-1.14	-0.03	5.37	-1.05	-0.06	-2.48	0.02
GNMA 30 yr.	0.76	-1.19	-0.01	1.30	-1.11	-0.01	-0.53	0.01
15-year MBS	3.52	-0.86	-0.03	3.26	-0.88	-0.03	0.26	0.00
Balloon	3.03	-0.54	-0.02	0.48	-0.73	0.00	2.55	-0.01
7.0 % <= COUPON < 8.0%								
Conventional	4.93	-2.10	-0.10	8.32	-2.79	-0.23	-3.39	0.13
GNMA 30-yr.	4.66	-3.20	-0.15	3.90	-2.82	-0.11	0.76	-0.04
15-year MBS	0.00	0.00	0.00	1.83	-1.92	-0.04	-1.83	0.04
Balloon	0.00	0.00	0.00	0.25	-1.98	-0.01	-0.25	0.01
8.0 % <= COUPON < 9.0%								
Conventional	5.14	-3.91	-0.20	2.26	-4.27	-0.10	2.87	-0.10
GNMA 30-yr.	0.00	0.00	0.00	1.71	-4.71	-0.08	-1.71	0.08
15-year MBS	0.00	0.00	0.00	0.31	-2.16	-0.01	-0.31	0.01
Balloon	0.00	0.00	0.00	0.00	-2.38	0.00	0.00	0.00
9.0 % <= COUPON < 10.0%								
Conventional	0.00	0.00	0.00	0.54	-6.64	-0.04	-0.54	0.04
GNMA 30-yr.	2.11	-7.24	-0.15	0.62	-6.05	-0.04	1.49	-0.12
15-year MBS	0.00	0.00	0.00	0.04	-1.61	0.00	-0.04	0.00
Balloon	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
COUPON >= 10.0%								
Conventional	0.00	0.00	0.00	0.10	-8.14	-0.01	-0.10	0.01
GNMA 30-yr.	0.00	0.00	0.00	0.17	-7.49	-0.01	-0.17	0.01
15-year MBS	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Balloon	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Subtotals								
Conventional	12.96		-0.34	16.6		-0.43	-3.64	0.09
GNMA 30-yr.	7.53		-0.31	7.70		-0.26	-0.16	-0.06
15-year MBS	3.52		-0.03	5.59		-0.07	-2.06	0.04
Balloon	3.03		-0.02	0.78		-0.01	2.25	-0.01
Totals	27.04		-0.70	30.67		-0.76	-3.62	0.07

or issuers. The portfolio report in Figure 9 helps elucidate this risk. The rightmost column of the figure shows the percentage of the portfolio's market value invested in each security. As the portfolio is relatively small, each bond makes up a noticeable fraction. In particular, there are two extremely large positions in corporate bonds, issued by GTE Corp. and Coca-Cola. With $50 million apiece, each of these two bonds represents more than 8% of the portfolio. A negative credit event associated with either of these firms (i.e., a downgrade) would cause large losses in the

Figure 8. **MBS Convexity Analysis**
Sample Portfolio vs. Aggregate Index, 9/30/98

	Portfolio			Benchmark			Difference	
MBS Sector	**% of Portfolio**	**Convexity**	**Cntrb. to Convexity**	**% of Portfolio**	**Convexity**	**Cntrb. to Convexity**	**% of Portfolio**	**Cntrb. to Convexity**
COUPON < 6.0%								
Conventional	0.00	0.00	0.00	0.00	-0.56	0.00	0.00	0.00
GNMA 30-yr.	0.00	0.00	0.00	0.00	-0.85	0.00	0.00	0.00
15-year MBS	0.00	0.00	0.00	0.14	-0.88	0.00	-0.14	0.00
Balloon	0.00	0.00	0.00	0.05	-0.48	0.00	-0.05	0.00
6.0% <= COUPON < 7.0%								
Conventional	2.90	-3.52	-0.10	5.37	-3.19	-0.17	-2.48	0.07
GNMA 30-yr.	0.76	-3.65	-0.03	1.30	-3.13	-0.04	-0.53	0.01
15-year MBS	3.52	-1.78	-0.06	3.26	-2.06	-0.07	0.26	0.00
Balloon	3.03	-1.50	-0.05	0.48	-1.11	-0.01	2.55	-0.04
7.0% <= COUPON < 8.0%								
Conventional	4.93	-3.39	-0.17	8.32	-2.60	-0.22	-3.39	0.05
GNMA 30-yr.	4.66	-2.40	-0.11	3.90	-2.88	-0.11	0.76	0.00
15-year MBS	0.00	0.00	0.00	1.83	-1.56	-0.03	-1.83	0.03
Balloon	0.00	0.00	0.00	0.25	-0.97	0.00	-0.25	0.00
8.0% <= COUPON < 9.0%								
Conventional	5.14	-1.27	-0.07	2.26	-1.01	-0.02	2.87	-0.04
GNMA 30-yr.	0.00	0.00	0.00	1.71	-0.56	-0.01	-1.71	0.01
15-year MBS	0.00	0.00	0.00	0.31	-0.93	0.00	-0.31	0.00
Balloon	0.00	0.00	0.00	0.00	-0.96	0.00	0.00	0.00
9.0% <= COUPON < 10.0%								
Conventional	0.00	0.00	0.00	0.54	-0.80	0.00	-0.54	0.00
GNMA 30-yr.	2.11	-0.34	-0.01	0.62	-0.36	0.00	1.49	-0.01
15-year MBS	0.00	0.00	0.00	0.04	-0.52	0.00	-0.04	0.00
Balloon	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
COUPON >= 10.0%								
Conventional	0.00	0.00	0.00	0.10	-0.61	0.00	-0.10	0.00
GNMA 30-yr.	0.00	0.00	0.00	0.17	-0.21	0.00	-0.17	0.00
15-year MBS	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Balloon	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Subtotals								
Conventional	12.96		-0.33	16.6		-0.42	-3.64	0.08
GNMA 30-yr.	7.53		-0.15	7.70		-0.16	-0.16	0.02
15-year MBS	3.52		-0.06	5.59		-0.10	-2.06	0.04
Balloon	3.03		-0.05	0.78		-0.01	2.25	-0.04
Totals	27.04		-0.59	30.67		-0.69	-3.62	0.10

Figure 9. **Portfolio Report: Composition of Sample Portfolio,** 9/30/98

#	Cusip	Issuer Name	Coup	Maturity	Moody	S&P	Sect	DurW	DurA	Par Val	%
1	057224AF	BAKER HUGHES	8.000	05/15/04	A2	A	IND	4.47	4.47	5,000	0.87
2	097023AL	BOEING CO	6.350	06/15/03	Aa3	AA	IND	3.98	3.98	10,000	1.58
3	191219AY	COCA - COLA ENTERPRISES I	6.950	11/15/26	A3	A+	IND	12.37	12.37	50,000	8.06
4	532457AP	ELI LILLY CO	6.770	01/01/36	Aa3	AA	IND	14.18	14.18	5,000	0.83
5	293561BS	ENRON CORP	6.625	11/15/05	Baa2	BBB+	UTL	5.53	5.53	5,000	0.80
6	31359MDN	FEDERAL NATL MTG ASSN	5.625	03/15/01	Aaa+	AAA+	USA	2.27	2.27	10,000	1.53
7	31359CAT	FEDERAL NATL MTG ASSN-G	7.400	07/01/04	Aaa+	AAA+	USA	4.66	4.66	8,000	1.37
8	FGG06096	FHLM Gold 7-Years Balloon	6.000	04/01/26	Aaa+	AAA+	FHg	2.55	1.69	20,000	3.03
9	FGD06494	FHLM Gold Guar Single F.	6.500	08/01/08	Aaa+	AAA+	FHd	3.13	1.95	23,000	3.52
10	FGB07098	FHLM Gold Guar Single F.	7.000	01/01/28	Aaa+	AAA+	FHb	3.68	1.33	32,000	4.93
11	FGB06498	FHLM Gold Guar Single F.	6.500	02/01/28	Aaa+	AAA+	FHb	5.00	2.83	19,000	2.90
12	319279BP	FIRST BANK SYSTEM	6.875	09/15/07	A2	A-	FIN	6.73	6.73	4,000	0.65
13	339012AB	FLEET MORTGAGE GROUP	6.500	09/15/99	A2	A+	FIN	0.92	0.92	4,000	0.60
14	FNA08092	FNMA Conventional Long T.	8.000	05/01/21	Aaa+	AAA+	FNa	2.56	0.96	33,000	5.14
15	31364FSK	FNMA MTN	6.420	02/12/08	Aaa+	AAA+	USA	2.16	3.40	8,000	1.23
16	345397GS	FORD MOTOR CREDIT	7.500	01/15/03	A1	A	FIN	3.62	3.62	4,000	0.65
17	347471AR	FORT JAMES CORP	6.875	09/15/07	Baa2	BBB-	IND	6.68	6.68	4,000	0.63
18	GNA09490	GNMA I Single Family	9.500	10/01/19	Aaa+	AAA+	GNa	2.69	1.60	13,000	2.11
19	GNA07493	GNMA I Single Family	7.500	07/01/22	Aaa+	AAA+	GNa	3.13	0.75	30,000	4.66
20	GNA06498	GNMA I Single Family	6.500	02/01/28	Aaa+	AAA+	GNa	5.34	3.14	5,000	0.76
21	362320AQ	GTE CORP	9.375	12/01/00	Baa1	A	TEL	1.91	1.91	50,000	8.32
22	458182CB	INT-AMERICAN DEV BANK-G	6.375	10/22/07	Aaa	AAA	SUP	6.76	6.76	6,000	1.00
23	459200AK	INTL BUSINESS MACHINES	6.375	06/15/00	A1	A+	IND	1.58	1.58	10,000	1.55
24	524909AS	LEHMAN BROTHERS INC	7.125	07/15/02	Baa1	A	FIN	3.20	3.20	4,000	0.59
25	539830AA	LOCKHEED MARTIN	6.550	05/15/99	A3	BBB+	IND	0.59	0.59	10,000	1.53
26	563469CZ	MANITOBA PROV CANADA	8.875	09/15/21	A1	AA-	CAN	11.34	11.34	4,000	0.79
27	58013MDE	MCDONALDS CORP	5.950	01/15/08	Aa2	AA	IND	7.05	7.05	4,000	0.63
28	590188HZ	MERRILL LYNCH & CO. - GLO	6.000	02/12/03	Aa3	AA-	FIN	3.77	3.77	5,000	0.76
29	638585BE	NATIONSBANK CORP	5.750	03/15/01	Aa2	A+	FIN	2.26	2.26	3,000	0.45
30	650094BM	NEW YORK TELEPHONE	9.375	07/15/31	A2	A+	TEL	2.43	3.66	5,000	0.86
31	654106AA	NIKE INC	6.375	12/01/03	A1	A+	IND	4.30	4.30	3,000	0.48
32	655844AJ	NORFOLK SOUTHERN CORP	7.800	05/15/27	Baa1	BBB+	IND	12.22	12.22	4,000	0.71
33	669383CN	NORWEST FINANCIAL INC.	6.125	08/01/03	Aa3	AA-	FIN	4.12	4.12	4,000	0.62
34	683234HG	ONT PROV CANADA-GLOBA	7.375	01/27/03	Aa3	AA-	CAN	3.67	3.67	4,000	0.65
35	744567DN	PUB SVC ELECTRIC + GAS	6.125	08/01/02	A3	A-	ELU	3.36	3.36	3,000	0.47
36	755111AF	RAYTHEON CO	7.200	08/15/27	Baa1	BBB	IND	12.61	12.61	8,000	1.31
37	761157AA	RESOLUTION FUNDING CORP	8.125	10/15/19	Aaa+	AAA+	USA	11.22	11.22	17,000	3.51
38	88731EAF	TIME WARNER ENT	8.375	03/15/23	Baa2	BBB-	IND	11.45	11.45	5,000	0.90
39	904000AA	ULTRAMAR DIAMOND SHAM	7.200	10/15/17	Baa2	BBB	IND	10.06	10.06	4,000	0.63
40	912810DB	US TREASURY BONDS	10.375	11/15/12	Aaa+	AAA+	UST	6.30	6.38	10,000	2.17
41	912810DS	US TREASURY BONDS	10.625	08/15/15	Aaa+	AAA+	UST	9.68	9.68	14,000	3.43
42	912810EQ	US TREASURY BONDS	6.250	08/15/23	Aaa+	AAA+	UST	13.26	13.26	30,000	5.14
43	912827XE	US TREASURY NOTES	8.875	02/15/99	Aaa+	AAA+	UST	0.37	0.37	9,000	1.38
44	912827F9	US TREASURY NOTES	6.375	07/15/99	Aaa+	AAA+	UST	0.76	0.76	4,000	0.61
45	912827R4	US TREASURY NOTES	7.125	09/30/99	Aaa+	AAA+	UST	0.96	0.96	17,000	2.59
46	912827Z9	US TREASURY NOTES	5.875	11/15/99	Aaa+	AAA+	UST	1.06	1.06	17,000	2.62
47	912827T4	US TREASURY NOTES	6.875	03/31/00	Aaa+	AAA+	UST	1.42	1.42	8,000	1.23
48	9128273D	US TREASURY NOTES	6.000	08/15/00	Aaa+	AAA+	UST	1.75	1.75	11,000	1.70
49	912827A8	US TREASURY NOTES	8.000	05/15/01	Aaa+	AAA+	UST	2.31	2.31	9,000	1.50
50	912827D2	US TREASURY NOTES	7.500	11/15/01	Aaa+	AAA+	UST	2.72	2.72	10,000	1.67
51	9128272P	US TREASURY NOTES	6.625	03/31/02	Aaa+	AAA+	UST	3.12	3.12	6,000	0.96
52	9128273G	US TREASURY NOTES	6.250	08/31/02	Aaa+	AAA+	UST	3.45	3.45	10,000	1.60
53	912827L8	US TREASURY NOTES	5.750	08/15/03	Aaa+	AAA+	UST	4.22	4.22	1,000	0.16
54	912827T8	US TREASURY NOTES	6.500	05/15/05	Aaa+	AAA+	UST	5.33	5.33	1,000	0.17
55	9128273E	US TREASURY NOTES	6.125	08/15/07	Aaa+	AAA+	UST	6.90	6.90	1,000	0.17
56	949740BZ	WELLS FARGO + CO	6.875	04/01/06	A2	A-	FIN	5.89	5.89	5,000	0.80
57	961214AD	WESTPAC BANKING CORP	7.875	10/15/02	A1	A+	FOC	3.34	3.34	3,000	0.49

portfolio, while hardly affecting the highly diversified benchmark. The Aggregate Index consisted of almost 7000 securities as of September 30, 1998, so that the largest U.S. Treasury issue accounts for less than 1%, and most corporate issues contribute less than 0.01% of the index market value. Thus, any large position in a corporate issue represents a material difference between portfolio and benchmark exposures that must be considered in a full treatment of risk.

The magnitude of the return variance that the risk model associates with a mismatch in allocations to a particular issue is proportional to the square of the allocation difference and to the residual return variance estimated for the issue. This calculation is shown in schematic form in Figure 10 and illustrated numerically for our sample portfolio in Figure 11. With the return variance based on the square of the market weight, it is dominated by the largest positions in the portfolio. The set of bonds shown includes those with the greatest allocations in the portfolio and in the benchmark. The large position in the Coca-Cola bond contributes 21 bp of the total non-systematic risk of 26 bp. This is due to the 8.05% overweighting of this bond relative to its position in the index and the 77 bp monthly volatility of non-systematic return that the model has estimated for this bond. (This estimate is based on bond characteristics such as sector, quality, duration,

Figure 10. **Calculation of Variance Due to Special Risk (Issue-specific Model)***

	Portfolio Weights	**Benchmark Weights**	**Contribution to Issue-Specific Risk**
Issue 1	w_{P_1}	w_{B_1}	$(w_{P_1} - w_{B_1})^2 \sigma^2_{\varepsilon_1}$
Issue 2	w_{P_2}	w_{B_2}	$(w_{P_2} - w_{B_2})^2 \sigma^2_{\varepsilon_2}$
. . .			
Issue *N-1*	$w_{P_{N-1}}$	$w_{B_{N-1}}$	$(w_{P_{N-1}} - w_{B_{N-1}})^2 \sigma^2_{\varepsilon_{N-1}}$
Issue *N*	w_{P_N}	w_{B_N}	$(w_{P_N} - w_{B_N})^2 \sigma^2_{\varepsilon_N}$
Total Issue-specific Risk			$\sum_{i=1}^{N} (w_{P_i} - w_{B_i})^2 \sigma^2_{\varepsilon_i}$

* w_{P_N} and w_{B_N} are weights of security N in the portfolio and in the benchmark as a percentage of total market value. $\sigma^2_{\varepsilon_i}$ is the variance of residual returns for security N. It is obtained from historical volatility of security-specific residual returns unexplained by the combination of all systematic risk factors. See Appendix 3 for details.

age, and amount outstanding. See Appendix 3 for details.) The contribution to the annualized tracking error is then given by $\sqrt{12 \times (0.0805 \times 77)^2} = 21$.

While the overweighting to GTE is larger in terms of percentage of market value, the estimated risk is lower due to the much smaller non-systematic return volatility (37 bp). This is mainly because the GTE issue has a much shorter maturity (12/2000) than the Coca-Cola issue (11/2026). For bonds of similar maturities, the model tends to assign higher special risk volatilities to lower-rated issues. Thus, mismatches in low-quality bonds with long duration will be the biggest contributors to non-systematic tracking error. We assume independence of the risk from individual bonds, so the overall non-systematic risk is computed as the sum of the contributions to variance from each security. Note that mismatches also arise due to bonds that are underweighted in the portfolio. Most bonds in the index do not appear in the portfolio, and each missing bond contributes to tracking error. However, the percentage of the index each bond represents is usually very

Figure 11. **Illustration of the Calculation of Non-systematic Tracking Error**

CUSIP	Issuer	Coupon	Maturity	Spec. Risk Vol. (bp/mo.)	% of Portf.	% of Bnchmrk.	Diff.	Contrib. Tracking Error (bp/mo.)
097023AL	BOEING CO	6.350	06/15/03	44	1.58	0.01	1.58	2
191219AY	COCA-COLA ENTERPRISES INC.	6.950	11/15/26	77	8.06	0.01	8.05	21
362320AQ	GTE CORP	9.375	12/01/00	37	8.32	0.01	8.31	11
532457AP	ELI LILLY CO	6.770	01/01/36	78	0.83	0.01	0.82	2
563469CZ	MANITOBA PROV CANADA	8.875	09/15/21	73	0.79	0.01	0.79	2
655844AJ	NORFOLK SOUTHERN CORP	7.800	05/15/27	84	0.71	0.02	0.70	2
755111AF	RAYTHEON CO	7.200	08/15/27	85	1.31	0.01	1.30	4
761157AA	RESOLUTION FUNDING CORP	8.125	10/15/19	19	3.51	0.12	3.39	2
88731EAF	TIME WARNER ENT	8.375	03/15/23	80	0.90	0.02	0.88	2
912810DS	U.S. TREASURY BONDS	10.625	08/15/15	17	3.43	0.18	3.25	2
912810EC	U.S. TREASURY BONDS	8.875	02/15/19	18	0.00	0.49	-0.49	0
912810ED	U.S. TREASURY BONDS	8.125	08/15/19	18	0.00	0.47	-0.47	0
912810EG	U.S. TREASURY BONDS	8.750	08/15/20	18	0.00	0.54	-0.54	0
912810EL	U.S. TREASURY BONDS	8.000	11/15/21	17	0.00	0.81	-0.81	0
912810EQ	U.S. TREASURY BONDS	6.250	08/15/23	19	5.14	0.46	4.68	3
912810FB	U.S. TREASURY BONDS	6.125	11/15/27	20	0.00	0.44	-0.44	0
FGB07097	FHLM Gold Guar. Single Fam. 30-yr.	7.000	04/01/27	16	0.00	0.56	-0.56	0
FGB07098	FHLM Gold Guar. Single Fam. 30-yr.	7.000	01/01/28	15	4.93	0.46	4.47	2
FNA06498	FNMA Conventional Long T. 30-yr.	6.500	03/01/28	15	0.00	1.16	-1.16	1
FNA07093	FNMA Conventional Long T. 30-yr.	7.000	07/01/22	16	0.00	0.65	-0.65	0
FNA07097	FNMA Conventional Long T. 30-yr.	7.000	05/01/27	16	0.00	0.69	-0.69	0
FNA08092	FNMA Conventional Long T. 30-yr.	8.000	05/01/21	17	5.14	0.24	4.90	3
GNA07493	GNMA I Single Fam. 30-yr.	7.500	07/01/22	16	4.66	0.30	4.36	2

small. Besides, their contributions to return variance are squared in the calculation of tracking error. Thus, the impact of bonds not included in the portfolio is usually insignificant. The largest contribution to tracking error stemming from an underweighting to a security is due to the 1998 issuance of FNMA 30-year 6.5% pass-throughs, which represents 1.16% of the benchmark. Even this relatively large mismatch contributes only a scant 1 bp to tracking error.

This non-systematic risk calculation is carried out twice, using two different methods. In the *issuer-specific* calculation, the holdings of the portfolio and benchmark are not compared on a bond-by-bond basis, as in Figures 10 and 11, but are first aggregated into concentrations in individual issuers. This calculation is based on the assumption that spreads of bonds of the same issuer tend to move together. Therefore, matching the benchmark issuer allocations is sufficient. In the *issue-specific* calculation, each bond is considered an independent source of risk. This model recognizes that large exposures to a single bond can incur more risk than a portfolio of all of an issuer's debt. In addition to credit events that affect an issuer as a whole, individual issues can be subject to various technical effects. For most portfolios, these two calculations produce very similar results. In certain circumstances, however, there can be significant differences. For instance, some large issuers use an index of all their outstanding debt as an internal performance benchmark. In the case of a single-issuer portfolio and benchmark, the issue-specific risk calculation will provide a much better measure of non-systematic risk. The reported non-systematic tracking error of 26.1 bp for this portfolio, which contributes to the total tracking error, is the average of the results from the issuer-specific and issue-specific calculations.

COMBINING COMPONENTS OF TRACKING ERROR

Given the origins of each component of tracking error shown in Figure 2, we can address the question of how these components combine to form the overall tracking error. Of the 52 bp of overall tracking error (TE), 45 bp correspond to systematic TE and 26 bp to non-systematic TE. The net result of these two sources of tracking error does not equal their sum. Rather, the squares of these two numbers (which represent variances) sum to the variance of the result. Next we take its square root to obtain the overall TE ($\sqrt{45.0^2 + 26.1^2} = 52.0$). This illustrates the risk-reducing benefits of diversification from combining independent (zero correlation) sources of risk.

When components of risk are not assumed to be independent, correlations must be considered. At the top of Figure 2, we see that the systematic risk is composed of 36.3 bp of term structure risk and 39.5 bp from all other forms of systematic risk combined (non-term structure risk). If these two were independent, they would combine to a systematic tracking error of 53.6 bp

($\sqrt{36.3^2 + 39.5^2} = 53.6$). The combined systematic tracking error of only 45 bp reflects negative correlations among certain risk factors in the two groups.

The tracking error breakdown report in Figure 2 shows the sub-components of tracking error due to sector, quality, etc. These sub-components are calculated in two different ways. In the first column, we estimate the *isolated tracking error* due to the effect of each group of related risk factors considered alone. The tracking error due to term structure, for example, reflects only the portfolio/benchmark mismatches in exposures along the yield curve, as well as the volatilities of each of these risk factors and the correlations among them. Similarly, the tracking error due to sector reflects only the mismatches in sector exposures, the volatilities of these risk factors, and the correlations among them. However, the correlations between the risk factors due to term structure and those due to sector do not participate in either of these calculations. Figure 12 depicts an idealized covariance matrix containing just three groups of risk factors relating to the yield curve (Y), sector spreads (S) and quality spreads (Q). Figure 12a illustrates how the covariance matrix is used to calculate the sub-components of tracking error in the isolated mode. The three shaded blocks represent the parts of the matrix that pertain to: movements of the various points along the yield curve and the correlations among them (Y x Y); movements of sector spreads and the correlations among them (S x S); and movements of quality spreads and the correlations among them (Q x Q). The unshaded portions of the matrix, which deal with the correlations among different sets of risk factors, do not contribute to any of the partial tracking errors.

Figure 12. **Illustration of "Isolated" and "Cumulative" Calculations of Tracking Error Subcomponents***

a. **Isolated Calculation of Tracking Error Components**

Y x Y	**Y x S**	**Y x Q**
S x Y	**S x S**	**S x Q**
Q x Y	**Q x S**	**Q x Q**

b. **Cumulative Calculation of Tracking Error Components**

Y x Y	**Y x S**	**Y x Q**
S x Y	**S x S**	**S x Q**
Q x Y	**Q x S**	**Q x Q**

* Y - **Y**ield curve risk factors; S - **S**ector spread risk factors; Q - Credit **Q**uality spread risk factors.

The next two columns of Figure 2 represent a different way of subdividing tracking error. The middle column shows the *cumulative tracking error*, which incrementally introduces one group of risk factors at a time to the tracking error calculation. In the first row, we find 36.3 bp of tracking error due to term structure. In the second, we see that if term structure and sector risk are considered together, while all other risks are ignored, the tracking error increases to 38.3 bp. The rightmost column shows that the resulting "change in tracking error" due to the incremental inclusion of sector risk is 2.0 bp. As additional groups of risk factors are included, the calculation converges toward the total systematic tracking error, which is obtained with the use of the entire matrix. Figure 12b illustrates the rectangular section of the covariance matrix that is used at each stage of the calculation. The incremental tracking error due to sector reflects not only the effect of the S x S box in the diagram, but the S x Y and Y x S cross terms as well. That is, the partial tracking error due to sector takes into account the correlations between sector risk and yield curve risk. It answers the question, "Given the exposure to yield curve risk, how much more risk is introduced by the exposure to sector risk?"

The incremental approach is intuitively pleasing because the partial tracking errors (the "Change in Tracking Error" column of Figure 2) add up to the total systematic tracking error. Of course, the order in which the various partial tracking errors are considered will affect the magnitude of the corresponding terms. Also, note that some of the partial tracking errors computed in this way are negative. This reflects negative correlations among certain groups of risk factors. For example, in Figure 2, the incremental risk due to the MBS Sector is –1.7 bp.

The two methods used to subdivide tracking error into different components are complementary and serve different purposes. The isolated calculation is ideal for comparing the magnitudes of different types of risk to highlight the most significant exposures. The cumulative approach produces a set of tracking error sub-components that sum to the total systematic tracking error and reflect the effect of correlations among different groups of risk factors. The major drawback of the cumulative approach is that results are highly dependent on the order in which they are computed. The order currently used by the model was selected based on the significance of each type of risk; it may not be optimal for every portfolio/benchmark combination.

OTHER RISK MODEL OUTPUTS

The model's analysis of portfolio and benchmark risk is not limited to the calculation of tracking error. The model also calculates the absolute return volatilities (sigmas) of portfolio and benchmark. *Portfolio sigma* is calculated in the same fashion as tracking error, but is based on the factor loadings (sensitivities to market factors) of the portfolio, rather than on the differences from the benchmark. Sigma represents the volatility of portfolio returns, just as tracking error represents the volatility of the return difference between portfolio and benchmark. Also like tracking error, sigma consists of systematic and non-systematic components, and the volatility of the benchmark return is calculated in the same way. Both portfolio and benchmark sigmas appear at the bottom of the tracking error report (Figure 2). Note that the tracking error of 52 bp (the annualized volatility of return difference) is greater than the difference between the return volatilities (sigmas) of the portfolio and the benchmark (440 bp - 417 bp = 23 bp). It is easy to see why this should be so. Assume a benchmark of Treasury bonds, whose entire risk is due to term structure. A portfolio of short term, high-yield corporate bonds could be constructed such that the overall return volatility would match that of the Treasury benchmark. The magnitude of the credit risk in this portfolio might match the magnitude of the term structure risk in the benchmark, but the two would certainly not cancel each other out. The tracking error in this case might be larger than the sigma of either the portfolio or the benchmark.

In our example, the portfolio sigma is greater than that of the benchmark. Thus, we can say that the portfolio is “more risky” than the benchmark—its longer duration makes it more susceptible to a rise in interest rates. What if the portfolio was shorter than the benchmark and had a lower sigma? In this sense, we could consider the portfolio to be less risky. However, tracking error could be just as big given its capture of the risk of a yield curve rally in which the portfolio would lag. To reduce the risk of underperformance (tracking error), it is necessary to match the risk exposures of portfolio and benchmark. Thus, the reduction of tracking error will typically result in bringing portfolio sigma nearer to that of the benchmark; but sigma can be changed in many ways that will not necessarily improve the tracking error.

It is interesting to compare the non-systematic components of portfolio and benchmark risk. The first thing to notice is that, when viewed in the context of the overall return volatility, the effect of non-systematic risk is negligible. To the precision shown, for both the portfolio and benchmark, the overall sigma is equal to its systematic part. The portfolio-level risk due to individual credit events is very small when compared to the total volatility of returns, which includes the entire exposure to all systematic risks, notably yield changes. The portfolio also has significantly more non-systematic risk (27 bp) than does the benchmark (4 bp), because the latter is much more diversified. In fact, because the benchmark exposures to any individual issuer are so close to zero, the non-systematic tracking error (26 bp) is almost the same as the non-systematic part of portfolio sigma. Notice that the non-systematic risk can form a significant component of the tracking error (26.1 bp out of a total of 52 bp) even as it is a negligible part of the absolute return volatility.

Another quantity calculated by the model is *beta*, which measures the risk of the portfolio relative to that of the benchmark. The beta for our sample portfolio is 1.05, as shown at the bottom of Figure 1. This means that the portfolio is more risky (volatile) than the benchmark. For every 100 bp of benchmark return (positive or negative), we would expect to see 105 bp for the portfolio. It is common to compare the beta produced by the risk model with the ratio of portfolio and benchmark durations. In this case, the duration ratio is 4.82 / 4.29 = 1.12, which is somewhat larger than the risk model beta. This is because the duration-based approach considers only term structure risk (and only parallel shift risk at that), while the risk model includes the combined effects of all relevant forms of risk, along with the correlations among them.

3. RISK MODEL APPLICATIONS

QUANTIFYING RISK ASSOCIATED WITH A VIEW

The risk model is primarily a diagnostic tool. Whatever position a portfolio manager has taken relative to the benchmark, the risk model will quantify how much risk has been assumed. This helps measure the risk of the exposures taken to express a market view. It also points out the potential unintended risks in the portfolio.

Many firms use risk-adjusted measures to evaluate portfolio performance. A high return achieved by a series of successful but risky market plays may not please a conservative pension plan sponsor. A more modest return, achieved while maintaining much lower risk versus the benchmark, might be seen as a healthier approach over the long term. This point of view can be reflected either by adjusting performance by the amount of risk taken or by specifying in advance the acceptable level of risk for the portfolio. In any case, the portfolio manager should be cognizant of the risk inherent in a particular market view and weigh it against the anticipated gain. The increasing popularity of risk-adjusted performance evaluation is evident in the frequent use of the concept of an *information ratio*—portfolio outperformance of the benchmark per unit of standard deviation of observed outperformance. Plan sponsors often diversify among asset managers with different styles, looking for some of them to take more risk and for others to stay conservative, but always looking for high information ratios.

RISK BUDGETING

To limit the amount of risk that may be taken by its portfolio managers, a plan sponsor or a corporate treasury can prescribe a maximum allowable tracking error. In the past, an asset management mandate might have put explicit constraints on deviation from the benchmark duration, differences in sector allocations, concentration in a given issuer, and total percentage invested outside the benchmark. Currently, we observe a tendency to constrain the overall risk versus the benchmark and leave the choice of the form of risk to the portfolio manager based on current risk premia offered by the market. By expressing various types of risk in the same units of tracking error, the model makes it possible to introduce the concept of opportunistic risk budget allocation. To constrain specific types of risk, limits can be applied to the different components of tracking error produced by the model. As described above, the overall tracking error represents the best way to quantify the net effect of multiple dimensions of risk in a single number.

With the model-specific nature of tracking error, there may be situations where the formal limits to be placed on the portfolio manager must be expressed in more objective terms. Constraints commonly found in investment policies include limits on the deviation between the portfolio and the benchmark, both in terms of Treasury duration and in spread duration contributions from various fixed-income asset classes. Because term structure risk tends to be best understood, many organizations have firm limits only for the amount of duration deviation allowed. For example, a portfolio manager may be limited to an one-year band around benchmark duration. How can this limit be applied to risks along a different dimension?

The risk model can help establish relationships among risks of different types by comparing their tracking errors. Figure 13 shows the tracking errors achieved by several different blends of Treasury and spread product indices relative to the Treasury Index. A pure Treasury composite (Strategy 1) with duration one year longer than the benchmark has a tracking error of 85 bp per year. Strategies 2 and 3 are created by combining the investment-grade Corporate Index with both intermediate and long Treasury Indices to achieve desired exposures to spread duration while remaining neutral to the benchmark in Treasury duration. Similar strategies are engaged to generate desired exposures to spread duration in the MBS and high-yield markets. As seen in the table, a one-year extension in pure Treasury duration (Strategy 1) is equivalent to 2.5 years of extension in corporate spread duration, or about 0.75 years of extension in high-yield spread duration. Our results with MBS spreads show that 1.0 year of MBS spread duration causes a tracking error of 58 bp, while 1.5 years of duration gives a tracking

Figure 13. **"Risk Budget": An Example Using Components of Treasury and Spread Indices Relative to a Treasury Benchmark**

Index	Treasury	Intermediate Treasury	Long Treasury	Corporate	MBS	High Yield
Duration	5.48	3.05	10.74	5.99	3.04	4.68
Spread Duration	0.00	0.00	0.00	6.04	3.46	4.58

Strategy No.	Risk Strategy	Tsy Dur. Diff. (years)	Spread Dur. Diff. (years)	% Interm. Treasury	% Long Treasury	% Sprd. Sector	Tracking Error vs. Tsy. Index (bp/yr)
	Treasury Index			68.40	31.60	0.00	0
1	Treasury Duration	1.0	0.00	55.40	44.60	0.00	85
2	Corp. Spread Duration	0.0	1.00	58.17	25.27	16.56	34
3		0.0	2.50	42.83	15.78	41.39	85
4	Tsy. Dur. & Corp. Sprd. Dur	0.5	1.25	49.12	30.19	20.70	51
5	MBS Spread Duration	0.0	1.00	39.46	31.64	28.90	58
6		0.0	1.47	25.99	31.65	42.36	85
7		0.0	1.50	24.99	31.66	43.35	87
8	High Yield Spread Duration	0.0	0.75	55.50	28.13	16.38	84
9		0.0	1.00	51.19	26.97	21.83	119

error of 87 bp. A simple linear interpolation would suggest that a tracking error of 85 bp (the magnitude of the risk of a 1-year duration extension) thus corresponds to an extension in MBS spread duration of approximately 1.47 years.

Of course, these are idealized examples in which spread exposure to one type of product is changed while holding Treasury duration constant. A real portfolio is likely to take risks in all dimensions simultaneously. To calculate the tracking error, the risk model considers the correlations among the different risk factors. As long as two risks along different dimensions are not perfectly correlated, the net risk is less than the sum of the two risks. For example, we have established that 2.5 years of corporate spread duration produces roughly the same risk as 1 year of Treasury duration, each causing a tracking error of about 85 bp. For a portfolio able to take both types of risk, an investor might allocate half of the risk budget to each, setting limits of 0.5 years on Treasury duration and 1.25 years on corporate spread duration. This should keep the risk within the desired range of tracking error. As shown in Figure 13, this combination of risks produces a tracking error of only 51 bp. This method of allocating risk under a total risk budget (in terms of equivalent duration mismatches) can provide investors with a method of controlling risk that is easier to implement and more conservative than a direct limit on tracking error. This macro view of risk facilitates the capablity to set separate but uniformly expressed limits on portfolio managers responsible for different kinds of portfolio exposures.

PROJECTING THE EFFECT OF PROPOSED TRANSACTIONS ON TRACKING ERROR

Proposed trades are often analyzed in the context of a 1-for-1 (substitution) swap. Selling a security and using the proceeds to buy another may earn a few additional basis points of yield. The risk model allows analysis of such a trade in the context of the portfolio and its benchmark. By comparing the current portfolio versus benchmark risk and the pro forma risk after the proposed trade, an asset manager can evaluate how well the trade fits the portfolio. Our portfolio analytics platform offers an interactive mode to allow portfolio modifications and immediately see the effect on tracking error.

For example, having noticed that our sample portfolio has an extremely large position in the Coca-Cola issue, we might decide to cut the size of this position in half. To avoid making any significant changes to the systematic risk profile of the portfolio, we might look for a bond with similar maturity, credit rating, and sector. Figure 14 shows an example of such a swap. Half the position in the Coca-Cola 30-year bond is replaced by a 30-year issue from Anheuser-Busch, another single-A rated issuer in the beverage sector. As shown in Figure 20, this transaction reduces non-systematic tracking error from 26 bp to 22 bp. While we have unwittingly produced a 1 bp increase

Figure 14. **A Simple Diversification Trade: Cut the Size of the Largest Position in Half**

	Issuer	Coupon	Maturity	Par Value ($000s)	MV ($000s)	Sector	Quality	Dur. Adj. (years)
SELL:	COCA- COLA ENTERPRISES INC.	6.95	11/15/2026	25000	27053	IND	A3	12.37
BUY:	ANHEUSER-BUSCH CO.,INC.	6.75	12/15/2027	25000	26941	IND	A1	12.86

in the systematic risk (the durations of the two bonds were not identical), the overall effect was a decrease in tracking error from 52 bp to 51 bp.

OPTIMIZATION

For many portfolio managers, the risk model acts not only as a measurement tool but plays a major role in the portfolio construction process. The model has a unique optimization feature that guides investors to transactions that reduce portfolio risk. The types of questions it addresses are: What single transaction can reduce the risk of the portfolio relative to the benchmark the most? How could the tracking error be reduced with minimum turnover? The portfolio manager is given an opportunity to intervene at each step in the optimization process and select transactions that lead to the desired changes in the risk profile of the portfolio and are practical at the same time.

As in any portfolio optimization procedure, the first step is to choose the set of assets that may be purchased. The composition of this investable universe, or *bond swap pool,* is critical. This universe should be large enough to provide flexibility in matching all benchmark risk exposures, yet it should contain only securities that are acceptable candidates for purchase. This universe may be created by querying a bond database (selecting, for instance, all corporate bonds with more than $500 million outstanding that were issued in the last three years) or by providing a list of securities available for purchase.

Once the investable universe has been selected, the optimizer begins an iterative process, known as *gradient descent* (see Appendix 4 for a detailed mathematical description of the algorithm), searching for 1-for-1 bond swap transactions that will achieve the investor's objective. In the simplest case, the objective is to minimize the tracking error. The bonds in the swap pool are ranked in terms of reduction in tracking error per unit of each bond purchased. The system indicates which bond, if purchased, will lead to the steepest decline in tracking error, but leaves the ultimate choice of the security to the investor. Once a bond has been selected for purchase, the optimizer offers a list of possible market-value-neutral swaps of this security against various issues in the portfolio (with the optimal transaction size for each pair of bonds), sorted in order of possible reduction in tracking error. Investors are free to adjust the model's recommendations, either selecting different bonds to sell or adjusting (e.g., rounding off) recommended trade amounts.

Figure 15 shows how this optimization process is used to minimize the tracking error of the sample portfolio. A close look at the sequence of trades suggested by the optimizer reveals that several types of risk are

Figure 15. **Sequence of Transactions Selected by Optimizer Showing Progressively Smaller Tracking Error,** $000s

Initial Tracking Error: 52.0 bp

Transaction # 1		
Sold:	31000 of COCA-COLA ENTERPRISES	6.950 2026/11/15
Bought:	30000 of U.S. TREASURY NOTES	8.000 2001/05/15
Cash Leftover:	-17.10	
New Tracking Error:	29.4 bp	
Cost of This Transaction:	152.500	
Cumulative Cost:	152.500	
Transaction # 2		
Sold:	10000 of LOCKHEED MARTIN	6.550 1999/05/15
Bought:	9000 of U.S. TREASURY NOTES	6.125 2007/08/15
Cash Leftover:	132.84	
New Tracking Error:	25.5 bp	
Cost of This Transaction:	47.500	
Cumulative Cost:	200.000	
Transaction # 3		
Sold:	4000 of NORFOLK SOUTHERN CORP	7.800 2027/05/15
Bought:	3000 of U.S. TREASURY BONDS	10.625 2015/08/15
Cash Leftover:	-8.12	
New Tracking Error:	23.1 bp	
Cost of This Transaction:	17.500	
Cumulative Cost:	217.500	
Transaction # 4		
Sold:	33000 of GTE CORP	9.375 2000/12/01
Bought:	34000 of U.S. TREASURY NOTES	6.625 2002/03/31
Cash Leftover:	412.18	
New Tracking Error:	19.8 bp	
Cost of This Transaction:	167.500	
Cumulative Cost:	385.000	
Transaction # 5		
Sold:	7000 of COCA-COLA ENTERPRISES	6.950 2026/11/15
Bought:	8000 of U.S. TREASURY NOTES	6.000 2000/08/15
Cash Leftover:	-304.17	
New Tracking Error:	16.4 bp	
Cost of This Transaction:	37.500	
Cumulative Cost:	422.500	

reduced simultaneously. In the first trade, the majority of the large position in the Coca-Cola 30-year bond is swapped for a 3-year Treasury. This trade simultaneously changes systematic exposures to term structure, sector, and quality; it also cuts one of the largest issuer exposures, reducing non-systematic risk. This one trade brings the overall tracking error down from 52 bp to 29 bp. As risk declines and the portfolio risk profile approaches the benchmark, there is less room for such drastic improvements. Transaction sizes become smaller, and the improvement in tracking error with each trade slows. The second and third transactions continue to adjust the sector and quality exposures and fine-tune the risk exposures along the curve. The fourth transaction addresses the other large corporate exposure, cutting the position in GTE by two-thirds. The first five trades reduce the tracking error to 16 bp, creating an essentially passive portfolio.

An analysis of the tracking error for this passive portfolio is shown in Figure 16. The systematic tracking error has been reduced to just 10 bp and the non-systematic risk to 13 bp. Once systematic risk drops below non-systematic risk, the latter becomes the limiting factor. In turn, further tracking error reduction by just a few transactions becomes much less likely. When there are exceptionally large positions, like the two mentioned in the above example, non-systematic risk can be reduced quickly. Upon completion of

Figure 16. **Tracking Error Summary**
Passive Portfolio vs. Aggregate Index, 9/30/98

	Tracking Error (bp/year)		
	Isolated	**Cumulative**	**Change**
Tracking Error Term Structure	7.0	7.0	7.0
Non-Term Structure	9.6		
Tracking Error Sector	7.4	10.5	3.5
Tracking Error Quality	2.1	11.2	0.7
Tracking Error Optionality	1.6	11.5	0.3
Tracking Error Coupon	2.0	12.3	0.8
Tracking Error MBS Sector	4.9	10.2	-2.1
Tracking Error MBS Volatility	7.2	11.1	0.9
Tracking Error MBS Prepayment	2.5	10.3	-0.8
Total Systematic Tracking Error		10.3	
Non-systematic Tracking Error			
Issuer-specific	12.4		
Issue-specific	13.0		
Total	12.7		
Total Tracking Error Return		16	

	Systematic	**Non-systematic**	**Total**
Benchmark Sigma	417	4	417
Portfolio Sigma	413	13	413

such risk reduction transactions, further reduction of tracking error requires a major diversification effort. The critical factor that determines non-systematic risk is the percentage of the portfolio in any single issue. On average, a portfolio of 50 bonds has 2% allocated to each position. To reduce this average allocation to 1%, the number of bonds would need to be doubled.

The risk exposures of the resulting passive portfolio match the benchmark much better than the initial portfolio. Figure 17 details the term structure risk of the passive portfolio. Compared with Figure 3, the overweight at the long end is reduced significantly. The overweight at the 25-year vertex has gone down from 1.45% to 0.64%, and (perhaps more importantly) it is now offset partially by underweights at the adjacent 20- and 30-year vertices. Figure 18 presents the sector risk report for the passive portfolio. The underweight to Treasuries (in contribution to duration) has been reduced from -0.77% to -0.29% relative to the initial portfolio (Figure 4), and the largest corporate overweight, to consumer non-cyclicals, has come down from +1.00% to +0.24%.

Minimization of tracking error, illustrated above, is the most basic application of the optimizer. This is ideal for passive investors who want their portfolios to track the benchmark as closely as possible. This method also aids investors who hope to outperform the benchmark mainly on the basis of

Figure 17. **Term Structure Risk Report for Passive Portfolio,** 9/30/98

	Cash Flows		
Year	**Portfolio**	**Benchmark**	**Difference**
0.00	1.33%	1.85%	-0.52%
0.25	3.75	4.25	-0.50
0.50	4.05	4.25	-0.19
0.75	3.50	3.76	-0.27
1.00	8.96	7.37	1.59
1.50	7.75	10.29	-2.54
2.00	8.30	8.09	0.21
2.50	10.30	6.42	3.87
3.00	5.32	5.50	-0.19
3.50	8.24	4.81	3.43
4.00	6.56	7.19	-0.63
5.00	5.91	6.96	-1.05
6.00	3.42	4.67	-1.24
7.00	5.75	7.84	-2.10
10.00	6.99	7.37	-0.38
15.00	4.00	3.88	0.12
20.00	2.98	3.04	-0.05
25.00	2.37	1.73	0.64
30.00	0.47	0.68	-0.21
40.00	0.08	0.07	0.01

Figure 18. **Sector Risk Report for Passive Portfolio,** 9/30/98

	Portfolio			Benchmark			Difference	
Detailed Sector	**% of Portfolio**	**Adj. Dur.**	**Contrib. to Adj. Dur.**	**% of Portfolio**	**Adj. Dur.**	**Contrib. to Adj. Dur.**	**% of Portfolio**	**Contrib. to Adj. Dur.**
Treasury								
Coupon	40.98	4.72	1.94	39.82	5.58	2.22	1.16	-0.29
Strip	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Agencies								
FNMA	4.12	3.40	0.14	3.56	3.44	0.12	0.56	0.02
FHLB	0.00	0.00	0.00	1.21	2.32	0.03	-1.21	-0.03
FHLMC	0.00	0.00	0.00	0.91	3.24	0.03	-0.91	-0.03
REFCORP	3.50	11.22	0.39	0.83	12.18	0.10	2.68	0.29
Other Agencies	0.00	0.00	0.00	1.31	5.58	0.07	-1.31	-0.07
Financial Institutions								
Banking	1.91	5.31	0.10	2.02	5.55	0.11	-0.11	-0.01
Brokerage	1.35	3.52	0.05	0.81	4.14	0.03	0.53	0.01
Financial Cos.	1.88	2.92	0.05	2.11	3.78	0.08	-0.23	-0.02
Insurance	0.00	0.00	0.00	0.52	7.47	0.04	-0.52	-0.04
Other	0.00	0.00	0.00	0.28	5.76	0.02	-0.28	-0.02
Industrials								
Basic	0.63	6.68	0.04	0.89	6.39	0.06	-0.26	-0.01
Capital Goods	2.89	7.88	0.23	1.16	6.94	0.08	1.73	0.15
Consumer Cycl.	2.01	8.37	0.17	2.28	7.10	0.16	-0.27	0.01
Consum. Non-cycl.	2.76	12.91	0.36	1.66	6.84	0.11	1.10	0.24
Energy	1.50	6.82	0.10	0.69	6.89	0.05	0.81	0.05
Technology	1.55	1.58	0.02	0.42	7.39	0.03	1.13	-0.01
Transportation	0.00	0.00	0.00	0.57	7.41	0.04	-0.57	-0.04
Utilities								
Electric	0.47	3.36	0.02	1.39	5.02	0.07	-0.93	-0.05
Telephone	3.69	2.32	0.09	1.54	6.58	0.10	2.15	-0.02
Natural Gas	0.80	5.53	0.04	0.49	6.50	0.03	0.31	0.01
Water	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Yankee								
Canadians	1.45	7.87	0.11	1.06	6.67	0.07	0.38	0.04
Corporates	0.49	3.34	0.02	1.79	6.06	0.11	-1.30	-0.09
Supranational	1.00	6.76	0.07	0.38	6.33	0.02	0.62	0.04
Sovereigns	0.00	0.00	0.00	0.66	5.95	0.04	-0.66	-0.04
Hypothetical	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Cash	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Mortgage								
Conventional 30-yr.	12.96	1.52	0.20	16.60	1.42	0.24	-3.64	-0.04
GNMA 30-yr.	7.53	1.23	0.09	7.70	1.12	0.09	-0.17	0.01
MBS 15-yr.	3.52	1.95	0.07	5.59	1.63	0.09	-2.07	-0.02
Balloons	3.02	1.69	0.05	0.78	1.02	0.01	2.24	0.04
OTM	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
European & International								
Eurobonds	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
International	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Asset Backed	0.00	0.00	0.00	0.96	3.14	0.03	-0.96	-0.03
CMO	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Other	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Totals	100.00		4.35	100.00		4.29	0.00	0.06

security selection, without expressing views on sector or yield curve. Given a carefully selected universe of securities from a set of favored issuers, the optimizer can help build security picks into a portfolio with no significant systematic exposures relative to the benchmark.

For more active portfolios, the objective is no longer minimization of tracking error. When minimizing tracking error, the optimizer tries to reduce the largest differences between the portfolio and benchmark. But what if the portfolio is meant to be long duration or overweighted in a particular sector to express a market view? These views certainly should not be "optimized" away. However, unintended exposures need to be minimized, while keeping the intentional ones.

For instance, assume in the original sample portfolio that the sector exposure is intentional but the portfolio should be neutral to the benchmark for all other sources of risk, especially term structure. The risk model allows the investor to keep exposures to one or more sets of risk factors (in this case, sector) and optimize to reduce the components of tracking error due to all other risk factors. This is equivalent to reducing all components of tracking error but the ones to be preserved. The model introduces a significant penalty for changing the risk profile of the portfolio in the risk categories designated for preservation.

Figure 19 shows the transactions suggested by the optimizer in this case.[6] At first glance, the logic behind the selection of the proposed transactions is not as clear as before. We see a sequence of fairly small transactions, mostly trading up in coupon. Although this is one way to change the term structure exposure of a portfolio, it is usually not the most obvious or effective method. The reason for this lies in the very limited choices we offered the optimizer for this illustration. As in the example of tracking error minimization, the investable universe was limited to securities already in the portfolio. That is, only rebalancing trades were permitted. Because the most needed cashflows are at vertices where the portfolio has no maturing securities, the only way to increase those flows is through higher coupon payments. In a more realistic optimization exercise, we would include a wider range of maturity dates (and possibly a set of zero-coupon securities as well) in the investable universe to give the optimizer more flexibility in adjusting portfolio cashflows. Despite these self-imposed limitations, the optimizer succeeds in bringing down the term structure risk while leaving the sector risk almost unchanged. Figure 20 shows the tracking error breakdown for the resulting portfolio. The term structure risk has been

[6] Tracking error does not decrease with each transaction. This is possible because the optimizer does not minimize the tracking error itself in this case, but rather a function that includes the tracking error due to all factors but sector, as well as a penalty term for changing sector exposures.

Figure 19. **Sequence of Transactions Selected by Optimizer, Keeping Exposures to Sector,** $000s

Initial Tracking Error: 52.0 bp

Transaction # 1

Sold:	2000 of COCA-COLA ENTERPRISES	6.950 2026/11/15
Bought:	2000 of NORFOLK SOUTHERN CORP	7.800 2027/05/15
Cash Leftover:	-235.19	
New Tracking Error:	52.1 bp	
Cost of This Transaction:	10.000	
Cumulative Cost:	10.000	

Transaction # 2

Sold:	2000 of COCA-COLA ENTERPRISES	6.950 2026/11/15
Bought:	2000 of NEW YORK TELEPHONE	9.375 2031/07/15
Cash Leftover:	-389.36	
New Tracking Error:	50.1 bp	
Cost of This Transaction:	10.000	
Cumulative Cost:	20.000	

Transaction # 3

Sold:	10000 of U.S. TREASURY BONDS	6.250 2023/08/15
Bought:	10000 of NEW YORK TELEPHONE	9.375 2031/07/15
Cash Leftover:	-468.14	
New Tracking Error:	47.4 bp	
Cost of This Transaction:	50.000	
Cumulative Cost:	70.000	

Transaction # 4

Sold:	2000 of COCA-COLA ENTERPRISES	6.950 2026/11/15
Bought:	2000 of FHLM Gold Guar. Single Fam.	7.000 2028/01/01
Cash Leftover:	-373.47	
New Tracking Error:	46.0 bp	
Cost of This Transaction:	10.000	
Cumulative Cost:	80.000	

Transaction # 5

Sold:	6000 of U.S. TREASURY BONDS	6.250 2023/08/15
Bought:	6000 of GNMA I Single Fam.	7.500 2022/07/01
Cash Leftover:	272.43	
New Tracking Error:	47.2 bp	
Cost of This Transaction:	30.000	
Cumulative Cost	110.000	

Transaction # 6

Sold:	1000 of NORFOLK SOUTHERN CORP	7.800 2027/05/15
Bought:	1000 of U.S. TREASURY NOTES	6.125 2007/08/15
Cash Leftover:	343.44	
New Tracking Error:	46.4 bp	
Cost of This Transaction:	5.000	
Cumulative Cost:	115.000	

Transaction # 7

Sold:	2000 of NORFOLK SOUTHERN CORP	7.800 2027/05/15
Bought:	2000 of ANHEUSER-BUSCH CO.,INC.	6.750 2027/12/15
Cash Leftover:	587.60	
New Tracking Error:	45.7 bp	
Cost of This Transaction:	10.000	
Cumulative Cost:	125.000	

Figure 20. **Summary of Tracking Error Breakdown for Sample Portfolios**

Tracking Error Due to:	Original Portfolio	Swapped Coca-Cola	Passive	Keep Sector Exposures
Term Structure	36	37	7	12
Sector	32	32	7	30
Systematic Risk	45	46	10	39
Non-systematic	26	22	13	24
Total	52	51	16	46

reduced from 36 bp to 12 bp, while the sector risk remains almost unchanged at 30 bp.

PROXY PORTFOLIOS

How many securities does it take to replicate the Lehman Corporate Index (containing about 4,500 bonds) to within 25 bp/year? How close could a portfolio of $50 million invested in 10 MBS securities get to the MBS index return? How many high yield securities does a portfolio need to hold to get sufficient diversification relative to the High Yield Index? How could one define “sufficient diversification” quantitatively? Investors asking any of these questions are looking for “index proxies”—portfolios with a small number of securities that nevertheless closely match their target indices. Proxies are used for two distinct purposes: passive investment and index analysis. Both passive portfolio managers and active managers with no particular view on the market at a given time might be interested in insights from index proxies. These proxy portfolios represent a practical method of matching index returns while containing transaction costs. In addition, the large number of securities in an index can pose difficulties in the application of computationally intensive quantitative techniques. A portfolio can be analyzed against an index proxy of a few securities using methods that would be impractical to apply to an index of several thousand securities. As long as the proxy matches the index along relevant risk dimensions, this approach can speed up many forms of analysis with only a small sacrifice in accuracy.

There are several approaches to the creation of index proxies. Quantitative techniques include stratified sampling or cell-matching, tracking error minimization, and matching index scenario results. (With limitations, replication of index returns can also be achieved using securities outside of indices, such as Treasury futures contracts.[7] An alternative way of getting index returns is entering into an index swap or buying an appropriately structured note.) Regardless of the means used to build a proxy

[7] *Replicating Index Returns with Treasury Futures,* Lehman Brothers, November 1997.

portfolio, the Lehman Brothers Risk Model can measure how well the proxy is likely to track the index.

In a simple cell-matching technique, a benchmark is profiled on an arbitrary grid that reflects the risk dimensions along which a portfolio manager's allocation decisions are made. The index contribution to each cell is then matched by one or more representative liquid securities. Duration (and convexity) of each cell within the benchmark can be targeted when purchasing securities to fill the cell. We have used this technique to produce proxy portfolios of 20-25 MBS passthroughs to track the Lehman Brothers MBS Index. These portfolios have tracked the index of about six hundred MBS generics to within 3 bp per month.[8]

To create or fine-tune a proxy portfolio using the risk model, we can start by selecting a seed portfolio and an investable universe. The tracking error minimization process described above then recommends a sequence of transactions. As more bonds are added to the portfolio, risk decreases. The level of tracking achieved by a proxy portfolio depends on the number of bonds included. Figure 21a shows the annualized tracking errors achieved using this procedure, as a function of the number of bonds, in a proxy for the Lehman Brothers Corporate Bond Index. At first, adding more securities to the portfolio reduces tracking error rapidly. But as the number of bonds grows, the improvement levels off. The breakdown between systematic and non-systematic risk explains this phenomenon. As securities are added to the portfolio, systematic risk is reduced rapidly. Once the corporate portfolio is sufficiently diverse to match index exposures to all industries and credit qualities, non-systematic risk dominates, and the rate of tracking error reduction decreases.

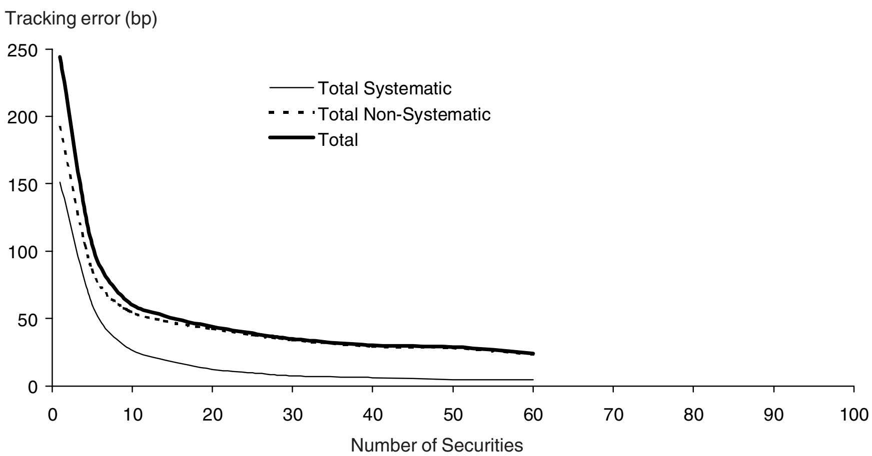
Figure 21b illustrates the same process applied to the Lehman Brothers High-Yield Index. A similar pattern is observed: Tracking error declines steeply at first as securities are added; tracking error reduction falls with later portfolio additions. The overall risk of the high-yield proxy remains above the investment-grade proxy. This reflects the effect of quality on our estimate of non-systematic risk. Similar exposures to lower-rated securities carry more risk. As a result, a proxy of about 30 investment-grade corporates tracks the Corporate Index within about 50 bp/year. Achieving the same tracking error for the High-Yield Index requires a proxy of 50 high-yield bonds.

[8] *Replicating the MBS Index Risk and Return Characteristics Using Proxy Portfolios*, Lehman Brothers, March 1997.

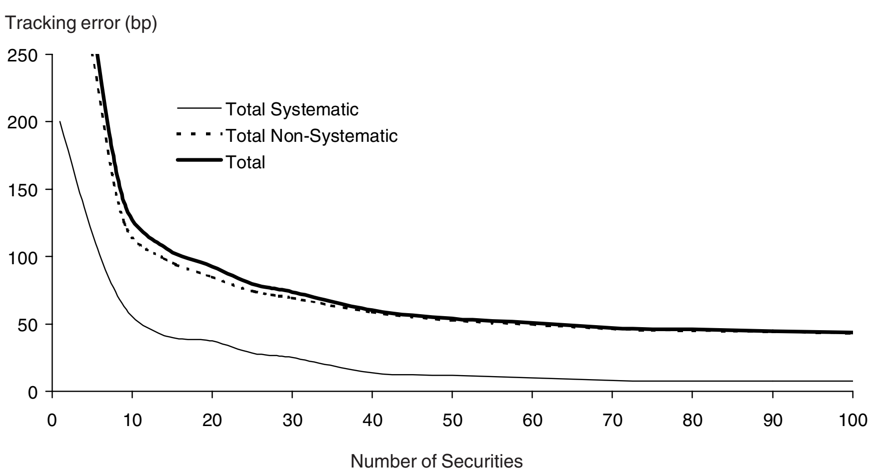
To demonstrate that proxy portfolios track their underlying indices, we analyze the performance of three proxies over time. The described methodology was used to create a corporate proxy portfolio of about 30 securities from a universe of liquid corporate bonds (minimum $350 million outstanding). Figure 22 shows the tracking errors projected at the start of each month from January 1997 through September 1998, together with the

Figure 21. **Corporate Proxy—Tracking Error as a Function of Number of Bonds (Effect of Diversification)**

a. **Proxy for Corporate Bond Index**



b. **Proxy for High Yield Index**



performance achieved by portfolio and benchmark. The return difference is sometimes larger than the tracking error. (Note that the monthly return difference must be compared to the monthly tracking error, which is obtained by scaling down the annualized tracking error by $\sqrt{12}$.) This is to be expected. Tracking error does not constitute an upper bound of return difference, but rather one standard deviation. If the return difference is normally distributed with the standard deviation given by the tracking error, then the return difference should be expected to be within ±1 tracking error about 68% of the time, and within ±2 tracking errors about 95% of the time. For the corporate proxy shown here, the standard deviation of the return difference over the observed time period is 13 bp, almost identical to the projected monthly tracking error. Furthermore, the result is within ±1 tracking error in 17 months out of 24, or about 71% of the time.

Figure 22. **Corporate Proxy Portfolio: Comparison of Achieved Results with Projected Tracking Errors**

Date	Annual Tracking Error (bp)	Monthly Tracking Error (bp)	Return (%/mo.) Proxy	Return (%/mo.) Index	Return Difference (bp/mo.)	Ret. Diff./ Monthly Tracking Error
Jan-97	48	14	0.15	0.14	0	0.03
Feb-97	48	14	0.37	0.42	-5	-0.34
Mar-97	48	14	-1.60	-1.56	-4	-0.30
Apr-97	47	14	1.60	1.52	8	0.60
May-97	48	14	1.14	1.13	1	0.04
Jun-97	48	14	1.42	1.42	0	0.03
Jul-97	47	14	3.62	3.66	-4	-0.27
Aug-97	48	14	-1.48	-1.48	0	-0.01
Sep-97	47	14	1.65	1.75	-10	-0.72
Oct-97	48	14	1.43	1.27	16	1.13
Nov-97	49	14	0.60	0.57	4	0.25
Dec-97	49	14	1.33	1.06	27	1.88
Jan-98	49	14	1.36	1.19	17	1.19
Feb-98	46	13	0.05	-0.03	8	0.59
Mar-98	46	13	0.39	0.37	2	0.16
Apr-98	45	13	0.75	0.63	12	0.93
May-98	44	13	1.22	1.19	3	0.24
Jun-98	45	13	0.79	0.74	6	0.42
Jul-98	45	13	-0.18	-0.10	-8	-0.63
Aug-98	44	13	0.76	0.47	29	2.26
Sep-98	44	13	3.62	3.24	38	2.99
Oct-98	46	13	-1.40	-1.54	15	1.11
Nov-98	45	13	2.04	1.88	16	1.20
Dec-98	47	14	0.17	0.29	-12	-0.87
Std. Dev.:					13	

	Number	Percentage
Observations within +/- 1 x tracking error	17	71%
Observations within +/- 2 x tracking error	22	92%
Total number of observations	24	

Figure 23 summarizes the performance of our Treasury, corporate, and mortgage index proxies. The MBS Index was tracked with a proxy portfolio of 20-25 generics. The Treasury index was matched using a simple cell-matching scheme. The index was divided into three maturity cells, and two highly liquid bonds were selected from each cell to match the index duration. For each of the three proxy portfolios, the observed standard deviation of return difference is less than the tracking error. The corporate portfolio tracks as predicted by the risk model, while the Treasury and mortgage proxies track better than predicted. The corporate index proxy was generated by minimizing the tracking error relative to the Corporate Index using 50-60 securities. Being much less diversified than the index of about 4,700 securities, the corporate proxy is most exposed to non-systematic risk. In the difficult month of September 1998, when liquidity in the credit markets was

Figure 23. **Summary of Historical Results of Proxy Portfolios for Treasury, Corporate, and MBS Indices,** in bp per month

	Treasury		Corporate		MBS	
	Tracking Error	Return Difference	Tracking Error	Return Difference	Tracking Error	Return Difference
Jan-97	5.5	-1.7	13.9	0.4	4.3	0.8
Feb-97	5.2	-0.6	13.9	-4.7	4.3	-0.3
Mar-97	5.5	-1.8	13.9	-4.2	4.0	2.9
Apr-97	5.5	1.7	13.6	8.2	4.3	-3.3
May-97	5.8	-0.3	13.9	0.6	4.0	1.6
Jun-97	6.6	3.5	13.9	0.4	4.0	-0.5
Jul-97	6.6	3.8	13.6	-3.7	4.0	-2.5
Aug-97	6.9	-3.8	13.9	-0.1	4.3	1.5
Sep-97	6.4	1.5	13.6	-9.8	4.3	-1.2
Oct-97	6.4	3.2	13.9	15.7	4.0	-0.6
Nov-97	6.1	-2.3	14.1	3.5	4.0	0.8
Dec-97	6.6	6.0	14.1	26.6	4.0	-2.4
Jan-98	6.6	1.0	14.1	16.9	4.3	1.8
Feb-98	6.6	-1.8	13.3	7.8	4.9	2.2
Mar-98	6.6	1.8	13.3	2.1	4.0	-1.9
Apr-98	6.6	-1.8	13.0	12.1	4.6	-0.9
May-98	6.6	3.8	12.7	3.1	4.6	-0.3
Jun-98	7.8	-1.4	13.0	5.5	4.9	0.4
Jul-98	7.5	-1.7	13.0	-8.2	4.3	-1.3
Aug-98	7.5	-0.6	12.7	28.7	4.3	-3.4
Sep-98	8.1	-6.1	12.7	38.0	4.0	-1.7
Oct-98	7.8	5.4	13.3	14.7	4.0	3.4
Nov-98	7.8	-4.9	13.0	15.6	4.6	-1.8
Dec-98	6.1	-2.7	13.6	-11.8	4.3	-1.6
Mean	**6.6**	0.0	**13.5**	6.6	**4.3**	-0.3
Std. Dev.		**3.2**		**12.5**		**1.9**
Min		-6.1		-11.8		-3.4
Max		6.0		38.0		3.4

severely stemmed, this resulted in a realized return difference three times the projected tracking error.

A proxy portfolio for the Lehman Brothers Aggregate Index can be constructed by building proxies to track each of its major components and combining them with the proper weightings. This exercise clearly illustrates the benefits of diversification. The aggregate proxy in Figure 24 is obtained by combining the government, corporate, and mortgage proxies shown in the same figure. The tracking error achieved by the combination is smaller than that of any of its constituents. This is because the risks of the proxy portfolios are largely independent.

When using tracking error minimization to design proxy portfolios, the choice of the "seed" portfolio and the investable universe should be considered carefully. The seed portfolio is the initial portfolio presented to the optimizer. Due to the nature of the gradient search procedure, the path followed by the optimizer will depend on the initial portfolio. The seed portfolio will produce the best results when it is closest in nature to the benchmark. At the very least, asset managers should choose a seed portfolio with duration near that of the benchmark. The investable universe, or bond swap pool, should be wide enough to offer the optimizer the freedom to match all risk factors. But if the intention is to actually purchase the proxy, the investable universe should be limited to liquid securities.

These methods for building proxy portfolios are not mutually exclusive, but can be used in conjunction with each other. A portfolio manager who seeks to build an investment portfolio that is largely passive to the index can use a combination of security picking, cell matching, and tracking error minimization. By dividing the market into cells and choosing one or more preferred securities in each cell, the manager can create an investable universe of candidate bonds in which all sectors and credit qualities are represented. The tracking error minimization procedure can then match index exposures to all risk factors while choosing only securities that the manager would like to purchase.

Figure 24. **Effect of Diversification—Tracking Error vs. Treasury, Corporate, MBS, and Aggregate**

Index	**No. of Bonds in Proxy**	**No. of Bonds in Index**	**Tracking Error** (bp/year)
Treasury	6	165	13
Government	39	1843	11
Corporate	51	4380	26
Mortgage	19	606	15
Aggregate	109	6928	10

BENCHMARK SELECTION: BROAD VERSUS NARROW INDICES

Lehman Brothers' development has been guided by the principle that benchmarks should be broad-based, market-weighted averages. This leads to indices that give a stable, objective and comprehensive representation of the selected market. On occasion, some investors have expressed a preference for indices composed of fewer securities. Among the rationales, transparency of pricing associated with smaller indices and a presumption that smaller indices are easier to replicate have been most commonly cited.

We have shown that it is possible to construct proxy portfolios with small numbers of securities that adequately track broad-based benchmarks. Furthermore, broad benchmarks offer more opportunities for outperformance by low-risk security selection strategies.[9] When a benchmark is too narrow, each security represents a significant percentage, and a risk-conscious manager might be forced to own nearly every issue in the benchmark. Ideally, a benchmark should be diverse enough to reduce its non-systematic risk close to zero. As seen in Figure 2, the non-systematic part of sigma for the Aggregate Index is only 4 bp.

DEFINING SPREAD AND CURVE SCENARIOS CONSISTENT WITH HISTORY

The tracking error produced by the risk model is an average expected performance deviation due to possible changes in all risk factors. In addition to this method of measuring risk, many investors perform "stress tests" on their portfolios. Here scenario analysis is used to project performance under various market conditions. The scenarios considered typically include a standard set of movements in the yield curve (parallel shift, steepening, and flattening) and possibly more specific scenarios based on market views. Often, though, practitioners neglect to consider spread changes, possibly due to the difficulties in generating reasonable scenarios of this type. (Is it realistic to assume that industrial spreads will tighten by 10 bp while utilities remain unchanged?) One way to generate spread scenarios consistent with the historical experience of spreads in the marketplace is to utilize the statistical information contained within the risk model.

For each sector/quality cell of the corporate bond market shown in Figure 25, we create a corporate sub-index confined to a particular cell and use it as a portfolio. We then create a hypothetical Treasury bond for each security in this sub-index. Other than being labeled as belonging to the Treasury sector and having Aaa quality, these hypothetical bonds are identical to their corresponding real corporate bonds. We run a risk model comparison between the

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[9] *Value of Security Selection versus Asset Allocation in Credit Markets: A "Perfect Foresight" Study*, Lehman Brothers, March 1999.

Figure 25. **Using the Risk Model to Define Spread Scenarios Consistent With History**

		Dur.	Annual Tracking Error (%)			Spread Volatility (bp)			
		(years)	Sector	Quality	Both	Sector	Quality	Both	Monthly
U.S. Agencies	Aaa	4.54	0.26	0.00	0.26	6	0	6	2
Industrials	Aaa	8.42	2.36	0.00	2.36	28	0	28	8
	Aa	6.37	1.72	0.57	2.03	27	9	32	9
	A	6.97	1.89	0.82	2.43	27	12	35	10
	Baa	6.80	1.87	1.36	2.96	27	20	43	13
Utilities	Aaa	7.34	1.62	0.13	1.65	22	2	22	6
	Aa	5.67	1.21	0.45	1.39	21	8	25	7
	A	6.03	1.33	0.63	1.67	22	10	28	8
	Baa	5.68	1.36	1.01	2.07	24	18	36	11
Financials	Aaa	4.89	1.41	0.00	1.41	29	0	29	8
	Aa	4.29	1.31	0.34	1.50	30	8	35	10
	A	4.49	1.31	0.49	1.65	29	11	37	11
	Baa	4.86	1.58	0.86	2.14	32	18	44	13
Banking	Aa	4.87	1.23	0.44	1.40	25	9	29	8
	A	5.68	1.43	0.62	1.72	25	11	30	9
	Baa	5.06	1.27	1.13	2.11	25	22	42	12
Yankees	Aaa	6.16	1.23	0.06	1.26	20	1	20	6
	Aa	5.45	1.05	0.49	1.27	19	9	23	7
	A	7.03	1.62	0.89	2.17	23	13	31	9
	Baa	6.17	1.51	1.36	2.60	24	22	42	12

portfolio of corporate bonds versus their hypothetical Treasury counterparts as the benchmark. This artificially forces the portfolio and benchmark sensitivity to term structure, optionality and any other risks to be neutralized, leaving only sector and quality risk. Figure 25 shows the tracking error components due to sector and quality, as well as their combined effect. Dividing these tracking errors (standard deviations of return differences) by the average durations of the cells produces approximations for the standard deviation of spread changes. The standard deviation of the overall spread change, converted to a monthly number, can form the basis for a set of spread change scenarios. For instance, a scenario of "spreads widen by one standard deviation" would imply a widening of 6 bp for Aaa utilities, and 13 bp for Baa financials. This is a more realistic scenario than an across-the-board parallel shift, such as "corporates widen by 10 bp."

HEDGING

Since the covariance matrix used by the risk model is based on monthly observations of security returns, the model cannot compute daily hedges. However, it can help create long-term positions that over time perform better than a naïve hedge. This point is illustrated by a historical simulation of a simple barbell versus bullet strategy in Figure 26, in which a combination of the 2- and 10-year on-the-run Treasuries is used to hedge the on-the-run 5-year. We compare two methods of calculating the relative weights of the two bonds in the hedge. In the first method, the hedge is rebalanced at the start of each month to match the duration of the 5-year Treasury. In the second, the model is engaged on a monthly basis to minimize the tracking error between the portfolio of 2- and 10-year securities and the 5-year benchmark. As shown in Figure 26, the risk model hedge tracks the performance of the 5-year bullet more closely than the duration hedge, with an observed tracking error of 19 bp/month compared with 20 bp/month for the duration hedge.

The duration of the 2- and 10-year portfolio built with the minimal tracking error hedging technique is consistently longer than that of the 5-year. Over the study period (1/94–2/99), the duration difference averaged 0.1 years. This duration extension proved very stable (standard deviation of 0.02) and is rooted in the shape of the historically most likely movement of the yield curve. It can be shown that the shape of the first principal component of yield curve movements is not quite a parallel shift.[10] Rather, the 2-year will typically experience less yield change then the 5- or 10-year. To the extent that the 5- and 10-year securities experience historically similar yield changes, a barbell hedge could benefit from an underweighting of the 2-year and an overweighting of the 10-year security. Over the 62 months analyzed in this study, the risk-based hedge performed closer to the 5-year than the duration-based hedge 59% of the time.

A similar study conducted using a 2- and 30-year barbell versus a 5-year bullet over the same study period (1/94-2/99) produced slightly more convincing evidence. Here, the risk-based hedge tracked better than the duration hedge by about 3 bp/month (33 bp/month tracking error versus 36 bp/month) and improved upon the duration hedge in 60% of the months studied. Interestingly, the duration extension in the hedge was even more pronounced in this case, with the risk-based hedge longer than the 5-year by an average of 0.36 years.

[10] *Managing the Yield Curve with Principal Component Analysis*, Lehman Brothers, November 1998.

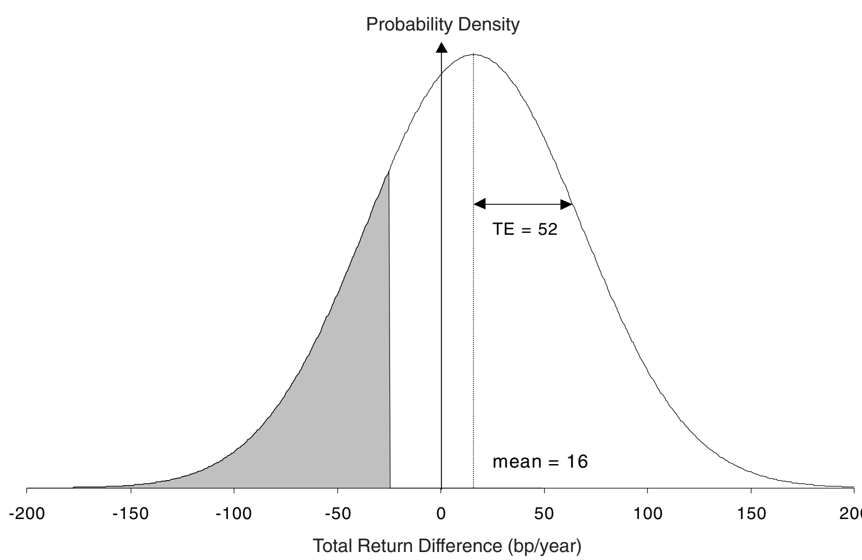
Figure 26. **Historical Performance of a Two-security Barbell vs. the 5-year On-the-run Treasury Bullet; Duration-based Hedge vs. a Tracking Error-based Hedge,** January 1994-February 1999.

		Difference				
		Duration Hedge		Tracking Error Hedge		% of Months
		Return	Duration	Return	Duration	Tracking Improved
2-10 vs. 5	Mean	0.03	0.00	0.03	0.10	59%
	Std. Dev.	0.20	0.00	0.19	0.02	
2-30 vs. 5	Mean	0.04	0.00	0.04	0.36	62%
	Std. Dev.	0.36	0.00	0.33	0.03	

ESTIMATING THE PROBABILITY OF PORTFOLIO UNDERPERFORMANCE

What is the probability that a portfolio will underperform the benchmark by 25 basis points or more over the coming year? To answer such questions, we need to make some assumptions about the distribution of the performance difference. We assume this difference to be distributed normally, with the standard deviation given by the tracking error calculated by the risk model. However, the risk model does not provide an estimate of the mean outperformance. Such an estimate may be obtained by a horizon total return analysis under an expected scenario (e.g., yield curve and spreads unchanged), or by simply using the yield differential as a rough guide. In the example of Figure 1, the portfolio yield exceeds that of the benchmark by 16 bp, and the tracking error is calculated as 52 bp. Figure 27 depicts the normal distribution with a mean of 16 bp and a standard deviation of 52 bp. The area of the shaded region, which represents the probability of underperforming by 25 bp or more, may be calculated as $N(((-25)-16)/52) = 0.215 = 21.5\%$ where N(x) is the standard normal cumulative distribution function. As the true distribution of the return difference may not be normal, this approach must be used with care. It may not be

Figure 27. **Projected Distribution of Total Return Difference** (in bp/year) **between Portfolio and Benchmark, Based on Yield Advantage of 16 bp and Tracking Error of 52 bp, Assuming Normal Distribution**



accurate in estimating the probability of rare events such as the “great spread sector crash” in August 1998. For example, this calculation would assign a probability of only 0.0033 or 0.33% to an underperformance of -125 bp or worse. Admittedly, if the tails of the true distribution are slightly different than normal, the true probability could be much higher.

MEASURING SOURCES OF MARKET RISK

As illustrated in Figure 2, the risk model reports the projected standard deviation of the absolute returns (sigma) of the portfolio and the benchmark as well as that of the return difference (tracking error). However, the detailed breakdown of risk due to different groups of risk factors is reported only for the tracking error. To obtain such a breakdown of the absolute risk (sigma) of a given portfolio or index, we can measure the risk of our portfolio against a riskless asset, such as a cash security. In this case, the relative risk is equal to the absolute risk of the portfolio, and the tracking error breakdown report can be interpreted as a breakdown of market sigma.

Figure 28 illustrates the use of this technique to analyze the sources of market risk in four Lehman Brothers indices: Treasury, (investment grade) Corporate, High-Yield Corporate, and MBS. The results provide a clear picture of the role played by the different sources of risk in each of these markets. In the Treasury Index, term structure risk represents the only

Figure 28. **Risk Model Breakdown of Market Risk (Sigma) to Different Categories of Risk Factors (Isolated Mode) for Four Lehman Brothers Indices,** as of 9/30/98, in percent per year

Index:	**Treasury**	**Corporate**	**High Yield**	**MBS**
Duration (years)	5.58	6.08	4.74	1.37
Convexity	0.69	0.68	0.20	-2.19
Term Structure Risk	5.25	5.81	5.58	3.25
Non-term Structure Risk	0.17	2.14	3.25	2.28
Risk Due to:				
Corp. Sector	0.00	1.50	1.44	0.00
Quality	0.00	0.84	2.08	0.00
Optionality	0.01	0.08	0.07	0.00
Coupon	0.17	0.01	0.05	0.00
MBS Sector	0.00	0.00	0.00	1.15
MBS Volatility	0.00	0.00	0.00	1.27
MBS Prepayment	0.00	0.00	0.00	0.73
Total Systematic Risk	5.26	5.47	4.93	2.69
Non-systematic Risk	0.04	0.08	0.17	0.09
Total Risk (std. dev. of annual return)	5.26	5.47	4.94	2.69

significant form of risk. In the Corporate Index, sector and quality risk add to term structure risk, but the effect of a negative correlation between spread risk and term structure risk is clearly visible. The overall risk of the Corporate Index (5.47%) is less than the term structure component alone (5.81%). This reflects the fact that when Treasury interest rates undergo large shocks, corporate yields often lag, moving more slowly in the same direction. The High Yield Index shows a marked increase in quality risk and in non-systematic risk relative to the Corporate Index. But, the negative correlation between term structure risk and quality risk is large as well, and the overall risk (4.94%) is less than the term structure risk (5.58%) by even more than it is for corporates. The effect of negative correlations among risk factors is also very strong in the MBS Index, where the MBS-specific risk factors bring the term structure risk of 3.25% down to an overall risk of 2.69%.

4. MODELING THE RISK OF NON-INDEX SECURITIES

The risk model calculates risk factor exposures for every security in the portfolio and the benchmark. As the model supports all securities in the Lehman Brothers Aggregate Index, the risk of the benchmark usually is fully modeled. Portfolios, however, often contain securities (and even asset classes) not found in the Aggregate Index. Our portfolio analytics platform has several features designed to represent out-of-index portfolio holdings. In addition, modeling techniques can be used to synthesize the risk characteristics of non-index securities through a combination of two or more securities.

Bonds—The analytics platform supports the modeling of all types of government and corporate bonds. User-defined bonds may contain calls, puts, sinking fund provisions, step-up coupon schedules, inflation linkage, and more. Perpetual-coupon bonds (and preferred stock) can be modeled as bonds with very distant maturity dates. Floating rate bonds are represented by a short exposure to term structure risk (as though the bond would mature on the next coupon reset date) and a long exposure to spread risk (the relevant spread factors are loaded by the bond's spread duration, which is based on the full set of projected cashflows through maturity).

Mortgage Passthroughs—The Lehman Brothers MBS Index is composed of several hundred "generic" securities. Each generic is created by combining all outstanding pools of a given program, pass-through coupon, and origination year (e.g., FNMA conventional 30-year 8.0% of 1993).[11] The index database contains over 3000 such generics, offering comprehensive coverage of the agency passthrough market, even though only about 600 meet the liquidity requirements for index inclusion. In addition to this database of MBS generics and their risk factor loadings, the analytics platform contains a lookup table of individual pools. This allows portfolios that contain mortgage pools to be bulk loaded based on either the pool CUSIP or the agency and pool number. For portfolio analytics, the characteristics of the appropriate generic are used as a proxy for the pool. This can lead to some inaccuracy for esoteric pools that differ considerably from the generic to which they are mapped, but adequately represents most mortgage portfolios in our experience.

[11] For a discussion of MBS Index composition and the relationship between pools and generics, see the Lehman Brothers report, *MBS Index Returns: A Detailed Look*, August 1998.

CMOs—CMOs are not included in the Lehman Brothers MBS Index because their collateral has already been included as passthroughs. At present, the portfolio analytics recognize and process structured securities as individual tranches, but do not possess deal-level logic to project tranche cashflows under different assumptions. Thus, each tranche is represented in the system by a fixed set of cashflows, projected using the Lehman Brothers prepayment model for the zero-volatility interest rate path calibrated to the forward curve. Risk factor loadings for these securities are calculated as a hybrid between the characteristics of the tranche and the underlying collateral. Term structure risk is assumed to follow the cashflows of the tranche. For PAC securities with less than 3 years to maturity (WAM), the model assigns no mortgage sector risk. For PACs with WAM greater than 10 years and for other types of tranches, the mortgage sector risk is assumed to be equal to that of a position in the underlying collateral with the same dollar duration. For PACs with WAM between 3 and 10 years, we use a prorated portion of the mortgage risk exposure of the collateral. This set of assumptions well represents tranches with stable cashflows, such as PACs trading within their bands. Tranches with extremely volatile cashflows, such as IOs and inverse floaters, can not be represented adequately in the current system. We hope to add this capability in our next-generation analytics. The mechanism of defining a "cashflow bond" (with an arbitrary fixed cashflow stream), with or without the additional treatment of mortgage risk, can be used to model many kinds of structured transactions.

Futures—A bond futures contract may be represented as a combination of a long position in the Treasury security that is the cheapest-to-deliver issue (CDI) and a short position in a cash instrument. To match the dollar duration of a Treasury futures position with a notional market value of N_f, the size of the position N_t in the CDI Treasury bond should satisfy

$$(P_t + A_t)N_t D_t = P_f N_f D_f,$$

where D_f is the option-adjusted duration of the futures contract. The negative holding N_c in the cash instrument has to offset the market value of the CDI:

$$(P_c + A_c)N_c + (P_t + A_t)N_t = 0.$$

If the cash instrument is priced at par and has no accrued interest, the amount needed is simply

$$N_c = -(P_t + A_t)N_t.$$

If the option-adjusted duration of the futures contract is not known, one could approximate N_t for a given CDI using the conversion factor CF_t:

$$N_t = N_f / CF_t,$$

(The conversion factor for each bond deliverable to a contract is published by the exchange when the contract is launched and does not change over time. It is the price ratio that governs the price to be paid on the delivery date for a given bond, $P_t = CF_t P_f$.) This approach ignores the optionality in the futures contract.

The disadvantage of a representation using a single CDI is that the notional values N_t and N_c need to be regularly maintained in order to properly reflect the risk of an unchanged position in futures. As yields change, the resulting changes in the delivery probabilities of different bonds will change the futures duration. A failure to update the portfolio frequently enough can lead to a discontinuity, especially around a switch in the CDI. A more sophisticated synthetic representation of a futures contract may involve more than one deliverable instrument weighted by the probability of delivery.

Index Swaps—The analytics platform provides a mechanism for including index swaps in portfolios. An individual security can be defined as paying the total return of a particular index, and a specific face amount of such a security can be included in a portfolio, corresponding to the notional value of the swap. These special securities have been created for all widely used Lehman Indices and are stored in the standard security database. Swaps written on other custom indices or portfolios can be modeled in a similar fashion. These capabilities, in conjunction with the dollar-based risk reporting described below, allow a comprehensive risk analysis of a portfolio of index swaps versus a hedge portfolio.

5. ADVANCED FEATURES OF THE RISK MODEL

EXPLORING AND MODIFYING THE COVARIANCE MATRIX

Lehman Brothers' portfolio management platform provides access to all details of the covariance matrix. Users can examine the full list of risk factors, their volatilities, and the correlation between any two. Figure 29a shows a selected set of risk factors together with their volatilities and correlations. Interpretation of these numbers is not always easy, because the effect of each risk factor on return is determined by multiplying the factor by the corresponding factor loading for each bond. As the loadings used for different factors have different meanings and units, so do the factors themselves.

As mentioned above, term structure risk factors can be interpreted as returns on Treasury zeros of different maturities. The matrix estimates the monthly return volatility of a 30-year zero at 6.787% and that of a 2-year zero at 0.654%, with a correlation of 0.75 between the two. The corporate risk factors, loaded by the negative of durations, represent spread changes. Thus, we see from the figure that the model implies a monthly spread volatility of 8.2 bp for the banking sector and 7.7 bp for consumer non-cyclicals. The correlations must also be interpreted with care; the 0.68 correlation between these two sectors represents a strong positive correlation between their spread movements and hence between their returns. However, the correlation of 0.33 between consumer non-cyclicals and 30-year Treasuries is a positive correlation between the spread change and the return on the 30-year zero. In return space, this implies that the sector return is negatively correlated with the yield curve return. This reflects the observation that corporate yields often move more slowly than Treasury yield.

The risk factors representing quality show a higher volatility for Baa spreads (6.3 bp/mo) than for Aa spreads (2.7 bp). There is a question, though, of why the matrix shows higher volatilities for sector risk than for quality risk. The reason is that every corporate bond contributes simultaneously to both sector and quality risk. In the calibration process, any corporate excess returns may be assigned either to sector or quality risk. Systematic movements of one sector independent of another (e.g., industrials tighten but utilities do not) are more prevalent than systematic movements of one quality group independently of others. The commonly held view is that the risk level is most dependent on quality rating. This is true due to the impact of special risk. The volatility of individual-security returns is much higher in the Baa sector than in Aa, and this is reflected in higher non-systematic errors for concentrations of the same size.

Figure 29e shows volatilities and correlations of selected MBS-related risk factors. These factors are more difficult to interpret, as they are not

Figure 29. **Selected Risk Factor Volatilities and Correlations, from Covariance Matrices Calibrated Using Data from Different Historical Time Periods**

Factor	Volatility (%/mo)	Correlations 2-yr.	30-yr.	Banking	Consumer Non-cycl.	Aa	Baa
a) 1/87-9/98							
2-year	0.654	X					
30-year	6.787	0.75	X				
Banking	0.082	0.24	0.26	X			
Consumer Non-cycl.	0.077	0.32	0.33	0.68	X		
Aa	0.027	0.17	0.18	0.27	0.38	X	
Baa	0.063	0.25	0.22	0.59	0.64	0.79	X
b) 1/87-7/98							
2-year	0.653	X					
30-year	6.784	0.75	X				
Banking	0.073	0.21	0.24	X			
Consumer Non-cycl.	0.076	0.29	0.31	0.61	X		
Aa	0.026	0.15	0.17	0.18	0.34	X	
Baa	0.060	0.22	0.20	0.49	0.59	0.78	X
c) 1/94-7/98							
2-year	0.558	X					
30-year	6.313	0.80	X				
Banking	0.040	-0.17	0.15	X			
Consumer Non-cycl.	0.026	0.23	0.40	0.65	X		
Aa	0.009	-0.05	-0.14	-0.11	-0.04	X	
Baa	0.016	-0.07	-0.08	-0.02	0.25	0.67	X
d) 1/94-9/98							
2-year	0.565	X					
30-year	6.333	0.80	X				
Banking	0.065	0.10	0.24	X			
Consumer Non-cycl.	0.038	0.33	0.40	0.86	X		
Aa	0.011	0.10	-0.01	0.38	0.39	X	
Baa	0.033	0.19	0.13	0.71	0.74	0.75	X

e) Selected Risk Factor Volatilities and Correlations for Selected MBS-related Factors, as of 10/1998

Factor	Volatility	Correlations 2-year	10-year	30-year Conv. Vol.	30-year Conv. Prep.	GNMA Prep.	15-year Prep.	Balloon Prep.
2-yr. Term Struct.	0.654	X						
10-yr. Term Struct.	2.763	0.87	X					
Conventional Vol	0.019	0.08	0.26	X				
Conventional Prep.	8.410	0.80	0.83	0.07	X			
GNMA Prep.	9.708	0.81	0.88	0.23	0.86	X		
15-yr. Prep.	10.73	0.64	0.59	0.17	0.68	0.66	X	
Balloon Prep.	17.29	0.07	0.01	-0.09	0.08	0.05	0.10	X

straightforward price or spread volatilities. The prepayment risk factors, for instance, represent a percentage change in prepayment rates. The high positive correlation of 0.88 between 30-year GNMA prepayment speeds and 10-year zero-coupon returns, for example, reflects the increase in refinancings that is typical when interest rates fall.

The numeric values in the covariance matrix depend on the time window of historical data used to calibrate the model. Our practice has not been to favor recent history over the more distant past, but to take the longest possible view of historical data. In models for overnight hedging, it may be appropriate to use only the most recent data. However, for estimating monthly or annual tracking error, older data should not be excluded or discounted without a clear economic reason. (For example, we exclude mortgage data from prior to 1991, when fundamental increases in refinancing efficiency caused a dislocation in the market.) Given the availability of our index data, the covariance matrix is built from historical data spanning the period from January 1987 to the present by default. All monthly observations are equally weighted—we do not use a time-decay mechanism that would discount older data. The volatilities and correlations of Figure 29a from October 1998 reflect market experience from January 1987 through September 1998.

Our analytics platform allows recalculation of the covariance matrix based on any subset of this time period. This allows us to study how risk estimates might be affected by the time window used for calibration. For example, Figure 29b shows the same set of volatilities and correlations from our default covariance matrix of August 1998. The long time window makes the matrix evolve very slowly; the values in Figure 29b are very close to those in Figure 29a. The effect of using a shorter time window is illustrated in Figure 29c, which reflects data from January 1994 through July 1998. Term structure risk shows slightly lower volatility and slightly higher correlation between the long and short ends of the curve (less twist). The most dramatic change is in corporate spread volatilities, which are less than half the values in Figure 29a. Spreads were fairly quiescent during this time period. This figure emphasizes the danger of calibrating a model to only the most recent time period. Had the estimate of spread risk been reduced due to the recent calm, the model would have been left more vulnerable to the violent spread movements of August and September 1998, unusually large by any measure. Figure 29d shows the effect of adding these two additional months of data to the matrix of Figure 29c. The sector spread volatilities increase, and many of the correlations exhibit drastic changes. The matrix built from the shorter time period, once these two months are included (Figure 29d), becomes much more similar to the long-term matrix (Figure 29a). This is perhaps due to the similarity of 1998 experience to the events of 1987-1991 (two U.S. equity market crashes in 1987 and 1989 and the 1990 gulf war and recession) already represented in the long-term matrix.

DOLLAR RISK

In most cases, the model analyzes risk in return space, measuring the deviation of projected portfolio returns from those of a benchmark, which can be either a broad market index or a reference portfolio. The model can also be used to analyze the risk of an asset portfolio versus a hedging portfolio in terms of the variance of the projected profit or loss, in dollars.

In the standard tracking error calculation, the market value of the benchmark, which usually is many times that of the portfolio, is not relevant. In the calculation of dollar-based risk for hedging purposes, differences in the market values of the two portfolios (more precisely, in their dollar durations) result in large tracking errors.

To illustrate the application of the risk model to a hedged portfolio, we isolated the corporate segment of the passive portfolio in Figure 16 and hedged this 27-bond corporate portfolio with positions in the 2-, 5-, 10-, and 30-year on-the-run Treasuries. The risk report for this portfolio, in dollar-based terms, is shown in Figure 30. We can see that the hedge is fairly well matched to the portfolio's market value ($163.7 million), duration (6.03 years), and dollar duration (the product of the two). The portfolio yield to worst (5.71%) is 1.25% higher than that of the hedge portfolio. The dollar risk entailed in this position, the (annualized) standard deviation of the overall position P&L, is calculated at $3.412 million. The tracking error breakdown report uses the same format as before to attribute this risk to the various categories of risk factors. Sector risk ($2.358 million) and quality risk ($1.234 million) are the largest sub-components by far, while the term structure risk has been mostly hedged out. The risk of systematic corporate spread movements is much greater than the non-systematic risk, given that any excessively large concentrations were already reduced during the formation of the passive portfolio.

The dollar-based risk report, like the returns-based report, expresses risk in annualized terms. To obtain the monthly number corresponding to the annualized tracking error of $3.4 million, we divide the tracking error by $\sqrt{12}$ to obtain $985 thousand. This number could be scaled down further to estimate the position's overnight value-at-risk (VAR). But this procedure is not recommended because the model was not calibrated to daily returns. (The assumption of independence between consecutive trading periods, which underlies such scaling of tracking errors, is borne out by monthly data much better than by daily data.) Unlike many VAR models, though, this model quantifies the risks due to spread movements (systematic and non-systematic) in addition to term structure risk.

ACCELERATED INDEX CALCULATIONS

The evaluation of risk factor exposures of any portfolio or benchmark requires the calculation of exposures for all constituent securities and the weighted averaging of all such exposures based on market capitalization. Ultimately, a portfolio or benchmark is represented as a vector of factor loadings for systematic risk factors and another vector of exposures to each security in the database for non-systematic risk. For commonly used indices, the repeated aggregation of a benchmark's risk exposure vectors is fairly time consuming. Therefore, the analytics platform allows the exposure vector of a benchmark to be saved in files as an alternative representation of the benchmark at a particular point in time. These accelerator files speed up the generation of risk reports, particularly when several different portfolios are compared to the same index. In addition, this capability enables the treatment of index swaps described above.

Figure 30. **"Dollar-based Risk"—Tracking Error Expressed in Terms of Difference in Monthly P&L between Portfolio and Benchmark (Hedge);**
Corporate Portfolio vs. a Hedge Portfolio of On-the-run Treasuries

	Portfolio	Benchmark
Number of Issues	27	4
Market Value ($000s)	163,728	163,562
Average Maturity/Average Life (years)	11.69	9.60
Internal Rate of Return (%)	6.02	4.70
Average Yield to Maturity (%)	5.79	4.46
Average Yield to Worst (%)	5.71	4.46
Average Option Adj. Convexity	0.77	0.81
Average OAS To Maturity (bp)	106	0
Average OAS To Worst (bp)	105	0
Portfolio Mod Adjust Duration (years)	6.03	6.05
Portfolio Average Price	107.39	105.89
Portfolio Average Coupon	7.24	5.21
Portfolio Mod Adj. $Duration ($000s)	986675	989077

Risk Characteristics
Estimated Total Tracking Error

Return	2.09%
Price ($000s)	3412
Portfolio Beta	0.893

Tracking Error Details (in $000s)

	Tracking Error		
	Isolated	Cumulative	Change in Cumulative
Tracking Error Term Structure	296	296	296
Non-Term Structure	3321		
Tracking Error Sector	2358	2344	2048
Tracking Error Quality	1234	3217	873
Tracking Error Optionality	43	3220	3
Tracking Error Coupon	237	3320	100
Tracking Error MBS Sector	0	3320	0
Tracking Error MBS Volatility	0	3320	0
Tracking Error MBS Prepayment	0	3320	0
Total Systematic Tracking Error	3320		
Non-Systematic Tracking Error			
Issuer-Specific	785		
Issue-Specific	785		
Total	785		
Total Tracking Error Price	3412		

Market Sigma	
Systematic	9213
Non-systematic	371
Total	9221
Portfolio Sigma	
Systematic	8834
Non-systematic	691
Total	8861

6. TESTING THE MODEL'S PERFORMANCE

What can be expected from a risk model? Certainly, unusual events in a given month (e.g., August 1998) can cause a portfolio to deviate from its benchmark by significantly more than the tracking error. But if a portfolio maintains a given tracking error, then expected performance will be within one tracking error about two-thirds of the time. We illustrated this concept in Figures 22 and 23 with a historical analysis of proxy portfolios. This analysis is relatively limited in that it looks only at positions with relatively small exposures and is based on less than two years of historical returns. To validate tracking error estimates of the model for more active positions, we analyzed historical results over a longer period of time. To avoid the subjectivity involved in building test portfolios for this purpose, we applied the model to the risk analysis of one index versus another. For example, if our benchmark is the Treasury Index, and we purchase the Corporate Index as the portfolio, what does the risk model calculate as the tracking error, and how is that borne out by historical index returns?

Figure 31 shows the results of this analysis over a two-year period for the Corporate and MBS Indices. These more active positions show that extreme events can occur, but the standard deviation of the return difference is once again close to the projected tracking error.

Figure 32 shows summary results for various index versus index combinations over a period of more than 7 years. The selection of pairs of indices was intended to test different aspects of the model individually. For example, the Long Treasury versus Intermediate Treasury Index tests the model's treatment of term structure risk. Pitting the different Corporate Index quality groups against the Treasury Index highlights the effect of sector and quality spread risk. The model's increase in projected risk for lower credit qualities parallels the observed dependence. In most cases, the tracking error estimated by the risk model predicts deviation of return differences quite well. The model tracking error tends to be a little larger than the observed tracking error. This is probably because the model is calibrated to historical data, which includes the late 1980s and early 1990s—a relatively volatile period. This makes the model's risk estimates somewhat conservative.

The performance of the model at predicting tracking error is the ultimate test of the volatility calibration scheme. In all of the results discussed above, tracking error was projected using the default covariance matrix, which is recalibrated each month based on a growing time window from January 1987 through the present. Figure 33 compares these results for the Corporate versus Treasury Indices to those obtained using a covariance matrix calibrated to a rolling 5-year window of returns. For example, the covariance matrix used in August 1998 was calibrated to data from August 1993 through

July 1998. The low volatility of sector spreads through much of the 1990s led to smaller projected tracking errors when using this rolling 5-year window. As a result, the tracking errors projected using this matrix are very low, especially when compared with the extreme return differences observed in late 1998. The tracking errors projected using the long-term covariance matrix were much more effective during this time period.

Figure 31. **Historical Comparison of Projected and Realized Tracking Errors for the Treasury Index vs. the Corporate and MBS Indices**
in bp per month

	Corporate		MBS	
	Tracking Error	**Return Difference**	**Tracking Error**	**Return Difference**
Jan-97	62.07	-4.20	52.83	-64.40
Feb-97	62.93	-29.70	52.25	-21.10
Mar-97	61.20	47.70	52.25	-13.90
Apr-97	61.20	-8.00	45.61	-15.80
May-97	61.20	-26.80	47.63	-11.60
Jun-97	61.49	-29.90	50.52	-4.90
Jul-97	60.62	-79.10	51.38	98.40
Aug-97	60.91	46.30	64.09	-77.70
Sep-97	58.31	-23.00	59.18	25.20
Oct-97	58.31	47.40	66.97	63.30
Nov-97	58.31	-4.40	74.19	19.30
Dec-97	57.74	-0.70	77.36	14.00
Jan-98	58.31	34.50	80.25	53.70
Feb-98	58.60	-26.60	87.76	-51.00
Mar-98	58.31	-9.60	84.87	-15.00
Apr-98	58.02	-18.60	81.98	-12.20
May-98	58.02	-14.80	81.98	37.50
Jun-98	57.74	41.30	86.89	67.50
Jul-98	57.74	25.20	87.47	-35.00
Aug-98	57.74	224.80	84.00	180.80
Sep-98	57.16	-45.30	96.42	158.10
Oct-98	62.07	122.30	113.16	-19.20
Nov-98	59.76	-190.70	97.28	-52.00
Dec-98	59.47	-9.40	96.42	-22.90
Mean	**59.47**	2.86	**73.86**	12.55
Std. Dev.		**73.65**		**64.83**
Min		-190.70		-77.70
Max		224.80		180.80

Figure 32. **Historical Back-testing of Risk Model on Various Index-Index Combinations,** August 31, 1991-October 31, 1998

Portfolio	Benchmark	Tracking Error (bp/mo.) Observed	Tracking Error (bp/mo.) Predicted	Within 1 St. Dev.	Within 2 St. Dev.
Long Treasury	Intermediate Treasury	158	184	77%	97%
Corp Aaa	Treasury	34	48	87	98
Corp Aa	Treasury	34	52	94	99
Corp A	Treasury	41	61	91	99
Corp Baa	Treasury	54	78	91	98
Corp BB	Treasury	113	127	91	98
Corp B	Treasury	157	180	82	98
Basic Industry	Corporate	20	37	91	100
FNMA	GNMA	11	31	98	100
Intermediate Corp	Intermediate Treasury	38	49	83	98
MBS	Intermediate Treasury	38	56	92	95

Figure 33. **Comparison of Risk Model Performance during 1998 Using Covariance Matrices Calibrated to Different Amounts of Historical Data; Projected Monthly Tracking Errors between Corporate and Treasury Indices Are Compared with Realized Return Differences**

	Return			Using Long-term Covariance Matrix		Using Rolling 5-year Covariance Matrix	
	Corp. (%/mo.)	Tsy. (%/mo.)	Diff. (bp/mo.)	Monthly Tracking Error (bp/mo.)	Ret Diff./ Monthly Tracking Error	Monthly Tracking Error (bp/mo.)	Ret Diff./ Monthly Tracking Error
Jan-98	1.19	1.53	-34	58	-0.58	29	-1.19
Feb-98	-0.03	-0.30	27	59	0.46	28	0.95
Mar-98	0.37	0.27	10	58	0.17	26	0.38
Apr-98	0.63	0.45	18	58	0.31	27	0.67
May-98	1.19	1.04	15	58	0.26	27	0.55
Jun-98	0.74	1.15	-41	58	-0.71	25	-1.67
Jul-98	-0.09	0.16	-25	58	-0.43	25	-1.00
Aug-98	0.47	2.72	-225	58	-3.90	27	-8.47
Sep-98	3.24	2.79	45	57	0.79	22	2.02
Oct-98	-1.54	-0.32	-122	62	-1.97	36	-3.41
Nov-98	1.88	-0.02	190	60	3.18	37	5.14
Dec-98	0.29	0.20	9	59	0.15	40	0.23
Jan-99	0.99	0.58	41	59	0.70	44	0.93
Percentage within +/- 1 x tracking error					77%		54%
Percentage within +/- 2 x tracking error					85%		69%

7. RELATIONSHIP WITH OTHER MODELS

Our risk model is closely related to several other models used in the financial services industry. The most closely related are the *value-at-risk* (VAR) models. Financial institutions with large offsetting positions in assets and liabilities face the risk that changes in market valuations will decrease the value of their assets relative to that of their liabilities. A similar consideration applies to the value of a long position versus its hedge.

In recent years, the increased attention of regulatory authorities to this type of risk has fostered widespread use of various models for the calculation of daily VAR. Some use a multi-factor approach similar to that of our model. Several key differences should be noted. VAR models usually measure the overall risk of an overnight position, in dollar terms. While our model compares a portfolio to its benchmark, VAR models compare short and long positions. Our model's tracking error is usually expressed in return space rather than in dollars and is calibrated to a monthly holding period (using monthly return data) as opposed to the daily time frame commonly used for VAR calculations. The estimates of volatilities and correlations used in VAR models are typically based on short windows of historical data, often with a time-decay mechanism to discount the contribution of older data. Our risk model uses a much longer time window and equal weighting, reflecting the longer horizon of the tracking error. While VAR models sometimes include the effect of operational risk and counterparty risk, our model is more thorough in its coverage of spread sector risk and non-systematic risk.

Another closely related model is the *mean-variance* approach commonly used for asset allocation. In this type of model, a covariance matrix is formed from the return volatilities of different asset classes and their correlations. Risk and tracking error of a portfolio are attributed simply to a weight differential relative to the benchmark in a given asset holding. Based on these weight differentials, the covariance matrix and a set of expected returns for each asset class, the model calculates an efficient frontier. This is the set of portfolios (expressed as allocation weights for each asset class) that give the minimum risk for any desired level of expected return. This model is similar to our risk model in that both rely on the use of a covariance matrix to estimate risk. However, asset allocation models rely on the return history of a given asset class as a whole, without any analysis of its source. They have difficulties with highly correlated asset classes such as Treasuries and corporates. While the multi-factor approach used in our model is able to recognize that corporate bonds have yield curve risk (shared with Treasuries) and spread risk (a separate factor), the asset allocation approach views corporates and Treasuries as two distinct asset classes with a very high correlation. The high correlations among asset class returns in the fixed income market can cause such models to be numerically unstable. One more obvious difference is that asset allocation models, by their nature, do not give any indication of the non-systematic risks due to individual securities held in a portfolio. They also cannot break down systematic risk into components beyond asset class definitions. A major advantage, however, of an asset allocation approach to risk measurement is that many diverse asset classes can be covered by a single model. The more independent the returns of the different asset classes considered, the more successful the model will be.

Principal components analysis[12] is often used to measure term structure risk. In this approach, an analysis of historical movements of the yield curve produces a set of characteristic shapes that such movements typically follow. The first three principal components are sufficient to represent a large portion of observed yield curve movements. These three components correspond roughly to a parallel shift in the curve, a twist (steepening or flattening of the curve), and a butterfly (or curvature) movement in which the center of the curve moves up or down relative to the wings. Our risk model could conceivably use the volatilities and correlations of these three components of yield curve movements as the risk factors for term structure, and sensitivities to these movements as the factor loadings. While this approach may be mathematically robust (since these risk factors are orthogonal to each other), we have found that our representation of a more detailed view of cashflows along the curve is better suited to most market practitioners.

One of the central assumptions underlying the risk model is that returns are normally distributed. The merit and the validity of using the normal distribution as an approximation for financial variables have been debated extensively in the literature. It is well known that the distributions of most financial quantities have fatter tails than the normal distribution. Thus, the occurrence of extreme cases ("rare events") is more frequent in the real world than would be predicted by the normal distribution. In other words, 3 sigma events are probably more common than suggested by our risk model. This implies that tracking error estimates work best for standard, ordinary environments. To understand and quantify the risks associated with extreme market changes, *scenario analysis* models can be used to project portfolio and benchmark returns under various worst case scenarios. This form of "stress testing" has been incorporated into the risk management processes of many firms and can form a valuable complement to the risk model.

[12] *Managing the Yield Curve with Principal Component Analysis*, Lehman Brothers, November 1998.

Finally, models for projecting risk *ex ante* are closely related to those that analyze the sources of portfolio and benchmark returns *ex post.* At Lehman Brothers, we have developed two such models, each with a slightly different focus. Our return attribution analysis[13] breaks down the return of each security to basic sources such as time return (rolldown and accretion), yield curve return (shift, twist, butterfly, and a residual), volatility return, and spread return. These results can be aggregated to the portfolio level, and a portfolio's return can be compared to that of the benchmark in each of these categories. In performance attribution,[14] each security's total return is considered as an indivisible whole. This model attempts to explain the difference between portfolio and benchmark returns in terms of differences in yield curve placement and sector allocation, with the remainder attributed to security selection. Both return attribution and performance attribution are related to the risk model. It is not necessary for the risk factors to match explanatory variables of the return and performance attribution models one-to-one, but at some level, components of return must be traceable to components of risk. In a portfolio with a small tracking error due to term structure, for example, we expect both models to show very little difference in the returns due to yield curve movement. When overall systematic risk is low, we expect almost all the return differences between portfolio and benchmark to come from security selection; the range that is expected for this return difference due to security selection corresponds roughly to the non-systematic tracking error.

[13] *The Lehman Brothers Return Attribution Model*, Lehman Brothers, May 1996.

[14] *Attribution of Portfolio Performance Relative to an Index*, Lehman Brothers, March 1998.

8. CONCLUSION

Since the early 1990s, we have applied the multi-factor risk model described in this report (with several revisions) to portfolio management tasks faced by fixed-income investors. Through many types of market conditions, the model has provided accurate forecasts of portfolio risk relative to an index. Passive portfolio managers have found that minimizing tracking error always leads to close replication of index returns, while active managers have been able to weigh the risks assumed against expected outperformance.

In retrospect, several modeling decisions have proven particularly useful:

First, the selected set of risk factors, while redundant (many of them are highly correlated), has high statistical significance and, most importantly, intuitive appeal to portfolio managers. Our experience shows that as long as *all major categories of risk* in a given market are accounted for, the resulting tracking errors have predictive power with any reasonable choice of risk sensitivities.

Second, by deriving risk factor realizations implied by individual security returns rather than simply observing them from market averages, the model is able to isolate the idiosyncratic or issue-specific part of the return. This allows the quantification of non-systematic risk and assignment of a proper penalty for insufficient diversification.

Third, the historical time frame was extended as far back as our database permitted for all asset classes, absent a compelling economic reason to limit the amount of history used (as with MBS). We chose not to follow the time-decay approach and do not assign a lower weight to older historical observations while constructing the default covariance matrix. This decision served the model well during the extreme market volatility of 1998. In the few instances when reduction in the length of historical observations was justified (MBS), the mathematical requirements of forming a covariance matrix were satisfied not by back-filling the missing history, but by perturbing the resulting matrix in a controlled fashion.[15]

While we are excited about the portfolio management capabilities demonstrated by the risk model over the past decade and the numerous ways in which it helped investors, much more needs to be done.

[15] See Appendix 5 for a detailed explanation.

As mathematically "pure" risk measures, such as key rate or principal component durations, gain acceptance among investors, we plan to revise our choice of risk factors, possibly using fewer of them. The incorporation of yield curve sensitivity measures adjusted for optionality will allow combining term structure and callability risk in a single set of factors. In the area of credit risk, we are looking for a few main factors of excess return that would retain an intuitive appeal to investors.

There is a pressing need to expand market coverage within the model to other asset classes included in the Lehman Brothers Global Family of Indices (emerging markets, euro-denominated bonds, etc.). The need is felt particularly strongly in the newly-formed Euro credit market. Many European portfolio managers are being confronted for the first time with the task of managing a portfolio of few securities with high non-systematic risk relative to a well-diversified index. An extension of the model to euro-denominated credit securities would greatly simplify the tasks of replicating such indices and controlling portfolio risk. As of the date of this report, the history of security returns for the post-EMU credit markets in Europe is still too short for a meaningful derivation of variances and correlations of the underlying risk factors. The instant popularity with investors of our macro indices (such as U.S. Universal or Global Aggregate) demands inclusion of an increasingly mixed set of asset classes in a single risk model framework.

Actual returns of fixed income securities are not normally distributed, but rather exhibit "fat tails." Events outside of one standard deviation are more probable than suggested by the normal distribution. Interpreting the model results according to the assumption of normality can sometimes understate the risk of large losses. Our inclusion of "old" historical observations of spread changes compensated for this to some extent in the period preceding the 1998 spread widening. It caused the model to project tracking errors higher than those realized in 1995-1997, but it served the model well in 1998.

Academic literature discusses alternative measures of portfolio risk relative to an index, such as semi-variance of the return difference or risk of *underperformance* rather than *deviation* from the benchmark. Practical applications to every day portfolio management based on this concept of risk will no doubt be built in the near future.

There are risk modeling methodologies used today that are not based on any historical observations of security returns, but rather on stochastic simulation of all possible *future* returns given the current yield curves, implied volatilities, and spreads. Ideally, this Monte Carlo simulation of portfolio returns should be used in tandem with the history-based analysis to provide the best possible estimate of risk.

APPENDIX 1. BASIC RISK MODEL MATHEMATICS

OVERVIEW

The primary goal of the risk model is to project how well a portfolio is likely to track its benchmark over the coming month. To accomplish this goal, the model establishes a relationship between individual security returns and a set of risk factors that drive them. This relationship forms the bridge by which market experience in the form of past returns can be applied to characterize the expected distribution of future returns. In this first appendix, the model is viewed as a probabilistic model for future returns. The difference between portfolio and benchmark returns over the coming period is represented by a random variable, and we characterize its distribution in terms of the distribution of the risk factors. In Appendix 5, the same relationship is viewed as an explanatory model of past returns of individual securities. Historical risk factor realizations are backed out by regressions, and the volatilities and correlations of these risk factors are computed.

The basic assumption of the model is that the covariance matrix composed of volatilities and correlations of historical risk factor realizations is a reasonable characterization of the risk factor distribution for the coming period. The model extrapolates only these second-moment statistics. It does not attempt to project expected values of portfolio return or outperformance ("alpha") based on historical returns.

MODELING RETURNS

Let us assume that our investment universe consists of a finite set of N securities. The performance of the entire universe over the coming month can then be represented by an $N \times 1$ random vector $\mathbf{r}$ of (unknown) individual security total returns. The multi-factor model attempts to explain the return r_i on any bond i in terms of broader market movements. A set of M risk factors ($M << N$) is chosen to represent the primary sources of risk (and return) to which a portfolio may be exposed. The extent to which bond i is exposed to a particular risk factor j is modeled by a fixed factor loading f_{ij}. The $1 \times M$ row vector $\mathbf{f}_i$ thus characterizes the exposure of security i to systematic risk.

The return of any bond i can be expressed in terms of the $M \times 1$ random factor vector $\mathbf{x}$ by

(1) $$r_i = \sum_{j=1}^{M} f_{ij} x_j + \varepsilon_i = \mathbf{f}_i \mathbf{x} + \varepsilon_i ,$$

where $\mathbf{f}_i = \{ f_{ij} \}$ is the known vector of factor loadings that characterizes bond i, and ε_i is the non-systematic random error. That is, ε_i is the portion of the return r_i that is not explained by the systematic risk model. This reflects the possibility of events specific to a given issue or issuer, such as a sudden demand for a particular Treasury security, or a takeover announcement by a particular corporate issuer.

If we let $\mathbf{F}$ be the $N \times M$ matrix containing one row for the factor loading vector of each of the N bonds in our universe, and denote by $\boldsymbol{\varepsilon}$ the $N \times 1$ vector of non-systematic random errors, we can restate Equation 1 in matrix form,

(2) $$\mathbf{r} = \mathbf{F}\mathbf{x} + \boldsymbol{\varepsilon}.$$

It then becomes clear that (to the extent that the non-systematic error vector is small, or $\boldsymbol{\varepsilon} << \mathbf{r}$) the factor vector $\mathbf{x}$ summarizes the holding period performance of our universe.

The distribution of possible returns on individual securities and portfolios can thus be expressed in terms of the distributions of values of the random factor vector $\mathbf{x}$ and the random error vector $\boldsymbol{\varepsilon}$. Specifically, the systematic risk can be expressed in terms of the $M \times M$ covariance matrix $\boldsymbol{\Omega} = \{\Omega_{jk}\}$, where $\Omega_{jk} = COV(x_j, x_k)$. (On the diagonal, $\Omega_{jj} = VAR(x_j)$.)

APPLICATION TO PORTFOLIO MANAGEMENT

We can represent a given portfolio p by a $1 \times N$ allocation vector $\mathbf{q}_p$, which states the proportion of the market value of the portfolio allocated to each of the N securities in our universe. The portfolio return r_p is then given by

(3) $$\begin{aligned} r_p &= \mathbf{q}_p \mathbf{r} \\ &= \mathbf{q}_p \mathbf{F}\mathbf{x} + \mathbf{q}_p \boldsymbol{\varepsilon}, \\ &= \mathbf{f}_p \mathbf{x} + \mathbf{q}_p \boldsymbol{\varepsilon} \end{aligned}$$

where $\mathbf{f}_p = \mathbf{q}_p \mathbf{F}$ is the factor loading vector that summarizes the systematic risk exposure of a portfolio as a weighted sum of the exposures of its constituent securities.

Of primary importance in assessing portfolio risk are the second-moment statistics, the return volatilities. The variances σ_p^2 and σ_b^2 of the portfolio and benchmark returns r_p and r_b may be expressed as

(4) $$\begin{aligned} \sigma_p^2 &= VAR(r_p) = \mathbf{f}_p \boldsymbol{\Omega} \mathbf{f}_p^{\mathbf{T}} + \mathbf{q}_p \boldsymbol{\Gamma} \mathbf{q}_p^{\mathbf{T}} \\ \sigma_b^2 &= VAR(r_b) = \mathbf{f}_b \boldsymbol{\Omega} \mathbf{f}_b^{\mathbf{T}} + \mathbf{q}_b \boldsymbol{\Gamma} \mathbf{q}_b^{\mathbf{T}} \end{aligned}$$

where the covariance matrix $\mathbf{\Omega}$ is the $M \times M$ matrix described above, which contains the covariances of the systematic risk factors, and $\mathbf{\Gamma}$ is a sparse $N \times N$ matrix, which contains the covariances of the security-specific residual risk terms, $\Gamma_{ij} = COV(\varepsilon_i, \varepsilon_j)$. (The details of the model of non-systematic risk, including the precise form of the matrix $\mathbf{\Gamma}$, will be addressed in Appendix 3.) The portfolio variance can be seen to be composed of one term due to systematic risk and another due to security-specific risk. There are no cross terms, due to our assumptions that the error vector $\boldsymbol{\varepsilon}$ and the systematic factor vector $\mathbf{x}$ are uncorrelated $\left(\mathbf{E}[\varepsilon_i x_j] = 0 \text{ for all } i, j\right)$, and that the errors have mean zero $\left(\mathbf{E}[\varepsilon_i] = 0 \text{ for all } i\right)$.

In the context of portfolio/benchmark comparison, we report the return volatilities σ_p and σ_b of the portfolio and benchmark, respectively, as given by Equation 4. In addition, we report the tracking error σ_{TE} and the β given by

(5)
$$\sigma_{TE}^2 = VAR(r_p - r_b) = (\mathbf{f}_p - \mathbf{f}_b)\mathbf{\Omega}(\mathbf{f}_p - \mathbf{f}_b)^T + (\mathbf{q}_p - \mathbf{q}_b)\mathbf{\Gamma}(\mathbf{q}_p - \mathbf{q}_b)^T$$
$$\beta = \frac{COV(r_p, r_b)}{VAR(r_b)} = \frac{1}{\sigma_b^2}\left(\mathbf{f}_p \mathbf{\Omega} \mathbf{f}_b^T + \mathbf{q}_p \mathbf{\Gamma} \mathbf{q}_b^T\right).$$

The tracking error measures the dispersion between portfolio and benchmark returns. The β measures the sensitivity of the portfolio return to changes in the benchmark return. From the definition of tracking error, it is obvious that the smaller the value of σ_{TE}, the closer the portfolio tracks the benchmark. If the portfolio and the benchmark are identically composed $(\mathbf{q}_p = \mathbf{q}_b)$, then r_p will be identical to r_b under all random outcomes, and we will have $\sigma_{TE} = 0$ and $\beta = 1$. This is the only way that a zero tracking error may be achieved; other portfolios, however, might achieve $\beta = 1$.

The β is closely related to both the tracking error σ_{TE} and the correlation coefficient ρ between portfolio and benchmark returns. These relationships may be expressed as

(6)
$$\rho = \frac{COV(r_p, r_b)}{\sigma_p \sigma_b} = \frac{\sigma_b}{\sigma_p}\beta$$
$$\sigma_{TE}^2 = \sigma_p^2 + \sigma_b^2 - 2\rho\sigma_p\sigma_b = \sigma_p^2 + \sigma_b^2 - 2\beta\sigma_b^2.$$

Thus, when $\beta = 1$, the variance of outperformance (σ_{TE}^2) reduces to a difference between the variances of the returns of the portfolio and the benchmark; in the other extreme, when $\beta = 0$, the tracking error becomes the sum of these variances. The correlation coefficient measures the extent to which portfolio and benchmark returns move in the same direction. It may take values from –1 to 1, and is unaffected by the relative magnitudes of the portfolio and benchmark risk. While the risk model does not report this quantity, it may be easily calculated from the reported β using Equation 6.

APPENDIX 2. RISK FACTORS AND FACTOR LOADINGS

The risk model represents the return of any bond as the sum of terms due to each of the risk factors, as seen in Equation 1. Each such term is the product of a factor loading f_{ij} and a risk factor x_j, and each such term must be in units of return. In our model, the factor loadings are directly defined based on bond characteristics, while the risk factors themselves are not directly specified. As the different factor loadings are denominated in different units, the units of the risk factors are different as well. Interpretation of the risk factors is based on the notion that when multiplied by the factor loading, it generates return. In this appendix, we detail the factor loadings for every factor in the model. Whenever possible, we provide interpretation of the risk factor itself, as well. Some risk factors, notably those for term structure, sector, and quality, have a much more straightforward interpretation than others.

TREASURY TERM STRUCTURE

The first set of risk factors is designed to capture yield curve risk. These factors correspond to the returns on Treasury cashflows at each of the 20 points in time, or vertices, shown in Figure A1.

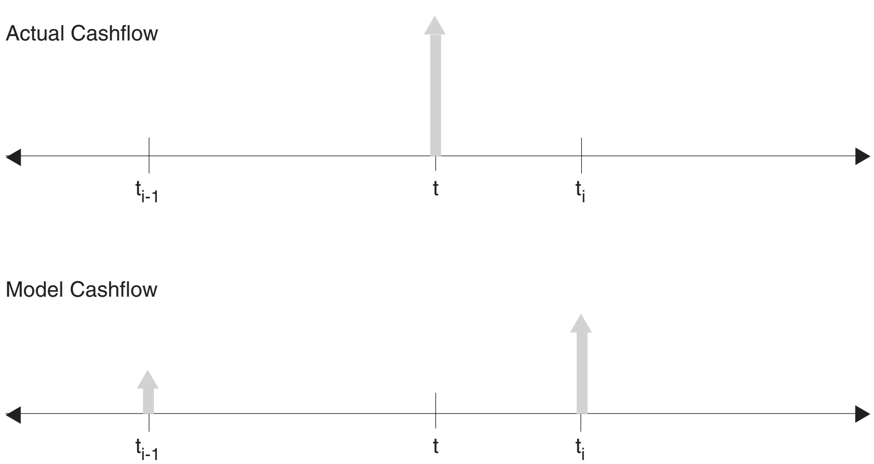
Figure A1. **Term Structure Risk Factors**

Factor	Term
1	0 (Cash)
2	3 months
3	6 months
4	9 months
5	1 year
6	1.5 years
7	2 years
8	2.5 years
9	3 years
10	3.5 years
11	4 years
12	5 years
13	6 years
14	7 years
15	10 years
16	15 years
17	20 years
18	25 years
19	30 years
20	40 years

The factor loadings of any bond for these factors are obtained by distributing the present value of a bond's actual cash flows among the above vertices. In this calculation, each of a bond's projected cashflows are apportioned between the vertices before and after the projected cashflow date. As illustrated in Figure A2, the single cashflow projected to occur at time t is represented by two cashflows on the left and right vertex dates t_{i-1} and t_i such that the total present value and duration of the two distributed cashflows match those of the actual cash flow. Once this process is carried out for all of a bond's cashflows, the resulting cashflow stream has cashflows only on the model's vertex dates, but matches the original cashflow stream quite closely. Its present value and duration match those of the original cashflow stream exactly, and the distribution of cashflows along the yield curve (and hence sensitivities to yield curve movements) is largely preserved. The factor loading for each of the vertices is then set equal to the percentage of the present value of the bond due to the cash flows placed on that vertex. (The sum of these factor loadings for each bond must therefore be equal to one.)

For a callable bond, these term structure factor loadings are derived from its cashflows to adjusted duration. This cashflow stream is calculated as the weighted blend of the cashflows to call CF_{call} and the cashflows to maturity

Figure A2. **Splitting a Cashflow between Two Neighboring Vertices**



Present value preservation: $PV\ (CF_{t_{i-1}}) + PV\ (CF_{t_i}) = PV\ (CF_t)$

Duration preservation: $PV\ (CF_{t_{i-1}})t_{i-1} + PV\ (CF_{t_i})t_i = PV\ (CF_t)t$

$CF_{mat,}$ with the weight w chosen such that the resulting duration matches the bond's option-adjusted duration DUR_{adj}:

(7)
$$CF_{adj} = wCF_{mat} + (1-w)CF_{call,}$$
$$w = \frac{DUR_{adj} - DUR_{call}}{DUR_{mat} - DUR_{call.}}$$

For mortgage-backed securities, the term structure factor loadings are obtained from the cashflows generated by the Lehman Brothers prepayment model for the zero-volatility path of forward short rates, discounted by a spread over the forward rate curve.

CORPORATE SECTOR RISK

A set of risk factors captures the exposure to movement of sector spreads. For a given corporate bond, one of these factors will be loaded by the negative of spread duration, and the rest will be zero. Note the absence of a Treasury sector; this is equivalent to setting the factor loading to zero for the Treasury sector.

The risk factors in this section are based on the Lehman Brothers industry classification scheme, which reflects our current view on how the corporate market should be broken down. The model uses 27 categories chosen from that classification scheme, at differing levels of detail, as shown in Figure A3.

A bond with an industry classification that does not fall neatly into any of the above categories (e.g., hypothetical) will be placed into one of them via default rules based on sector (but has been excluded from the risk model calibration process described in Appendix 4).

QUALITY RISK

A third set of risk factors captures the exposure to movement of credit quality spreads. For a given bond, one of these factors will be loaded by the negative of spread duration, and the rest will be zero. Bonds rated AAA+ and AAA will have zeros for all quality factor loadings.

The model classifies bonds according to a quality score mechanism that considers the ratings from all three rating agencies that we follow; a bond with a split rating may have its duration contribution prorated into two adjacent quality categories. There are seven quality categories, spanning the range of investment grade and high yield securities, as shown in Figure A4. An unrated bond will be placed into one of these buckets based on a comparison of its option-adjusted spread with averages for bonds of the same sector in each quality category. (But unrated bonds do not participate in the risk model calibration. See Appendix 5 for the detail of the historical calibration process.)

OPTIONALITY RISK

Six risk factors capture the risk exposure due to optionality. They compare the extent of optionality between the portfolio and benchmark from several different (and somewhat redundant) angles. These factors are summarized

Figure A3. **Sector Spread Risk Factors**

Factor	Classification
1	FNMA
2	FHLB
3	FHLMC
4	Refcorp
5	Other Agencies
6	Banking
7	Brokerage
8	Finance Companies
9	Insurance
10	Other Financial Institutions
11	Basic Industry
12	Capital Goods
13	Consumer Cyclical
14	Consumer Non-cyclical
15	Energy
16	Technology
17	Transportation
18	Electric Utilities
19	Telephone Utilities
20	Natural Gas Utilities
21	Water Utilities
22	Canadian Yankees
23	Corporate Yankees
24	Supranational Yankees
25	Sovereign Yankees
26	Eurobonds and International
27	Asset-backed

Figure A4. **Quality Spread Risk Factors**

Factor	Quality
1	Aa
2	A
3	Baa
4	Ba
5	B
6	C
7	Worse than C

in Figure A5. A callable or putable bond will load one of the first four of these factors, depending on whether it is considered to be trading to redemption or to maturity, and on whether its time to maturity is above or below 10 years. This factor will be loaded by the difference between the adjusted duration and the duration of the bond to redemption or to maturity, respectively. This quantity, also referred to as “reduction due to call (or put),” is a measure of the extent to which the options embedded in a bond make its behavior different from that of an otherwise equivalent bullet bond. The determination of whether a bond is treated as trading to redemption or to maturity for purposes of risk analysis is based on how the bond’s adjusted duration compares to its durations to redemption and to maturity.

The other two factors are loaded for every callable bond with the delta (δ) and gamma (γ) of the embedded option. These are, respectively, the first and second derivatives of option value (OV) with respect to price of an equivalent bullet bond (P_{bullet}).

(8)

$$\delta = \frac{\partial OV}{\partial P_{bullet}} = \frac{\partial}{\partial P_{bullet}}\left(P_{bullet} - P\right) = 1 - \frac{\partial P}{\partial P_{bullet}} = 1 - \frac{\frac{\partial P}{\partial y}}{\frac{\partial P_{bullet}}{\partial y}} = 1 - \frac{P \cdot D}{P_{bullet} \cdot D_{bullet}}$$

$$\gamma = \frac{\partial \delta}{\partial P_{bullet}}$$

Delta will be close to 1 for options deep in the money (e.g., bonds trading strongly to call) and close to zero for out of the money options (bonds trading strongly to maturity). Gamma, conversely, will be near zero for both of these extreme cases, and will reach its highest level when the option is at the money (e.g., when it is least clear which way the bond is trading). For risk analysis purposes, these quantities are computed using the option-adjusted duration and convexity of a bond and the duration and convexity of the associated bullet bonds to call and to maturity.

Figure A5. **Risk Factors Relating to Optionality**

Factor	Description
1	Long trading to maturity
2	Long trading to redemption
3	Intermediate trading to maturity
4	Intermediate trading to redemption
5	Option delta
6	Option gamma

This category of risk factors could largely be eliminated if the treatment of term structure risk was carried out using fully option-adjusted yield curve sensitivities instead of cashflows.

COUPON RISK

The coupon level of a particular bond, relative to the rest of the market, can make it more or less attractive to investors. To the extent that the value placed on coupon level responds to changes in tax codes, accounting standards, and the like, this effect represents an additional source of risk. Zero-coupon bonds, favored by asset-liability managers, often trade differently from the rest of the market.

Two factors capture the risk related to coupon level for non-MBS securities, as shown in Figure A6.

One factor, representing a zero coupon spread, is loaded by the modified adjusted duration DUR_{adj} for all zero coupon bonds (except Treasury STRIPS). A second is loaded by $\left(C - C_{avg}\right) \cdot DUR_{adj}$ for all coupon bonds (except Treasuries), where C is the coupon of a given bond, and C_{avg} is an average coupon of bonds in the Government-Corporate Index. This risk factor will be most significant for bonds with unusually high or low face coupon rates.

TREASURY-SPECIFIC RISK FACTORS

This group of factors captures the risk related to the Treasury market.

For all non-callable Treasury coupon bonds, one factor is loaded by $C \cdot DUR$ (coupon times duration). This represents the risk of a systematic change in the yields of Treasury coupon bonds in which the yield change is proportional to the coupon level.

The most recently issued ("on-the-run") Treasury securities of each maturity are generally considered to be the most liquid, and often command a liquidity premium as a result. This may cause on-the-runs to have lower yields and "special" repo rates. A single factor represents the risk of a systematic change in on-the-run yield spreads, and is loaded by modified duration for all on-the-run Treasuries.

Figure A6. **Risk Factors Relating to Coupon Level**

Factor	Description
1	Agency and Corporate Zeros
2	High / Low Coupons

For STRIPS, the risk due to the spread between the spot and STRIPS curves is modeled by two factors, which essentially parametrize a quadratic approximation to the spread between these two curves as a function of time to maturity. These two factors are loaded by t and $t^2/100$, respectively, where t is time to maturity. The Treasury-specific risk factors are summarized in Figure A7.

This completes the risk factors for bonds; the remaining factors have nonzero factor loadings only for mortgage-backed securities.

MBS RISK FACTORS

As described above, mortgage-backed securities load the set of 20 cashflow factors with their zero volatility cashflows. The resulting estimate of term structure risk for a mortgage passthrough thus reflects the exposure to movement of Treasury yields that would be felt by a bond paying these fixed cashflows.

The modeling of MBS risk is completed by the three categories of risk shown in Figure A8: sensitivities to prepayment speeds, volatilities, and spreads. To reflect the fact that changes in these quantities might differ by sector, each of these categories is represented by four risk factors, one for each of the broad sectors shown in Figure A9. There are thus a total of 12 MBS-specific risk factors.

Figure A7. **Treasury-specific Risk Factors**

Factor	Description
1	Non-Callable Treasury Coupon Level
2	On-the-run Liquidity Spread
3	STRIPS risk (linear in term)
4	STRIPS risk (quadratic in term)

Figure A8. **MBS Risk Factors, Each Split into 4 Sector Buckets**

Factor	Description	Securities Most Affected
1	Prepayment Sensitivity	Deep Discounts and Premiums
2	Volatility Sensitivity	Cusp Coupons
3	Sector Spread Sensitivity	All MBS

MBS PREPAYMENT

A set of four risk factors is used to represent the risk due to changes in prepayment speeds for the four broadly defined sectors of mortgage-backed securities shown in Figure A9. Within each sector, the prepayment risk factor is loaded by the sensitivity to a change in prepayment speeds across the sector, $PSA \cdot (\partial P / \partial PSA)$. In this expression, PSA is the yield-equivalent prepayment speed—that is, the speed for which the cashflows projected by the PSA prepayment model (the market standard, from the Bond Market Association) will correspond to the same yield as those projected by our zero-volatility model. The prepayment duration, $\partial P / \partial PSA$ is the sensitivity of price to changes in this prepayment speed. This factor loading, proportional to both the prepayment duration and the prepayment speed, implies that the type of systematic change expected would not be a parallel shift, in which all prepayment speeds increase by 10. Rather, the risk represented is a relative change, such as one in which all PSA prepayment speeds in a sector increase by 3%. The separation of this factor into four sector buckets allows the risk model to express the relationship between yield movement and prepayment differently for each sector.

The loadings on these risk factors for a mortgage portfolio will be strongly influenced by the premium/discount composition of the portfolio, as prepayment sensitivity is positive for discounts and negative for premiums.

This risk factor is negatively correlated with the term structure risk factors. It is well-known that MBS prepayment speeds increase as yields decrease, and that this strongly influences the movement of MBS prices as yields change. The precise nature of this relationship is the subject of the prepayment models used to calculate option-adjusted durations for MBS. The risk model was developed independently of such option-adjusted models. Term structure exposures are modeled using static, not option-adjusted measures. The relationship between yield movement and prepayment is represented by the negative correlations between these risk factors.

Figure A9. **MBS Sector Breakdown for Risk Analysis**

Sector	Description
1	Conventional (FNMA and FHLB) 30-year MBS
2	GNMA 30-year MBS
3	All 15-year MBS
4	All Balloon MBS

MBS SECTOR SPREAD

A set of four risk factors represents the risk due to changes in MBS sector spreads for the four sectors defined in Figure A9. These spread factors are loaded by spread duration.

MBS VOLATILITY AND CONVEXITY

A set of four risk factors represents the risk due to changes in interest rate volatility. This factor will be loaded for each of the four sectors defined in Figure A9 by the sensitivity to interest rate volatility, $\partial P / \partial \sigma$ where σ is the current volatility of mortgage rates. The partial derivative, $\partial P / \partial \sigma$ is not calculated numerically, but is approximated using a closed form to derive it from convexity. The expression used to calculate this quantity is

$$\frac{\partial P}{\partial \sigma} = \sigma_{BW10} \cdot CONV \cdot P, \tag{9}$$

where σ_{BW10} is the monthly volatility of the 10-year Treasury rate, and $CONV$ is the option-adjusted convexity of this security.

This set of risk factors should capture the exposure of a mortgage portfolio to cusp coupon securities, which tend to have the largest-magnitude convexities.

Convexity as a risk sensitivity measure accounts for the impact of market volatility or realized volatility of interest rates. MBS securities are also sensitive to changes in implied volatility, which is measured by vega. The option-adjusted convexity of an MBS security is computed by giving small parallel shift shocks to the yield curve and measuring the resulting changes in duration. Vega, on the other hand, is computed by giving a small shock to the term structure of volatilities used in the simulation of interest rates. Computation of a vega for each MBS security in the index became a real possibility only recently with advances in volatility calibration techniques and computational efficiency. Historically, we could not have used vega as one of the risk factor sensitivities.

APPENDIX 3. SECURITY-SPECIFIC RISK

In Appendix 1, we saw that portfolio variance and tracking error relative to a benchmark may each be expressed as the sum of two terms, one due to systematic risk and one due to security-specific risk (also known as non-systematic, idiosyncratic, or "special" risk). In terms of our basic risk model (Equation 2), this is the risk due to the security-specific residual terms ε_i. At the heart of the special risk calculations in Equations 4 and 5 is the $N \times N$ covariance matrix $\mathbf{\Gamma}$, which contains the pairwise covariances $\Gamma_{ij} = COV(\varepsilon_i, \varepsilon_j)$ between these residuals. These can be expressed as

(10) $$\Gamma_{ij} = \rho_{ij} \sigma_{\varepsilon_i} \sigma_{\varepsilon_j},$$

where $\sigma_{\varepsilon_i} \geq 0$ is the standard deviation of the residual ε_i, and $-1 \leq \rho_{ij} \leq 1$ is the correlation coefficient between the residuals ε_i and ε_j.

In building our risk model, we paid close attention to two issues in calculating special risk: the estimation of the special risk volatility σ_{ε_i} to be assigned to each bond $1 \leq i \leq N$, and the assumptions to be used to set the correlation coefficients ρ_{ij}.

ESTIMATING SPECIAL RISK VOLATILITY

The special risk model uses a linear model to estimate the variance of each bond's residual ε_i based on the characteristics of bond i. We focus on bond properties that indicate risks not captured by our systematic risk model. Some of the selected characteristics, like bond age and amount outstanding, are measures related to liquidity. More liquid securities are more likely to follow the market; less liquid securities are more likely to be affected by idiosyncratic effects particular to a single bond or sector.

To project the special risk variance $\sigma^2_{\varepsilon_i}$ of a Treasury or agency bond,

(11) $$\sigma^2_{\varepsilon_i} = a_0^T + a_1^T D_i + a_2^T \frac{A_i}{OM_i} + a_3^T OUT_i + a_4^T 1_{\{C_i > 0\}} \left(C_i - C_{BW} \right)$$
$$+ a_5^T 1_{\{C_i = 0\}} + a_6^T 1_{\{i = agency\}} + a_7^T 1_{\{i = callable\}},$$

where the parameters a_n^T are a set of fixed coefficients calibrated to these markets (see Appendix 5), and the notation $1_{\{condition(i)\}}$ represents the indicator function (or "dummy"), which is equal to 1 if the condition is true for bond i, and 0 otherwise. The relevant characteristics of bond i are thus its adjusted duration D_i, the ratio of age A_i to original maturity OM_i, the amount outstanding OUT_i, the difference between the bond's coupon C_i and that of the 10-year bellwether (C_{BW}), and whether it is a zero, an agency, or a callable bond.

For corporates, we have

(12) $$\sigma_{\varepsilon_i}^2 = a_0^C + a_1^C 1_{\{C_i>0\}}\left(C_i - C_{avg}\right) + a_2^C 1_{\{C_i=0\}} + \sum_{s \in sectors} a_s^C 1_{\{sector(i)=s\}} + \sum_{q \in qualities} a_q^C D_i 1_{\{quality(i)=q\}}.$$

The characteristics of a corporate bond used in the model are its adjusted duration D_i, the difference between its coupon C_i and the average coupon C_{avg} of the government/corporate universe, and its sector and quality. For this purpose, corporate bonds are divided into five sectors, broadly defined as industrials, utilities, finance, Canadian, and foreign. (This notion of "sector" is not as detailed as the classification scheme used to define systematic sector risk factors.) In addition, they are split into seven quality buckets, for Aa, A, Baa, Ba, B, C, and lower. The fact that the bonds are divided here among the same set of quality buckets used in the systematic risk calculation should not be viewed as a double-counting of the same effect, as the special risk volatility is inherently different. The quality effect captured by the systematic risk calculation is that in a given month, spreads within a given quality range may tend to change across the board, causing a particular quality range to outperform or underperform the market. In the special risk calculation, which focuses on individual bond effects, the division into qualities is meant to reflect wider variances of residuals among bonds of lower quality than in higher quality ranges.

For MBS, we have

(13) $$\sigma_{\varepsilon_i}^2 = a_0^M + a_1^M D_i + a_2^M OUT_i + a_3^M \frac{A_i}{OM_i} + a_4^M 1_{\{i=balloon\}} + a_5^M 1_{\{i=gold\}}.$$

As for governments, we use the adjusted duration, the amount outstanding, and the ratio of age to original maturity for MBS (for balloons, original maturity is the time until the balloon payment). In addition, flags indicate whether a security is a balloon or a Gold. Unlike governments, coupon level does not play a role in the calculation of special risk volatility for mortgages. This is because coupon effects play such a major role in the mortgage market that they are considered in the very definition of securities and in the various systematic risk factors for MBS.

MODELING SPECIAL RISK CORRELATIONS

One of the basic goals of the multifactor risk model is to decrease the dimensionality of portfolio risk, from the number of bonds N down to the dimensionality M of the covariance matrix. Thus, the appearance of an $N \times N$ covariance matrix of residuals might seem to defeat the purpose of the entire exercise. In fact, this is not the case. The dimensionality of the special risk is reduced drastically by assuming zero correlations among the residuals, with a few exceptions.

Let us take a close look at the special risk component σ_{SR}^2 of the tracking error of Equation 5,

$$(14) \qquad \sigma_{SR}^2 = (\mathbf{q}_p - \mathbf{q}_b)\mathbf{\Gamma}(\mathbf{q}_p - \mathbf{q}_b)^T = \sum_{i=1}^{N}\sum_{j=1}^{N}(q_i^p - q_i^b)(q_j^p - q_j^b)\rho_{ij}\sigma_{\varepsilon_i}\sigma_{\varepsilon_j}\ .$$

This quantity is composed of $N \times N$ terms. However, most of the off-diagonal terms can be assumed to be zero, as residuals as a rule should be uncorrelated ($\rho_{ij} = 0$) by their very nature. As the exceptions to this rule, we have used the notion of "shelves" to designate sets of related bonds, whose residuals are likely to be positively correlated. For instance, members of a given shelf can be corporates of the same issuer, or mortgage-backed securities quoted off the same generic. Let $S_k, k = 1, \ldots, K$ be a set of K such shelves. Then Equation 10 can be rewritten as

$$(15) \qquad \sigma_{SR}^2 = \sum_{k=1}^{K}\sum_{i \in S_k}\sum_{j \in S_k}(q_i^p - q_i^b)(q_j^p - q_j^b)\rho_{ij}\sigma_{\varepsilon_i}\sigma_{\varepsilon_j}\ .$$

Let us assume that the correlations ρ_{ij} may be characterized by

$$(16) \qquad \rho_{ij} = \begin{cases} 1 & i = j \\ \rho & \text{bonds } i \text{ and } j \text{ are two different securities on the same shelf} \\ 0 & \text{otherwise} \end{cases}$$

where ρ is a model constant determining how we will treat corporates of the same issuer and MBS quoted off the same generic. With this assumption in place, the special risk covariance matrix $\mathbf{\Gamma}$ can be seen to be a block-diagonal matrix, with one block for each corporate issuer and MBS quoted generic. Let us examine two special cases for the correlation ρ between two different bonds from the same issuer. If we assume perfect correlation, or $\rho = 1$, then the expression for special risk simplifies to

(17) $$\sigma^2_{SR,issuer} = \sum_{k=1}^{K}\left(\sum_{i\in S_k} q_i^p \sigma_{\varepsilon_i} - \sum_{i\in S_k} q_i^b \sigma_{\varepsilon_i}\right)^2 .$$

This assumption calculates non-systematic risk based on exposures to issuers. Since the non-systematic returns of two bonds of the same issuer are assumed to be perfectly correlated, an overweight in one bond can be offset by an underweight in another. The model therefore accumulates the total exposures of the portfolio and the benchmark to a given issuer, and squares the difference between them to obtain the contribution to special risk. If we instead assume $\rho = 0$, treating all residuals as uncorrelated, our expression simplifies even more, to

(18) $$\sigma^2_{SR,bond} = \sum_{i=1}^{N}\left(q_i^p - q_i^b\right)^2 \sigma^2_{\varepsilon_i} .$$

In this expression, every single difference between portfolio and benchmark weights to individual bonds contributes a positive amount to our estimate of special risk. If indeed no two bonds are correlated in their special risks, there is no way to make up for an overweight in a particular bond via a position in any other.

Each of these forms of the special risk calculation has some intuitive appeal. The first, based on issuer-level allocations, represents the broadly held view that the primary source of event risk is the exposure to issuers. The second, which assigns risk to specific security allocations, ensures that the model sees a portfolio that exactly matches the benchmark as having less event risk than one that uses a single issue to represent an issuer. This is of particular interest when dealing with a portfolio or benchmark that is concentrated within a single issuer or group of issuers (such as the portfolio of all debt of a given issuer).

Of course, the best setting for ρ is probably somewhere between 0 and 1. Portfolio returns will be affected both by issuer-level events as well as security-level phenomena, neither of which are covered under the systematic risk model. In a statistical study using pricing data from bonds of 20 large issuers, we found a correlation coefficient of about 0.2 between residuals of bonds of the same issuer. This low correlation is probably due to the fact that major issuer-level events are quite rare, and the majority of the residuals represent just pricing noise, which is likely to take place at the individual security level. (A higher value, such as 0.5, may be chosen to remain a bit more conservative with regard to issuer risk.) This would seem to necessitate a much more expensive calculation than either of the above special cases; however, it can be shown that the correct result is achieved by a weighted average of the two:

$$\sigma^2_{SR,both} = (1-\rho)\sigma^2_{SR,bond} + \rho\sigma^2_{SR,issuer} \tag{19}$$

Examination of the calculations in Equations 17 through 19 reveals that our assumptions on correlations of residuals brings the complexity of our special risk calculations (i.e., the approximate number of multiplication and addition operations needed to calculate the tracking error) down to order N, as opposed to complexity N^2, which might have been inferred from Equation 14.

APPENDIX 4. OPTIMIZATION

The optimizer in the Lehman Brothers risk model uses an iterative procedure to minimize tracking error. At each step, the optimizer guides the investor to select the one-for-one bond swap transaction that will reduce the tracking error the most. To calculate the effect that a given swap will have on the tracking error, we begin by rewriting the formula for tracking error (Equation 5) directly in terms of the security allocation vectors $\mathbf{q}_p$ and $\mathbf{q}_b$ for the portfolio and benchmark, respectively:

(20) $$\sigma_{TE}^2 = (\mathbf{q}_p - \mathbf{q_b}) \cdot \mathbf{F} \cdot \mathbf{\Omega} \cdot \mathbf{F} \cdot (\mathbf{q_p^T} - \mathbf{q_b^T}) + (\mathbf{q}_p - \mathbf{q_b}) \cdot \mathbf{\Gamma} \cdot (\mathbf{q_p^T} - \mathbf{q_b^T})$$

The gradient $\mathbf{G}$ of σ_{TE}^2 with respect to $\mathbf{q_p}$ is the row vector of partial derivatives of σ_{TE}^2 with respect to the portfolio allocations q_i^p to each security in the universe, given by

(21) $$\mathbf{G} = \nabla_{\mathbf{q_p}} \sigma_{TE}^2 = \left(\frac{\partial \sigma_{TE}^2}{\partial q_1^p}, \frac{\partial \sigma_{TE}^2}{\partial q_2^p}, \cdots, \frac{\partial \sigma_{TE}^2}{\partial q_N^p} \right)$$

$$= 2 \cdot (\mathbf{q_p} - \mathbf{q_b}) \cdot \mathbf{F} \cdot \mathbf{\Omega} \cdot \mathbf{F} + 2 \cdot (\mathbf{q_p} - \mathbf{q_b}) \cdot \mathbf{\Gamma} \,.$$

Let us investigate the effect on tracking error of a small change $\mathbf{\Delta}$ in portfolio weights. If we denote the new portfolio allocation vector by $\mathbf{q_p'} = \mathbf{q_p} + \mathbf{\Delta}$, we can see that the tracking error for the new portfolio is:

(22) $$\begin{aligned} \sigma_{TE}^{\prime 2} &= (\mathbf{q_p'} - \mathbf{q_b}) \cdot \mathbf{F} \cdot \mathbf{\Omega} \cdot \mathbf{F} \cdot (\mathbf{q_p'^T} - \mathbf{q_b^T}) \\ &\quad + (\mathbf{q_p'} - \mathbf{q_b}) \cdot \mathbf{\Gamma} \cdot (\mathbf{q_p'^T} - \mathbf{q_b^T}) \\ &= \sigma_{TE}^2 + 2 \cdot (\mathbf{q_p} - \mathbf{q_b}) \cdot \mathbf{F} \cdot \mathbf{\Omega} \cdot \mathbf{F} \cdot \mathbf{\Delta^T} \\ &\quad + 2 \cdot (\mathbf{q_p} - \mathbf{q_b}) \cdot \mathbf{\Gamma} \cdot \mathbf{\Delta^T} + \mathbf{\Delta} \cdot \mathbf{F} \cdot \mathbf{\Omega} \cdot \mathbf{F} \cdot \mathbf{\Delta^T} + \mathbf{\Delta} \cdot \mathbf{\Gamma} \cdot \mathbf{\Delta^T} \,. \end{aligned}$$

We can express the resulting change of tracking error in terms of the change $\mathbf{\Delta}$ in portfolio weights and the gradient $\mathbf{G}$, as

(23) $$\Delta \sigma_{TE}^2 = \sigma_{TE}^{\prime 2} - \sigma_{TE}^2 = \mathbf{G} \cdot \mathbf{\Delta^T} + \mathbf{\Delta}(\mathbf{F} \cdot \mathbf{\Omega} \cdot \mathbf{F} + \mathbf{\Gamma})\mathbf{\Delta^T} \,.$$

Gradient descent optimization is a well-known minimization technique in which small steps are taken along the gradient of the function until a minimum is found. However, each step in such a procedure might involve changes to every security weight q_i^p, which corresponds to small changes in the size of each position in the portfolio. This is clearly impractical. We have therefore chosen to constrain the set of transactions considered to one-for-one swaps between two bonds that preserve the total market value of the portfolio. Thus, the transaction to sell bond m and purchase an equivalent amount of bond l would be represented by a vector $\mathbf{\Delta}$ whose elements are given by

(24) $$\Delta_i = \begin{cases} 0 & i \neq l, m \\ x & i = l \\ -x & i = m, \end{cases}$$

where x is the size of the swap, which is the percentage market value of both the purchase and the sale relative to the total portfolio market value. For this special type of transaction, with only two non-zero entries in the change vector $\mathbf{\Delta}$, the expression for the change in tracking error squared can be simplified dramatically. For instance, the first term of Equation 23, which gives the first-order effect of the change, can be seen as simply the difference between the sensitivities to the two securities in the swap:

(25) $$\mathbf{G} \cdot \mathbf{\Delta}^{\mathbf{T}} = \sum_{i=1}^{N} G_i \cdot \Delta_i = x \cdot \left(G_l - G_m\right),$$

where G_i is the i th element of the gradient vector $\mathbf{G}$.

The second term of Equation 23 incorporates the second-order effects of changing the weights, as well as the cross-effects of changing more than one weight at a time. This term, which considers both systematic and non-systematic risk, can be simplified as well, and the change in tracking error squared can be expressed as

(26) $$\Delta\sigma_{TE}^2 = x \cdot (G_l - G_m) + x^2 \cdot \left[(\mathbf{f_l} - \mathbf{f_m}) \cdot \mathbf{\Omega} \cdot (\mathbf{f_l^T} - \mathbf{f_m^T}) + (1 - \rho_{lm}) \cdot (\sigma_{\varepsilon_l}^2 + \sigma_{\varepsilon_m}^2) + \rho_{lm} \cdot (\sigma_{\varepsilon_l} - \sigma_{\varepsilon_m})^2\right]$$

where $\mathbf{f_i}$ is the factor loading vector of bond i, σ_{ε_i} is the special risk volatility of bond i, and ρ_{lm} is the correlation between the special risk of bonds l and m, as per Equation 16.

For a given set of bonds to swap, Equation 26 gives the change in tracking error as a quadratic function of x,

(27) $$\Delta\sigma_{TE}^2 = bx + ax^2,$$

where

(28) $$b = G_l - G_m$$

$$a = \left[(\mathbf{f_l} - \mathbf{f_m}) \cdot \mathbf{\Omega} \cdot (\mathbf{f_l^T} - \mathbf{f_m^T}) + (1-\rho_{lm}) \cdot (\sigma_{\varepsilon_l}^2 + \sigma_{\varepsilon_m}^2) + \rho_{lm} \cdot (\sigma_{\varepsilon_l} - \sigma_{\varepsilon_m})^2\right]$$

To find the optimal market value x to swap, we set $d\Delta\sigma_{TE}^2 / dx = 0$, and find that $\Delta\sigma_{TE}^2$ has a minimum at

(29) $$x = -\frac{b}{2a}.$$

(This is clearly a minimum because $\mathbf{\Omega}$ is the covariance matrix, which is positive semi-definite, guaranteeing that $a \geq 0$.) The resulting change in tracking error (as long as $a \neq 0$) is

$$\Delta\sigma_{TE\,\min}^2 = -\frac{b^2}{4a}\cdot$$

Thus, for any selected pair of bonds, it is straightforward to calculate the optimal transaction size as well as the resulting tracking error. In principle, the best such transaction could be found by a brute force search across all pairs of bonds in the universe, selecting the one that gives the largest reduction in tracking error.

However, for N bonds there are (N-1)N/2 possible pairs of bonds, making this a time-consuming operation. Additionally, investors may have other reasons to favor one bond over another. We have therefore designed a procedure in which we rank the bonds in the bond swap pool in order of their effect on tracking error.

From the above equations, it is easy to conclude that the more negative G_l is, the lower $\Delta\sigma_{TE}^2$ is; and the more positive G_m is, the lower $\Delta\sigma_{TE}^2$ is. Thus, we would want to buy the bond with the most negative gradient and sell the bond with the most positive gradient, thus the name "*gradient descent.*"

In our risk model optimization, the first step is to compute the gradients of all the bonds. If the investor wishes to first pick the bond to buy, then we present the bonds in the bond swap pool in ascending order of their gradients. If the investor prefers to first select the bond to sell, we present the bonds in the portfolio in descending order of their gradients. Once the user chooses a bond, the model computes the amount to buy (or sell) and the resulting tracking error for swaps between this bond and all other N-1 bonds. This list is then presented with the lowest tracking error first. The investor can then select the second bond in the swap according to personal preference. The optimal par values in the swap may be overridden, for example, to transact round lots. This procedure is repeated until the portfolio has reached the desired level of tracking error.

APPENDIX 5. HISTORICAL CALIBRATION OF RISK MODEL

The statistical parameters defining the risk model are obtained from historical data. For a set of quoted bonds in a given month, we have an $N \times 1$ vector of observed returns $\mathbf{r}$, and a fixed $N \times M$ factor loading matrix $\mathbf{F}$. Linear regression can then be used to find the factor vector $\mathbf{x}$ that minimizes the residual vector $\boldsymbol{\varepsilon}$ in Equation 2. The general form of this minimization is

$$\min \sum_{i=1}^{N} w_i \varepsilon_i^2, \tag{30}$$

where w_i is the weight assigned to security i in the minimization. The standard least-squared linear regression applies equal weights $w_i = 1$ to all securities. When unequal weights are assigned, the regression will result in a realization of the factor vector $\mathbf{x}$ in which securities with larger weights have smaller residuals. That is, these securities play the dominant role in defining how the "market" moved in this month. One possibility would be to use each bond's market capitalization as its weight. However, this would bias the model, implying that larger issues have less non-systematic risk than smaller ones. Also, a credit event at a single large issuer might be reflected as a sector-wide event in such a model. For these reasons, we prefer to use equal weights rather than market weights in our regression procedure. The one type of weighting that is used is to choose carefully which bond prices participate in the regression. The only securities whose returns are included are those that are trader-quoted at both the beginning and the end of the month. Bonds whose returns have extreme values (outliers) are excluded as well. Our regression procedure can therefore be characterized as using equal weights $(w_i = 1)$ over a calibrating set of security returns, with $w_i = 0$ for all excluded bonds.

The regression procedure uses a multi-stage approach. At each stage, a different subset of the quoted bonds is brought into play, and a different set of factors is calibrated. This technique ensures, for example, that the risk factors representing yield curve movement are fit only to the returns of U.S. Treasuries, and can not be affected by term-related changes in corporate bond spreads.

The stages are as follows:

1. Returns on noncallable Treasury coupon bonds are used to define the values of the yield-curve-related factors.

2. Returns on Treasury zero-coupon instruments are used to define factors relating to the STRIPS curve.
3. Returns on noncallable investment-grade coupon-bearing corporates (excluding asset-backed) are used to define the factors relating to sector and quality spreads.
4. Returns on investment-grade zero-coupon corporate instruments are used to define factors relating to these issues.
5. Returns on investment-grade callable bonds are used to define the factors relating to optionality.
6. Returns on high yield bonds are used to define quality spread factors for the high yield market.
7. Returns on asset-backed securities are used to define the single risk factor representing asset-backed spreads.
8. Returns on quoted MBS generics are used to define all the MBS-related risk factors.

At each stage of the above procedure, those factors already calculated in previous stages are held fixed. These can be used to provide a "model return" for any bond over the month in question. The difference between these returns and those actually achieved become the input to the regression at the next stage, which seeks to best account for this as yet unexplained return by the introduction of a new set of risk factors.

For example, in the first step, the instantiation of the yield curve related factors are obtained by regressing the returns r_i^{treas} of Treasury coupon bonds on the term structure risk factors to find the values of x_j that minimize the error terms ε_i in the least-squared sense:

(31) $$r_i^{treas} = \sum_{j=1}^{20} CF_{ij} x_j^{treas} + \varepsilon_i \, ,$$

where the factor loading CF_{ij} is the percentage of present value of bond i due to cashflows at vertex j, as indicated in Appendix 2. Note that the factor realizations x_j^{treas} are not measured directly. Rather, the factor loadings are defined, and the factor realizations are obtained by regression against realized security returns.

The realization of corporate sector and quality risk factors is then obtained by regressing them against the excess returns of corporate bonds over the returns implied by the established term structure risk factors:

$$r_i^{corp} - \sum_{j=1}^{20} CF_{ij} x_j^{treas} = \sum_{s \in sectors} (-D_i) 1_{\{sector(i)=s\}} x_s + \sum_{q \in qualities} (-D_i) 1_{\{quality(i)=q\}} x_q + \varepsilon_i \quad (32)$$

By step 5, we have already accounted for systematic returns due to changes in the yield curve, as well as those due to changes in sector and quality spreads. With the values of these factors fixed, the model can estimate the returns of callable corporates quite well. The differences between these model returns and the actual returns of callable corporates are used to calibrate the factors relating to callability:

$$r_i^{corp-call} - \sum_{j=1}^{20} CF_{ij} x_j^{treas} - (-D_i) x_{sector(i)} - (-D_i) x_{quality(i)} = \sum_{j=1}^{4} 1_{\{option\_type(i)=j\}} x_j^{redn\_call} + \delta_i x_\delta + \gamma_i x_\gamma + \varepsilon_i \quad (33)$$

These factors represent systematic changes in the way the market values different types of optionality, and include the effects of changes in volatility.

The twelve mortgage factors are calculated in stage 8 by regression against the excess returns of mortgage generics over those implied by the Treasury returns on their zero-volatility cashflows:

$$r_i^{MBS} - \sum_{j=1}^{20} CF_{ij} x_j^{treas} = \sum_{s \in MBS\_sectors} 1_{\{sector(i)=s\}} \left(PSA_i \frac{\partial P_i}{\partial PSA_i} x_s^{prepay} + \frac{\partial P_i}{\partial \sigma} x_s^{vol} + D_i x_s^{spread} \right) + \varepsilon_i \quad (34)$$

The result of this procedure is a single instantiation of the random vector $\mathbf{x}$, which represents the performance of the fixed income marketplace over a single given month. As we apply this technique sequentially to monthly returns for T months $t=1,\ldots,T$, we obtain a time series of factor vectors $\mathbf{x}(t)$. These factor vectors are then adjusted by subtracting the prevailing risk-free single period return (one month of interest at the short term rate at time t) from each of the first 20 factors. (Recall that these factors represent total returns on risk-free cashflows of various terms. The adjusted factors then represent the excess returns over the risk-free rate.) We can then estimate the mean of the distribution of the random factor vector $\mathbf{x}$ by

$$\hat{\mathbf{E}}[x_j]=\frac{1}{T}\sum_{t=1}^{T}x_j(t), \tag{35}$$

where the notation $\hat{a}$ is used to denote a data-driven estimate of an unknown parameter a. We can form a similar estimate of the covariance matrix $\mathbf{\Omega}$, in which each matrix element is estimated by

$$\begin{aligned}\hat{\Omega}_{jk} &= COV(x_j,x_k)\\ &= \hat{\mathrm{E}}\left[\left(x_j-\hat{\mathrm{E}}[x_j]\right)\left(x_k-\hat{\mathrm{E}}[x_k]\right)\right]\\ &= \frac{1}{T-1}\sum_{t=1}^{T}\left(x_j(t)-\hat{\mathrm{E}}[x_j]\right)\left(x_k(t)-\hat{\mathrm{E}}[x_k]\right)\end{aligned} \tag{36}$$

These historical estimates of factor covariance comprise the matrix $\hat{\mathbf{\Omega}}$ that is used in our tracking error calculations to characterize the distribution of systematic market movements over a given month.

Unfortunately, the time series $\mathbf{x}(t)$ is not complete. In some months, some of the risk factors are not loaded by any bonds in our data sample, and no estimate is available for those components of $\mathbf{x}(t)$. (For example, data on mortgage-backed securities prior to October 1990 is unavailable.) To build our matrix, each parameter was estimated over that subset of the months for which the relevant data was available. The volatilities of each risk factor were calculated based on all of the available data for that risk factor; the correlations between any two factors were calculated from the subset of monthly observations for which data was available for both factors. The computed matrix $\hat{\mathbf{\Omega}}$ thus consists of terms computed from different data sets. As a consequence, this matrix is not necessarily positive semi-definite. This mathematical property of a covariance matrix guarantees that when any vector is used to multiply this matrix on both sides, as in Equations 4 and 5, the result is a non-negative number, ensuring a positive tracking error. An additional feature of a positive semi-definite matrix is that it has no negative eigenvalues. To restore this property, we use the Jacobi[1] method to express the matrix as a product of its eigenvalues and eigenvectors, replace any negative eigenvalues with zeros, and reconstruct the matrix using this modified set of eigenvalues. In practice, we have found that this procedure causes only minor adjustments in the numeric values within the matrix.

The special risk portion of the calculations can be similarly based on historical data. When the multistage regression for the factor vector $\mathbf{x}(t)$ is complete in a given month, a residual ε_i remains unexplained for each bond i that participated in the regression. While we do not expect these residuals to carry any more information, we can perhaps learn something from which types of bonds have the residuals of the largest magnitudes. The goal of the special risk model is to provide the best predictor of each bond's special risk volatility σ_{ε_i} based on bond characteristics. Since a basic property of residuals is that $\mathbf{E}[\varepsilon_i] = 0$, the goal of the special risk model is to estimate $\sigma_{\varepsilon_i}^2 = E[\varepsilon_i^2]$. Thus, a second set of three regressions is run to help determine the coefficients for the special risk model. The first of these regressions, for Treasuries, finds the parameters a_n^T, which make the estimated special risk variances $\sigma_{\varepsilon_i}^2$ from Equation 11 closest (in the least-squared sense) to the squares of the observed residuals ε_i^2 for government bonds. Similar regressions are then carried out to find the parameters a_n^C and a_n^M of the corporate and MBS special risk models of Equations 12 and 13, respectively. The coefficients thus obtained are then averaged over all months of available data to compute the coefficients that will be used in the risk model to estimate the special risk volatilities of individual securities.

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[1] Golub, Gene H., and Charles F. Van Loan. *Matrix Computations*, 2nd ed. Baltimore: Johns Hopkins University Press, 1989.

APPENDIX 6. GLOSSARY OF TERMS

Risk factor—a market change that affects returns of all securities in a certain market segment (e.g., changes in interest rates, changes in sector spreads, changes in volatility of interest rates, changes in prepayments, etc.).

Factor loading—sensitivity of a given security (or portfolio) to a particular risk factor (e.g., percentage of cash flow in a given segment of the yield curve, spread duration, convexity, PSA duration, etc.). Portfolio factor loadings are calculated as market-weighted averages of factor loadings for individual securities.

Active portfolio exposures—differences between portfolio and benchmark factor loadings with respect to a given risk factor.

Covariance matrix—matrix combining volatilities of risk factors with correlations between them, calibrated to historical data.[2]

Systematic risk—risk due to effect of risk factors.

Non-systematic risk—risk not explained by the combination of all systematic risk factors. Represents risk due to events that affect individual issues and/or issuers (also known as special risk, specific risk, idiosyncratic risk, concentration risk).

Issuer-specific risk—non-systematic risk calculated under the assumption that the residuals of bonds from the same issuer are perfectly correlated. Measures risk due to concentrations in specific issuers.

Issue-specific risk—non-systematic risk calculated under the assumption that the residuals of all bonds are totally uncorrelated. Measures risk due to concentrations in specific issues.

Tracking error—standard deviation of the difference between portfolio and benchmark returns (usually annualized). Is projected based on active portfolio exposures and covariance matrix.

Isolated tracking error—method of calculating the partial tracking error due to a single group of risk factors in isolation; no other forms of risk are considered.

[2] To be more precise, its diagonal elements equal historical variances of risk factors and off-the-diagonal elements equal their co-variances (or correlations times the pair-wise product of the standard deviations).

Cumulative tracking error—tracking error due to cumulative effect of several groups of risk factors. Used iteratively to calculate the incremental effect of risk of one group of risk factors (partial tracking error) considering its correlations with other risks.
Gradient descent optimization—an iterative mechanism for minimizing tracking error one transaction at a time by seeking the transaction that will give the steepest reduction in tracking error.

Bond swap pool—set of securities to be considered as candidates for purchase by gradient descent optimization procedure.

Portfolio sigma—projected standard deviation of absolute portfolio return (not relative to the benchmark).

Market sigma—projected standard deviation of the absolute return of the benchmark.

Beta—sensitivity of the portfolio return to benchmark return. If $\beta = 0.9$, then the model projects that if market forces cause a benchmark return of 100 bp, the portfolio return is likely to be 90 bp.