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SOME INSIGHTS ON DURATION AND CONVEXITY

The extreme volatility of bond markets in the late 1990s has led to heightened awareness of the significance of sound risk management practices for bond portfolios. One important aspect of bond portfolio risk management is monitoring duration exposure and understanding how duration exposure is affected by shifts in important market parameters such as yield. Duration and convexity pertain to relative price changes (price returns), not absolute price changes, which complicates their relationship. This short piece is designed to elucidate some of these complexities and describe the relation between convexity and the sensitivity of duration to changes in yield. We focus on these relationships as they apply to bullet securities. This discussion, therefore, will be more relevant to the corporate market than to mortgages.

Convexity and the Sensitivity of Duration to Yield

Consider the standard two-term approximation for expressing changes in bond value as a function of yield:

$$P(y+\Delta_y)-P(y) \approx P'(y)\,\Delta_y + (1/2)P''(y)\,\Delta_y^2 \tag{1}$$

where P(y) is the bond value expressed as a function of yield and Δ_y is the change bond yield. Equation (1) expresses the change in bond value as a function of the change in bond yield. Portfolio managers in bond markets generally find it more useful to express bond performance in terms of returns, rather than profit and loss. By dividing both sides of Equation (1) by price, we can re-express Equation (1) in terms of return

$$\frac{P(y+\Delta_y)-P(y)}{P(y)} \approx \frac{1}{P(y)}P'(y)\,\Delta_y + (1/2)\frac{1}{P(y)}P''(y)\,\Delta_y^2 \tag{2}$$

which yields the relationship:

$$\text{Return} \approx -\text{Duration}\,\Delta_y + (1/2)\text{Convexity}\,\Delta_y^2 \tag{3a}$$

with the formulas[1]

$$\text{Duration} = -\frac{1}{P(y)}P'(y) \tag{3b}$$

$$\text{Convexity} = \frac{1}{P(y)}P''(y)\,. \tag{3c}$$

In Equation (3a), duration and convexity are in positions parallel to P'(y) and P"(y) in Equation (1). Just as P"(y) is the sensitivity of P'(y) to changes in bond yield, it is tempting to think that convexity is the sensitivity of duration to changes in bond yield. However, the sensitivity of duration to yield changes turns out to be quite different. As will be shown below, the actual formula is

$$D'(y) = D^2 - C \tag{4}$$

where D is duration and C is convexity. To see the magnitude of the potential discrepancies, consider the current 30-year on-the-run Treasury. As of the close of trading on April 6, 2000, its duration was 13.644 years, and its convexity was 288.[2] The true sensitivity of duration to yield changes was $(13.64)^2 - 288 = -102$ which is strikingly different from -288; the result obtained if one ignored the duration term in Equation (4). A 100 bp increase in bond yield will reduce duration by approximately 1.02 years, not 2.88 years.

Given the parallel structure between Equation (1) and Equation (3a), why isn't convexity the sensitivity of duration to yield changes? The answer comes from the fact that Equation (1) is written in terms of bond *values,* whereas the duration and convexity approximation is written in terms of bond *returns*. Duration is the sensitivity of price *return* to changes in yield. A comparison of Equations (1) and (2) shows that expressing performance in terms of returns, rather than values, causes the duration formula to be 1/P(y) times -P'(y), rather than -P'(y) alone. The magnitude of 1/P(y) is increasing in yield, while the magnitude of P'(y) is decreasing in yield. Without further analysis, it is not clear whether duration should increase or decrease as a result of an increase in bond yield.

[1] Throughout, duration will refer to modified duration. The return in Equation (3) is the instantaneous return due to the change in yield. Because Equation (3) is the instantaneous return, the time return is zero.

[2] Convexity is typically stated in hundreds. The convexity of this bond stated in standard units (hundreds) is 2.884. See more detailed discussion below.

The extra 1/P(y) in the duration formula requires use of the product rule to determine the derivative of duration with respect to yield.

$$D'(y) = \frac{1}{P(y)^2}\left(P'(y)\right)^2 - \frac{1}{P(y)}P''(y) \qquad (5)$$

The second term on the right-hand side of the above equation is convexity. The first term on the right-hand side arises from the fact that 1/P(y) is increasing in yield. It is equivalent to D^2. As described earlier, the two effects work in opposite directions, causing them to enter into the D'(y) formula with opposite signs. Despite the fact that yield changes push 1/P(y) and P'(y) in opposite directions, it can be shown for bullet bonds that duration is always decreasing in yield (see Appendix).[3]

While convexity is not the sensitivity of duration to changes in yield, it is the case that dollar convexity is the magnitude of the sensitivity of dollar duration to bond yield. Dollar duration is bond value times duration. This reduces to -P'(y). Dollar convexity is bond value times convexity. This reduces to P"(y).

Figure 1 graphs value per dollar par value for a 30-year, 6% coupon bond as a function of its yield. This graph has the familiar downward sloping, convex shape. The slope of this curve is the change in bond value per unit change in yield P'(y), not return per unit change in yield. The convex shape of the graph of price as a function of yield implies that the magnitude of P'(y) is decreasing in yield. But it doesn't directly imply anything about duration: the sensitivity of return to increases in yield.

By plotting -P'(y) and duration on the same graph, Figure 2 brings the distinction between -P'(y) and duration into sharper focus. Figure 2 graphs -P'(y), the slope of the bond value curve, against the right hand axis and graphs the duration of the same 30-year, 6% bond against the left hand axis.[4]

Note that the duration graph is much closer to linear than the -P'(y) graph. The flatness of the duration graph relative to the -P'(y) graph is a general property. Duration multiplies -P'(y) by 1/P(y). Since the bond value function P(y) is convex, the magnitude of P'(y) is decreasing in bond yield. However, P(y) is also decreasing in bond yield. Thus, for low yields, the magnitude of P'(y) is large, but P(y) is also large, causing duration to be moderate. On the other hand, when yields are high, the magnitude of P'(y) is small, but P(y) is also small, once again causing duration to be moderate. Thus, changes in duration per unit change in yield will be much less extreme than changes in -P'(y) per unit change in yield, as can be seen in Figure 3.

[3] While Equations (3a,b,c) and Equation (4) hold for nonbullets, D^2 - C does not have to be negative for callable bonds or mortgages.

[4] For small yield changes, the duration graph can be linearly rescaled to provide the return graph per unit yield change for small changes in bond yield.

Figure 1. **Value of a 30-Year, 6% Coupon Bond as a Function of Yield**

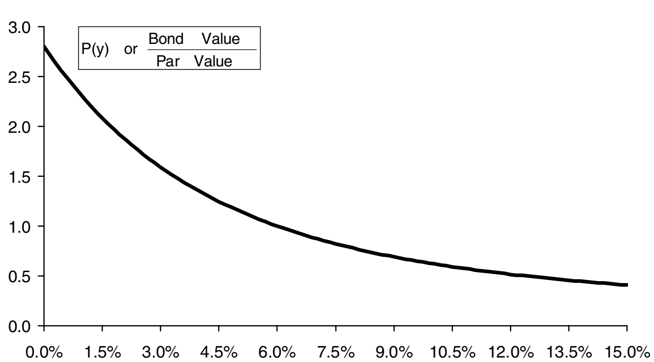
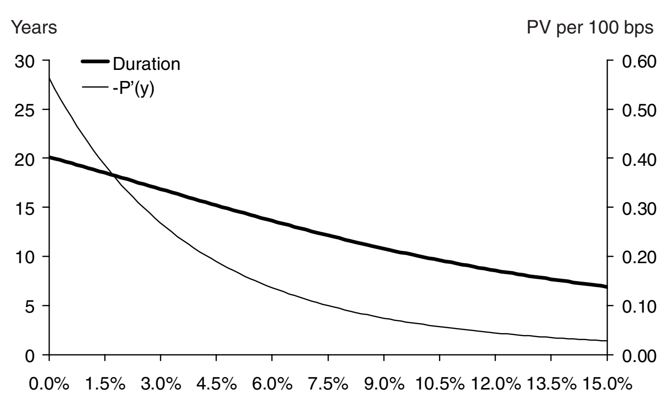


Figure 2. **Duration vs. Change in Price per Change in Yield: 30-Year, 6% Bond**



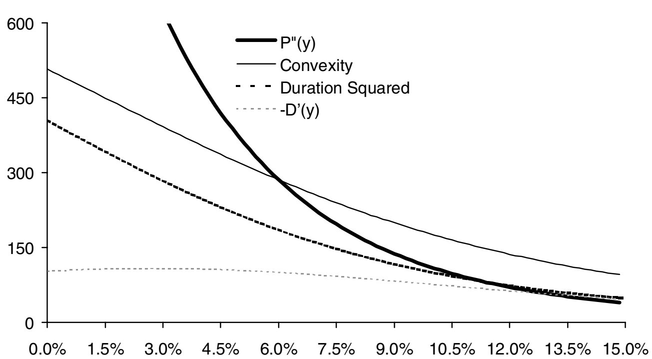
P"(y) is the slope of the P'(y) graph. Of the various lines plotted in Figure 3, P"(y) has, by far, the most variability. For example, when bond yield is 3%, P"(y) is 632. This drops to 71 for a yield of 12%. The dashed line in Figure 3 is convexity. Note that convexity is more moderate and flatter than the P"(y) curve. The reason for this is the same as the reason that the graph of duration is more moderate than the graph of -P'(y). Convexity is P"(y) multiplied by 1/P(y) and both P"(y) and P(y) are large when bond yields are low and both P"(y) and P(y) are low when bond yields are high.

From Figure 3, we also see that while the variability in convexity may be much more moderate than the variability in P"(y), it is still quite large relative to the variability in -D'(y), the magnitude of the slope of the duration curve. Figure 3 demonstrates the importance of using the correct formula for the sensitivity of duration to yield curve changes: D^2 - C. For a long bond, such as the 30-year bond graphed in the figure, both D^2 and C will be very large numbers. Each will have a magnitude several times larger than -D'(y). Using -C alone will lead to large magnitude errors in one's assessment of the sensitivity of duration to bond yield.

Example Using Lehman's PC Product

Consider the 6 1/8%, August 2029 U.S. Treasury, the current 30-year bellwether. By selecting "Bond Calculator" under the calculator menu in PC Product, the user gets the following information for this bond as of the close of trading on April 6, 2000:

Figure 3. **Sensitivity of Duration to Yield: 30-Year, 6% Bond**



- Price: 102.844
- Yield: 5.919%
- Modified Duration: 13.644 years
- Convexity (in hundreds): 2.884

PC Product reports convexities in hundreds. Thus, a reported convexity of 2.884 equals 288.4 in the above formulas. Consider an increase in bond yields of 25 bp. The duration/convexity equation approximates that the return on the bond due to this yield shift would be $-(13.644)(.0025) + (1/2)(288.4)(.0025)^2 = -0.0332$ or -3.32%.

Most risk managers prefer to renormalize yield and return units so that 1.00 denotes a one percent return, rather than a 100 percent return. In this case, the 25 bp increase would be denoted 0.25, rather than 0.0025. With these units, the previous convexity number must be divided by 100 and enters the duration /convexity approximation as 2.884.[5] For these units, the duration/convexity return approximation is written as follows: $-(13.644)(.25) + (1/2)(2.884)(.25)^2 = -3.32$, which corresponds to a bond return of -3.32%. In general, these units are easier to work with and provide the motivation behind the standard convention reporting convexity in hundreds.

Alternatively, the user could have PC Product re-price the bond assuming a 25 bp increase in bond yield.[6] With this shifted Treasury curve, PC Product returns a new bond price of 99.397, implying that a 25 bp increase in yield would lead to a -3.35% bond return: (99.397 - 102.844)/102.844 = -3.35%, which is reasonably close to the predicted -3.32% return from the duration/convexity approximation.

[5] Let r and y be return and yield respectively measured such that 1.00 represents a return or yield of 100%. The standard duration/convexity approximation is
$r \approx - D\,\Delta_y + (1/2)\text{ Convexity }(\Delta_y)^2$.
Multiplying both sides by 100,
$100r \approx - D\,(100\Delta_y) + (1/2)\text{ Convexity }(100)\,(\Delta_y)^2$
$100r \approx - D\,(100\Delta_y) + (1/2)\,(\text{Convexity}/100)\,(100\Delta_y)^2$
Let R = 100r and Y = 100y. R and Y are measured in units where 1.00 corresponds to 1%, rather than 100%.
$R \approx - D\,(\Delta_Y) + (1/2)\,(\text{Convexity}/100)\,(\Delta_Y)^2$.
Note that with these units, duration is unchanged, but the original convexity number must be divided by 100.

[6] This can be implemented by entering 6.169 in the Yield field in the Bond Calculator and hitting the green "GO" button.

While it is easier to work with convexity reported in hundreds for approximating bond returns, the analyst must use the formal version of convexity (PC Product convexity multiplied by 100) to apply the formula for the sensitivity of duration to yield changes: $D'(y) = D^2 - C$. In the current example, duration is 13.644 years, and convexity, as reported by PC Product, is 2.884. Therefore, $D'(y) = (13.644)^2 - 288.4 = -102.2$. The duration sensitivity formula predicts that duration will fall by 1.022 years per 100 bp increase in yield.

We can check the accuracy of this prediction by having PC Product re-evaluate the duration of this bond at 1) a 25 bp increase in yield and 2) a 25 bp decrease in yield. A 25 bp increase in yield results in a modified duration of 13.389. For a 25 bp decrease in yield, PC Product reports the bond duration at 13.900. The realized change in bond duration per unit change in bond yield was (13.389 - 13.900)/(.5%) = -102.2, an exact match to the number predicted by the formula $D^2 - C$.

Conclusion

Duration and convexity are among the most important tools available to managers of bond portfolios. This piece examined some of their intricacies. Duration is the sensitivity of return, not price, to yield shift. For this reason, duration sensitivity to yield shift is not given by convexity, but instead comes from the expression provided in Equation (4). Dollar convexity, on the other hand, is the sensitivity of dollar duration to a shift in yield.

Appendix

This appendix establishes that duration is strictly decreasing in yield for bullet bonds.

From Equation (4) in the text, the sensitivity of duration to yield change is

$$D'(y) = D^2 - C \tag{A1}$$

$$w_i = \frac{1}{P}\frac{\text{Bond Cashflow at time } t_i}{(1+y)^{t_i}}.$$

w_i is the proportion of bond value coming from the ith cash flow. From the definitions of modified duration and convexity for bullets,

$$D = \frac{1}{(1+y)}\sum_i w_i t_i \tag{A2}$$

$$C = \frac{1}{(1+y)^2}\sum_i w_i\left(t_i^2 + t_i\right) = \frac{1}{(1+y)^2}\sum_i w_i t_i^2 + \frac{1}{(1+y)}D \tag{A3}$$

Since the weights w_i are all nonnegative and sum to one, Jensen's Inequality implies

$$\sum_i w_i t_i^2 \geq \left(\sum_i w_i t_i\right)^2.$$

Therefore,

$$\frac{1}{(1+y)^2}\sum_i w_i t_i^2 \geq D^2 \qquad \text{(from Eqn (A2)).}$$

Combining this result with Eqns (A1) and (A3) yields

$$D'(y) = D^2 - C \leq -\frac{1}{(1+y)}D < 0.$$

Since $D'(y)$ is always negative, duration is strictly decreasing in yield for bullet bonds.