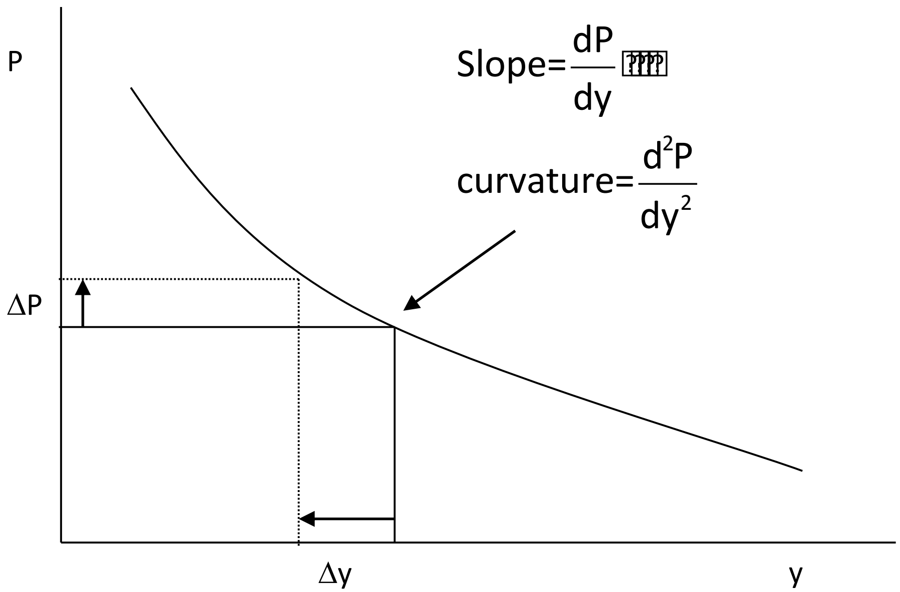
Bond Risk and Portfolios

Under Redevelopment

Interest Rate Risk



$$DP = \frac{dP}{dy} Dy + \frac{1}{2!}\frac{d^2P}{dy^2}(Dy)^2 + \frac{1}{3!}\frac{d^3P}{dy^3}(Dy)^3 + \ldots$$

$$\frac{DP}{P} = \frac{1}{P} \cdot \frac{DP}{Dy} \cdot Dy + \ldots$$

$$D^* \equiv -\frac{1}{P} \cdot \frac{DP}{Dy}$$ ‘Modified’ Duration

$$\frac{\Delta P}{P} = -D^* \cdot \Delta y$$

$$\Delta P = -P \cdot D^* \cdot \Delta y$$

Modified Duration

$$\frac{\Delta P}{P} = -D^{*} \cdot \Delta y$$

If a bond's modified duration, D^{*}, is equal to 4, and the yield decreases by 1% (Δy = -1%) then the first order estimate for the change in bond price is a 4% increase.

$$\frac{\Delta P}{P} = -4 \cdot -1\% = 4\%$$

Modified Duration Calculation

and as $\Delta y \rightarrow dy$

$$\frac{dP}{P} = \frac{1}{P} \cdot \frac{dP}{dy} dy + \frac{1}{2} \cdot \frac{1}{P} \cdot \frac{d^2P}{dy^2} (dy)^2 + \ldots$$

$$P = \sum_{i=1}^{M} \frac{CF_i}{\left(1 + \frac{y}{m}\right)^i} = \sum_{i=1}^{M} CF_i \cdot \left(1 + \frac{y}{m}\right)^{-i}$$

$$\frac{dP}{dy} = -\sum_{i=1}^{M} CF_i \times \frac{i}{m} \times \left(1 + \frac{y}{m}\right)^{-i-1}$$

$$\frac{dP}{dy} = \frac{-1}{\left(1 + \frac{y}{m}\right)} \sum_{i=1}^{M} \frac{i}{m} \cdot \frac{CF_i}{\left(1 + \frac{y}{m}\right)^i}$$

Modified Duration Calculation

$$\frac{1}{P}\cdot\frac{dP}{dy}=\frac{-1}{\left(1+\frac{y}{m}\right)}\sum_{i=1}^{M}\frac{i}{m}\cdot\frac{CF_i}{P\cdot\left(1+\frac{y}{m}\right)^i}$$

$$D\equiv\sum_{i=1}^{M}\frac{i}{m}\cdot w_i$$

Macaulay Duration

$$\frac{1}{P}\cdot\frac{dP}{dy}=\frac{-1}{\left(1+\frac{y}{m}\right)}\sum_{i=1}^{M}\frac{i}{m}\cdot w_i$$

$$\frac{1}{P}\cdot\frac{dP}{dy}=\frac{-D}{\left(1+\frac{y}{m}\right)}$$

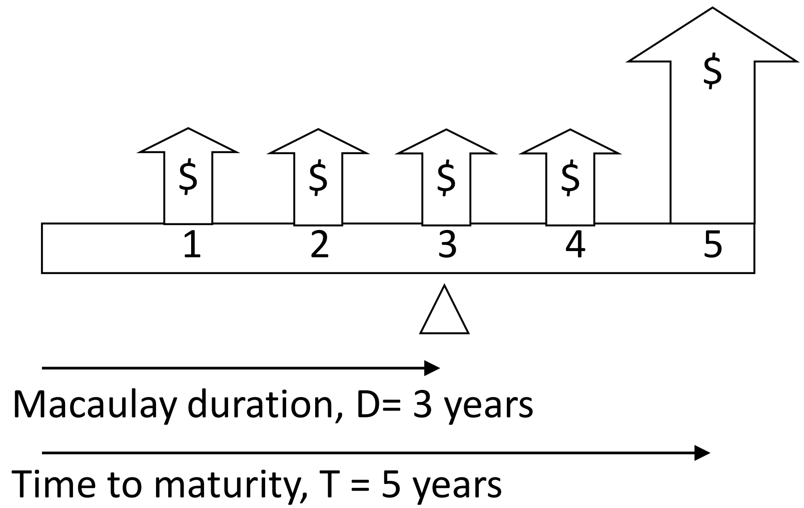
$$w_i\equiv\frac{CF_i}{P\cdot\left(1+\frac{y}{m}\right)^i}$$

Modified Duration Calculation

$$D^* \equiv -\frac{1}{P}\frac{dP}{dy} = \frac{D}{\left(1+\frac{y}{m}\right)}$$

$$D^* \equiv \frac{D}{\left(1+\frac{y}{m}\right)}$$

$$\frac{\Delta P}{P} = -D^* \cdot \Delta y = -\frac{D}{\left(1+\frac{y}{m}\right)} \cdot \Delta y$$



The Macaulay duration is the weighted average time to maturity of all of the bonds cash flows

Bond Duration

Bond duration

From Wikipedia, the free encyclopedia

In finance, the **duration** of a financial asset that consists of fixed cash flows, for example a bond, is the weighted average of the times until those fixed cash flows are received. When an asset is considered as a function of yield, duration also measures the price sensitivity to yield, the rate of change of price with respect to yield or the percentage change in price for a parallel shift in yields.[1][2][3]

The dual use of the word "duration", as both the weighted average time until repayment and as the percentage change in price, often causes confusion. Strictly speaking, Macaulay duration is the name given to the weighted average time until cash flows are received, and is measured in years. Modified duration is the name given to the price sensitivity and is the percentage change in price for a unit change in yield. When yields are continuously compounded Macaulay duration and modified duration will be numerically equal. When yields are periodically compounded Macaulay and modified duration will differ slightly, and there is a simple relation between the two. Modified duration is used more than Macaulay duration.

For bonds with fixed cash flows a price change can come from two sources:

1. The passage of time (convergence towards par). This is of course totally predictable, and hence not a risk.
2. A change in the yield. This can be due to a change in the benchmark yield, and/or change in the yield spread.

Duration Calculation

	Price						Duration	
i	t		CF	DF		DCF	w	w·i/m
1	1.0	$	100.00	0.9009	$	90.09	0.0957	0.0957
2	2.0	$	100.00	0.8116	$	81.16	0.0862	0.1725
3	3.0	$	100.00	0.7312	$	73.12	0.0777	0.2331
4	4.0	$	100.00	0.6587	$	65.87	0.0700	0.2800
5	5.0	$	100.00	0.5935	$	59.35	0.0631	0.3153
6	6.0	$	100.00	0.5346	$	53.46	0.0568	0.3409
7	7.0	$	100.00	0.4817	$	48.17	0.0512	0.3583
8	8.0	$	100.00	0.4339	$	43.39	0.0461	0.3689
9	9.0	$	100.00	0.3909	$	39.09	0.0415	0.3738
10	10.0	$	1,100.00	0.3522	$	387.40	0.4116	4.1165
				P	$	941.11	D	6.6549
							D^*	5.9954

Closed form: Annual coupons

$$D = \frac{1+y}{y} - \frac{(1+y) + N \cdot (c-y)}{c \cdot [(1+y)^N - 1] + y}$$

$$D = \frac{1+11\%}{11\%} - \frac{(1+11\%) + 10 \cdot (10\% - 11\%)}{10\% \cdot [(1+11\%)^{10} - 1] + 11\%} = 6.6549 \text{ years}$$

$1000 bond 10% annual coupon, 10 yrs to maturity, 11% yield

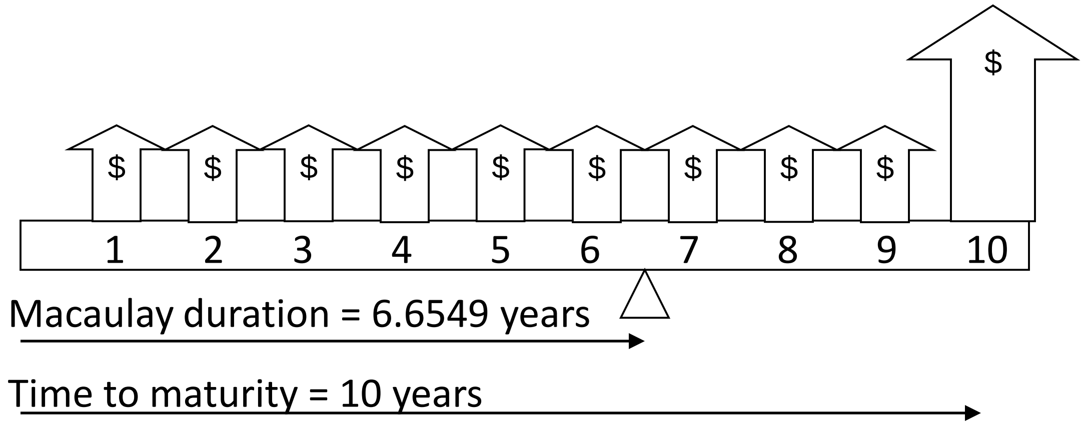
$$D^* = \frac{D}{(1+y)} = \frac{6.6549}{(1+.11)} = 5.9954$$

$$\begin{aligned}\Delta P &= -P \cdot D^* \cdot \Delta y \\ &= -\$941.11 \cdot 5.9954 \cdot 0.01 \\ &= -\$56.42\end{aligned}$$

$$\begin{aligned}\bar{P} &= P - \Delta P \\ &= \$941.11 - \$56.42 \\ &= \$884.68\end{aligned}$$

(exact price at $y = 11\%$ is $887.00)

Duration



When interest rates increase,
bond price decreases,
but coupons can get reinvested at the higher interest rate

The Macaulay duration is the time at which the two effects offset each other (first order accuracy)

The Macaulay duration is also an indicator of interest rate risk, but the modified duration is better

Duration Example

$1000 bond with coupon rate of 10% paid annually. The yield was 11% until just after the t=0 coupon was paid, then the yield jumped to 11.1%. The duration at y=11% was 6.6549 years. Now with a change of yield, the change of total value is neutral at about 6.65 years.

Bond Portfolios

- Portfolio Duration

$$D_P = \sum_{i=1}^{NB} w_i \cdot D_i$$

  - Exact if all bonds have the same yield to maturity
- Bond Funds
- Active v. Passive Portfolio Management
- Portfolio Immunization
- Cash Flow Matching / Bond Ladder

Duration Calculation: Semi Annual Coupon

$1000 bond, 7% coupon yield w/ semi annual coupon, and 12% yield

		Price			Duration	
i	t	CF	DF	DCF	w	w·i/m
1	0.5	$ 35.00	0.9434	$ 33.02	0.0398	0.0199
2	1.0	$ 35.00	0.8900	$ 31.15	0.0375	0.0375
3	1.5	$ 35.00	0.8396	$ 29.39	0.0354	0.0531
4	2.0	$ 35.00	0.7921	$ 27.72	0.0334	0.0668
5	2.5	$ 35.00	0.7473	$ 26.15	0.0315	0.0788
6	3.0	$ 35.00	0.7050	$ 24.67	0.0297	0.0892
7	3.5	$ 35.00	0.6651	$ 23.28	0.0280	0.0982
8	4.0	$ 35.00	0.6274	$ 21.96	0.0265	0.1058
9	4.5	$ 1,035.00	0.5919	$ 612.61	0.7381	3.3216
			P	$ 829.958	D	3.8709
					D^*	3.6518

Closed form: Periodic coupons

$$D = \frac{1+\frac{y}{m}}{\frac{y}{m}} - \frac{\left(1+\frac{y}{m}\right) + M\cdot\left(\frac{c}{m}-\frac{y}{m}\right)}{\frac{c}{m}\cdot\left[\left(1+\frac{y}{m}\right)^{M}-1\right]+\frac{y}{m}}$$

duration in periods, not years

$$D = \frac{1+\frac{12\%}{2}}{\frac{12\%}{2}} - \frac{\left(1+\frac{12\%}{2}\right) + 9\cdot\left(\frac{7\%}{2}-\frac{12\%}{2}\right)}{\frac{7\%}{2}\cdot\left[\left(1+\frac{12\%}{2}\right)^{9}-1\right]+\frac{12\%}{2}}$$

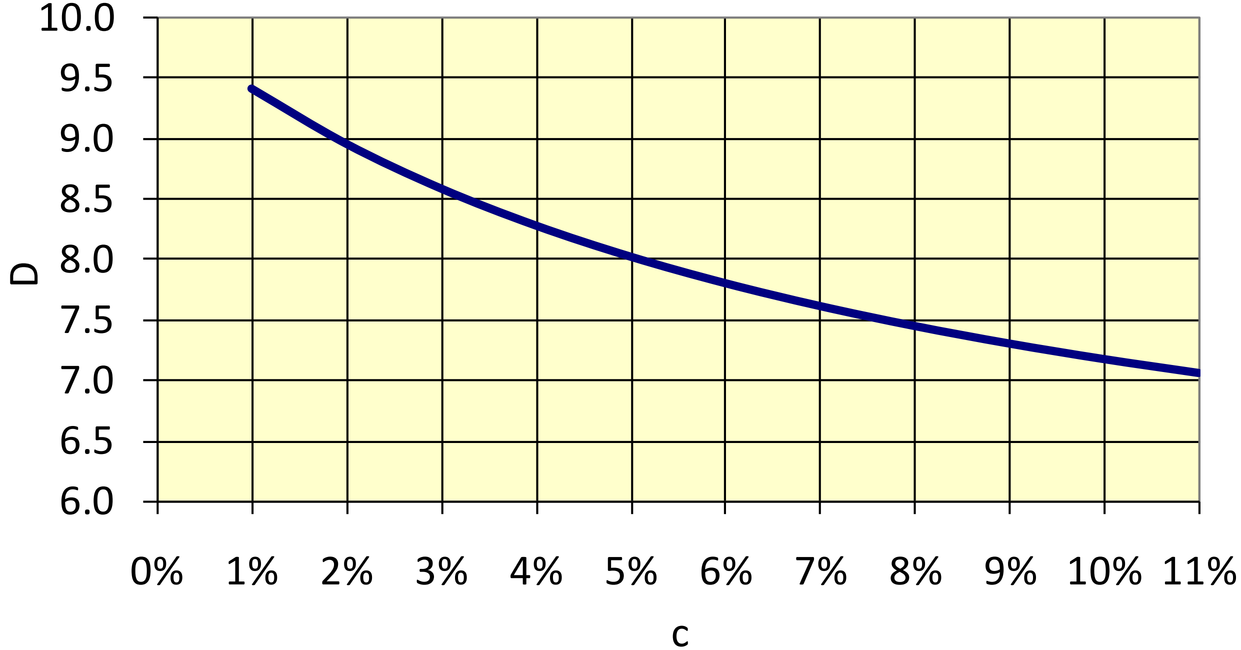
$$= 7.7418 \text{ semi annual periods}$$

$$D = \frac{7.7414}{2} = 3.8709 \text{ years}$$

$$D^* = \frac{3.8709}{\left(1+\frac{12\%}{2}\right)} = 3.6518$$

Duration Determinants

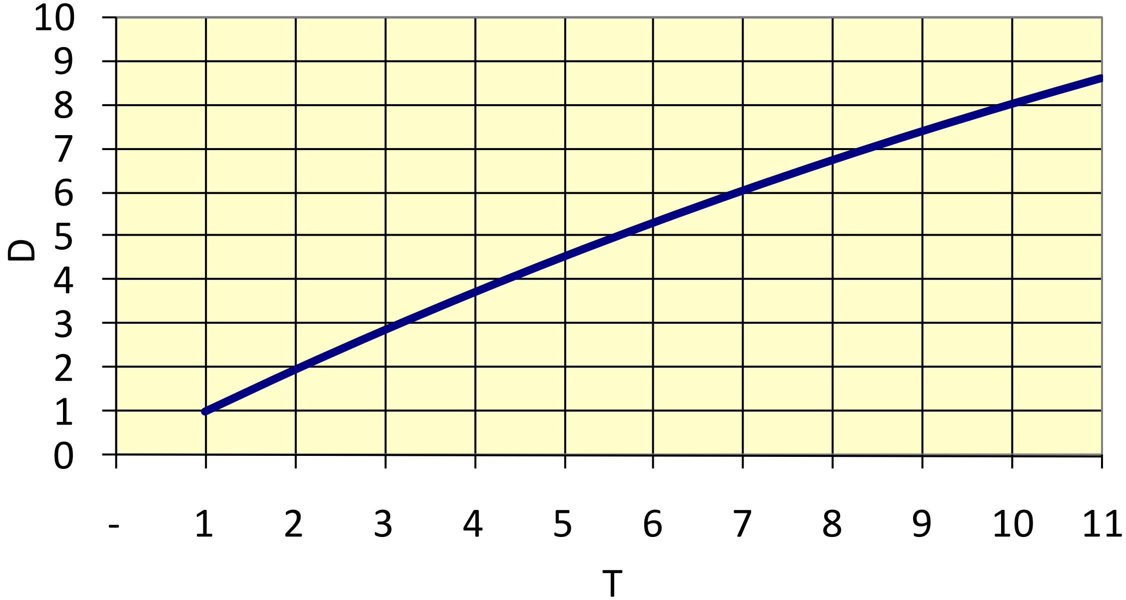
Higher coupon yield decreases duration and interest rate risk



Constant: F=$1000, T=10 yrs, y=7%, m=1

Duration Determinants

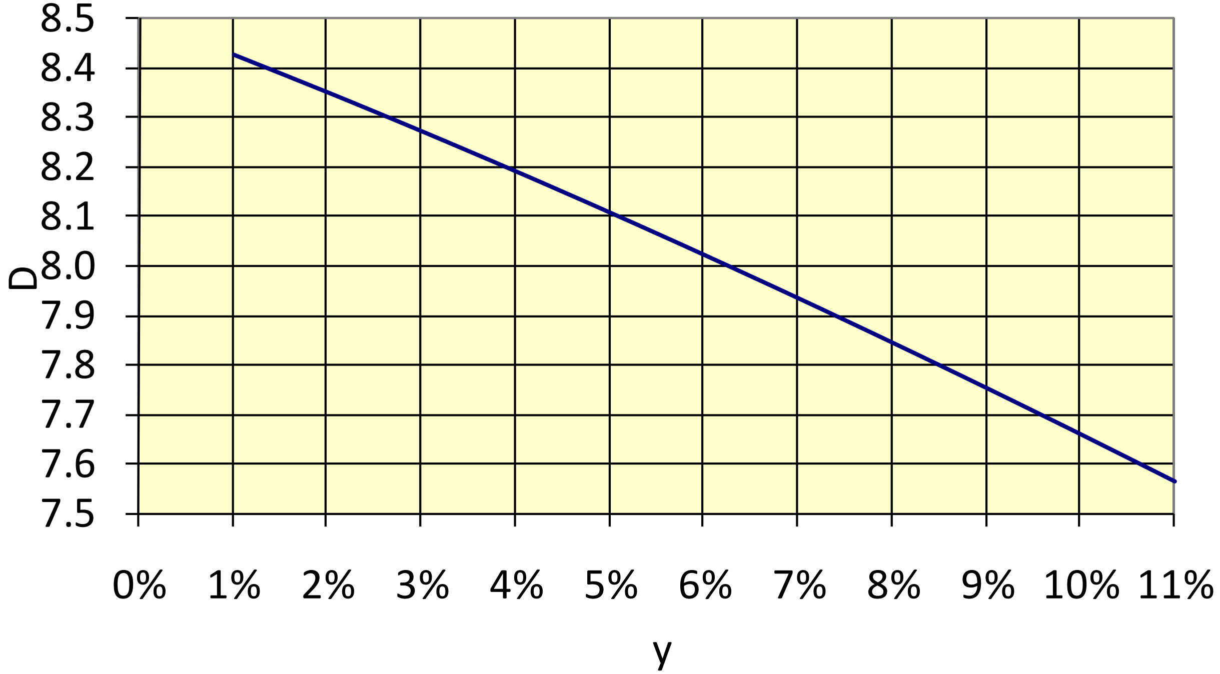
Higher time to maturity increases duration and interest rate risk



Constant: F=$1000, c=5%, y=6%, m=1

Duration Determinants

Higher yield to maturity decreases duration and interest rate risk



Constant: F=$1000, T=10 yrs, c=5%, m=1

Convexity

$$\frac{\Delta P}{P} = -D^* \cdot \Delta y + \frac{1}{2}\frac{d^2P}{dy^2}(\Delta y)^2 + \ldots$$

$$C \equiv \frac{1}{P}\frac{d^2P}{dy^2}$$

Convexity

$$\Delta P = -P \cdot D^* \cdot \Delta y + \frac{1}{2} \cdot P \cdot C \cdot (\Delta y)^2 + \ldots$$

$$\frac{dP}{dy} = -\sum_{i=1}^{M} CF_i \cdot \frac{i}{m} \cdot \left(1 + \frac{y}{m}\right)^{-i-1}$$

$$\frac{d^2P}{dy^2} = \sum_{i=1}^{M} CF_i \cdot \frac{i \cdot (i+1)}{m^2} \cdot \left(1 + \frac{y}{m}\right)^{-i-2}$$

$$\frac{d^2P}{dy^2} = \frac{1}{\left(1 + \frac{y}{m}\right)^2} \sum_{i=1}^{M} \frac{i \cdot (i+1)}{m^2} \cdot \frac{CF_i}{\left(1 + \frac{y}{m}\right)^i}$$

$$\frac{1}{P} \cdot \frac{d^2P}{dy^2} = \frac{1}{\left(1 + \frac{y}{m}\right)^2} \sum_{i=1}^{M} \frac{i \cdot (i+1)}{m^2} \cdot \frac{CF_i}{P \cdot \left(1 + \frac{y}{m}\right)^i}$$

$$C = \frac{1}{P} \cdot \frac{d^2P}{dy^2} = \frac{1}{\left(1 + \frac{y}{m}\right)^2} \sum_{i=1}^{M} \frac{i \cdot (i+1)}{m^2} \cdot w_i$$

Convexity Example

		Price				Duration		Convexity	
i	t	CF	DF	DCF		w	$w \cdot i/m$	$w \cdot i \cdot (i+1)/m^2$	
1	1.0	$ 100.00	0.9009	$ 90.09		0.0957	0.0957	0.1915	
2	2.0	$ 100.00	0.8116	$ 81.16		0.0862	0.1725	0.5174	
3	3.0	$ 100.00	0.7312	$ 73.12		0.0777	0.2331	0.9323	
4	4.0	$ 100.00	0.6587	$ 65.87		0.0700	0.2800	1.3999	
5	5.0	$ 100.00	0.5935	$ 59.35		0.0631	0.3153	1.8918	
6	6.0	$ 100.00	0.5346	$ 53.46		0.0568	0.3409	2.3860	
7	7.0	$ 100.00	0.4817	$ 48.17		0.0512	0.3583	2.8661	
8	8.0	$ 100.00	0.4339	$ 43.39		0.0461	0.3689	3.3198	
9	9.0	$ 100.00	0.3909	$ 39.09		0.0415	0.3738	3.7385	
10	10.0	$ 1,100.00	0.3522	$ 387.40		0.4116	4.1165	45.2810	
			P	$ 941.11		D	6.6549	62.5243	
						D^*	5.9954	50.7461	C

$1000 bond 10% annual coupon, 10 yrs to maturity, 11% yield

$$\Delta P = -P \cdot D^* \cdot \Delta y + \frac{1}{2} \cdot P \cdot C \cdot (\Delta y)^2$$

$$= -\$941.11 \cdot 5.9954 \cdot .01 + .5 \cdot 941.11 \cdot 50.7461 \cdot 0.0001$$

$$= -\$56.42 + \$2.39 = -\$54.03$$

$$\overset{=}{P} = P + \Delta P$$

$$= \$941.11 - \$54.03 = \$887.07$$

Exact price: $887.00

Convexity: Semi Annual Coupon

		Price			Duration		Convexity	
i	t	CF	DF	DCF	w	w·i/m	$w \cdot i \cdot (i+1)/m^2$	
1	0.5	$ 35.00	0.9434	$ 33.02	0.0398	0.0199	0.0199	
2	1.0	$ 35.00	0.8900	$ 31.15	0.0375	0.0375	0.0563	
3	1.5	$ 35.00	0.8396	$ 29.39	0.0354	0.0531	0.1062	
4	2.0	$ 35.00	0.7921	$ 27.72	0.0334	0.0668	0.1670	
5	2.5	$ 35.00	0.7473	$ 26.15	0.0315	0.0788	0.2363	
6	3.0	$ 35.00	0.7050	$ 24.67	0.0297	0.0892	0.3122	
7	3.5	$ 35.00	0.6651	$ 23.28	0.0280	0.0982	0.3926	
8	4.0	$ 35.00	0.6274	$ 21.96	0.0265	0.1058	0.4763	
9	4.5	$ 1,035.00	0.5919	$ 612.61	0.7381	3.3216	16.6079	
			P	$ 829.958	D	3.8709	18.3747	
					D^*	3.6518	16.3534	C

$1000 bond, 7% coupon yield w/ semi annual coupons, and 12% yield

$$\Delta P = -P \cdot D^* \cdot \Delta y + \frac{1}{2} \cdot P \cdot C \cdot (\Delta y)^2$$

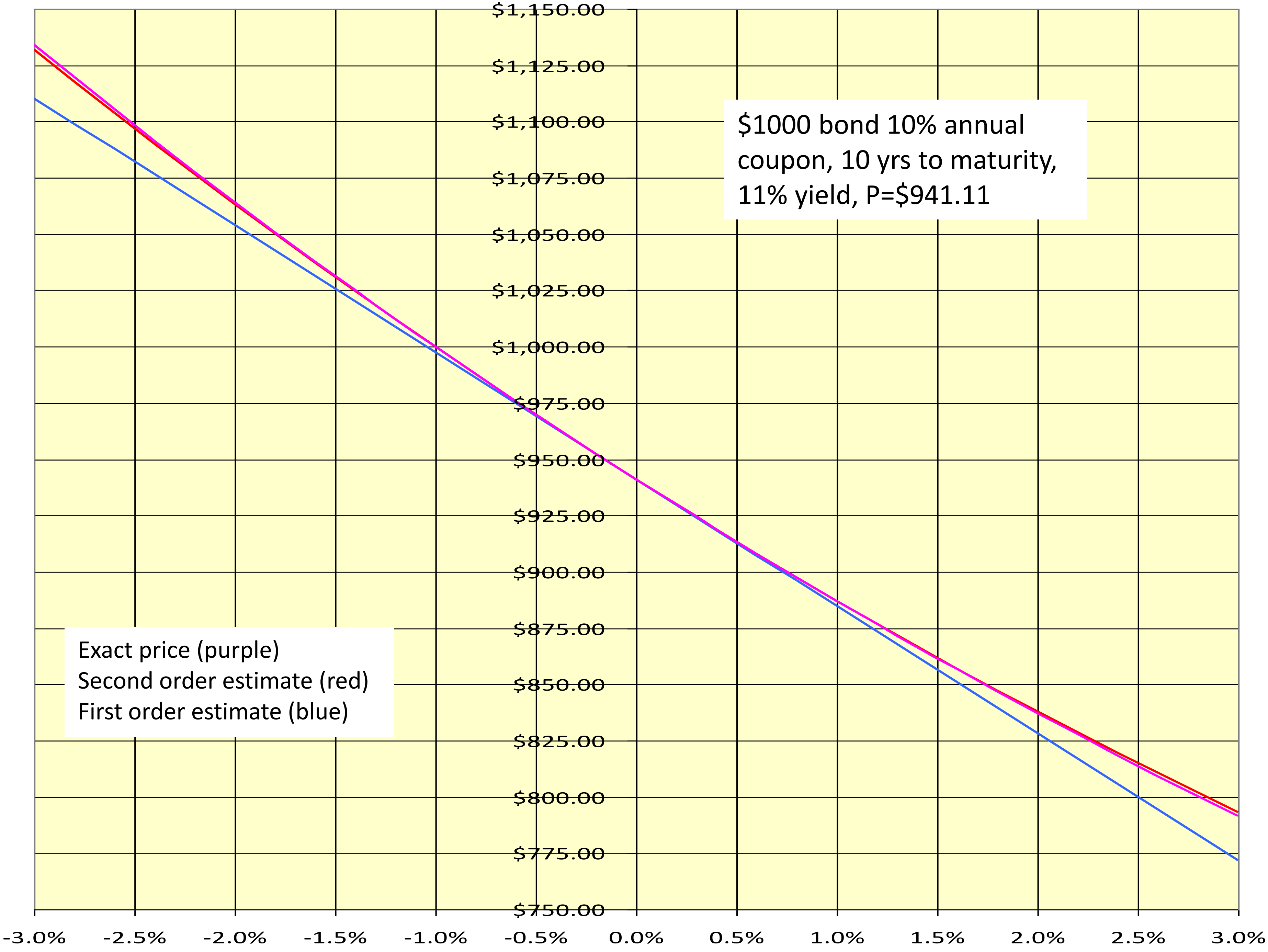
$$= -\$829.96 \cdot 3.6518 \cdot .01 + .5 \cdot \$829.96 \cdot 16.3534 \cdot 0.0001$$

$$= -\$30.31 + \$0.68 = -\$29.63$$

$$\bar{\bar{P}} = P + \Delta P$$

$$= \$829.96 - \$29.63 = \$800.33$$

(exact price is $800.32)


$1000 bond 10% annual coupon, 10 yrs to maturity, 11% yield, P=$941.11
Exact price (purple)
Second order estimate (red)
First order estimate (blue)
$1,150.00
$1,125.00
$1,100.00
$1,075.00
$1,050.00
$1,025.00
$1,000.00
$975.00
$950.00
$925.00
$900.00
$875.00
$850.00
$825.00
$800.00
$775.00
$750.00
-3.0%
-2.5%
-2.0%
-1.5%
-1.0%
-0.5%
0.0%
0.5%
1.0%
1.5%
2.0%
2.5%
3.0%

Convexity

- Note from previous slide
  - Convexity benefits the price increase when yields fall
  - Convexity minimizes the price decrease when yields rise
  - Thus convexity is a positive characteristic

Alternate Calculation of D and C

$$D^* \equiv \frac{1}{\left(1+\frac{y_{nom}}{m}\right)} \sum_{i=1}^{M} \frac{i}{m} \cdot \frac{CF_i}{P \cdot \left(1+\frac{y_{nom}}{m}\right)^i}$$

$$C = \frac{1}{\left(1+\frac{y_{NOM}}{m}\right)^2} \sum_{i=1}^{M} \frac{i \cdot (i+1)}{m^2} \cdot \frac{CF_i}{P \cdot \left(1+\frac{y_{NOM}}{m}\right)^i}$$

$$\frac{i}{m} = t_i \qquad \frac{i+1}{m} = t_i + \Delta t$$

$$t_i = \frac{i}{m} \qquad y_{EFF} = \left(1+\frac{y_{NOM}}{m}\right)^m - 1$$

$$C = \frac{1}{\left(1+\frac{y_{nom}}{m}\right)^2} \sum_{i=1}^{M} t_i \cdot (t_i + \Delta t) \cdot \frac{CF_i}{P \cdot (1+y_{eff})^{t_i}}$$

$$D^* \equiv \frac{1}{\left(1+\frac{y_{nom}}{m}\right)} \sum_{i=1}^{M} t_i \cdot \frac{CF_i}{P \cdot (1+y_{eff})^{t_i}}$$

For semi-annual coupons m=2

$$C = \frac{1}{(1+y_{eff})} \sum_{i=1}^{M} t_i \cdot (t_i + \Delta t) \cdot \frac{CF_i}{P \cdot (1+y_{eff})^{t_i}}$$

Alternate Calculation of D and C

		Price			Duration		Convexity	
i	t	CF	DF	DCF	w	w·t	w·t·(t+Δt)	
1	0.5	$ 35.00	0.9434	$ 33.02	0.0398	0.0199	0.0199	
2	1.0	$ 35.00	0.8900	$ 31.15	0.0375	0.0375	0.0563	
3	1.5	$ 35.00	0.8396	$ 29.39	0.0354	0.0531	0.1062	
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5	2.5	$ 35.00	0.7473	$ 26.15	0.0315	0.0788	0.2363	
6	3.0	$ 35.00	0.7050	$ 24.67	0.0297	0.0892	0.3122	
7	3.5	$ 35.00	0.6651	$ 23.28	0.0280	0.0982	0.3926	
8	4.0	$ 35.00	0.6274	$ 21.96	0.0265	0.1058	0.4763	
9	4.5	$ 1,035.00	0.5919	$ 612.61	0.7381	3.3216	16.6079	
			P	$ 829.96	D	3.8709	18.3747	
					D^*	3.6518	16.3534	C

F=$1000

c=7% with semi-annual coupons, m=2

C=$35

y_{NOM}=12%

Δt=0.5 yrs

T=4.5 yrs

$$y_{EFF} = \left(1 + \frac{12\%}{2}\right)^2 - 1 = 12.36\%$$

Active v. Passive Portfolio Management

- Passive
  - Match a bond portfolio index such as those maintained by Lehman Brothers
    - http://www.lehman.com/fi/indices/factsheets.htm#
    - Example Fund
- Active
  - Interest rate forecasting or anticipation
  - Credit spread forecasting or anticipation

Portfolio Immunization

A bond fund manager must have a portfolio valued at $1M after two years and can buy any number of

- Bond A: $1000 par value zero coupon bond with 1 year to maturity
- Bond B: $1000 par value 4.5% annual coupon bond with 3 years to maturity

The yield-to-maturity, y, is 4% for both bonds.

$$D_P = \sum_{i=1}^{M} w_i \cdot D_i$$

	Bond A	Bond B
c		4.5%
y	4%	4%
N	1	3
F	$ 1,000	$ 1,000
C	$ -	$ 45.00
P	$ 961.54	$ 1,013.88
D	1.0000	2.8736

$$\pi_0 = \frac{\$1{,}000{,}000}{(1+4\%)^2} = \$924{,}556.21$$

$$w_A + w_B = 1$$

$$w_A \cdot D_A + w_B \cdot D_B = 2.0$$

$$w_A \cdot 1.0000 + w_B \cdot 2.8736 = 2.0$$

$$w_A \cdot 1.0000 + (1 - w_A) \cdot 2.8736 = 2.0$$

$$w_A = 46.6271\%$$

$$w_B = 53.3729\%$$

Portfolio Immunization

$$\pi_{A0} = 46.6271\% \cdot \$924{,}556.21 = \$431{,}093.81$$

$$\pi_{B0} = 53.3729\% \cdot \$924{,}556.21 = \$493{,}462.41$$

$$n_A = \frac{\$431{,}093.81}{\$961.54} = 448$$ type A bonds

$$n_B = \frac{\$493{,}462.41}{\$1{,}013.88} = 487$$ type B bonds

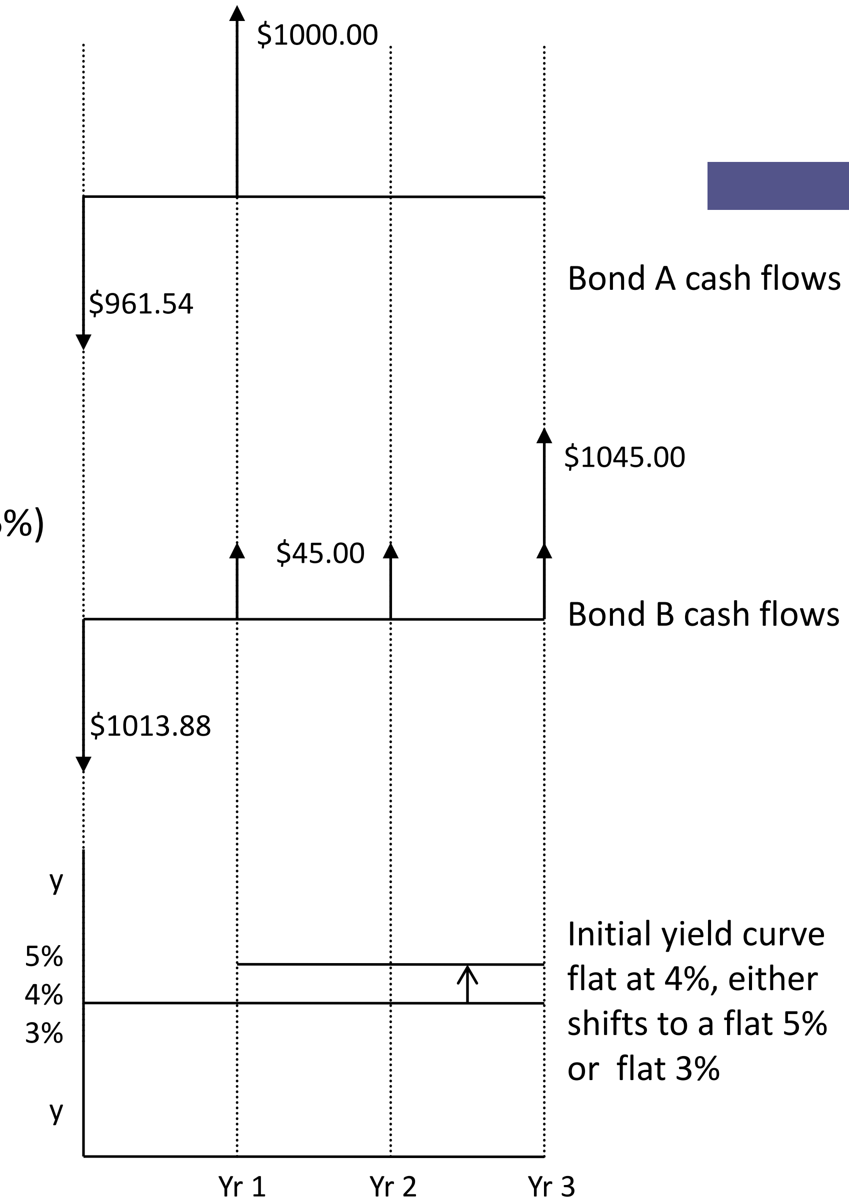
Portfolio Immunization

$$\pi_{A2} = n_A \cdot F_A \cdot (1+y)$$
$$= 448 \cdot \$1000 \cdot (1+5\%) = \$470{,}574$$

$$\pi_{B0} = n_B \cdot C_B \cdot (1+y) + n_B \cdot C_B + n_B \cdot (F_B + C_B)/(1+y)$$
$$= 487 \cdot \$45 \cdot (1+5\%) + 487 \cdot \$45 + 487 \cdot \$1045/(1+5\%)$$
$$= \$22{,}997 + \$21{,}902 + \$484{,}391 = \$529{,}290$$

$$\Pi_2 = \Pi_{A2} + \Pi_{B2} = \$470{,}754 + \$529{,}290$$
$$= \$1{,}000{,}045$$

	P_{A2}	P_{B2}	P_2
3.00%	$ 461,788	$ 538,258	$ 1,000,046
3.25%	$ 462,909	$ 537,117	$ 1,000,026
3.50%	$ 464,029	$ 535,982	$ 1,000,011
3.75%	$ 465,150	$ 534,853	$ 1,000,003
4.00%	$ 466,271	$ 533,729	$ 1,000,000
4.25%	$ 467,392	$ 532,611	$ 1,000,003
4.50%	$ 468,513	$ 531,499	$ 1,000,011
4.75%	$ 469,634	$ 530,392	$ 1,000,025
5.00%	$ 470,754	$ 529,290	$ 1,000,045



Cash Flow Matching

A bond portfolio manager has required minimum payouts for the next six years. She may choose any number of the 10 bonds labeled A – J. To maximize return she should create the portfolio at the lowest cost possible. Each bond has annual coupons.

Year	Reqd Payout
1	$ 15,000
2	$ 17,000
3	$ 19,000
4	$ 20,000
5	$ 23,000
6	$ 27,000

Bond	T	c	y
A	6	10%	9%
B	6	7%	8%
C	5	8%	8%
D	5	6%	7%
E	4	7%	8%
F	4	5%	6%
G	3	4%	5%
H	3	3%	4%
I	2	3%	3%
J	1	0%	3%

Cash Flow Matching

$$\pi_0 = \text{minimize} \sum_{i=1}^{M} P_i \cdot n_i$$

where

P_i , Π	price of each bond i and the bond portfolio
n_i	number of bond i in portfolio
M	number of available bonds from which to choose (10 in this example)

subject to these constraints:

$CF_j \geq CF_j^{reqd}$	for each year j from 1 to 6 in this example
$n_i \geq 0$	no shorting of bond
n_i is an integer	no fractional bonds allowed

Cash Flow Matching

Payout of each type of bond over six years

Bond	A	B	C	D	E	F	G	H	I	J
1	$ 100	$ 70	$ 80	$ 60	$ 70	$ 50	$ 40	$ 30	$ 30	$ 1,000
2	$ 100	$ 70	$ 80	$ 60	$ 70	$ 50	$ 40	$ 30	$ 1,030	
3	$ 100	$ 70	$ 80	$ 60	$ 70	$ 50	$ 1,040	$ 1,030		
4	$ 100	$ 70	$ 80	$ 60	$ 1,070	$ 1,050				
5	$ 100	$ 70	$ 1,080	$ 1,060						
6	$ 1,100	$ 1,070								

Solution from linear programming and integer programming in Excel

Flt and Int Solutions

A	B	C	D	E	F	G	H	I	J	Cost
24.55	0.00	19.02	0.00	14.98	0.00	13.44	0.00	11.10	9.10	$ 92,165
25.00	0.00	18.00	2.00	15.00	0.00	14.00	0.00	11.00	9.00	$ 93,899

Year	Portfolio Payout	Reqd Payout
1	$ 15,000	$ 15,000
2	$ 17,000	$ 17,000
3	$ 19,670	$ 19,000
4	$ 20,110	$ 20,000
5	$ 24,060	$ 23,000
6	$ 27,500	$ 27,000
Portfolio Cost		$ 93,899

Essential Points