Investigating the Role of Systematic and Firm-Specific Factors in Default Risk: Lessons from Empirically Evaluating Credit Risk Models

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Abstract

This paper proposes and empirically investigates a family of credit risk models driven by a two-factor structure for the short-interest rate and an additional third factor for firm-specific distress, using the reduced-form framework of Duffie and Singleton (1999). The set of firm-specific distress factors analyzed in the study include leverage, book-to-market, profitability, equity-volatility, and distance-to-default. Our estimation approach and performance yardsticks show that interest rate risk is of first-order importance for explaining variations in single-name defaultable coupon bond yields and credit spreads. When applied to low-grade bonds, a credit risk model that takes leverage into consideration reduces absolute yield mispricing by as much as 30% relative to a competing model that ignores leverage. None of the distress factors improve performance for high-grade bonds. A strategy relying on traded Treasury instruments is surprisingly effective in dynamically hedging credit exposures for firms in our sample.

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I Introduction

The paradigm that default occurs when a continuous-process such as firm-value reaches a default boundary has long-standing in finance (see Merton (1974) and the refinements in Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001)). In this conceptually elegant structural approach, default is modeled using a predictable stopping time and default probabilities are systematically connected to firm-leverage. There is wide consensus, however, that structural models are difficult to implement. One, the capital structure of the firm is often far too complex to specify recovery to all claimants in the event of default. Two, at the empirical level, structural models generate counterfactually low short-term credit spreads for high quality borrowers (Duffee (1999), Collin-Dufresne and Goldstein (2001), and Longstaff and Schwartz (1995)).

Building credit risk models as the basis for evaluating default exposures remains a fundamental issue. Theoretical research continues to shed wisdom on the qualitative nature of credit spreads and their dependencies on essential features of the defaultable contract such as credit rating of the participating parties and firm-specific/systematic default characteristics. For instance, Jarrow and Turnbull (1995) propose a framework where the underlying asset or the counterparty may default. Duffie and Singleton (1997, 1999), Lando (1998), and Collin-Dufresne, Goldstein, and Hugonnier (2004) treat default as an unpredictable event governed by the instantaneous probability of default; Madan and Unal (1998) analytically decompose the risk of default into components related to timing and recovery. Each of these contributions view default as occurring at a surprise stopping time. In related work, Jarrow, Lando, and Turnbull (1997) develop theoretical models where the bankruptcy process obeys a discrete-state space Markov chain in credit rating.

While theoretical advances have been made in interpreting credit risks and in parameterizing the price of credit sensitive securities there is a relative paucity of empirical studies that investigate the relevance of leverage, distance-to-default, and other systematic/firm-specific factors in the surprise stopping time default approach. Which characteristics capture variations in default risk? Which credit risk model is suitable for marking-to-market defaultable securities? Which model performs the best in hedging dynamic credit exposures? Are single-name valuation errors correlated, and, if so, what is the possible source of this covariation? Empirical investigations of credit risk models attempting to analytically capture patterns of structural dependencies on theoretically interpretable grounds have become even more desirable in light of Basle committee recommendations on managing default risk.

The class of credit risk models we empirically analyze share some features in common. One, our characterization of credit risk relies on the Duffie-Singleton (1999) assumption that recovery is proportional to the pre-defaultable debt value. Two, complementing the empirical study of Duffee

(1999), we develop a set of three-factor credit risk models that depend on systematic and firm-specific distress characteristics. In one specific model we link the instantaneous likelihood of default to the short interest-rate, the stochastic long-run interest rate, and firm-specific leverage where the leverage dynamics is modeled in both level and logs. These models are empirically tractable and account for both interest rate risk and firm-specific default risk. The latter appealing feature inherits the flavor of the structural models of Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001). To broaden empirical content we adopt alternative plausible models for firm-specific distress based on book-to-market, profitability, distance-to-default, and equity-volatility, and consider a general risk premium specification for the interest rate as in Duffee (2002). Having firm-specific characteristics in a default model, in theory, offers flexibility in explaining the behavior of single-name credit spreads.

Econometric implementation of default-theoretic models provides crucial insights into the sources of firm-level credit risk. First, incorporating a stochastic mean interest-rate feature into the working of credit risk models lead to more realistic credit curves. Thus, substantial care must be taken in specifying the factor structure of default-free bonds. Second, our decomposition shows that interest-rate risk captures the first-order impact of movements up to BBB-rated corporate credit yields. Third, controlling for interest-rate considerations in a credit risk model the incremental pricing improvement with leverage is economically large and statistically significant for low-grade bonds. Differentiating our work from existing studies, the leverage generalization produces a 30% reduction in absolute yield errors relative to a model that excludes leverage. The estimation strategy shows that the credit risk model with leverage appears to be least misspecified among our three-factor default risk models. Overall, these conclusions hold irrespective of whether the performance of credit risk models is based on defaultable yields or on credit spreads. On balance, credit risk models have quantitative pricing performance that could be adequate for marking risk exposures. In the spirit of Collin-Dufresne, Goldstein, and Martin (2001), we investigate the relationship between default model mispricing and a set of economy-wide systematic factors, and find their joint explanatory power to be fairly small for the majority of single-names.

Hedge portfolios that combine a short-position in the defaultable bond and a long-position in the risk-free bond of the same maturity using two Treasury discount bonds and the firm's equity remains sensitive to changes in debt and book values. Consistent with the pricing exercise, we observe that systematic risk exposures are crucial to the profit/loss accounts of the hedger. Given the mapping that exists between the price of corporate bonds, Treasury bonds and credit default swaps, our analysis potentially suggests that, for our sample of firms, the price of credit protection can be effectively hedged using Treasury bonds. This hedge is possible when credit exposures are tied to the Treasury markets.

Section II presents the theoretical framework to price defaultable coupon bonds and relates credit yields to systematic and firm-specific default characteristics. The defaultable coupon bond sample and empirical issues are described in Section III. Section IV applies Kalman filtering technique to a cross-section and time-series of Treasury yields. Section V employs a maximum-likelihood procedure jointly consistent with risk-neutral and physical dynamics of firm-specific distress and the cross-correlation between bonds of distinct maturities. In Section VI we present the pricing and hedging results and conduct specification diagnostics and GMM tests. Concluding remarks are provided in Section VII.

II Reduced-Form Defaultable Coupon Bond Models

This section outlines the Duffie and Singleton (1999) model for pricing risky debt and the assumptions that lead to the class of credit risk models pursued empirically.

A Valuation Framework

Defaultable coupon bonds, like other debt contracts, are defined by their promised stream of cash flows through time. Typically these consist of a promised face value, F, to be paid at maturity T and a stream of coupon payments to be paid in the interim. To accommodate both discrete and continuous coupon payments we denote by $C(t)$ the non-decreasing function of cumulated coupon payments until time t. In addition to specifying the promised payments, defaultable debt recognizes that there is a random time $\mathcal{T}$ at which default occurs. At this time, a payment $y(\mathcal{T})$ is made in fulfillment of the debt obligation. The recovery, $y(\mathcal{T})$, if any, is generally far below the value of remaining promised payments. We associate with $\mathcal{T}$, the unit step function $\phi(t)$:

$$\phi(t) \quad = \quad \begin{cases} 1 & t \geq \mathcal{T} \\ 0 & \text{Otherwise.} \end{cases} \tag{1}$$

The defaultable debt contract can now be defined by the entities: $(F,T,C(t),y(t))$ that are presumed adapted to a Brownian sub-filtration $(\mathcal{G}_t, 0 \leq t \leq T)$ of the information filtration $(\Im_t, 0 \leq t \leq T)$ of a probability space $(\Omega, \Im, P)$ satisfying the usual technical conditions. The filtration $\Im_t$ extends $\mathcal{G}_t$ by the knowledge of the default time process $\phi(t)$.

Suppose $b(t) \equiv \exp\left(\int_0^t r(s)\,ds\right)$ is the accumulation of the money market account where the spot interest rate is $r(t)$. According to Duffie (1996), the absence of arbitrage is ensured by the existence of a probability measure Q equivalent to P under which the money market discounted gains processes for all assets are martingales. It follows that the time t price of the defaultable coupon bond with maturity τ periods from time t, denoted $P(t,\tau)$, is:

$$\begin{aligned} P(t,\tau)\, 1_{\mathcal{T}>t} &= E^Q \left\{ \int_t^{t+\tau} \frac{b(t)}{b(u)} \left(1-\phi(u)\right) dC(u) + \frac{b(t)}{b(t+\tau)} \left(1-\phi(t+\tau)\right) F \right. \\ &\quad \left. + \int_t^{t+\tau} \frac{b(t)}{b(u)} (1-\phi(u))\, y(u)\, d\phi(u) \,|\, \mathcal{G}_t \right\}, \end{aligned} \tag{2}$$

where E^Q is expectation operator under the probability measure Q. Henceforth, we will abbreviate the left-hand side of equation (2) to $P(t,\tau)$ with the understanding this price is only observed assuming that default has not yet occurred. The first integral in equation (2) accounts for the stream of coupon payments received as long as there is no default and stopped at the default time. The second term accounts for the promised face value given no default and the last integral accounts for the single recovery at the default time (note that $d\phi(u)$ equals the Kronecker delta function at $u = \mathcal{T}$, and is zero at times other than the default time when it is one). The formulation of default in (2) is consistent with the surprise stopping time approach (Duffie and Singleton (1999), Lando (1998), and Collin-Dufresne, Goldstein, and Hugonnier (2004)).

For a surprise default time that is a stopping time, there exists a positive process $h(t)$, called the hazard rate process, such that $\phi(t) - \int_0^t (1-\phi(u))\, h(u)\, du$ is a martingale. When we have a martingale under probability Q, we refer to $h(t)$ as the risk-neutral hazard rate process. Heuristically speaking, $h(t)\, dt$ models the probability of default in the interval $(t, t+dt)$, the instantaneous likelihood of default. The process $h(t)$ is adapted to $\mathcal{G}_t$. In this case we may write,

$$E^Q \left[(1-\phi(u)) \,|\, \mathcal{G}_u \right] = \exp\left(- \int_0^u h(s)\, ds \right). \tag{3}$$

Using (3) and iterated expectations, first computing expectations with respect to $\mathcal{G}_u$ and then again with respect to $\mathcal{G}_t$, it follows that

$$\begin{aligned} P(t,\tau) &= E^Q \left\{ \int_t^{t+\tau} \frac{b(t)}{b(u)} \exp\left(- \int_t^u h(s)\, ds \right) dC(u) + \frac{b(t)}{b(t+\tau)} \exp\left(- \int_t^{t+\tau} h(s)\, ds \right) F \right. \\ &\quad \left. + \int_t^{t+\tau} \frac{b(t)}{b(u)} \exp\left(- \int_t^u h(s)\, ds \right) y(u)\, h(u)\, du \,|\, \mathcal{G}_t \right\}. \end{aligned} \tag{4}$$

Unlike equation (2), the pricing equation (4) eliminates reference to the discontinuous process $\phi(t)$ and reduces the problem of pricing defaultable debt to that of pricing non-defaultable debt with an altered discount rate and cash flow claim.

B Duffie-Singleton (1999) Model

If we now follow Duffie and Singleton (1999) and define recovery as a proportion, $\nu(t)$, of the pre-default value of the defaultable debt so that

$$y(t) = \nu(t)\, P\,(t_{-}, \tau)\,, \tag{5}$$

then the defaultable debt price is (see Duffie and Singleton (1999) for intermediate steps):

$$\begin{aligned} P(t,\tau) \;\; = \;\; & E^{Q}\left\{ \int_{t}^{t+\tau} \exp\left(- \int_{t}^{t+u} [r(s) + h(s)\,(1-\nu(s))]\, ds \right) dC(u) + \right. \\ & \left. F \exp\left(- \int_{t}^{t+\tau} [r(s) + h(s)\,(1-\nu(s))]\, ds \right) \mid \mathcal{G}_t \right\}. \end{aligned} \tag{6}$$

Because it is not possible in the Duffie-Singleton approach to separate the effects of the hazard rate process $h(t)$ from that of the loss process $(1-\nu(t))$, the aggregate **defaultable discount rate** can be defined as:

$$R(t) \equiv r(t) \,+\, h(t)\,[1-\nu(t)]. \tag{7}$$

This discount rate consolidates time value, default arrival rates and recovery. With deterministic and continuous coupon rate $c(t)$, the debt equation can be simplified as:

$$P(t,\tau) = \int_{0}^{\tau} c(t+u) \times P^{*}(t,u)\, du \,+\, F \times P^{*}(t,\tau), \tag{8}$$

where

$$P^{*}(t,u) = E^{Q}\left\{ \exp\left(- \int_{t}^{t+u} R(s)\, ds \right) \mid \mathcal{G}_t \right\}, \tag{9}$$

is the price of the unit face defaultable zero-coupon bond with maturity $t+u$.

C Specification of the Defaultable Discount Rate

In this subsection we specify $R(t)$ that lead to empirically testable closed-form models for the price of defaultable coupon debt. Consider the family of aggregate defaultable discount rate models shown below:

$$R_n(t) = \Lambda_{0,n} \,+\, \Lambda_{r,n}\, r(t) \,+\, \Lambda_{s,n}\, S_n(t), \qquad\qquad n = 1, \cdots, N, \tag{10}$$

where $S_n(t)$ surrogates firm-specific distress indexed by n and postulated either as the level or the logarithmic of the variable. While incorporating both an economy-wide and a firm-specific variable, the linearity of $R_n(t)$ in $r(t)$ and $S_n(t)$ affords analytical tractability. To keep a parsimonious factor structure we assume distress is driven by a single-factor $S_n(t)$. However, our characterization of $R_n(t)$ is versatile to accommodate a $\mathcal{K}$-factor model of $S_n(t)$. Such an angle is pursued further in the empirical section.

Equation (10) indicates that cross-sectional variations in credit risk are due to cross-sectional variations in $\Lambda_{0,n}$, $\Lambda_{r,n}$, $\Lambda_{s,n}$, and S_n. Λ_0 measures the unconditional instantaneous credit yield. If $\Lambda_r > 0$, defaultable yields are positively related to interest rates; existing evidence on co-movements between Treasury and corporate yield curves is indicative of $\Lambda_r > 0$ (see, e.g., Duffee (1998) and Collin-Dufresne, Goldstein, and Martin (2001)). Assuming $S_n(t)$ is positively associated with firm-specific distress, the credit quality of the firm deteriorates when distress rises (provided $\Lambda_{s,n} > 0$).

The parameters of $R_n(t)$ can vary systematically with credit rating as modeled by Jarrow, Lando, and Turnbull (1997). Two special cases of (10) are of relevance. **CASE 1:** Setting $\Lambda_0 = \Lambda_s = 0$ and $\Lambda_r = 1$ gives the term structure of default-free bonds. **CASE 2:** If $\Lambda_{s,n} = 0$ one obtains a class of credit risk models driven solely by systematic factors (Duffie and Singleton (1997)).

D Term Structure of Default-Free Bonds

Let $z(t)$ represent the long-run mean of the short rate and $X(t) \equiv \begin{pmatrix} r(t) \\ z(t) \end{pmatrix}$. The following two-factor model is adopted on empirical and theoretical grounds:

$$dX(t) \;=\; \Pi^* \left[\Theta^* - X(t)\right] dt + \sigma\, d\widetilde{W}(t), \tag{11}$$

where $\Pi^* \equiv \begin{pmatrix} \kappa_r + \lambda_r & -\kappa_r \\ 0 & \kappa_z + \lambda_z \end{pmatrix}$, $\Theta^* \equiv \begin{pmatrix} \frac{\kappa_r(\kappa_z\,\mu_z - \lambda_{0,z})}{(\kappa_r+\lambda_r)(\kappa_z+\lambda_z)} - \frac{\lambda_{0,r}}{(\kappa_r+\lambda_r)} \\ \frac{\kappa_z\mu_z - \lambda_{0,z}}{(\kappa_z+\lambda_z)} \end{pmatrix}$, $\sigma \equiv \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_z \end{pmatrix}$, and $\widetilde{W}(t) \equiv (\widetilde{\omega}_r, \widetilde{\omega}_z)'$ are standard Brownian motions under the risk-neutral measure with correlation $\rho_{r,z}$. κ_r (κ_z) is the mean-reversion rate for $r(t)$ ($z(t)$) and μ_z is the long-run mean for $z(t)$. The risk-neutral dynamics (11) results by assuming that the risk premium associated with $X(t)$ is $\begin{pmatrix} \lambda_{0,r} \\ \lambda_{0,z} \end{pmatrix} + \begin{pmatrix} \lambda_r & 0 \\ 0 & \lambda_z \end{pmatrix} X(t)$, and the $X(t)$ dynamics under the physical measure is as specified in (24). Like Duffee (2002), this four-parameter risk premium specification permits the risk-neutral long-term mean and the coefficient of mean-reversion to depart from their statistical counterparts.

Solving a standard valuation equation, the price of default-free discount bond is:

$$B(t,\tau) \equiv E^Q\left\{\exp\left(-\int_t^{t+\tau} r(s)\,ds\right)\right\} = \exp\left[-\overline{\alpha}(\tau) - \overline{\beta}(\tau)\,r(t) - \overline{\gamma}(\tau)\,z(t)\right], \tag{12}$$

where $\overline{\beta}(\tau) \equiv \frac{1-\exp[-(k_r+\lambda_r)\,\tau]}{k_r+\lambda_r}$, $\overline{\gamma}(\tau) \equiv \frac{1-\exp[-(\kappa_z+\lambda_z)\tau]}{\kappa_z+\lambda_z} + \frac{\exp[-(\kappa_z+\lambda_z)\,\tau]-\exp[-(\kappa_r+\lambda_r)\,\tau]}{\kappa_z+\lambda_z-\kappa_r-\lambda_r}$ and $\overline{\alpha}(\tau) \equiv -\frac{1}{2}\,\sigma_r^2 \int_0^\tau \overline{\beta}^2(u)\,du - \frac{1}{2}\,\sigma_z^2 \int_0^\tau \overline{\gamma}^2(u)\,du + (\kappa_z\,\mu_z - \lambda_{0,z}) \int_0^\tau \overline{\gamma}(u)\,du - \rho_{r,z}\,\sigma_r\,\sigma_z \int_0^\tau \overline{\beta}(u)\,\overline{\gamma}(u)\,du - \lambda_{0,r} \int_0^\tau \overline{\beta}(u)\,du$. The yield-to-maturity, to be used in the Kalman filtering estimation, is:

$$y(t,\tau) \quad = \quad \frac{\overline{\alpha}(\tau)}{\tau} + \frac{\overline{\beta}(\tau)}{\tau}\,r(t) + \frac{\overline{\gamma}(\tau)}{\tau}\,z(t). \tag{13}$$

Buhler, Uhrig, Walter, and Webber (1999) and Dai and Singleton (2000) make the point that the one-factor model is statistically rejected in favor of a two-factor model. Moreover, Litterman and Scheinkman (1991), Chen and Scott (1993, 2003) and Duffee (1998) show that movements in the level and slope of the yield curve capture a large fraction of the Treasury term structure variations.

E Term Structure of Single-Name Defaultable Bonds

For its analytical tractability, we assume, under the risk-neutral measure, that the distress factor obeys:

$$dS_n(t) = [\kappa_s\,\mu_s - (\kappa_s\,+\,\lambda_s)\,S_n(t)\,]\,dt\,+\,\sigma_s\,d\widetilde{\omega}_s(t), \tag{14}$$

where $\widetilde{\omega}_s$ is a standard Brownian motions under the risk-neutral measure and $\lambda_s\,S(t)$ is the risk premium for $S(t)$. Let $\rho_{r,s} \equiv \mathrm{Cov}_t\,(\widetilde{\omega}_r, \widetilde{\omega}_s)$ and $\mathrm{Cov}_t\,(\widetilde{\omega}_s, \widetilde{\omega}_z) = 0$. The form of (14) is a robust four-parameter specification for the distress factor.

Consider the price of a unit face defaultable discount bond with τ periods to maturity. Using the risk-neutral dynamics of $r(t)$ and $S_n(t)$ and solving (9), we have:

$$P^*(t,\tau) = \exp\left[-\alpha(\tau) - \beta(\tau)\,r(t) - \gamma(\tau)\,z(t) - \theta(\tau)\,S(t)\right], \tag{15}$$

where:

$$\beta(\tau) \quad \equiv \quad \frac{\Lambda_r\,[1 - e^{-(\kappa_r+\lambda_r)\tau}]}{\kappa_r+\lambda_r}, \tag{16}$$

$$\gamma(\tau) \quad \equiv \quad \frac{\kappa_r\Lambda_r\,[1 - e^{-(\kappa_z+\lambda_z)\tau}]}{(\kappa_r+\lambda_r)(\kappa_z+\lambda_z)} + \frac{\kappa_r\Lambda_r\,[e^{-(\kappa_z+\lambda_z)\tau} - e^{-(\kappa_r+\lambda_r)\tau}]}{(\kappa_r+\lambda_r)[(\kappa_z+\lambda_z)-(\kappa_r+\lambda_r)]}, \tag{17}$$

$$\theta(\tau) \quad \equiv \quad \frac{\Lambda_s\,[1 - e^{-(\kappa_s+\lambda_s)\tau}]}{\kappa_s+\lambda_s}, \tag{18}$$

and

$$\begin{aligned}\alpha(\tau) \quad \equiv \quad & \Lambda_0\,\tau - \frac{1}{2}\,\sigma_r^2 \int_0^\tau \beta^2(s)\,ds + \kappa_s\,\mu_s \int_0^\tau \theta(s)\,ds - \frac{1}{2}\,\sigma_z^2 \int_0^\tau \gamma^2(s)\,ds - \frac{1}{2}\,\sigma_s^2 \int_0^\tau \theta^2(s)\,ds \\ & + \kappa_z\,\mu_z \int_0^\tau \gamma(s)\,ds - \rho_{r,s}\,\sigma_r\,\sigma_s \int_0^\tau \beta(s)\,\theta(s)\,ds - \rho_{r,z}\,\sigma_r\,\sigma_z \int_0^\tau \beta(s)\,\gamma(s)\,ds.\end{aligned} \tag{19}$$

We first observe that $P^*(t,\tau)$ is exponential affine in three state variables: the short interest rate, the stochastic long-run mean interest rate and the firm-specific distress factor. The bond price is driven by 18 structural parameters: there are 3 parameters in the defaultable discount rate specification, 10 in the interest rate process and 5 associated with the dynamics of firm-specific distress. Second, under the positivity of Λ_r, the defaultable discount bond price is negatively related to $r(t)$ and $z(t)$:

$$\Delta_r^*(t,\tau) \equiv \frac{\partial P^*(t,\tau)}{\partial r} = -\beta(\tau)\, P^*(t,\tau) < 0, \tag{20}$$

$$\Delta_z^*(t,\tau) \equiv \frac{\partial P^*(t,\tau)}{\partial z} = -\gamma(\tau)\, P^*(t,\tau) < 0. \tag{21}$$

The bond price is also negatively associated with the distress factor, as seen by

$$\Delta_s^*(t,\tau) \equiv \frac{\partial P^*(t,\tau)}{\partial S} = -\theta(\tau)\, P^*(t,\tau) < 0, \tag{22}$$

provided $\Lambda_s > 0$. The expressions for the local risk exposures can be employed to develop hedges for marked-to-market risks.

Third, the yield-to-maturity of the defaultable discount bond, for maturity τ, is

$$Y^{**}(t,\tau) \equiv -\frac{\log[P^*(t,\tau)]}{\tau} = \frac{\alpha(\tau) + \beta(\tau)\, r(t) + \gamma(\tau)\, z(t) + \theta(\tau)\, S(t)}{\tau} \tag{23}$$

and decomposes single-name credit yield into a systematic component and a firm-specific component. Letting $\tau \to \infty$, the asymptotic defaultable yield is: $Y^{**}(t,\infty) = \Lambda_0 - \frac{\Lambda_r^2\sigma_r^2}{2\kappa_r^2} + \mu_z\Lambda_r - \frac{\rho_{r,z}\sigma_r\sigma_z\Lambda_r^2}{\kappa_r\kappa_z} - \frac{\sigma_z^2\Lambda_r^2}{2\kappa_z^2} + \Lambda_s\mu_s - \frac{\sigma_s^2\Lambda_s^2}{2\kappa_s^2} - \frac{\rho_{r,s}\sigma_r\sigma_s\Lambda_s\Lambda_r}{\kappa_r\kappa_s}$ and $\lim_{\tau\to 0} Y^{**}(t,\tau) = \Lambda_{0,n} + \Lambda_{r,n}\, r(t) + \Lambda_{s,n}\, S_n(t)$. The three-factor model offers the flexibility to produce various yield curve shapes including double humped. The yield curve of the defaultable coupon bond, $Y^*(t,\tau)$, inherits the same structure as displayed by the zeros. Compared to yield dynamics (23), the hazard rate specification in Duffee (1999) is based on two latent-factors for the interest rate process plus a single-factor that is independent of the interest rate process.

Five candidates for $S_n(t)$ are selected for their empirical plausibility. As discussed below, each choice leads to a distinct testable model of credit risk.

1. Assume S_n is the level or the logarithmic of firm-leverage. Standard corporate finance theory suggests that leverage captures firm-level distress. Leverage is also a key ingredient in the structural models of Merton (1974), Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001). We will refer to this model as the **Leverage Model**.

2. Let S_n be the firm's book value divided by market value (i.e., Book-to-Market) or its logarithmic transformation. According to Fama and French (1992), firms with high book-to-market are relatively more distressed with poor cash flow prospects. This is the **B/M Model**.

3. The next candidate for S_n is one minus profitability ratio and reflects the firms' ability to honor debt obligations out of its operating income (Titman and Wessels (1988)). Credit risk models that incorporate profitability concerns is the **Profitability Model**.

4. For the fourth model, S_n is the level or the logarithmic of stock volatility and proxies asset volatility. Based on Merton (1974) this measure of business risk is included in practitioner models of default prediction such as KMV (i.e., credit spreads rise with volatility in Merton). Campbell and Taksler (2003) show that equity volatility explains cross-sectional variation in corporate bond yields. The resulting credit risk model is the **Stock Volatility Model**.

5. Finally, S_n is distance-to-default model of KMV. It captures the number of standard deviations the firm asset-value is away from the default boundary (Crosby and Bohn (2003)). KMV defines distance-to-default as net asset-value normalized by asset-volatility, where the default event is identified as the first time the firm asset-value falls below the default boundary. We call this credit risk model as the **Distance-to-Default Model**.

Depending upon theoretical plausibility and analytical convenience, equations (10) and (14) offer the flexibility to model the observable distress variable in levels or in logs. Guided by Collin-Dufresne and Goldstein (2001), we expand on the set of default risk models by including a logarithmic transformation for leverage, B/M, and volatility (these variables are strictly positive). Our later empirical exercises also present the results of two additional tests. First, we investigate if the chosen dynamics in level or in logs is more consistent with the Gaussian specification for the physical process (31). Second, we assess model fitting-errors with both level and log transformation.

In what follows, the model $\Lambda_s = 0$ will be used to benchmark performance and our focus will be on reduced-form models with identifiable factors that may or may not be tradable. As default is triggered only at maturity, the Merton (1974) model cannot be easily adapted to price defaultable coupon bonds. Unless jumps are added these models imply counterfactually low credit spreads for short-maturity bonds; Collin-Dufresne and Goldstein (2001) establish that one-factor structural models have undesirable long-run yield properties. In addition, Jones, Mason, and Rosenfeld (1984), Wei and Guo (1997), and Eom, Helwege, and Huang (2004) present evidence rejecting such models.

III Single-Name Defaultable Coupon Bonds

For this study we use non-convertible coupon debt with principal in excess of one million dollars taken from Lehman Brothers Fixed Income Database. This database contains the amount of coupon and principal, the month-end flat price, the accrued interest and the maturity date. Each bond is assigned a credit rating and debt issues are classified as callable, putable, subordinated and by sinking fund provisions.

Several exclusion filters are imposed to construct the single-name bond sample. First, trader bid quotes are used (ask quotes are not recorded) and matrix quotes are eliminated. Second, we delete pass-through, asset-backed securities and debt with embedded options. Next, we include non-callable bonds with semi-annual coupons that have maturity greater than one year. As few non-callable bonds were issued prior to March 1989 we limit attention to the sample between March 1989 and March 1998. Bonds with time-to-maturity closest to 2, 7, and 15 years are selected to represent short-term, medium-term and long-term bonds and this classification produces a time-series of monthly bond prices. 183 firms satisfy all the above requirements with no missing observations.

To test credit risk models, five distress factor proxies are constructed:

Leverage: $\mathrm{LEV}_n(\mathrm{t})$, is long-term book value of debt (COMPUSTAT quarterly item 51) divided by firm value. The firm value is the sum of long-term debt and the market capitalization of common equity, M;

Book-to-Market: $\mathrm{B}_n(\mathrm{t})/\mathrm{M}_n(\mathrm{t})$, is computed as the book value of equity (COMPUSTAT quarterly item 59) divided by the market value of equity;

Profitability: $\mathrm{PROFIT}_n(\mathrm{t})$, is one minus operating income (COMPUSTAT quarterly item 21) divided by net sales (COMPUSTAT quarterly item 2);

Volatility: To avoid overlapping observations, $\mathrm{VOL}_n(t)$, is the standard deviation of the daily stock returns during the month;

Distance-to-Default: is based on estimates of asset value, default point, and asset volatility. The exact KMV procedure for constructing distance-to-default is proprietary so we appeal to the measure in Vassalou and Xing (2004). The data is kindly made available to us by Yuhang Xing.

While matching firm sample with CRSP and COMPUSTAT 80 firms did not have data on equity prices and 27 firms have missing data on leverage, book-to-market or profitability. The final sample of 76 firms consists of 16,518 coupon bond observations with no violation of absolute priority rules.

Long-term debt and book values are recorded at the quarterly frequency while leverage and book-to-market series are monthly. To circumvent any look-ahead biases we use debt and book values from the previous quarter to proxy leverage and book-to-market for the next three months. A cubic spline is used to approximate quarterly profitability measure into a monthly profitability measure.

Bonds are classified into credit rating categories using Standard and Poor's ratings: bonds with (numerical) rating score up to 9 are designated as A-rated and with score higher than 9 are designated as BBB-rated (or below). Since the fraction of bonds rated above BBB is 69%, our bond-sample is skewed towards high-grade bonds.

IV Estimation of Interest Rate Process

Based on considerations outlined below, Kalman filtering technique and a panel of Treasury yields are chosen to estimate the interest rate process. First, the unobservable nature of the instantaneous interest rate and its long-run drift makes Kalman filtering estimation appropriate for the task at hand. Second, our pricing equation (15) suggests that interest rate parameters can be estimated using defaultable coupon bonds. However, Dai and Singleton (2003) and Duffee (1999) reason that the joint estimation of default and default-free process across individual firms can make the overall dimensionality of the optimization problem rather large making estimation computationally difficult. Third, the full parameter set can be estimated separately for each firm but has the undesirable implication that the interest-rate process is different in the firm cross-section.

A Kalman Filtering Estimation Approach

To make proper connections, we require an assumption on the evolution of $X(t) \equiv (r(t), z(t))'$ under the physical measure, that is:

$$dX(t) \quad = \quad \Pi\left[\Theta - X(t)\right] dt + \sigma\, dW(t), \tag{24}$$

where $\Pi \equiv \begin{pmatrix} \kappa_r & -\kappa_r \\ 0 & \kappa_z \end{pmatrix}$, $\Theta \equiv \begin{pmatrix} \mu_z \\ \mu_z \end{pmatrix}$, $\sigma \equiv \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_z \end{pmatrix}$, $W(t) \equiv \begin{pmatrix} \omega_r(t) \\ \omega_z(t) \end{pmatrix}$ and ω_r, and ω_z are standard Brownian motions with $\mathrm{Cov}_t\left(\omega_r, \omega_z\right) \equiv \rho_{r,z}$. This assumption has the implication that risk-neutral density and physical density of $X(t)$ share the same parametric form. Integrating the stochastic differential equation (24),

$$X(s) \;=\; \left(I - e^{-\Pi(s-t)}\right)\Theta \;+\; e^{-\Pi(s-t)}\,X(t) \;+\; \sigma \int_t^s e^{-\Pi(s-u)}\,dW(u), \tag{25}$$

where I is a 2 x 2 identity matrix. It follows that $\int_t^s e^{-\Pi(s-u)}\,\Sigma\,dW(u)$ is a normally distributed vector with zero mean and variance-covariance matrix $V \equiv \int_t^s e^{-\Pi(s-u)}\,\Sigma\,e^{-\Pi'(s-u)}\,du$, where $\Sigma \equiv \sigma \begin{pmatrix} 1 & \rho_{r,z} \\ \rho_{r,z} & 1 \end{pmatrix} \sigma'$. The conditional moments of $X(t)$ given $X(t-1)$ are:

$$E_{t-1}[X(t)] \;=\; \left(I - e^{-\Pi\Delta t}\right)\Theta + e^{-\Pi\Delta t}X(t-1) \tag{26}$$

$$E_{t-1}[(X(t) - E_{t-1}[X(t)])(X(t) - E_{t-1}[X(t)])'] \;=\; \int_0^{\Delta t} e^{-\Pi u}\,\Sigma\,e^{-\Pi' u}du \tag{27}$$

where Δt is the time interval between $t-1$ and t. Exact discrete-time state-space $X(t)$ formulation is then based on the assumption that $x(0)$, $x(1)$, ..., $x(t)$ is a Markov process with initial state $x(0) \sim p^*(x(0))$ and $x(t)|x(t-1) \sim p^*(x(t)|x(t-1))$.

Let $Y(t) = (y_1(t), ..., y_K(t))'$ be the time-t observed Treasury yields where K denotes the number of yields employed in the estimation. As in Chen and Scott (2003), Duffee (1999) and Babbs and Nowman (1999), the measurement equation for observed yields can be expressed as:

$$Y(t) \;=\; \mathcal{A}_t \;+\; \mathcal{B}_t\,X(t) \;+\; \varepsilon(t), \qquad\qquad t = 1, ..., T, \tag{28}$$

where $\varepsilon(t) \sim \mathcal{N}(0, \mathcal{H}_t)$ and $\mathcal{H}_t$ is an $N \times N$ diagonal matrix with element $\mathcal{H}_{i,i} = \eta_i^2$. $\mathcal{A}_t$ is an $N \times 1$ vector with i-th element $\mathcal{A}_{t,i} = -\frac{\overline{\alpha}(\tau_i)}{\tau_i}$ and $\mathcal{B}_t$ is an $N \times 2$ matrix with i-th row $(\frac{\overline{\beta}(\tau_i)}{\tau_i}, \frac{\overline{\gamma}(\tau_i)}{\tau_i})$. The transition equation for the unobserved state-variable X_t follows:

$$X(t) \;=\; \mathcal{J}_t \;+\; \mathcal{F}_t\,X(t-1) \;+\; u(t), \tag{29}$$

where $E_{t-1}[u(t)] = 0$ and $E_{t-1}[u(t)u(t)'] = \mathcal{R}_t$ is a 2 x 2 matrix given by (27). $\mathcal{J}_t$ is a 2 x 1 vector $(I - e^{-\Pi\Delta t})\,\Theta$ and $\mathcal{F}_t$ is a 2 x 2 matrix $e^{-\Pi\Delta t}$.

The normality assumption on $\varepsilon(t)$ and $u(t)$ allows us to implement a Kalman filter recursion based on the results of Harvey (1991). Briefly let $\widehat{X}(t|t-1)$ denote the optimal estimate of $X(t-1)$ based on $t-1$ information and $\mathcal{P}_{t|t-1}$ denote the 2 x 2 covariance matrix of the estimation error. In this setting the conditional distribution of $Y(t)$ is normal with mean $\widetilde{Y}(t|t-1) = \mathcal{A}_t + \mathcal{B}_t\,\widehat{X}(t|t-1)$, and the prediction errors $v_t = Y(t) - \widetilde{Y}(t|t-1) = \mathcal{B}_t\left[X(t) - \widehat{X}(t|t-1)\right] + \varepsilon(t)$, t = 1, 2, ..., T, are independently and normally distributed with mean zero and variance-covariance $\mathcal{Z}_t =$

$\mathcal{B}_t \mathcal{P}_{t|t-1} \mathcal{B}_t' + \mathcal{H}_t$.[1] The log-likelihood function is accordingly,

$$\mathcal{L} \;=\; -\frac{N\,T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log|\mathcal{Z}_t| - \frac{1}{2}\sum_{t=1}^{T} v_t{}'\,\mathcal{Z}_t^{-1}\,v_t \tag{30}$$

which can be maximized to generate the state-space system parameters. In our optimization routine we set the initial value for $X(0)$ to its unconditional moment $[\mu_z,\, \mu_z]'$ and the initial value for $\mathcal{P}_0$ to $\sigma \begin{pmatrix} 1 & \rho_{r,z} \\ \rho_{r,z} & 1 \end{pmatrix} \sigma'$.

B Interpretation of Results

For the Kalman filter estimation we select a broad cross-section of Treasury yields with maturity of 6-months, 2-years, 5-years, and 10-years. In the discussion to follow, we impose the restriction $\lambda_{0,r} \equiv 0$ as the maximum-likelihood procedure failed to converge when $\lambda_{0,r}$ is freely determined as in Duffee (2002). Panel A of Table 1 reports restricted estimation results with $\lambda_{0,z} \equiv 0$ (labeled SET 1) as well as for the specification that accommodates non-zero $\lambda_{0,z}$ (labeled SET 2). The key observation to make from the two estimations is that the log-likelihood, $\mathcal{L}$, barely changes from 2180.99 to 2181.05. Thus, the three-parameter risk premium specification for $X(t)$ fares no better than the two-parameter counterpart. Given the insensitivity of $\mathcal{L}$ to $\lambda_{0,z}$, the remainder of the paper focuses on SET 1 that maintains $\lambda_{0,r} = \lambda_{0,z} = 0$ (the defaultable yields and credit spreads were also insensitive to estimated $\lambda_{0,z}$ with no impact on performance statistics).

The parameter estimates with SET 1 are reasonable and in line with economic intuition. Take the long-run interest rate, μ_z, as an example which is estimated at 8.7% and this measure of central tendency compares with 7.28% estimated by Babbs and Nowman (1999). Our results indicate strong mean-reversion for the $r(t)$ process but a near random walk behavior for the $z(t)$ process. Specifically the estimated $\kappa_r = 0.5301$ and $\kappa_z = 0.0518$ imply a half-life of 1.31 and 13.38 years for the $r(t)$ and $z(t)$ process, respectively.

As would be expected $r(t)$ and $z(t)$ co-move positively as reflected in the estimated $\rho_{r,z}$ value of 0.356 (see also Jegadeesh and Pennacchi (1996)). Because the risk-neutral drift of the short-rate is $(\kappa_r + \lambda_r)\left(\frac{\kappa_r\, z(t)}{\kappa_r + \lambda_r} - r(t)\right)$, a negative market price of interest rate risk, λ_r, has the impact of increasing the risk-neutral drift relative to the physical drift. To the contrary a positive λ_z decreases the risk-neutral drift of $z(t)$. In general the estimated parameters are several fold larger than their standard errors suggesting statistical significance.

[1] In the absence of normality, the error vector is still a zero mean vector and its covariance matrix at time t is still $\mathcal{B}_t\, \mathcal{P}_{t|t-1}\, \mathcal{B}_t' + \mathcal{H}_t$. In addition the errors can be shown to be serially uncorrelated.

The goodness-of-fit of the interest rate model is formally assessed in Panel B of Table 1. One, we report the standard deviation of the measurement errors which compare favorably with existing studies (e.g., Chen and Scott (2003), Duffee (1999) and Babbs and Nowman (1999)). For instance, the estimated standard deviation of the measurement error for 6-month, 2-year, 5-year, and 10-year Treasury yields is, respectively, 0, 13, 0, and 14 basis points. Two, the model provides desirable fitting-errors (actual minus model yield) across the yield curve: the median in-sample absolute errors for 6-month, 2-year, 5-year, and 10-year yield is 14bp, 23bp, 21bp and 19bp.

The log-likelihood and the p-value from $\chi^2(4)$-distributed likelihood-ratio statistic validate a two-factor specification over a restricted specification with $z(t)$ non-stochastic. Therefore, allowing for stochastic $z(t)$ enhances the ability of the three-factor credit risk models to describe credit curves more realistically. Demonstrating the plausibility of the two-factor interest-rate model from a different perspective (i) the correlation between the Kalman filter predicted $r(t)$ ($z(t)$) and the 6-month (10-year) market yield is 0.997 (0.872) and (ii) the slope of the yield curve (10-year market yield minus 6 month yield) has a correlation of 0.993 with the model generated slope. Overall the two factors reasonably explain the cross-section of yields as well as yield curve movements through time.

V Estimation of Single-Name Defaultable Process

In the spirit of Chen and Scott (2003), Duffee (2002), Duffie and Singleton (1997), Duffie, Pedersen and Singleton (2003), and Pearson and Sun (1994) we suppose that for each bond the model price deviates from the true market price and pricing errors are correlated in the cross-section of bond maturities. To incorporate these restrictions into our econometric procedure assume, under the physical probability measure, that the firm-specific distress variable $S_n(t)$ is governed by dynamics (14) with $\lambda_s \equiv 0$:

$$dS_n(t) \quad = \quad \kappa_s \, [\, \mu_s - S_n(t) \,] \, dt + \sigma_s \, d\omega_s(t), \tag{31}$$

where ω_s is a standard Brownian motion. Since $S_n(t)$ is normally distributed conditional on $S_n(t-1)$, the transition density of $S_n(t)$ satisfies

$$f(S_n(t)|S_n(t-1)) = \frac{1}{\sqrt{2\pi Q_n(t)}} \, \exp\left[-\frac{(S_n(t) - \mu_n(t))^2}{2\, Q_n(t)} \right] \tag{32}$$

where $\mu_n(t)$ and $Q_n(t)$ represent the first and second conditional moment of $S_n(t)$.

Recall by selection there are three corporate coupon bonds outstanding each month with time-to-maturity respectively close to 2, 7, and 15 years. Let the 3×1 vector

$$\varepsilon_n^*(t;\vartheta) \equiv \overline{Y}_n^*(t) - Y_n^*(t;\vartheta_n) \tag{33}$$

be the difference between the market bond yields and model-determined yields of single-name bond n at time t and

$$\vartheta_n \equiv (\Lambda_0, \Lambda_r, \Lambda_s, \kappa_s, \lambda_s, \mu_s, \sigma_s, \rho_{r,s}) \tag{34}$$

denote the parameter vector characterizing the default process. Suppose $\varepsilon_n^*(t;\vartheta)$ is serially uncorrelated but jointly normally distributed with zero mean and variance-covariance matrix Ω_ε. Under these assumptions the log-likelihood function for a sample of observations on corporate coupon bond prices for $t = 2, \ldots, T$ is:[2],

$$\begin{aligned} \max_{\vartheta_n, C_n} \quad \mathcal{L}^* &= -2(T-1)\log(2\pi) - \frac{1}{2}\sum_{t=2}^{T}\log(Q_n(t)) - \frac{1}{2}\sum_{t=2}^{T}\frac{(S_n(t)-\mu_n(t))^2}{Q_n(t)} \\ & -\frac{T-1}{2}\log|\,\Omega_\varepsilon| - \frac{1}{2}\sum_{t=2}^{T}\varepsilon_n^*(t;\vartheta)'\,\Omega_\varepsilon^{-1}\,\varepsilon_n^*(t;\vartheta). \end{aligned} \tag{35}$$

To ensure that the variance-covariance matrix Ω_ε is positive semi-definite assume, following Duffee (2002), that Ω_ε satisfies the Cholesky decomposition $\Omega_\varepsilon = C_n C_n'$ where C_n is a 3×3 matrix with non-zero elements C_{11}, C_{22}, C_{33}, C_{21}, and C_{32}.

The estimation procedure (35) is applied to credit risk models with defaultable discount rate specification for firm n given by:

1. Interest Rate Model	$R_n(t) = \Lambda_{0,n} + \Lambda_{r,n}\, r(t)$
2. Leverage Model	$R_n(t) = \Lambda_{0,n} + \Lambda_{r,n}\, r(t) + \Lambda_{s,n}\, \mathrm{LEV}_n(t)$
3. B/M Model	$R_n(t) = \Lambda_{0,n} + \Lambda_{r,n}\, r(t) + \Lambda_{s,n}\, \frac{B_n(t)}{M_n(t)}$
4. Profitability Model	$R_n(t) = \Lambda_{0,n} + \Lambda_{r,n}\, r(t) + \Lambda_{s,n}\, \mathrm{PROFIT}_n(t)$
5. Stock Volatility Model	$R_n(t) = \Lambda_{0,n} + \Lambda_{r,n}\, r(t) + \Lambda_{s,n}\, \mathrm{VOL}_n(t)$
6. Distance-to-Default Model	$R_n(t) = \Lambda_{0,n} + \Lambda_{r,n}\, r(t) + \Lambda_{s,n}\, \text{Distance-to-Default}_n(t)$

For each model, the corporate discount (coupon) bond price can be determined from equation (15) ((8)). The econometric approach, thus, uses the desired information in the cross-section of

[2] One can also include the unconditional likelihood to construct the exact likelihood function. Given that (i) the conditional MLE and the exact MLE have the same large sample distributions and (ii) the conditional MLE provides consistent estimates under some regularity conditions while the same is not true for the exact MLE, we chose the conditional MLE method (see Hamilton (1994)).

corporate bond prices and the underlying transition density of $S_n(t)$. For each firm we omit the last 12 months in the estimation to conduct out-of-sample inference. Motivated by the approach of Collin-Dufresne and Goldstein (2001), we also implement credit risk models that depend on the logarithmic transformation of leverage, B/M, and volatility.

Table 2 reports the results from applying the estimation procedure to the firm-sample and presents the average parameters for BBB-rated and A-rated bonds. Let us focus on the leverage model (*in level*) where the average Λ_s is 0.033 (0.015) with BBB-rated (A-rated) bonds. Hence, consistent with the work of Merton (1974), Longstaff and Schwartz (1995), and Collin-Dufresne and Goldstein (2001), the marginal impact of leverage is to enhance the magnitude of $R_n(t)$ and credit spreads and this effect is stronger for lower-rated debt.

We can observe that introducing leverage as an additional factor into the interest rate model tends to lower the magnitude of Λ_r: for BBB-rated and A-rated bonds the estimates becomes 0.799 and 0.955 versus 1.018 and 0.985 with the interest rate model. As observed in Duffee (1999), the positive estimates substantiate the notion that an upward shift in the short-rate raises the defaultable discount rate (see also Longstaff and Schwartz (1995)). The near-unity sensitivity coefficients suggest that bond yields are sensitive to $r(t)$ and more so for high-grade debt. With average Λ_0 close to 0.6% for BBB-rated and 0.4% for A-rated bonds, the instantaneous credit yield is higher with worsening credit quality.

Reported κ_s indicates that leverage is a mean-reverting stochastic process and $\lambda_s < 0$ has the interpretation that mean-reversion in leverage is stronger under the physical probability measure. In accordance with dynamics (14), a negative leverage risk premium increases the long-run drift of $S_n(t)$ which lowers corporate bond prices and increases credit spreads. Not at odds with intuition, the estimate of long-run leverage, μ_s, under the physical measure is 34.5% for BBB-rated bonds and 32.1% for A-rated bonds. It must be mentioned that BBB-rated bonds have a volatility parameter, σ_s, of 0.10 which is substantially higher than its $r(t)$ and $z(t)$ counterparts in Table 1. Finally, the correlation coefficient $\rho_{r,s}$ is estimated as 0.238 for BBB-rated bonds. This estimate implies that leverage induced distress is more severe in a rising interest rate environment possibly due to the impact of declining equity values.

In the last column we examine the restriction that $\Lambda_s = 0$ via a likelihood ratio test computed as twice the difference between the unrestricted and the restricted model log-likelihood. This test statistic is distributed $\chi^2(6)$ and the reported 1_{χ^2} is the fraction of firms for which the likelihood ratio test is rejected. Providing support for the inclusion of leverage in the defaultable discount rate specification our results indicate that the restriction $\Lambda_s = 0$ is rejected for 100% (96%) of BBB-rated (A-rated) single-name firms.

When distress is proxied by competing factors such as book-to-market, profitability, stock volatility and distance-to-default, the parameter estimates have interpretation similar to that described in the case of the leverage model. However, based on a comparison of average log-likelihood across models, leverage provides a superior fit to the objective function (35) and especially improves log-likelihood relative to interest rate and distance-to-default models. Fixing the distress factor, the log-likelihood values are typically higher with the level specification for BBB-rated firms while the log specification yields a slightly better description of A-rated firms. However, we recognize that the comparison of log-likelihood of non-nested models can be inexact.

VI Measures of Model Misspecification and Performance

A Determinants of Credit Risk

Because credit risk models are often employed in marking-to-market other illiquid securities, evaluating model performance is of broad interest (Dai and Singleton (2003)). Does a three-factor credit risk model with leverage improve upon a two-factor counterpart with firm-specific distress considerations absent? Is a credit risk model based on leverage less misspecified compared to the alternatives of book-to-market, profitability, volatility, and distance-to-default? To address these questions and to model credit risk determinants, Table 3 presents four measures of performance for each model: (i) the absolute yield valuation errors (in basis points), (ii) the absolute percentage valuation errors (in %), (iii) the Schwarz-Bayes information criterion, and (iv) the absolute credit spread errors (in basis points) for maturity-matched spread of corporate over Treasury. Each measure provides a perspective on the effectiveness of a model to explain the term structure of defaultable yields and credit spreads.

Based on the results for yield errors on BBB-rated bonds, a three-factor model with leverage (*in level*) is attractive relative to the two-factor interest rate model. In the row labeled "ALL" note that incorporating leverage in the defaultable discount rate specification leads to yield errors of 36 bp with leverage (level) model compared to 45 bp with interest rate model. This strong improvement over the base model is, in fact, observed at all maturities with each model performing slightly worse at the short-end of the defaultable yield curve. From a different angle, the documented fitting errors suggest that systematic variables have a first-order impact on defaultable term structures with leverage providing a sizeable next-order explanatory ability.

Compared with the three-factor setting of Duffee (1999) and the single-factor intensity process specification of Longstaff, Mittal, and Neis (2004), the average yield error is -7 bp (not reported in the table to save on space) making the leverage model reasonably accurate for marking-to-market credit risks. Although the focus of academic work has mostly been on default factors and our results are encouraging, realize that omitted nondefault attributes such as illiquidity of corporate bonds and actual spread volatility can cause model prices to systematically deviate from observed market prices even in the presence of properly specified default factors (see the practitioner perspective outlined in Dignan (2003) and Kao (2000)).

Table 3 imparts the insight that the interest rate model and the leverage (in level or logs) model are virtually indistinguishable for high-grade bonds at all maturities. This seemingly counterintuitive result can be understood to mean that high-rated bonds are mostly determined by systematic factors and rather insensitive to variations in firm-specific leverage. Although Λ_s has the theoretically correct positive sign in the leverage model that raises default probability for firms with higher leverage, the estimated magnitude is insufficiently strong to impact the pricing performance for high-grade bonds. Consistent with Λ_0 and Λ_r of the interest rate model we may write defaultable discount rate $R_n(t) = 0.8\% + 0.985\, r(t)$ implying that fixed-income markets value high-grade bonds using a translated default-free process. Thus, a more complex model that accounts for firm-specific distress need not necessarily perform better for high-grade bonds. As would be expected, each credit risk model (in particular the interest rate model) is less misspecified for high-grade bonds than its low-grade counterpart. Shifts in interest rate risk play a fundamental role in the valuation of single-name high-grade bonds (see, e.g., Kwan (1996) and Collin-Dufresne, Goldstein, and Martin (2001)).

According to empirical results for BBB-rated bonds, the leverage (*in level*) model performs the best followed in turn by the B/M (in level) model, the profitability model, the stock volatility model, and the distance-to-default model. The interest rate model is the last place performer. To fix main ideas, consider long-term bonds and the level specification: the percentage pricing errors (yield errors) are 2.66% (33 bp), 3.07% (39 bp), 3.24% (41 bp), 3.18% (40 bp), 3.20% (39 bp) and 3.37% (42 bp) respectively for the leverage, B/M, profitability, volatility, distance-to-default, and interest rate, models. It must be noted that this model rank-ordering does not perfectly coincide with the average log-likelihood ratio based ordering presented in Table 2. The reason is that the objective function in the estimation is not to minimize the sum-of-squared pricing errors; instead it is derived from the conditional density of $S_n(t)$ and the assumed cross-correlation structure of the valuation errors to give realistic single-name empirical moments. Given the large notional value of fixed income securities the documented superiority of the leverage model is economically meaningful.

Another way to judge performance is to compare the SIC statistic which rewards goodness-of-fit while penalizing the dimensionality of the model. Across most maturities, the SIC of the interest rate model exceeds the SIC of the leverage model for BBB-rated bonds while the reverse is true for high-grade bonds. This empirical yardstick thus favors the more heavily parameterized leverage model over the interest rate model for BBB-rated bonds. However, for high-grade bonds, simplicity is a more desirable modeling element. It is worth noting that in the case of BBB-rated firms, the SIC of the B/M and distance-to-default models exceed the SIC of the leverage model and therefore perform no better. Issues related to model differentiation based on statistical significance are formalized in the next subsection.

Relevant to our findings on high-grade bonds, Collin-Dufresne, Goldstein, and Martin (2001) argue that while variation in corporate yields is predominantly due Treasury yields, the credit spread is a much harder entity to predict. Are credit spreads mainly driven by the short interest rate or firm-specific distress variables? To answer this question, we provide additional information on the absolute difference between the observed and model generated credit spreads and report them as ACE in Table 3. The chief result that emerges from this exercise is that absolute credit spread errors have a large interest rate component, and accounting for leverage and other firm-specific distress is important. Consider BBB-rated bonds where the credit spread errors can be reduced from 40 bp in the case of interest rate model to 33 bp in the case of leverage model (*in level*). Sharing this feature with AYE, adding a firm-specific distress variable to the credit risk model does not help to lessen absolute credit spread errors for high-grade bonds. Thus, the performance metrics that rely on yields (i.e., AYE) and credit spreads (i.e., ACE) provide similar conclusions.

To gauge the robustness of the findings, Table 4 displays valuation results based on a holdout sample of one-year. The standard argument favoring such an exercise is that model performance need not improve out-of-sample for a more complex valuation model: Extra parameters have identification problems and may penalize accuracy. In implementation, we compute theoretical prices evaluated at the Kalman filter forecast of $r(t+1)$ and $z(t+1)$ and the optimal forecast of $S_n(t+1)$ based on the parametric model (31). The results are more striking in the out-of-sample context for BBB-rated bonds. First, comparing the entries in Table 3 and Table 4, there is a general deterioration in the performance of credit risk models and particularly the interest rate model. Thus, the interest rate model is most misspecified out-of-sample especially at the short-end of the low-grade yield curve. Second, the leverage model enhances the overall effectiveness of the interest rate model by 16 bp, and the B/M (distance-to-default) model by 8 (9) bp. Third, the results tell us that the relative ranking between credit risk models is unaltered in-sample versus out-of-sample.

Based on in-sample and out-of-sample absolute yield errors and the SIC statistic, we generally observe that the model relying on the distress variable specified in level does slightly better than the log counterpart. For instance, the absolute yield error in Table 4 for BBB firms with leverage (in level) is 40 bp versus 49 bp with leverage (in logs). Similar improvement is documented for

SIC in Table 3 which is -7.53 for leverage (level) and -7.32 for leverage (logs). To confirm whether the level or the log of leverage is more consistent with the assumed Gaussian physical dynamics, we also computed the Shapiro-Wilk normality statistic applied to firm-level leverage series. This test shows that moving from level of leverage to the logarithmic of leverage improves the Shapiro-Wilk statistic for 31% of the firms, but worsens for 69% of the firms. In summary, the level of leverage conforms to the normality hypothesis better than the log of leverage. However, the log B/M appears more consistent with the Gaussian dynamics for 63% of the firms.

B GMM Test of Model Comparisons and Specification Tests

The ask price is not reported in the Lehman database so we are unable to assess the extent of the improvement relative to the bid-ask spreads. However for any two defaultable coupon bond models, define the disturbance terms:

$$\mathcal{W}_t\left(\mu^{int}, \mu^{lev}\right) \equiv \begin{pmatrix} \frac{1}{3}\sum_{i=1}^{3} |\epsilon_i^{int}(t)| - \mu^{int} \\ \frac{1}{3}\sum_{i=1}^{3} |\epsilon_i^{lev}(t)| - \mu^{lev} \end{pmatrix}, \tag{36}$$

and

$$E[\mathcal{W}_t\left(\mu^{int}, \mu^{lev}\right)] = 0, \tag{37}$$

where μ^{int} and μ^{lev} represent the mean monthly absolute yield errors (averaged across short, medium, and long) for the interest rate model and the leverage model (in level).

Statistics presented in Table 5 are helpful for judging whether incremental explanatory power of a model is statistically significant relative to other models. First, we report the rejection rate for the null hypothesis $\mu^{int} = \mu^{lev}$ versus the alternative $\mu^{int} \neq \mu^{lev}$. Taking the weighting matrix from the unconstrained estimation this test is done by performing a restricted estimation with $\mu^{int} = \mu^{lev}$. Then, T times difference in the restricted and the unrestricted criterion functions is asymptotically $\chi^2(1)$-distributed. Two, we present the rejection rate for the one-sided test $\mu^{int} > \mu^{lev}$ using the Newey-West method to adjust for heteroskedasticity and autocorrelation. Lastly we compute $\Phi_t \equiv \frac{|\epsilon^{int}(t)| - |\epsilon^{lev}(t)|}{|\epsilon^{int}(t)|}$ (or, $\overline{\Phi}_t$), which reflects the average absolute yield error (credit spread error) reduction in moving from the interest rate model (the base model) to the leverage model.

Consider low-grade bonds: valuation improvement is statistically large by adding leverage (in level) to the defaultable discount rate specification of the interest rate model. Overall, leverage results in a 30.1%, 14.3%, 25.2%, 33.8%, and 15.2% relative improvement over the interest rate, the

B/M, profitability, volatility, and distance-to-default models. Between leverage and interest rate models, the two-sided (one-sided) rejection rate is 70% (67%) implying statistical improvement for the majority of the single-names (and likewise for credit spreads).

The B/M (log) generalization to the interest rate model also delivers economically and statistically significant performance enhancement, albeit by a lesser extent relative to the leverage model (we focus on log as it is more consistent with the Gaussian hypothesis although the result are similar with level). The relative improvement is 18.4% over the interest rate model with a one-sided rejection rate of 46%. With high-grade bonds, however, there is no discernible improvement over the interest rate model with either leverage or B/M. In this regard, Agarwal, Elton, Gruber, and Mann (2001) reason that differences in default are insufficient to explain yield spreads observed across rating classes. Such a result also parallels a calibration-based finding of Huang and Huang (2003) that default risk accounts for only a small fraction of aggregate credit spreads, and this result can now be extended to high-grade single-name credit spreads. Allowing for a richer stochastic structure beyond the interest rate model in the firm-specific dimension is of greater relevance for yield errors and credit spread error of low-grade bonds.

To further study properties of model mispricing, we appeal to Collin-Dufresne, Goldstein, and Martin (2001) and investigate whether pricing-errors from the leverage model are contemporaneously correlated. Specifically, we focus on four systematic factors: (i) the default spread between low-rated and high-rated corporate bonds (denoted DEFAULT), (ii) the spread between the 10-year and 2-year Treasury yield (denoted SLOPE), (iii) the size premium (SMB), and (iv) the VIX volatility index from CBOE. The following regression is performed for each firm (letting $\epsilon_n^{lev}(t) \equiv \frac{1}{3}\sum_{i=1}^{3}|\epsilon_{n,i}^{lev}(t)|$):

$$\epsilon_n^{lev}(t) = a_0 + a_1\,\mathrm{DEFAULT}(t) + a_2\,\mathrm{SLOPE}(t) + a_3\,\mathrm{VIX}(t) + a_4\,\mathrm{SMB}(t) + e(t), \qquad n = 1,\cdots,N. \tag{38}$$

Three findings are relevant to our modeling approach. First, we find that the average adjusted-R^2 across our sample of firms is 9%, implying that single-name pricing-errors show little time-covariation with the chosen systematic factors (50% of the firms had R^2 less than 5%). Second, the sensitivity coefficients are typically positive. In economic terms, this means that model mispricing tends to be greater during periods of widening default spreads or steepening Treasury-curves (or equivalently when VIX/size premium are high). Third, the t-statistics (corrected for heteroskedasticity and autocovariance) are greater for 2 for at most 19% of the firms. For the majority of the firms, the pricing-errors are, thus, statistically unrelated to the systematic-factors. Extant research has found that corporate bond markets are subject to high transaction cost and low volume. Therefore, it appears likely that bond market illiquidity has contributed to model mispricing. More empirical work is needed to study the role of bond market liquidity along the lines of Collin-Dufresne, Goldstein, and Martin (2001). If there are market-wide default and liquidity factors, they are perhaps not adequately captured by the set of included systematic variables.

Additionally, we performed a principal component analysis of absolute yield errors and credit spread errors of each model by binning the data into 9 groups (by short, medium, long, and by AA, A, and BBB credit rating). The results establish that there is a dominant component for each of the models, with first two components explaining the bulk of the variation. Take the leverage model as an example, where the first component explains 83.6% of the total variation. Together, the first two components capture 92.7% of the variation. As shown by the regression analysis, none of the systematic variables appear to be a good proxy for the predominant factors.

Tables 3 and 4 have the unfortunate implication that credit risk models are misspecified albeit to various degrees. One key issue that remains to be addressed is whether cross-sectional variations in model misspecification are related to excluded firm-specific distress risk? Indeed, there is no theoretical basis for picking a firm-specific distress variable successively in a default model. In doing so, we were guided by parsimony and ease of implementation: a four-factor generalization driven by (31) has 4 additional parameters and two correlation terms requiring the estimation of 22 parameters (plus 5 Cholesky decomposition constants). How can a four-factor credit risk model with two firm-specific factors expected to fare relative to a three-factor credit risk model? To informally attend to this concern two regression specifications are explored using the entire cross-section. First, we regress the absolute yield errors of the leverage model on B/M ($n = 1, \cdots, N$) and obtained:

$$\mathrm{AYE}_n\,|_{\mathrm{Lev}} \;=\; \underset{(15.2)}{0.262} + \underset{(4.86)}{0.08}\,\frac{\mathrm{B}_n}{\mathrm{M}_n} + e_n, \quad R^2{=}2.7\%$$

where the reported coefficients are time-series pooled averages and the t-statistics are based on GMM. Second, we regress B/M mispricing on leverage:

$$\mathrm{AYE}_n\,|_{\mathrm{B/M}} \;=\; \underset{(14.7)}{0.288} + \underset{(3.13)}{0.098}\,\mathrm{LEV}_n + e_n, \quad R^2{=}0.4\%.$$

Based on the sensitivity coefficients one may conjecture that it is possible to lower the pricing errors of defaultable bonds: the valuation errors of B/M model are statistically higher for firms with high leverage even after accounting for B/M (and vice versa). Owing to burdensome MLE implementation of higher-dimensional models and the rather low R^2 explanatory power of $\frac{\mathrm{B}_n}{\mathrm{M}_n}$ and LEV_n in the aforementioned cross-sectional estimations, we leave a formal analysis of four-factor credit risk models to a later study.

C Hedging Dynamic Credit Exposures

This subsection concentrates on hedging a portfolio consisting of a short-position in a corporate bond and a long position in a Treasury bond with matched maturity. Notice there is an arbitrage link between the corporate bond market, the Treasury market and the credit default swap market. According to Duffie (1999), Duffie and Liu (2001) and Houweling and Vorst (2001), holding a τ-period par corporate bond and buying credit protection through a credit default swap is approximately risk-free (adjusting for floating-rate feature and coupon equivalence). For this reason the price of single-name credit protection is the same as reference entity credit spread. Our approach, as shown, can be useful in understanding which traded fixed-income instruments can most effectively hedge dynamic credit exposures and, to first-order, potentially hedge the risks of writing credit protection. This connection arises since sellers of credit default swaps can hedge their exposure by shorting corporate bonds (see Longstaff, Mittal, and Neis (2004, p.21)) or vice versa.

Fix the credit risk model as the interest rate model. In this case the combined position can be written as:

$$-\frac{1}{\tau}\log\left[P(t,\tau)/B(t,\tau)\right]+\{w_0(t)+w_1(t)\,B(t,\tau_1)+w_2(t)\,B(t,\tau_2)\} \tag{39}$$

which hedges credit exposure using two risk-free discount bonds with short-maturity τ_1 and long-maturity τ_2. Taking the target as the log relative of the corporate coupon bond price to the Treasury bond price and normalizing it by τ imparts it a yield interpretation. To dynamically hedge $r(t)$ and $z(t)$ risks we take the required partial derivatives and derive the positioning as:

$$w_1(t) = \frac{\frac{1}{\tau}\left[\frac{\Delta_z(t,\tau)}{P(t,\tau)}-\frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial z}\right]\times\frac{\partial B(t,\tau_2)}{\partial r}-\frac{1}{\tau}\left[\frac{\Delta_r(t,\tau)}{P(t,\tau)}-\frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial r}\right]\times\frac{\partial B(t,\tau_2)}{\partial z}}{\frac{\partial B(t,\tau_1)}{\partial z}\times\frac{\partial B(t,\tau_2)}{\partial r}-\frac{\partial B(t,\tau_1)}{\partial r}\times\frac{\partial B(t,\tau_2)}{\partial z}}, \tag{40}$$

$$w_2(t) = \frac{\frac{1}{\tau}\left[\frac{\Delta_z(t,\tau)}{P(t,\tau)}-\frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial z}\right]\times\frac{\partial B(t,\tau_1)}{\partial r}-\frac{1}{\tau}\left[\frac{\Delta_r(t,\tau)}{P(t,\tau)}-\frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial r}\right]\times\frac{\partial B(t,\tau_1)}{\partial z}}{\frac{\partial B(t,\tau_1)}{\partial r}\times\frac{\partial B(t,\tau_2)}{\partial z}-\frac{\partial B(t,\tau_1)}{\partial z}\times\frac{\partial B(t,\tau_2)}{\partial r}}, \tag{41}$$

with $\frac{\partial B(t,\tau)}{\partial r}$ and $\frac{\partial B(t,\tau)}{\partial z}$ determined from (12). The hedge portfolio is made self-financing by setting $w_0(t)=\frac{1}{\tau}\log[\overline{P}(t,\tau)/\overline{B}(t,\tau)]-w_1(t)\,B(t,\tau_1)-w_2(t)\,B(t,\tau_2)$. Local risk exposures for the defaultable coupon bond are the aggregated face and coupon exposures:

$$\Delta_r(t,\tau) \equiv F\,\Delta_r^*(t,\tau)+\int_0^\tau c(t+u)\,\Delta_r^*(t,u)\,du, \tag{42}$$

$$\Delta_z(t,\tau) \equiv F\,\Delta_z^*(t,\tau)+\int_0^\tau c(t+u)\,\Delta_z^*(t,u)\,du, \tag{43}$$

where $\Delta_r^*(t,\tau)$ and $\Delta_z^*(t,\tau)$ are displayed in (20)-(21). The combined position in the defaultable bond and the replicating portfolio is liquidated at time $t+\Delta t$. Dollar hedging errors are defined as:

$$\begin{aligned} H(t+\Delta t) &= w_0(t)e^{r(t)\Delta t} + w_1(t)\, B(t+\Delta t, \tau_1 - \Delta t) + w_2(t)\, B(t+\Delta t, \tau_2 - \Delta t) \\ &\quad -\frac{1}{\tau}\log[\overline{P}(t+\Delta t, \tau - \Delta t)/\overline{B}(t+\Delta t, \tau - \Delta t)]. \end{aligned} \tag{44}$$

Selecting τ as 2-years and 7-years the hedging strategy is executed each month and for each single-name bond. We compute absolute dollar hedging errors each month as:

$$\mathrm{AHE}(t+\Delta t) \equiv \frac{1}{K}\sum_{k=1}^{K} \mid H_k(t+\Delta t) \mid \tag{45}$$

and absolute percentage hedging errors as:

$$\mathrm{PHE}(t+\Delta t) \equiv \frac{1}{K}\sum_{k=1}^{K} \left| \frac{H_k(t+\Delta t)}{\frac{1}{\tau}\log[\overline{P}(t,\tau)/\overline{B}(t,\tau)]} \right|. \tag{46}$$

Implementing a dynamic hedging strategy based on leverage and B/M models involves non-traded leverage and book values and only a partial hedge is feasible. To hedge $r(t)$, $z(t)$ risks we employ two Treasury discount bonds and single-name equity to dynamically hedge the market component of leverage. However, this leaves the long-term debt component of leverage unhedged. Notice that we can write leverage as $\mathrm{LEV}(t) = \frac{D(t)/q_0}{\mathcal{M}(t)+D(t)/q_0}$ where q_0 is the number of shares outstanding and D(t) is debt value. Assuming the stock price $\mathcal{M}(t)$ follows geometric Brownian motion we derive $w_1(t)$ and $w_2(t)$ as shown in (40)-(41) and $w_3(t) = -\frac{1}{\tau}\Delta_s \times \frac{D(t)/q}{(\mathcal{M}(t)+D(t)/q)^2}$ where $\Delta_s(t,\tau) \equiv F\,\Delta_s^*(t,\tau) + \int_0^\tau c(t+u)\,\Delta_s^*(t,u)\,du$.

The results reported in Panel A and B of Table 6 reveal that leaving the debt (book) component of leverage (B/M) unaccounted in the hedging strategy can impair dynamic ability of three-factor credit risk models. Despite the flexibility of an additional instrument, the absolute hedging errors from the leverage model and the B/M model are no better than that of the two-factor interest rate model. From the perspective of absolute hedging errors interest rate movements are dominant. When averaged over BBB-rated single-name bonds the absolute hedging error with the leverage (book-to-market) model is \$0.107 (\$0.107) compared to \$0.106 with the interest rate model (assuming \$100 face value and for $\tau = 2$ years). Credit risk models relying on purely traded factor tend to produce better hedging effectiveness. Regardless of maturity, the two Treasury instruments do a fairly reasonable job neutralizing dynamic movements in single-name credit exposures for firms in our sample. We also investigated the relation of single-name hedging-errors using (38) and found them to be characterized by a lack of covariation.

Although not reported, a robustness check shows that the same conclusion applies when the target is a short-position in a corporate bond. Our results open up the wider implication that short-positions in credit default swaps could be equivalently hedged using Treasury instruments up to BBB-rated single-name reference entities.

VII Conclusions

This paper has presented a theoretical and empirical framework for analyzing variations in defaultable bond yields and credit spreads using the approach of Duffie and Singleton (1999). Relying on a sample of single-name defaultable coupon bonds, the empirical investigation posed two questions of economic interest: Which credit risk models are suitable for marking-to-market default risks and for hedging credit exposures? What type of default factors (systematic or firm-specific) are relevant for explaining credit risk? Under appropriate assumptions on the interest rate process, the default process, and the market price of risks, our estimation strategy lead to several new insights. First, interest-rate consideration is of first-order prominence in explaining credit curves for our sample of investment grade bonds. Second, accounting for the impact of interest-rates the incremental pricing improvement from modeling leverage is economically large and statistically significant for low-grade bonds. Third, the credit risk model with leverage appears least misspecified and has pricing performance adequate for marking risk exposures. Finally the hedging analysis opens up the possibility that traded Treasury bonds may be effective in hedging dynamic credit exposures.

Since callability, putability and convertibility are common features of defaultable bonds, more effort is needed to theoretically and empirically characterize optionality. Future research should also explore ways to incorporate such nondefault components as liquidity and spread volatility into the working of risky debt models. These extensions are worthy of a follow-up study.

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Table 1: Kalman Filtering Estimation of the Interest Rate Process

The reported parameters of the default-free process in Panel A are based on Kalman filtering. Under the risk-neutral measure and a four-parameter risk premium specification (i.e., $\{\lambda_{0,r}, \lambda_{0,z}, \lambda_r, \lambda_z\}$) of Duffee (2002), the dynamics of the two-factor interest rate model is:

$$\begin{aligned} dr(t) &= [\kappa_r\, z(t) - \lambda_{0,r} - (\kappa_r + \lambda_r)\, r(t)]\; dt + \sigma_r\, d\widetilde{\omega}_r(t), \\ dz(t) &= [\kappa_z\, \mu_z - \lambda_{0,z} - (\kappa_z + \lambda_z)\, z(t)]\; dt + \sigma_z\, d\widetilde{\omega}_z(t), \end{aligned}$$

and the counterpart dynamics under the physical probability measure is specified in equation (24). The estimation uses a monthly time-series of Treasury yields with maturity of 6-months, 2-years, 5-years and 10-years. The asymptotic standard errors in parenthesis are calculated from the Hessian of the log-likelihood function. We set $\lambda_{0,r} \equiv 0$ due to the lack of convergence. **SET 1** additionally imposes $\lambda_{0,z} \equiv 0$ and **SET 2** accommodates a specification where $\lambda_{0,z}$ is a freely-determined parameter. In the final column, the maximized log-likelihood function is reported as $\mathcal{L}$ and the restricted likelihood function setting $\mu_z = \lambda_z = \kappa_z = \sigma_z = 0$ (in square-brackets) is $\mathcal{L}^R$. The p-value for the likelihood ratio test distributed $\chi^2(4)$ is in curly brackets. Using parameters from SET 1, Panel B reports the (i) median in-sample pricing errors (observed yield minus model yield, in bp), (ii) median absolute pricing errors (in bp), (iii) the root mean squared pricing errors (in bp), and (iv) the estimated standard deviation of the measurement error (standard errors in parenthesis) for each Treasury yield. The sample period of the estimation is March 1989 through March 1998.

Panel A: Interest Rate Parameters

Parameter $i = \{r, z\}$	κ_i	σ_i	μ_i	λ_i	$\rho_{r,z}$	$\lambda_{0,i}$	$\mathcal{L}$ $[\mathcal{L}^R]$ $\{\chi^2(4)\}$
SET 1							
r	0.5301 (0.0465)	0.0079 (0.0006)		-0.1418 (0.0382)	0.3564 (0.0828)	0	2180.99 [1804] {0.00}
z	0.0518 (0.0087)	0.0116 (0.0012)	0.0870 (0.0082)	0.0196 (0.0079)		0	
SET 2							
r	0.5352 (0.0459)	0.0079 (0.0006)		-0.1470 (0.0390)	0.3519 (0.0870)	0	2181.05 [1804] {0.00}
z	0.0133 (0.0125)	0.0115 (0.0011)	0.0869 (0.0087)	0.05848 (0.0156)		0.0033 (0.0013)	

Panel B: Fitting Errors and Standard Deviation of Measurement Errors

	6-months	**2-years**	**5-years**	**10-years**
Median Absolute Pricing Errors (bp)	14	23	21	19
Median Pricing Errors (bp)	-5	-3	-6	-4
Squared-root of Mean Squared Errors (bp)	22	32	28	29
Std Dev. of Measurement Errors	0.0000 (0.0002)	0.0013 (0.0000)	0.0000 (0.0001)	0.0014 (0.0001)

Table 2: Estimation of Defaultable Bond Models

This estimation procedure uses three corporate coupon bonds with maturity close to 2-years, 7-years and 15-years. Let 3×1 vector $\varepsilon_n^*(t) \equiv \overline{Y}_n^*(t) - Y_n^*(t;\vartheta)$ measure the difference between the market observed corporate bond yields and the model-determined yields of firm n in month t. It is assumed that $\varepsilon_n(t)$ are (i) jointly normally distributed with zero mean and variance-covariance matrix Ω_ε and (ii) the pricing errors are serially uncorrelated over time. The log-likelihood function to be maximized is:

$$\max\ \mathcal{L}^* \equiv -2(T-1)\log(2\pi) - \frac{1}{2}\sum_{t=2}^{T}\log(Q_n(t)) - \frac{1}{2}\sum_{t=2}^{T}\frac{(S_n(t)-\mu_n(t))^2}{Q_n(t)}$$
$$-\frac{T-1}{2}\log|\Omega_\varepsilon| - \frac{1}{2}\sum_{t=2}^{T}\varepsilon_n^*(t)'\,\Omega_\varepsilon^{-1}\,\varepsilon_n^*(t),$$

where $\mu_n(t)$ and $Q_n(t)$ represent the first two conditional moments of the distress factor $S_n(t)$. For each credit risk model with parameter vector ϑ the reported values are the respective averages across BBB-rated bonds and A-rated bonds. 1_{χ^2} is the fraction of firms (in %) for which the model restriction $\Lambda_s = 0$ is rejected based on a likelihood ratio test (distributed $\chi^2(6)$). The average log-likelihood value is shown as $\mathcal{L}^*/T$. The construction of distance-to-default is as outlined in Section III. Results for (i) leverage model, (ii) B/M model, and (iii) stock volatility model are reported for both the level (denoted "Level") and the logarithmic (denoted "Logs") specification.

Model	Sample	Λ_0	Λ_r	Λ_s	κ_s	σ_s	μ_s	λ_s	$\rho_{r,s}$	$\mathcal{L}^*/T$	1_{χ^2}
Interest Rate (Base Model)	BBB	0.013	1.018							10.28	
	A	0.008	0.985							10.84	
Leverage (Level)	BBB	0.006	0.799	0.033	0.465	0.100	0.345	-0.275	0.238	12.85	100
	A	0.004	0.955	0.015	1.016	0.112	0.321	-0.631	-0.080	12.77	96
Leverage (Logs)	BBB	0.010	0.890	0.013	0.376	0.125	0.227	-0.185	0.418	12.68	96
	A	0.007	0.969	0.001	0.806	0.090	0.307	-0.243	0.272	12.93	100
B/M (Level)	BBB	0.007	0.800	0.023	1.078	0.553	0.680	-0.956	0.022	11.15	75
	A	0.006	0.902	0.010	1.175	0.609	0.630	-0.793	-0.116	11.56	76
B/M (Logs)	BBB	0.005	0.836	0.026	0.406	0.111	0.373	-0.171	0.250	12.46	100
	A	0.005	0.952	0.007	0.859	0.094	0.449	-0.394	0.172	12.80	100
Profitability	BBB	0.007	0.840	0.012	0.983	0.148	0.709	-0.634	0.125	12.38	92
	A	0.005	0.961	0.005	0.889	0.061	0.710	-0.541	0.002	13.09	98
Volatility (Level)	BBB	0.010	0.851	0.036	2.084	0.244	0.171	-1.360	-0.068	11.63	96
	A	0.007	0.927	0.026	2.887	0.299	0.245	-0.998	-0.056	11.18	86
Volatility (Logs)	BBB	0.014	0.910	0.008	2.266	0.959	0.421	-1.509	-0.023	11.46	89
	A	0.007	0.964	0.003	2.238	0.961	0.289	-1.637	0.131	11.15	86
Distance to Default	BBB	0.006	0.767	0.032	0.268	0.097	0.325	-0.162	0.188	10.93	94
	A	0.005	0.966	0.006	0.879	0.091	0.371	-0.312	0.230	11.33	88

Table 3: Valuation Errors for Defaultable Coupon Bond Models

For each credit risk model we compute **absolute yield error** (AYE, in basis points) as $\left|\overline{Y}_n^*(t) - Y_n^*(t;\vartheta)\right|$, where $\overline{Y}_n^*(t)$ is observed yield and $Y_n^*(t;\vartheta)$ is model-determined yield, and a similar calculation applies to **absolute percentage valuation error** (APE, in %). The model defaultable bond price is based on the estimated interest rate process and the defaultable process. ACE is **absolute credit spread error** (in basis points) computed as the absolute difference between maturity-matched observed and model credit spreads. SIC is Schwarz-Bayes information criterion. Results for (i) leverage model, (ii) B/M model, and (iii) stock volatility model are reported for both the level (denoted "Level") and the logarithmic (denoted "Logs") specification.

		Interest Rate		**Leverage (Level)**		**Leverage (Logs)**		**B/M (Level)**		**B/M (Logs)**		**Profit Model**		**Volatility (Level)**		**Volatility (Logs)**		**Distance to Default**	
		BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A
ALL	AYE	45	27	36	28	38	26	38	29	40	26	41	27	41	29	43	27	40	25
	APE	2.43	1.43	1.80	1.43	2.03	1.36	1.98	1.48	2.08	1.35	2.18	1.37	2.18	1.53	2.40	1.39	2.17	1.34
	SIC	-6.96	-7.80	-7.53	-7.90	-7.32	-7.93	-7.33	-7.84	-7.31	-7.95	-7.15	-7.91	-7.15	-7.82	-7.12	-7.91	-7.17	-7.94
	ACE	40	18	33	21	35	18	35	21	37	18	38	18	39	21	39	21	37	18
Short	AYE	49	29	38	30	39	28	41	31	41	28	39	29	41	29	41	29	42	28
	APE	1.21	0.77	0.99	0.78	1.03	0.74	1.15	0.79	1.05	0.73	0.98	0.78	1.02	0.78	1.13	0.75	1.12	0.74
	SIC	-8.59	-9.20	-8.83	-9.21	-8.79	-9.27	-8.65	-9.19	-8.73	-9.29	-8.77	-9.18	-8.78	-9.17	-8.73	-9.23	-8.65	-9.31
	ACE	43	22	33	22	35	19	38	22	31	22	35	20	40	22	39	22	39	20
Med.	AYE	45	27	36	28	38	26	39	28	40	26	42	26	45	26	45	27	40	25
	APE	2.40	1.47	1.88	1.54	2.05	1.43	2.02	1.58	2.19	1.42	2.25	1.45	2.42	1.44	2.45	1.45	2.17	1.39
	SIC	-7.17	-7.90	-7.63	-7.93	-7.46	-7.97	-7.43	-7.89	-7.41	-7.99	-7.28	-7.94	-7.20	-7.96	-7.23	-7.97	-7.36	-8.01
	ACE	39	16	30	18	32	15	33	19	35	15	38	16	40	16	37	17	35	15
Long	AYE	42	27	33	27	37	25	39	25	39	24	41	25	40	27	42	25	39	23
	APE	3.37	2.15	2.66	2.08	2.95	1.91	3.07	1.94	3.09	1.89	3.24	1.91	3.18	2.10	3.34	1.97	3.20	1.88
	SIC	-6.39	-7.18	-6.96	-7.34	-6.73	-7.39	-6.72	-7.36	-6.72	-7.41	-6.54	-7.37	-6.55	-7.28	-6.56	-7.37	-6.57	-7.38
	ACE	38	19	36	22	37	20	38	21	38	20	41	20	40	22	40	23	37	20

Table 4: Out-of-Sample Valuation Errors Based on a Hold-Out Sample

The results are based on a hold-out sample of one-year for each firm. For each credit risk model we compute **absolute yield error** (AYE, in basis points) as $\left|\overline{Y}_n^*(t) - Y_n^*(t;\vartheta)\right|$ and a similar calculation applies to **absolute percentage valuation error** (APE, in %). In implementation, the model defaultable bond price is evaluated at the Kalman filter forecast of $r(t+1)$ and $z(t+1)$, the optimal forecast of $S_n(t+1)$ based on the parametric model (31), and the estimated parameters of the interest rate and the defaultable process.

		Interest Rate		**Leverage (Level)**		**Leverage (Logs)**		**B/M (Level)**		**B/M (Logs)**		**Profit Model**		**Volatility (Level)**		**Volatility (Logs)**		**Distance to Default**	
		BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A
ALL	AYE	56	27	40	29	49	26	49	26	47	27	54	28	55	26	51	29	48	31
	APE	2.62	1.39	2.08	1.45	2.49	1.34	2.46	1.30	2.39	1.32	2.72	1.37	2.78	1.36	2.70	1.53	2.49	1.53
Short	AYE	66	28	38	26	47	24	46	23	43	25	48	26	49	25	45	27	44	29
	APE	1.34	0.66	0.80	0.60	0.98	0.56	0.98	0.53	0.92	0.57	0.98	0.61	1.03	0.60	0.94	0.62	0.96	0.68
Med.	AYE	56	28	40	30	49	27	46	27	46	25	53	26	55	28	50	30	47	33
	APE	2.69	1.45	1.97	1.57	2.39	1.36	2.33	1.38	2.29	1.31	2.60	1.36	2.80	1.42	2.58	1.55	2.46	1.77
Long	AYE	46	26	43	31	52	27	54	28	51	26	60	27	61	26	57	31	51	29
	APE	3.62	2.06	3.41	2.45	4.10	2.08	4.16	2.20	3.97	2.04	4.71	2.10	4.77	2.06	4.57	2.41	4.00	2.28

Table 5: Generalized Method of Moments Tests of Statistical Significance

For each firm we conduct a pair-wise valuation comparison between (i) the leverage model (level) versus other models and (ii) the book-to-market model (logs) versus remaining models. Take the leverage model and fix the interest rate model as the base model. Define,

$$\mathcal{W}_t\left(\mu^{int}, \mu^{lev}\right) \equiv \begin{pmatrix} \frac{1}{3}\sum_{i=1}^{3} |\epsilon_i^{int}(t)| - \mu^{int} \\ \frac{1}{3}\sum_{i=1}^{3} |\epsilon_i^{lev}(t)| - \mu^{lev} \end{pmatrix},$$

and the moment condition:

$$E[\mathcal{W}_t\left(\mu^{int}, \mu^{lev}\right)] = 0,$$

where μ^{int} and μ^{lev} respectively represent the mean monthly absolute yield errors for the interest rate model and the leverage model. In the first row we report the **Two-sided Rejection** rate that tests the null hypothesis $\mu^{int} = \mu^{lev}$ versus the alternative $\mu^{int} \neq \mu^{lev}$. Next we report the **One-Sided Rejection** rate which is the fraction of firms with $\mu^{int} > \mu^{lev}$ at the 95% confidence level. Lastly, we report a measure of relative model **yield improvement**, computed as:

$$\Phi_t \equiv \frac{|\epsilon^{int}(t)| - |\epsilon^{lev}(t)|}{|\epsilon^{int}(t)|},$$

and the **spread improvement**, $\overline{\Phi}_t$, is analogously based on credit spread errors. The Newey-West method is used to adjust for heteroskedasticity and autocorrelation.

		Leverage Model (Level) vs					**B/M Model (Logs) vs**			
		Interest Rate	**B/M** (Logs)	**PROF.**	**VOL** (Logs)	**Distance to Default**	**Interest Rate**	**PROF.**	**VOL** (Logs)	**Distance to Default**
BBB	Two-Sided Rejection	70.8	50.0	54.2	75.0	46.7	45.8	54.2	54.2	33.3
	One-Sided Rejection	66.7	45.8	54.2	66.7	40.0	45.8	37.5	33.3	20.0
	Yield Improvement, Φ_t (%)	30.1	14.3	25.2	33.8	15.2	18.4	12.7	22.8	-2.1
	Spread Improvement, Φ_t (%)	29.3	12.7	25.8	37.1	7.8	19.0	14.9	28.0	-0.4
A	Two-Sided Rejection	24.5	22.4	18.4	12.2	13.0	20.4	10.2	14.3	21.7
	One-Sided Rejection	22.4	8.2	10.2	6.1	8.7	20.4	10.2	6.1	8.7
	Yield Improvement, Φ_t (%)	-5.4	-11.4	-9.0	-6.1	2.9	5.4	2.1	4.7	2.2
	Spread Improvement, $\overline{\Phi}_t$ (%)	-8.3	-15.1	-12.4	-1.8	8.7	5.9	2.4	11.6	10.4

Table 6: Hedging Credit Exposures

In what follows we fix the maturity, τ, of the target to be hedged as either 2-years or 7-years. Choose the interest rate model and the target as $\frac{1}{\tau}\log[P(t,\tau)/B(t,\tau)]$. In the hedging strategy as many instruments as sources of risks are employed to create a hedge in all dimensions which leads to $w_1(t) = \frac{\frac{1}{\tau}[\frac{\Delta_z(t,\tau)}{P(t,\tau)} - \frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial z}]\times\frac{\partial B(t,\tau_2)}{\partial r} - \frac{1}{\tau}[\frac{\Delta_r(t,\tau)}{P(t,\tau)} - \frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial r}]\times\frac{\partial B(t,\tau_2)}{\partial z}}{\frac{\partial B(t,\tau_1)}{\partial z}\times\frac{\partial B(t,\tau_2)}{\partial r} - \frac{\partial B(t,\tau_1)}{\partial r}\times\frac{\partial B(t,\tau_2)}{\partial z}}$ and $w_2(t) = \frac{\frac{1}{\tau}[\frac{\Delta_z(t,\tau)}{P(t,\tau)} - \frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial z}]\times\frac{\partial B(t,\tau_1)}{\partial r} - \frac{1}{\tau}[\frac{\Delta_r(t,\tau)}{P(t,\tau)} - \frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial r}]\times\frac{\partial B(t,\tau_1)}{\partial z}}{\frac{\partial B(t,\tau_1)}{\partial r}\times\frac{\partial B(t,\tau_2)}{\partial z} - \frac{\partial B(t,\tau_1)}{\partial z}\times\frac{\partial B(t,\tau_2)}{\partial r}}$ and $w_0(t) = \frac{1}{\tau}\log[\overline{P}(t,\tau)/\overline{B}(t,\tau)] - w_1(t)\,B(t,\tau_1) - w_2(t)\,B(t,\tau_2)$. For each credit risk model and single-name bond compute the **absolute dollar hedging errors** (AHE) as: $|w_0(t)e^{r(t)\Delta t} + w_1(t)\,B(t+\Delta t,\tau_1-\Delta t) + w_2(t)\,B(t+\Delta t,\tau_2-\Delta t) - \frac{1}{\tau}\log[\overline{P}(t+\Delta t,\tau-\Delta t)/\overline{B}(t+\Delta t,\tau-\Delta t)]|$ and a similar calculation applies to **absolute percentage hedging errors** (PHE). In the leverage model and the B/M model the debt component and the book component of leverage and B/M are left unhedged for one month. All results are based on the face value amount of $100.

Panel A: 2-Year Credit Exposures

	Interest Rate		**Leverage Model**		**B/M Model**	
	BBB	A	BBB	A	BBB	A
AHE	$0.106	$0.099	$0.107	$0.100	$0.107	$0.100
PHE	4.18%	4.07%	4.20%	4.09%	4.19%	4.09%

Panel B: 7-Year Credit Exposures

	Interest Rate		**Leverage Model**		**B/M Model**	
	BBB	A	BBB	A	BBB	A
AHE	$0.033	$0.057	$0.034	$0.057	$0.034	$0.057
PHE	4.76%	8.14%	4.87%	8.26%	4.87%	8.26%