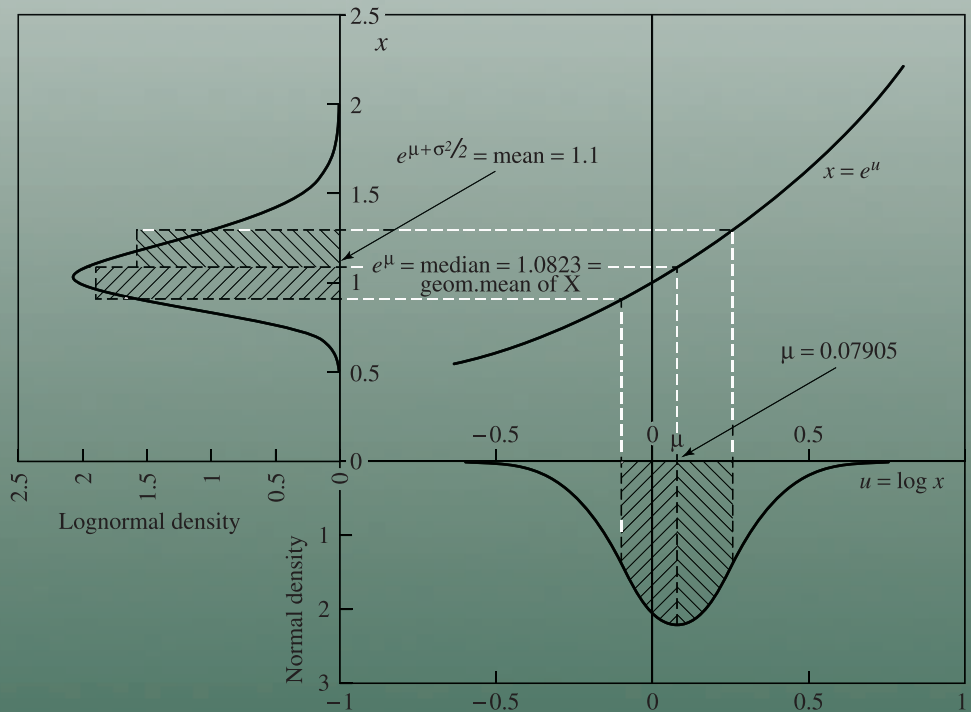


# Bond Pricing and Portfolio Analysis

## Protecting Investors in the Long Run



Olivier de La Grandville

# **BOND PRICING AND PORTFOLIO ANALYSIS**



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Olivier de La Grandville

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*For my wife, Ann, and our children,  
Diane, Isabelle, and Henry,  
and in loving memory of my mother,  
Anika Pataki*



# CONTENTS

Introduction	ix
1 A First Visit to Interest Rates and Bonds	1
2 An Arbitrage-Enforced Valuation of Bonds	25
3 The Various Concepts of Rates of Return on Bonds: Yield to Maturity and Horizon Rate of Return	59
4 Duration: Definition, Main Properties, and Uses	71
5 Duration at Work: The Relative Bias in the T-Bond Futures Conversion Factor	95
6 Immunization: A First Approach	143
7 Convexity: Definition, Main Properties, and Uses	153
8 The Importance of Convexity in Bond Management	171
9 The Yield Curve and the Term Structure of Interest Rates	183
10 Immunizing Bond Portfolios Against Parallel Moves of the Spot Rate Structure	209
11 Continuous Spot and Forward Rates of Return, with Two Important Applications	221
12 Two Important Applications	237
13 Estimating the Long-Term Expected Rate of Return, Its Variance, and Its Probability Distribution	267
14 Introducing the Concept of Directional Duration	295
15 A General Immunization Theorem, and Applications	307
16 Arbitrage Pricing in Discrete and Continuous Time	327
17 The Heath-Jarrow-Morton Model of Forward Interest Rates, Bond Prices, and Derivatives	359
18 The Heath-Jarrow-Morton Model at Work: Applications to Bond Immunization	383
By Way of Conclusion: Some Further Steps	405



<b>Answers to Questions</b>	411
<b>Further Reading</b>	437
<b>References</b>	441
<b>Index</b>	447

# INTRODUCTION

DUKE OF SUFFOLK. I will reward thee for this venturous deed.

*Henry VI*

The last two decades have witnessed an unusually high volatility of interest rates, making the choice of so-called fixed-income instruments a difficult but potentially rewarding task. The reasons for such increasing volatility can be traced back to the end of the system of fixed exchange rates set up at Bretton Woods. Perhaps the dividing date in the postwar period is August 15, 1971, when the United States announced that it would not commit itself further to selling gold to foreign central banks at \$35 per ounce. Since that date, foreign exchange markets have been shaken by high turbulence, which has had direct consequences on the price of loanable funds throughout the world.

It may be useful to remember how the whole problem started. In the twenty years following the end of World War II, the U.S. dollar—after being scarce for some time—became relatively abundant because of the increasing deficit of the American balance of payments. This deficit had three main causes: relatively higher inflation in the United States led to a deficit of the U.S. trade balance; direct investment made by American firms abroad far outweighed foreign direct investments in the United States; and important aid abroad contributed to increasing the deficit. In order to maintain the quasi-fixed parities of their currencies with the U.S. dollar, foreign monetary authorities were compelled to intervene in the foreign exchange markets in order to buy the excess supply of dollars; this excess supply was quickly compounded by expectations of a downfall of the system when it appeared, in the early 1970s, that the amount of foreign exchange held by foreign central banks was nearly four times the amount of the gold reserves held by the United States.

In the years that followed, the major economic powers were unable to set up a durable system of fixed exchange rates. As a result, huge capital movements, anticipating increases or falls in exchange rates, led to strong fluctuations in rates of interest. And monetary authorities, in order to protect the external value of their currencies or to prevent them from rising, took a number of measures that affected those interest rates. Such volatility

was increased by two major factors: the two oil shocks of 1973 and 1979, and inflation spells, varying in intensity from country to country. Finally, we should keep in mind that today most countries, following the example set by the United States in 1979, have decided to define their monetary policies on the basis of specified monetary targets in lieu of pursuing the even more illusory objective of fixing interest rates—a definite blow to stability.

### Analytical background

In finance, as in any science, entirely novel ideas have been relatively scarce, but each time they appeared they have had a profound influence on our way of thinking, and wide applications. In this study on the valuation of bonds, we will make use of some remarkable discoveries that span more than two centuries: they range from Turgot de L'Aulne's *Réflexions* (1766) to the Heath-Jarrow-Morton (1992) stochastic model of the interest rate term structure. Let us now say a brief word about those developments.

To the best of our knowledge, we owe to Turgot de L'Aulne the first explanation of the *equilibrium* price difference between a high-quality asset with given risk and a low-quality asset bearing the same risk. Specifically, his purpose was to determine the equilibrium prices of good lands and bad lands. In his *Réflexions sur la formation et la distribution des richesses*<sup>1</sup> he gave the correct answer: through arbitrage, from an initial disequilibrium situation that would enable arbitrageurs to step in, prices would move to levels such that the *expected return* on both assets (on both types of land) would be the same. He added that the yields—and consequently the prices—on all types of investments bearing different risks were linked together (in his days, the asset bearing the least risk, land ownership, was followed by loans, and finally by assets in trade and industry). The equilibrium levels of the expected returns were separated by a risk premium, as the levels of two liquids of different densities in the two branches of a siphon would always remain separated from each other; one

<sup>1</sup>A. Turgot de L'Aulne, *Réflexions sur la formation et la distribution des richesses*, 1776. Reprinted by Dalloz, Paris, 1957. See particularly propositions LXXXII through XCIX, pp. 128–139.

level could not increase, though, without the other one moving up as well.<sup>2</sup>

It would take more than a century for Turgot de L'Aulne's basic idea, corresponding to a given risk level, to be formalized into an equation, which we owe to Irving Fisher (1892).<sup>3</sup> Fisher showed that in a risk-free world, equilibrium prices corresponded to the equality between the rate of interest and the sum of the direct yield of the capital good and its relative increase in price. Fisher had obtained his equation through an "integral" approach, by equating the price of an asset to the sum (an integral in continuous time) of the discounted cash flows of the asset and differentiating the equality with respect to time. It is interesting to note that, to the best of our knowledge, the first time this equation was obtained through a differential, and an arbitrage, point of view was in Robert Solow's fundamental 1956 "Contribution to the theory of economic growth."<sup>4</sup>

The power and generality of Fisher's equation can hardly be overstated. Since optimal economic growth theory is the theory of optimal investment over time, it is of no surprise that the Fisher equation is central to the solution of variational problems in growth theory. For instance, a typical problem of the calculus of variations is to determine the optimal *path* of investment to maximize a sum, over time, of utility flows generated in a given economy; the problem can be extended to the optimal allocation of exhaustible resources. It can be shown that the Euler equations governing the optimal paths of renewable stocks of capital or those of nonrenewable resources are nothing else than extensions of the Fisher equation.<sup>5</sup>

<sup>2</sup>We cannot resist the pleasure of quoting Turgot in his beautiful eighteenth-century style: "Les différents emplois des capitaux rapportent donc des produits très inégaux : mais cette inégalité n'empêche pas qu'ils s'influencent réciproquement les uns les autres et qu'il ne s'établisse entre eux une espèce d'équilibre, comme entre deux liquides inégalement pesants, et qui communiqueraient ensemble par le bas d'un siphon renversé, dont elles occuperaient les deux branches ; elles ne seraient pas de niveau, mais la hauteur de l'une ne pourrait augmenter sans que l'autre montât aussi dans la branche opposée" (Turgot de L'Aulne, 1776, op. cit., p. 130).

<sup>3</sup>Irving Fisher's original work on the equation of interest was published in *Mathematical Investigations in the Theory of Value and Price* (1892); *Appreciation and Interest* (1896); in "Reprints of Economic Classics" (New York: A. M. Kelley, 1965).

<sup>4</sup>Robert M. Solow, "A contribution to the theory of economic growth," *The Quarterly Journal of Economics*, 70 (1956), pp. 439–469.

<sup>5</sup>The interested reader may refer to Olivier de La Grandville, *Théorie de la croissance économique* (Paris: Masson, 1977), and to "Capital theory, optimal growth and efficiency conditions with exhaustible resources," *Econometrica*, 48, no. 7 (November 1980), pp. 1763–1776.

In the present study, the Fisher equation will quickly be put to work. We will ask the following question: suppose that interest rates do not change within a year (or, equivalently, that within a year, interest rates have come back to their initial level) and that interest rates are described by a flat structure. Consider a coupon-bearing bond without default risk, with a remaining maturity of a few years; suppose, finally, that the bond was initially, and therefore still is after one year, under par. At what speed, within one year, has it come back toward par (in other words, what was the relative change in price of the bond over the course of one year)? We will show that the Fisher equation gives an immediate answer: the relative rate of change will always be equal to the difference between the rate of interest and the bond's simple (or direct) yield, defined as the ratio between the coupon and the bond's initial value.

A central question, however, set up by Turgot remained yet unsolved: for any two assets bearing different risks, what are the equilibrium risk premia born by each asset which lead to equilibrium asset prices? A first answer to this question would have to wait for nearly two centuries: it was not until the 1960s that William Sharpe and Jack Lintner would develop independently their famous capital asset pricing model on the basis of optimal portfolio diversification principles set by Harry Markovitz about a decade earlier. The importance of their discovery is that it yields, at the same time, a measure of risk and the asset's risk premium.

It would take the development of organized markets of derivative products for other major advances to be made: there was first the Black-Scholes<sup>6</sup> and Merton<sup>7</sup> valuation formula of European options (1973); then the recognition by Harrison and Pliska<sup>8</sup> (1981) that the absence of arbitrage was intimately linked to the existence of a martingale probability measure. And a major discovery was finally made by David Heath, Robert Jarrow, and Andrew Morton in 1992;<sup>9</sup> it is of central importance to our subject, since it deals with the stochastic properties of the term

<sup>6</sup>Fischer Black and Myron Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, 81 (1973), pp. 637–654.

<sup>7</sup>Robert Merton, "Theory of rational option pricing," *Bell Journal of Economics and Management Science*, 4 (1973), pp. 141–183.

<sup>8</sup>Michael Harrison and Stanley Pliska, "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and their Applications*, 11 (1981), pp. 215–260.

<sup>9</sup>David Heath, Robert Jarrow, and Andrew Morton, "Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation," *Econometrica*, 60 (1992), pp. 77–105.

structure of interest rates. Heath, Jarrow, and Morton's fundamental discovery is the following: arbitrage-free markets imply that, if a Wiener process drives the forward interest rate, the drift term of the stochastic differential equation cannot be independent from its volatility term; on the contrary, it will be a deterministic function of the volatility. Today this surprising result is central to bond pricing and hedging in the context of a stochastic term structure.

### **An overview of the book**

Volatility of interest rates has direct consequences on the returns of bonds, because of both a price effect and an effect on the reinvestment of coupons. This naturally leads the money manager to play two different roles. Often he will choose to have an active style of management, trying to use maximum information to forecast interest rates and take positions on the bond market; or he can try to protect the investor from the up-and-down swings of rates of interest and try to immunize bond portfolios against those changes.

Our purpose in this book is to discuss some of the most important concepts pertaining to bond analysis and management, with major emphasis on the protection of investors against changes in interest rates. In our introductory chapter, we pay a first visit to bonds and interest rates. In particular, we explain how the various types of bonds are quoted in the financial press by defining basic concepts related to those fixed-income securities, and we introduce the various ways of defining interest rates. But we also ask the rather innocuous question: why do interest rates exist in the first place? The concept of forward rates gives us an opportunity to meet the concept of arbitrage. In the final part of our first chapter, we describe the main features of a particularly important nonstandard type of bond—the convertible. In chapter 2, we take up the valuation of a bond. One special feature of this chapter is to relate the valuation of bond to the arbitrage equation of interest; in particular, we show that the price of the bond and its rate of change must be such that this equation always holds. In chapter 3, we survey the most relevant measures of rates of return on bonds, namely the yield to maturity and the horizon rate of return.

We are then able to turn to the central concept of duration, its main properties and uses (chapters 4 and 5). The all-important concept of

immunization (protection of the investor against fluctuations in interest rates) is introduced in chapter 6. In chapters 7 and 8, the concept of convexity is introduced and related to that of duration. With concrete examples, we show its role in bond management. Chapter 9 deals with the question of the term structure of interest rates and its significance in estimating what the future structure may look like.

The fundamental result of immunization of bond portfolios for *parallel changes* in the term structure of interest rates is demonstrated in chapter 10. In the next chapter, we discuss spot and forward rates of return in continuous time and apply the central limit theorem to justify the workhorse of today's financial theory, the lognormal distribution of the dollars returns and asset prices. This leads us, in chapter 12, to a theoretical justification of the lognormal distribution of asset prices and to an examination of recent methods used to determine the term structure of interest rates. In chapter 13, we take up a subject that seems to us little known and particularly important to bond investors: estimating the long-term expected rate of return, its variance, and its probability distribution. We introduce in chapter 14 a new concept, directional duration, which generalizes the Macaulay and Fisher-Weil concepts.

We then offer, in chapter 15, a general immunization theorem, which we put to the test of *any* change received by the term structure of interest rates. This theorem generalizes the result demonstrated in chapter 10. Arbitrage in discrete and continuous time, which is so central to modern financial analysis, is described in chapter 16. Here we present the concept of probability measure change. In chapter 17, we discuss the famous Heath-Jarrow-Morton model of forward interest rates and bond prices, and in chapter 18 we put this model to work by applying it in some detail to bond portfolio immunization.

### **The target audience**

We have tried to keep the level of technicality of this study as low as possible. For the most part, mathematical developments have been given in appendixes, and only quite simple ones remain in the main text. The first ten chapters can be used for core studies in an undergraduate course; the last part of the book, which requires some familiarity with probability and stochastic calculus, can be used together with a quick review of the first part for a graduate course.

### What is new in this book?

We can promise a systematic valuation of bonds through arbitrage as well as some (nice) surprises and hopefully novel results.

- For instance, the value of a bond can always be expressed as a weighted average between 1 and the ratio of the coupon rate to the rate of interest, with weights that are moving through time. The formula we suggest may be very useful in understanding how the bond's value will evolve over time toward par for any given rate of interest (chapter 2).
- An explanation for the apparently strange phenomenon of the indeterminacy between an increase in a bond's maturity and its duration; we will also give an economic interpretation of the value of the duration of a perpetual bond, whatever its coupon rate may be (chapter 4).
- An introduction of the concept of the *relative* bias regarding the most important contract in financial futures markets, the Chicago Board of Trade T-bond futures contract (chapter 5).
- The importance of convexity in bond management and ways of improving the convexity of a bond portfolio (chapters 7 and 8).
- A proof of the immunization theorem for parallel changes in the spot rate curve, based on a variational approach (chapter 10).
- A discussion of the exact relationship between spot and forward rates of return. In particular, we show that a spot rate structure *cannot* increase and reach a plateau without the forward rate *decreasing* on an interval before that stage (chapter 11).
- Recent methods in deriving the structure of interest rates from coupon bond prices (chapter 12).
- Exact formulas for the expected value and variance of the long-term rate of return on bonds and, more generally, on any investment, directly expressed as functions of one-year estimates, and an explanation of the probability distribution governing the one-year return and the long-horizon return (chapter 13).
- A new concept of duration, based on the directional derivative: *directional duration* (chapter 14). We use this to cope with nonparallel shifts in the spot rate structure.



- A general immunization theorem, applicable to *any* kind of shock the spot rate curve may receive (chapter 15).
- A geometrical interpretation of a martingale probability construction (chapter 16).
- An intuitive demonstration of the Cameron-Martin-Girsanov theorem (chapter 16).
- A complete discussion of the one-factor Heath-Jarrow-Morton model of interest term structure and its applications to bond portfolio duration and immunization. In particular, we give the full demonstration of the HJM one-factor model in the all-important case of the Vasicek volatility coefficient (chapter 17).
- We discuss in detail how the HJM model duration and immunization results differ from—and generalize—the classical results (chapter 18).

In order for this text to be as user-friendly as possible, each chapter is followed by a series of questions, problem sets, and projects, and detailed solutions for all of these can be found at the back of the book.

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Although, for no apparent reason, he has not yet made the team, his favorite baseball club is still the San Francisco Giants.

# 1

## A FIRST VISIT TO INTEREST RATES AND BONDS

SHYLOCK. ... and he rails ... on my well-won thrift  
Which he calls interest.

*The Merchant of Venice*

MACBETH. Cancel and tear to pieces that great bond  
Which keeps me pale.

*Macbeth*

### Chapter outline

<b>1.1 A first look at interest rates or rates of return</b>	<b>3</b>
1.1.1 Horizons shorter than or equal to one year	4
1.1.2 Horizons longer than one year	5
1.1.2.1 Horizons longer than one year and integer numbers	5
1.1.2.2 Horizons longer than one year and fractional	5
<b>1.2 Forward rates and an introduction to arbitrage</b>	<b>6</b>
<b>1.3 Creating synthetically a forward contract</b>	<b>9</b>
<b>1.4 But why do interest rates exist in the first place?</b>	<b>10</b>
<b>1.5 Kinds of bonds, their quotations, and a first approach to yields on bonds</b>	<b>13</b>
1.5.1 Treasury bills	13
1.5.2 Treasury notes	17
1.5.3 Treasury bonds	18
<b>1.6 Bonds issued by firms</b>	<b>18</b>
<b>1.7 Convertible bonds</b>	<b>19</b>

### Overview

A bond is a certificate issued by a debtor promising to repay borrowed money to a lender or, more generally, to the owner of the certificate, at fixed times. The following glossary will be immediately useful.

- The amount borrowed is called the *principal*; it is often referred to as the *par value*, or *face value*, of the security.
- The time span until reimbursement will be called the *maturity* of the bond. Normally, “maturity” designates the date at which the principal is due; in this book, however, we will often consider maturity to be the difference between the date of reimbursement and today.
- The contract underlying the security may involve a number of payments of equal size. Those cash flows to be received at regular intervals

by the lender are called *coupons*; they may be paid yearly (as is the case frequently in Europe), twice a year, or on a quarterly basis (as in the United States).

- The *coupon rate* is the yearly value of cash flows paid by the bond issuer divided by the par value.
- If no coupons are to be paid, the bond is called a *zero-coupon bond*. A zero-coupon bond is said to be a *pure-discount security*. Notice that all coupon-bearing bonds automatically become pure-discount securities once the last coupon and par value are all that remain to be paid out by the issuer.

If the borrower does not default on his promises, the lender can count on fixed payments throughout the lifetime of the security. But even in this case, owning such a financial instrument does not guarantee a fixed yearly *yield* on invested money.

The first and foremost reason yields on bonds are likely to fluctuate comes from an essential property of the bond: as a promise on future cash flows, it has a value that is most likely to move up or down, because each of those promises can fluctuate in value—a fact which, in turn, needs to be explained.

Suppose a large company with an excellent credit rating (soon to be defined) borrows today at a 5% coupon rate by issuing 20-year maturity bonds. The company has done this in a financial market where a supply of loanable funds meets a demand for such funds; presumably, the numbers of lenders and borrowers are such that none of them, individually, is able to have a significant impact on this market. We can reasonably imagine that if the company has tried to obtain, and has been able to secure, a 5% loan in that peculiar securities market, it is because:

1. It would have been unable to obtain it at a lower coupon rate (4.75% for instance); competition from other borrowers for relatively scarce loanable funds has pushed up the coupon rate to 5%.
2. It would have been foolish for firms to offer a higher coupon rate; for a 5.25% coupon rate, borrowers would have faced an excess supply of loanable funds, which would have driven down the coupon rate to 5%.

Suppose now, a few weeks later, conditions on that financial market have considerably changed. Due to good economic prospects, demand for loans from the same category of firms has increased; however, supply has

not changed. On the 20-year maturity market, the coupon rate is bound to increase, perhaps to 5.50%. Obviously, something momentous will happen to our 5% coupon rate security; its price will fall. Indeed, there exists a market for that bond, and the initial buyer may want to sell it. It is evident, however, nobody will want to buy a “5% coupon” at par when for the same price the buyer can get a “5.5%.” Therefore, both the seller and the buyer of the previously issued security may agree on a price that will be lower than par. If a few days later market conditions change again (this time in the opposite direction, perhaps because large inflows of capital movements from abroad have increased the supply of loanable funds on that market, entailing a decrease in the coupon rate from 5.5% to 5.25%), the price of the security is bound to move again, this time upward.

We will determine in chapter 2, on bond valuation, exactly how large those fluctuations will be. For the time being, our aim is to recognize that bond values will move up or down according to conditions of the financial market, and this creates the first factor of uncertainty for the bond holder.

To understand a second reason for uncertainty, suppose that our investment horizon, in buying the 20-year bond, is a few years—or even 20 years. The various cash flows received will be reinvested at rates we do not know today. So even if we buy a bond at a given price and hold onto it until maturity, we cannot be certain of the yield of that investment. This uncertainty is referred to as *reinvestment risk*.

There are thus two reasons why assets that do not carry any default risk are still prone to uncertainty. First, even for zero-coupon bonds there is some price risk if bonds are not held until maturity; second, there is reinvestment risk. These two sources of risk find their origins in the general conditions of supply and demand of loanable funds, which usually are liable to change, in most cases in an unpredictable way. Naturally, some fund managers will try to know more, and quicker, than other managers about such changes in order to benefit from bond price increases, for instance.

### 1.1 A first look at interest rates or rates of return

Until now we have not yet referred to, much less defined, *rate of interest*. The different concepts of rate of interest can be distinguished according to

- the time of the inception of the contract;
- the time at which the loan begins;
- the time at which the loan ends;
- the mode of calculation of the interest between the beginning and the end of the loan.

A loan agreed upon on a certain date, starting at that date will give rise to a *spot* interest rate. A loan starting after the date of the contract is a *forward* contract. In the following discussion we will refer to the *horizon* as the length of time between the start of the loan and the time of its reimbursement. This horizon will also be referred to as the *trading period*.

The easiest way to define a spot rate is to say it is equal to the (yearly) return of a zero-coupon with a remaining maturity equal to the investor's horizon. Therefore, it is the yearly rate of return that transforms a zero-coupon value into its par value. Unfortunately, this definition is not quite precise, because there are many different ways to calculate a “per year rate.” Generally, however, the following rules are followed: Call  $i$  the rate of interest,  $B_t$  the value of the zero-coupon bond at time  $t$ , and  $B_T$  the par value of the zero-coupon bond (or the reimbursement value of the bond at time  $T$ ). The investor's horizon is  $h = T - t$ .

### 1.1.1 Horizons shorter than or equal to one year

For horizons  $T - t$  equal to or shorter than one year, the rate of interest is the relative rate of increase between the bond's value at  $t$ ,  $B_t$ , and the par value  $B_T$ , divided by the horizon's length  $T - t$ . This is the simplest definition that can exist, since it corresponds to capital gain per time unit (per year, if the time unit is a year). For example, suppose a zero-coupon with par value equal to  $B_T = 100$  is worth 98 at time  $t$  (today, three months before maturity). The rate of interest for the three-month period, expressed on a yearly basis, is

$$i = \frac{B_T - B_t}{B_t} \bigg/ (T - t) = \frac{100 - 98}{98} \bigg/ \frac{1}{4} = 8.163\% \text{ per year} \quad (1)$$

Notice there is an alternate way to define the same concept: rearrange the above expression and write it as

$$i(T - t) = \frac{B_T - B_t}{B_t}$$

or, equivalently,

$$B_t + i(T - t)B_t = B_t[1 + i(T - t)] = B_T \quad (2)$$

The short-term spot interest rate is thus seen as one that determines the interest due on the loan in the following way: The interest due,  $i(T - t)B_t$ , is directly proportional to the amount loaned  $B_t$  and to the horizon of the loan  $(T - t)$ . In other words, the interest to be paid for fixed  $B_T$  is a *linear* function of the horizon  $(T - t)$ .

### 1.1.2 Horizons longer than one year

We will consider two cases: First, the horizon is an integer number of years; second, the horizon is not an integer (for instance,  $T - t = 3.45$  years).

#### 1.1.2.1 Horizons longer than one year and integer numbers

For horizons longer than one year *and* integer numbers, another way of determining and computing rates of interest is generally used. Since over a number of years the amount borrowed often does not imply payment of interest at the end of each year but only at the maturity of the loan, the amount due at  $T$ , after  $T - t$  years, is the following, applying the well-known principle of compound interest:

$$B_t(1 + i)^{T-t} = B_T \quad (3)$$

Therefore, the rate of interest  $i$ , or the rate of return on a loan starting at  $t$  and reimbursed  $T - t$  years later, is

$$i = \left( \frac{B_T}{B_t} \right)^{1/(T-t)} - 1 \quad (4)$$

For instance, consider a zero-coupon bond with three years of maturity, par value  $B_T = 100$ , and value today  $B_t = 78.48$ . Applying (4), the rate of return on that default-free bond is 8.413%. And we can infer that the three-year spot rate, or the rate of interest for three-year loans, is 8.413%.

#### 1.1.2.2 Horizons longer than one year and fractional

We can tackle the case of a horizon  $T - t$  larger than one year and fractional in the following way. When horizons are longer than one year and not an integer number (for instance, 3.45 years), there are two possibil-

ities: Either directly apply formula (4), replacing  $T - t$  by 3.45, which leads to 7.277%; or, logically, consider applying a combination of formula (4) (compound interest), the integer number of years (3 years) and formula (2) (simple interest), the remaining fraction of a year (0.45 of a year). This implies that our rate of interest would be calculated as follows. When  $\overline{T - t}$  designates the integer part of the horizon and  $\alpha$  the remaining fraction of the year (we have  $T - t = \overline{T - t} + \alpha$ ; in our example,  $T - t = 3.45$  years;  $\overline{T - t} = 3$  years and  $\alpha = 0.45$  year),  $i$  would be such that

$$B_t(1 + i)^{\overline{T - t}} + B_t(1 + i)^{\overline{T - t}}\alpha i = B_t(1 + i)^{\overline{T - t}}(1 + \alpha i) = B_T \quad (5)$$

Unfortunately, equation (5) cannot be solved analytically for  $i$ , and only trial and error (or numerical methods) can be used. Applying such methods would lead to a value of  $i = 7.2576\%$ .<sup>1</sup> Since this method is cumbersome, formula (4) is usually preferred.

There are many other ways to determine a rate of return or rate of interest. First, an interest can be calculated and paid once a year, or a number of times per year. (It can be paid, for instance, twice a year, four times a year, every month; for reasons that will become clear later in this book, dealt with in detail in chapter 11, it can even be considered to be calculated an infinite number of times per period—equivalently, it can then be considered as being paid continuously.)

## 1.2 Forward rates and an introduction to arbitrage

On the other hand, as we mentioned earlier, interest rates can be agreed upon today for loans starting only at a future date, for a fixed horizon or trading period; those are forward rates. Call  $f(0, t, T)$  a forward rate decided upon today, at time 0, for a loan starting at  $t$  and reimbursed at  $T$ . A spot rate (for a loan that starts at time 0 and is reimbursed at any time  $u$ ) is a particular case of a forward rate. We denote it  $f(0, 0, u) \equiv i(0, u)$  for a maturity  $u$ , or  $f(0, 0, t) \equiv i(0, t)$  for a maturity  $t$ . The arbitrage-enforced links between forward rates and spot rates are very strong, and we will briefly explain here why.

There is in fact an arbitrage-driven equivalency either in borrowing or in lending directly at rate  $i(0, T)$ , on the one hand, or via the shorter rate

<sup>1</sup>The reason we get a lower rate of return applying equation (5) instead of equation (4) is that  $1 + \alpha i$  is *larger* than  $(1 + i)^\alpha$  when  $0 < \alpha < 1$ ; the converse is true for  $\alpha > 1$ .



$i(0, t)$  and the forward rate  $f(0, t, T)$ , on the other. This equivalency must translate via this equation:

$$[1 + i(0, t)]^t [1 + f(0, t, T)]^{T-t} = [1 + i(0, T)]^T \quad (6)$$

because arbitrageurs would otherwise step in, and their very actions would establish this equilibrium. We will now demonstrate this, but first we must define what we mean by *arbitrage*.

Arbitrage consists of taking financial positions that require no investment outlay and that guarantee a profit. At first sight, it may seem strange that a profit may be generated without any upfront personal outlay or expenditure, but in many circumstances this is precisely the case. We will now present such a case, supposing to that effect that the left-hand side of (6) is larger than the right-hand side. We would thus have

$$[1 + i(0, t)]^t [1 + f(0, t, T)]^{T-t} > [1 + i(0, T)]^T$$

This means it is more rewarding to lend and more expensive to borrow over a period  $T$  via two contracts (the spot, short one, and the forward contract) than via the unique long-term contract. Arbitrageurs would immediately borrow \$1 on the long contract, invest this dollar in the short contract, and *at the same time* invest the proceeds to be received at  $t$ ,  $[1 + i(0, t)]$ , on the forward market for a time period  $T - t$ . The forward rate being  $f(0, t, T)$ , they would receive  $[1 + i(0, t)]^t [1 + f(0, t, T)]^{T-t}$  at time  $T$ , which is larger than their disbursement  $[1 + i(0, T)]^T$  by hypothesis. Thus, without any initial outlay on their part, they would reap at time  $T$  a sure profit equal to  $[1 + i(0, t)]^t [1 + f(0, t, T)]^{T-t} - [1 + i(0, T)]^T > 0$ .

What would be the consequences of such actions? First, demand would increase on the long spot market, and the price of loanable funds on that market,  $i(0, T)$ , would start to increase; second, supply both on the short spot market and on the forward market would start to increase, entailing a decrease in the prices of funds on those markets,  $i(0, t)$  and  $f(0, t, T)$ , respectively. This would take place until, barring transaction costs, equilibrium was restored.

But arbitrageurs would not be the only ones to ensure a quick return to equilibrium. Consider the “ordinary” operators on these three markets; by “ordinary” operators we mean the persons, firms, or government agencies that are the lenders or the borrowers acting on those markets as simple investors or debtors; they simply carry out the operation for its own sake. Suppliers, or lenders, have no reason to lend on the long spot

market when, at no additional risk, they can get higher rewards on the short spot market and on the forward market; thus their supply will shrink on the long spot market, contributing to the increase of  $i(0, T)$ . Supply will therefore increase on the short spot market and on the forward market, thus driving down  $i(0, t)$  and  $f(0, t, T)$ . And of course, symmetrically, borrowers have no reason to take the two-market route when they can borrow directly on the long spot market; demand will decline on the former markets and will increase on the latter, thus contributing to the increase of  $i(0, T)$  and the decrease of  $i(0, t)$  and  $f(0, t, T)$ .

One can easily see what would happen if initially the converse were true; arbitrageurs, if they reacted quickly enough, could reap the (positive) difference between the right-hand side of (6) and its left-hand side by borrowing on the shorter markets and lending the same amount on the long one. The rest of the analysis would be the same, and all results would be symmetrical. We thus can conclude that, barring special transaction costs, equation (6) will be maintained by forces generated both by arbitrageurs and ordinary operators on those financial markets.

Equation (6) permits us to express the forward rate  $f(0, t, T)$  as a function of the spot rates,  $i(0, t)$  and  $i(0, T)$ . Indeed, we can rearrange (6) to obtain

$$f(0, t, T) = \frac{[1 + i(0, T)]^{T/(T-t)}}{[1 + i(0, t)]^{t/(T-t)}} - 1 \quad (7)$$

Any interest rate *plus* one is usually called the “dollar return”; it is equal to the coefficient of increase of a dollar when invested at the yearly rate  $i(0, t)$ . For instance, if a rate of interest (spot or forward) is 7% per year, 1.07 is the dollar return corresponding to that rate of interest. We can deduce from (6) an important property of the spot dollar return  $1 + i(0, T)$ . Taking (6) to the  $1/T$  power, it can then be written as

$$1 + i(0, T) = [1 + i(0, t)]^{t/T} [1 + f(0, t, T)]^{(T-t)/T} \quad (8)$$

Remembering that the spot rate  $i(0, t)$  can be considered as the forward rate at time 0 for a loan starting immediately, written  $f(0, 0, t)$ , we can see immediately that the  $T$ -horizon spot dollar return,  $1 + i(0, T)$ , is the

weighted geometric average<sup>2</sup> of the dollar forward returns, the weights being the shares of the various trading periods for the forward rates in the total trading period  $T$ .

Of course, what we have demonstrated through arbitrage and traditional trading in equations (6), (7), and (8) generalizes to any arbitrary partitioning of the time interval  $[0, T]$ ; let  $t_1, t_2, \dots, t_n$  be an arbitrary partitioning of  $[0, T]$  into  $n$  intervals, those intervals being not necessarily the same size. We then have the arbitrage-enforced equation

$$[1 + i(0, t_1)]^{t_1} [1 + f(0, t_1, t_2)]^{t_2 - t_1} \dots [1 + f(0, t_{n-1}, t_n)]^{t_n - t_{n-1}} = [1 + i(0, t_n)]^{t_n} \quad (9)$$

The  $t_n$  horizon spot dollar return is the geometric average of all dollar forward rates:

$$1 + i(0, t_n) = \prod_{j=1}^n [1 + f(0, t_{j-1}, t_j)]^{(t_j - t_{j-1})/t_n} \quad (10)$$

Forward rates will be discussed again in chapter 9 of this book; we will deal in more detail with the way interest rates are calculated in chapter 11. Finally, forward rates will be used extensively in chapters 14 and 15.

### 1.3 Creating synthetically a forward contract

We will now show that we are always able to create synthetically a forward contract from spot rates. Indeed, suppose you want to borrow \$1 at the future date  $t$ , to be reimbursed at  $T$ .

If you want to receive \$1 at  $t$ , you can borrow on the long market an amount (to be determined) and lend it on the short market; the amount to be lent must be such that, at rate  $i(0, t)$ , it becomes 1 at  $t$ . It must therefore be equal to  $1/[1 + i(0, t)]^t$ . If this is the amount you borrow at time 0, you owe  $\{1/[1 + i(0, t)]^t\}[1 + i(0, T)]^T$  at time  $T$ . You have incurred no disbursement at time 0, because at the same time you have borrowed and lent  $1/[1 + i(0, t)]^t$ ; at time  $t$  you have received 1, and at time  $T$  you reimburse the amount just mentioned; notice, of course, this corresponds

<sup>2</sup>Recall that the geometric weighted average of  $x_1, \dots, x_n$  with weights  $f_1, \dots, f_n$  (such that  $\sum_{j=1}^n f_j = 1$ ) is  $G = \prod_{j=1}^n x_j^{f_j}$ . We will make use of this concept again in chapter 13, when dealing with the estimation of the long-term expected return of a bond or bond portfolios.

exactly to the amount we would have obtained applying the forward rate defined implicitly in (6) and explicitly in (7). Indeed, applying that rate to \$1 borrowed on the forward market, we would have to repay at  $T$

$$\begin{aligned} [1 + f(0, t, T)]^{T-t} &= \left\{ 1 + \frac{[1 + i(0, T)]^{T/(T-t)} - 1}{[1 + i(0, t)]^{t/(T-t)}} \right\}^{T-t} \\ &= \frac{[1 + i(0, T)]^T}{[1 + i(0, t)]^t} \end{aligned} \quad (11)$$

which is exactly the amount to be reimbursed in this synthetic forward contract made up of two spot rate contracts.

#### 1.4 But why do interest rates exist in the first place?

Until now we have examined a variety of interest rates, depending on the date of the contract and the trading period. In particular, we were led to distinguish between spot rates and forward rates. However, there is an important question that we could have asked in the first place: why do interest rates exist? The answer to this question is not so obvious and merits some comments.

Most people, when asked this apparently innocuous question, respond that since inflation is a pervasive phenomenon common to all places and times, it is therefore only natural that lenders will want to protect themselves against rising prices; borrowers might well agree with that, therefore establishing the existence of an equilibrium rate of interest.

It is rather strange that inflation is so often the first explanation given for the existence of interest rates. In fact, positive interest rates have always been observed in countries where there was practically no inflation—or even a slight decrease in prices. Although inflation is certainly a contributor to the magnitude of interest rates, it is in no way their primary determinant. There are two other principal causes.

First, and foremost, an interest rate exists because one monetary unit can be transformed, through fixed capital, into more than one unit after a certain period of time. Some people are able, and willing, to perform that rewarding transformation of one monetary unit into some fixed capital good. On the other hand, some are willing to lend the necessary amount, and it is only fitting that a price for that monetary unit be established

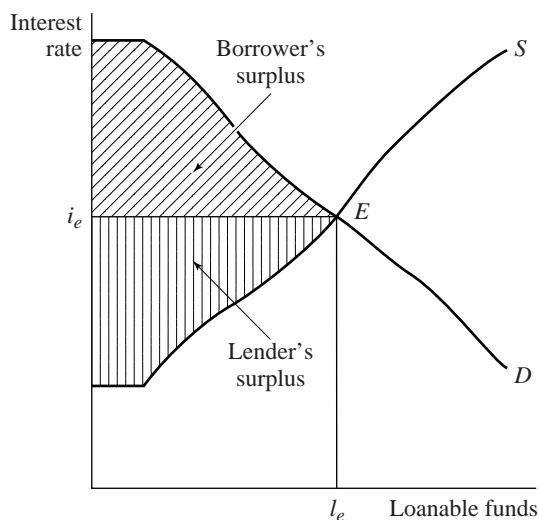
between the borrowers and the lenders. Let us add that on any such market *both* borrowers and lenders will find a benefit in this transaction. Let us see why.

Suppose that at its equilibrium the interest rate sets itself at 6%. Generally, lots of borrowers would have accepted a transaction at 7%, or even at much higher levels, depending on the productivity of the capital good into which they would have transformed their loan. The difference between the rate a borrower would have accepted and the rate established in the market is a benefit to him. On the other hand, consider the lenders: they lent at 6%, but lots of them might well have settled for a lower rate of interest (perhaps 4.5%, for instance). Both borrowers and lenders derive benefits (what economists call *surpluses*) from the very existence of a financial market, in the same way that buyers and suppliers derive surpluses from their trading at an equilibrium price on markets for ordinary goods and services.

Second, an interest rate exists because people usually prefer to have a consumption good now rather than later. For that they are prepared to pay a price; and as before, others may consider they could well part with a portion of their income or their wealth today if they are rewarded for doing so. There again, borrowers and lenders will come to terms that will be rewarding for both parties, in a way entirely similar to what we have seen in the case of productive investments.

Together with *expected* inflation, these reasons explain the existence of interest rates. The level of interest rates will depend on the attitudes of the parties regarding each kind of transaction, and of course on the perceived risk of the transactions involved.

An important point should be made here that, to the best of our knowledge, has received short shrift until now. Anyone can agree that, for a certain category of loanable funds, there exists at any point in time a supply and a demand that lead to an equilibrium price, that is, an equilibrium rate of interest. We should add here that, as in any commodity market, the mere existence of an equilibrium interest rate set at the intersection of the supply-and-demand curves for loanable funds entails two major consequences: First, the surplus of society (the surplus of the lenders and the borrowers) is maximized (shaded area in figure 1.1); second, the amount of loanable funds that can be used by society is also maximized (figure 1.1).

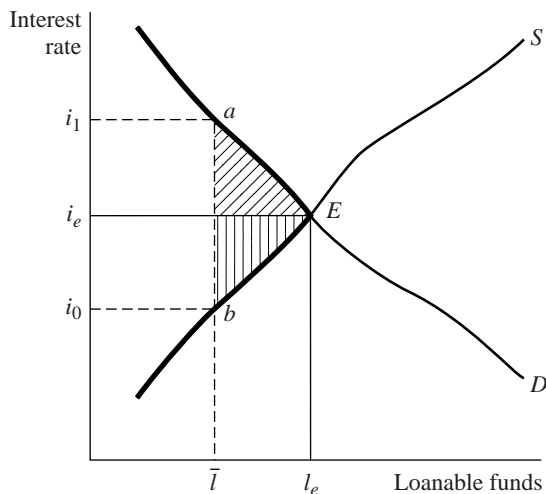


**Figure 1.1**

**An equilibrium rate of interest on any financial market maximizes the sum of the lenders' and the borrowers' surpluses, as well as the amount of loanable funds at society's disposal.**

Suppose now that for some reason interest rates are fixed at a level different from the equilibrium level ( $i_e$ ). Two circumstances, at least, can lead to such fixed interest rates. First, government regulations could prevent interest rates from rising above a certain level (for instance  $i_0$ , below  $i_e$ ). Alternatively, a monopoly or a cartel among lenders could fix the interest above its equilibrium value  $i_e$  (for instance, at  $i_1$ ).

If interest rates are fixed in such a way, the funds actually loaned will always be the smaller of either the supply or the demand. Indeed, the very existence of a market supposes that one cannot force lenders to lend whatever amount the borrowers wish to borrow if interest rates are below their equilibrium value—and vice versa if interest rates are above that value. We can then draw the curve of effective loans when interest rates are fixed—the kinked, dark line in figure 1.2. Should the interest rate be fixed either at  $i_0$  or at  $i_1$ , the loanable funds will be  $\bar{l}$  (less than  $l_e$ ), and the loss in surplus for society will be the shaded area  $abE$ . Innocuous as these properties of financial markets may seem, they are either unknown or ignored by governments when they nationalize banks—or when they close their eyes to cartels established among banks (nationalized or not).



**Figure 1.2**

Any artificially fixed interest rate reduces both the amount of loanable funds at society's disposal and the surplus of society (the hatched area).

## 1.5 Kinds of bonds, their quotations, and a first approach to yields on bonds

There are two main ways to distinguish between bonds: First, according to their maturity, starting with personal saving deposits or very short-term deposits, going to long-term (for instance, 30-year maturity securities); and second, by separating bonds according to their level of riskiness, from entirely risk-free bonds issued by the U.S. Treasury, for example, to so-called “junk” bonds. Since this book is concerned mainly with defaultless bonds, we will consider bonds according to their maturity. But we will also say a word about ranking bonds according to their riskiness.

### 1.5.1 Treasury bills

Treasury bills have a maturity of one year or less and of course are issued on a discount basis; their par value is usually \$10,000. In table 1.1 (see the last part of the table), six features of Treasury bills are presented:

par = par value of the Treasury bill (for instance: par = \$10,000)

**Table 1.1**  
**Information about U.S. Treasury Bonds, notes, and bills**

# U.S. TREASURY ISSUES

**Monday, June 7, 1999**

Representative and indicative Over-the-Counter quotations based on transactions of \$1 million or more.  
Treasury bond, note and bill quotes are as of mid-afternoon. Columns in bond and note bid-and-asked quotes represent 32nds; 101:01 means 101 1/32. Net changes in 32nds. Treasury bill quotes in hundredths, quoted on terms of a rate of discount. Days to maturity calculated from settlement date. All yields are based on a one-day settlement and calculated on the offer quote. Current 13-week and 26-week bills are boldfaced.  
For bonds callable prior to maturity, yields are computed to the earliest call date for issues quoted above par and to the maturity date for issues quoted below par. n-Treasury note, w-When issued; daily change is expressed in basis points.  
Source: Dow Jones Telerate/Cantor Fitzgerald

GOVT. BONDS & NOTES					Maturity					Mat.							
Maturity	Rate	Mo/Yr	Bid	Asked	Chg. Yld.	Rate	Mo/Yr	Bid	Asked	Chg. Yld.	Rate	Mo/Yr	Bid	Asked	Chg. Yld.		
6 1/4	Jun 99n	100:02	100:04	-.384	5 7/8	Nov 01n	100:16	100:18	.....	5.63	11 1/4	Nov 09-14	142:17	142:23	+4 6.14		
6 1/4	Jun 99n	100:03	100:05	-.1	4.05	61 1/4	Dec 01n	101:01	101:03	.....	5.66	11 1/4	Feb 15	149:31	150:05	+5 4.90	
6 1/2	Jul 99n	100:05	100:07	-.1	4.12	14 1/4	Jan 02n	101:11	101:13	.....	5.67	105 1/2	Aug 15	144:06	144:12	+4 6.22	
5 7/8	Jul 99n	100:04	100:06	.....	4.49	14 1/4	Feb 02	120:31	121:05	.....	5.65	97 1/2	Nov 15	134:28	137:02	+3 4.24	
6 1/2	Jul 99n	100:08	100:10	-.1	4.61	61 1/4	Feb 02n	101:12	101:14	.....	5.67	91 1/4	Feb 16	130:20	130:26	+4 6.25	
6	Aug 99n	100:07	100:09	.....	4.41	6 1/2	Mar 02n	102:11	102:13	.....	5.68	71 1/4	May 16	110:10	110:16	+4 6.24	
8	Aug 99n	100:19	100:21	-.1	4.37	7 1/2	May 02n	104:24	104:26	.....	5.69	71 1/2	Nov 16	113:01	113:05	+4 6.25	
5 1/2	Aug 99n	100:07	100:09	.....	4.56	6 1/2	May 02n	102:03	102:05	.....	5.70	63 1/4	May 17	126:16	126:22	+4 6.25	
6 1/2	Aug 99n	100:14	100:16	-.1	4.58	6 1/2	Jun 02n	101:14	101:16	.....	5.71	87 1/2	Aug 17	128:02	128:08	+5 6.25	
5 1/4	Sep 99n	100:08	100:10	.....	4.68	3 1/2	Jul 02l	99:25	99:26	.....	3.69	91 1/2	May 18	131:11	131:17	+3 6.26	
7 1/2	Sep 99n	100:21	100:23	-.1	4.72	6	Jul 02n	100:23	100:25	.....	5.72	9	Nov 18	130:12	130:18	+2 6.26	
6	Oct 99n	100:11	100:13	.....	4.79	6 1/2	Aug 02n	101:24	101:26	.....	5.74	87 1/2	Feb 19	129:06	129:12	+3 6.26	
5 1/2	Oct 99n	100:06	100:08	.....	4.95	6 1/4	Aug 02n	101:13	101:15	.....	5.74	81 1/2	Aug 19	121:01	121:07	+2 6.26	
7 1/2	Oct 99n	100:29	100:31	-.1	4.95	6 1/4	Sep 02n	100:10	100:12	.....	5.75	81 1/2	Feb 20	125:21	125:27	+3 6.26	
5 1/2	Nov 99n	100:10	100:12	-.1	4.97	5 1/2	Oct 02n	99:30	100:00	.....	5.75	83 1/4	May 20	128:22	128:28	+2 6.26	
7 1/2	Nov 99n	101:06	101:08	-.1	4.91	11 1/2	Nov 02	117:22	117:26	.....	5.83	83 1/4	Aug 20	128:26	129:00	+2 6.26	
5 1/2	Nov 99n	100:08	100:10	-.1	4.95	5 1/4	Nov 02n	99:27	99:29	.....	5.78	77 1/2	Feb 21	118:19	119:01	+2 6.26	
7 1/4	Nov 99n	101:08	101:10	-.1	4.93	5 1/2	Dec 02n	99:15	99:17	.....	5.77	81 1/2	May 21	121:31	122:05	+3 6.25	
5 1/2	Dec 99n	100:08	100:10	-.1	5.05	5 1/2	Jan 03n	99:01	99:03	.....	5.78	81 1/2	Aug 21	122:02	122:08	+2 6.26	
7 1/4	Dec 99n	101:13	101:15	-.1	5.05	6 1/4	Feb 03n	101:14	101:16	.....	5.79	8	Nov 21	120:23	120:29	+1 6.25	
6 1/4	Jan 00n	100:23	100:25	.....	5.03	10 1/4	Feb 03	115:30	116:02	.....	5.84	71 1/4	Aug 22	112:01	112:05	.....	6.25
5 1/2	Jan 00n	100:06	100:08	.....	4.97	5 1/2	Feb 03n	98:30	99:00	.....	5.80	77 1/2	Nov 22	116:26	116:30	+1 6.24	
7 1/4	Jan 00n	101:21	101:23	-.1	5.00	5 1/2	Mar 03n	98:30	99:00	.....	5.79	61 1/4	Aug 23	110:31	111:03	+1 6.22	
5 1/2	Feb 00n	100:15	100:17	-.1	5.07	5 1/4	Apr 03n	99:25	99:27	.....	5.79	71 1/2	Nov 24	116:09	116:13	+1 6.21	
8 1/2	Feb 00n	102:08	102:10	-.1	5.02	5 1/2	May 03n	98:27	98:29	.....	5.81	77 1/2	Feb 25	118:01	118:05	+1 6.20	
5 1/2	Feb 00n	100:08	100:10	.....	5.05	5 1/2	Jun 03n	98:14	98:16	.....	5.79	67 1/2	Aug 25	108:16	108:20	+1 6.20	
5 1/2	Feb 00n	101:12	101:14	-.1	5.07	5 1/4	Jun 03n	97:29	97:31	.....	5.80	6	Feb 26	97:21	97:23	.....	6.17
5 1/2	Mar 00n	100:07	100:09	.....	5.13	5 1/4	Aug 03n	99:21	99:23	.....	5.82	61 1/4	Aug 26	107:05	107:07	.....	6.20
5 1/2	Mar 00n	101:10	101:12	.....	5.11	11 1/2	Aug 03	119:04	119:08	.....	5.87	61 1/2	Nov 26	104:00	104:02	.....	6.19
6 1/2	Apr 00n	100:05	100:07	.....	5.23	4 1/4	Nov 03n	93:31	94:01	.....	5.79	67 1/2	Feb 27	105:24	105:26	.....	6.18
5 1/2	Apr 00n	100:09	100:11	.....	5.22	11 1/2	Nov 03	122:29	123:03	.....	5.88	61 1/2	Aug 27	102:19	102:21	.....	6.17
6 1/4	Apr 00n	101:08	101:10	.....	5.22	4 1/4	Feb 04n	95:27	95:28	.....	5.76	3 1/2	Nov 27	99:16	99:18	.....	6.16
6 1/2	May 00n	100:30	100:30	.....	5.26	5 1/2	Feb 04n	100:06	100:08	+1	5.81	5 1/2	Apr 28	95:07	95:08	+7	3.90
6 1/2	May 00n	100:06	100:08	+1	5.19	5 1/4	May 04n	97:31	98:00	+1	5.72	5 1/2	Nov 28	91:22	91:24	.....	6.11
6 1/4	May 00n	100:28	100:30	.....	5.25	7 1/4	May 04n	105:28	105:30	+1	5.85	5 1/4	Nov 28	88:23	88:24	.....	6.07
5 1/2	Jun 00n	100:01	100:03	.....	5.28	12 1/2	May 04	127:05	127:11	.....	5.90	5 1/4	Feb 29	90:00	90:01	.....	5.97
5 1/2	Jun 00n	100:17	100:19	.....	5.29	7 1/4	Aug 04n	106:01	106:03	+1	5.87	3 1/2	Apr 29	99:15	99:16	+5	3.90
5 1/2	Jul 00n	99:31	100:01	.....	5.34	13 1/4	Aug 04	134:10	134:16	.....	5.93						
6 1/2	Jul 00n	100:26	100:28	.....	5.32	7 1/2	Nov 04n	109:02	109:04	.....	5.89						
6	Aug 00n	100:21	100:23	.....	5.36	11 1/2	Nov 04	126:01	126:07	.....	5.91						
6 1/4	Aug 00n	103:26	103:28	.....	5.34	7 1/2	Feb 05n	107:20	107:22	+2	5.89						
5 1/2	Aug 00n	99:20	99:22	.....	5.38	6 1/2	May 05n	102:31	103:01	+2	5.89						
6 1/4	Aug 00n	100:30	101:00	.....	5.39	8 1/4	May 00-05	102:18	102:20	.....	5.33						
4 1/2	Sep 00n	98:25	98:27	+1	5.42	12	May 05	129:26	130:00	+1	5.93						
6 1/2	Sep 00n	100:26	100:28	.....	5.42	6 1/2	Aug 05n	102:31	103:01	+1	5.91						
4	Oct 00n	98:01	98:03	+1	5.43	10 1/4	Aug 05	124:12	124:18	+1	5.95						
5 1/4	Oct 00n	100:11	100:13	.....	5.44	5 1/2	Nov 05n	99:24	99:26	+1	5.91						
5 1/4	Nov 00n	100:11	100:13	+1	5.45	5 1/2	Feb 06n	98:09	98:11	+1	5.93						
8 1/2	Nov 00n	104:03	104:05	.....	5.44	9 1/2	Feb 06	118:22	118:26	+1	5.93						
4 1/2	Nov 00n	98:23	98:27	+1	5.45	6 1/2	May 06n	105:05	105:07	+2	5.94						
5 1/2	Nov 00n	100:05	100:07	.....	5.47	7	Jul 06n	105:30	106:00	+2	5.95						
4 1/2	Dec 00n	98:21	98:23	.....	5.49	6 1/2	Oct 06n	103:04	103:06	+1	5.96						
5 1/2	Dec 00n	99:30	100:00	.....	5.50	3 1/4	Jan 07l	96:23	96:24	+1	3.87						
4 1/2	Jan 01n	98:13	98:15	+1	5.48	4 1/4	Feb 07n	101:24	101:26	+1	5.95						
5 1/4	Jan 01n	99:18	99:20	.....	5.49	7 1/2	Feb 02-07	103:31	104:01	.....	5.98						
5 1/2	Feb 01n	99:24	99:26	.....	5.49	6 1/2	Mar 07n	104:03	104:05	+1	5.96						
7 1/4	Feb 01	103:16	103:18	.....	5.51	4 1/4	Aug 07n	101:01	101:03	+2	5.95						
11 1/4	Feb 01	109:27	109:29	+1	5.52	7 1/2	Nov 02-07	105:26	105:28	.....	5.96						
5	Feb 01n	99:03	99:05	.....	5.51	3 1/2	Jan 08l	97:29	97:30	+1	3.91						
5 1/2	Feb 01n	100:04	100:06	.....	5.50	5 1/2	Feb 08n	97:07	97:09	.....	5.90						
4 1/2	Mar 01n	98:26	98:28	.....	5.53	5 1/2	May 08n	97:31	98:01	+2	5.91						
6 1/2	Mar 01n	101:11	101:13	+1	5.54	8 1/2	Aug 03-08	108:20	108:22	.....	6.00						
5	Apr 01n	99:00	99:01	.....	5.54	4 1/4	Nov 08n	91:26	91:27	+2	5.89						
6 1/4	Apr 01n	101:06	101:08	.....	5.54	8 1/2	Nov 03-08	110:11	110:15	.....	6.02						
5 1/2	May 01n	100:02	100:04	.....	5.55	3 1/2	Jan 09l	99:28	99:29	+1	3.89						
8	May 01n	104:11	104:13	.....	5.56	5 1/2	May 09n	97:23	97:24	.....	5.80						
13 1/4	May 01	113:19	113:23	-2	5.54	9 1/4	May 04-09	112:25	112:29	.....	6.06						
5 1/4	May 01n	99:14	99:15	.....	5.54	10 1/2	Nov 04-09	119:20	119:24	.....	6.05						
6 1/2	Jun 01n	101:22	101:24	.....	5.57	11 1/4	Feb 05-10	126:26	127:00	.....	6.07						
6 1/2	Jun 01n	101:31	102:01	.....	5.57	10	May 05-10	119:05	119:09	.....	6.0						
7 1/4	Aug 01n	104:18	104:20	.....	5.60	12 1/2	Nov 05-10	135:00	135:06	.....	6.04						
13 1/4	Aug 01	115:22	115:26	-2	5.59	13 1/2	Nov 06-11	143:12	143:18	.....	6.08						
6 1/2	Aug 01n	101:24	101:26	+1	5.59	14	Nov 06-11	144:12	144:23	.....	6.09						
6 1/2	Sep 01n	101:18	101:20	.....	5.61	10 1/2	Nov 07-12	127:16	127:22	.....	6.12						
6 1/4	Oct 01n	101:11	101:13	.....	5.61	12	Aug 06-12	140:17	140:23	+1	6.14						
7 1/2	Nov 01n	104:04	104:06	.....	5.63	12 1/2	May 09-14	152:00	152:06	+1	6.15						
15 1/4	Nov 01	122:20	122:26	-2	5.99	12 1/2	Aug 06-14	147:10	147:16	+2	6.16						

TREASURY BILLS					
Maturity	Rate	Mo/Yr	Bid	Asked	Chg. Yld.
Jun 10 '99	2	4.34	4.26	+1.01	4.32
Jun 17 '99	9	4.17	4.09	-0.02	4.15
Jun 24 '99	16	4.16	4.08	-0.08	4.14
Jul 01 '99	23	3.96	3.88	-1.06	3.94
Jul 08 '99	30	4.09	4.05	-0.06	4.12
Jul 15 '99	37	4.13	4.09	-0.03	4.16
Jul 22 '99	44	4.25	4.21	-0.04	4.29
Jul 29 '99	51	4.30	4.26	-0.02	4.35
Aug 05 '99	58	4.34	4.30	-0.01	4.39
Aug 12 '99	65	4.39	4.37	-0.03	4.47
Aug 19 '99	72	4.43	4.41	.....	4.51
Aug 26 '99	79	4.39	4.37	-0.02	4.47
Sep 02 '99	86	4.40	4.39	-0.07	4.50
Sep 09 '99	93	4.50	4.48	.....	4.60
Sep 16 '99	93	4.52	4.51	-0.01	4.63
Sep 23 '99</					



$d_b$  = the *bid discount rate* applied by dealers when they buy a Treasury bill

$p_b$  = the dealers' *price bid* when buying a Treasury bill, resulting from the discount bid rate

$n$  = the number of days remaining until the T-bill's maturity

360 = the number of days in a year arbitrarily chosen when linking  $d_b$  to  $p_b$ , and when calculating the “ask yield”

$\theta = n/360 \equiv$  fraction of an “accounting year” remaining until the T-bill's maturity

The formula linking the discount bid  $d_b$  (quoted in financial newspapers) to the price offered by the dealers  $p_b$  is then calculated in the following natural way:

$$d_b = \frac{\text{par} - p_b}{\text{par}} \bigg/ \theta = \frac{\text{par} - p_b}{\text{par}} \cdot \frac{360}{n} \quad (12)$$

Conversely,  $p_b$  can be calculated from (12) as

$$\theta \cdot \text{par} \cdot d_b = \text{par} - p_b$$

Therefore,

$$p_b = \text{par}(1 - \theta d_b) = \text{par} \left( 1 - \frac{n}{360} d_b \right) \quad (13)$$

In consequence, any quoted bid discount translates immediately into an offered price  $p_b$  by the dealer. For instance, consider the T-bill on the last line of table 1.1. Its remaining maturity is 352 days, and its bid discount is 4.80% per accounting year. Therefore, applying (13), the price offered by the dealers is

$$p_b = \$10,000 \left[ 1 - \frac{352}{360} (0.048) \right] = \$9530.67$$

for a transaction of \$1 million or more.

Let us now introduce the following additional notation:

$d_a$  = the *asked discount* rate applied by dealers when the dealers *sell* a Treasury bill

$p_a$  = the dealers' asked price when they *sell* a Treasury bill

Of course, the asked discount rate  $d_a$  is related to the dealers asked price  $p_a$  by a formula entirely symmetrical to (1):

$$d_a = \frac{\text{par} - p_a}{\text{par}} \bigg/ \theta = \frac{\text{par} - p_a}{\text{par}} \cdot \frac{360}{n} \quad (14)$$

Conversely, the asked price is linked to the quoted asked discount by

$$p_a = \text{par}(1 - \theta d_a) = \text{par} \left( 1 - \frac{n}{360} d_a \right) \quad (15)$$

Let us consider the same Treasury bill as before. Its asked discount is 4.79% per accounting year; therefore, the price asked by the dealers is

$$p_a = \$10,000 \left[ 1 - \frac{352}{360} (0.0479) \right] = \$9531.64$$

Suppose now that we want to know the yield  $y$  received by the buyer if he keeps his Treasury bill until maturity. He will of course make a calculation with a 365-day year. His yield will be

$$y = \frac{\text{par} - p_a}{p_a} \bigg/ \frac{n}{365} = \left( \frac{\text{par}}{p_a} - 1 \right) \frac{365}{n}$$

In our example, the yield received by the bond's buyer is

$$y = \left( \frac{10,000}{9531.64} - 1 \right) \frac{365}{352} = 5.09\%/\text{year}$$

However, the “ask yield” printed in the *Wall Street Journal* (5.03%) still takes into account the “accounting” year of 360 days, and not the calendar year of 365 days. This is why they get

$$\begin{aligned} y_a \equiv \text{printed asked yield} &= \left( \frac{10,000}{9531.64} - 1 \right) \frac{360}{352} \\ &= 5.03\% \text{ per accounting year} \end{aligned}$$

It is easy enough to see why the yield for the buyer will always be higher than the asked discount. We can see that

$$y > d_a$$

since the above inequality translates as

$$\frac{\text{par} - p_a}{p_a} \cdot \frac{365}{n} > \frac{\text{par} - p_a}{\text{par}} \cdot \frac{360}{n}$$

This inequality leads to

$$\frac{365}{p_a} > \frac{360}{\text{par}}$$

which is always true, since  $p_a < \text{par}$ . In fact, the ratio yield/asked discount is equal to

$$\frac{y}{d_a} = \frac{\text{par}}{p_a} \cdot \frac{365}{360} > 1$$

a product of two numbers that are both larger than one. In our example, this ratio is 1.0637.

### 1.5.2 Treasury notes

These are bonds with maturities between one and 10 years; they could be called “medium-term” bonds. They usually pay semi-annual coupons in the United States, contrary to many European bonds with the same maturity that pay only one coupon yearly.

For those coupon-bearing, longer-term (more than a year) bonds, the quotations are made on quite a different basis than the Treasury bills; instead of quoting a bid or asked discount, professionals, together with the financial newspapers, quote both a bid *price* and an asked price (*ask price*). In turn, those prices have two main features: First, they are not quoted in dollars, but in percent of par value; second, the figure following those percents is not a decimal but  $\frac{1}{32}$ . For instance, a Treasury note with given maturity (between 1 and 10 years) and an ask price of 105.17 means that the ask price is  $105\frac{17}{32} = 105.53125\%$  of its nominal value ( $\frac{1}{32}$  translates into 3.125% of one percent<sup>3</sup>).

The *ask yield* of that bond requires that we explain in detail what a bond’s yield to maturity is. We do this in chapter 3, where the central concepts of yield to maturity and horizon rate of return are discussed.

<sup>3</sup>In financial parlance,  $\frac{1}{100} \cdot \frac{1}{100} = \frac{1}{10,000}$  = one ten-thousandth and is called “one basis point”; so  $\frac{1}{32}$  of 1% is 3.125 basis points. For instance, if an interest rate, initially at 4%, increases by 50 basis points, it means that it has gone from 0.04 to 0.045, or from 4% to 4.5%.

However, for those who are already familiar with the concept of the *internal rate of return* on an investment (IRR), suffice it to say that the bond's ask yield exactly corresponds to the IRR of the investment consisting of buying this bond, with one proviso: the price to be paid by the investor is not just the ask price; the buyer has to pay the ask price *plus* whatever accrued interest has incurred on this bond that has not yet been paid by the issuer of the debt security. Consider, for instance, that an investor decides to buy a bond with an 8% coupon. As is often the case in the United States, this bond pays the coupon semi-annually, that is, 4% of the par value every six months. Suppose three months have elapsed since the last semi-annual coupon payment. The holder of the bond will receive the ask price, *plus* the accrued interest, which is 2% of the bond's par value. So the cost to the investor, which is the price he will have to pay and consider in the calculation of his internal rate of return, is the quoted ask price (translated in dollars by multiplying the ask price by the par value), *plus* the incurred interest.<sup>4</sup> This is the cost the investor will have to take into account for the calculation of his internal rate of return, or, in bond parlance, its ask yield to maturity.

### 1.5.3 Treasury bonds

These bonds are long-term: their maturity is longer than 10 years. Some of them have an embedded feature, in the sense that they are “callable”; they can be reimbursed at a date that is usually fixed between 5 and 10 years before maturity. For the Treasury, this feature enables it to take into account a drop in interest rates in order to reissue bonds with smaller coupons.

Table 1.1, extracted from the *Wall Street Journal*, summarizes much of this information regarding the quotation of bonds.

## 1.6 Bonds issued by firms

The bonds issued by firms, also called *corporate bonds*, generally have characteristics very close to those described above—in particular, they might be callable by the issuer. Their trademark, of course, is that they

<sup>4</sup>The quoted price is also called the *flat* price. The quoted, or flat, price *plus* incurred interest is called the *full*, or *invoice* price. In England, British government securities are called gilts; their prices carry special names: the quoted, or flat, price is called “clean price”; the full, or invoice, price is the “dirty price.”

**Table 1.2**  
**Standard & Poor's debt rating definitions**

---

<b>AAA</b>	Debt rated "AAA" has the highest rating assigned by Standard & Poor's. Capacity to pay interest and repay principal is extremely strong.
<b>AA</b>	Debt rated "AA" has a very strong capacity to pay interest and repay principal and differs from the higher-rated issues only in small degree.
<b>A</b>	Debt rated "A" has a strong capacity to pay interest and repay principal although it is somewhat more susceptible to the adverse effects of changes in circumstances and economic conditions than debt in higher-rated categories.
<b>BBB</b>	Debt rated "BBB" is regarded as having an adequate capacity to pay interest and repay principal. Whereas it normally exhibits adequate protection parameters, adverse economic conditions or changing circumstances are more likely to lead to a weakened capacity to pay interest and repay principal for debt in this category than in higher-rated categories.
<b>BB, B, CCC, CC, C</b>	Debt rated "BB," "B," "CCC," "CC," and "C" is regarded, on balance, as predominantly speculative with respect to capacity to pay interest and repay principal in accordance with the terms of the obligation. "BB" indicates the lowest degree of speculation and "C" the highest degree of speculation. While such debt will likely have some quality and protective characteristics, these are outweighed by large uncertainties or major risk exposures to adverse conditions.
<b>CI</b>	The rating CI is reserved for income bonds on which no interest is being paid.
<b>D</b>	Debt rated D is in payment default.

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Source: Standard & Poor's Bond Guide, June 1992, p. 10.

are usually more risky than Treasury bonds. Their degree of riskiness is evaluated by two large organizations: Standard & Poor's and Moody's. In table 1.2 the reader will find Standard & Poor's debt rating definition; table 1.3 gives Moody's corporate rankings.

## 1.7 Convertible bonds

Convertible bonds are hybrid financial instruments that lie between straight bonds and stocks. They are bonds that give their holder the right (but not the obligation) to convert the bond (issued generally by a company) into the company's shares at a predetermined ratio. This ratio (the conversion ratio)  $r$  is the number of shares to which one bond entitles its owner. For instance, in January 1999, an on-line bookseller issued a 10-year convertible with a 4.75% coupon. Each \$1000 bond could be converted into 6.408 shares. (For the story of that issue, see the excellent article by Mitchell Martin in the *New York Herald Tribune* of February 13–14, 1999.) Suppose that the price at which the convertible bond was issued was exactly par; it then follows that its holder would have converted

**Table 1.3**  
**Moody's corporate bond ratings**

---

**Aaa**

Bonds that are rated **Aaa** are judged to be of the best quality. They carry the smallest degree of investment risk and are generally referred to as "gilt edge." Interest payments are protected by a large or exceptionally stable margin, and principal is secure. While the various protective elements are likely to change, the changes that can be visualized are most unlikely to impair the fundamentally strong position of such issues.

**Aa**

Bonds that are rated **Aa** are judged to be of high quality by all standards. Together with the **Aaa** group they comprise what are generally known as high-grade bonds. They are rated lower than the best bonds because margins of protection may not be as large as in **Aaa** securities; or the fluctuation of protective elements may be of greater amplitude; or other elements may be present that make the long-term risks appear somewhat larger than in **Aaa** securities.

**A**

Bonds that are rated **A** possess many favorable investment attributes and are considered to be upper medium-grade obligations. Factors giving security to principal and interest are considered adequate but elements may be present which suggest a susceptibility to impairment sometime in the future.

**Baa**

Bonds that are rated **Baa** are considered to be medium-grade obligations, i.e., they are neither highly protected nor poorly secured. Interest payments and principal security appear adequate for the present but certain protective elements may be lacking or may be characteristically unreliable over any great length of time. Such bonds lack outstanding investment characteristics and in fact have speculative characteristics as well.

**Ba**

Bonds that are rated **Ba** are judged to have speculative elements; their future cannot be considered as well assured. Often the protection of interest and principal payments may be very moderate and thereby not well safeguarded during future good and bad times. Uncertainty of position characterizes bonds in this class.

**B**

Bonds that are rated **B** generally lack characteristics of the desired investment. Assurance of interest, principal payments, or maintenance of other terms of the contract over any long period of time may be small.

**Caa**

Bonds that are rated **Caa** are of poor standing. They may be in default or elements of danger may be present with respect to principal or interest.

**Ca**

Bonds that are rated **Ca** represent obligations that are highly speculative. They are often in default or have other marked shortcomings.

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it into shares if the share price had been more than the par/conversion ratio =  $\$1000/6.408 = \$156.05$ . Of course, the price of the stock that day was lower than that benchmark: it was  $\$122.875$ .

The possibility given to the owner of the bond to convert it into shares has two drawbacks: first, the coupon rate is always lower than the level corresponding to the kind of risk the market attributes to the issuer; second, convertibles rank behind straight bonds in case of bankruptcy of the firm.

In order to get a clear picture of the option value embedded in the convertible, let us consider that a firm has  $S$  shares outstanding and has issued  $C$  convertible bonds with face value  $B_T$ . Suppose also that the bonds are convertible only at maturity  $T$ . Let us first determine what share ( $\lambda$ ) of the firm the convertible bond holders will own if they convert at maturity. This share will be equal to the number of securities they were allowed to convert for their  $C$  convertibles,  $rC$ , divided by the total number of securities outstanding after the conversion:  $S + rC$ . So our bondholders have access to the following fraction of the firm:

$$\lambda = \frac{rC}{S + rC}$$

At maturity, suppose that the firm's value is  $V$  and that  $V$  is above the par value of all convertibles issued,  $CB_T$ . Convertible bonds' holders have two choices: either they convert, or they do not. If they do convert, their portfolio of stocks is worth  $\lambda V$  (the share  $\lambda$  of the firm they own times the firm's value); if they do not convert, they will receive from the firm the par value of their bonds, that is,  $CB_T$ . So the decision to convert will hinge on the firm's value at maturity. Conversion will occur if and only if

$$\lambda V > CB_T$$

or

$$V > \frac{CB_T}{\lambda}$$

and the value of the convertible portfolio will be a linear function of the firm's value,  $\lambda V$ . If, on the other hand,  $\lambda V < CB_T$ , conversion will not take place; convertible holders will stick to their bonds and will receive a value equal to par ( $B_T$ ) times the number of convertible bonds ( $C$ ),  $CB_T$ . This quantity is independent from the firm's value.

**Table 1.4**  
**Convertible's value at maturity  $T$**

Firm's value	Convertible's portfolio value	Convertible's value
$V < CB_T$	$V$	$V/C$
$CB_T < V < \frac{CB_T}{\lambda}$	$CB_T$	$B_T$
$V > \frac{CB_T}{\lambda}$	$\lambda V$	$\lambda V/C$

Finally, suppose that the firm's value at maturity is less than  $CB_T$ . In that unfortunate case, as sole creditors of the firm the convertible holders are entitled to the firm's entire value, that is, they will receive  $V$ .

Table 1.4 summarizes the convertible's portfolio value and the payoff to an individual convertible as functions of the firm's value.

So the convertible's value at maturity is just its par value if the firm's value is between  $CB_T$  and  $CB_T/\lambda$ . If the firm's value is outside those bounds, the convertible's value is a linear function of the firm's value:  $V/C$  if  $V < CB_T$  and  $\lambda V/C$  if  $V > \frac{CB_T}{\lambda}$ . These functions are depicted in figure 1.3(a). In figure 1.3(b), the option (warrant) part of the convertible is separated from its straight bond part. We can write:

- value of straight bond:  $\min(\frac{V}{C}, B_T)$
- value of warrant:  $\min(0, \frac{\lambda V}{C} - B_T)$

This last value is the value of a call on the part of the firm with an exercise price (per convertible) of  $B_T/\lambda$ .

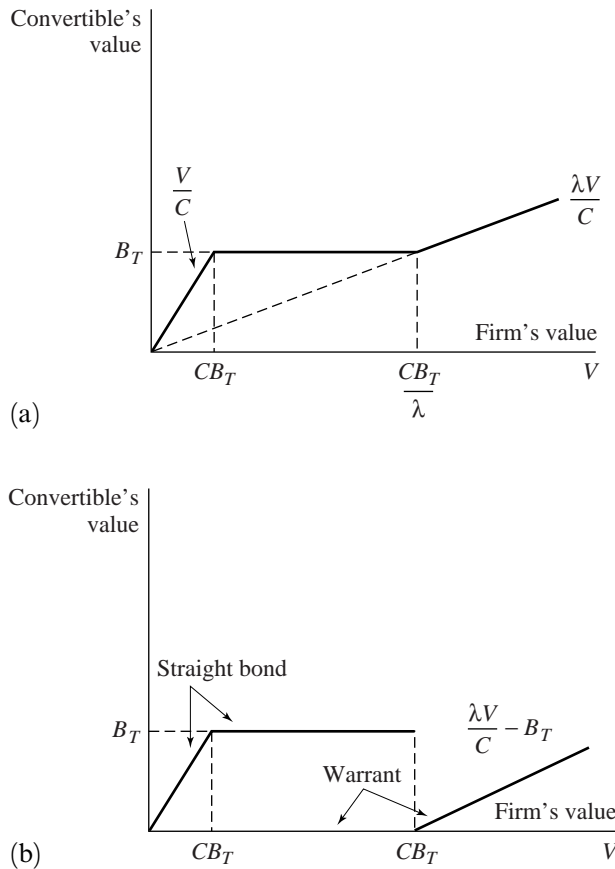
Most convertible bonds have more complicated features than the one depicted above. In particular, conversion can be performed before maturity. Furthermore, often the firm can call back the convertible, and the holder can as well put it to the issuer under specified conditions. Thus a convertible has a number of imbedded options: a warrant, a written (sold) call feature, and a put feature.

The first valuation studies of convertibles were carried out by Brennan and Schwartz<sup>5</sup> and Ingersoll<sup>6</sup> in the case where interest rates are non-

<sup>5</sup>M. J. Brennan and E. S. Schwartz, "The Case for Convertibles," *Journal of Applied Corporate Finance*, 1 (1977): 55–64.

<sup>6</sup>J. E. Ingersoll, "An Examination of Convertible Securities," *Journal of Financial Economics*, 4 (1977): 289–322.



**Figure 1.3**

(a) A convertible's value at maturity if stocks and convertibles are the sole financial components of its financial structure. (b) Splitting the convertible's value at maturity into its components: straight bond and warrant.

stochastic. The extension to a stochastic term structure was carried out by Brennan and Schwartz<sup>7</sup>.

## Questions

- 1.1. Suppose a default-free zero-coupon bond expires in four months and is worth \$97 today. What is the annualized return for its owner at maturity?
- 1.2. A zero-coupon bond has a 6% annualized return per year and will be redeemed in six months at \$100. Supposing it is default-free, what is its price today?
- 1.3. Suppose a zero-coupon bond has a maturity equal to 2.65 years; its par value is \$100, and its value today is \$87. What is its annualized return?
- 1.4. What is the price today of a zero-coupon bond that will be redeemed at \$1000 in 3.2 years, if its annualized rate of return is 6%?
- 1.5. Consider a two-year spot rate of 5% and a five-year spot rate of 6%. What is the (equilibrium) implied forward rate for a loan starting in two years and maturing three years later? What is the interpretation of the five-year dollar spot return in terms of the two-year dollar spot return and the dollar forward rate corresponding to the forward return you have just calculated?
- 1.6. Suppose that today a six-year spot rate is 7% per year, and a forward rate starting in four years and maturing two years later is 7.5% per year. What is the implied four-year spot rate?

<sup>7</sup>M. J. Brennan and E. S. Schwartz, "Analyzing Convertible Bonds," *Journal of Financial and Quantitative Analysis*, 15 (1980): 907–929.

# 2 AN ARBITRAGE-ENFORCED VALUATION OF BONDS

LADY MACBETH. Thy letters have transported me beyond  
This ignorant present, and I feel now  
The future in the instant.

*Macbeth*

## Chapter outline

<b>2.1</b>	<b>An arbitrage-driven valuation of a zero-coupon bond</b>	<b>26</b>
2.1.1	The case of an undervalued zero-coupon bond	27
2.1.2	The case of an overvalued bond	28
<b>2.2</b>	<b>An arbitrage-enforced valuation of a coupon-bearing bond</b>	<b>29</b>
2.2.1	The case of an undervalued coupon-bearing bond	30
2.2.2	The case of an overvalued coupon-bearing bond	32
<b>2.3</b>	<b>Discount factors as prices</b>	<b>34</b>
<b>2.4</b>	<b>Evaluating a bond: a first example</b>	<b>36</b>
<b>2.5</b>	<b>Open- and closed-form expressions for bonds</b>	<b>39</b>
<b>2.6</b>	<b>Understanding the relative change in the price of a bond through arbitrage on interest rates</b>	<b>44</b>
<b>2.7</b>	<b>A global picture of a bond's value</b>	<b>52</b>
<b>2.8</b>	<b>How can we really price bonds?</b>	<b>54</b>

## Overview

In this chapter we evaluate a defaultless bond as a sum of cash flows to be received in the future. Those future cash flows can be discounted in two ways: either by assuming that all spot interest rates are the same, whatever the maturity—which is very rarely the case—or by using for each maturity a well-defined spot rate. Each method will provide a theoretical value of the bond that may be compared with the market price; in that sense it is possible to estimate a possible overvaluation or undervaluation of the defaultless bond.

A defaultless bond may see its value change for two reasons. By far the more important one is a change in rates of interest; a secondary one is the passage of time. When interest rates move up, a bond's price goes down, and vice versa. Those movements can be quite large and difficult to predict, which makes bond portfolio management both a rewarding and difficult task. On the other hand, should a bond not be priced at par, the

bond price will tend toward par after some time has elapsed, until it reaches par at maturity. This result can be derived from the fact that a bond's value, as a percentage of its par value, can be shown to be the weighted average between the coupon rate divided by the rate of interest on the one hand, and one, on the other. The first weight decreases through time, which implies that the second weight increases through time, so that the bond's price converges toward 100%.

Define the direct yield of a bond as the ratio of the coupon to the price of the bond. The capital gain on the bond in percentage terms will always be the difference between the rate of interest and the direct yield. This result stems from arbitrage on interest rates, because in the absence of any movement in interest rates, no one would ever buy a bond whose total return (direct yield plus capital gain) would not equal the current rate of interest.

It has become commonplace to say the present value of a future (certain) cash flow is the size of that cash flow discounted appropriately by a discount factor. However, familiarity with that concept tends to mask the true reason for that equation, which is the absence of arbitrage opportunities. Our intent in this chapter is to show how the future cash flows of a bond can be valued today through arbitrage mechanisms, and how the forces driven by arbitrage will lead to equilibrium prices. We will start with the valuation of a zero-coupon bond. We will then consider the valuation of a coupon-bearing bond, which is nothing other than a portfolio of zero-coupon bonds, since it rewards its owner with a stream of cash flows at precise dates.

## 2.1 An arbitrage-driven valuation of a zero-coupon bond

Consider that today, at time 0, the  $T$ -horizon spot rate is  $i(0, T)$  and that a zero-coupon bond pays at maturity  $T$  the par value  $B_T$ . Is there a way to determine its arbitrage-enforced value today? If a market exists for rates of interest with maturity  $T$  on which it is possible to lend or borrow at rate  $i(0, T)$ , there is definitely a way to find an arbitrage-driven equilibrium price for this zero-coupon bond. We will now show that any price other than  $B_0 = B_T[1 + i(0, T)]^{-T}$  would trigger arbitrage, which would drive the bond's price toward  $B_0$ . We will consider two cases: First, the zero-coupon bond is undervalued with respect to  $B_0 = B_T[1 + i(0, T)]^{-T}$ ; second, it is overvalued.

**Table 2.1**  
**Arbitrage in the case of an undervalued zero-coupon bond;  $V_0 < B_0 = B_T[1 + i(0, T)]^{-T}$**

Time 0		Time $T$	
Transaction	Cash flow	Transaction	Cash flow
Borrow $V_0$ at rate $i(0, T)$	$+V_0$	Reimburse loan	$-V_0[1 + i(0, T)]$
Buy bond	$-V_0$	Receive par value of zero-coupon bond	$+B_T$
	Net cash flow: 0		Net cash flow: $B_T - V_0[1 + i(0, T)]^T > 0$

### 2.1.1 The case of an undervalued zero-coupon bond

Suppose first that the market value of the bond, denoted  $V_0$ , is below  $B_0$ ; we have the inequality

$$V_0 < B_0 = B_T[1 + i(0, T)]^{-T} \quad (1)$$

Let us show why this bond is undervalued. Suppose you borrow, at time 0,  $V_0$  at rate  $i(0, T)$  for period  $T$ ; you will use the loan to buy the bond. At time 0, your net cash flow is zero. Consider now your position at time  $T$ . You owe  $V_0[1 + i(0, T)]^T$ , and you receive  $B_T$ . Observe that (1) implies also

$$B_T > V_0[1 + i(0, T)]^T \quad (2)$$

Inequality (2) implies that  $B_T$  (what you receive at  $T$ ) is larger than what you owe,  $V_0[1 + i(0, T)]^T$ . Your net profit at  $T$  is thus the positive net cash flow  $B_T - V_0[1 + i(0, T)]^T$ . Table 2.1 summarizes these transactions and their related cash flows.

The important question to ask now is: what will happen on the financial market if such transactions are carried out? There is of course no reason why only one arbitrageur would be ready to cash in a positive amount at time  $T$  with no initial financial outlay. Many individuals would presumably be willing to play such a rewarding part. The result of such an arbitrage would be to *increase* the value  $V_0$  of the zero-coupon bond at time 0 (since there will be excess demand for it), and to *increase* as well the interest rate of the market of loanable funds with maturity  $T$ , because in order to perform their operations, arbitrageurs will of course set those markets in excess demand. Furthermore, anybody who had the idea of lending on that market will prefer buying the bond, and all those who

**Table 2.2**  
**Arbitrage in the case of an overvalued zero-coupon bond;  $V_0 > B_0 = B_T[1 + i(0, T)]^{-T}$**

Time 0		Time $T$	
Transaction	Cash flow	Transaction	Cash flow
Borrow bond	0	Receive proceeds of loan	$+V_0[1 + i(0, T)]^T$
Sell bond at $V_0$	$+V_0$	Buy back bond	$-B_T$
Invest proceeds of bond's selling	$-V_0$	Give back bond to its owner	0
Net cash flow: 0		Net cash flow: $V_0[1 + i(0, T)]^T - B_T > 0$	

wanted to issue bonds on that market will prefer borrowing instead. This will contribute to pushing price  $V_0$  and interest rate  $i_0$  further up. These moves will contribute to transforming inequality (1) (equivalently, inequality (2)) into an equality, barring any transaction costs that prevent an equality from being established.

### 2.1.2 The case of an overvalued bond

It is useful to consider in some detail the opposite case of an overvalued bond, because it is not entirely symmetrical to the case that has just been treated. Suppose indeed that we now start with the inequality

$$V_0 > B_0 = B_T[1 + i(0, T)]^{-T} \quad (3)$$

Our first idea, of course, is if an arbitrageur should *buy* an undervalued security (as he has just done in our first case), he should *sell* an overvalued security, as is the case now. But how can he sell a security he does not own? (Remember that if he initially *owns* the security, and if he sells it because it is overvalued, he is *not* doing arbitrage; by definition, arbitrage means earning something without investing any of one's own resources. Since he is putting his money up front, he is investing his own resources.)

This is the point that makes the two cases asymmetrical: the arbitrageur will have to take one step more than in the first case. First, the arbitrageur has to *borrow* the zero-coupon bond; this does not involve any cash flows. He will then sell it for  $V_0(> B_0 = B_T[1 + i(0, T)]^{-T})$  and invest the proceeds at rate  $i(0, T)$  for period  $T$ . At  $T$ , he will receive  $V_0[1 + i(0, T)]^T$ . Now (3) can also be written

$$V_0[1 + i(0, T)]^T > B_T \quad (4)$$

So, with his earnings  $V_0[1 + i(0, T)]^T$  at time  $T$ , he can buy the bond for its par value  $B_T$ , give it back to the person he borrowed it from, and make a profit  $V_0[1 + i(0, T)]^T - B_T$ . Table 2.2 recapitulates these transactions and their associated cash flows.

Of course, if the operations are not entirely symmetrical to those of the first case (one operation more is needed at time 0 and at time  $T$ : borrowing the asset at time 0 and giving it back at  $T$ ), the effect on the markets are exactly opposite to those we described earlier: the bond's price  $V_0$  will be driven down by excess supply, and the interest rate  $i(0, T)$  will also be reduced by excess supply, until inequalities (3) and (4) are restored to equalities.

We conclude from this that the present value of the zero-coupon bond with par value  $B_T$  is indeed an arbitrage-enforced value. Should any difference prevail between the market value of the bond at time 0,  $V_0$ , and  $B_T[1 + i(0, T)]^{-T}$ , arbitrage would drive the value toward its equilibrium value. We observe also that if this is true, then a zero-coupon bond value and a rate of interest are two different ways of representing the *same* concept: the time value of a monetary unit between time 0 and time  $T$ .

## 2.2 An arbitrage-enforced valuation of a coupon-bearing bond

We now turn to valuing a portfolio of zero-coupon bonds in the form of a single coupon-bearing bond. However, our analysis could be immediately extended to a portfolio of bonds.

Suppose there are  $T$  markets of loanable funds, each establishing a price that is the spot interest rate  $i(0, t)$ ;  $t = 1, \dots, T$ . Let  $c_t$  designate the bond's cash flow at time  $t$ ; for  $t = 1, \dots, T - 1$ ,  $c_t$  is simply the coupon,  $c$ ; for  $t = T$ , the (last) cash flow is the sum of the last coupon paid,  $c$ , and the bond's par value  $B_T$ . We will now show that the equilibrium, arbitrage-enforced value of the bond  $B_0$  is the sum of all its discounted cash flows, the rates of discount being the rates of interest  $i(0, t)$ ,  $t = 1, \dots, T$ :

$$B_0 = \sum_{t=1}^T c_t [1 + i(0, t)]^{-t} \quad (5)$$

As before, we will distinguish between the case of a bond with an initial value  $V_0$  below  $B_0$ , and the case of an overvalued bond.

### 2.2.1 The case of an undervalued coupon-bearing bond

Suppose we have the inequality

$$V_0 < B_0 = \sum_{t=1}^T c_t [1 + i(0, t)]^{-t} \quad (6)$$

Since we know both  $V_0$  and  $B_0$ , we can also determine a coefficient  $\alpha$  ( $0 < \alpha < 1$ ) such that  $V_0 = \alpha B_0$ ;  $\alpha$  is thus the ratio of the bond's market value  $V_0$  to its arbitrage-enforced value  $B_0$  ( $\alpha = V_0/B_0 < 1$ ). We will now show how an arbitrageur can profit from this situation.

She will first borrow the amount  $V_0 = \alpha \sum_{t=1}^T c_t [1 + i(0, t)]^{-t} = \sum_{t=1}^T \alpha c_t [1 + i(0, t)]^{-t}$ , splitting this loan into the  $T$  loanable funds markets. On the first market (the market with maturity of one year), she will borrow  $\alpha c_1 [1 + i(0, 1)]^{-1}$ ; on the 2-year maturity market, she will borrow  $\alpha c_2 [1 + i(0, 2)]^{-2}$ , and so forth. On the  $t$ -year market, her loan will be  $\alpha c_t [1 + i(0, t)]^{-t}$ ; on the  $T$ -year market, she will borrow  $\alpha$  times the present value of the cash flow she will receive at that time (which is  $c_T$ , the last coupon plus the par value  $B_T$ ), so she will borrow  $\alpha c_T [1 + i(0, T)]^{-T}$ . Her net cash flow at time 0 is nil, since she has borrowed the amount  $V_0$ , which covers the price of the bond she has to pay.

After one year, she will have to reimburse  $\alpha c_1$  and will receive  $c_1$ ; after the  $t$ th year, she will have to reimburse  $\alpha c_t$  and will cash in  $c_t$ , and so forth. At the bond's maturity, she will disburse  $\alpha c_T$  and receive  $c_T$ . In all, she will receive a series of positive net cash flows each year, equal to  $c_t - \alpha c_t = (1 - \alpha)c_t$ ,  $t = 1, \dots, T$ , without having committed a single cent of his own money in the process. The stream of the net cash flows  $(1 - \alpha)c_t$ ,  $t = 1, \dots, T$  can be evaluated in terms of time 0 (in present value), in terms of time  $T$  (in future value), or in terms of any year  $t$  for that matter. For instance, let us evaluate it in present value. It will be equal to

$$\begin{aligned} \text{Net cash flow in present value} &= \sum_{t=1}^T (1 - \alpha)c_t [1 + i(0, t)]^{-t} \\ &= (1 - \alpha) \sum_{t=1}^T c_t [1 + i(0, t)]^{-t} \quad (7) \end{aligned}$$



From

$$B_0 = \sum_{t=1}^T c_t [1 + i(0, t)]^{-t} \quad (8)$$

and  $V_0 = \alpha B_0$ , the arbitrageur's net cash flow is thus

$$(1 - \alpha) \sum_{t=1}^T c_t [1 + i(0, t)]^{-t} = B_0 - \alpha B_0 = B_0 - V_0 > 0 \quad (9)$$

Our arbitrageur has earned, throughout the period  $[0, T]$ , a net profit equal, in present value, to exactly the difference between the theoretical value of the bond ( $B_0$ ) and its market value ( $V_0$ ). Table 2.3 summarizes these transactions and their associated cash flows.

We now need to demonstrate that the theoretical value of the bond is indeed the equilibrium, arbitrage-enforced value of the bond; that is, the prices  $V_0$  and  $B_0$  will be driven toward each other by arbitrage forces.

**Table 2.3**

**Arbitrage in the case of an undervalued coupon-bearing bond;  $V_0 < B_0 = \sum_{t=1}^T c_t [1 + i(0, t)]^{-t}$ ;  $V_0 = \alpha B_0$ ,  $0 < \alpha < 1$**

Time	Transaction	Cash flow
0	Borrow, on each of the $t$ -year markets ( $t = 1, \dots, T$ ), $\alpha c_t [1 + i(0, t)]^{-t}$	$+V_0 = \alpha \sum c_t [1 + i(0, t)]^{-t}$
	Buy bond for $V_0$	$-V_0$
		Net cash flow: 0
1	Pay back $\alpha c_1$	$-\alpha c_1$
	Receive first cash flow ( $c_1$ )	$+c_1$
		Net cash flow: $(1 - \alpha)c_1$
...	...	...
$t$	Pay back $\alpha c_t$	$-\alpha c_t$
	Receive $t$ th cash flow ( $c_t$ )	$+c_t$
		Net cash flow: $(1 - \alpha)c_t$
...	...	...
$T$	Pay back $\alpha c_T$	$-\alpha c_T$
	Receive $T$ th cash flow	$+c_T$
		Net cash flow: $(1 - \alpha)c_T$
Total net cash flows in present value at time 0: $(1 - \alpha) \sum_{t=1}^T c_t [1 + i(0, t)]^{-t} = (1 - \alpha)B_0 = B_0 - \alpha B_0 = B_0 - V_0 > 0$		

This is easy to do, similar to what we have seen in the case of zero-coupon bonds. First, when arbitrageurs are borrowing on each of the  $t$ -term markets, they are creating an excess demand on each of those markets, thus increasing each rate of interest  $i(0, t)$ ,  $t = 1, \dots, n$ . This will drive down the theoretical price of the bond,  $B_0$ . Second, when buying the bond, they are also creating an excess demand on the bond's market, driving up its price  $V_0$ . Barring transaction costs, this will take place until  $V_0 = B_0$ . We can of course add that if the magnitude of the transactions on the bond is small because the existing stock of those bonds is small, equilibrium will be reached only through the upward move of  $V_0$ , not by a significant drop in  $B_0$ , since the rates of interest  $i(0, t)$  are not liable to move upward in any significant way. The same kind of remark would have applied before when we considered the market for undervalued zero-coupon bonds.

### 2.2.2 The case of an overvalued coupon-bearing bond

We will now consider how arbitrage will bring back an overvalued coupon-bearing bond to its equilibrium value. We suppose here that initially the market value of the bond is higher than its theoretical value  $B_0$  :  $V_0 > B_0$  or  $V_0 = \alpha B_0$ , with  $\alpha > 1$ . We have here

$$V_0 > B_0 = \sum_{t=1}^T c_t [1 + i(0, t)]^{-t}$$

or, equivalently,

$$V_0 = \alpha B_0 = \alpha \sum_{t=1}^T c_t [1 + i(0, t)]^{-t}, \quad \alpha > 1$$

This is how our arbitrageur will proceed. He will first borrow the overvalued bond, sell it for  $V_0 = \alpha B_0$ , and invest the proceeds in the following way: On the one-year maturity market, he will invest  $\alpha c_1 [1 + i(0, 1)]^{-1}$  at rate  $i(0, 1)$ ; on the  $t$ -year market, he will invest  $\alpha c_t [1 + i(0, t)]^{-t}$  at rate  $i(0, t)$ , and so forth; on the  $T$ -year market, he will invest  $\alpha c_T [1 + i(0, T)]^{-T}$  at rate  $i(0, T)$ .

Consider now what will happen on each of those terms  $t$  ( $t = 1, \dots, T$ ). At time 1, he will receive  $\alpha c_1$  from his loan. This is more than  $c_1$ , since  $\alpha > 1$ , and he will have no problem in paying the first coupon  $c_1 = c$  to

the initial bond's owner; he will thus make at time 1 a profit  $\alpha c_1 - c_1 = c_1(\alpha - 1)$ . The same operation will be done on each term  $t$ ; at time  $t$ , he will cash in  $\alpha c_t$  and pay out the bond's owner  $c_t$ , making a profit  $c_t(\alpha - 1)$ .

At maturity our arbitrageur has two possibilities: either he buys back the bond just before the issuer pays the last coupon  $c$  plus its par value  $B_T$ , and hands the bond to its initial owner (in which case his cash outflow is  $-c_T = -(c + B_T)$ ); or he can also pay directly the residual value of the bond to its owner and incur the same cash outflow of  $-c_T$ . On the other hand, he has received  $\alpha c_T$  from the loan he made on the  $T$ -maturity market. Whatever route he, or the bond's initial owner, may choose, our arbitrageur will receive at maturity a positive cash flow  $c_T - \alpha c_T = c_T(1 - \alpha)$ . Table 2.4 summarizes these operations and their related cash flows.

**Table 2.4**

**Arbitrage in the case of an overvalued coupon-bearing bond;  $V_0 > B_0 = \sum_{t=1}^T c_t[1 + i(0, t)]^{-t}$ ;  $V_0 = \alpha B$ ,  $\alpha > 1$**

Time	Transaction	Cash flow
0	Borrow bond	0
	Sell bond for $V_0$	$+V_0 = \alpha \sum_{t=1}^T c_t[1 + i(0, t)]^{-t}$
	Loan, on each $t$ -term markets, $\alpha c_t[1 + i(0, t)]^{-t}$ , $t = 1, \dots, T$	$-V_0$
		Net cash flow: 0
1	Receive reimbursement on first loan, $\alpha c_1$	$+\alpha c_1$
	Pay first coupon to bond's owner	$-c_1$
		Net cash flow: $(\alpha - 1)c_1 > 0$
...	...	...
$t$	Receive reimbursement on $t$ -year loan, $\alpha c_t$	$+\alpha c_t$
	Pay $t$ -year coupon to bond's owner	$-c_t$
		Net cash flow: $(\alpha - 1)c_t > 0$
...	...	...
$T$	Receive reimbursement on $T$ -year loan, $\alpha c_T = \alpha(c + B_T)$	$+\alpha c_T$
	Pay last coupon plus par to bond's owner	$-c_T$
		Net cash flow: $(\alpha - 1)c_T$
Total net cash flow in present value at time 0: $(\alpha - 1) \sum_{t=1}^T c_t[1 + i(0, t)]^{-t} = (\alpha - 1)B_0 = \alpha B_0 - B_0 = B_0 - V_0 > 0$		

Consider now the sum of all these positive cash flows in present value. We have

$$\begin{aligned}
 \text{Arbitrage profits in present value} &= \sum_{t=1}^T (\alpha - 1) c_t [1 + i(0, t)]^{-t} \\
 &= (\alpha - 1) \sum_{t=1}^T c_t [1 + i(0, t)]^{-t} \\
 &= (\alpha - 1) B_0 = \alpha B_0 - B_0 \\
 &= V_0 - B_0 > 0
 \end{aligned}$$

The present value of the arbitrageur's profit will be exactly equal to the overvaluation of the bond at time 0:  $V_0 - B_0$ . This result is exactly symmetrical to the one we reached previously: the arbitrageur could pocket the precise amount of the undervaluation of the bond.

What will happen now in the financial markets is quite obvious: Arbitrageurs will create an excess supply on the bond's market, thus driving down the bond's price  $V_0$ . On the other hand, the systematic lending by arbitrageurs on each  $t$ -market will create an excess supply on those markets as well, driving down all rates  $i(0, t)$ ,  $t = 0, \dots, T$ , pushing up the theoretical price  $B_0$  until  $V_0$  reaches  $B_0$ . Thus the equilibrium bond's value will have been truly arbitrage-driven.

### 2.3 Discount factors as prices

An understanding of bond valuation—and any investment for that matter—can be gained through an economic interpretation of the discount factors  $1/(1 + i_t)^t$ ,  $t = 1, \dots, n$  as *prices*. Consider first the cash flow pertaining to the first year, which we call  $c_1$ . In present value terms, we know that arbitrage arguments lead us to write this as  $c_1/(1 + i_1)$ , or

$$c_1/(1 + i) = c_1 \cdot \frac{1}{1 + i_1}$$

Let us consider this present value of  $c_1$  as a product, and let us examine the units in which each term of the product is expressed. First,  $c_1$  is in monetary units of year one; for instance, it could be dollars of year one.

The ratio  $\frac{1}{1+i_1}$  is in  $\frac{\$ \text{ of year zero}}{\$ \text{ of year one}}$ ; so this is an *exchange rate*: \$1 of year zero

can be exchanged for  $\$(1 + i_1)$  of year one. This is nothing but a *price*: the price of \$1 of year one expressed in dollars of year zero. Our present value is therefore a number of units of dollars of year one ( $c_1$ ) times the price of each dollar of year one in terms of dollars of year zero. We have

$$\frac{c_1}{1 + i_1} = c_1 \cdot \frac{1}{1 + i_1} = \$ \text{ of year one} \cdot \frac{\$ \text{ of year zero}}{\$ \text{ of year one}} = \$ \text{ of year zero}$$

Our discounting factor  $(1 + i_1)^{-1}$  really acts as a price, which serves to translate dollars of year one into dollars of year zero.

This of course generalizes to any discount factor such as the ones we have used in this chapter. The present value of cash flow  $c_t$ , to be received at time  $t$ , expressed in dollars of year zero, is

$$c_t(1 + i_t)^{-t} = c_t \cdot \frac{1}{(1 + i_t)^t}$$

As before, the right-hand side of this equation should be viewed as a product of dollars of year  $t$  times the price of a dollar of year  $t$  expressed in dollars of year zero  $\left[\frac{1}{(1 + i_t)^t}\right]$ .

In terms of bonds, these prices  $(1 + i_t)^{-t}$  can be viewed as the arbitrage-driven prices of zero-coupon bonds maturing in  $t$  years, with par value equal to one dollar. Further in this text (first in chapter 9), these prices of zero-coupon bonds with par value 1 will become extremely useful. So we will call them  $b(0, t)$ , and we will define them as

$$\frac{1}{(1 + i_t)^t} = b(0, t)$$

Inversely, the spot interest rate  $i_t$  can of course be expressed in terms of the zero-coupon bond's price  $b(0, t)$ . From the above expression, we derive

$$i_t = \left( \frac{1}{b(0, t)} \right)^{1/t} - 1$$

An economic interpretation of this expression is interesting; in other words, the answer can be derived just by reasoning, without recourse to any calculation. For this purpose, the definition of *gross return*, also called the *dollar return*, is useful. The gross, or dollar, return is the ratio between what a dollar becomes after a given period and the dollar

invested; it is the principal (the dollar invested) plus interest paid, divided by the dollar invested. This is exactly  $(1 + i_t)^t$ , equal to  $1/b(0, t)$ . The per year gross return, or per year dollar return is the  $t$ th root of this, minus one, that is,  $[(1 + i_t)^t]^{1/t} - 1 = i_t = [1/b(0, t)]^{1/t} - 1$ . Hence no calculation is required to express the spot rate as a function of the zero-coupon bond price.

## 2.4 Evaluating a bond: a first example

Bonds promise to pay their holder coupons at regular intervals and to reimburse the principal at a maturity date. As we have seen, their value is none other than the present value of all cash flows that will be received by their owner. We deal here mainly with a defaultless bond; this implies that the nominal cash flows to be received can be considered completely risk-free. We will first evaluate a bond as a sum of cash flows and then indicate formulations in open form whereby it will be possible to see at a glance how a bond is a decreasing function of any interest rate, be it an interest rate common to all maturities or a spot rate pertaining to any given maturity. We will then indicate a closed-form formula that will prove useful to understanding how the price of a bond will always ultimately reach its par value at maturity.

The following question naturally arises: what kind of discounting factor should be used to calculate the present value of any cash flow to be received  $t$  years from now? Normally, we would use the riskless spot rate for horizon  $t$ , as described below. Some simplified notation will be useful here. Let us call

- $B_0$  = the present value of the bond
- $B_T$  = the face (or the reimbursement) value of a bond (example: \$1000)
- $c$  = the annual coupon of the bond (example: \$70)
- $c/B_T$  = the coupon rate of the bond (in this example, the coupon rate is  $\$70/\$1000 = 7\%$  per year)
- $i_t$  = the spot rate prevailing today for a contract of length equal to  $t$  years;  $i_t$  represents the term structure of interest rates. For example,  $i_1 = 4.5\%$ ;  $i_2 = 4.75\%$ ;  $i_{10} = 5.5\%$ .
- $T$  = maturity of the bond; this is the number of years remaining until maturity (example: 10 years).

**Table 2.5**  
**An example of a term structure**

Term (years)	1	2	3	4	5	6	7	8	9	10
Spot rate (%)	4.5	4.75	4.95	5.1	5.2	5.3	5.4	5.45	5.5	5.5

In summary, the bond we have considered in this example has a face value of \$1000; its coupon rate is 7% per year; it will be reimbursed in 10 years. Suppose that currently the one-year to 10-year spot rates exhibit a slightly increasing structure: they range between 4.5% and 5.5%. Presumably, since the coupon rate (7%) is significantly above these spot rates, it can be inferred that the bond was issued in earlier times, when interest rates were higher. Now the present value of the bond is calculated as follows:

$$B = \frac{c}{1 + i_1} + \frac{c}{(1 + i_2)^2} + \cdots + \frac{c}{(1 + i_t)^t} + \cdots + \frac{c + B_T}{(1 + i_T)^T} \quad (10)$$

Suppose that, for the 10 years remaining in the bond's life in our example, the term structure is that shown in Table 2.5. This term structure is supposed to be slightly increasing, leveling off for long maturities, in the same way as has often been observed. We will show in chapter 9 exactly how this spot rate term structure can be derived from a set of coupon-bearing prices.

The present value of the bond, using the above term structure, would be

$$\begin{aligned} B &= \frac{70}{1.045} + \frac{70}{1.0475^2} + \frac{70}{1.0495^3} + \frac{70}{1.051^4} + \frac{70}{1.052^5} \\ &\quad + \frac{70}{1.053^6} + \frac{70}{1.054^7} + \frac{70}{1.0545^8} + \frac{70}{1.055^9} + \frac{1070}{1.055^{10}} \\ &= \$1118.25 \end{aligned}$$

This value is above par, a result that confirms our intuition; since the rates of interest corresponding to the various terms of the bond are below the coupon rate, we expect the price of the bond to exceed its par value. Any holder of this bond, which promises to pay a coupon rate above any of the interest rates available on the market, will accept parting from his bond only if he is paid a price above par; and any buyer will accept paying at least a bit more than par to acquire an asset that will earn him a return above the available interest rates.

This valuation process, therefore, constitutes a natural way to define the intrinsic value of a defaultless bond. Indeed, any price mismatch between the market value of the bond and its theoretical price would lead to buying the asset, in the case of undervaluation by the market, and to selling it, in the case of overvaluation. There is, however, one difficulty, which has to do with the estimation of the term structure of the rate of interest. We generally have no direct knowledge of the set of spot rates such as the one postulated above, because there are not enough financial markets that would have exactly the horizons required to construct such a profile of spot rates. Indeed, one way of obtaining these rates would be to consider a whole series of zero-coupon bonds, the maturity of which would correspond to each term.

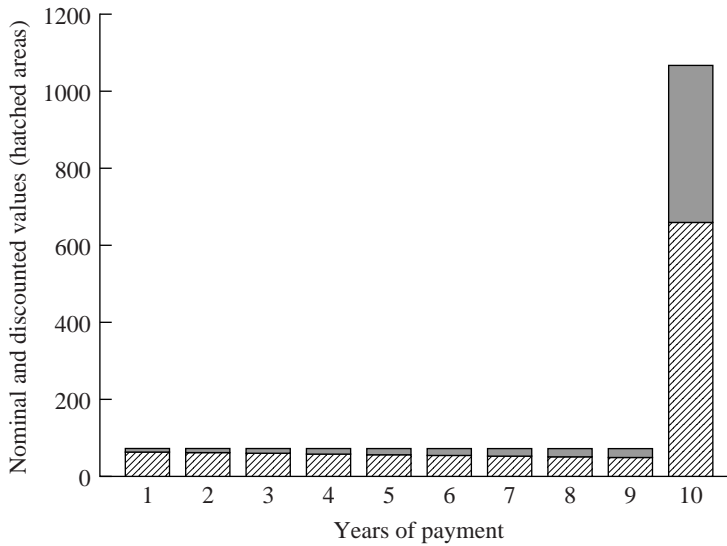
To illustrate, remember that a zero-coupon bond promises to pay a single cash flow—the face value (perhaps \$1000)—in  $t$  years. Let  $Z_0$  be the value today of that bond, and let  $F$  be the face value of the bond. To determine the interest rate (the spot rate)  $i_t$  implied by  $Z_0$ ,  $F$ , and the time span  $t$ ,  $i_t$  must be such that it transforms  $Z_0$  into  $F$  after  $t$  years, with compounded interest. That means  $Z_0(1 + i_t)^t = F$ , which implies in turn that the spot rate is simply  $i_t = (F/Z_0)^{1/t} - 1$ .

Unfortunately, we do not have the series of zero-coupon bonds required for that purpose. We are then faced with the problem of evaluating the term structure of interest rates. We will have more to say about this later, but already we can discern that the value of a bond is inversely related to any increase in this structure, because if every rate of interest happened to increase, every cash flow provided by the bond would see its present value diminish. Therefore, it may be rewarding to make the simplification that the term structure is flat (the spot rates are the same whatever the horizon may be), so that we can infer the changes in the theoretical price of a bond from a change in this single interest rate, which we will choose to call  $i$ . If we do make this simplification, then  $i_t = i$  for all  $t = 1, \dots, T$ , and the evaluation formula (10) becomes

$$B(i) = \frac{c}{1+i} + \frac{c}{(1+i)^2} + \cdots + \frac{c}{(1+i)^t} + \cdots + \frac{c + B_T}{(1+i)^T} \quad (11)$$

It will be very useful to visualize the sum of these discounted cash flows. In figure 2.1, the upper rectangles represent the cash flows of our bond in current value, and the lower rectangles are their present (or discounted) value. The value of the bond is simply the sum of these discounted cash





**Figure 2.1**

The value of a bond as the sum of discounted cash flows (sum of hatched areas; coupon: 7%; interest rate: 5%).

flows. If the coupon rate is 7%, the interest rate 5%, and the maturity 10 years, the bond's value is \$1154.

## 2.5 Open- and closed-form expressions for bonds

Sums like (11) are not expressed in a direct analytical form that would allow us to infer its change in value from a change in the parameters it contains. For instance, formula (11) establishes clearly that the bond's value is a decreasing function of  $i$ , because it is a sum of functions that all are decreasing functions of  $i$ ; but what about the dependency between  $B$  and  $T$ ? Suppose that maturity  $T$  decreases by one year. Will  $B$  decrease, increase, or stay constant? We cannot answer this question merely by considering (11); indeed, if  $T$  diminishes by one unit and the par value  $B_{T-1} = B_T$ , the last fraction increases, but you have one coupon less—so you cannot make a conclusion. It will therefore be useful to express (11) in analytical form. It turns out that (11) simplifies to a very convenient form, as we shall demonstrate.

From the value of a bond given by (11), set  $\frac{c}{1+i}$  in factor; we get

$$B = \frac{c}{1+i} \left[ 1 + \frac{1}{1+i} + \cdots + \frac{1}{(1+i)^{T-1}} \right] + \frac{B_T}{(1+i)^T} \quad (12)$$

The term in brackets is a geometrical series of the form  $1 + a + \cdots + a^{T-1}$ , where  $a = 1/(1+i)$ . Its sum<sup>1</sup> is  $\frac{1-a^T}{1-a}$ , and therefore we have:

$$B = \frac{c}{1+i} \left[ \frac{1 - (\frac{1}{1+i})^T}{1 - \frac{1}{1+i}} \right] + \frac{B_T}{(1+i)^T} = \frac{c}{i} \left[ 1 - \frac{1}{(1+i)^T} \right] + \frac{B_T}{(1+i)^T} \quad (13)$$

Let us first check that it has the right value at maturity (when the remaining maturity  $T$  becomes zero). For  $T = 0$ ,  $1 - 1/(1+i)^T$  is equal to zero, and  $B_T/(1+i)^{-T}$  becomes  $B_T$ . So indeed the value of the bond is at its par, as it should be.

Now look at what happens when there is no reimbursement (or, equivalently, when maturity is infinity). The bond, in this case, is called a *perpetual*, or a *consol*; when  $T \rightarrow \infty$ ,  $B$  tends toward  $c/i$ . We then have

$$\text{Value of a perpetual: } \lim_{T \rightarrow \infty} B = \frac{c}{i} \quad (14)$$

The reader may wonder whether formula (13), derived through some simple calculations, has a direct economic interpretation: in other words, is it possible to obtain it directly through some economic reasoning? The answer is definitely positive if we keep in mind that the value of a perpetual is  $c/i$ . We are going to show that the price of any coupon-bearing bond with finite maturity is in fact a portfolio of three bonds, two held long and one held short. We can first consider the value of the coupons until maturity  $T$ : this is the difference between a perpetual today (at time 0) (equal to  $\frac{c}{i}$ ) and a perpetual whose coupons start being paid at time  $T+1$ . Its value at time  $T$  is  $\frac{c}{i}$ , and its value today is just  $\frac{c}{i}(1+i)^{-T}$ , so the present value of the coupons of the bond is a perpetual held long today,

<sup>1</sup> Call  $S_{T-1}$  the sum  $S_{T-1} = 1 + a + a^2 + \cdots + a^{T-1}$ . We then can write  $aS_{T-1} = a + a^2 + a^3 + \cdots + a^T$ . Now subtract  $aS_{T-1}$  from  $S_{T-1}$ : you get  $S_{T-1} - aS_{T-1} = S_{T-1}(1-a) = 1 - a^T$ ; therefore  $S_{T-1} = (1 - a^T)/(1 - a)$ .

less a perpetual truncated from 0 to  $T$  held from  $T$  onward. The total value of that portfolio today is  $\frac{c}{i} - \frac{c}{i}(1+i)^{-T} = \frac{c}{i}[1 - (1+i)^{-T}]$ , and this is the first part of formula (13). Now we simply have to add the reimbursement in present value,  $B_T(1+i)^{-T}$ , which may be considered as the value of a zero-coupon, and we get our formula for the coupon-bearing bond as a portfolio of a perpetual and a zero-coupon held long, less a perpetual held at time  $T$ .

In practice, the bond's value is usually quoted per unit of par value  $B_T$ . Let us divide  $B$  by  $B_T$ .

$$\boxed{\frac{B}{B_T} = \frac{c/B_T}{i} \left[ 1 - \frac{1}{(1+i)^T} \right] + \frac{1}{(1+i)^T}} \quad (15)$$

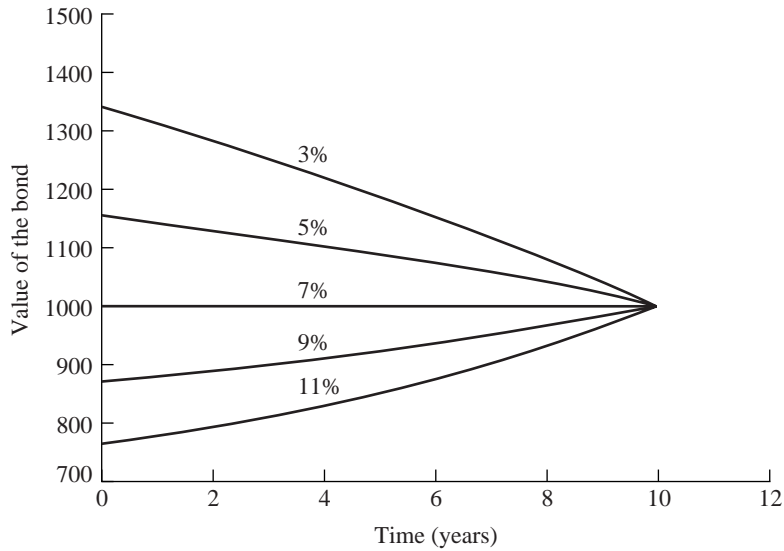
This formula is very useful, because it can be immediately seen that the value of a bond (relative to its par value) is simply the weighted average between the coupon rate divided by the rate of interest, on the one hand, and the number 1, on the other; the weights are just the coefficients  $1 - (1+i)^{-T}$  and  $(1+i)^{-T}$ , which add up to 1, as they should.

Either formula—(11) or (15)—shows that the value of a defaultless bond depends on two variables that are not in the control of its owner: the rate of interest  $i$  and the remaining maturity  $T$ .

The advantage of formula (11) is that it can be seen immediately that the value of the bond is a decreasing function of  $i$ . But (15) readily shows two fundamental properties of bonds:

- First, since the bond's value (relative to its par value) is always between  $\frac{c/B_T}{i}$  and 1, the price of the bond will be *above par* if and only if the coupon rate is above the rate of interest. Conversely, it will be *below par* if and only if the coupon rate is below the rate of interest.
- Second, when maturity  $T$  diminishes, the weight of 1—that is,  $\frac{1}{(1+i)^T}$ —becomes larger and larger, so the value of the bond tends toward its par value, to become equal to 1 when maturity  $T$  becomes zero. This process will be accomplished by a *decrease* in value if initially the bond is above par and by an *increase* in value if initially the bond is below par.

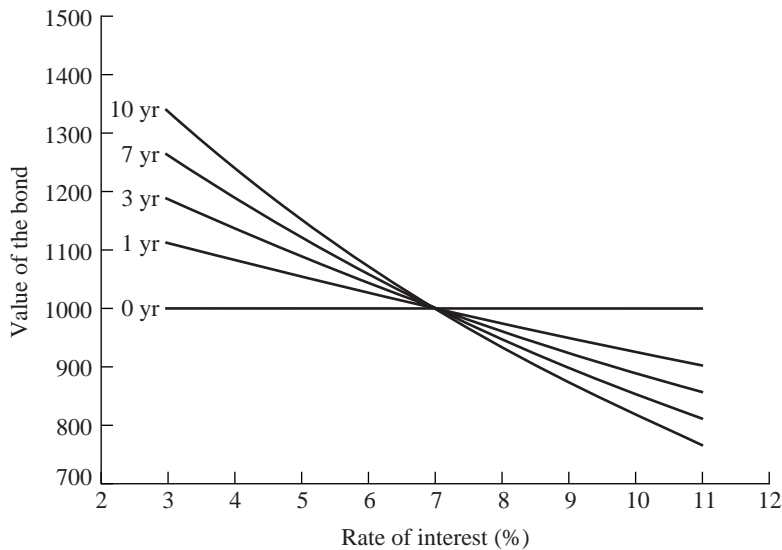
The evolution of our bond through time has been pictured in figure 2.2, for various rates of interest. Should the rate of interest change during



**Figure 2.2**  
Evolution of the price of a bond through time for various interest rates (coupon: 7%; maturity: 10 years).

this process, the value of the bond would jump from one curve to another. Suppose, for instance, that the interest rate is 5% and that the time elapsed in our bond's life is four years. (Time to maturity is therefore six years.) Should interest rates drop to 3%, our bond's trajectory would immediately jump up from 5% to 3%, and then follow that path until maturity—unless other changes also occur in the interest rates.

Consider now the relationship between the rate of interest and the price of the bond. From (11), as we have just said, we can see that the bond's value is a decreasing function of the rate of interest. This relationship is pictured for our bond in figure 2.3, where we consider various maturities. (We have chosen 10, 7, 3, 1, and 0 years.) As could be foreseen from the previous paragraph, the shorter the maturity, the closer the bond price is to its par value. Also, for  $T = 0$  (when the bond is at maturity), its price is at par irrespective of the rate of interest. Naturally, any asset that is about to be paid at its face value without default risk would have precisely this

**Figure 2.3**

**Value of a bond as a function of the rate of interest and of its maturity (coupon: 7%; maturities: 10, 7, 3, 1, and 0 year).**

value, whatever the interest rate may be. All curves, each corresponding to a given maturity, exhibit a small convexity.<sup>2</sup>

This convexity is a fundamental attribute of a bond value and, more generally, of bond portfolios. It is so important that we will devote an entire chapter to it. Already we can immediately figure out that the more convexity a bond exhibits, the more valuable it will become in case of a drop in the interest rates, and the less severe will be the loss for its owner if interest rates rise. We will also ask, in due course, what makes a bond, or a portfolio, more convex than others.

In figure 2.2, we have already noticed that the value of a bond converged toward its par value when maturity decreased. The natural ques-

<sup>2</sup> From your high school days, you remember that a function's convexity is measured by its second derivative. The function is *convex* if its second derivative is *positive* (if its slope is increasing). This is the case in figure 2.3: each curve sees its slope increase when  $i$  increases (its value decreases in absolute value but *increases* in algebraic value). A curve is *concave* if its second derivative is negative. A curve is neither convex nor concave if its second derivative is zero; this implies that the first derivative is a constant and therefore that the function is represented by a straight line (an affine function).

tion to ask now is: at what rate will the decrease or the increase in value take place? This is an important question; if the owner of a bond takes it for granted that for such a smart person as himself his asset gains value through time, he will have some difficulty accepting that his asset is doomed to take a plunge (if only a small one) year after year when his bond is above par.

The key to answering this question is to realize that, in the absence of any change in the interest rate, the owner of a bond simply cannot receive a return on his investment different from the prevailing interest rate. Whenever the coupon rate is above the interest rate, the investor will see the value of his bond, which he may have bought at  $B_0$ , diminish one year later to a value  $B_1$ , such that the total return on the bond, defined by  $c/B_0 + (B_1 - B_0)/B_0$ , will be exactly equal to the rate of interest. We will show this rigorously, but here is first an intuitive explanation.

Suppose that the coupon  $c$  divided by the initial price  $B_0$  ( $c/B_0$ ) is above the rate of interest, and that the value of the bond one year later has not fallen to  $B_1$  but slightly above it. The return to the bond's owner is then above the rate of interest, and it is in the interest of the owner to try to sell it at time 1. However, possible buyers will find the bond overvalued compared to the prospective future value of the bond (ultimately, the bond is redeemed at par value), and this will drive down the price of the bond until  $B_1$  is reached. On the other hand, had the price fallen below  $B_1$ , owners of the bond would resist selling it, and possible buyers would attempt to buy it, because it would be undervalued. Its price would then go up, until it reached  $B_1$ .

Let us now look at this matter a little closer; we will then be led to the all-important equation of arbitrage on interest rate, which plays a central role in finance and economics.

## 2.6 Understanding the relative change in the price of a bond through arbitrage on interest rates

In the two first sections of this chapter, we showed how the valuation of a bond could be obtained through direct arbitrage. In this section, we want to show how the change in price of a bond can be derived from arbitrage on rates of interest. To that effect, we will derive the Fisher equation,

using for that purpose two very different kinds of assets: a financial asset (a zero-coupon bond) and a capital goods asset in the form of a vineyard. We have deliberately chosen two assets of an entirely different nature; by doing so, we will clearly indicate the remarkable generality of Fisher's equation. Also, we will postulate the absence of uncertainty. But how can we escape uncertainty when considering a vineyard?

The answer is that we may transform a risky asset into a riskless one by supposing the existence of forward markets. (Let us not forget that forward and futures markets were created—thousands of years ago—precisely in order to remove uncertainty from a number of future transactions.) In our case, we will suppose that an investor buys a vineyard (or a piece of it), rents it for a given period, and resells it after that period; thus, any uncertainty is removed from the outcomes of these transactions. Let us also suppose that the rental fee is paid immediately, and that the reselling of the vineyard is done in a forward contract with guarantees given by the buyer such that all uncertainty is removed from the outcome of that forward contract.

Let us first ask the question, what can an investor do with \$1? Suppose he has two possibilities: invest it in a tangible asset (a vineyard, for instance) or in a financial asset. For the time being, suppose that both assets are riskless. This means that the financial asset (perhaps a zero-coupon bond maturing in one year) will bring with certainty a return equal to the interest rate  $i$ . It also means that the income that may be generated by the vineyard, both in terms of net income and capital gains, can be foreseen with certainty. This certainty hypothesis is much less of a constraint than it may appear; we will suppress it later, but for the time being we prefer to maintain this certainty for clarity's sake.

Let us take up the possibility of investing in the financial market, at the riskless rate. The owner of \$1 will then receive after one year  $1 + i$ . Turn now to the investor in the vineyard, and suppose that the price of this vineyard today is  $p_0$ . With \$1, he can buy a piece  $\frac{1}{p_0}$  of it. (Of course, it is of no concern to us that the asset is supposed to be divisible in such a convenient way; should this not be the case, we would suppose that the money at hand for the investor is  $\$p_0$  instead of \$1, and he would then compare the reward of investing  $p_0$  in the financial market, at the riskless interest  $i$ , with buying the whole vineyard at price  $p_0$ .)

Coming back to our investor, he now holds a piece  $\frac{1}{p_0}$  of the vineyard, and he may very well want to rent it. Suppose that the yearly rent he can

find for such a vineyard is  $q$  (the rental rate  $q$  is expressed in dollars per vineyard, per year). After one year, he will receive a rental of  $q \frac{1}{p_0} = \frac{q}{p_0}$ . The price of the vineyard will be  $p_1$  at that time, and hence he will be able to sell his piece of vineyard for  $\frac{1}{p_0} \cdot p_1$ . In total, investing \$1 in the vineyard will have earned him exactly  $\frac{q}{p_0} + \frac{p_1}{p_0} = \frac{q+p_1}{p_0}$ , and we suppose this amount could have been forecast by the investor with certainty. So there are two investment possibilities, each one earning for the investor either  $1 + i$  (if he chose the financial asset) or  $\frac{q+p_1}{p_0}$  (if he went for the vineyard) with certainty. The annual rate of return is  $\frac{1+i-1}{1} = i$  in the case of the financial asset and,  $\frac{\frac{q+p_1}{p_0} - 1}{1} = \frac{q+p_1-p_0}{p_0} = \frac{q+\Delta p}{p_0}$  in the case of the vineyard, where  $\Delta p \equiv p_1 - p_0$ .

We will now show why both assets should earn the same return, and how markets are going to attain this result through price adjustments. Suppose first that the vineyard earns more than the financial asset (either in dollar terms or in terms of annual return). Then immediately pure arbitrage would be in order. Investors would increase their demand for vineyards; on the financial market, an excess demand would appear, due to both a decrease in supply of loanable funds (which prefer to look for a higher reward in vineyards) and an increase in demand for funds from pure arbitrageurs who, at no cost, would borrow on the financial market in order to invest in vineyards and earn a higher return there. The consequences would be to *increase* the rate of interest, on the one hand, and to *drive up* the price of vineyards, on the other; such an increase in the price  $p_0$  of the vineyards would *decrease* the rate of return on the vineyard. (The denominator of this rate of return  $\frac{q+\Delta p}{p_0} = \frac{q+p_1}{p_0} - 1$  increases, and thus the rate of return decreases.) We can see that both rates of return will converge toward the same value, assuming there are no transaction costs, because it will be in the interest of investors to enter a market with the higher return and leave the market with the lower return; this process will continue until both returns are equal. Hence we have the equality

$$i = \frac{q}{p_0} + \frac{\Delta p}{p_0} \quad (16)$$

which is called the equation of arbitrage on interest rates, or the Fisher equation in honor of Irving Fisher, who published it at the end of the



nineteenth century. It may be interesting to note that about two centuries ago Turgot de l'Aulne came up with this relationship, showing how the equilibrium prices of capital assets, namely poor lands and fertile lands, were determined from it (as we mentioned in our introduction).

The *current price of the asset*  $p_0$  makes this equality hold. In the case we just considered, we supposed the left-hand side of (16) to be lower than the right-hand side; this led to an increase in the price of the vineyard, a decrease in the return on the vineyard, and an increase in the interest rate. Should the contrary situation have prevailed initially, and the return on the vineyard have been lower than the interest rate, there would have been a net supply of vineyards and a net supply of loanable funds on the financial market. This would have led to a drop in the price of vineyards, an increase in their rates of return, and a decrease in the interest rate until equilibrium was restored.

Until now we have followed our initial hypothesis that both kinds of assets were riskless, or equivalent from the point of view of our conclusions, which rendered the investor neutral to risk. The investment was riskless if the investor knew with certainty the exact amounts of his rental rate  $q$  and the future price  $p_1$  of his investment (perhaps because he had been paid his rent in advance and sold his investment in a forward transaction). If the investor is neutral to risk, that means he makes a forecast of  $q$  and  $p_1$  (which are random variables) and considers the resulting expected value  $E(q + p_1)$  as if it were a certainty. In either case, the investor does not attach any weight to the riskiness of his investment. In reality, we know that certainty can be postulated only if we assume the existence of forward or futures markets, and very seldom can an investment be considered as completely riskless; only a handful of individuals would be truly risk-neutral. Therefore, the Turgot-Fisher equation of arbitrage  $i = \frac{q + \Delta p}{p_0}$  holds (and permits us to determine the price of capital assets) only under very special circumstances.

Since investors attach weight to riskiness, they will require a risk premium over the riskless rate to induce them to hold the risky asset at price  $p_0$ . In other words, the (expected) rate of return  $\frac{E(q + \Delta p)}{p_0}$  will have to be higher than  $i$ , by a margin called the risk premium, for price  $p_0$  to be an equilibrium price. Therefore, the current price of the asset  $p_0$  will have to be *lower* than the level considered earlier for the following equality to be valid. Call  $\pi$  the risk premium; we will then have:

$$i + \pi = \frac{E(q + p_1)}{p_0} - 1 \quad (17)$$

and therefore  $p_0$  has to be below its initial value for this equation to be valid. Figure 2.4 illustrates this point. If  $p_0$  designates the value of  $p_0$ , which solves  $i = \frac{E(q+p_1)}{p_0} - 1$ , then  $p'_0$  ( $p'_0 < p_0$ ) solves  $i + \pi = \frac{E(q+p_1)}{p'_0} - 1$ .

We can of course calculate readily  $p'_0$  as  $\frac{E(q+p_1)}{1+i+\pi}$  and see that the new price of the asset, taking into account a risk premium, is an increasing function of the expected rental and the future price, while it is a decreasing function of the interest rate and the risk premium.

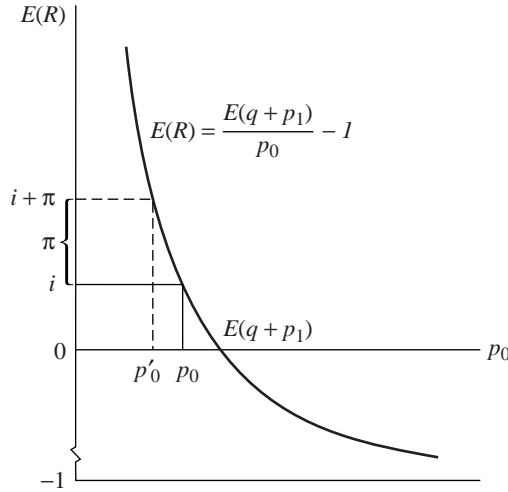
The all-important question remains regarding what might be the order of magnitude for the risk premium  $\pi$ . An entire branch of economic and finance research during these last three decades was devoted to the quest for an answer to this far from trivial question. In 1990, the Nobel Prize in economics was awarded to William Sharpe, who in 1964, with his capital asset pricing model, published the first attempt to capture this elusive but essential concept.<sup>3</sup>

In his path-breaking contribution, Sharpe showed two things. First, the correct measure of riskiness of an investment was its  $\beta$ , that is, the covariance of the asset's return with the market return  $R_M$  divided by the variance of the market return. This is the slope of the regression line between the market return (the abscissa) and the asset return (the ordinate). Second, the risk premium  $\pi$  is the product of  $\beta$  and the difference between the expected market return  $E(R_M)$  and the risk-free rate  $i$ . According to the capital asset pricing model,  $\pi$  is therefore given by

$$\pi = [E(R_M) - i]\beta$$

When dealing with risky bonds, we must tackle the difficult issue of the risk premium attributed by the market to that kind of asset if we want to know whether the price of the risky bond observed on the market is an equilibrium price or not. In this text, we deal mainly with bonds bearing no default risk, so we can safely return to our riskless model for the time

<sup>3</sup> See William F. Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *Journal of Finance*, 19 (September 1964): 425–442, or any investment text (for instance: William F. Sharpe, Gordon J. Alexander, Jeffrey V. Bailey, *Investments*, 6th ed., Englewood Cliffs, N.J.: Prentice Hall International, 1999).

**Figure 2.4**

**Determination of the price of the asset  $p'_0$ , corresponding to the arbitrage equation of interest with a risk premium;  $p'_0$  is a decreasing function of  $\pi$ .**

being. However, we will not dodge the problem of the riskiness posed by possible changes in interest rates.

Coming back, therefore, to our bond market, we realize that the vineyard can simply be replaced by a coupon-bearing bond, with the income stream  $q$  generated by the vineyard (the coupon), while the price of the vineyard  $p$  is changed to the price of the bond  $B$ . So we must have

$$i = \frac{c}{B} + \frac{\Delta B}{B} \quad (18)$$

This relationship shows that the return on the bond must be equal to the rate of interest. In this relationship,  $c/B$  will always be called the *direct yield* of the bond (not to be confused with the coupon rate  $c/B_T$  we introduced earlier). The coupon rate  $c/B_T$  is the coupon divided by the par value (7% in our previous example). The direct yield is the coupon divided by the current price of the bond  $B$ . If this relationship is true, the price of the bond will see its price change by the amount  $\Delta B/B = i - c/B$ , which, in our case, is

$$\frac{\Delta B}{B} = i - \frac{c}{B} = 5\% - \frac{70}{1154.44} = 5\% - 6.06\% = -1.06\%$$

Let us verify that this is indeed true. When there are 10 years to maturity, the bond is worth  $B_0 = 1154.44$ ; one year later, when maturity is reduced to 9 years, the new price, which can be handily calculated from formula (13), is 1142.16. Hence the relative price change is  $\frac{\Delta B}{B} = \frac{1142.16 - 1154.44}{1154.44} = -1.06\%$ , as expected from the equation of arbitrage on interest.

Before proving rigorously that relationship (16) is true, and that our numerical results are not just the fruit of some happy coincidence, let us notice that the direct yield is 6.06%, which is above the rate of interest (5%). (This is precisely the reason why the owner of the bond will make a capital loss after one period; there would be a “free lunch” otherwise.) Notice a general property is that the *direct yield*  $c/B$  (not only the coupon rate  $c/B_T$ ) will *always* be above the interest rate  $i$  if the bond is above par (equivalently, if the coupon rate is above the rate of interest); this property is not quite obvious, and it needs to be demonstrated. Once more, formula (15) will prove to be very useful. From (15), we know that  $\frac{B}{B_T}$  is just a weighted average between  $\frac{c/B_T}{i}$  and 1. So we can always write, when  $\frac{c/B_T}{i}$  is above 1,

$$\frac{c/B_T}{i} > \frac{B}{B_T} > 1$$

From the first inequality above, we can write  $\frac{c}{i} > B$ , and hence  $\frac{c}{B} > i$ . The converse, of course, would be true if the coupon rate was below the interest rate; with  $c/B_T < i$  we would have  $\frac{c}{B} < i$ .

Now, to prove relationship (18), we just have to calculate  $\frac{\Delta B}{B}$ . Suppose that  $B_t$  represents the value of the bond at time  $t$  with maturity  $T$ , and that  $B_{t+1}$  is the value of the bond at time  $t + 1$  when maturity has been reduced to  $T - 1$ . All we have to do is calculate  $\frac{\Delta B}{B} = \frac{B_{t+1} - B_t}{B_t}$ .  $B_{t+1}$  is then the value of the same bond maturing at time  $T$ , in  $T - 1$  years. Applying the closed-form expression (13) from section 2.5, we then have

$$\begin{aligned} B_t &= \frac{c}{i} [1 - (1 + i)^{-T}] + B_T (1 + i)^{-T} \\ &= \frac{c}{i} + (1 + i)^{-T} \left[ B_T - \frac{c}{i} \right] \end{aligned} \tag{19}$$

$$B_{t+1} = \frac{c}{i} + (1 + i)^{-(T-1)} \left[ B_T - \frac{c}{i} \right] \tag{20}$$

The change in value of the bond when maturity decreases from  $T$  to  $T - 1$  is

$$\begin{aligned} B_{t+1} - B_t &= \left( B_T - \frac{c}{i} \right) [(1+i)^{-T+1} - (1+i)^{-T}] \\ &= \left( B_T - \frac{c}{i} \right) [i(1+i)^{-T}] \end{aligned} \quad (21)$$

Let us determine the value of this last expression. From the former definition of  $B_t$  [first part of (19)], we can write

$$\begin{aligned} B_t &= \frac{c}{i} - \frac{c}{i}(1+i)^{-T} + B_T(1+i)^{-T} \\ iB_t &= c - c(1+i)^{-T} + i(1+i)^{-T}B_T \\ &= c + i(1+i)^{-T} \left( B_T - \frac{c}{i} \right) \end{aligned}$$

and also

$$\left( B_T - \frac{c}{i} \right) [i(1+i)^{-T}] = iB_t - c \quad (22)$$

Consequently, we get

$$B_{t+1} - B_t = iB_t - c \quad (23)$$

and, finally,

$$\frac{\Delta B}{B} = \frac{B_{t+1} - B_t}{B_t} = i - \frac{c}{B_t} \quad (24)$$

Hence (18) is true.

Of course, as the reader who is already familiar with the concept of continuous interest rates may well surmise, this relationship holds also when the bond provides a continuous coupon. It would be written

$$i(t) = \frac{c(t)}{B(t)} + \frac{1}{B(t)} \frac{dB(t)}{dt} \quad (25)$$

or, equivalently,

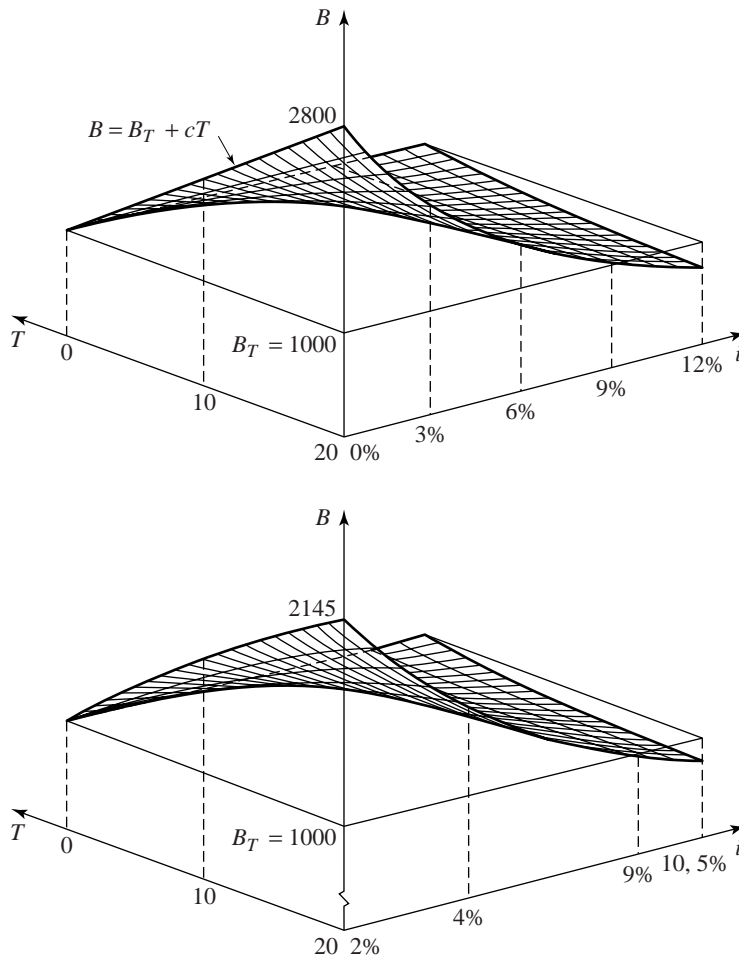
$$\frac{1}{B(t)} \frac{dB}{dt} = i - \frac{c(t)}{B(t)}$$

We may even add that the demonstration in continuous time is much easier and much more direct than the demonstration in discrete time. We will show this in chapter 11, where we deal with continuous spot and forward rates of return.

## 2.7 A global picture of a bond's value

We now have a firm grasp of two fundamental properties of bonds. First, there is a negative relationship between the rate of interest and the price of the bond; this was pictured in figure 2.3. Second, even in the presence of fluctuations in the interest rate, the price of the bond will ultimately move toward par. This convergence of the price toward par was pictured in figure 2.2. We may now want to have a complete picture of these two phenomena in order to describe how, in the course of its life, the price of the bond may jump up and down, above and below its par value, but nevertheless will ultimately end up at its par value. For that purpose, just consider that either formula (11) or (13) for the bond value is a function of the two variables  $i$  and  $T$ , denoted  $B(i, T)$ , and hence represented by a surface in a 3-D space where  $i$  and  $T$  are on the horizontal plane, while the value of the bond  $B(i, T)$  is on the vertical axis (figure 2.5). We have chosen for that purpose a long bond (20 years) and a coupon of \$90, the par value being 1000.

At any point in time, all the curves representing the value of the bond as a function of the rate of interest are sections of this surface by planes parallel to plane  $(B, i)$ . To illustrate, when the remaining maturity is 20 years, the bond value is 1000, if the rate of interest is 9%. (The bond is then at par.) But should the rate of interest fall as low as 2%, the price of the bond would climb as high as 2145. (See the lower half of figure 2.5.) For a 4% rate of interest, the price of the bond would be 1680. The curve connecting 2145, 1680, and 1000 is the section of  $B(i, T)$  for  $T = 20$ ; it represents the value of the bond for all values of the rate of interest between 2% and 10.5%, when the maturity is 20 years. For shorter maturities, the relevant parallel sections are drawn on the same diagram; they depict the value of the bond for the same range of interest rates. They are similar to the curves that were represented in figure 2.3. The reader can visually verify the convexity of these curves, which become flatter and flatter as time passes; ultimately, when the bond is at maturity (when  $T = 0$ ) the curve  $B(i)$  degenerates to the horizontal line, the height of



**Figure 2.5**  
The price of a bond as a function of its maturity and the rate of interest.

which is the par value. The economic interpretation of this horizontal line is, of course, that whatever may be the interest rate prevailing at the time the bond reaches maturity, its value will be at par.

The reader can see from the same diagram the evolution of the bond for any given rate of interest when maturity decreases from  $T$  to zero. The types of curves that were pictured in figure 2.2 are represented for this bond (9% coupon rate; maturity: 20 years) by sections of the surface in the direction of the time axis. They are, of course, above the horizontal plane at height \$1000 when the rate of interest is below the coupon rate of 9%, and they lie under it for interest rates above 9%. They all converge toward the par value of \$1000.

We can now finally understand the apparent erratic evolution of the price of a bond through time. If a bond jumps up and down around its par value, ending finally at its par value, it is simply because the interest rate has an erratic behavior. Suppose that the evolution of the rate of interest is represented by a curve in the horizontal  $i, T$  plane, its behavior being such that it is sometimes above and sometimes below 9%. Also suppose that its fluctuations are more or less steady through time. (In terms of statistics, we would say that its historical variance is constant.) Fluctuations of the price of the bond can be understood by watching the corresponding curve on the surface  $B(i, T)$ . It can be seen that the fluctuations of the price of the bond, which are due solely to those of the rate of interest, will dampen as maturity approaches, because we are more and more in the area where the surface gets flatter. Although the displacements on the  $i$  axis keep more or less the same amplitude, the variations in the bond's price will become smaller and smaller. Thus it becomes clear that the historical variance of the price of a bond is not constant, but gets smaller, tending toward zero when maturity is about to be reached. This fact will be important if we want to evaluate a bond with a call option attached to it, which is the case if the issuer of a bond has the right to buy it back from its owner if the price of the bond increases beyond a certain fixed level (or, equivalently, if rates of interest drop sufficiently).

## 2.8 How can we really price bonds?

A first review of this chapter may leave the reader with the impression bonds are simply priced as portfolios of zero-coupon bonds, each being



discounted by a rate of interest that acts erratically and unpredictably. First, we considered a “given” term structure, and then we simplified this to a horizontal (constant) term structure. In reality, however, discount factors and the term structure are themselves computed from the universe of bond prices in a number of ways that will be explained later in the book (in chapter 9). We will also address some of the difficulties researchers have met in doing so.

This leads us to another problem. Imagine that a sophisticated econometric method has been devised to obtain from the universe of bond prices a complete series of spot rates (the entire term structure). We should then raise the following issue: if bond prices are equilibrium prices, then the corresponding term structure can certainly be considered one. In theory, this information could then be used to determine if any given triple-A bond is under- or overpriced. But how can we be sure these are equilibrium values? And even if we knew for sure what the true equilibrium values are, we would not yet be safe. As John Maynard Keynes once pointed out, what we need to know is not what assets’ true equilibrium values are, but what the market *thinks* the equilibrium values are! What Keynes said of stocks is of course equally relevant for bonds, and we are still a long way from making sure bets on the direction that bond prices or spot rates will take in the future, and even further from guessing then exactly.

### Key concepts

- The value of a bond is the sum of the discounted cash flows of the bond
- A bond is above par if its coupon rate is above interest rates, and vice versa.
- The value of a default-free bond will always decrease if interest rates rise, and vice versa.
- The fundamental determinant of changes in the price of a bond is movement in interest rates; a secondary cause is shortening maturity

### Basic formulas

- Notation:  $B$  = value of bond today;  $T$  = maturity;  $B_T$  = par or redemption value;  $c$  = coupon value;  $i_t$  = spot rate corresponding to term  $t$ ;

$i$  = interest rate pertaining to all maturities;  $c_t$  = cash flow received by owner at time  $t$ .

•  $B(i_t) = \sum_{t=1}^T c_t(1+i_t)^{-t}$ ,  $t = 1, \dots, T$  ( $i_t$  defines the “term structure of interest rates”)

•  $B(i)/B_T = \frac{c/B_T}{i} \left[ 1 - \frac{1}{(1+i)^T} \right] + \frac{1}{(1+i)^T}$  (if  $i_t$  is constant for all terms  $t$ , that is, the term structure is flat).

•  $\frac{\Delta B}{B(i)} = i - \frac{c}{B(i)}$

### Questions<sup>4</sup>

2.1. Consider a bond whose par value is \$1000, its maturity 10 years, and its coupon rate 9%. (Assume that the coupon is paid once a year.) Suppose also that the spot interest rates for each of those 10 years is constant (the term structure of interest rates is said to be flat), and consider that their level is either 12%, 9%, or 6%.

a. Without doing any calculations, explain whether the bond’s value will be increasing or decreasing when interest rates decrease from 12% to 9% to 6%. In which of these three cases can the bond’s value be predicted without any calculations?

b. In which case will the bond’s value be the farthest from its par value, and why?

c. Confirm your answers to (a) and (b) by determining the bond’s value in each case.

2.2. Consider two bonds ( $A$  and  $B$ ) with the same maturity and par value. Bond  $B$ , however, has a coupon (or a coupon rate) which is 20% lower than that of  $A$ . Will the price of bond  $B$  be

a. lower than  $A$ ’s price by more than 20%?

b. lower than  $A$ ’s price by 20%?

c. lower than  $A$ ’s price by less than 20%?

Explain your answer. Can you figure out a case in which the price of  $B$  would be lower than  $A$ ’s price by exactly 20%?

<sup>4</sup> In the questions for this chapter and the next ones, simply consider that coupons are paid once a year.

2.3. In a riskless world, suppose that the future (or futures or forward) price of an asset with a one-year maturity contract is \$100,000; the (certain) rental rate of that asset is \$5000, while the riskless interest rate is 6% per year. What is the equilibrium expected rate of return on this asset? What is the current price of this asset?

2.4. Using the same data as in question 2.3, we now assume the market applied a 10% risk premium to that kind of investment. What would be the equilibrium price of the asset in this risky world?

2.5. Suppose that one of the following events for an investment occurs:

- a. The expected rental of the asset increases.
- b. The expected future price of the asset increases.
- c. The current price of the asset increases.

If the risk premium of the asset does not change, what are the consequences of these events on the expected rate of return of the investment?



# 3

## THE VARIOUS CONCEPTS OF RATES OF RETURN ON BONDS: YIELD TO MATURITY AND HORIZON RATE OF RETURN

SHYLOCK. Is it so nominated in the bond?

...

I cannot find it; 'tis not in the bond!

*The Merchant of Venice*

### Chapter outline

3.1	Yield to maturity	60
3.2	Yield to maturity for bonds paying semi-annual coupons	61
3.3	The drawbacks of the yield to maturity as a measure of the bond's future performance	62
3.4	The horizon rate of return	62
Appendix 3.A.1	Yield to maturity for semi-annual coupons, and any remaining maturity (not necessarily an integer number of periods)	66

### Overview

There are two fundamental rates of return that can be defined for a bond: the yield to maturity and the horizon rate of return. The yield to maturity of a bond is entirely akin to the internal rate of return of an investment project: it is the discount rate that makes equal the cost of buying the bond at its market price and the present value of the future cash flows generated by the bond. Equivalently, it can be viewed as the rate of return an investor would make on an investment equal to the cost of the bond if all cash flows could be reinvested at that rate.

The concept of future value is central to that of horizon rate of return. If all the proceeds of a bond are reinvested at certain rates until a given horizon, then a future value of investment can be calculated with respect to that horizon. It is made up of two parts: first, the future value of the cash flows reinvested at various rates (which have to be determined); second, the remaining value of the bond at that horizon, which consists of the remaining coupons and par value expressed in terms of the horizon year chosen. From that future value at a given horizon, it is possible to define the horizon rate of return as the yearly rate of return, which would transform an investment equal to the cost of the bond into that future value.

There are advantages and some inconveniences in using each kind of rate of return. The advantage of using the internal rate of return is that it is very easy to calculate (all software programs perform this calculation

handily); the problem with that concept is that we have to suppose, implicitly, that all proceeds of the bond will be reinvested at that same rate—which, of course, is not always the case. This is why the concept of horizon rate of return was introduced: to take into account the fact that the entire structure of interest rates is permanently moving. The increase in realism has its cost; we have to be able to forecast what the future spot rates will be for all the cash flows reinvested until the horizon year, and to forecast as well the structure of interest rates prevailing at the horizon year for the remaining cash flows to be received at that time—no light task indeed.

### 3.1 Yield to maturity

The *yield to maturity* of an investment that buys and holds a bond until its maturity is entirely akin to the concept of the *internal rate of return* of a physical investment. It shares its simplicity but also, unfortunately, its drawbacks.

Suppose that an investment—be it physical or financial—is characterized by a series of cash flows that are supposed to be known with certainty; this is precisely the case of a defaultless bond. Suppose also that we know the price today to be  $B_0$ , and that the bond will be kept to its maturity. The question then arises regarding the rate of return this bond will earn for its owner. Traditionally, one way to answer this question is to define the bond's yield to maturity in the following way: it is the rate of interest that will make the present value of the bond's cash flows exactly equal to the price of the bond. Thus the yield to maturity, denoted here  $i^*$ , will be the rate of interest such that

$$B_0 = \frac{c}{1+i^*} + \frac{c}{(1+i^*)^2} + \cdots + \frac{c+B_T}{(1+i^*)^T} \quad (1)$$

Written in more compact form,  $i^*$  is such that

$$B_0 = \sum_{t=1}^T c_t (1+i^*)^{-t} \quad (1')$$

where  $c_t (t = 1, \dots, T)$  are the cash flows of the bond.

How do we calculate this value  $i^*$ ? Analytically, in the general case, it is not possible to calculate a closed-form expression for  $i^*$  depending on  $B_0$ ,

$c$ , and  $B_T$ . Indeed, should we try to do so, we would soon be on our way to calculating the roots of a polynomial of order  $T$ —and for  $T > 4$ , no closed form of these roots exists. Therefore, only a trial-and-error process can be used, such as those developed today in pocket calculators or in any financial software; for one initial value of  $i$ , the computer calculates the present value of the sum of the cash flows. If it happens to be above  $B_0$ , the computer tries a higher value for  $i$ ; should the present value be lower than  $B_0$  this time, the computer will choose a somewhat lower value for  $i$ . This process continues very quickly until a value of  $i$  gives  $B_0$  with a sufficiently high accuracy; that will be  $i^*$ , the yield to maturity of the bond.

Defining the yield to maturity in such a way, as that value  $i^*$  that equates both sides of equation (1), has the drawback of concealing the true significance of the return that is supposed to accrue to the bond's owner. Note that the yield to maturity  $i^*$  is also the return to the investor, which makes it equivalent to buying the bond, receiving and reinvesting all coupons at rate  $i^*$ , and investing a lump-sum  $B_0$  at rate  $i^*$  for  $T$  years.

This can be seen by multiplying both sides of (1) by  $(1 + i^*)^T$  (we just transform present values on both sides into future values at horizon  $T$ ). We get

$$B_0(1 + i^*)^T = c(1 + i^*)^{T-1} + c(1 + i^*)^{T-2} + \cdots + c + B_T \quad (2)$$

which translates as

$$\begin{aligned} \text{Future value of an investment } B_0 \\ = \text{future value of all cash flows reinvested at rate } i^* \end{aligned}$$

### 3.2 Yield to maturity for bonds paying semi-annual coupons

As we mentioned in chapter 1, the price the investor in a Treasury note or bond receives is not just the ask price, but the ask price *plus* any accrued interest (AI), if applicable; that is if at the time of the transaction some time has elapsed since the last coupon payment or since the bond's issuance (as in most cases, of course), when no coupon has yet been made. We call "total price" (TP) the sum of the ask price (AP) and accrued interest (AI). So we have  $TP = AP + AI$ . Let  $n$  be the number of periods (for instance, six-month periods) remaining in the bond's life—or until the first call date established by the bond's issuer. Let  $f$  be the fraction of the period remaining until the next coupon. For a bond paying a semi-annual coupon, the yield to maturity of that bond—its internal rate to its

potential investor—is the rate  $y$ , which equates its cost to the present value of all future discounted cash flows. (In economic parlance, the interest rate makes its net present value equal to zero.) The yield to maturity is thus  $y$ , such that it solves the equation:

$$TP = \frac{f(c/2)}{(1 + y/2)^f} + \sum_{t=1}^n \frac{c/2}{(1 + y/2)^{t+f}} + \frac{\text{par}}{(1 + y/2)^{n+f}}$$

As before, only trial-and-error or numerical methods can solve this equation. Numerous software programs and even pocket calculators make such calculations quite easy; as inputs, one simply needs the date of purchase, the maturity (or first call date), and the yearly coupon to obtain the yield to maturity.

### 3.3 The drawbacks of the yield to maturity as a measure of the bond's future performance

The central problem with the concept of yield to maturity is simply that it is a return for the owner of the bond if we assume that all cash flows will be reinvested at the same rate  $i^*$ , which is hardly realistic, precisely because we know that interest rates do move around. After all, when we defined the yearly return on a bond, this was equal to  $\frac{c}{B} + \frac{\Delta B}{B} \cdot \frac{c}{B}$ . The direct yield was known with certainty, but  $\Delta B$ , the future price change in the bond, was not known, precisely because the future rates of interest, one year from now, were unknown. It is therefore no surprise that the yield to maturity of a bond, which usually involves an investment period longer than one year, does not accurately predict what the yield for the investor will be. A better measure is needed here, and this is precisely why the concept of horizon rate of return (also called the *holding period rate of return*) has been introduced, and is fortunately more and more widespread.

### 3.4 The horizon rate of return

The purpose of the horizon rate of return is to measure the future performance of the investor if he keeps his bond for a number of years (called the *horizon*). Defined in the most general way, the horizon rate of return, denoted as  $r_H$ , is the return that transforms an investment bought today at price  $B_0$  by its owner into a future value at horizon  $H$ ,  $F_H$ . Thus  $r_H$  is such that



$$B_0(1 + r_H)^H = F_H \quad (3)$$

and  $r_H$  is equal to

$$r_H = \left( \frac{F_H}{B_0} \right)^{1/H} - 1 \quad (4)$$

As the reader may well surmise, the difficulty here lies in calculating the future value  $F_H$  of the investment when this investment is a coupon-bearing bond. Indeed, all coupons to be paid are supposed to be regularly reinvested until horizon  $H$  (which may well be, of course, shorter than maturity  $T$ ). Therefore, we have to be in a position to evaluate the future rates of interest at which it will be possible to reinvest these coupons. This implies a *forecast* of future rates of interest, which is no easy task. For the time being, to keep matters clear, consider the situation of an investor who has just bought our bond valued at  $B(i_0)$  when the interest rate common to all terms was equal to  $i_0$ . Suppose the following day the interest rate moves up to  $i$  ( $i \neq i_0$ ); suppose also that an investor holds onto his investment for  $H$  years. What will be his  $H$ -year horizon return if we suppose that interest rates will remain fixed for that time span? The new value of the bond, when  $i_0$  moves to  $i$ , will be  $B(i)$ , and its future value will be  $F_H = B(i)(1 + i)^H$ . Hence, applying our formula (4), we get

$$r_H = \left( \frac{B(i)}{B_0} \right)^{1/H} (1 + i) - 1 \quad (5)$$

As an example, consider the situation of our investor who has bought the bond with a remaining term to maturity of 10 years and a coupon rate of 7%, when interest rates were at 5%. He has paid  $B_0 = B(i_0) = \$1154.44$ . Suppose now that the next day interest rates move up to 6%. His bond will drop in value to \$1073.60. If he holds his bond for 5 years, and if interest rates stay at 6% for that span of years, his horizon return will be

$$r_H = \left( \frac{1073.60}{1154.44} \right)^{1/5} (1.06) - 1 = 4.47\%$$

We observe immediately that, although the rate of interest at which he can reinvest his coupons (6%) is higher than initially, his overall yearly

performance will be *smaller* than 5%. This apparently strange phenomenon can be explained by the capital loss our investor has incurred when the interest rate moved up from 5% to 6%, and by the fact that the reinvestment of coupons at a higher rate (6%) than the previous one (5%) has not been sufficient to compensate for a capital loss that still hurts after 5 years. When such an increase in interest rates happens, two conflicting phenomena result: first, there is a capital loss and, second, there is some gain from the reinvestment of coupons at a higher rate. Which of these two phenomena will prevail? In other words, will the horizon rate of return be higher or lower than the initial yield at which the bond was bought? From what we have said already (and for the time being!), nobody can tell, unless we make the direct calculations as indicated by formula (5). But one thing is sure: the longer the horizon and the longer the replacement of coupons at a higher rate, the greater the chances that the investor will outperform the initial yield of 5%.

This will be true for two reasons: First, the reinvestment of coupons for a long period of time will tend to boost the horizon rate of return; second, the longer the horizon, the closer to its par value the bond will be, thereby offsetting the initial loss in value due to the rise in the interest rate. Of course, the reverse would be true if interest rates fall. Suppose that interest rates fall from 5% to 4% immediately after the purchase of the bond. The investor will realize an important capital gain, which will erode itself as time passes. On the other hand, the investor with a short horizon will not have to care too much about the reinvestment of his coupons at a lower rate (4%) than the original rate (5%). Thus, in the event of a decrease in interest rates, the shorter the horizon, the higher the performance (there are important capital gains and little to lose in coupon reinvestment); and the longer the horizon, the smaller the performance. (The capital gain will dampen out through time, and the loss from smaller returns on coupon reinvestment will pinch more.)

Before getting a little deeper into this subject, let us show, as an example, what the horizon return would be in the three following scenarios: no change in the rate of interest (5%), an increase of the rate of interest to 6%, and a decrease to 4%. The bond is the one we have already considered in our example: coupon 7%; maturity: 10 years; par value: 1000.

Our intuitive grasp of the properties of the horizon rate of return are confirmed when reading table 3.1. The longer the horizon, the higher the horizon return if interest rates move up immediately after the bond's purchase, and the lower the horizon return if interest rates have gone

**Table 3.1**

**Horizon return (in percentage per year) on the investment in a 7% coupon, 10-year maturity bond bought 1154.44 when interest rates were at 5%, should interest rates move immediately either to 6% or to 4%**

Horizon (years)	Interest rates		
	4%	5%	6%
1	12.01	5	-1.42
2	7.93	5	2.22
3	6.60	5	3.47
4	5.95	5	4.09
5	5.55	5	4.47
6	5.29	5	4.73
7	5.11	5	4.92
7.7	5.006	5	5.006
8	4.97	5	5.04
9	4.86	5	5.15
10	4.77	5	5.23

down. Of course, the reader will want to have sound proof of this proposition by considering formula (5). If interest rates go up,  $B(i)/B(i_0)$  is smaller than one, and the  $H$ th root of a number smaller than one is also a number smaller than one, and it is an increasing function of  $H$ . Conversely, if interest rates go down,  $B(i)/B(i_0)$  is larger than one, and its  $H$ th root is a decreasing function of  $H$ . Therefore, the horizon rate of return, defined by (5), is an increasing function of the horizon  $H$  if  $i > i_0$ , and a decreasing function of  $H$  if  $i < i_0$  (it is equal to  $i_0$  and independent from  $H$  if  $i$  stays at  $i_0$ ).

We now need to notice another remarkable property of these horizon rates of return. We know that in each interest scenario, the horizon return may be smaller than 5% or higher than that value, depending on the horizon itself. It turns out that in *both* scenarios there exists a horizon for which the horizon return will be extremely close to 5%. This horizon seems to be between seven and eight years, and, at first inspection of the table, perhaps closer to eight years than seven. Indeed, some calculations show that if the horizon is 7.7 years, then the horizon rate of return will be slightly above 5% (5.006%) whether the interest rate falls to 4% or increases to 6%. Immediately, the question arises: Is this a coincidence stemming from the special features of our bond, or would it be possible for any bond to have a horizon such that the horizon return is equal to the original rate of return whatever happens to the rates of interest? This question is important. We will show later on that this unique horizon does

in fact exist for any bond or bond portfolio; its name is the *duration* of the bond. This concept and its properties are so important that we will now devote a special chapter to it.

### Appendix 3.A.1 Yield to maturity for semi-annual coupons, and any remaining maturity (not necessarily an integer number of periods)

The calculation of the yield to maturity of a bond paying semi-annual coupons, and whose next coupon will be paid in less than half a year, can be made in a number of alternative ways with the same result. We find the simplest way to be the following.

Let “one period” denote six months, and assume that one coupon is paid at the end of each period.

Let 0 designate the point in time at which the last coupon was paid or the date of the bond’s issuance if no coupon has yet been paid. We will call this point in time “initial time.”

Let  $\theta$  be the fraction of a period elapsed since the last coupon payment or since the date of the bond’s issuance if no coupon has yet been paid. Thus, if a coupon is paid twice a year, we always have  $0 \leq \theta < 1$  (see figure 3.1).

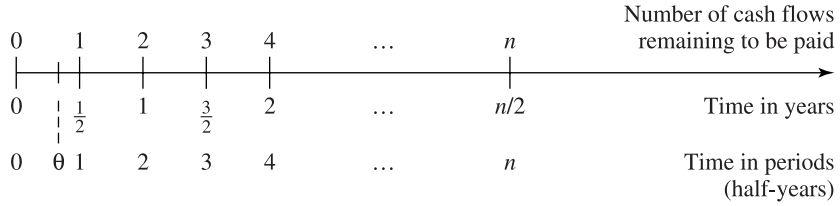
Let  $n$  be the number of coupons remaining to be paid. From our definitions and figure 3.1, this implies that the next coupon will be paid in  $1 - \theta$  half-years. If  $i$  is the annual rate of interest used to discount all cash flows (coupons and principal) to be paid, the present value of those cash flows as viewed from the initial time is

$$\text{Value of bond at time 0} = \sum_{j=1}^n \frac{c_j}{(1 + i/2)^j} \quad (6)$$

where  $c_j$  designates the  $j$ th cash flow and  $j$  designates the *periods*.

Now this value at point 0 should be translated in value at point  $\theta$ . We do this simply by multiplying the above expression by  $(1 + \frac{i}{2})^{-\theta}$ , where  $\theta$  is the fraction of the six-month period elapsed since initial time 0. (If 31 days have elapsed in a 182-day six-month period, then  $\theta$  is simply  $\frac{31}{182} = 0.1703$ .) We have then

$$\text{Value of bond at time } \theta = \left(1 + \frac{i}{2}\right)^{-\theta} \sum_{j=1}^n \frac{c_j}{(1 + i/2)^j} \quad (7)$$



Time  $\theta$  is expressed in half-years ( $0 < \theta < 1$  half-year). It is the time since the last coupon has been paid (or the time since the bond was emitted, if no coupon has yet been paid). For instance, on this figure,  $\theta = \frac{2}{3}$  half-years (or 121 days;  $\theta = \frac{121}{182} = \frac{2}{3}$  half-years).

**Figure 3.1**

**Correspondence between time and the number of cash flows remaining to be paid for a bond paying coupons twice a year.**

Now remember how an internal rate of return is calculated: it is the rate of interest that equates the cost of an investment to its discounted net cash flows; equivalently, it is the rate of interest that makes the net present value of the investment equal to zero. Here the cost of the investment is the cost of the bond (the amount paid by the buyer), and it is equal to the quoted price of the bond *plus* the accrued interest on the bond. This accrued interest belongs to the initial owner of the bond at the time he sells it; so the buyer redeems it for  $\theta c$ . Thus, the cost of acquiring the bond is

$$\text{Cost to the buyer} = \text{ask price} + \theta c$$

The yield to maturity is thus the rate of interest  $i$  such that the following equality holds:

$$\text{Cost to the buyer} = \text{present value of cash flows at time } \theta$$

$$\begin{aligned} &= \text{ask price} + \theta c = (1 + i/2)^\theta \sum_{j=1}^n \frac{c_j}{(1 + i/2)^j} \\ &= (1 + i/2)^\theta \left[ \frac{2c}{i} \left( 1 + \frac{1}{(1 + i/2)^n} \right) + \frac{B_T}{(1 + i/2)^n} \right] \end{aligned}$$

As before, this equation does not generally have a closed-form solution, and only trial-and-error solutions, usually embedded in pocket calculators or spreadsheets, are available. This yield to maturity is called “ask yield”

because it is based on the price that is *asked* to the potential customer (the price he will have to pay, in addition to the accrued interest  $\theta c$ ).

Note that if the value of the bond at time  $\theta$  had been calculated by multiplying  $\sum_{j=1}^n c_j(1+i/2)^{-j}$  by  $1+\theta i/2$  instead of  $(1+i/2)^\theta$  (that is, by using a simple-compounding interest formula instead of a compounded one), nothing much would have changed because  $1+(i/2)\theta$  is just the first-order Taylor approximation of  $(1+i/2)^\theta$  at point  $i=0^1$ ; also  $i/2$  is a small number compared to zero. Consider, for instance, our former example, where one month has elapsed since the last coupon has been paid out or since the bond was emitted. In that case,  $\theta = \frac{31}{182} = 0.1703$ . Suppose also that  $i = 5\%$ /year. Applying  $(1+i/2)^\theta$ , we get  $1.025^{31/182} = 1.0042$ ; on the other hand,  $1+(i/2)\theta$  yields  $1+(0.025)\frac{31}{182} = 1.0042$ ; the first four digits are caught by the McLaurin approximation. If we consider the *longest* period that  $\theta$  could ever take, that is  $\theta = 181/182$ , we would have  $(1+i/2)^\theta = 1.02486$  and  $1+(i/2)\theta = 1.02486$  (even the first five decimals would be caught). In the case of the *shortest* period possible for  $\theta$ ,  $1/182$ , we would have  $(1+i/2)^{1/182} = 1.00013$  and  $1+(i/2)(1/182) = 1.00013$ , no difference to speak of either.

### Key concepts

- Yield to maturity is the interest rate that makes the price of the bond equal to the sum of its cash flows, discounted at this single interest rate.
- Horizon rate of return is the interest rate that transforms the present cost of an investment into a future value.

### Basic formulas

- Yield to maturity:  $i^*$  such that

$$B_0 = \sum_{t=1}^T c_t(1+i^*)^{-t}$$

where  $B_0$  is the bond's value today and  $c_t$  are the annual cash flows.

<sup>1</sup> Indeed, we have  $(1+i/2)^\theta \approx 1+(i/2)\theta$  as a first-order MacLaurin approximation (the Taylor approximation at point  $i/2=0$ ).

- Horizon rate of return: with  $F_H \equiv$  future value of the investment,  $r_H$  is such that

$$B_0(1 + r_H)^H = F_H$$

$$r_H = \left(\frac{F_H}{B_0}\right)^{1/H} - 1$$

### Questions

- 3.1. Why did we say, in section 3.1, that in order to calculate analytically the yield to maturity of a coupon-bearing bond with maturity  $T$ , we would have to solve a polynomial equation of order  $T$ ?
- 3.2. What is the horizon rate of return on a bond if the interest rate does not move and stays fixed at  $i_0$ ?
- 3.3. A bond with 9% coupon rate, \$1000 par value, and 30-year maturity was bought when the (flat) interest structure was 8%. The next day, the interest rates jumped to 9% and stayed at that level for 10 years. What are the horizon returns on that bond for the following horizons  $H$ ?
  - a.  $H = 1$  year
  - b.  $H = 4$  years
  - c.  $H = 10$  years
- 3.4. Consider the scenario corresponding to section 3.3. What would be the infinite horizon rate of return on an investment in such a scenario? Give an answer from analytical considerations, and then give an answer that does not require any calculations.
- 3.5. Using a spreadsheet perhaps, confirm the results established in table 3.1. (Suppose as usual that coupons are paid yearly.)
- 3.6. Referring to section 3.3, can you explain intuitively the existence of a horizon leading to the same rate of return whenever interest rates either increase from 5% to 6% or decrease to 4%?





# 4

## DURATION: DEFINITION, MAIN PROPERTIES, AND USES

PORTIA. . . . 'tis to peize the time<sup>1</sup>  
*The Merchant of Venice*

### Chapter outline

<b>4.1</b>	<b>Definition and calculation</b>	<b>72</b>
<b>4.2</b>	<b>Duration as a center of gravity</b>	<b>75</b>
<b>4.3</b>	<b>Is duration an important concept?</b>	<b>75</b>
<b>4.4</b>	<b>Duration as a measure of a bond's sensitivity to changes in yield to maturity</b>	<b>76</b>
<b>4.5</b>	<b>The concept of modified duration</b>	<b>78</b>
<b>4.6</b>	<b>Properties of duration</b>	<b>79</b>
4.6.1	Relationship between duration and coupon rate	80
4.6.2	Relationship between duration and yield	81
4.6.3	Relationship between duration and maturity	83
<b>4.7</b>	<b>Duration of a bond portfolio</b>	<b>86</b>
<b>4.8</b>	<b>Measuring the riskiness of foreign currency-denominated bonds</b>	<b>87</b>

### Overview

There are two main reasons why duration is an essential concept in bond analysis and management: it provides a useful measure of the bond's riskiness; and it is central to the problem of immunization, that is, the process by which an investor can seek protection against unforeseen movements in interest rates.

Duration is simply defined as the weighted average of the times of the bond's cash flow payments; the weights are proportions of the cash flows in present value. This concept has a nice physical interpretation, as the center of gravity of the cash flows in present value. This interpretation proves to be very useful when we want to know how duration

<sup>1</sup>French-speaking readers will have little difficulty understanding the verb "to peize" used by Portia, because the Latin *pensare* gave the French language both *penser* (to think) and *peser* (to weigh). It is now interesting for both French- and English-speaking readers to look up this verb in *The Shorter Oxford English Dictionary* (2 Vol., Oxford University Press, 1973, p. 1540). We find: "3. To keep or place in equilibrium." Placing time in equilibrium is nothing short of determining duration, as we will see in this chapter. Once again Shakespeare had said it all.

will be modified if either the interest rate, the coupon rate, the maturity of the bond, or any combination of those three factors are modified.

The definition of the duration of a bond extends immediately to the definition of the duration of a bond portfolio: this is simply the weighted average of all bonds' durations, the weights being the shares of each bond in the portfolio.

Duration can be shown to be equal to the relative increase, by linear approximation, in the bond's price multiplied by minus the factor rate of capitalization (defined as one plus the rate of interest). This gives a fair approximation of the riskiness of the bond, in the sense that we immediately know from the bond's duration the order of magnitude by which the bond's price will decrease (in percentage) when the rate of interest increases by one percent.

Duration is significantly dependent on the coupon rate, the rate of interest, and maturity. An increase in the coupon rate will lead to a decrease in duration. (Such an increase in the coupon rate will enhance the importance of early payments; therefore, the weights of the first terms of payment will be increased relatively to the last one—which includes the principal—and consequently, the center of gravity of these weights, which is equal to duration, will move to the left.) An increase in the rate of interest will have a similar effect on duration, because the present value of the payments will be significantly diminished for late payments, much less so for early ones; thus the center of gravity of those payments will also move to the left. The effect of increasing the maturity of a bond on its duration is more complex, and contrary to what our intuition would suggest to us, increasing maturity will not necessarily increase duration. In special cases, for above-par bonds and very long maturities, the converse will be true. We will explain why this is so.

Finally, it should be emphasized that, in the case of foreign currency-denominated bonds, the greater part of the risk lies in modifications of the exchange rate.

#### 4.1 Definition and calculation

- Definition: the *duration* of a bond is the *weighted average of the times of payment* of all the cash flows generated by the bond, the weights being the shares of the bond's cash flows in the bond's present value.

Not surprisingly, this definition extends to the duration of a portfolio of bonds. If we consider a portfolio of bonds with the same yield to maturity, the duration will be the weighted average of the times of payment of all the cash flows generated by the portfolio (the weights being the shares of the cash flows in present value). We will show later that this duration is the weighted average of the durations of the bonds making up the portfolio.

As seen in chapter 1, there are several ways to calculate the present value of any cash flow received in the future. One way is to calculate the present value of the cash flows (the sum of which will equal the bond's price). If we use the internal rate of return as the discount rate of the cash flows, the duration will be called *Macauley's duration*, in honor of Frederick Macauley who developed this concept in 1938.<sup>2</sup> The concept of duration was independently rediscovered by Paul Samuelson (1945), John Hicks (1946), and F. M. Redington (1952); Samuelson and Redington invented the concept in their analysis of the sensitivity of financial institutions' assets and liabilities to changes in interest rates.<sup>3</sup> For a historical presentation of the subject, as well as its development, see Gerald Bierwag (1987) and G. Bierwag, G. Kaufman and A. Toevs (1983).<sup>4</sup> If, on the other hand, we use spot rates to evaluate the cash flows, we will get another set of weights for our average (although the result will, generally, be very close to that obtained with the Macauley method). If we use spot rates, we will get a duration referred to as the *Fisher-Weil duration*.<sup>5</sup> We will start with the Macauley concept and then consider the Fisher-Weil approach when dealing, later on, with the immunization process.

<sup>2</sup>Frederick Macauley, "Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1865," *National Bureau of Economic Research*, 1938.

<sup>3</sup>The reader should refer to P. A. Samuelson, "The Effect of Interest Rate Increases on the Banking System," *The American Economic Review* (March 1945): 16–27; T. R. Hicks, *Value and Capital* (Oxford: Clarendon Press, 1946), and F. M. Redington, "Review of the Principle of Life Office Valuations," *Journal of the Institute of Actuaries*, 18 (1952): 286–340.

<sup>4</sup>G. Bierwag, *Duration Analysis: Managing Interest Rate Risk* (Cambridge, Mass.: Ballinger Publishing Company, 1987), and G. Bierwag, G. Kaufman, and A. Toevs, "Recent Developments in Bond Portfolio Immunization Strategies," in *Innovations in Bond Portfolio Management*, ed. G. Kaufman, G. Bierwag, and A. Toevs (Greenwich, Conn.: TAI Press, 1983).

<sup>5</sup>Lawrence Fisher and Roman Weil, "Coping with the Risk of Interest-rate Fluctuations: Return to Bondholders from Naive and Optimal Strategies," *The Journal of Business*, 44, no. 4 (October 1971).

**Table 4.1**  
**Calculation of the duration of a bond with a 7% coupon rate, for a yield to maturity  $i = 5\%$**

(1)	(2)	(3)	(4)	(5)
Time of payment $t$	Cash flows in current value	Cash flows in present value ( $i = 5\%$ )	Share of cash flows in present value in bond's price	Weighted time of payment (col. 1 $\times$ col. 4)
1	70	66.67	0.0577	0.0577
2	70	63.49	0.0550	0.1100
3	70	60.47	0.0524	0.1571
4	70	57.59	0.0499	0.1995
5	70	54.85	0.0475	0.2375
6	70	52.24	0.0452	0.2715
7	70	49.75	0.0431	0.3016
8	70	47.38	0.0410	0.3283
9	70	45.12	0.0391	0.3517
10	1070	656.89	0.5690	5.6901
Total	1700	1154.44	1	7.705 = duration

The Macauley duration is calculated as follows. When  $i$  denotes the yield to maturity of the bond, since it is the weighted average of all times of payment  $t = 1, 2, \dots, T$ , we can write

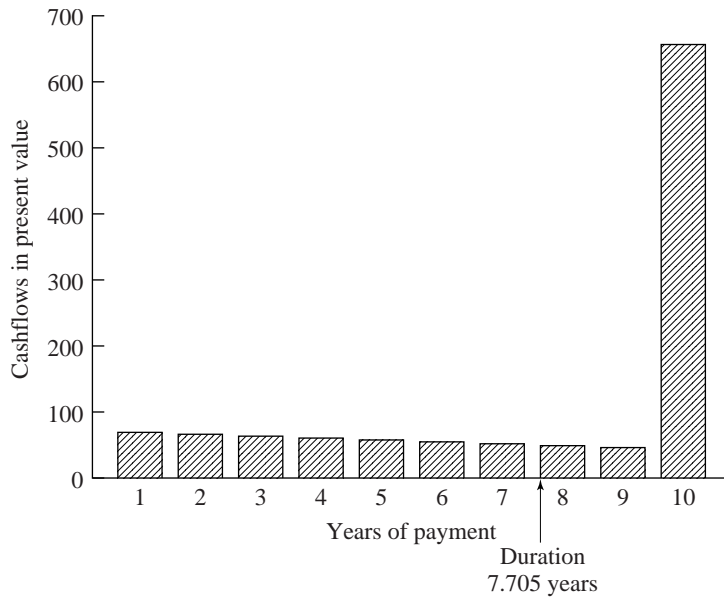
$$D = 1 \frac{c/B}{1+i} + 2 \frac{c/B}{(1+i)^2} + \dots + T \frac{(c + B_T)/B}{(1+i)^T} \quad (1)$$

or, more compactly,

$$D = \sum_{t=1}^T t \frac{c_t/B}{(1+i)^t} = \frac{1}{B} \sum_{t=1}^T t c_t (1+i)^{-t} \quad (2)$$

where  $B$  designates the value of the bond,  $c_t$  is the cash flow of the bond to be received at time  $t$ , and  $i$  is the bond's yield to maturity (which we had previously called  $i^*$  but now simply call  $i$  for convenience).

Table 4.1 shows how the duration of the bond can be calculated for our 7% coupon bond, when all the spot interest rates are 5% (alternatively, when the yield to maturity is 5% and when the price of the bond is \$1154.44). Thus, the bond has a duration of 7.7 years. This means that the weighted average of the times of payment for this bond is 7.7 years when the weights are the cash flows in present value.



**Figure 4.1**

**Duration of a bond as the center of gravity of its cash flows in present value (coupon: 7%; interest rate: 5%).**

## 4.2 Duration as a center of gravity

Duration has a neat physical interpretation. As a weighted average, it is none other than a center of gravity; it will be the center for gravity for the cash flows in present value. If these cash flows, represented as the hatched rectangles in figure 2.1 (chapter 2), are considered as weights, then the duration of our bond is the center of gravity of the (weightless) ruler that would support them. Figure 4.1 presents this physical analogy, which will be very useful because we will be able to derive from it some important properties of duration.

## 4.3 Is duration an important concept?

When asked what they know about duration, most people will respond that it constitutes a measure of the riskiness of the bond with respect to changes in the yield to maturity of the bond. This does not, however, constitute the main interest of the concept. Duration can, indeed, be used

as a proxy of the bond price sensitivity to changes in the yield. But, supposing that knowing this sensitivity with regard to this statistic (the deficiencies of which have already been shown) is important, computers or even hand calculators can almost immediately give the *exact* sensitivity of the price to changes in yield.

The true reason why such a concept is so important is that, originally devised by actuaries, it is today at the heart of the *immunization* process, a strategy designed to protect the investor against changes in interest rates. We will devote an entire chapter to this strategy (chapter 6).

Duration and the immunization process are also highly relevant to banks, pension funds, and other financial institutions. In chapter 7, we will give some important applications of these concepts, and we will address the problem of matching future assets and liabilities (defined for the first time by F.M. Redington in 1952). We will then turn to hedging a financial position with futures. Each time duration will play a central role. Finally, as we will show in the next chapter, duration can help clarify a rather arcane concept for the nonspecialist: the bias introduced in the T-bond futures conversion factor. However, for the time being, we will merely analyze how and why duration is a measure of bond price sensitivity to interest rates. We will then proceed to analyze the properties of duration because they and their economic interpretations will prove to be very interesting, often surprising, and highly useful.

#### 4.4 Duration as a measure of a bond's sensitivity to changes in yield to maturity

Let us now show that duration is a measure of the riskiness of the bond. More precisely, we will show that duration (denoted  $D$ ) is, in linear approximation, the relative rate of decrease in the value of the bond when the yield to maturity  $i$  increases by 1%, multiplied by  $1 + i$ . We have

$$D = -\frac{1+i}{B} \frac{dB}{di} \quad (3)$$

The simplest way to show this is to write the value of the bond as

$$B = \sum_{t=1}^T c_t(1+i)^{-t} \quad (4)$$

and take the derivative of this expression with respect to  $i$ . (Remember that  $i$  can be considered either as the yield to maturity of the bond, or a spot rate common to all maturities  $t = 1, \dots, T$ .) We will have

$$\frac{dB}{di} = \sum_{t=1}^T (-t)c_t(1+i)^{-t-1} = -\frac{1}{1+i} \sum_{t=1}^T tc_t(1+i)^{-t} \quad (5)$$

Dividing (5) by  $B$ , we get

$$\frac{1}{B} \frac{dB}{di} = -\frac{1}{1+i} \sum_{t=1}^T tc_t(1+i)^{-t} / B \quad (6)$$

We recognize that the right-hand side of (6) contains the expression of duration given by (2). We can now write

$$\boxed{\frac{1}{B} \frac{dB}{di} = -\frac{D}{1+i}} \quad (7)$$

which shows that the relative increase in the value of the bond is *minus* its duration divided by 1 plus the yield to maturity. Equivalently, we have

$$D = -\frac{1+i}{B} \frac{dB}{di} \quad (8)$$

which is what we wanted to show. Both expressions, (7) and (8), will be of utmost importance.

Let us consider some examples. Suppose a bond is at par ( $B = B_T = 100$ ); its coupon is 9%, and therefore the yield to maturity is 9%. (When you consider equation (13) in chapter 1 remember that  $i = c/B_T$  is the only way that  $B = B_T$ , or  $B/B_T = 1$ .) Its duration can be calculated as 6.995 years. Suppose that the yield increases by 1%; we then have, in linear approximation,

$$\frac{\Delta B}{B} \approx \frac{dB}{B} = -\frac{D}{1+i} di = -\frac{6.99}{1.09} \cdot 1\% = -6.4\%$$

(Notice that the result is indeed expressed as a percentage, because the time units of duration  $D$ —expressed in years—cancel out those of  $di$ , which are percent per year, and  $1+i$  is a pure number.)

This result is highly useful, because duration gives us a very quick way to approximate the change in the value of the bond if interest rates move by 1%: it suffices to divide duration by  $1 + i$  to get the approximate decrease of the bond's price when  $i$  moves up by 1%.

How good is the approximation? It is a rather satisfactory one, because the relationship between  $B$  and  $i$  is nearly a straight line, and therefore the derivative of  $B$  with respect to  $i$  is very close to its exact rate of increase. Indeed, should we calculate the latter, we would get in this case

$$\frac{\Delta B}{B} = \frac{B(i_1) - B(i_0)}{B(i_0)} = \frac{B(10\%) - B(9\%)}{B(9\%)} = \frac{93.855 - 100}{100} = -6.145\%$$

as opposed to  $-6.4\%$ . Notice, however, that the linear approximation to the curve  $B(i)$  conceals the all-important phenomenon of the convexity of that curve. Duration analysis tells us that the bond will either increase or decrease *approximately* by  $6.4\%$  in case of a decrease or a  $1\%$  increase in yield  $i$ , respectively. However, in exact value the bond will decrease by *less* than  $6.4\%$  (in fact, by  $6.145\%$ ), and it will increase by *more* than that amount if  $i$  moves down to  $8\%$ ; the bond's price increase will be  $\frac{B(8\%) - B(9\%)}{B(9\%)} = 6.71\%$ . This, of course, is the direct result of convexity of bonds (a property we have already mentioned in chapter 1), a topic so important we will devote an entire chapter to it.

#### 4.5 The concept of modified duration

Since duration is so closely linked to the change in price of a bond,  $\frac{D}{1+i}$  has a special name: *modified duration*, and we shall denote it by  $D_m$ . Modified duration is simply duration divided by one plus the yield to maturity. We have

$$\text{Modified duration} \equiv D_m = \frac{\text{duration}}{1 + i}$$

and therefore,

$$\frac{\Delta B}{B} \approx \frac{dB}{B} = -\text{modified duration} \times di = -D_m di$$



In our current example, modified duration is 6.4 years. Coming back to the example we considered earlier (a bond with coupon rate 70, par 1000, and yield 5%), duration was calculated as 7.7 years; modified duration is then  $7.7/1.05 = 7.33$  years. This means that a change in yield from 5% to either 6% or 4% entails a relative change in the bond's price approximately equal to  $-7.33\%$  or  $+7.33\%$ , respectively.

It is therefore quite natural to consider duration to be a measure of the interest rate risk of the bond, because if a high duration for a bond implies that its owner will highly benefit from a decrease in the interest rates, he will also suffer a big loss if rates of interest increase. More precisely, let us notice (for those readers who are statistically inclined) that the relationship between  $dB/B$  and  $di$ ,

$$\frac{dB}{B} = -D_m di$$

is a linear one between the two random variables  $dB/B$  and  $di$ . The variance of  $(dB/B)$  is therefore,

$$\boxed{VAR(dB/B) = D_m^2 VAR(di)} \quad (9)$$

and therefore the standard deviation of  $dB/B$  is

$$\boxed{\sigma(dB/B) = D_m \sigma(di)} \quad (10)$$

From (10), we can see immediately that the standard deviation of the relative change in the bond's price is a linear function of the standard deviation of the changes in interest rates, the coefficient of proportionality being none other than the modified duration.

#### 4.6 Properties of duration

We will now review the three main properties of duration, namely the relationship between duration and

- coupon rate
- yield to maturity
- maturity

#### 4.6.1 Relationship between duration and coupon rate

Suppose that the coupon rate of a bond increases, the yield and maturity remaining constant. The effect of this change in coupon rate on duration can be analyzed from two points of view. First, we can make use of the physical interpretation of duration, which we mentioned earlier, as the weighted average of the discounted cash flows. We can immediately see, for instance, from figure 4.1, that doubling the coupons will double all the first  $n - 1$  cash flows, but will not double the last one (because it contains not only the coupon, but also the par value, which remains unaffected by the change). Therefore, the distribution of the weights will be tilted to the left, and the center of gravity will also move to the left; the duration of the bond will decrease. This result can be verified by considering the expression of duration in closed form. We can now show that duration in such a closed form can be expressed as:

$$D = 1 + \frac{1}{i} + \frac{T(i - c/B_T) - (1 + i)}{(c/B_T)[(1 + i)^T - 1] + i} \quad (11)$$

Indeed, it is often convenient to use a closed form for duration, as opposed to an open form using the sign  $\sum$ , in the same way that closed forms for the value of a bond are often more useful than open ones. To obtain this closed form for duration, we will make use of the fact that it is equal to

$$D = -\frac{1 + i}{B} \frac{dB}{di}$$

and that the bond's value is

$$B = \frac{c}{i}[1 - (1 + i)^{-T}] + B_T(1 + i)^{-T}$$

Dividing both sides by  $B_T$ , we get

$$\frac{B}{B_T} = \frac{c/B_T}{i}[1 - (1 + i)^{-T}] + (1 + i)^{-T}$$

which can be written as the following product:

$$\frac{B}{B_T} = \frac{1}{i} \{ (c/B_T)[1 - (1 + i)^{-T}] + i(1 + i)^{-T} \}$$

Taking the logarithm, we have

$$\log(B/B_T) = -\log i + \log\{(c/B_T)[1 - (1+i)^{-T}] + i(1+i)^{-T}\}$$

The derivative with respect to  $i$  is

$$\begin{aligned} \frac{d \log(B/B_T)}{di} &= \frac{d \log B}{di} = \frac{1}{B} \frac{dB}{di} \\ &= -\frac{1}{i} + \frac{(c/B_T)T(1+i)^{-T-1} + (1+i)^{-T} + i(1+i)^{-T-1}(-T)}{(c/B_T)[1 - (1+i)^{-T}] + i(1+i)^{-T}} \end{aligned}$$

Duration  $D$  is  $-\frac{(1+i)}{B} \frac{dB}{di}$ . Multiplying the expression just above by  $-(1+i)$ , we finally get expression (11) after simplifications.

We can check if the coupon rate  $c/B_T$  increases, whether there will be a definite decrease in duration, because the numerator of the far-right fraction in (11) will decrease, and the denominator will increase. (As a word of caution, this result cannot be obtained as easily from the open-form expression of duration (2), because whenever the coupon rate changes, the value of the bond  $B$  changes too; further analysis is then in order. This is precisely the reason why it is simpler to use the closed-form of duration (11) as we have done.)

#### 4.6.2 Relationship between duration and yield

Should the yield to maturity increase, we can surmise from the diagram picturing the center of gravity of the bond's cash flows that the masses will be relatively more alleviated on the right-hand side of the diagram than on the left-hand side. This must be so, because the discount factor is  $(1+i)^t$ ; changing  $i$  will modify relatively more the high-order terms (those with high  $t$ 's) than the low-order ones. Therefore the center of gravity will move to the left and duration will be reduced. Of course, this property needs to be confirmed analytically.

This time, the easiest way to proceed is to take the derivative of the open-form of duration (2) with respect to  $i$ . We get a result that will be most useful in our chapter on convexity. The result is

$$\boxed{\frac{dD}{di} = -(1+i)^{-1}S} \quad (12)$$

where  $S$  is the dispersion, or variance of the payment times of the bond, and as such is always positive. Hence  $\frac{dD}{di}$  is always negative, and duration decreases with an increase in yield to maturity.

Let us now establish this result. From the expression of duration

$$D = \frac{1}{B} \sum_{t=1}^T tc_t(1+i)^{-t}$$

we can write

$$\frac{dD}{di} = \frac{-1}{B^2} \left[ \sum_{t=1}^T t^2 c_t (1+i)^{-t-1} \cdot B(i) + \sum_{t=1}^T tc_t (1+i)^{-t} \cdot B'(i) \right]$$

Factoring  $(1+i)^{-1}$  and multiplying each term in the bracket by  $1/B^2$ , we get

$$\frac{dD}{di} = -(1+i)^{-1} \left[ \frac{\sum_{t=1}^T t^2 c_t (1+i)^{-t}}{B(i)} + (1+i) \frac{B'(i)}{B(i)} \frac{\sum_{t=1}^T tc_t (1+i)^{-t}}{B(i)} \right]$$

The second term in the bracket is simply  $-D^2$ . We may thus write

$$\frac{dD}{di} = -(1+i)^{-1} \left[ \frac{\sum_{t=1}^T t^2 c_t (1+i)^{-t}}{B(i)} - D^2 \right]$$

We recognize in the bracket the *variance*, or *dispersion*, of the times of payment  $t$ ; indeed, it is equal to the weighted average of the squares of the differences between the times  $t$  and their average  $D$ . For simplicity, let us denote

$$w_t = \frac{c_t(1+i)^{-t}}{B(i)}$$

so that  $\sum_{t=1}^T w_t = 1$ , and  $D = \sum_{t=1}^T tw_t$ . The bracket then becomes equal to

$$\begin{aligned} \sum_{t=1}^T t^2 w_t - D^2 &= \sum_{t=1}^T t^2 w_t - 2D \sum_{t=1}^T tw_t + D^2 \sum_{t=1}^T w_t \\ &= \sum_{t=1}^T w_t (t^2 - 2tD + D^2) = \sum_{t=1}^T w_t (t - D)^2 \end{aligned}$$

and this last term is none other than the variance of the times of payment, which we will denote as  $S$ . Therefore equation (12) is true.

In the case of duration as well as in the case of convexity, the weights  $w_t$  are just the shares of the bond's cash flows (in present value) in the bond's value. Of course, the dispersion  $S$  is measured in years<sup>2</sup> and is always positive. We have thus proved that  $dD/di$  is always negative.

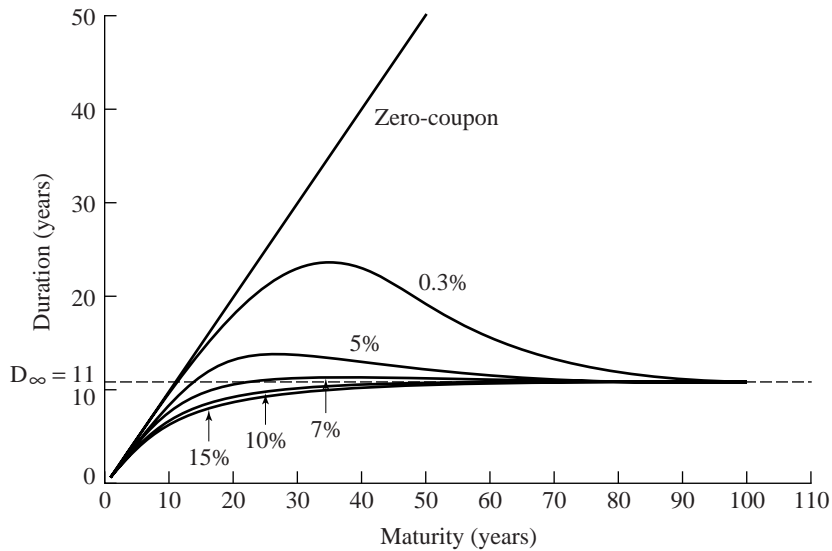
#### 4.6.3 Relationship between duration and maturity

Contrary to our intuition and to what some superficial analysis might lead us to believe, there is not always a positive relationship between duration and maturity, in the sense that an increase in maturity does not necessarily entail an increase in duration. Let us first state the results.

- *Result 1.* For zero-coupon bonds, duration is always equal to maturity. Therefore, duration increases with maturity.
- *Result 2.* For all coupon-bearing bonds, duration tends toward a limit given by  $1 + \frac{1}{i}$  when maturity increases infinitely. This limit is independent of the coupon rate.
- *Result 3.* For bonds with coupon rates *equal* and *above* yield to maturity (equivalently, for bonds above par) an increase in maturity entails an increase in duration toward the limit  $1 + \frac{1}{i}$ .
- *Result 4.* For bonds with coupon rates strictly *positive* and *below* yield to maturity (equivalently, for bonds below par), an increase in maturity has the following consequences: duration will first increase; it will pass through a maximum; and then it will decrease toward the limit  $1 + \frac{1}{i}$ .

If result 1 is obvious, results 2, 3, and 4 are quite surprising, to say the least. Let us tackle these in order.

In result 2, the duration of all coupon-bearing bonds tends toward a common limit. This can be seen from the expression of duration in closed form (11); indeed, the numerator in the large fraction on the right-hand side of (11) is an affine function of  $T$ , and the denominator of the same fraction is a positively-sloped exponential function of  $T$ . When  $T$  goes to infinity, the exponential will ultimately predominate over the affine, and the fraction will go to zero, letting duration reach a limit equal to  $1 + \frac{1}{i}$ . (For example, if  $i = 10\%$ , duration tends toward  $1 + \frac{1}{0.1} = 11$  years.) This surprising result, together with its delightful simplicity, compels us to seek an explanation. Why does duration tend toward  $1 + \frac{1}{i}$  when the maturity

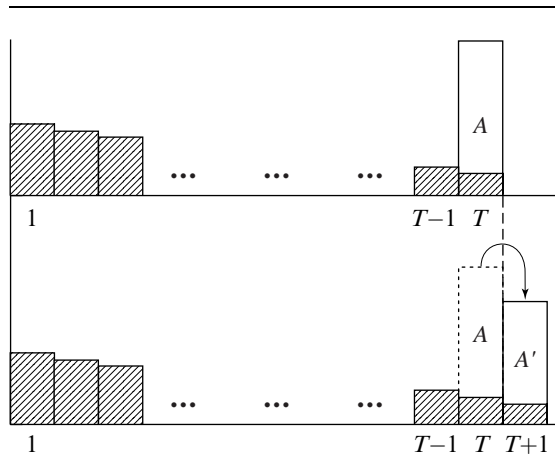


**Figure 4.2**  
Duration of a bond as a function of its maturity for various coupon rates ( $i = 10\%$ ).

of the bond, whatever its coupon, goes to infinity? We suggest the following interpretation, which permits us to get to this result without recourse to any calculation.

We merely have to think about two concepts—duration and the rate of interest—and see how they are related. Duration can be thought of as the average time you have to wait to get back your money from the bond issuer. The rate of interest is the percent of your money you get back per unit of time; hence the *inverse of the rate of interest* is the amount of time you have to wait to recoup 100% of your money. Thus, you will have to wait  $\frac{1}{i}$  years to recoup your money; however, since payment of interest is not continuous but discontinuous (you have to wait one year before receiving anything from your borrower), you will have to wait  $1 \text{ year} + \frac{1}{i}$  years, which is exactly the limit of the duration when maturity extends to infinity. (The reader who is curious enough to evaluate a bond with a *continuously* paid coupon will find that its duration is  $\frac{1}{i}$ , just as foreseen.)

Results 3 and 4 reflect the ambiguity of the effect of an increase of maturity on duration. For large coupons (for  $c/B_T > i$ ), duration increases unambiguously, but for small coupons, duration will first in-



Adding one additional cash flow, at  $T + 1$ , has two effects on the right end of the scale:

1. It *adds weight* at the end of the scale, first by adding a new coupon at  $T + 1$ , and second by increasing the length of the lever corresponding to the reimbursement of the principal.
2. It *removes weight* at the end of the scale by replacing  $A$  with a smaller quantity,  $A' = A/(1 + i)$ .

Therefore the total effect on the center of gravity (on duration) is ambiguous and requires more detailed analysis.

**Figure 4.3**

An intuitive explanation of the reason why the relationship between duration and maturity is ambiguous.

crease and then decrease. In order to understand the origin of this ambiguity, our analogy with the center of gravity will prove helpful once more.

Let us first see why our intuition leads us in a false direction. If maturity increases by one year, we are not always adding some weight to the right end of our scale, as we might be tempted to think. What we are in fact doing is *increasing* the right-end weight by adding at time  $T + 1$ , in present value, the par value  $B_T$  plus an additional coupon; but we are *alleviating* the same end since the present value of par  $B_{T+1}$  is smaller

than the present value of  $B_T$  at time  $T$  (see figure 4.3). Therefore, we are not sure what the final outcome of this process will be. We can easily surmise, however, that if the coupon is small relative to the rate of interest, we will remove some weight from the right end of the scale, and therefore the center of gravity will move to the left (the duration will decrease)—which is exactly what happens.

This is a good example of where further analysis is required. Strangely, only in 1982 was the analysis of the relationship between duration and maturity completely carried out (by G. Hawawini).<sup>6</sup> The derivations for results 3 and 4 are unfortunately rather tedious to obtain<sup>7</sup> and are not given here; instead we refer the interested reader to their original source.

#### 4.7 Duration of a bond portfolio

Let us now consider the duration of a portfolio of bonds. First, for simplicity, restrict the number of bonds to two, whose prices are denoted  $B_1(i)$  and  $B_2(i)$ , respectively. Consider a portfolio of  $N_1$  bonds “1” and  $N_2$  bonds “2”.

We remember that the duration of a coupon-bearing bond  $B_i$  was just  $D = \sum_{t=1}^T tc_t(1+i)^{-t}/T$ , and we showed it was equal to  $-\frac{1+i}{B} \frac{dB}{di}$ . It is only natural to treat a single coupon-bearing bond as a portfolio of bonds; a portfolio  $P$  of coupon-bearing bonds is of course such that its duration, denoted  $D_P$ , will be equal to  $-\frac{1+i}{P} \frac{dP}{di}$ . Let us therefore determine this last expression.

The portfolio's value is

$$P = N_1 B_1(i) + N_2 B_2(i) = P(i)$$

Taking the derivative of  $P(i)$  with respect to  $i$ , we get

$$\frac{dP(i)}{di} = N_1 \frac{dB_1(i)}{di} + \frac{N_2 dB_2(i)}{di}$$

<sup>6</sup>G. Hawawini, “On the Mathematics of Macauley's Duration,” in: G. Hawawini, ed., *Bond Duration and Immunization: Early Developments and Recent Contributions* (New York and London: Garland Publishing, Inc., 1982).

<sup>7</sup>To make matters worse, in the case of the curves representing duration of under-par bonds as a function of maturity, the abscissa of their maximum cannot be tracked with an explicit expression and therefore has to be given implicitly.



Multiply each member of the above equation by  $-(1+i)/P$ . Then multiply and divide the first term of its right-hand side by  $B_1(i)$  and the second term by  $B_2(i)$ , respectively. We get

$$-\frac{1+i}{P(i)} \frac{dP(i)}{di} = \frac{N_1 B_1(i)}{P(i)} \left( -\frac{1+i}{B_1(i)} \frac{dB_1(i)}{di} \right) + \frac{N_2 B_2(i)}{P(i)} \left( -\frac{1+i}{B_2(i)} \frac{dB_2(i)}{di} \right)$$

The above expression may then be written as

$$D_P = \frac{N_1 B_1(i)}{P(i)} D_1 + \frac{N_2 B_2(i)}{P(i)} D_2$$

Denoting  $N_1 B_1(i)/P(i) \equiv X_1$  and  $N_2 B_2(i)/P(i) \equiv X_2$  (with  $X_1 + X_2 = 1$ ),  $D_P$  reduces to

$$D_P = X_1 D_1 + X_2 D_2 = X_1 D_1 + (1 - X_1) D_2$$

and therefore the duration of a portfolio is just the weighted average of the bond's durations, the weights being the shares of each bond in the portfolio's value.

The extension of this rule to an  $n$ -bond portfolio is immediate, and we have

$$D_P = \sum_{j=1}^n X_j D_j, \quad X_j \equiv \frac{N_j B_j(i)}{P(i)} \quad \text{and} \quad \sum_{j=1}^n X_j = 1$$

An important caveat is in order here. From a practical point of view, it is incorrect to add or average durations (or modified durations) of bonds that do not carry the same yield to maturity. Furthermore, we have dealt only with obligations that do not carry any significant default likelihood (which is definitely not the case with some corporate securities, even less with some bonds issued in emerging markets, either by governments or firms). In these latter cases, duration has little to do with any significant measure of the investor's risk.

#### 4.8 Measuring the riskiness of foreign currency-denominated bonds

Before leaving this chapter on duration and its properties, let us consider the risk of foreign bonds. As the reader may guess, their risk (apart from

any default from the issuer) stems from the following sources: volatility of the foreign interest rate, duration, and foreign exchange risk. It will be interesting to see how these three kinds of risk are related and to assess their relative importance.

Let  $B$  represent the value of a foreign bond (for instance, a Swiss bond held by an American owner). Let  $e$  be the exchange rate for the Swiss franc (number of dollars per Swiss franc). If  $V$  is the value of the Swiss bond expressed in dollars, we have

$$V = eB \quad (13)$$

Taking logs on both sides, and then the differential, we get

$$d \ln V = d \ln B + d \ln e \quad (14)$$

which may be written also as

$$\boxed{\frac{dV}{V} = \frac{dB}{B} + \frac{de}{e}} \quad (15)$$

All three relative increases involved in (15) are random variables; *in linear approximation* the relative increase of the Swiss bond, expressed in dollars, is the sum of the relative increase of the Swiss bond, expressed in Swiss francs, and the relative increase of the Swiss franc. Should the Swiss franc rise, for instance, this will increase the performance of the Swiss bond from the American point of view.

If  $i$  designates the Swiss rate of interest, we have, using the property of duration we have just seen (equation (7)),

$$\frac{dB}{B} = \frac{1}{B} \frac{dB}{di} di = -\frac{D}{1+i} di \quad (16)$$

and therefore we include both the duration of the foreign bond and possible changes in the foreign rate of interest in equation (15), which becomes

$$\frac{dV}{V} = -\frac{D}{1+i} di + \frac{de}{e} \quad (17)$$

Let us now take the variance of the relative change in value of the foreign

bond; we get

$$VAR\left(\frac{dV}{V}\right) = \left(\frac{D}{1+i}\right)^2 VAR(di) + VAR\left(\frac{de}{e}\right) - 2\frac{D}{1+i} COV\left(di, \frac{de}{e}\right)$$

(18)

The covariance between changes in interest rates and modifications in the exchange rates is usually very low, and the bulk of the variance of changes in foreign bonds' value stems mainly from the variance of the exchange rate. Empirical studies<sup>8</sup> show the share of the exchange rate variance is easily two-thirds of the total variance; for instance, for an American investor, this share was 61% for U.K. bonds, 86% for Swiss bonds, and 94% for Belgian bonds.

An important caveat is in order here. We have underlined the fact that equation (15) gives the relative change in the Swiss bond's value from the point of view of an American investor, *in linear approximation*. This is a perfectly acceptable approximation if the changes both in the bond's value and in the exchange rate are small (on the order of a few percent). However, it is *not* a good approximation if those changes are large, as they often are in emerging markets.

Many practitioners fall into the trap of using this formula for international investments in stocks or bonds even when those rates of changes are large (for instance, 20% or 40%); they simply add up the relative change in the domestic value of the investment with the relative change in the exchange rate. Even a highly respectable and otherwise excellent periodical (whose name we will not divulge) made this mistake when, in 1998, it published tables on the declines of emerging East Asian markets from the point of view of an international investor. This even led them to draw a diagram in which they stated that the loss on the Indonesian stock market had been 123%. This result of course does not make sense; even if you invest in Martian stocks, you cannot lose more than your investment,

<sup>8</sup>See the study by Steven Dym, "Measuring the Risk of Foreign Bonds," *The Journal of Portfolio Management* (Winter 1991): 56–61. For more recent data on volatility and correlation of bond markets, see Bruno Solnik, Cyril Boucrelle, and Yann Le Fur, "International Market Correlation and Volatility," *Financial Analysts Journal* (September/October 1996): 17–34.

which is 100%. Likewise, if you bet your shirt in Indonesia and lose it, nobody will ask you to throw in your tie as well.

Where does the error come from? We will now show how the true loss, in exact value, should be calculated.

As before, let  $B$  denote the domestic value of the international investment (in that case, the investment in Indonesian rupees), and let  $e$  designate the exchange rate (number of dollars per Indonesian rupee). Let  $V = Be$  designate the value of the investment in Indonesia expressed in dollars. Let us determine the relative rate of change (which, of course, can be negative) in the investment's value  $V$ . Suppose that  $B$  changes by  $\Delta B$ , and  $e$  moves by  $\Delta e$ . We then have

$$\begin{aligned}\frac{\Delta V}{V} &= \frac{(B + \Delta B)(e + \Delta e) - Be}{Be} \\ &= \frac{Be + e\Delta B + B\Delta e + \Delta B\Delta e - Be}{Be}\end{aligned}$$

so, finally,

$$\boxed{\frac{\Delta V}{V} = \frac{\Delta B}{B} + \frac{\Delta e}{e} + \frac{\Delta B}{B} \cdot \frac{\Delta e}{e}} \quad (19)$$

So the rate of change in the international investment is exactly equal to the sum of the relative change in the domestic investment plus the relative change in the exchange  $(\Delta B/B + \Delta e/e)$ , to which you have to add the product of the relative change of the domestic investment and the relative change in the exchange rate,  $(\Delta B/B) \cdot (\Delta e/e)$ .

An example will illustrate this. Let us come back to the case we just mentioned. Suppose that the loss on the Jakarta stock market was 60% in a given period and that the rupee lost 60% of its value against the dollar in the same period. The rate of change in the investment's value from the point of view of an American investor is of course not minus 120%. It is

$$\begin{aligned}\frac{\Delta V}{V} &= -60\% + (-60\%) + (-60\%)(-60\%) \\ &= -120\% + 36\% = -84\%\end{aligned}$$

This result is not only exact but makes good sense, as can easily be seen in the following illustration. Suppose your little sister has just baked a delicious chocolate cake in the shape of a rectangle. In the short time she has her back turned, you first slice off 60% of the length of the cake, swallow it with delight, and cannot resist stealing another 60% of its width. After your first mischief, 40% of the cake is left; your second prank has taken away  $60\% \times 40\% = 24\%$  of what remained, so that only  $40\% - 24\% = 16\%$  is left. (Equivalently, you could say that 40% of 40%, that is, 16%, is left.) Your sister has lost indeed 84% of her cake. You can, however, try to comfort her by explaining that, although you were mischievous—for which you deeply apologize—you did not take away 120% of the cake, as many people would unjustly accuse you of doing.

The good news is that the true value is easy to calculate; the bad news is that the expected value, variance, and probability distribution of an expression like (19) are very difficult to analyze. Hence the popularity of the linear approximation we mentioned, which is valid, however, only for small changes in asset values and exchange rates.

### Key concepts

- Duration is the weighted average of the times to payment of all the cash flows generated by the bond; the weights are the cash flows in present value.
- Duration can be viewed as the center of gravity for all the bond's discounted cash flows, if these are set on a horizontal time axis.
- An important property of duration is the bond's sensitivity to interest rate risk; more fundamentally, duration is at the heart of the immunization process (a method of protecting an investor against fluctuations in the rate of interest).
- Duration decreases with coupon rate and interest rate; it always increases with maturity for above-par bonds (i.e., when the coupon rate is above the interest rate). For below-par bonds, except zero-coupon bonds, duration first increases with maturity and then decreases.
- In all cases except for zero-coupons, duration tends toward a limit when maturity increases to infinity.

### Basic formulas

- Duration =  $D = \sum_{t=1}^n tc_t(1+i)^{-t}/B = 1 + \frac{1}{i} + \frac{T(i-c/B_T)-(1+i)}{(c/B_T)[(1+i)^T-1]+i}$
- Duration as a measure of the sensitivity of the bond's price to interest rate (or yield to maturity):

$$D = -\frac{1+i}{B} \frac{dB}{di}$$

- Modified duration  $\equiv D_m = \frac{D}{1+i} = -\frac{1}{B} \frac{dB}{di}$
- Relation between duration and yield:

$$\frac{dD}{di} = -(1+i)S$$

where  $S$  is the variance (the dispersion) of the bond's times of payment

- Limit of duration when  $T \rightarrow \infty : \frac{1}{i} + 1$ .
- Duration of bond portfolio (valid only for bonds of same yield to maturity);

$$D_p = \sum_{j=1} X_j D_j$$

- Relative rate of increase in value of a foreign-denominated bond:

$$\frac{\Delta V}{V} = \frac{\Delta B}{B} + \frac{\Delta e}{e} + \frac{\Delta B}{B} \cdot \frac{\Delta e}{e}$$

where  $e$  is the exchange rate.

### Questions

- 4.1. Consider equation (6) of section 4.4. In which units are both sides of the equation measured?
- 4.2. What are the measurement units of modified duration?
- 4.3. Confirm that equation (11) in section 4.6.1 is exact.
- 4.4. In each of the following cases, what is the duration of a bond with par value: \$1000; coupon rate: 9%; maturity: 10 years?
  - a.  $i = 9\%$
  - b.  $i = 8\%$

Comment on your results.

4.5. Two bonds,  $A$  and  $B$ , have the following characteristics:

	Bond $A$	Bond $B$
Maturity	15 years	7 years
Coupon rate	10%	10%
Par value	\$1000	\$1000

Suppose that the rate of interest increases from 12% to 14%. Determine the relative change in the bond's values, and comment on your results. The relative change in the bond's value should be determined in exact value (and denoted  $\Delta B/B$ ) and linear approximation (denoted  $dB/B$ ). Will these relative changes be more pronounced in exact value or in linear approximation?

4.6. Consider question 4.5 again, this time using the following data:

	Bond $A$	Bond $B$
Maturity	20 years	20 years
Coupon rate	12%	8%
Par value	\$1000	\$1000

Comment on your results, which you can indicate in exact value or in linear approximation.

4.7. Consider question 4.5 again, this time using the following data:

	Bond $A$	Bond $B$
Maturity	20 years	7 years
Coupon rate	10%	14%
Par value	\$1000	\$1000

Comment on your results. How can you compare these results to those you obtained in the original question 4.5?

4.8. Repeat question 4.5, using the following data:

	Bond $A$	Bond $B$
Maturity	15 years	11 years
Coupon rate	10%	5%
Par value	\$1000	\$1000

4.9. Finally, consider question 4.5 with the following data:

	Bond <i>A</i>	Bond <i>B</i>
Maturity	40 years	20 years
Coupon rate	2%	2%
Par value	\$1000	\$1000

What strikes you in your results?

4.10. Verify that equation (18) follows from equation (17). (See section 4.8.)



# 5

## DURATION AT WORK: THE RELATIVE BIAS IN THE T-BOND FUTURES CONVERSION FACTOR

DION. For present comfort, and for future good ...  
*The Winter's Tale*

KING LEAR. ... that future strife  
May be prevented now.  
*King Lear*

### Chapter outline

<b>5.1</b>	<b>Forward and futures markets: a brief survey</b>	<b>96</b>
5.1.1	The forward contract and its deficiencies	96
5.1.2	Introducing futures markets	99
5.1.3	Futures trading	100
5.1.4	The actors in futures markets	100
5.1.4.1	Hedgers	100
5.1.4.2	Speculators	101
5.1.4.3	Arbitrageurs	101
5.1.5	The organization of safeguards: the role of the clearinghouse and the margin requirements	102
5.1.6	The system of daily settlements and margins	102
5.1.7	The riskiness of a “naked” futures position	106
5.1.8	Closing out a futures position	107
5.1.8.1	Delivery	107
5.1.8.2	Cash settlement	109
5.1.8.3	Offsetting positions or reversing trades	110
5.1.8.4	Exchange of futures for physicals (EFP agreement)	110
5.1.9	Consequences of failure to comply	111
<b>5.2</b>	<b>Pricing futures through arbitrage</b>	<b>111</b>
5.2.1	Convergence of the basis to zero at maturity	112
5.2.2	Properties of the basis before maturity, $F(t, T) - S(t)$ : introduction	112
5.2.3	A simple example of arbitrage when the asset has no carry cost other than the interest	116
5.2.4	Arbitrage with carry costs	117
5.2.4.1	The case of an overvalued financial futures contract	121
5.2.4.2	The case of an undervalued financial futures contract	122
5.2.5	Possible errors in reasoning on futures and spot prices	123

<b>5.3 The relative bias in the T-bond futures contract</b>	<b>125</b>
5.3.1 The various types of options imbedded in the T-bond futures contract	125
5.3.1.1 The wild card (time to deliver) option	125
5.3.1.2 The quality option	126
5.3.2 The conversion factor as a price	127
5.3.3 The “true” or “theoretical” conversion factor	128
5.3.4 The relative bias in the conversion factor	129
5.3.5 The profit of delivering and its biases	134
<b>Appendix 5.A.1 The calculation of the conversion factor in practice</b>	<b>139</b>

## Overview

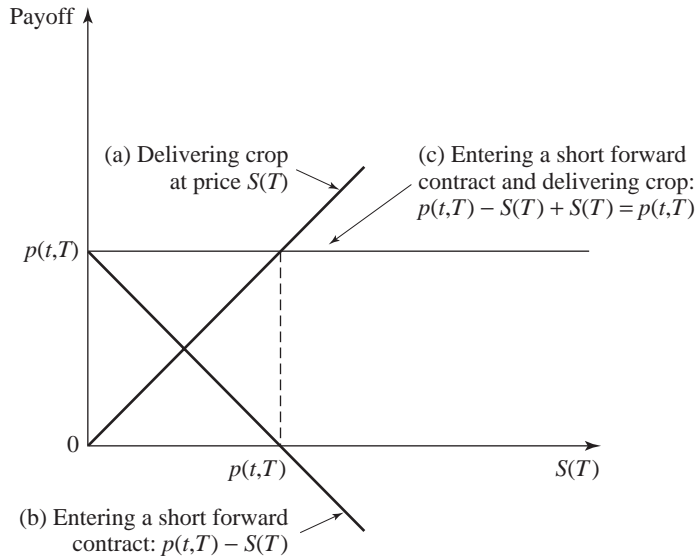
As we will show in this book, duration has many practical applications. Perhaps one of the most surprising is its application as a measure of the relative bias that is introduced in one of the most important financial contracts on organized markets, the 30-year Treasury bond futures contract, set up by the Chicago Board of Trade (CBOT). Before showing this, we will review briefly what forward and futures markets are about and then describe the various futures contracts on bonds. This is a long chapter, and the first section can be skipped by the reader who is already familiar with forward and futures markets.

## 5.1 Forward and futures markets: a brief survey

Before examining the characteristics of a futures contract, it is important to review those of the simpler forward contract. We will thus see how the necessity of setting up a system of futures contracts stems in a natural way from some drawbacks of the forward contract.

### 5.1.1 The forward contract and its deficiencies

Uncertainty has important consequences for all agents who have to make transactions in the future. Consider, for instance, a producer of wheat (a farmer) and a user of wheat (a mill). Each of those agents incurs a risk with regard to the price at which he will exchange wheat in the future. Suppose each agent is willing to secure a fixed price, set today, for the exchange of wheat that will take place in, say, six months. That price can be denoted  $p(t, T)$ , the forward price set today (at time  $t$ ) for a transaction that will occur at time  $T$ . In such a case we say that the farmer and



**Figure 5.1**  
Possible monetary outcomes for the farmer (the short).

the mill enter into a forward contract to exchange a commodity of certain specifications, in a given amount, for a price  $p(t, T)$ . Without any hedging, the farmer runs the risk of having to sell at a price below  $p(t, T)$ ; on the other hand, he might make an unexpected profit if the spot price at time  $T$ , denoted  $S(T)$ , is above  $p(t, T)$ . The situation is symmetrical for the mill: it runs the risk of having to pay more than  $p(t, T)$ , and it may also cash in on an unexpected profit if the price happens to be lower than  $p(t, T)$ .

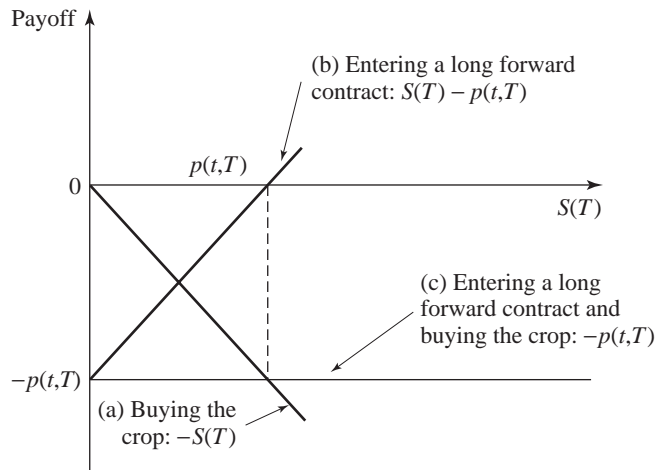
Suppose that each of those agents is willing to forego the possible advantage that uncertainty may bring about by securing a transaction at  $p(t, T)$ .

Consider first the farmer's position. He will be holding the wheat and is willing to sell it; he is said to have a *short* position (and he is called *the short*). Figure 5.1 represents the farmer's possible payoffs at time  $T$ , according to every possible future spot price  $S_T$ . His proceeds will be, in terms of price, the 45° line (equal to  $S(T)$ ). He incurs considerable risk if the price is very low. Suppose now that at time  $t$  he enters a forward contract (he takes a so-called short position on each unit of wheat) at  $p(t, T)$ . He therefore has the obligation of selling one unit of wheat at

time  $T$  at a price  $p(t, T)$  established at time  $t$ . What will his payoff be if he does not own the wheat? Suppose for the time being that anyone entering such a contract has to deliver one unit of a certain quality of wheat at time  $T$ . His payoff will be  $p(t, T)$  minus the cost of acquiring this unit, that is, the spot price  $S(T)$ ; it will thus be  $p(t, T) - S(T)$ . The payoff for this contract will be the negatively sloped line going through  $p(t, T)$  on the  $S_T$  axis. Indeed, if at time  $T$  the spot price  $S(T)$  is lower than  $p(t, T)$ , he receives the positive amount  $p(t, T) - S(T)$ ; if, on the contrary, the spot price is higher than  $p(t, T)$ , he loses  $S(T) - p(t, T)$  (or receives  $-[S(T) - p(t, T)] = p(t, T) - S(T)$ , a negative amount). In any case, he always receives  $-S(T) + p(t, T)$ , an amount that may be positive or negative. Now his total position, if at time  $T$  he owns the commodity and at the same time receives the proceeds of his futures position, is simply  $S(T) - S(T) + p(t, T) = p(t, T)$ , the horizontal line at height  $p(t, T)$ . Indeed, he has now secured a comfortable position; no matter what the future spot price may be, he will cash in  $p(t, T)$ . He has bought this safety by foregoing any possible benefit he might have incurred if the future spot case  $S(T)$  had been above  $p(t, T)$ .

Let us now turn to the mill. If it does not enter a futures contract, it will have to buy one unit at price  $S(T)$ ; its cash outflow is represented by the negatively sloped line  $-S(T)$  going through the origin (see figure 5.2). This entails incurring the risk of having to buy wheat at a very high price, thus disbursing a high amount  $S_T$ . On the other hand, suppose the mill takes a long position at  $p(t, T)$ ; the payoff from this futures contract is the positively sloped, 45° line going through  $p(t, T)$ , equal to  $S(T) - p(t, T)$ . The total payoff, corresponding to buying the wheat and buying the futures contract, is simply  $-S(T) + S(T) - p(t, T) = -p(t, T)$ . So the mill is going to incur a cash outflow of  $-p(t, T)$ , no matter what happens on the wheat spot market at time  $t$ . The mill pays this security by foregoing the possibility of buying wheat at lower prices than  $p(t, T)$  if the market spot price drops below  $p(t, T)$  at time  $T$ . Notice also that, both agents being considered together, the disbursements of one agent (the mill) will be equal to the receipts of the other (the farmer). This is basically the mechanism of a forward contract.

However, many problems would arise with such a contract, the most important one being the possibility of default by one of the parties. A second problem is that neither party would be sure of obtaining the best possible deal by entering into this agreement; the seller might well think



**Figure 5.2**  
Possible monetary outcomes for the mill (the long).

he should have looked around among other potential buyers to see if he could have received a better price, while the buyer could also regret not having shopped around more.

A third drawback of a forward agreement is that no party can change his mind during the period of the contract. For instance, should the short party (the party with the obligation to sell) change his mind, the only possibility would be for him to try to sell his contract to a third party, and this could be very costly.

The reader should not infer from the above that forwards can't trade. On the contrary, they do trade in large volumes. This comes from the fact that standardized contracts (as futures contracts are, as we will see) may be not flexible enough for the participants' wishes. Also, liquidity may be higher in some forward deals. For instance, there is a huge volume of bank-to-bank over-the-counter deals, where liquidity can be higher than in exchanged traded contracts like futures.

### 5.1.2 Introducing futures markets

As the reader will notice, each of the three major drawbacks of a forward contract carries a given type of risk: the risk of default, the risk of making the transaction at an over- (or under-) stated price, and the risk of a party

bearing a large loss if that party wants to change its mind. And such a contract was originally intended to remove risk from the transaction.

For these reasons, and others that we will briefly review, forward markets have been progressively developed into organized “futures” markets, the main features of which are the following: (a) futures contracts are traded on organized exchanges (a clearinghouse makes certain that each party’s obligation is fulfilled); (b) each contract is standardized; (c) through a system of margin payments and daily settlement, the position is cleared every day, as we will explain.

### 5.1.3 Futures trading

In the United States, the first organized futures exchanges were trading grains in the nineteenth century; other categories of commodities came to be traded on such exchanges. Today, soybeans, livestock, energy products (from heating oil to propane), metals, textiles, forest products, and food-stuff futures are traded on such exchanges as the Chicago Board of Trade or the New York Mercantile Exchange. Financial instruments were first traded on futures markets in the 1970s, and they have come to play a crucial role. It has become customary to distinguish three kinds of financial futures: interest-earning assets, foreign currencies, and indices. In this text we will be interested primarily in interest-earning assets. Note that the common wording for futures on those assets is *interest rate futures*, although the traded object is of course the underlying financial asset (for example, a short-term (one year) Treasury bill or a long-term (20 year) T-bond).

### 5.1.4 The actors in futures markets

It has become customary to classify the operators in futures markets into three groups: hedgers, speculators, and arbitrageurs.

#### 5.1.4.1 Hedgers

In our introduction, we dealt at length with the hedging motive some agents may have. Some have good reason to hedge against a low future price (those who own the commodity or who will own it—perhaps because they will be producing it). Others hedge against a high future price (those who will have to buy the commodity, presumably because it enters their own chain of production).

Also, as we will see later, it is quite possible to hedge against mishaps to a given commodity while entering a futures market of a slightly different commodity, if the two commodities happen to be sufficiently related that their prices are highly correlated. One refers to such transactions on the futures market as *cross-hedging*.

#### 5.1.4.2 Speculators

“To speculate” comes from the Latin *speculare*, “to forecast.” The Latin word also conveys an overtone of divination, especially in the noun “speculatio.” Many agents try to forecast what the future spot price of any commodity will be and will compare their forecast to the futures price being established on the futures market. Those agents, who do not hold the commodity and do not intend to hold it, take considerable risk. They can very quickly lose their money if shorts see the futures price climb or if longs witness a plunge. Of course, their risk is rewarded by substantial profits if, on the contrary, things turn out their way. In order to make forecasts, some will rely on fundamental analysis; they will try to forecast future general conditions for the supply and demand of the commodity and infer an expected spot price. If the difference between that estimate and the futures price is large enough, they will take a position on the futures market. In any case, of course, spot and futures prices are fundamentally driven by the same forces, and there is a definite, strong relationship between the two prices, which we will examine very soon.

Positions held by speculators can be held for variable periods: from a few months for long-term speculators to a few seconds for the so-called “scalpers” (individuals who take bets on the fact that sell orders, for instance, may well be soon followed by buy orders). Scalpers profit from a small difference between the price at which they sell the buy orders and the price at which they bought the sell orders.

#### 5.1.4.3 Arbitrageurs

Arbitrageurs are traders who profit from mismatches between spot and futures prices and between futures prices with different maturities. In order to give some examples of such arbitrages, in the next section we discuss the all-important question of the determination of futures prices. We should stress now that we will show what the links are between spot and futures prices today, at a given point in time; of course, we are not able to know what the spot or the futures prices will be at any distant point in time.

### 5.1.5 The organization of safeguards: the role of the clearinghouse and the margin requirements

Contrary to a forward contract, where each party has to organize itself against possibilities of default from its counterpart, exchanges set themselves between the parties; each party deals separately with the exchange. Also, in contrast to a forward agreement, the specifications of a futures contract are always standardized: the unit amount of the contract, the exact quality of the product to be traded, and the time and location of delivery are all set in a unique and precise way. In addition, the exchange will not generally permit daily fluctuations of the futures price that would exceed certain limits.

Each futures market has a clearinghouse that is closely associated with it, either a part of the exchange or a separate corporation. In either case, the clearinghouse buys the futures contract from the short party and sells it to the long. Thus the short and the long are obligated only to the clearinghouse (through their respective brokers). Also, the clearinghouse runs practically no risk for the following two reasons. First, for each trade it makes with any given party, it makes the opposite trade with another party, and thus holds no net (positive or negative) position; second, it requires from each trading party security deposits (or margins) to ensure that if ever one party defaulted its obligations to the clearinghouse, it would be covered against that kind of default. We will now explain what kind of margins are required both from the short and the long.

### 5.1.6 The system of daily settlements and margins

Anyone entering a futures agreement commits himself at time  $t$  to buy or deliver a given commodity or a financial product at some point of time  $T$  in the future, for a given futures price  $F(t, T)$ .<sup>1</sup> The system of daily settlements and margins, which we will now describe, is a very efficient way to make sure each participant in the market will indeed honor its obligations, on the one hand, and on the other enable each participant to end its commitment at any point in time.

For clarity, let us go back to the case of a forward contract. Suppose you set up a system of guarantee deposits by which you make sure that any participant in a forward market will fulfill its obligations. How much

<sup>1</sup> Throughout this book, we denote a futures price as  $F(t, T)$  and a forward price as  $p(t, T)$ .



would you require as a guarantee from an individual A who, for instance, commits himself at time  $t$  to sell one unit of a given commodity at a certain time  $T$ , at price  $p(t, T)$ ? At time  $T$ , the spot price of the commodity  $S(T)$  may very well be under  $p(t, T)$  or above it. In the first case, A earns the difference between the two. If A is a speculator, he can buy the commodity at  $T$ , pay  $S(T)$ , sell it at the higher price  $p(t, T)$ , and pocket the difference. If he owns the commodity, he also receives the difference  $p(t, T) - S(T)$ . The risk of default arises if  $S(T)$  is above the forward price  $p(t, T)$ , because the speculator or the owner of the commodity will incur a loss on that contract equal to  $S_T - p(t, T)$ . Indeed, the speculator may simply not buy the commodity at price  $S(T)$  and sell it at  $p(t, T)$ , and instead disappear. The opposite risk appears in the case of party B, who has committed himself to buying the commodity at price  $p(t, T)$ ; should  $S(T)$  be lower than the agreed on price  $p(t, T)$ , B will certainly not be pleased at the prospect of buying a commodity at a price higher than that of the spot market, since he will incur a loss  $p(t, T) - S(T)$  when he sells it. Also, if B is a user of the product, he may very well want to buy it on the spot market instead of fulfilling his commitment to buy it at the higher price  $p(t, T)$ .

You observe that the maximum loss that any of the two parties A and B can incur is not the whole forward price  $p(t, T)$ , but only the difference between  $p(t, T)$  and  $S(T)$  in absolute value. Therefore you will require from both parties a guarantee that will be in the order of magnitude of this possible difference, which you do not know at time  $t$  and which you will have to forecast for date  $T$ . Furthermore, you will want to resettle the contracts at regular intervals between  $t$  and  $T$ , perhaps every day, so that each party has an easy way to exit the market. Finally, you observe that the required guarantee will certainly diminish if it has to cover default risk over a much shorter period (one day) than  $T - t$  (which may be nine months). The market will become more liquid, and fewer funds will be tied up in the guarantee process.

Thus you will establish futures markets, such that every participant has to permanently meet a minimum safeguard called a *maintenance margin*. In no circumstances can the deposit fall lower than that margin. (Note that each exchange establishes a minimum maintenance margin, but stockbrokers can set higher minimal values for some clients whom they suspect to be more likely to default their obligations than others.) Above this, clearinghouses—or the stockbrokers—will require from their clients

an amount that will permit them to settle immediately any daily loss. This is why, when a futures contract is initiated, each party will be required to make an initial deposit, called the *initial margin*, which will be above the maintenance margin by an amount of the order of about one third. (The maintenance margin will therefore be about three fourths of the initial margin.) The funds deposited may be posted in cash, in U.S. Treasury Bills, or in letters of credit; of course, some of this collateral may earn interest, which remains in the owners' hands. Should the trader's account exceed the initial margin because daily settlements have been favorable to the trader, he has the right to withdraw at any time the amount exceeding the initial margin. On the other hand, should the account fall to the level of the maintenance margin (or below), the trader will immediately receive a call to replenish his account up to the level of the initial margin. Should he not comply with this demand, he will have to face dire consequences, which we will detail (in section 5.1.9) after we have explained how, normally, a trader can exit the futures market.

We now present an example of such a system of daily settlements. We consider the case of a trader who, early on Tuesday, July 1, buys a September gold futures at price  $F(t, T) = \$400$  an ounce, and we trace the evolution of his long position through the next nine trading days, ending on July 11, according to the daily evolution of the price of the September futures contract for gold. We suppose that one futures contract on gold entails the obligation to buy (or sell) 100 ounces of gold at futures price expressed in dollars per ounce of gold. The initial margin is set at \$2000 and the maintenance margin at \$1500. This implies that in all likelihood, the daily variation is not expected to be above \$500 per contract, or \$5 per ounce.

On the first day, when our buyer and our seller enter their contracts with the exchange, they are asked to deposit \$2000 each as initial margin requirement. At the end of the first day, suppose that the settlement price turns out to be \$405. (In fact, prices are quoted to the cent in this contract; for simplicity of exposition, we limit ourselves to dollars.) The September futures price is now \$405. Our long buyer benefits from this increase in the futures price; now operators on the market are willing to buy gold at \$405, while he is the owner of a contract specifying a price of \$400. His contract is thus worth  $\$5 \times 100 = \$500$ . In a way, if the contract were of the "forward" type, he could sell it for \$500; indeed, anybody wanting to buy gold at the futures maturity date for \$405 would be

indifferent in entering the market with a \$405 commitment, or paying \$5 for a \$400 commitment, which is what it would amount to if he bought for \$5 the initial contract set at \$400.

The clearinghouse then credits the long with \$500, taken from the account of the unfortunate short, who had contracted to sell 100 ounces of gold (one contract) at the futures price of \$400, which is now valued at \$405. If his contract were a forward contract, he would have to pay someone \$500 to buy his contract. If the short's position is "marked to the market," he is considered as having lost \$5 per ounce, and he suffers a cash outflow of  $\$5 \times 100 = \$500$ . Having been so adjusted, both the short and the long positions are considered cleared, and both traders could depart the futures market. If, for instance, the short stays in it, he will now be considered to have a short position on that September contract, whose maturity is now one day closer, with a price  $F(t+1, T) = \$405$ . Our subsequent example refers to the cash flows of the long trader; for the short trader, all those cash flows correspond to cash flows of identical magnitude but opposite sign.

At the end of the first day, the long has a cumulative gain of \$500 and a final cash balance of \$2500 (column 5 of table 5.1), which is the next day's  $(t+1)$  initial balance (column 2). The next day, things go the long's way again: the futures price rises to \$407; he is credited with a cash inflow of \$200; his cumulative gain is \$700; and his final cash balance—the initial cash balance for  $t+2$ —is now \$2700. Unfortunately for our long, the

**Table 5.1**  
The futures buyer's position; initial futures price:  $F(t, T) = \$400/\text{ounce}$  on July 1

Date	(1) Settle- ment price	(2) Initial cash balance	(3) Mark to market cash flow	(4) Cumulative gain	(5) Final cash balance*	(6) Mainte- nance margin call
7/1	405	2000	+500	500	2500	—
7/2	407	2500	+200	700	2700	—
7/3	401	2700	−600	100	2100	—
7/4	394	2100	−700	−600	2000	600
7/7	392	2000	−200	−800	1800	—
7/8	391	1800	−100	−900	1700	—
7/9	387	1700	−400	−1300	2000	700
7/10	390	2000	+300	−1000	2300	—
7/11	391	2300	+100	−900	2400	—

\* This final cash balance takes into consideration margin calls, if any.

futures price declines on day  $t + 3$  to \$401, entailing a cash flow of  $-\$600$ , therefore bringing his cumulative gain to only \$100. Thus the final cash balance is \$2100. On the fourth day, the settlement price falls to \$394. This means a loss of \$700 for our long. If that sum were taken from his cash account, its balance would fall to  $\$2100 - \$700 = \$1400$ . This amount is below the maintenance margin (which was set at \$1500), and such a situation triggers a maintenance margin call whose purpose is to bring back the cash balance to \$2000; therefore, the amount of the maintenance margin call is \$600. Over the next three days the long's situation deteriorates, because on July 9 the price takes a dive to \$387; the cash balance at the beginning of that day was \$1700; it is depleted now by a loss of \$400 and would thus fall to \$1300, below the \$1500 maintenance margin. This triggers a second maintenance margin call in the amount of \$700.

In the last two trading days the long makes up a little for his losses, because the futures price has risen somewhat.

At the end of trading on July 11, the long's situation is as follows. The futures price is at \$391. His loss can be determined in any of the following four ways:

1. (change in futures price)  $\cdot 100 = [F(t + 8, T) - p(t, T)] \cdot 100 = (391 - 400)100 = -900$
2. sum of mark to market cash flows  $= -900$  [sum of column (3)]
3. cumulative net gain  $= -900$  [last entry of column (4)]
4. final cash balance [last entry of column (5)] minus initial cash balance [first entry of column (2)] minus all maintenance margin calls  $= 2400 - 2000 - 600 - 700 = -900$ .

### 5.1.7 The riskiness of a “naked” futures position

Notice how extremely risky any such futures position is,<sup>2</sup> be it a long's position or a short's. Our long has invested “only” the \$2000 initial margin; he has lost \$900 after nine days of trading. Had the futures price been \$360, corresponding to a 10% decline in the futures price of gold, he would have lost \$4000, 200% of his investment! If you buy a commodity

<sup>2</sup> Such a position, not being covered by the present or future holding of the underlying commodity (gold in this case), is said to be a “naked” position. Here we suppose that neither the short holds the commodity, nor does the long intend to buy the commodity at maturity of the contract; thus, both are supposed to be speculators.

or an asset on the spot market, the maximum you can lose is 100% of your investment, whereas on a derivatives market such as the futures market, you can lose many times your investment. In the example of the gold futures market, let  $r$  be the (possible) rate of relative decrease of the futures price between the time you bought the contract ( $t$ ) and time  $t + d$ . We have:  $r = [F(t, T) - F(t + d, T)]/F(t, T)$ ; the new futures price is  $F(t + d, T)$ , and your loss is  $[F(t, T) - F(t + d, T)] \cdot 100 = rF(t, T)100$ . Suppose the relative decrease of the futures price is 20% and  $F(t, T)$  is \$400. The long loses \$8000; as a percentage of his initial margin, he loses  $\frac{rF(t, T)100}{2000} \cdot 100 = 5rF(t, T)\%$ . If the futures price declines by 50% and  $F(t, T)$  is 400, he loses 1000% of his initial investment! In case of an increase in the futures price, the same is true for the short, for whom  $r$  now represents the rate of increase of the futures price, and who risks losing  $5rF(t, T)\%$  of his investment in only a few days of futures trading.

### 5.1.8 Closing out a futures position

It is now time to turn toward the all-important question of the means by which a trader can voluntarily close out his position. As we will see, there are basically four ways to accomplish this.

The vast majority of futures contracts do not result in actual delivery of the underlying commodity or financial product. Why is this so? For the good reason that the majority of traders find it much more convenient to fulfill their contracts through reversing trade, which simply offsets their positions. We will review all possibilities in turn, starting with the delivery process. Indeed, it turns out that delivery may still be quite a rewarding process, as is the case in the most important futures contract existing at the time of this writing, the T-bond futures contract.

A trader can terminate a futures contract in four ways: (a) delivery, (b) cash settlement, (c) reversing trade, and (d) exchange for physicals. We will examine each in turn.

#### 5.1.8.1 Delivery

Many futures contracts allow the short (the seller) to deliver the commodity in various *grades* and at various *times* within a time span (say, one month). Why is this so? The futures exchanges have established such provisions in order to facilitate the task of the short, by making the commodity's market more liquid. Indeed, if the contract was established so that only a very short period (one day, for instance) was allowed for de-

livery and furthermore only a very specific kind or grade of the commodity could be delivered, such stringent conditions would have two possible consequences. First, they could kill the market altogether, since no one would be willing to commit himself to taking such a specific commitment; second, those conditions could entice some to try to “corner” the market, by which we mean that an individual or a group of individuals could engage in the following operation: they would buy on the spot market an extraordinarily high proportion of the said grade of the commodity, thus forcing the shorts to buy it back at a very high price when obligated by their contract to make delivery.

Futures markets are regulated precisely to avoid such “corners”; quite a number of grades are deliverable, at different locations, over a rather long time span. Also, any attempt to corner a market is a felony under the Trading Act of 1921; civil and criminal suits can be launched against traders who try to engage in such manipulations. Finally, the exchange could force manipulators to settle at a fixed, predetermined price. This is what happened in the famous Hunt manipulation of the silver market at the beginning of the 1980s.

More recently (in 1989), this also happened in a famous squeeze involving an important soybean processor. A long trader on the futures market, the processor had also accumulated a large part of the deliverable soybeans. The squeeze started to drive prices up steeply, both on the cash and the futures markets. Of course, the company’s line of defense was to argue that it was simply hedging its operations.<sup>3</sup>

Since only one futures price is quoted, while a set of deliverable grades of the commodity are deliverable, exchanges have set up a system of conversion factors whose purpose it is to equate, at least theoretically,

<sup>3</sup> On this, see for instance: S. Blank, C. Carter, and B. Schmiesing, *Futures and Options Markets* (Englewood Cliffs, N.J.: Prentice Hall, 1991), p. 384; and the articles they cite: Kilman, S. and S. McMurray, “CBOT damaged by last week’s intervention,” *The Wall Street Journal*, July 1987, Sec. C, pp. 1 and 12; Kilman, S. and S. Shellenberger, “Soybeans fall as CBOT order closes positions,” *The Wall Street Journal*, July 13, 1989, Sec. C, pp. 1 and 13. See also: Chicago Board of Trade, *Emergency Action, 1989 Soybeans*, Chicago, Board of Trade of the City of Chicago, 1990.

We should add that, unfortunately, fixed-income markets have not been exempt from cornering attempts. For instance, a few bond dealers were accused in 1991 of cornering an auction of two-year Treasury notes. On this, see S. M. Sunderasan, *Fixed Income Markets and Their Derivatives* (Cincinnati: South-Western, 1997), pp. 67 and 69, as well as the references contained therein, especially N. Jegadeesh, “Treasury Auction Bids and the Salomon Squeeze,” *Journal of Finance* 48 (1993): 1403–1419.

the attractiveness of delivering a particular grade. Consider first, as an example, the gold futures contract. Gold is available in various finesse grades, and the conversion factor is simply the finesse percentage of each grade. Will this distort the delivery process? If grade prices are exactly proportional to finesse—if the market just prices the gold *content* of any grade—there will be no distortion, because the possible profit or loss of any party remains as it should, proportional to the party's position. However, this is not what happens in the gold market: prices are not exactly proportional to gold content, and the short party, when making delivery, will always have to account for the conversion factor and the cost of acquiring gold to maximize his profit. Per unit of gold, this profit will be equal to the quoted price times the conversion factor of the delivered grade minus the cost of the delivered grade.

Before leaving this topic, we should underline that the futures contract on Treasury bonds allows the delivery of quite a number of bonds and that the conversion factors used by the Chicago Board of Trade (CBOT) entail substantive differences in profit for the short party. This issue is so important that we will devote the third section of this chapter to it; there we will derive a number of surprising results.

#### **5.1.8.2 Cash settlement**

The second way to close up a futures contract is through cash settlement. This may occur if the contract is written on a financial index or if a commodity is priced on an index. For instance, this is the case with the feeder cattle contract of the Chicago Mercantile Stock Exchange, whereby a cash settlement price is established as the average of prices of a number of auctions held throughout the United States over the seven days preceding settlement day. As another example, let us consider a long contract on the most important U.S. stock index future: the Standard & Poor's 500 futures contract. The S&P Index is a value-weighted index of 500 major stocks; as such it acts as the price of a basket of securities. One can set up a contract on that basket, requiring from parties a future exchange of the basket (the content of the index) for \$450; of course, this contract would be undeliverable because one could not trade fractions of stocks. It would also be untradable because no one would ever sustain the costs of acquiring so many stocks. Thus, it is quite natural that such a contract on an index will be closed not through delivery but through settlement. In fact, the minimum position on such a contract is the value of the index

times \$500. The cash settlement the long would pay is the difference between the futures price he has agreed to pay and the spot value of the index at maturity times \$500. Suppose, for instance, that a party enters a long futures contract at 400 and that the index falls to 350; he pays  $400 \times \$500 - 350 \times \$500 = \$25,000$ . Had the index risen to 420, he would have received  $20 \times \$500 = \$10,000$ .

#### **5.1.8.3 Offsetting positions or reversing trades**

One of the reasons for organizing futures markets with contracts marked to the market was to enable participants to enter and exit forward agreements easily. Should any party want to exit a forward position early, he would have to find another party willing to trade the specific commodity or financial product for the initial settlement date, which might be cumbersome if not impossible. Also, the party would have to carry two forward contracts (the long and the short) in his books until maturity. Futures markets remedy both of these drawbacks at once by permitting the long party wanting to get out of his futures contract to sell his long position to another party. As a result, on the day the long party (A) wants to get rid of his futures contract (which had been matched by a short, (B)), he simply enters a short contract with the same specifications as the initial one. His new (short) position will now be matched with the long position of a third party, called C. Party A has met all his obligations toward the clearing house; he has received his gains or sustained his losses. Now C has replaced A in its obligations toward B. By far the majority of futures contracts are settled in this way because it is much less expensive and often more convenient for all parties to operate in this way.

#### **5.1.8.4 Exchange of futures for physicals (EFP agreement)**

In the case where transportation of commodities is very costly (for example, in the case of energy products like gasoline or heating oil), the futures trading parties may try to negotiate delivery arrangements between themselves. Not only the location of delivery, but also its date and the exact grade of the commodity, as well as its price, can be negotiated between two parties. The parties may know each other, or the trade can be facilitated by the exchange itself. The procedure is then as follows: once all provisions of the agreement (including price) have been agreed upon, A (the long, for instance) will sell his long contract to B (the short) in exchange for the cash commodity. B does the same; through the ex-



change, he will buy A's long contract in exchange for delivery of the cash commodity. Both parties, A and B, now hold two positions (a long and a short) with the exchange, which recognizes this and closes out A and B.

### 5.1.9 Consequences of failure to comply

We have described the various ways a trader would voluntarily close out his position. A trader may also renege his obligation to replenish his account to its initial margin, when called by his broker to do so at some time  $t_1$  ( $t < t_1 < T$ ). Let us now consider what this trader's fate will be. What would happen to a trader who suffered losses, who saw his margin account depleted beyond the maintenance margin, and who then refused to post up the additional amount required at some time  $t_1$  to bring his account back to the initial margin level? Do not forget that at time  $t_1$ , the amount lost per unit by the (long) trader, for instance, is the difference between the initial futures price,  $F(t, T)$ , and the futures price  $F(t_1, T)$ ; he had committed himself to buying one unit of the commodity or the financial product at price  $F(t, T)$ , and now (at time  $t_1$ ), the futures price has fallen to  $F(t_1, T) < F(t, T)$ . He has therefore lost  $F(t, T) - F(t_1, T)$ , and that amount has been withdrawn by the broker from his initial deposit. The broker has the right to (and will) sell the client's contract at the prevailing price, which is  $F(t_1, T)$ . What will happen next?

Suppose that A is the defaulting party. Initially, his commitment to buy was matched by the exchange against B's willingness to sell. The broker has the power to transfer A's long contract to any other party (call this other party C) at the prevailing price  $F(t_1, T) < F(t, T)$ . Party C now becomes a long party, committing himself to buying the commodity or the financial product at the market futures price,  $F(t_1, T)$ . B's position was short and was matched initially to A's. Now C replaces A, so B and C are matched together. The broker will refund to A the balance of his account (on which A has lost  $F(t, T) - F(t_1, T)$ —B's gain), less a sizable commission.

## 5.2 Pricing futures through arbitrage

An important relationship exists between a futures price and a spot price. After all, for a perfectly defined product, the spot price paid today for said commodity obviously has a strong relationship to the price set today for the same commodity for a transaction that will take place in, say, six

months, since the only difference in the contract is the time of delivery. We will first explain intuitively the kind of arbitrage that could take place if a large difference were to be observed between the futures and the spot price. We will then be much more precise and evaluate exactly what that difference between the futures and the spot price, called the basis, should be in equilibrium.

### 5.2.1 Convergence of the basis to zero at maturity

Let us first observe that at maturity  $T$  the futures price  $F(T, T)$  must be equal to the spot rate  $S(T)$ . Indeed, any difference would immediately trigger arbitrage, which would force both prices toward the same level. Consider the following example. Suppose that at maturity the futures price is higher than the spot price,  $F(T, T) > S(T)$ . We will show that arbitrageurs would immediately step in, buying the commodity on the spot market, selling it on the futures market, and pocketing the difference. They could do so through pure arbitrage, that is, without committing to this operation any money of their own; they would simply borrow  $S(T)$  for a day, buy the commodity, and sell the futures. At delivery, they would cash in  $F(T)$  and pay back  $S(T)$  plus interest—a very small interest. Of course both supply and demand on each market would move in such a way that these adjustments would be nearly instantaneous. On the futures market, demand would shrink and supply would expand, thus causing the futures price to fall, while on the spot market the contrary would be true—supply would decrease and demand would increase, entailing an increase in the spot price.

### 5.2.2 Properties of the basis before maturity, $F(t, T) - S(t)$ : introduction

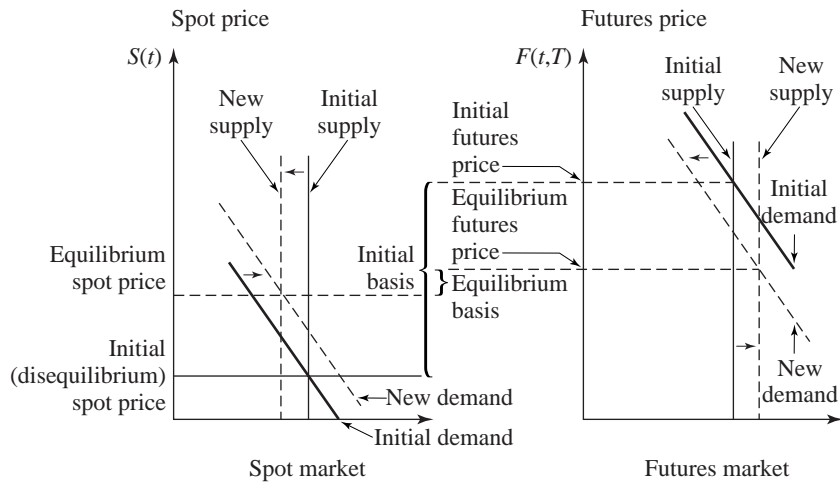
Let us now consider that we are at time  $t$ , before  $T$  ( $T - t = 6$  months, for instance). Suppose you observe a very large positive basis—a huge difference between the futures price and the spot price. You could well exploit this difference by buying the commodity on the spot market and selling the futures.

To understand arbitrage on a futures contract, start with arbitrage on the simpler forward contract. Suppose that the forward price is much higher than the spot price. An arbitrageur would step in, sell the forward contract, and buy the commodity on the spot market. At the maturity of the contract, he would make delivery and pocket the difference between the forward price and the spot price. This would be his gross profit, from

which a number of costs should be deducted: the interest he has to pay on the loan he contracted to buy the commodity on the spot market, and perhaps some storage, insurance, and transport costs. All these costs are referred to as *carrying costs*, because they pertain to holding the commodity during the whole period  $(T - t)$ . Notice that if our arbitrageur did not undertake a “pure” arbitrage, by borrowing  $p(t, T)$ , but, on the contrary, used his own funds, he would still bear an interest cost, which we would call an *opportunity cost*, because by investing his money in this operation he loses the opportunity to invest it in some other vehicle. We supposed that the forward price was way above the spot price, and therefore we could well imagine that his gross profit would still be above those costs. We could also suppose that the underlying asset is not a commodity, but an income generating asset such as a stock (which might bring a dividend during the holding period) or a bond (which might pay a coupon). In such cases, the arbitrageur’s costs would be reduced by such revenues, and we can say that arbitrage will occur whenever the forward price exceeds the spot price by more than these net costs (carrying costs less possible income receipts from holding the asset).

We can now leave the case of arbitrage in the forward market to consider the case of arbitrage on the difference between a futures price and the spot price of an asset. Let us not forget that since futures contracts are marked to the market, the arbitrageur who has sold a futures contract and who intends to deliver will not receive the initial futures price at delivery time, but the futures price at maturity, that is, the spot price at maturity. Indeed, any gain or loss at maturity will already have been settled through the daily cash flows he has received or paid out; what counts at maturity is the futures price at  $T$ , which equals the spot price,  $F(T, T) = S(T)$ . We will have the same outcome as in the case of a forward contract, but the calculation of the arbitrageur’s profit is somewhat different and goes through the daily resettlements of the contract. Here is the process by which an arbitrage on a forward contract and on a futures contract will have the same outcome, if we do not take into account the possible interest income flows generated by the daily settlements of the futures contract.

When the futures contract matures, the short collects the net profits he made on his futures contract: these are the net cash flows on his margin account. As we have seen before, they add up to  $F(t, T) - F(T, T) = F(t, T) - S(T)$ . This amount is positive if the futures price contracted at



**Figure 5.3**

**Adjustment of the basis toward its equilibrium value. In the case of an exceedingly high initial basis, supply and demand will shift both on the spot and futures markets. As a result, basis will shrink to its equilibrium, equal to the cost of carry net of any possible income flows generated by the asset.**

time  $t$  is larger than the futures price at  $T$ , equal to  $S(T)$ ; the net cash flows are negative if  $F(t, T) < S(T)$ . Thus, the arbitrageur may very well make a profit on the futures contract if  $F(t, T) > S(T)$ , or suffer a loss if  $S(T) > F(t, T)$ . However, since in this case he has bought initially, at time  $t$ , the underlying commodity at  $S(t)$ , which supposedly was much lower than  $F(t, T)$ , he will certainly make a profit.

Let us see why. At maturity  $T$ , his payoff is made with two positions: first, his futures position,  $F(t, T) - S(T)$ , and second, his “cash” position. Choosing to deliver, he will receive  $F(T, T) = S(T)$  and will remit the commodity he has bought at price  $S(t)$ ; his cash position is  $S(T) - S(t)$ . His payoff is thus  $F(t, T) - S(T) + S(T) - S(t) = F(t, T) - S(t)$ , a difference that corresponds to what he would have received in a forward market and that is bound to be larger than the costs the arbitrageur will incur. These costs are exactly the same as those that would prevail in an arbitrage on a forward contract: carrying costs, less any income flow from the stored asset. Since we have supposed that the difference  $F(t, T) - S(t)$  was very large, it will admittedly be larger than those net costs. What will be the outcome of such arbitrage? It will displace to the right the demand curve on the spot market and also move to the right the supply curve on

the futures market. This will tend to raise the price on the spot market and lower it on the futures market, thereby decreasing the basis.

However, the basis will not necessarily wait for such an arbitrage to shrink. Other forces will be at play. First, why should anybody buy the commodity at such a highly priced futures or become a supplier in such a bad spot market? Indeed, what will happen is that the demand will shrink to the left on the futures market, because those who intended to hold the commodity for six months will now consider buying it on the spot market, and those who considered selling it now will try to postpone the sale. This will have the same effect as the arbitrage we described before. Also, other operators on both markets will reinforce the possible action of the arbitrageurs. Holders of the commodity will increase their supply on the futures market at the expense of the supply on the spot market. So, arbitrageurs may well step into both markets (spot and futures) and lower the basis by these actions, while the operators on any of those markets may also contribute to the same kind of change.

The same kind of reasoning would apply if the basis were initially extremely small or negative. Suppose, for instance, that it is zero ( $F(t, T) = S(t)$ ). What would be the outcome of such a situation? Reverse processes such as the following could well take place. Those who own the commodity could sell it and invest the proceeds at the risk-free rate. On the other hand, they would buy a futures contract. At maturity time, they would cash in their investment plus interest; they would then be able to buy back their commodity at exactly the price of their investment and keep the interest as net benefit. This would tend to move to the right both the supply curve on the spot market and the demand curve on the futures market, thus exerting downward pressure on the spot price and upward pressure on the futures price.

This is not, of course, the whole story. Arbitrageurs may very well step in and borrow the commodity from owners of the good, short sell it, and with the proceeds do what we have just described. After receiving the commodity at the maturity of the futures contract, they would give back the commodity to the owners. This supposes of course that it is indeed possible to “short sale” the commodity (“short selling” means borrowing an asset and selling it, with the promise to restitute it at some later day). Of course, normal transaction costs will be incurred by the arbitrageur, and, as before, we should take into consideration any possible income flow generated by the asset.

### 5.2.3 A simple example of arbitrage when the asset has no carry cost other than the interest

In frictionless markets such as these, let us first suppose that there are no storage costs on commodities and that financial assets that would be traded on such spot and futures markets do not pay, within the time period considered, any dividend (if these assets are stocks), nor do they promise to pay any coupon (if the assets are bonds). We can now easily see the theoretical relationship that will exist between the spot price and the futures price. Let us go back to the situation where the futures price is significantly higher than the spot price multiplied by  $1 + i(0, T)T$ .

By entering the series of transactions we described earlier, which we summarize in table 5.2, we observe that with no negative cash flow at time 0, an arbitrageur could generate a positive cash flow  $F(t, T) - S(t) \cdot (1 + i(0, T)T)$ , since we supposed that  $F(t, T) > S(t) \cdot (1 + i(0, T)T)$ .

This cannot last, since this arbitrageur would be joined by many others, pushing up  $S(0)$  and driving down  $F(0, T)$  until there could be no such free lunch anymore; we would then have the basic relationship between  $F(0, T)$  and  $S(0)$ :

$$F(0, T) = S(0) \cdot (1 + i(0, T)T) \quad (1)$$

Such a relationship could have been obtained in a straightforward way, just by being careful with the units of measurement. Indeed,  $F(0, T)$  is the price of the commodity in terms of dollars to be paid at time  $T$ .  $S(0)$  is the price of the same commodity at time  $t$ . Thus, the futures price can be none other than the spot price, where dollars at 0 are converted into dollars at

**Table 5.2**  
The futures buyer's positions: initial futures price:  $F(t, T) = \$400/\text{ounce}$  on July 1

Time 0		Time $T$	
Transaction	Cash flow	Transaction	Cash flow
Borrow $S(0)$ at (simple) interest rate $i_{0, T}$	$+S(0)$	Reimburse loan with interest	$-S(0) \cdot (1 + i_{0, T})T$
Buy asset	$-S(0)$	Sell asset at $F(0, T)$	$F(0, T)$
Sell futures at $F(0, T)$	0		
Net cash flow	0	Net cash flow	$F_{t, T} - S_t(1 + i_{0, T} \cdot T)$

$T$ . We must have a relationship of the form

$$\frac{\$ \text{ at time } T}{\text{commodity}} = \frac{\$ \text{ at time } 0}{\text{commodity}} \times \frac{\$ \text{ at time } T}{\$ \text{ at time } 0}$$

The last expression on the right-hand side,  $(\$ \text{ at time } T)/(\$ \text{ at time } 0)$ , is indeed a price: it is the price of \$1 at time 0 expressed in dollars at  $T$ , and this is equal to  $1 + i(0, T)T$ .

Consider now the reverse situation, where the futures price  $F(0, T)$  is smaller than the future value of the spot price  $S(0)[1 + i(0, T)T]$ . An arbitrageur could step in and proceed as follows. Let us suppose that if he could dispose of all the proceeds of a short sale, he would borrow the commodity (or the financial asset), sell it on the spot market, and invest the proceeds  $S(0)$  at rate  $i(0, T)$  during period  $T$ . At the same time he would perform this short sale, he would buy the commodity on the futures market. At time  $T$  he would receive the result of his futures position, which would be  $F(T, T) - F(0, T) = S(T) - F(0, T)$ . He would also collect the reimbursement of his loan plus interest, an amount equal to  $S(0)[1 + i(0, T)T]$ ; on the other hand, he would have to give back the asset to the initial owner; he would have to buy the commodity on the spot market for  $S(T)$  and give it back to its owner. In total his position would be  $S(T) - F(0, T) + S(0)(1 + i(0, T)T) - S(T) = S(0)[1 + i(0, T)T] - F(0, T)$ . This operation is called *reverse cash and carry*.

We have supposed that  $S(0)[1 + i(0, T)T]$  was larger than  $F(0, T)$  and have just shown that an arbitrageur could step in and pocket, as pure arbitrage profit, that very difference. Of course, all participants would realize that excess supply would appear on the spot market, forcing its price  $S(0)$  down; similarly, excess demand would build up pressure on the futures market, driving its price  $F(0, T)$  up until equilibrium, corresponding to  $F(0, T) = S(0)[1 + i(0, T)T]$ , would be restored.

Table 5.3 summarizes the initial and final positions of an arbitrageur who undertook these operations. As before, note that this is pure arbitrage, in the sense that no money outflows are required from the arbitrageur at time 0, although a net profit is obtained at the futures contract maturity.

## 5.2.4 Arbitrage with carry costs

In our introduction to futures pricing, we mentioned that the choice an individual makes between taking positions either in the spot market or in

**Table 5.3****Transactions and corresponding cash flows at time 0 (today) and at time  $T$  in a simple reverse cash-and-carry arbitrage**

Time 0		Time $T$	
Transaction	Cash flow	Transaction	Cash flow
Borrow asset	0	Receive proceeds of futures short contract*	$+S(T) - F(0, T)$
Sell asset	$+S(0)$	Collect loan plus interest	$+S(0)[1 + i(0, T)T]$
Lend proceeds of asset's sale	$-S(0)$	Buy asset on the spot market and remit it to its owner	$-S(T)$
Sell futures at $F(0, T)$	0		
Net cash flow	0	Net cash flow	$S(0)[1 + i(0, T)T] - F(0, T) > 0$

\*Sum of net cash flows received during time span  $(0, T)$ .

the futures market has to take into account the costs and the possible benefits of holding commodities or assets. Costs of holding assets always include interest. According to the nature of assets (which may be financial assets or commodities ranging from precious metals to livestock), many sorts of carry costs may be incurred, including storage costs, insurance costs, and/or transportation costs. If the assets are of a financial nature, they may or may not carry a return. Finally, many commodities traded on the futures markets enter a production process as intermediary goods (e.g., gold, semiprecious metals, agricultural products). When held, these goods generate costs and also a *convenience return* that is difficult to measure. This convenience return corresponds to the benefits any processor of such goods receives by holding them in his inventories. Indeed, let us suppose that the spot price of an intermediate good such as copper is much too high compared to its futures price. Normally, what could happen is this: pure arbitrageurs could step in, borrow copper from their owners, short sell it, invest the proceeds at a risk-free interest rate, and buy a futures contract; at maturity, they could receive (or pay) proceeds of futures  $S(T) - F(0, T)$ , collect proceeds of loan  $S(0)[1 + i(0, T)T]$ , and buy asset  $[-S(T)]$ , for a (positive) net cash flow of  $S(0)[1 + i(0, T)T] - F(0, T)$ . Also, owners of copper could themselves sell it on the spot market, lend its proceeds, and buy a futures contract; at maturity, they would receive the same cash flow as the pure arbitrageur. However, these operations, which we may call pure arbitrage and quasi-arbitrage, respectively, may not occur at all if owners of copper choose not to part with



**Table 5.4**  
**Types of carry costs and carry income, with their corresponding futures contracts**

Type of cost or income	Type of futures contract where they typically apply
Carry costs	
Interest	All futures
Storage	Commodities
Insurance	Commodities
Transportation	Commodities
Carry income	
Dividend	Stocks and stock indices
Coupons	Bonds
Convenience income	Commodities

their inventories, which may be trimmed down to their optimum level, enabling their owners to meet any unforeseen excess demand in their final product, and thus preventing them from losing customers. Therefore, inventories, which are assets, do bring a convenience income to their owner. Although difficult to measure, we know that it exists, and it could well prevent arbitrage or quasi-arbitrage in the event of a spot price that is relatively high compared to a futures price.

In summary, we can classify carry costs and carry income as shown in table 5.4, which indicates the kind of futures contract where each type of cost or income typically applies.

Since in this book we deal mainly with futures on interest rates (on bonds), we will not take into consideration this convenience income. Thus, our main evaluation equation can be derived in a very natural way, as follows. Remember that originally all futures markets were designed in such a way as to enable future buyers and sellers of a commodity to hedge against possible price fluctuations, obviating all possible shortcomings of a forward contract (i.e., a contract set on an unorganized market). Thus, for any long party on the futures market, one unit of the commodity at time  $T$  may be acquired in two ways. The first is to take a position on the futures market; the second is to acquire the commodity immediately (say at time 0), incur carrying costs, and receive any carry income. The net costs corresponding to both possibilities should be the same.

Let us take them in order. If our individual enters the futures market with a long position, he will pay  $p(0, T) - S(T)$  at time  $T$ , which is

the net cash flow of his futures position in future value  $p(0, T) - S(T)$ , the sum of positive and negative cash flows paid out during the futures' contract life. We will assume that this is the future value of all these cash flows. Of course, each cash flow should be evaluated in future value, since the owner of the contract may incur positive and negative cash flows. The error made by neglecting the interest corresponding to these cash flows between their receipt and  $T$  is not very significant. Furthermore, the long party will buy the asset on the spot market and pay  $S(T)$ . In all, he will disburse  $p(0, T) - S(T) + S(T) = p(0, T)$ .

Consider now the other way of acquiring the commodity at  $T$  and the cost of that operation, evaluated at  $T$ . Our long party can buy the commodity outright on the spot market and pay  $S(0)$ . The first carry costs are interest, equal to  $i(0, T)S(0)T$ ; these are evaluated with respect to time  $T$  and are equal to the interest the long party has to pay out at  $T$  if he has borrowed to pay the asset at price  $S(0)$ . He will have opportunity costs as well if he has used his own funds to buy the asset (he has had to forego that interest). He may also incur minor carry costs such as the various fees he will pay his broker, for instance, to keep the asset for him and collect whatever income the asset can earn. We will call all those carrying costs in future value CCFV (carrying costs in future value). On the other hand, if the asset generates some income, in the form of dividends or coupons, those payments have to be evaluated at time  $T$ ; call them CIFV (carry income in future value). The net cost in future value of buying the asset outright on the spot market is therefore  $S(0) + CCFV - CIFV$ . Equilibrium requires that both prices be equal, and we should have a fundamental equilibrium relationship:

$$F(0, T) = S(0) + CCFV - CIFV \quad (2)$$

Note that this equation holds equally well for both the potential buyer and the potential seller of the asset. Should there be any discrepancy between the left-hand side and the right-hand side of this equation, arbitrage through the cash-and-carry mode or through reverse cash-and-carry can be performed along the lines we described earlier, putting pressure on both the spot price  $S(0)$  and the futures price  $F(0, T)$ , until equilibrium is restored. We will explain next how arbitrageurs can operate in both cases.

#### 5.2.4.1 The case of an overvalued financial futures contract

The case where the futures price is overvalued can be written as  $F(0, T) > S(0) + CCFV - CIFV$ . An arbitrageur would step in and take a short position on the futures contract, selling the asset at time 0 at the futures price  $F(0, T)$ . At the same time he would borrow  $S(0)$  in order to buy the asset. None of these transactions would imply any net positive or negative cash flow at time 0. At time  $T$ , our arbitrageur would liquidate his futures short position; he would then receive a net cash flow equal to  $F(0, T) - F(T, T) = F(0, T) - S(T)$ . He would also sell the asset and receive  $S(T)$ , and then he would reimburse his loan of  $S(0)$ , including the interest  $S(0)i(0, T)T$  plus fees to his broker (interest plus fees equal the carry costs in future value,  $CCFV$ ). Finally he would collect the income from coupons or dividends received during the holding period of the asset, which the arbitrageur invested until the maturity of the futures contract. In all, these income flows are valued at  $CIFV$  at time  $T$ . (See table 5.5 for a summary of these transactions and their corresponding cash flows.) Thus at time  $T$  he would receive  $F(0, T) - S(T) + S(T) - S(0) - CCFV + CIFV = F(0, T) - S(0) - CCFV + CIFV$ , a positive amount since we initially supposed that  $F(0, T) > S(0) + CCFV - CIFV$ . Once more we can see that an arbitrageur could turn any discrepancy between

**Table 5.5**

**Transactions and corresponding cash flows in the case of a cash-and-carry arbitrage on an asset with carry costs and carry income. Initial situation: futures price overvalued:  $F(0, T) > S(0) + CCFV - CIFV$**

Time 0		Time $T$	
Transaction	Cash flow	Transaction	Cash flow
Sell futures at $F(0, T)$	0	Liquidate futures short position	$+F(0, T) - S(T)$
Borrow asset value on spot market	$+S(0)$	Sell asset on spot market	$+S(T)$
Buy asset on spot market	$-S(0)$	Reimburse principal on loan	$-S(0)$
		Pay interest on loan, plus fees to stockbroker	$-CCFV$
		Collect income generated by asset	$+CIFV$
Net cash flow	0	Net cash flow	$F(0, T) - S(0) - CCFV + CIFV(>0)$

two different prices for the same object into a net profit. Also we observe that selling the asset on the futures market and buying it on the spot market pulls down the futures price and pushes up the spot price of the asset until equilibrium (as described by equation (1)) is restored.

Are there any uncertain elements in equation (1)? The only uncertainty pertains to CIFV, the carry income in future value, since it does not represent coupons paid by default-free bonds, such as Treasury bonds. Indeed, if the financial instrument is a stock or an index on stocks, we have to replace CIFV by its expected value, denoted by  $E(\text{CIFV})$ ; probably the market will attach a risk premium (denoted  $\pi$ ) in equation (1), in the sense that this risk premium would have to be *added* to  $E(\text{CIFV})$  for (1) to hold. Thus, with uncertain cash income, we would have in equilibrium:

$$F(0, T) = S(0) + CCFV - E(\text{CIFV}) - \pi \quad (3)$$

As always, the problem arises of evaluating this risk premium; matters become complicated since transaction costs, and more generally carry costs, are not the same for all actors in futures markets.

#### 5.2.4.2 The case of an undervalued financial futures contract

The case where the futures price is undervalued corresponds to  $F(0, T) < S(0) + CCFV - \text{CIFV}$ . In such circumstances arbitrageurs would buy the futures for  $F(0, T)$ ; they would then short sell the asset (they would borrow the asset, sell it on the spot market at  $S(0)$ , and invest the proceeds). At the maturity of the futures contract, arbitrageurs would settle their long position, thus paying out  $F(0, T) - S(T)$  or receiving  $S(T) - F(0, T)$ . They would have to go on the spot market and buy the asset for  $S(T)$ , collecting interest on  $S(0)$  (presumably an amount very close to the carry cost in future value that we mentioned earlier). The difference between the borrowing rate and the lending rate times  $S(0)$  (the difference between the interest paid out by a short arbitrageur and the interest received by a long arbitrageur) is more or less equal to the broker's fees that the short arbitrageur had to pay. In other words, interest earned by long arbitrageur = interest paid by short arbitrageur + broker's fee = CCFV. Finally, arbitrageurs would have to pay the asset's owner the

**Table 5.6**

**Transactions and corresponding cash flows in the case of a reverse cash-and-carry arbitrage on an asset with carry costs and carry income. Initial situation: futures price undervalued:  $F(0, T) < S(0) + CCFV - CIFV$**

Time 0		Time $T$	
Transaction	Cash flow	Transaction	Cash flow
Buy futures at $p(0, T)$	0	Liquidate futures long position	$-F(0, T) + S(T)$
Borrow asset and sell asset at $S(0)$	$+S(0)$	Buy asset on spot market and return it to its owner	$-S(T)$
Lend asset's value $S(0)$	$-S(0)$	Collect principal and interest on loan	$+S(0) + CCFV$
		Pay to the owner of the asset income generated by the asset	$-CIFV$
Net cash flow	0	Net cash flow	$-F(0, T) + S(0) + CCFV - CIFV > 0$

carry income in future value generated during time span  $(0, T)$  by the asset borrowed at the outset,  $CIFV$ .

Table 5.6 summarizes these transactions and their outcome. At the bottom line, the long arbitrageur would receive  $S(T) - F(0, T) - S(T) + S(0) + CCFV - CIFV$ , equal to  $-F(0, T) + S(0) + CCFV - CIFV$ , a positive amount, since we have supposed that  $F(0, T) < S(0) + CCFV - CIFV$ .

As before, the arbitrageur nets a profit equal to the difference between two values when they are out of equilibrium. Upward pressure on the future price and downward pressure on the spot price will reestablish the equilibrium relationship (1) between the two prices so long as there is no uncertainty as to the future value of the carry income generated by the asset.

### 5.2.5 Possible errors in reasoning on futures and spot prices

If the reader hesitates in his understanding of the various relationships between futures prices, spot prices, and future spot prices, he may find solace in the fact that one of the most distinguished economists of the twentieth century, John Maynard Keynes, erred on the subject; what's more, it took seven decades for researchers to discover the error. Keynes'

blunder on the subject was made in his *Treatise on Money* (1930); to the best of our knowledge, this error was reported for the first time by Darrel Duffie in his remarkable book *Futures Markets* (1989), where it is very well documented.

In his *Treatise on Money*, Keynes addresses the issue of the sign of the basis (the difference between the forward price and the spot price,  $F(0, T) - S(0)$ ). He defines “backwardation” as a situation in which the spot price exceeds the forward price; in modern language, “backwardation” means that the basis, as we have defined it, is negative. For Keynes, this is a normal situation, which he justifies as follows:

It is not necessary that there should be an abnormal shortage of supply in order that a backwardation be established. If supply and demand are balanced, the spot price must exceed the forward price by the amount which the producer is ready to sacrifice in order to “hedge” himself, i.e., to avoid the risk of price fluctuations during his production period. Thus in normal conditions the spot price exceeds the forward price, i.e., there is backwardation.<sup>4</sup>

There are some severe flaws in this reasoning. First, there is no such risk premium for the producer to sacrifice in order to hedge himself; forward and then futures markets have been developed precisely to enable both the buyer and the seller of a commodity to protect themselves against uncertainty without any cost. As we have seen, both agents have to “pay” something when they enter a forward or a futures market; they have to forego possible gains at maturity  $T$  of the contract, but that is the difference between  $S(T)$  and  $F(t, T)$  (if it turns out to be positive) for the seller, and the difference between  $F(t, T)$  and  $S(T)$  (if it is positive) for the buyer. However, this has nothing to do with the basis at  $t$ ,  $F(t, T) - S(t)$ , which may very well be positive or negative.

Furthermore, we can see easily that the only way to determine the sign of the basis is writing equation (1) the following way:

$$F(t, T) - S(t) = CCFV - CIFV$$

In other words, in equilibrium the sign of the basis is just the sign of the difference between carry costs and carry income in future value. When no

<sup>4</sup> John Maynard Keynes, *A Treatise of Money*, Vol. II, chapter 2a, “The Applied Theory of Money,” part V, “The Theory of the Forward Market”; quoted by Darrell Duffie in his *Futures Markets* (Englewood Cliffs, N.J.: Prentice Hall, 1989), p. 99.

seasonal shortage or excess supply exists or is predictable, there are many commodities for which carry costs are definitely larger than carry income, and this is why under normal conditions, contrary to what Keynes ascertained, the normal situation is that of a positive basis.

### 5.3 The relative bias in the T-bond futures contract

#### 5.3.1 The various types of options imbedded in the T-bond futures contract

We should underline that the short trader, who has committed himself to selling an eligible T-bond (with at least 15 years maturity or 15 years to first callable date) on a delivery day, holds a number of valuable options. First, he has what is called a *wild card* option. This in turn can be broken up into two options: the first is the *time to deliver* option; the second is the *choice of the bond* to be delivered (also called a *quality* option).

##### 5.3.1.1 The wild card (time to deliver) option

The delivery of the T-bond by the short party must be done in any business day of the delivery month. However, the various operations needed for that span a three-day period:

- The first day is called *position day*; it is the first permissible day for the trader to *declare his intent* to deliver. The first possible position day is the penultimate business day preceding the delivery month. The importance of this “position day” is considerable: on that day at 2 P.M. the *settlement price* and therefore the invoice amount to be received by the short are determined (hence the day’s name). Now the short can wait until 8 P.M. to announce her intent to deliver. Clearly, during the six hours, she has the option of doing precisely this, thereby reaping some profits (or not). If interest rates rise between 2 P.M. and 8 P.M., the bond’s price will fall. She will then earn a profit from the difference between the actual price she will receive for the bond (called the settlement price), set at 2 P.M., and the prices he will have to pay to acquire the bond, which will be lower than the settlement price because of the possible rise in interest rates between 2 P.M. and 8 P.M.
- The second day has only administrative importance; it is the *notice of intention day*. During this day the clearing corporation identifies the short and the long trader to each other.

- The third day is the *delivery day*. During this day, the short party is obligated to deliver the T-bond to the long party, who will pay the settlement price fixed at 2 P.M. on position day.

The short trader has the option to exercise this “time to deliver” option from the penultimate business day before the start of the delivery month and the eighth to the last business day of the delivery month, called the *last trading day*. On that day at 2 P.M. the last settlement price is established for the remainder of the month. This still leaves our short with a special option, called an *end-of-the-month option*, which we examine now.

Suppose our short has not elected to declare her intention to deliver until the last trading day, either because interest rates have not increased or because they have not increased enough to her taste. The last trading day has now come, and the settlement price she will receive when delivering the bond is now fixed. She can still wait five days to declare her intention. This leaves her with the possibility of earning some profits if, for instance, she buys the bond at the last trading day settlement price, keeps it for a few days, and therefore earns accrued interest on it. She will receive a profit if the accrued interest is higher than the interest she will pay in order to finance the purchase of the bond.

The various options offered to the short party carry value; this value added to the futures price must equal the present value of the bond plus carry costs in order to yield equilibrium values both for the bond and the futures. In the last 20 years, many studies have been undertaken to capture this rather elusive option value. Robert Kolb made an excellent synthesis of these researches in his text *Understanding Futures Markets*.<sup>5</sup>

We will now take up the issue of analyzing in depth the relative bias introduced by the Chicago Board of Trade in the T-bond futures conversion factor.

### 5.3.1.2 The quality option

A short party has the benefit of exercising another important option: he can choose to deliver a T-bond among a list of bonds, established regularly by the Chicago Board of Trade, which have at least 15 years of maturity from their delivery date until their first call date. Since the bonds can vary widely in maturity and coupon rate, the CBOT normalizes their

<sup>5</sup> R. Kolb, *Understanding Futures Markets* (Kolb Publishing Company, 1994).



value by applying to each a *conversion factor* (described in detail in the appendix to this chapter).

It is well known that when yields are above 8%, the Chicago Board of Trade conversion factor introduces a bias that tends to favor the delivery of low coupon, long maturity bonds; the opposite is true when yields are below 8%. In this section we will show why this is so. The main reason for this is that any conversion factor system (as set up by the Chicago Board of Trade, or any other exchange organizing a futures market on long-term government securities along the same lines) introduces a *relative* bias that is an *affine transformation* of the duration of the bond to be remitted. The transformation is a positive one when rates are above 8% and a negative one when rates are below 8%.

Since the introduction of the CBOT T-bond futures contract, excellent studies of the bias entailed by the conversion factor have been carried out (see in particular M. Arak, L. S. Goodman, and S. Ross (1986); T. Killcollin (1982); J. Meisner and J. Labuszewski (1984)).<sup>6</sup> However, these researchers concentrated on the simple difference between the CBOT conversion factor and the “fair,” or theoretical, factor. It is of crucial importance to consider, in fact, the *relative* difference between the CBOT factor and the fair factor, because this relative bias can be shown to be an affine transformation of the remitted bond’s duration—giving the key to profit maximization in the delivery process, at least in an environment of flat term structures.

### 5.3.2 The conversion factor as a price

The conversion factor calculated by the CBOT is basically the value of a \$1 principal bond, carrying a coupon  $c$  (as a percentage) and a maturity  $T$  equal to those of the bond remitted by the seller, when the yield to maturity is 6%.<sup>7</sup> In order to understand the economic meaning and the financial implications of converting the futures price into an invoice amount the long party will have to pay, the following notation is useful:

<sup>6</sup> Marcelle Arak, Laurie S. Goodman, and Susan Ross, “The Cheapest to Deliver Bond on the Treasury Bond Futures Contract,” *Advances in Futures and Options Research*, Volume 1, Part B (Greenwich, Conn.: JAI Press, 1986), pp. 49–74; Thomas E. Killcollin, “Difference Systems in Financial Futures Contracts,” *Journal of Finance* (December 1982): 1183–1197; James F. Meisner and John W. Labuszewski, “Treasury Bond Futures Delivery Bias,” *Journal of Futures Markets* (Winter 1984): 569–572.

<sup>7</sup> The Chicago Board of Trade began calculating the conversion factor this way on March 1, 2000.

$B_{c,T}(i)$  = value of a \$100 principal bond, with coupon  $c$  (in %) and maturity  $T$ , when the structure of interest rates is flat and equal to  $i$ ; coupon is paid semi-annually.

For instance,  $B_{9\%,20}(6\%)$  is the value of a bond with 9% semi-annual coupon, 20-year maturity, when the rate of interest is 6%. In such a case,  $B_{9\%,20}(6\%)$  is 137.90. Using this notation, we can write the conversion factor  $CF_{c,T}$  as

$$\text{Conversion factor} \equiv CF_{c,T} = \frac{B_{c,T}(6\%)}{100}$$

(Note that the conversion factor depends solely on the coupon and maturity of the delivered bond and not on the rate of interest; we denote it  $CF_{c,T}$ .) In the case of the bond we just considered, the conversion factor would be 1.3790. For bonds with a maturity not equal to an integer number of years, the conversion factor can be determined from Appendix 5.A.1, a portion of the conversion table of the Chicago Board of Trade.

The conversion factor can thus be seen as the ratio of *two*  $T$ -maturity bond prices: the price of a  $c$ -coupon evaluated at 6% and the price of a 6%-coupon evaluated at 6% (which is equal to 100). The ratio of these two prices is therefore an exchange rate between a  $(6\%, 20)$  bond, and a  $(c, T)$  bond when  $i$  is 6%. We have

$$CF = \frac{B_{c,T}(6\%)}{B_{6\%,20}(6\%)} = \frac{\frac{\$}{(c,T) \text{ bond}}}{\frac{\$}{(6\%,20) \text{ bond}}} = \frac{\text{number of } (6\%, 20) \text{ bonds}}{\text{one } (c, T) \text{ bond}}$$

Being an exchange rate, the conversion factor is therefore a *price*: the price of one  $(c, T)$  bond, expressed in 6%-coupon bonds when rates are equal to 6%. If one remits a  $(c, T)$ , one must count it as if it were a number  $CF$  of  $(6\%, 20)$  bonds, which themselves are evaluated at the futures prices.

### 5.3.3 The “true” or “theoretical” conversion factor

The above-mentioned property of the conversion factor suggests that the conversion factor may well introduce a bias as soon as the rate of interest is not 6%. Consider, for instance, the case where the interest rate structure is at 4%, and where the seller chooses to remit a  $(10\%, 20)$  bond. The conversion factor is entirely independent from the interest rate structure; it is fixed at  $CF_{c,T} = \frac{B_{10\%,20}(6\%)}{100} = 1.4623$ . However, let us now introduce

the concept of a “true” (or theoretical) conversion factor, denoted as  $TCF_{c,T}(i)$ , the number of (6%, 20) bonds that correspond to one (10%,  $T$ ) bond. Such a true conversion factor should be defined as follows:

$$\begin{aligned} TCF_{c,T}(i) &= \frac{\text{number of (6\%, 20) bonds at 4\%}}{\text{one (10\%, 20) bond at 4\%}} \\ &= \frac{\text{price of a (10\%, 20) bond at } i = 4\%}{\text{price of a (6\%, 20) bond at } i = 4\%} \\ &= \frac{B_{10\%,20}(4\%)}{B_{6\%,20}(4\%)} = \frac{182.07}{127.36} = 1.4296 \end{aligned}$$

for a relative difference  $(CF - TCF)/TCF$  of 2.28%.

More generally, the theoretical conversion factor for delivering a  $(c\%, T)$  bond when the interest rate is  $i$  should be equal to the *number* of (6%, 20) bonds that a  $(c\%, T)$  bond is worth. Hence it is equal to the *price* of a  $(c\%, T)$  bond divided by the *price* of a (6%, 20) bond when rates are  $i$ . We should have

$$\begin{aligned} TCF_{c,T}(i) &= \frac{\text{number of (6\%, 20) bonds}}{\text{one (c\%, T) bond}} \\ &= \frac{\text{price of (c\%, T) bond at rate } i}{\text{price of (6\%, 20) bond at rate } i} \\ &= \frac{B_{c\%,T}(i)}{B_{6\%,20}(i)} \end{aligned}$$

### 5.3.4 The relative bias in the conversion factor

We now introduce the concept of the *relative* bias in the CBOT conversion factor, defined as the difference between the conversion factor and the true conversion factor, divided by the true conversion factor:

$$\text{Relative bias} \equiv b = \frac{CF - TCF}{TCF} = \frac{CF}{TCF} - 1 \equiv \frac{\frac{B_{c\%,T}(6\%)}{B_{6\%,20}(6\%)}}{\frac{B_{c\%,T}(i)}{B_{6\%,20}(i)}} - 1 \quad (4)$$

Of course, there is no bias if  $i = 6\%$ , because then  $TCF = CF$ . However, this is not the only case where the conversion factor does not intro-

duce a bias, although this is seldom realized. We will show that there is in fact a whole set of values (rate of interest  $i$ , coupon  $c$ , and maturity of the deliverable bond  $T$ ) that lead to a zero relative bias; these are all the values of  $i$ ,  $c$ , and  $T$  for which the equation  $b = 0$  is satisfied. To get a clear picture of this, it is useful first to determine, for any given value of  $i$ , what are the limits of the relative bias introduced by the conversion factor when the maturity of the bond tends toward either infinity or zero. Although deliverable bonds must have a maturity longer than 15 years and are usually shorter than 30 years, we gain in our understanding of the relative bias by considering a broader set of bonds and then retaining only those of interest.

Consider first what would be the bias in the conversion factor when the maturity of the deliverable bond goes to zero. Since  $\lim_{T \rightarrow 0} B_{c\%, T}(6\%) = \lim_{T \rightarrow 0} B_{c\%, T}(i) = 100$ , we can write, from (4),

$$\lim_{T \rightarrow 0} b = \frac{B_{6\%, 20}(i)}{100} - 1$$

For example, if  $i = 3\%$ , the relative bias tends toward  $+44.9\%$ ; if  $i = 9\%$ , the bias is  $-27.6\%$ . This limit is positive if and only if  $i < 6\%$  and negative if and only if  $i > 6\%$ .

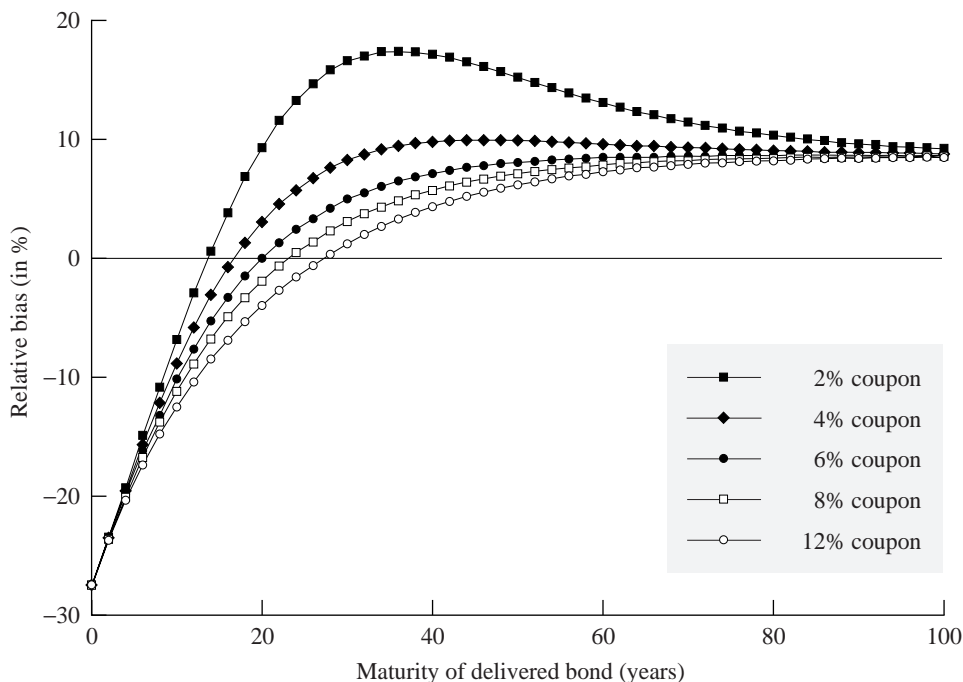
We can see that this limit depends solely on  $i$  and not on the coupon  $c$ . It implies that all curves representing the bias for different coupons will intersect on the ordinate (for  $T = 0$ ). This is shown in figures 5.4 and 5.5, which correspond to possible biases for various coupons and maturities, and for interest rates equal to  $i = 9\%$  and  $i = 3\%$ , respectively.

Let us now see what the relative bias is if we deliver a bond with infinite maturity. Since  $\lim_{T \rightarrow \infty} [B_{c\%, T}(6\%)/B_{c\%, T}(i)] = \frac{c}{6\%}/\frac{c}{i} = \frac{i}{6\%}$ , we have

$$\lim_{T \rightarrow \infty} b = \frac{i}{6\%} \frac{B_{6\%, 20}(i)}{100} - 1$$

Denoting  $1/(1 + i/2)^{40}$  as  $\theta$ , we have, after simplifications,  $\lim_{T \rightarrow \infty} b = \theta(\frac{i}{6\%} - 1)$ . Thus  $\lim_{T \rightarrow \infty} b$  is positive if and only if  $i > 6\%$ , and negative if and only if  $i < 6\%$ .

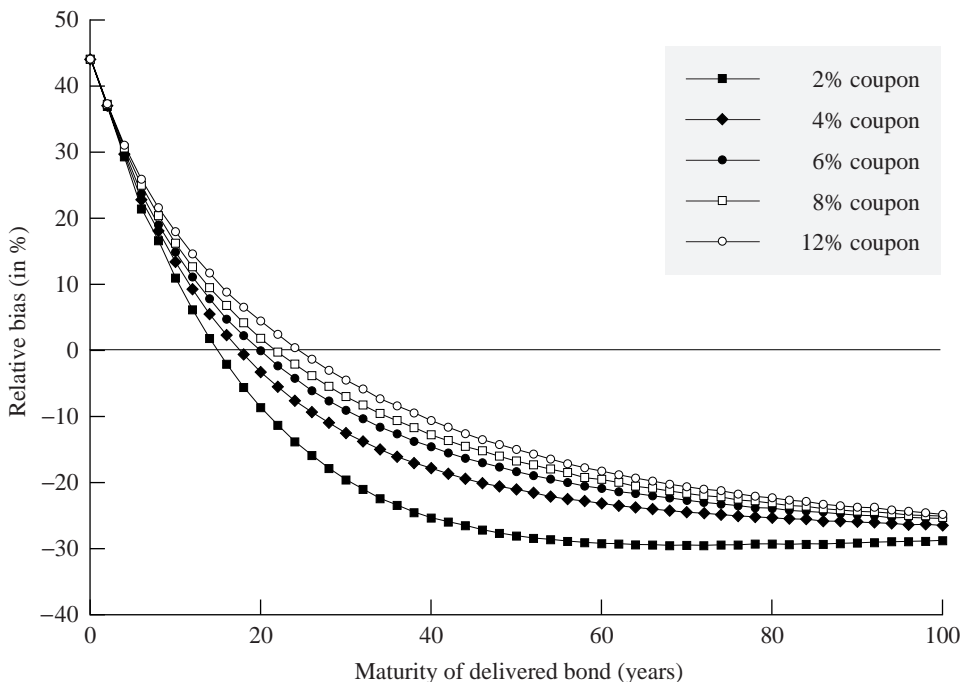
For instance, if  $i = 3\%$ , the limit of the bias is  $-27.6\%$ ; if  $i = 9\%$ , it is equal to  $8.6\%$ . This limit does not depend on the coupon rate either. It implies that the relative bias of each bond in (maturity, bias) space will be represented by a series of curves that have nodes (intersections) at the origin and at infinity. It is important to realize that, for  $i > 6\%$ , the ordi-



**Figure 5.4**  
Relative bias of conversion factor in T-bond contract, with  $i = 9\%$ .

nates of these nodes are respectively *negative* (for  $T \rightarrow 0$ ) and *positive* (for  $T \rightarrow \infty$ ). The function  $b$  being continuous in  $T$ , we can apply Rolle's theorem and conclude that  $b$  will *at least once* be equal to zero (its curve will cross the abscissa at least once). This is confirmed by figure 5.4: for any coupon  $c$ , and for  $i = 9\%$ , the relative bias is zero for one value of  $T$ , which depends upon  $c$ . The same is true when  $i < 6\%$ : the ordinates of the nodes are this time respectively positive (when  $T \rightarrow 0$ ) and negative (when  $T \rightarrow \infty$ ). Therefore,  $b$  will cross the abscissa at least once, as is confirmed in figure 5.5, corresponding to  $i = 3\%$ . For each deliverable bond, there will be one maturity for which the CBOT conversion factor does not entail any bias, even if  $i$  is different from  $6\%$ .

Figure 5.4 represents relative biases for bonds with semi-annual coupons of 2, 4, 6, 8, and 12%, when the rate of interest is equal to 9%. For all bonds with a coupon lower than the rate of interest (for all below-par bonds), the relative bias always first increases, goes through a maximum, and then tends toward the common limit mentioned above. For all bonds



**Figure 5.5**  
Relative bias of conversion factor in T-bond contract, with  $i = 5\%$ .

above par (in this case, the 8% and 12% coupon bonds), the movement toward the limit is monotonous; the bias constantly increases along a curve that is flattening out.

Do these curves on figure 5.4 look familiar? They may, because this kind of behavior is fundamentally that of a duration. Indeed, we will now show that the relative bias of the conversion factor in the Chicago Board of Trade T-bond contract (or any contract of that type) is an affine transformation of the duration of the deliverable bond; it is a *negative* transformation when  $i < 6\%$ , and it is a *positive* one when  $i > 6\%$ . (From a geometric point of view, a negative affine transformation means that the graph of the relative bias is essentially the graph of the duration of the deliverable bond, upside down; the scaling is modified both linearly and by an additive constant.)

First, let us denote the *modified* duration of a bond with value  $B_{c,T}(i_0)$ , as  $D_{c,T}(i_0)$ . From the properties of modified duration, we know that it is

equal, *in linear approximation*, to the relative change in the bond's value per unit change in the rate of interest.

$$\frac{B_{c,T}(i) - B_{c,T}(i_0)}{B_{c,T}(i_0)} \approx \frac{dB}{B} = -D_{c,T}(i_0) \cdot (i - i_0) \quad (5)$$

Therefore, changing  $i - i_0$  into  $i_0 - i$ , the ratio of two values of the bond, corresponding to two different rates of interest, can be written as

$$\frac{B_{c,T}(i)}{B_{c,T}(i_0)} \approx 1 + D_{c,T}(i_0) \cdot (i_0 - i) \quad (6)$$

Let us now come back to the CBOT conversion factor. Its relative bias is equal to

$$b = \frac{\frac{B_{c\%,T}(6\%)}{B_{6\%,20}(6\%)}}{\frac{B_{c\%,T}(i)}{B_{6\%,20}(i)}} - 1 \quad (7)$$

which can be written equivalently as

$$b = \frac{\frac{B_{c,T}(6\%)}{B_{c,T}(i)}}{\frac{B_{6\%,20}(6\%)}{B_{6\%,20}(i)}} - 1 \approx \frac{1 + D_{c,T}(i) \cdot (i - 6\%)}{1 + D_{6\%,20}(i) \cdot (i - 6\%)} - 1 \quad (8)$$

or, more compactly, as

$$b = mD_{c,T}(i) + n$$

where

$$m \equiv \frac{(i - 6\%)}{1 + D_{6\%,20}(i) \cdot (i - 6\%)}$$

and

$$n \equiv \frac{1}{1 + D_{6\%,20}(i) \cdot (i - 6\%)} - 1$$

We can check that the bias is zero whenever  $i = 6\%$ . Also, we can easily show that for any relevant value of  $i$ , the denominator ( $m$ ) cannot become

negative. Therefore, the relative bias essentially behaves like an affine transformation of the duration of the deliverable bond: a *positive* one ( $m > 0$ ) whenever  $i > 6\%$  and a *negative* one ( $m < 0$ ) when  $i < 6\%$ . This explains the shapes of the bias that correspond to an interest rate either above 6% (9%; figure 5.4) or below 6% (3%; figure 5.5).<sup>8</sup>

There are many lessons to be learned from these diagrams. Consider first the case  $i > 6\%$ . Notice that the conversion factor unequivocally favors bonds with high duration. This stems from the fact that, as we have just shown, the bias in this case is an affine transformation of the duration of the delivered bond. It is crucial to observe here that contrary to the generally held view, this does *not* mean that the bias favors high *maturities* bonds. Indeed, it is well known that for under-par bonds (bonds such that their coupon is below the rate of interest), duration goes through a maximum (whose abscissa cannot be given in analytic form, but which may very easily be computed). It then converges toward the common limit equal to  $\frac{1}{i} + \alpha$ , where  $\alpha$  is the fraction of a year remaining until payment of next coupon. (Here  $\alpha$  is  $\frac{1}{2}$ .) Consider, for example, a very low coupon bond (2%), with  $i = 9\%$ . The relative bias for such a bond will go through a *maximum* for a maturity of 36 years (rounded to the full year), and we can very well surmise that the profit of delivering that bond will also go through a maximum. This is what we will show in the next section.

If we now turn to the case where  $i$  is below 6% ( $i = 3\%$ ; figure 5.5), the opposite is true: the bias tends to favor high-coupon, low-duration bonds because the bias behaves like the *negative* of a duration. This implies that, in a symmetrical way to what we have just seen for  $i > 6\%$ , low-coupon bonds will see their bias go through a minimum before tending toward their common limit. This implies that we will have to be careful about all these factors when analyzing the various profits that will correspond to all deliverable bonds.

### 5.3.5 The profit of delivering and its biases

A bias in the calculation of the amount to be paid by the long party in the delivery process may well entail some bias in the profit earned by the short party. We will first recall what this profit is equal to. We will then

<sup>8</sup> It would be possible to increase the exactness of the relationship between the relative bias and the duration measure by adding (positive) convexity terms both in the numerator and in the denominator of the fraction in the right-hand side of (8). The conclusions would of course remain the same.



show the considerable biases that are introduced by the conversion factor, as well as the kind of surprises they carry with them.

The profit of the short is basically equal to the amount of the invoice paid by the long to the short, less the cost of acquiring the deliverable bond. Referring to a  $c$ -coupon,  $T$ -maturity deliverable bond, and again keeping the hypothesis of a flat structure rate  $i$ , the profit  $\Pi_{c,T}(i)$  of delivering the bond is a function of the three variables  $c$ ,  $T$ , and  $i$ , equal to

$$\Pi_{c,T}(i) = F \cdot CF_{c,T} - B_{c,T}(i) \quad (9)$$

where  $F$  denotes the futures price at maturity.

It is useful to write out this function. Remember that  $F$  and  $CF$  are given by

$$F = B_{6\%,20}(i)$$

and

$$CF = \frac{B_{c,T}(6\%)}{100}$$

respectively.

Thus we have

$$\Pi_{c,T}(i) = B_{6\%,20}(i) \cdot \frac{B_{c,T}(6\%)}{100} - B_{c,T}(i) \quad (10)$$

Replacing the three functions on the right-hand side of (10) by their expressions, we have

$$\begin{aligned} \Pi_{c,T}(i) = & \left[ \frac{6}{i} \left( 1 - \frac{1}{(1+i/2)^{40}} \right) + \frac{100}{(1+i/2)^{40}} \right] \\ & \cdot \left[ \frac{c/100}{0.06} \left( 1 - \frac{1}{1.03^{2T}} \right) + \frac{1}{1.03^{2T}} \right] \\ & - \left[ \frac{c}{i} \left( 1 - \frac{1}{(1+i/2)^{2T}} \right) + \frac{100}{(1+i/2)^{2T}} \right] \end{aligned} \quad (11)$$

This function of  $i$ ,  $c$ , and  $T$  has some important properties, which can best be described through figures 5.6 and 5.7.

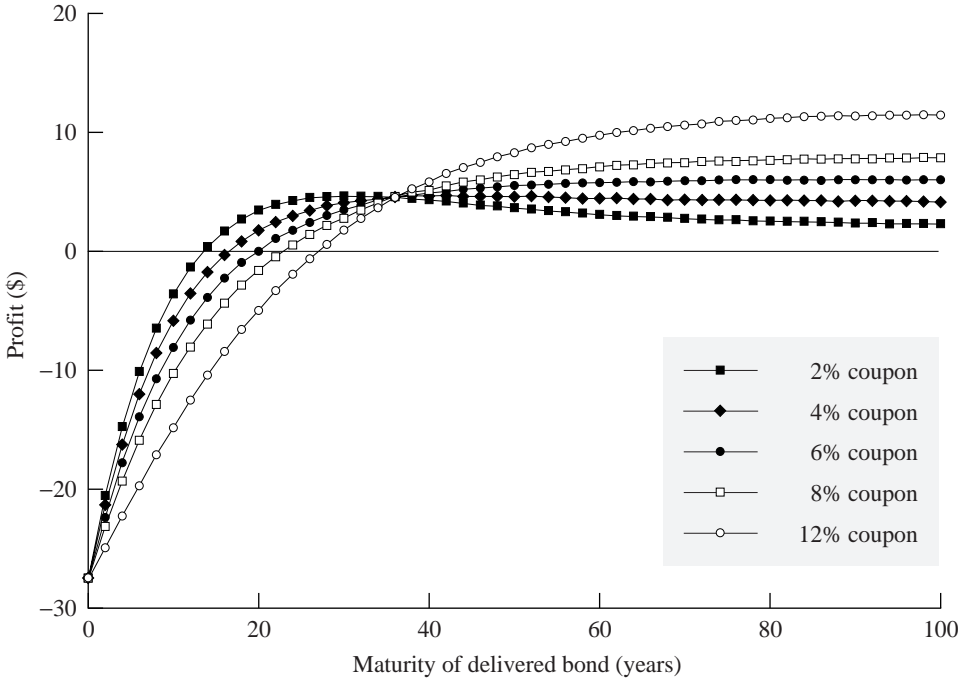
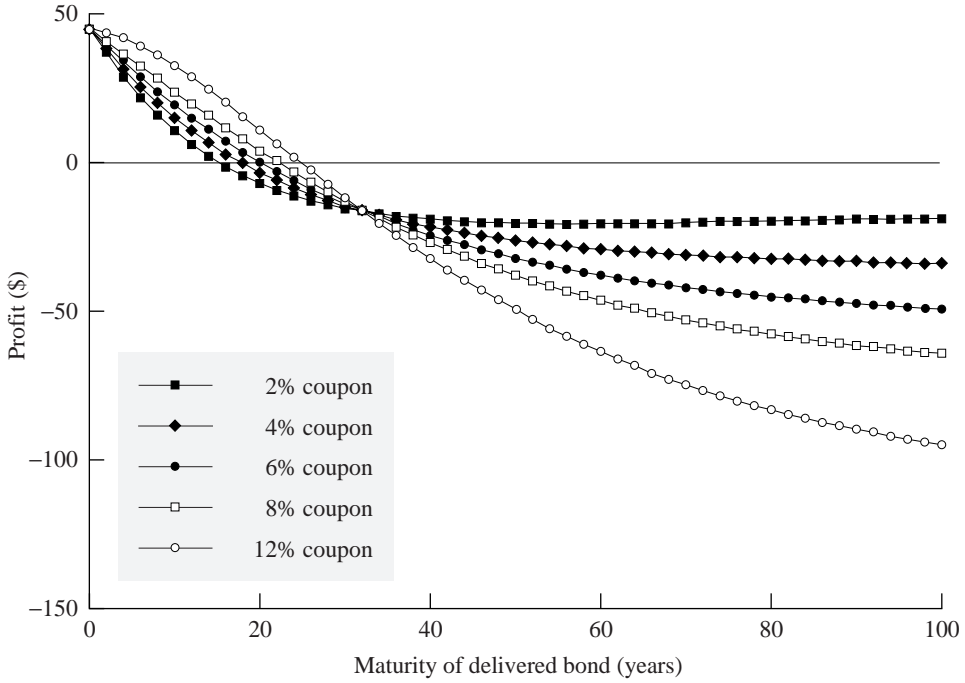


Figure 5.6

Profit from delivery with T-bond futures;  $F(T) = 75.93$ ;  $i = 11\%$ . Note that the relative bias of the conversion factor for a 6% coupon, 20-year maturity bond is zero, as it should be.

The first property to notice is that there is a whole set of values  $(c, T)$  that, for given  $i$ , lead to a zero profit. Those are the values we defined before as entailing a zero relative bias in the conversion factor. Suppose, indeed, that  $c$  and  $T$  are such that the relative bias is zero. This means that the conversion factor is equal to the true conversion factor, which we had found to be equal to the number of  $(6\%, 20)$  bonds exchangeable for one  $(c\%, T)$  bond, that is, the ratio of the *price* of a  $(c\%, T)$  bond divided by the *price* of a  $(6\%, 20)$ , both evaluated at rate  $i$  ( $TCF = B_{c\%, T}(i) / B_{6\%, 20}(i)$ ). Let us now replace in the profit function  $[\Pi_{c, T}(i)$  in equation (9)] the conversion factor  $CF$  by the true conversion factor. We thus get a profit equal to zero, as foreseen:

$$\begin{aligned}
 \Pi_{c, T}(i) &= F(T) \cdot TCF - B_{c, T}(i) \\
 &= B_{6\%, 20}(i) \cdot \frac{B_{c, T}(i)}{B_{6\%, 20}(i)} - B_{c, T}(i) = 0
 \end{aligned} \tag{12}$$



**Figure 5.7**

Profit from delivery with T-bond futures;  $F(T) = 137.65$ ;  $i = 5\%$ . Note that, as in the preceding case where  $i$  was equal to 9%, the relative bias of the conversion factor when remitting a 6%, 20-year bond is zero, as it should be.

The second important property to notice is that not all profit functions are monotonously increasing or decreasing; this is partly due to the non-monotonicity of the conversion factor, which, as we saw earlier, is basically an affine transformation of duration, itself a nonmonotonic function for all bonds with nonzero coupons below the rate of interest.

Consider first the case  $i = 11\%$  ( $i > 6\%$ ). All profits for coupons equal to or higher than 11% are always increasing (this is the case in figure 5.6 for  $c = 12$ ). On the other hand, all profits for coupons below 11% are increasing in a first stage, going through a maximum, and then decreasing. For any coupon, each profit tends toward its own limit, given by

$$\lim_{T \rightarrow \infty} \Pi_{c,T}(i) = c \left[ \frac{B_{6\%,20}(i)}{6} - \frac{1}{i} \right] \quad (13)$$

which (contrary to the corresponding limit for the relative bias) depends on the coupon. For example, if  $c = \$12$ , this limit is \$4.81.

We now turn toward the case  $i = 3\%$  ( $i < 6\%$ ). The profit functions are always decreasing for coupons *below* 5%; for coupons *above* 5%, they first decrease, reach a minimum value, and then increase. As before, for any coupon, each profit tends toward its own limit, given by (13).

It is crucial to recognize that all profit curves intersect at two points (nodes). The first node is fixed at  $T = 0$ ; the second one, for usual values of  $i$ , is not very sensitive to  $i$  and lies between 31.226 years for  $i = 5\%$  and 34.793 years for  $i = 11\%$  (see figures 5.7 and 5.6 respectively).

The general shape of these functions and the second node in the profit curves implies *possible reversals* in the ranking of the cheapest (or most profitable) to deliver bonds. Consider first an environment of high interest rates. Until now, as we noted in our introduction, it was believed that in such a case the most profitable bond to deliver was the one with the lowest coupon and the longest maturity. This is clearly not always true. Consider, for instance, a low coupon bond ( $c = 2\%$ ). The profit from delivering it, if  $i = 11\%$ , goes through a maximum when its maturity is (in full years) 24 years and is equal to \$3.2058. This is larger than the profit of remitting the same coupon bond with a 30-year maturity, which would be \$2.9206 (a relative difference of 9.8%). From what we said earlier, we might guess that the 24-year bond has a *higher* duration than the 30-year bond. (This is indeed true:  $D_{2\%,24}(11\%) = 12.07$  years, and  $D_{2\%,30}(11\%) = 11.75$  years.) So if a rule of thumb is to be used, we should always rely on *duration*, not on *maturity*.

The second observation we make is quite surprising and is due to the existence and characteristics of the second node in the profit functions. We can see that should the maturity of deliverable bonds be beyond the second node, *profit reversals would be the rule whatever the rates of interest* (with the exception, of course, of  $i = 6\%$ ). For any maturity beyond the second node, the bond that is cheapest to deliver would be the one with the highest coupon when  $i > 6\%$  and the one with the lowest coupon when  $i < 6\%$ . This property would become crucial if some foreign treasury issued bonds with maturities exceeding 30 years and if those bonds were deliverable in a futures market. This would be of particular importance if interest rates were above 6% (see figure 5.6, where a high coupon bond's profits will dominate the profits that can be made from delivering any other bond).

Naturally, this analysis applies when we consider a flat term structure. For completeness, it should be extended to nonflat structures. This would be a formidable task—but we would certainly not be surprised to discover results similar to those obtained with flat structures.

### Appendix 5.A.1 The calculation of the conversion factor in practice

The conversion factor used by the Chicago Board of Trade is the theoretical value of a \$1 nominal,  $c\%$  coupon,  $T$ -maturity bond when its yield to maturity is 6%. We will now show how this conversion factor is calculated.

First, the remaining maturity in months above a given number of years for the deliverable bond is rounded down to the nearest three-month period. Two cases may arise:

1. The remaining maturity is an integer number of years  $T$  plus a number of months strictly between 0 and 3, or between 6 and 9. In this case, the retained maturity (in years) for the calculation will be  $T$  or  $T + \frac{1}{2}$ , respectively; the number of six-month periods will then be  $n = 2T$  or  $n = 2T + 1$ , respectively. Formula (14) will then apply for a \$1 bond maturing in number  $n$  of six-month periods, which will be equal to either  $2T$  or  $2T + 1$ . In this case the conversion factor ( $CF$ ) will be

$$CF = \sum_{t=1}^n \frac{c}{2} (1.03)^{-t} + (1.03)^{-n}, \quad n = 2T \quad \text{or} \quad 2T + 1 \quad (14)$$

We then have

$$CF = \frac{c/2}{0.03} \left( 1 - \frac{1}{1.03^n} \right) + 1.03^{-n}, \quad n = 2T \quad \text{or} \quad 2T + 1 \quad (15)$$

For example, if a 9% yearly coupon rate bond has a remaining maturity equal to 24 years and 8 months, rounding down to the nearest three-month period gives 24 years and 6 months, which is equivalent to 49 six-month periods. Hence  $n = 49$  in formulas (14) and (15), and the conversion factor is equal to  $CF = 1.3825$ , as can be verified from the table published by the Chicago Board of Trade (see table 5.A.1).

2. The remaining maturity is an integer number of years  $T$  plus a number of months between 3 and 6, or between 9 and 12. In such circumstances,

**Table 5.A.1**

**Extract from the conversion table used by the Chicago Board of Trade for the Treasury Bond Futures contract (as of March 1, 2000)**

Coupon	Years—Months							
	25—9	25—6	25—3	25—0	24—9	24—6	24—3	24—0
8	1.2604	1.2595	1.2583	1.2573	1.2560	1.2550	1.2537	1.2527
$8\frac{1}{8}$	1.2767	1.2757	1.2744	1.2734	1.2720	1.2710	1.2696	1.2685
$8\frac{1}{4}$	1.2930	1.2920	1.2906	1.2895	1.2880	1.2869	1.2854	1.2843
$8\frac{3}{8}$	1.3093	1.3082	1.3067	1.3055	1.3040	1.3028	1.3013	1.3000
$8\frac{1}{2}$	1.3256	1.3244	1.3229	1.3216	1.3200	1.3188	1.3172	1.3158
$8\frac{5}{8}$	1.3419	1.3406	1.3390	1.3377	1.3361	1.3347	1.3330	1.3316
$8\frac{3}{4}$	1.3582	1.3568	1.3552	1.3538	1.3521	1.3506	1.3489	1.3474
$8\frac{7}{8}$	1.3744	1.3730	1.3713	1.3699	1.3681	1.3666	1.3647	1.3632
9	1.3907	1.3893	1.3875	1.3859	1.3841	1.3825	1.3806	1.3790
$9\frac{1}{8}$	1.4070	1.4055	1.4036	1.4020	1.4001	1.3985	1.3965	1.3948
$9\frac{1}{4}$	1.4233	1.4217	1.4198	1.4181	1.4161	1.4144	1.4123	1.4106
$9\frac{3}{8}$	1.4396	1.4379	1.4359	1.4342	1.4321	1.4303	1.4282	1.4264
$9\frac{1}{2}$	1.4559	1.4541	1.4520	1.4503	1.4481	1.4463	1.4441	1.4422
$9\frac{5}{8}$	1.4722	1.4704	1.4682	1.4664	1.4641	1.4622	1.4599	1.4580
$9\frac{3}{4}$	1.4884	1.4866	1.4843	1.4824	1.4801	1.4782	1.4758	1.4738
$9\frac{7}{8}$	1.5047	1.5028	1.5005	1.4985	1.4961	1.4941	1.4917	1.4895
10	1.5210	1.5190	1.5166	1.5146	1.5121	1.5100	1.5075	1.5053
$10\frac{1}{8}$	1.5373	1.5352	1.5328	1.5307	1.5282	1.5260	1.5234	1.5211
$10\frac{1}{4}$	1.5536	1.5515	1.5489	1.5468	1.5442	1.5419	1.5392	1.5369
$10\frac{3}{8}$	1.5699	1.5677	1.5651	1.5628	1.5602	1.5578	1.5551	1.5527
$10\frac{1}{2}$	1.5861	1.5839	1.5812	1.5789	1.5762	1.5738	1.5710	1.5685
$10\frac{5}{8}$	1.6024	1.6001	1.5974	1.5950	1.5922	1.5897	1.5868	1.5843
$10\frac{3}{4}$	1.6187	1.6163	1.6135	1.6111	1.6082	1.6057	1.6027	1.6001
$10\frac{7}{8}$	1.6350	1.6326	1.6297	1.6272	1.6242	1.6216	1.6186	1.6159
11	1.6513	1.6488	1.6458	1.6432	1.6402	1.6375	1.6344	1.6317

Source: Chicago Board of Trade. Website: <http://www.cbot.com/ourproducts/financial/US6p.html>

the Chicago Board of Trade considers that the maturity (in years) is either  $T$  plus three months, or  $T + \frac{1}{2}$  plus three months; hence the bond is considered to have  $n = 2T$  six-month periods *plus* three months, or  $n = 2T + 1$  six-month periods *plus* three months respectively. The calculation for the \$1 face value bond then amounts to performing the following four evaluating operations:

- evaluate the bond at a three-month horizon from now;
- add the six-month coupon to be received at that time;
- discount that sum to its present value;
- subtract (pay off) half a six-month coupon, considered as accrued interest on the bond (since it does not belong to the bond's owner).

These four operations are summarized in the following formula:

$$CF = \frac{\frac{c/2}{0.03} \left(1 - \frac{1}{1.03^n}\right) + 1.03^{-n} + \frac{c}{2}}{1.03^{1/2}} - \frac{c}{4} \quad (16)$$

As an example, consider a 9% yearly coupon rate bond with an (observed) remaining maturity of 24 years and five months, which is then rounded down to 24 years and three months. Hence, the number of six-month periods is 48, and the retained maturity is 48 periods plus three months. Formula (16) applies, yielding a conversion factor  $CF = 1.3806$ , as shown in table 5.A.1.

#### Main formulas

- Futures price:

$$F(0, T) = S(0) + CCFC - CIFV$$

where

$$CCFV \equiv \text{carry costs in future value}$$

$$CIFV \equiv \text{carry income in future value}$$

- Conversion factor:

$$CF_{c,T} = \frac{B_{c,T}(6\%)}{100}$$

- “True” or “theoretical” conversion price:

$$TCF_{c,T} = \frac{B_{c,T}(i)}{B_{6\%,20}(i)}$$

- Relative bias in the conversion factor:

$$b = \frac{CF - TCF}{TCF} = \frac{\frac{B_{c\%,T}(6\%)}{B_{6\%,20}(6\%)}}{\frac{B_{c\%,T}(i)}{B_{6\%,20}(i)}} - 1$$

$$\approx mD_{c,T}(i) + n$$

where

$$m = \frac{i - 6\%}{1 + D_{6\%,20}(i) \cdot (i - 6\%)}$$

and

$$n = \frac{1}{1 + D_{6\%,20}(i) \cdot (i - 6\%)} - 1$$

## Questions

The following exercises should be considered as projects, to be done perhaps with a spreadsheet.

- 5.1. Determine the relative bias in the conversion factor and the profit from delivering T-bonds with coupons 2%, 4%, 6%, 8%, and 12% when the rate of interest is at 10%. Draw a chart of your results, similar to figures 5.4 and 5.6.
- 5.2. Determine the relative bias in the conversion and the profit of delivering the same bonds as in (5.1), when the interest rate is 5%. Draw charts of your results, similar to figures 5.5 and 5.7.



# 6

## IMMUNIZATION: A FIRST APPROACH

ALONSO. Irreparable is the loss; and patience  
Says it is past her cure.

*The Tempest*

GHOST OF LADY ANNE. Thou quiet soul, sleep thou a quiet sleep  
Dream of success and happy victory!

*Richard III*

### Chapter outline

<b>6.1</b>	<b>An intuitive approach</b>	<b>145</b>
<b>6.2</b>	<b>A first proof of the immunization theorem</b>	<b>148</b>
6.2.1	A first-order condition	149
6.2.2	A second-order condition	150

### Overview

Holding bonds may be very rewarding in the short run if interest rates go down; but if they go up, the owner of a bond portfolio may suffer substantial losses. This is the short-run picture. However, the opposite conclusions apply if we consider long horizons: a decrease in interest rates results in a decrease in the total return on a bond, because the value of the bond, close to maturity, will have come back close to its par value, and the coupons will have been reinvested at a lower rate. On the other hand, if interest rates go up, the value of the bond may not be affected in the long run (because its value will have gone back to its par value), but the coupons will have been reinvested at a higher rate. A natural question then arises: is there an “intermediate horizon” for any bond, between the “short term” and the “long term,” such that any move of the interest rate immediately after the purchase of the bond will be of no consequence at all for the investor’s performance? This intermediate horizon does exist, and we will show in this chapter that it is equal to the duration of the bond.

If an investor has a horizon equal to the duration of the bond, his investment will be protected (or “immunized”) against parallel shifts in the structure of interest rates. Of course, the second question that comes to mind is: what if the investor’s horizon differs from the duration of the bonds he is willing to buy because they seem favorably valued and they are reasonably liquid? The answer is to construct a portfolio of bonds such that the duration of the portfolio will be equal to the investor’s

horizon. The duration of the portfolio is simply the weighted average of the durations of the individual bonds, and the weights are the proportions of each bond in the portfolio.

Immunization is then defined as the set of bond management procedures that aim at protecting the investor against changes in interest rates. They amount essentially to making the duration of the investor's portfolio equal to his horizon. Immunization is definitely a dynamic set of procedures, because the passage of time and changes in interest rates will modify the portfolio's duration by an amount that will not necessarily correspond to the steady and natural decline of the investor's horizon. On the one hand, if interest rates do not change, the simple passage of a year, for instance, will reduce duration of the portfolio by less than one year; the money manager will have to change the composition of the portfolio so that duration is reduced by a whole year. But, changes in interest rates will also modify the portfolio's duration; as we saw in chapter 3, a decrease in interest rates increases duration, and higher interest rates entail lower duration. Hence, a decrease of interest rates from one year to another will reinforce the necessity of readjusting the portfolio so that the portfolio's duration matches the investor's horizon.

There are a few caveats: the immunization procedure works well only if all interest rates shift together (in other words, only if the interest rate structure undergoes a parallel shift). And, as usual, investing in a bond portfolio and rebalancing it for duration adjustment to the investor's needs requires that the market for those bonds be large and highly liquid.

At the end of chapter 2, we encountered a remarkable property: in the case of either a drop or a rise in interest rates, when the horizon was properly chosen, the horizon rate of return for the bond's owner was about the same as if interest rates had not moved. In chapter 2, we hinted that this horizon was the duration of the bond; this is why we have devoted a chapter to defining this concept and describing its fundamental properties. We will now demonstrate rigorously why duration does indeed, in some circumstances that still need to be defined, protect the investor against moves in rates of interest. We will call "immunization" the process by which an investor can protect himself against interest rate changes by suitably choosing a bond or a portfolio of bonds such that its duration is equal to his horizon.

We will proceed as follows. In the first section, we will introduce the "highly probable" existence of such an immunizing horizon, thus rein-

forcing the first insight we had earlier. In the second section, we will demonstrate rigorously the immunization theorem.

## 6.1 An intuitive approach

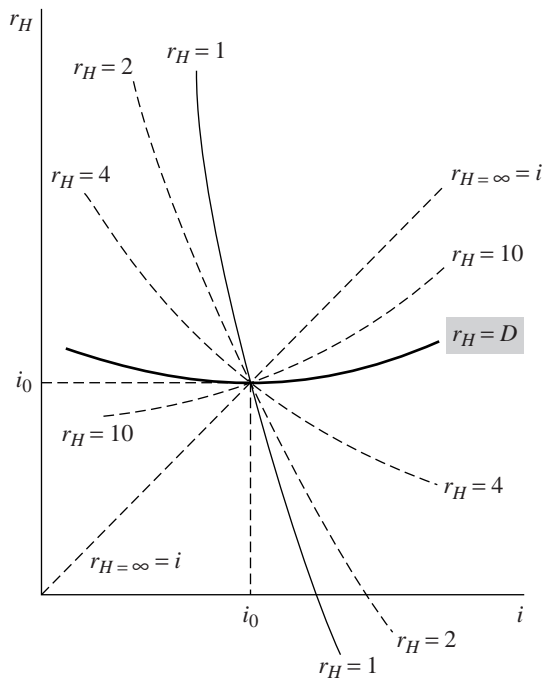
Suppose that today we buy a 10-year maturity bond at par and that the structure of spot rates is flat and equal to 9%. Then tomorrow the interest rate structure drops to 7%, and we do not see any reason why it would move from this level in the coming years. Is this good or bad news? From what we know, we can say that everything depends on the horizon we have as investors. If our horizon is short, it is good news, because we will be able to exploit the huge capital increase generated by the drop in interest rates. At the same time, we will not worry about coupon reinvestment at a low rate, because this part of the return will be negligible in the total return. On the other hand, if we have a long horizon, we will not benefit from a major price appreciation (because the bond will be well on its way back to par), and we will suffer from the reinvestment of coupons for a long time at lower rates. So our conclusion is that with a short horizon it is good news, but with a long horizon it is bad news, and for some intermediate horizon (duration of course, what else?) it may be no news, because the change in interest rates will have very little impact, if any, on the investor's performance (the capital gain will have nearly exactly compensated for the loss in coupon reinvestment).

It is time now to recall our formula of the horizon rate of return. We will immediately see why the problem we are addressing is not trivial. From chapter 3, equation (5), we have

$$r_H = \left( \frac{B(i)}{B_0} \right)^{1/H} (1 + i) - 1$$

This expression depends on two variables,  $i$  and  $H$ . It is essentially the product of a *decreasing* function of  $i$  ( $B(i)$ ) and an *increasing* one ( $1 + i$ ). Note that the decreasing function of  $i$  is taken to the power  $1/H$ . So the resulting function  $r_H$  is a rather complicated function of  $i$ , with  $H$  playing the role of a parameter inversely weighting the decreasing function of  $i$ .

In order to have a clear picture of the possible outcomes open to the investor, we will first consider all different possible scenarios for an investor with a short horizon, and all possible moves of the interest rate. We



**Figure 6.1**

The horizon rate of return is a decreasing function of  $i$  when the horizon is short and an increasing one for long horizons. For a horizon equal to duration, the horizon rate of return first decreases, goes through a minimum for  $i = i_0$ , and then increases by  $i$ .

will then progressively lengthen the investor's horizon and see what happens.

Consider first a short horizon (one year or less). We already know that a drop in interest rates is good news for our investor; symmetrically, it is bad news if the rates increase: he will suffer from a severe capital loss and will not have the opportunity to invest his coupons at higher rates, since he has such a short horizon. This one-year horizon rate of return is depicted in figure 6.1.

Suppose now that our investor's horizon is a little longer (for instance two years). How will his two-year horizon rate of return look if interest rates drop? It will be lower than previously (in the one-year horizon case), for two reasons:

1. The capital appreciation earned from the bond's price rise will be smaller than before, since the price will be closer to par. Indeed, after one year the bond will already have started going back to par; but after two years, it will be farther down in this process. (From chapter 2, we know that each year the bond drops by  $\frac{\Delta B}{B} = i_1 - \frac{c}{B} [< 0 \text{ in our case}]$ , where  $i_1$  is the new value of the rate of interest immediately after its change from  $i_0$ .)
2. The investor now has the worry (not to be taken too seriously, however) of reinvesting one coupon at a lower rate than in the one-year horizon case.

We can be sure then that for these two reasons the two-year horizon return will be below the one-year horizon return. Similarly, if interest rates rise, the two-year horizon return will be *above* the one-year horizon return for these two symmetrical reasons:

1. The capital loss will be less severe with the two-year horizon than with the one-year, since the bond's value will have come back closer to par after two years than after one year.
2. The two-year investor will be able to benefit from coupon reinvestment at a higher rate, which was not the case with the one-year investor.

Therefore, the curve depicting the two-year horizon rate will turn counterclockwise around point  $(i_0, i_0)$  compared to the one-year horizon rate of return curve. More generally, we can state that *all* horizon return curves will go through the fixed point  $(i_0, i_0)$ , where  $i_0$  represents the initial rate of interest, before any change has taken place; in other words, whatever the horizon, the rate of return will always be  $i_0$  if  $i$  does not move away from this value. Why is this so? The answer is not quite obvious and deserves some attention. When the bond was bought, the interest rate level was at  $i_0$ . Therefore, the price was  $B(i_0) = B_0$ . If the rate of interest does not move, the future value of the bond, at any horizon  $H$ , is  $B_0(1 + i_0)^H$ , and the rate of return  $r_H$  transforming  $B_0$  into  $B(1 + i_0)^H$  is such that  $B_0(1 + r_H)^H = B_0(1 + i_0)^H$ ; it is therefore  $i_0$ .<sup>1</sup> Hence *all* curves depicting the horizon rate of return as a function of  $i$  will go through point  $(i_0, i_0)$  in the  $(r_H, i)$  space, whatever the horizon, be it short or long.

<sup>1</sup>For an alternate demonstration of this property, see question 6.1 at the end of this chapter.

Now that we have seen that the curve depicting the slightly longer horizon rate of return would turn counterclockwise around point  $(i_0, i_0)$ , we may wonder what the limit of this process would be if the horizon becomes very large. If  $H$  tends toward infinity, we can see immediately that the horizon rate of return,  $r_H = \left[\frac{B(i)}{B_0}\right]^{1/H} (1+i) - 1$  tends toward  $i$ . So the straight line at  $45^\circ$  in  $(r_H, i)$  space is this limit. Consider now a long horizon, perhaps 10 years—the maturity of the bond. If the interest rate is lower than  $i_0$ , the bond will be reimbursed at par, and all coupons will be invested at a rate lower than  $i_0$ . Obviously, the horizon rate of return will be lower than  $i_0$ . If the interest rate is higher than  $i_0$ , coupons will be reinvested at a higher rate than  $i_0$ , and the horizon rate of return will be higher than  $i_0$ , increasing also by  $i$ .

We sense, however, that if the horizon is somewhat shorter (nine years, for instance), then the curve will rotate clockwise this time, for similar reasons to those we discussed earlier. If we consider the case  $i < i_0$ , then the shorter horizon means the bond will still be above par, and the coupons will suffer reinvestment at a lower rate for a shorter time. For these two reasons the return will be higher. If, on the contrary,  $i > i_0$ , the bond will still be somewhat below par, and the coupons will be reinvested at the higher rate for a shorter time. Again, these two factors will entail a lower rate of return.

We now have a complete picture of the behavior of the rate of return for various horizons. We understand that the curve describing these rates, as a function of the structure of interest rates, will definitely be moving counterclockwise around the fixed point  $(i_0, i_0)$  when the horizon increases, and we can well surmise that there might exist a horizon such that the curve will be either flat or close to flat. That horizon does indeed exist, and is none other than duration. Our purpose in section 6.2 will be to demonstrate this rigorously.

## 6.2 A first proof of the immunization theorem

We will now prove the following theorem: if a bond has duration  $D$ , and if the term structure of the interest rates is horizontal, then the owner of the bond is immunized against any parallel change in this structure if his horizon  $H$  is equal to duration  $D$ . This theorem can be generalized to the case of *any* term structure of interest rates, if the change in this structure is a parallel one (we will prove this generalization in chapter 10).

For the time being let us prove the simpler theorem above. What we want to show is that there exists a horizon  $H$  such that the rate of return for such a horizon goes through a minimum at point  $i_0$ .

### 6.2.1 A first-order condition

Let us recall that the rate of return at horizon  $H$  is such that

$$B_0(1 + r_H)^H = B(i)(1 + i)^H = F_H \quad (1)$$

where  $F_H$  is the future value of the bond, or the bond portfolio, and therefore  $r_H$  is equal to

$$r_H = \left( \frac{F_H}{B_0} \right)^{1/H} - 1 \quad (2)$$

Minimizing  $r_H$  is equivalent to minimizing any positive transformation of it, and therefore is equivalent to minimizing  $F_H$ . It suffices then to evaluate the derivative of  $F_H$  with respect to  $i$ . We have

$$\frac{dF_H}{di} = \frac{d}{di} [B(i)(1 + i)^H] = B'(i)(1 + i)^H + HB(i)(1 + i)^{H-1} = 0 \quad (3)$$

Multiplying (3) by  $(1 + i)^{1-H}$ , we get

$$B'(i)(1 + i) + HB(i) = 0 \quad (4)$$

We want  $\frac{dF_H}{di} = 0$  to hold at point  $i = i_0$ . Therefore, we must have

$$B'(i_0)(1 + i_0) + HB(i_0) = 0 \quad (5)$$

and

$$H = -\frac{(1 + i_0)}{B(i_0)} B'(i_0) \quad (6)$$

We have just demonstrated that the required horizon must be equal to  $-\frac{1+i_0}{B(i_0)} B'(i_0)$ ; one of the central properties of duration  $D$ , evaluated at  $i_0$ , is that it is precisely equal to this magnitude (see equation (3) in chapter 4). Therefore, the horizon must be equal to duration at the initial rate of interest (or return) for  $F_H$  (and  $r_H$ ) to run through a minimum. The size of

this minimum is  $r_{H=D} = i_0$ , because the minimum occurs at  $i = i_0$ , and for that value of  $i$ ,  $r_H = i_0$ .

### 6.2.2 A second-order condition

We have now a first-order condition for  $r_H$  to go through a minimum. In order to ensure a second-order condition for a global minimum at  $i = i_0$ , it is relatively easy to prove that  $\ln F_H(i)$  has a positive second derivative. This is what we will show now.

We have

$$\ln F_H = \ln B(i) + H \ln(1 + i) \quad (7)$$

$$\begin{aligned} \frac{d \ln F_H}{di} &= \frac{d \ln B(i)}{di} + \frac{H}{1+i} = \frac{1}{B(i)} \frac{dB(i)}{di} + \frac{H}{1+i} \\ &= \frac{-D}{(1+i)} + \frac{H}{1+i} = \frac{1}{1+i} (-D + H) \end{aligned} \quad (8)$$

Equating this expression to zero leads to  $D = H$  (as before). The second derivative of  $\ln F_H$  is

$$\frac{d^2 \ln F_H}{di^2} = \frac{1}{(1+i)^2} \left[ -\frac{dD}{di} (1+i) + D - H \right]$$

Since  $D = H$ , we have

$$\frac{d^2 \ln F_H}{di^2} = -\frac{1}{1+i} \frac{dD}{di} \quad (9)$$

We had shown previously that  $\frac{dD}{di} = -(1+i)^{-1}S$ , where  $S$  is the variance of the bond.

We finally get

$$\frac{d^2 \ln F_H}{di^2} = -\frac{1}{1+i} [-(1+i)^{-1}S] = \frac{S}{(1+i)^2}$$

and this expression is always positive, whatever the initial value  $i_0$ . Consequently  $F_H$  and  $r_H$  go through a global minimum at point  $i = i_0$  whenever  $H = D$ , and the immunization proof in the simplest case is now complete.



**Main formulas**

- Future value of bond (flat term structure):

$$F_H = B(i)(1+i)^H$$

- Rate of return at horizon  $H$ :

$$r_H = \left(\frac{F_H}{B_0}\right)^{1/H} - 1 = \left[\left(\frac{B(i)}{B_0}\right)^{1/H}\right](1+i) - 1$$

- First-order condition for minimizing  $r_H$ :

$$\frac{dr_H}{di} = 0 \Rightarrow H = -\frac{1+i_0}{B(i_0)} B'(i_0) = D(i_0)$$

- Second-order condition for minimizing  $r_H$ :

$$\frac{d^2 r_H}{di^2} > 0 \Rightarrow \frac{S}{(1+i)^2} > 0$$

**Questions**

- 6.1. Show that all curves  $r_H = r_H(i)$  for various horizons  $H$  ( $H = 1, 2, \dots, \infty$ ) go through point  $(i_0, i_0)$  using an alternate demonstration. In other words, show that  $(i_0, i_0)$  is a fixed point for all curves  $r_H(i)$ . (Hint: Use formula (5) from section 3.4 of chapter 3.)
- 6.2. In our intuitive approach to section 6.1, would anything have changed if we had assumed that initially we had bought the bond below par or above par? (Hint: The answer is yes; be careful in your reasoning by turning to an analytic argument.)
- 6.3. At the end of section 6.2, what is meant by “the variance of the bond”?
- 6.4. What are the measurement units of equation (9) in section 6.2.2?



# 7

## CONVEXITY: DEFINITION, MAIN PROPERTIES, AND USES

EARL OF KENT. Seeking to give losses their remedies . . .

*King Lear*

FRIAR FRANCIS. Let this be so, and doubt not but success  
Will fashion the event in better shape  
Than I can lay it down in likelihood.

*Much Ado About Nothing*

### Chapter outline

<b>7.1</b>	<b>Definition and calculation of convexity</b>	<b>155</b>
<b>7.2</b>	<b>Improving the measurement of a bond's interest rate risk through convexity</b>	<b>157</b>
<b>7.3</b>	<b>Convexity of bond portfolios</b>	<b>160</b>
<b>7.4</b>	<b>What makes a bond (or a portfolio) convex?</b>	<b>162</b>
<b>7.5</b>	<b>Some important applications of duration, convexity, and immunization</b>	<b>163</b>
7.5.1	The future assets and liabilities problem: the Redington conditions	164
7.5.2	Immunizing a position with futures	168

### Overview

As we saw in chapter 1, the relationship between the interest rate and the bond's price is a convex curve. In our usage of the term we refer not to the convexity of this curve, but to the convexity of the bond itself. The more convex a bond, the more rewarding it is to its owner, because its value will increase more if interest rates drop and will decrease less if interest rates rise.

The convexity of a bond depends positively on its duration and on its dispersion. The dispersion of a bond is simply the variance of its times of payment, the weights being the cash flows of the bond in present value.

It should be emphasized that substantial gains based on convexity can be made on bond portfolios only in markets that are highly liquid and where participants have not already attributed a premium to this convexity.

We have noted already, in chapter 2 that the value of a bond is a convex function of the rate of interest, and we just saw in chapter 6 that duration is very closely linked to the relative rate of change in the price of the bond per unit change in interest rate (it was equal to  $-\frac{(1+i)}{B} \frac{dB}{di}$ ).

**Table 7.1**  
**Characteristics of bonds *A* and *B***

	Coupon Rate	Maturity	Price	Yield to Maturity	Duration
Bond <i>A</i>	9%	10 years	\$1000	9%	6.99 years
Bond <i>B</i>	3.1%	8 years	\$673		

Consider now two bonds that have the same duration, for instance 6.99 years. This is the case for bond *A*, already encountered: it is valued at par, offers a 9% coupon rate, and has a maturity of 10 years. If we want to find another bond, called *B*, that has the same duration and the same yield to maturity (9%) as *A*, we can tinker with the other parameters that make up the value of duration, namely the coupon rate and the maturity. For instance, a lower coupon rate will increase duration, and it probably suffices to decrease maturity at the same time to keep duration at 6.99 years. This is precisely what we will do: take a coupon rate of 3.1% and a maturity of eight years for bond *B*; it will have a duration of 6.99. If the yield to maturity is to be 9% for both bonds, the price of *B* will be way below par, since its coupon rate is below  $i = 9\%$ ; in effect, its price will be 673 (rounding to the closest unit). The characteristics of bonds *A* and *B* are summarized in table 7.1.

Consider now two investments of equal value in each bond: the first (portfolio  $\alpha$ ) consists of 673 bonds *A* bought for the amount of \$673,000, and the second (portfolio  $\beta$ ) is made up of 1000 bonds *B* costing \$673 each—an equivalent investment. At first glance an investor could be indifferent as to which portfolio he chooses: they are worth exactly the same amount, they offer the same yield to maturity, and they seem to bear the same interest rate risk, since they have the same duration. However, the investor would be wrong to believe this. If we calculate the value of each portfolio if interest rates move up or down from 9%, portfolio  $\alpha$  always outperforms portfolio  $\beta$  (see table 7.2). This is a surprising outcome, and it is the direct result of the difference in convexities between the two bonds. In effect, the value of each portfolio can be pictured as in figure 7.1, investment in *A* (and bond *A*) being more convex than investment in *B* (and bond *B*). The precise analysis of convexity is the purpose of this chapter. In particular, we will ask what makes any bond more convex than another one (as in this case) if they have the same duration,

**Table 7.2**  
**Value of portfolios  $\alpha$  and  $\beta$  for various rates of interest**

$i$ (in %)	Portfolio $\alpha$ Value of Investment in $A$ (in \$1000)	Portfolio $\beta$ Value of Investment in $B$ (in \$1000)
5	881	877
6	822	820
7	768	767
8	719	718
9	673	673
10	632	631
11	594	593
12	559	558
13	527	525

and how we can increase the convexity of a portfolio of bonds, given its duration.

## 7.1 Definition and calculation of convexity

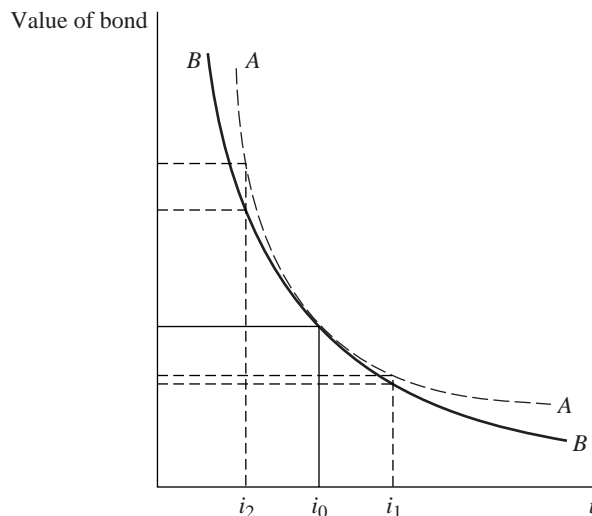
Convexity of a curve is always defined as *the rate of change of its slope*, that is, by its second derivative. The more quickly the slope of a curve changes in one direction (for instance, the steeper it becomes), the more convex is the curve. For bonds it is customary to define convexity as follows:

$$\text{Convexity} = \frac{1}{B(i)} \frac{d}{di} \left( \frac{dB}{di} \right) \equiv \frac{1}{B} \frac{d^2 B}{di^2}$$

Thus convexity is equal to the second derivative of  $B(i)$  divided by the value  $B(i)$ . The reason we divide the second derivative by  $B$  will soon become clear.

Remember from chapter 4 on duration that we calculated the first derivative  $\frac{dB}{di}$  as

$$\frac{dB}{di} = \sum_{t=1}^T (-t) c_t (1+i)^{-t-1} \quad (1)$$



**Figure 7.1**

The effect of a greater convexity for bond *A* than for bond *B* enhances an investment in *A* compared to an investment in *B* in the event of change in interest rates. Investment in *A* will gain more value than investment in *B* if interest rates drop and it will lose less value if interest rates rise.

The second derivative of *B* with respect to *i*, divided by *B*, will then be equal to

$$C = \frac{1}{B} \frac{d^2 B}{di^2} = \frac{1}{B(1+i)^2} \sum_{t=1}^T t(t+1)c_t(1+i)^{-t} \quad (2)$$

Before giving an example for the calculation of convexity, we should notice immediately two important properties of convexity:

- Property 1: A bond's convexity is always positive.
- Property 2: The dimension of a bond's convexity is (years<sup>2</sup>). It is very useful to remember this property, though it is often overlooked. It can be reached either by the fact that the dimension of  $\frac{1}{B} \frac{d^2 B}{di^2}$  is  $1/(1/t)^2 = t^2$  or by the fact that convexity is equal to the weighted average of all products  $t(t+1)$ , each of those having dimension (years<sup>2</sup>). The average is then

**Table 7.3**  
**Calculation of the convexity of bond *A* (coupon: 9%; yield to maturity: 10 years)**

(1)	(2)	(3)	(4)	(5)
Time of payment $t$	$t(t+1)$	Cash flow in nominal value $c_t$	Share of the discounted cash flows in bond's value $c_t(1+i)^{-t}/B$	$t(t+1)$ times share of discounted cash flows = (2) $\times$ (4) $t(t+1)c_t(1+i)^{-t}/B$
1	2	9	0.0826	0.165
2	6	9	0.0758	0.456
3	12	9	0.0695	0.834
4	20	9	0.0638	1.275
5	30	9	0.0585	1.755
6	42	9	0.0537	2.254
7	56	9	0.0492	2.757
8	72	9	0.0452	3.252
9	90	9	0.0414	3.729
10	110	109	0.4604	50.647
Convexity: total of (5) $\times \frac{1}{(1.09)^2} = 56.5$ (years <sup>2</sup> )				

divided by the pure number  $(1+i)^2$ ; hence dimension of convexity is indeed (years<sup>2</sup>).

As an example, let us calculate the convexity of our bond *A*. Table 7.3 summarizes those calculations; the result is 56.5 (years)<sup>2</sup>.

## 7.2 Improving the measurement of a bond's interest rate risk through convexity

In chapter 4 on duration we saw how the change in the bond's value could be approximated linearly, thanks to the duration. We will now see how we can improve upon this with a second-order approximation. Writing down the first two terms of Taylor's series, we have

$$\boxed{\frac{\Delta B}{B} \approx \frac{1}{B} \left[ \frac{dB}{di} di + \frac{1}{2} \frac{d^2 B}{di^2} (di)^2 \right]} \quad (3)$$

We will also consider here an increase of 1% in the rate of interest. Hence we get  $di = 1\% = 0.01$ . Earlier we had reached the part  $\frac{1}{B} \frac{dB}{di} di$  of this expression; this was equal to  $-\frac{1}{1+i} D di = -\frac{1}{1.09} (6.995 \text{ years}) (1\%$

**Table 7.4**  
**Improvement in the measurement of a bond's price change by using convexity**

Change in the rate of interest	Change in the bond's price		
	in linear approximation (using duration)	in quadratic approximation (using both duration and convexity)	in exact value
+1%	-6.4%	-6.135%	-6.14%
-1%	+6.4%	+6.700%	+6.71%

per year) = -6.4% (a pure number). This approximation made use of only the first term of Taylor's expansion. If we wish to get a better approximation, we simply add the second term, equal to

$$\frac{1}{2} \frac{1}{B} \frac{d^2 B}{di^2} di^2 = \frac{1}{2} [56.5(\text{years})^2] (1\%)^2 / (\text{year})^2 = 0.28\%$$

(This is a pure number, of course. The reader can now see why convexity was defined as  $(1/B) d^2 B/di^2$  and also why convexity *must* be expressed in  $(\text{years})^2$ . Otherwise it would not be compatible with Taylor's expansion of  $\frac{\Delta B}{B}$ . Since  $\frac{\Delta B}{B}$  is a pure number,  $(di)^2$  being expressed in  $1/(\text{years})^2$ , convexity  $\frac{1}{B} \frac{d^2 B}{di^2}$  must have dimension  $(\text{years})^2$ .)

Consequently, the relative increase in the bond's value is given, in quadratic approximation, by

$$\begin{aligned} \frac{\Delta B}{B} &\approx \frac{1}{B} \frac{dB}{di} di + \frac{1}{2} \frac{1}{B} \frac{d^2 B}{di^2} (di)^2 \\ &= -6.4176 + 0.2825 = -6.135\% \end{aligned}$$

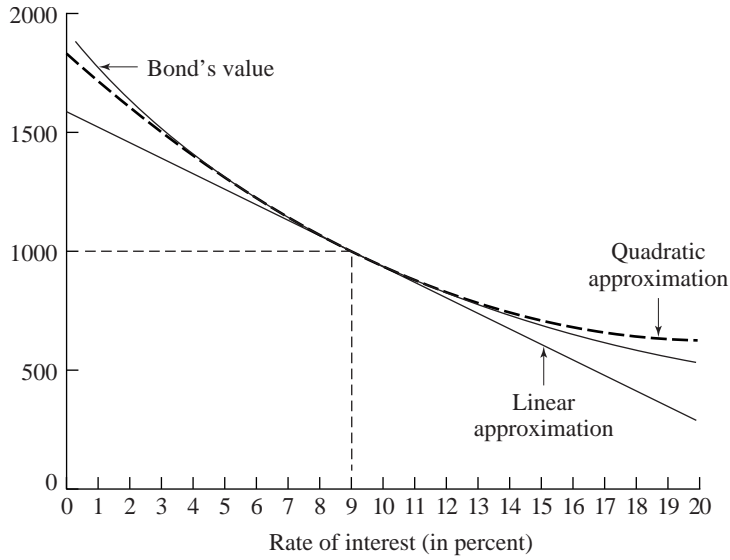
If, on the other hand,  $i$  decreases by 1%, we have, with  $di$  equal this time to -1%,

$$= +6.4176 + 0.2825 = 6.700\%$$

How good is the approximation now that we have gone to a second-order one? In exact value, if the interest rate increases or decreases by 1%, the bond's value will either go down by 6.14% or up by 6.71%, respectively. The approximation is in fact highly improved by the use of convexity. Table 7.4 summarizes these results.

In figure 7.2 we showed how well convexity improves the linear approximation to a bond's value. We now give the relevant equations we used





**Figure 7.2**  
Linear and quadratic approximations of a bond's value.

to draw this diagram, that is, the formulas of the straight line and the parabola tangent to the curve of the bond's value, which are the linear and quadratic approximations to the curve, respectively.

The linear approximation is given by the following equation, in space  $(\Delta B/B, di)$ ,

$$\frac{\Delta B}{B_0} = \frac{1}{B_0} \frac{dB}{di} di = -D_m di \quad (4)$$

Replacing  $\Delta B$  by  $B - B_0$  and  $di$  by  $i - i_0$ , we have in space  $(B, i)$ , after simplifications,

$$B(i) = B_0[(1 + D_m i_0) - D_m i] \quad (5)$$

The quadratic approximation is given, in space  $(\Delta B/B, di)$ , by

$$\frac{\Delta B}{B_0} = \frac{1}{B_0} \frac{dB}{di} di + \frac{1}{2} \frac{1}{B_0} \frac{d^2 B}{di^2} (di)^2 = -D_m di + \frac{1}{2} C(di)^2 \quad (6)$$

or, equivalently, in space  $(B, i)$ , by

$$B(i) = B_0 \left[ \frac{C}{2} i^2 - (D_m + Ci_0)i + 1 + D_m i_0 + \frac{C}{2} i_0^2 \right] \quad (7)$$

Note that, in space  $(\Delta B/B, di)$ , the minimum of the parabola has an abscissa equal simply to modified duration divided by convexity  $(D_m/C)$ . This means that if  $B(i)$  displays little convexity, to make our approximation, we essentially make use of a part of the parabola that is relatively far from its minimum.

Most financial services that provide the value of convexity for bonds divide the value of  $\frac{1}{B} \frac{d^2 B}{di^2}$  by 200 (in this case, convexity would be  $\frac{1}{B} \frac{d^2 B}{di^2} \frac{1}{200} = 0.28(\text{years})^2$ ). The reason for this is that this figure can readily be added to the term containing modified duration. Thus one can obtain immediately the (nearly exact) figures of  $-6.12\%$  and  $+6.68\%$  just by adding this “modified convexity number” to minus or plus the modified duration 6.4. Sometimes authors also refer to convexity as the value  $\frac{1}{2} \frac{1}{B} \frac{d^2 B}{di^2} \frac{1}{100}$ , for analogous reasons. In this text we will stick to the original definition given by  $\frac{1}{B} \frac{d^2 B}{di^2}$ .

### 7.3 Convexity of bond portfolios

Remember how convexity of a bond was defined: it was the second derivative of  $B(i) = \sum_{t=1}^T c_t(1+i)^{-t}$ , divided by  $B(i)$ . In fact, we see from this very definition that the cash flow  $c_t$  received at time  $t$  can be made of any sum of cash flows received at that time. Hence, the definition of convexity of a bond will immediately extend to bond portfolios. For that matter, we can also observe that any coupon-bearing bond can be considered a portfolio of bonds; for instance, it can be viewed as a series of  $T$  zero-coupon bonds (corresponding to each of the  $T$  cash flows to be received before and at reimbursement time). It can also be considered a combination of zero-coupon bonds and of coupon-bearing bonds. Therefore, it is not surprising that the convexity of a portfolio of bonds is defined simply as the second derivative of the portfolio’s value  $P$ , divided by  $P$ :

$$\text{Convexity of } P(i) \equiv C_P \equiv \frac{1}{P_i} \frac{d^2 P(i)}{di^2} \quad (8)$$

For simplicity, let us first restrict the number of bonds to 2; we then have

$$P = N_1 B_1 + N_2 B_2$$

$$P(i) = N_1 B_1(i) + N_2 B_2(i) \quad (9)$$

Taking the first derivative of (9) with respect to  $i$ , we get

$$\frac{dP(i)}{di} = N_1 \frac{dB_1}{di}(i) + N_2 \frac{dB_2}{di}(i) \quad (10)$$

The second derivative of  $P(i)$ , divided by  $P(i)$ , is

$$\frac{1}{P(i)} \frac{d^2 P(i)}{di^2} = \frac{N_1}{P(i)} \frac{d^2 B_1}{di^2}(i) + \frac{N_2}{P(i)} \frac{d^2 B_2}{di^2}(i) \quad (11)$$

Let us then multiply and divide each term in the right-hand side of (11) by  $B_1$  and  $B_2$ , respectively:

$$\frac{1}{P(i)} \frac{d^2 P(i)}{di^2} = \frac{N_1 B_1}{P} \frac{1}{B_1} \frac{d^2 B_1}{di^2} + \frac{N_2 B_2}{P} \frac{1}{B_2} \frac{d^2 B_2}{di^2} \equiv X_1 C_1 + X_2 C_2 \quad (12)$$

where  $C_1$  and  $C_2$  denote the convexities of bonds 1 and 2, respectively.

Therefore, the convexity of the bond portfolio is simply the weighted average of the bonds' convexities, the weights being the shares of each bond in the portfolio. In familiar terms, we could say that the convexity of the portfolio is the portfolio of the convexities. We can therefore write

$$C_P = X_1 C_1 + X_2 C_2$$

(13)

As the reader may well surmise, this extends to a portfolio of  $n$  bonds, and the demonstration of this property is immediate. We have

$$P(i) = \sum_{j=1}^n N_j B_j(i) \quad (14)$$

Taking the second derivative of (6) and dividing by  $P(i)$ , we have

$$\frac{1}{P(i)} \frac{d^2 P(i)}{di^2} = \sum_{j=1}^n \frac{N_j B_j}{P(i)} \frac{1}{B_j} \frac{d^2 B_j}{di^2} = \sum_{j=1}^n X_j \frac{1}{B_j} \frac{d^2 B_j}{di^2} \quad (15)$$

and therefore we have

$$C_P = \sum_{j=1}^n X_j C_j \quad (16)$$

This property will be very useful when our aim is to enhance convexity, in the case of either defensive or active bond management.

We should add here for convexity the same caveat as the one we pointed out earlier in chapter 4 on duration. First, it would be incorrect to calculate the convexity of a portfolio by averaging the convexities of bonds with different yields to maturity. Moreover, we should not forget that we are dealing here with *AAA* bonds, whose default risk is practically nil. This is not the case with a number of corporate securities, not to mention bonds issued in some emerging countries, either by governments or private firms. For those kinds of bonds, neither duration nor convexity could ever provide a meaningful guide for investment.

#### 7.4 What makes a bond (or a portfolio) convex?

We are now ready to determine what factors make a bond or a portfolio of bonds more or less convex.

In order to make the expression of convexity  $\frac{1}{B} \frac{d^2 B}{di^2}$  appear, let us take the derivative of  $D$  expressed in the form  $-\frac{1+i}{B} \frac{dB}{di}$  and equate the result to  $-(1+i)^{-1}S$ . We can write

$$D = -\frac{1+i}{B(i)} B'(i) \quad (17)$$

$$\frac{dD}{di} = -\left[ \frac{B - (1+i)B'}{B^2} \right] B'(i) - \frac{1+i}{B} B''$$

We have already shown, in chapter 4 on duration, that  $\frac{dB}{di}$  is equal to  $-(1+i)^{-1}S$ , where  $S$  is the dispersion, or variance, of the terms of the bond. Thus we can equate the right-hand side of (17) to  $-(1+i)^{-1}S$ . Hence, after simplification, we have

$$+\frac{1}{B} [1+D] B'(i) + \frac{1+i}{B} B'' = (1+i)^{-1} S \quad (18)$$

Denoting  $B'/B = -D/(1+i)$  and  $B''/B = C$ , and multiplying the last relationship by  $1+i$ , we finally get

$$-D(D + 1) + (1 + i)^2 C = S$$

which can be conveniently written as

$$C = \frac{1}{(1 + i)^2} [S + D(D + 1)] \quad (19)$$

This expression is quite important. It reveals that convexity depends on both dispersion and duration; the relationship with duration turns out to be a particularly strong one, since it is quadratic. Thus, a portfolio will be highly convex when, for any given duration, the variance of its times of payment is high.

## 7.5 Some important applications of duration, convexity, and immunization

Protecting an investment against fluctuations in interest not only is important to the individual investor but also is highly relevant to insurance companies, banks, pension funds, and other financial institutions. It is in fact those institutions that initiated research in this area. The first contribution came from Frederik Macaulay, whose article “Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the U.S. Since 1856” appeared in 1938. The first application to immunization, appearing in 1952, is attributed to F. M. Redington.<sup>1</sup>

In order to show the importance of immunization and the central role of duration played in this process, we will take up two issues of considerable breadth. First, how should a pension fund, or an insurance company, set up its assets portfolio in such a way as to be practically certain it will be able to meet its payment obligations in the future? This issue was first addressed by Redington (see reference above); we will call it *the future asset and liabilities problem*. Second, we will show the close links between duration and hedging with financial futures.

<sup>1</sup>Frederik Macaulay, “Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the U.S. Since 1856” (New York: National Bureau of Economic Research, 1938); F. M. Redington, “Review of the Principle of Life Office Valuations,” *Journal of the Institute of Actuaries*, 18 (1952): 286–340. Some of those early texts have been reprinted in G. Hawawini, *Bond Duration and Immunization* (New York: Garland, 1982).

### 7.5.1 The future asset and liabilities problem: the Redington conditions

Suppose that a company can make reasonably accurate predictions about its cash outflows in the next  $T$  years, denoted  $L_t$ ,  $t = 1, \dots, T$ , and about its inflows  $A_t$ ,  $t = 1, \dots, T$ . Suppose as before that the interest term structure is flat, equal to  $i$ ; the present value of these future liabilities and assets are, respectively,  $L_0$  and  $A_0$ :

$$L = \sum_{t=1}^T L_t(1+i)^{-t} \quad (20)$$

and

$$A = \sum_{t=1}^T A_t(1+i)^{-t} \quad (21)$$

Initially, Redington assumes that the initial net value of the future cash flows,  $N = A - L$ , is zero. The first question is, how should one choose the structure of the assets such that this net value does not change in the event of a change in interest rates? From our knowledge of duration we know that a first-order condition for this to happen is that the sensitivities of  $L$  and  $A$  have to  $i$  match—equivalently, that their durations are equal. First, we want  $N = A - L$  to be insensitive to  $i$ ; in other words, we want the condition

$$\frac{dN}{di} = \frac{d}{di} [\Sigma(A_t - L_t)(1+i)^{-t}] = 0 \quad (22)$$

to be fulfilled.

Equation (22) leads to

$$\begin{aligned} \frac{dN}{di} &= \frac{1}{1+i} \sum_{t=1}^T t(L_t - A_t)(1+i)^{-t} = \frac{1}{1+i} (D_L L - D_A A) \\ &= \frac{A}{1+i} (D_L - D_A) = 0 \end{aligned} \quad (23)$$

(since  $A = D$ ), where  $D_L = \sum_{t=1}^T tL_t(1+i)^{-t}/L$  and  $D_A = \sum_{t=1}^T tA_t(1+i)^{-t}/A$ . The first-order condition is indeed that  $D_L = D_A$ , as we had surmised. (This is called the first Redington property.)

A second-order condition can also be deduced relatively easily. Since we do not want the net value of assets to become negative,  $N(i)$  must be

such that for all possible values within an interval of  $i_0$ ,  $N$  remains positive. This will happen if  $N(i)$  is a *convex* function of  $i$  within that interval (note that this is a sufficient condition, not a necessary one). We know that convexity of a function is measured essentially by the second derivative of the function (for bonds, we divided this second derivative by the initial value of the functions, for reasons that we have seen). Here we have to deal with  $A(i) - L(i)$ , the difference between two functions; hence, to be convex its second derivative must be positive. Since the second derivative of a difference between two functions is simply the difference of the second derivative of each function, we infer that the second derivative of the assets with respect to  $i$  must be larger than the second derivative of the liabilities ( $\frac{d^2A}{di^2} > \frac{d^2L}{di^2}$ ). This is called the second Redington condition.

From this chapter (section 7.4, equation (19)), we know that once duration is given, convexity depends positively on the dispersion  $S$  of the cash flows. Hence a necessary condition for the second Redington condition to be met is that the *dispersion* of the inflows be larger than the dispersion of the outflows. Given our knowledge of duration and convexities for bonds, this condition is not difficult to meet, so that the net position of the firm will always remain positive in the advent of changes in interest rates.

As a practical example where not even the first of Redington's conditions was satisfied, we can cite the well-known disaster of the savings and loan associations in the United States in the early 1980s. These associations used to have inflows (deposits) with short maturities (and durations); on the other hand, their outflows (their loans) had very long durations, since they financed mainly housing projects. So the first Redington condition was not met; this situation spelled disaster. Everything would have been fine if interest rates had fallen; unfortunately, they climbed steeply, causing the net worth of the savings and loan associations to fall drastically—a hard-learned lesson.

As a direct application of Redington's conditions, let us see how the riskiness of a financial deal can be modified simply by maneuvering duration of assets or liabilities.<sup>2</sup> Suppose that a financial institution is investing \$1 million in a 20-year, 8.5% coupon bond. This investment is supposed to be financed with a nine-year loan carrying an 8% interest rate. Assuming that the firm manages to borrow exactly \$1 million at that

<sup>2</sup>We draw here and in the next section from S. Figlewski, *Hedging with Financial Futures for Institutional Investors* (Cambridge, Mass.: Ballinger, 1986), pp. 105–111. We have changed the examples somewhat.

rate, both assets and liabilities have the same initial present value ( $V_A$  and  $V_L$ , respectively); however, they have different exposures to interest changes. Here, since we assume a flat term structure, the riskiness of assets and liabilities is measured by making use of their Macaulay duration ( $D_A$  and  $D_L$ , respectively). We have

$$\frac{\Delta V_A}{V_A} \approx -\frac{D_A}{1+i_A} di_A = -9.165 di_A \quad (24)$$

and

$$\frac{\Delta V_L}{V_L} \approx -\frac{D_L}{1+i_L} di_L = -6.095 di_B \quad (25)$$

Indeed, the durations of the assets and liabilities can be calculated using the closed-form formula when cash flows are paid  $m$  times a year, and when the first cash flow is to be paid in a fraction of a year equal to  $\theta$  ( $0 < \theta < 1$ ):

$$D = \frac{1}{i} + \theta + \frac{\frac{N}{m}(i - c/B_T) - (1 + i/m)}{(c/B_T)[(1 + i/m)^N - 1] + i} \quad (26)$$

where

$\theta \equiv$  time to wait for the next coupon to be paid, as a fraction of a year  
( $0 \leq \theta \leq 1$ )

$m \equiv$  number of times a payment is made within one year

$N \equiv$  total number of coupons remaining to be paid

(In our case,  $\theta = \frac{1}{2}$ ;  $m = 2$ ;  $N$  is 40 for the 20-year bond and 18 for the nine-year loan made to the financial firm.)

Note that the above formula generalizes well formula (11) of chapter 4, which corresponds to the case where coupons were paid annually ( $m = 1$ ) and the remaining time until the next coupon payment was one year ( $\theta = 1$ ); also  $N$  was equal to  $T$ , the maturity of the bond (in years). Note also how this formula enables you to know immediately that whatever the frequency of the coupons to be paid, duration tends toward a limit when  $N$  goes to infinity (when the bond becomes a perpetual or a consol). In the last term of this expression (the large fraction), the numerator is an *affine* function of  $N$ , while the denominator is an *exponential* function of  $N$ . Therefore, its limit is zero when  $N \rightarrow \infty$ ; duration then tends toward  $\frac{1}{i} + \theta$  when the bond becomes a perpetual. Not only can we determine this



limit, but this limit makes excellent economic sense, and the reasoning is exactly the same as the one we proposed in chapter 4. The time you have to wait to recoup 100% of your investment is  $\frac{1}{i}$ , to which you have to add  $\theta$ , the time for the first payment to be made to you. As an application, you can verify that if a bond pays a *continuous* coupon at constant rate  $c(t)$  such that after one year you receive the sum  $c$ —a constant—(you have  $\int_0^1 c(t) dt = \int_0^1 c dt = c$ ), then the duration of a consol paying a continuous coupon is  $\frac{1}{i} + 0 = \frac{1}{i}$ .

In our example, the durations are

$$D_A = 9.944 \text{ years}$$

$$D_L = 6.583 \text{ years}$$

and the modified durations are

$$D_{mA} = 9.165 \text{ years}$$

$$D_{mL} = 6.095 \text{ years}$$

The conclusion we draw from these figures is the following: although the net initial position of the financial firm is sound (the investment is fully funded), its risk exposure presents a net duration of  $D_A - D_L = 3.361$  years and a net modified duration of  $D_{mA} - D_{mL} = 3.070$  years. This has important consequences. Since we have

$$\frac{\Delta V_A}{V_A} \approx \frac{dV_A}{V_A} = -D_{mA} di_A$$

and

$$\frac{\Delta V_L}{V_L} \approx \frac{dV_L}{V_L} = -D_{mL} di_L$$

the firm's net position for this project (denoted  $V_p$ ), equal to  $V_p = V_A - V_L$ , is such that

$$\frac{\Delta V_p}{V_p} = \frac{\Delta V_A}{V_A} - \frac{\Delta V_L}{V_L} \approx -(D_{mA} di_A - D_{mL} di_L)$$

Suppose for the time being that  $i_A$  and  $i_L$  receive the same increment,  $di$  ( $di = di_A = di_L$ ). We would have

$$\frac{\Delta V_p}{V_p} \approx -(D_{mA} - D_{mL}) di = -3.070 di$$

This implies that for this project the firm has a positive net exposure measured by a net modified duration of 3.070 years. It means that, in linear approximation, if interest rates increase by one percent (100 basis points), the net value of the project will diminish by 3.070%; in other words, the firm has a project that is equivalent, for instance, to buying a zero-coupon bond with a maturity equal to 3.070 years.

There is nothing wrong with that if the financial manager estimates that interest rates have a good chance of falling. But what if he is not so sure of that happening? On the other hand, if he wants to hedge his project against a rise in interest rates, he may choose to bring this net modified duration to zero by restructuring either his assets, his liabilities, or both. More generally, should his forecasts of  $di_A$  and  $di_L$  differ, he will want to bring the durations of his assets and his liabilities ( $D_A$  and  $D_L$ ) to levels such that the following equation is verified:

$$D_{mA} di_A = D_{mL} di_L$$

In doing so, the financial manager may want to pay special attention to the convexity of his assets and liabilities as well. This is an issue we will address in later chapters. Before dealing with it, however, we should remind ourselves of some important points. First, immunization is a *short-term* series of measures destined to match sensitivities of assets and liabilities. As time passes, these sensitivities continue to change, because duration does not generally decrease in the same amount as the planning horizon. Second, whenever interest rates change, duration also changes. There may be cases where duration should be decreased because of the shortening of the horizon, and at the same time interest rates have gone up in such a way that they have effectively reduced duration in the right proportion, but this is of course an exceptional case, not the general rule. Finally, we should keep in mind that until now we have considered flat term structures and parallel displacements of them. More refined duration measures and analysis are required if we do not face such flat structures, as we will see in our next chapter, and particularly in the last two chapters of this book, where we consider interest rates as following stochastic processes.

### 7.5.2 Immunizing a position with futures

When the financial manager is faced with some liquidity problems in the bonds market, he may well find a way out by adding a futures position to

his portfolio. What he will want to do is to take a position of  $n_f$  futures contracts on  $T$ -bonds, for instance, such that the total duration of his portfolio is zero. His exposure to interest rate risk would then be reduced to nil, subject to the qualifications we mentioned in our last section.

Suppose that  $f$  is the value of a futures contract; the value of his position in futures is then the number of contracts  $n_f$  multiplied by  $f$ , that is,  $n_f f$ . If the sensitivity of his portfolio of futures assets and liabilities has to be zero, the number of contracts he must buy (or sell),  $n_f$ , must be such that the duration of his portfolio must be zero; if  $D_f$  is the duration of a futures contract, this is tantamount to the equation

$$D_f n_f F + D_A V_A - D_L V_L = 0 \quad (28)$$

which leads to

$$n_f = \frac{D_L V_L - D_A V_A}{D_f F} \quad (29)$$

Some attention simply must be paid to the determination of the futures contract's duration  $D_f$ , which is calculated in the following way:

- The futures price  $F$  is the futures settlement price multiplied by the relevant deliverable bond's conversion price (established in the United States by the Chicago Board of Trade, as explained earlier). This gives the *adjusted price*.
- The relevant yield to maturity of the futures is calculated by using the adjusted price and the deliverable bond's cash flows as viewed from delivery date.

From the above formula giving  $n_f$ , we can see that the position in futures will be positive (negative) if and only if  $D_L V_L$  is larger (smaller) than  $D_A V_A$ .

#### Main formulas

- Convexity:

$$C = \frac{1}{B} \frac{d^2 B}{di^2} = \frac{1}{(1+i)^2} \sum_{t=1}^T t(t+1) c_t (1+i)^{-t} / B$$

- Quadratic approximation of relative change in bond's value:

$$\frac{\Delta B}{B} \approx D_m di + \frac{1}{2} C \cdot (di)^2$$

- Convexity of bond portfolios (for bonds with same return to maturity):

$$C_P = \sum_{j=1}^n X_j C_j$$

- Relationship between convexity, duration, and bond's dispersion (or variance of times of payment  $S$ ):

$$C = \frac{1}{(1+i)^2} [D(D+1) + S]$$

## Questions

7.1. Assuming that coupon payments are made once per year, confirm one of the results of table 7.1, for instance that the convexity of a ten-year maturity, 6% coupon rate is 62.79 (years<sup>2</sup>) when the bond's rate of return to maturity is 9%.

7.2. Looking at figure 7.2, you will notice that the quadratic approximation curve of the bond's value lies *between* the bond's value curve and the tangent line to the left of the tangency point and *outside* (above) these lines to the right of the tangency point. The fact that a quadratic (and hence "better") approximation behaves like this is not intuitive: we would tend to think that a "better approximation" would *always* lie between the exact value curve and its linear approximation. How can you explain this apparently nonintuitive result?

7.3. In section 7.3, what, in your opinion, is the crucial assumption we are making when calculating the convexity of a 2-bond or  $n$ -bond portfolio?

7.4. In section 7.4, make sure you can deduce equation (18) from equation (17).

# 8

## THE IMPORTANCE OF CONVEXITY IN BOND MANAGEMENT

2ND LORD. And how mightily sometimes we make us  
comforts of our losses!

*All's Well that Ends Well*

### Chapter outline

<b>8.1 Comparing the benefits of holding two coupon-bearing bonds with the same duration</b>	<b>172</b>
8.1.1 The case of defensive management (immunization)	173
8.1.2 The case of active management	174
<b>8.2 Example of a zero-coupon bond compared to a coupon-bearing bond</b>	<b>174</b>
8.2.1 The case of defensive management	176
8.2.2 The case of active management	176
<b>8.3 Looking for convexity in building a bond portfolio</b>	<b>176</b>
8.3.1 Setting up a portfolio with a given duration and higher convexity than an individual bond	177
8.3.2 Results of enhanced convexity	178

### Overview

If the bond's market is highly liquid, the search for convexity can substantially improve the investor's performance. If we consider the two basic styles of management, active and defensive, we will see that convexity will help the money manager.

Consider first the active management style. If bonds have been bought with a short-term horizon and with the prospect that interest rates will fall, the rate of return will be higher with more convex bonds or portfolios. Suppose now that the interest rates move in the opposite direction from what was anticipated and rise; the loss will be smaller for highly convex bonds than for those that exhibit less convexity.

On the other hand, consider a defensive style of management, such as an immunization scheme. In such a case, any movement in interest rates that occurs immediately after the purchase of the portfolio will translate into higher returns if the portfolio is highly convex.

Of course, the same caveats that we pointed to previously when discussing immunization apply here: convexity will definitely enhance a portfolio's performance only in those markets that are highly liquid.

**Table 8.1**  
Duration and convexity for two types of bonds

Coupon ( $c$ )	Type I bond Maturity: 10 years		Type II bond Maturity: 20 years	
	Duration (years) $D = -\frac{1+i}{B} \frac{dB}{di}$	Convexity (years <sup>2</sup> ) $CONV = \frac{1}{B} \frac{d^2B}{di^2}$	Duration (years) $D = -\frac{1+i}{B} \frac{dB}{di}$	Convexity (years <sup>2</sup> ) $CONV = \frac{1}{B} \frac{d^2B}{di^2}$
0	10	92.58	20	353.5
1	9.31	84.34	15.9	251.5
2	8.79	78.02	13.81	215.98
3	8.37	73.02	12.59	188.85
4	8.03	68.96	11.78	170.8
5	7.75	65.6	11.21	158
6	7.52	62.79	10.77	148.4
7	7.32	60.38	10.44	140.94
8	7.15	58.31	10.17	134.98
9	6.99	56.50	9.95	130.10
10	6.86	54.9	9.76	126.04
11	6.76	53.49	9.61	122.61
12	6.64	52.23	9.48	119.7
13	6.55	51.10	9.36	117.12
13.5	6.50	50.58	9.31	115.97
14	6.46	50.08	9.29	114.89
15	6.38	49.16	9.18	112.93

Our purpose in this chapter is to stress the role of convexity in any bond's performance, in both defensive and active management styles. We will first consider two examples. The first one will make use of two coupon-bearing bonds; in the second we will deal with a coupon-bearing bond and a zero-coupon. We will then indicate how we can enhance the convexity of a bond portfolio.

### 8.1 Comparing the benefits of holding two coupon-bearing bonds with the same duration

Let us consider two types of bonds (I and II), differing by their maturity (10 years for type I, 20 years for type II). Each type has various coupons, ranging from zero to 15, with a return to maturity equal to 9%. The corresponding duration and convexity for each bond are presented in table 8.1. We notice that increasing the coupon decreases both duration and convexity. We know why duration decreases; with regard to convexity, we cannot say a priori whether it will increase or not with the coupon, be-

**Table 8.2**  
**Characteristics of bonds *A* and *B***

	Bond <i>A</i>	Bond <i>B</i>
Maturity	10 years	20 years
Coupon	1	13.5
Duration	9.31 years	9.31 years
Convexity	84.34 years <sup>2</sup>	115.97 years <sup>2</sup>

cause the effect on dispersion is undetermined. Since convexity is a positively sloped function of both duration and dispersion, there is no way to tell except to do the cumbersome exercise of the comparative statistics on convexity. It turns out that the duration factor in convexity will generally outperform the dispersion factor; therefore, convexity will diminish with an increase of the coupon.

Let us now consider two bonds, one from each family, with the same duration but differing in their convexity. Both bonds, *A* and *B*, have the same duration (9.31 years). Bond *A* has a coupon of 1 and a convexity of 84.34 years<sup>2</sup>; bond *B* has a coupon of 13.5 and a convexity of 115.97 years<sup>2</sup>. Table 8.2 summarizes the characteristics of both bonds.

Bond *A*, of course, will be way below par (48.66), and bond *B* will be above par (141.08). Their durations are the same, but their convexities are not. This implies significantly different rates of return, in both defensive and active management.

### 8.1.1 The case of defensive management (immunization)

Suppose that we want to protect our investor against parallel changes in the interest rate structure, this structure being flat. Should the investor's horizon be 9.3 years (at least for part of his portfolio!), we know that he is protected by purchasing either *A* or *B*. But the fact that *B* exhibits more convexity has a number of advantages that will become immediately apparent. Suppose that initial rates are 9% and that they quickly move up by 1 or 2%—or that they drop by the same amount, staying there for the remaining investment period. Table 8.3 indicates the 9.31 horizon rate of return for both bonds.

The reader can see that the order of magnitude of the increase in the rate of return over 9% for the more convex bond (*B*) is tenfold that of the bond with lower convexity. Suppose, as a matter of illustration, that \$1 million is invested in each bond. Should the rates of interest decrease to 7%, the future value of each investment would be:

**Table 8.3**

**Rates of return for  $A$  and  $B$  with horizon  $D = 9.31$  years when rates move quickly from 9% to another value and stay there**

	Scenario				
	$i = 7\%$	$i = 8\%$	$i = 9\%$	$i = 10\%$	$i = 11\%$
Bond $A$	9.008%	9.002%	9%	9.002%	9.008%
Bond $B$	9.085%	9.021%	9%	9.020%	9.079%

- investment in  $A$ :  $1,000,000(1 + 0.9008)^{9.31} = 2,232,222$
- investment in  $B$ :  $1,000,000(1 + 0.9085)^{9.31} = 2,246,245$

which implies a difference of \$14,023 for no trouble at all, except looking up the value of convexity.

### 8.1.2 The case of active management

The vast majority of investment decisions in bonds are still made in an active management framework. If interest rates are forecast to decrease, managers will take a long position in bonds, hoping for a quick rise in the bond's value. The drawback, of course, is an error in the forecast, implying a loss in value of the investment. What we want to illustrate here is how convexity can enhance the performance of such a management style, if successful, and protect the investor slightly by cushioning a bit the loss incurred if a forecasting error has been made. Table 8.4 shows the 1-, 2-, and 3-year horizon rate of return in the same scenarios considered earlier.

The results are quite spectacular, and even more so when the horizon is shorter. For instance, an increase of nearly one percentage point in the one-year performance rate can be obtained if bond  $B$  is chosen rather than bond  $A$ . And in the unhappy circumstance of an increase in interest rates from 9% to 11%, the convexity of  $B$  will cushion the loss from 6.2% to 5.7%.

## 8.2 Example of a zero-coupon bond compared to a coupon-bearing bond

Let us now compare the following two investments. Both bonds have the same duration. The dispersion of the zero-coupon, however, will be zero (and its convexity will be limited to  $(1 + i)^{-2} \cdot D(D + 1)$ ); on the other hand, the dispersion of the coupon-bearing bond will be positive, and thus its convexity will be higher, being equal to  $(1 + i)^{-2}[D(D + 1) + S]$ . The characteristics of our two bonds are summarized in table 8.5(a).



**Table 8.4**

**Rates of return when  $i$  takes a new value immediately after the purchase of bond  $A$  or bond  $B$  (in percentage per year)**

Horizon (in years) and rates of return for $A$ and $B$	Scenario				
	$i = 7\%$	$i = 8\%$	$i = 9\%$	$i = 10\%$	$i = 11\%$
$H = 1$					
$R^A$	27.2	17.1	9	1.0	-6.2
$R^B$	28.1	17.9	9	1.2	-5.7
$H = 2$					
$R^A$	16.7	12.7	9	5.4	2.0
$R^B$	17.1	12.8	9	5.5	2.3
$H = 3$					
$R^A$	8.9	8.9	9	9.1	9.1
$R^B$	8.9	8.9	9	9.1	9.2

**Table 8.5**

**(a) Characteristics of bonds  $A$  and  $B$**

	Bond $A$ (zero-coupon)	Bond $B$
Coupon	0	9
Maturity	10.58 years	25 years
Duration	10.58 years	10.58 years
Convexity	103.12 years <sup>2</sup>	159.17 years <sup>2</sup>

**(b) Rates of return of  $A$  and  $B$  when  $i$  moves from  $i = 9\%$  to another value after the bond has been bought**

	Scenario				
	$i = 7\%$	$i = 8\%$	$i = 9\%$	$i = 10\%$	$i = 11\%$
Bond $A$ (zero-coupon)	9	9	9	9	9
Bond $B$	9.14	9.04	9	9.02	9.08

**Table 8.6**  
Rates of return when  $i$  takes a new value immediately after the purchase of bond  $A$  or bond  $B$   
(in percentage per year)

Horizon and rates of return for $A$ and $B$	Scenario				
	$i = 7\%$	$i = 8\%$	$i = 9\%$	$i = 10\%$	$i = 11\%$
$H = 1$					
$R^A$	30.2	19.1	9	-0.01	-8.4
$R^B$	31.9	19.5	9	0.02	-7.7
$H = 2$					
$R^A$	18.0	13.4	9	-0.001	-0.08
$R^B$	18.8	13.6	9	4.9	1.2
$H = 3$					
$R^A$	8.9	8.9	9	9.1	9.1
$R^B$	8.9	8.9	9	9.1	9.2

### 8.2.1 The case of defensive management

In the case of immunization against changes in interest rates, we would get the results you see in table 8.5(b). Note, of course, that the horizon rate of return of the zero-coupon  $B$  is *perfectly* horizontal and exactly equal to 9% in every scenario, as opposed to the rate for the coupon-bearing bond. This is due to the fact that if the zero-coupon bond is held to maturity, its terminal value,  $B_T$ , is completely independent of any reinvestment of the coupon, since none occurred. The horizon  $T = D$  rate of return,  $r_T$ , is defined by  $B_0(1 + r_T)^T = B_T$ ;  $r_T$  is then independent from  $i$  and equal to  $r_T = i_0$  (9% in our example).

### 8.2.2 The case of active management

We sometimes hear that if interest rates are almost definitely going down, the investor should have the longest maturity possible (always, of course, in the range of defaultless bonds). In fact, the investor's portfolio's *duration* and *convexity* should be as high as possible. This means that between  $A$  and  $B$ , the investor must choose  $B$ ; he will therefore dismiss the zero-coupon  $A$ . Table 8.6 shows that the zero-coupon is beaten by 1.7 percentage points if interest rates go down to 7% on a one-year horizon.

## 8.3 Looking for convexity in building a bond portfolio

Suppose we are unable to find a bond with the same duration and higher convexity as the one we are considering buying. We can easily solve this

**Table 8.7**  
**Summary of duration and convexity values for bonds and portfolios**

	Price	Duration (years)	Convexity (years <sup>2</sup> )
Bond 1	105.96	5	28.62
Bond 2	102.8	1	1.75
Bond 3	97.91	9	95.72
Portfolio	105.96	5	48.73

problem by building a portfolio that will have the same duration but higher convexity.

Before we start, it will be useful to remind ourselves of the following three results, developed in the preceding chapters:

- The duration of a portfolio is the portfolio of durations.
- The convexity of a portfolio is the portfolio of convexities.
- The convexity  $C$  of a bond, or a bond portfolio, is linked to its duration  $D$  and to its dispersion  $S$  by the following formula:

$$C = \frac{1}{(1+i)^2} [D(D+1) + S]$$

### 8.3.1 Setting up a portfolio with a given duration and higher convexity than an individual bond

Consider a bond 1 with such characteristics that its duration is exactly 5 years and its convexity 28.6 years<sup>2</sup> (see table 8.7). Bonds 2 and 3 are such that the equally weighted average of their durations is 5, and the same average of their convexity is above that of bond 1, which is 28.6 years<sup>2</sup>; it is in fact 48.73 years<sup>2</sup>.

We will thus form a portfolio having the same value and the same duration as bond 1. (Since we describe the principle of building such a portfolio here, we will not bother considering fractional quantities of bonds. In order to avoid that problem, we would simply consider an initial value equal, for instance, to 100 times the value of bond 1.)

Let  $N_2$  and  $N_3$  be the number of bonds 2 and 3 we will choose. Let  $B_1$ ,  $B_2$ , and  $B_3$  be the respective present values of each bond.  $N_2$  and  $N_3$  must be such that:

- The value of the portfolio is equal to  $B_1$ :

$$N_2 B_2 + N_3 B_3 = B_1 \quad (1)$$

- The duration of the portfolio is that of 1; thus the weighted average of the durations  $D_2$  and  $D_3$  must be equal to  $D_1$ . We must have

$$\frac{N_2 B_2}{B_1} D_2 + \frac{N_3 B_3}{B_1} D_3 = D_1 \quad (2)$$

Since  $D_1 = 5$ ,  $D_2 = 1$ , and  $D_3 = 9$ , we must have

$$\frac{N_2 B_2}{B_1} + 9 \frac{N_3 B_3}{B_1} = 5$$

Let  $X_2 \equiv \frac{N_2 B_2}{B_1}$ ,  $1 - X_2 = X_3 \equiv \frac{N_3 B_3}{B_1}$ . From  $X_2 + 9(1 - X_2) = 5$ , we get  $X_1 = X_2 = 0.5$  (as we could have figured out). We can then determine the number of bonds  $N_2$  and  $N_3$  to be bought, as

$$N_2 = 0.5 \frac{B_1}{B_2} = 0.5 \frac{105.96}{102.8} = 0.5153$$

and

$$N_3 = 0.5 \frac{B_1}{B_3} = 0.5 \frac{105.96}{97.91} = 0.5411$$

More generally, of course, solving (1) and (2) simultaneously would have given us directly the number of bonds as

$$N_2 = \frac{B_1}{B_2} \left( \frac{D_3 - D_1}{D_3 - D_2} \right) \quad (3)$$

$$N_3 = \frac{B_1}{B_3} \left( \frac{D_1 - D_2}{D_3 - D_2} \right) \quad (4)$$

Table 8.7 summarizes the duration and convexity of our bonds and those of the portfolio that we have built up.

### 8.3.2 Results of enhanced convexity

Thus, while the single bond 1 with a duration of 5 years had a convexity of about 29 years<sup>2</sup>, the portfolio with the same duration has more convexity. This results solely from the fact that we have been able to get more dispersion in the times of payment. As the reader may surmise, this

**Table 8.8**  
**Values of bond 1 and of portfolio  $P$  for various values of interest rate**

$i$	$B_1(i)$	$B_P(i)$
4%	122.28	123.48
5%	116.50	116.99
6%	111.06	111.18
7%	105.96	105.96
8%	101.16	101.25
9%	96.64	97.00
10%	92.38	93.15

**Table 8.9**  
**5-year horizon rates of return for bond 1 and portfolio**

$i$	$r_{H=5}^1$	$r_{H=5}^P$
4%	7.023%	7.230%
5%	7.010%	7.100%
6%	7.003%	7.025%
7%	7%	7%
8%	7.003%	7.024%
9%	7.010%	7.092%
10%	7.023%	7.203%

enhanced convexity will result in higher values for the portfolio than for bond 1, for any interest rate different from 7%, as table 8.8 shows.

Let us now compare the performance of the bonds. Suppose we have an immunization policy, and therefore our horizon is duration, that is, 5 years. As table 8.9 shows, the increase in convexity significantly increases the performance of the portfolio.

As we observed before, the order of magnitude of the relative gain above 7% is about tenfold for the portfolio compared to the single bond.

Suppose now that we are interested in short-term performance, with an active management when interest rates are forecast to decrease. Table 8.10 clearly indicates that the performance of the portfolio is significantly better if interest rates do decrease and that losses are somewhat contained if interest rates increase.

We can conclude from these observations that a natural way to increase convexity is to try to replace any given portfolio by another one that will exhibit more convexity. Formally, we could look for the shares  $X_1, \dots, X_n$  of a portfolio of  $n$  bonds that would maximize the convexity of the port-

**Table 8.10**  
**One-year horizon rates of return for bond 1 and portfolio**

$i$	$r_{H=1}^1$	$r_{H=5}^P$
4%	20.019%	21.194%
5%	15.443%	15.935%
6%	11.108%	11.225%
7%	7%	7%
8%	3.105%	3.203%
9%	-0.590%	-0.214%
10%	-4.098%	-3.294%

folio,  $C_P$ , equal to

$$C_P = \sum_{i=1}^n X_i C_i$$

(where the  $C_i$ 's are the individual bonds' convexities) under the constraints

$$\sum_{i=1}^n X_i = 1, \quad X_i \geq 0 \quad (i = 1, \dots, n)$$

and, if we are using defensive strategy, a duration constraint

$$\sum_{i=1}^n X_i D_i = \bar{D}$$

where  $\bar{D}$  is our chosen duration.

It would be rather simple to show that the solution would be to choose a portfolio of two bonds, one with maximal duration and the other with minimal duration. However, a number of problems are involved in such a procedure. On the one hand, liquidity problems might arise when we concentrate large investments on a restricted number of assets. On the other hand, it might be costly to update the portfolio frequently, since one of the bonds would be systematically short-lived. Nevertheless, keeping an eye on the convexity of any bond portfolio is clearly a good idea. We should add here that inasmuch as many investors share this point of view, the possibility of convexity enhancing will entail arbitrage opportunities and a price to pay for convexity.

**Key concepts**

- Convexity measures how fast the slope of the bond's price curve changes when the interest rate is modified.
- Convexity can be rewarding for the bond's owner: the greater the convexity, the higher the return if interest rates drop; the smaller the loss if interest rates rise.
- Dispersion of a bond is the variance of the *times of payment* of the bond, the weights being the discounted cash flows (duration being the average of those times of payment).
- Convexity depends strongly and positively on duration (it increases with the square of duration) and on the dispersion of the bond.

**Basic formulas**

- Convexity  $\equiv \frac{1}{B} \frac{d^2 B}{di^2} = \frac{1}{(1+i)^2} [D(D+1) + S]$
- Dispersion  $\equiv S = \sum_{t=1}^T (t-D)^2 c_t (1+i)^{-t} / B$

**Questions**

These exercises can be considered as projects.

8.1. Assuming that bonds pay coupons once a year, determine the duration and convexity values contained in tables 8.1 and 8.3. (Hint: Remember that you can approach the exact values of duration and convexity at any desired precision level by applying the approximations

$$D = -\frac{1+i}{B(i)} \frac{dB}{di} \approx -\frac{1+i}{B(i)} \frac{\Delta B}{\Delta i}$$

and

$$C = \frac{1}{B^2(i)} \frac{d^2 B}{di^2} \approx \frac{1}{B^2(i)} \frac{\Delta}{\Delta i} \left( \frac{\Delta B}{\Delta i} \right)$$

with  $\Delta i$  sufficiently small to obtain the desired precision level for  $D$  and  $C$ . The advantage of this method is that you can rely solely on closed-form formulas.)

8.2. Assuming that bonds pay coupons once per year, verify the results contained in tables 8.7 through 8.10.

8.3. Consider the following bonds:

	Bond <i>A</i>	Bond <i>B</i>
Maturity	15 years	11 years
Coupon rate	10%	5%
Par value	\$1000	\$1000

(The data are from exercise 4.8 of chapter 4.)

- Determine the convexities of each bond.
- Suppose you have a defensive strategy, and that you want to immunize the investor. What is each bond's rate of return at horizon  $H = D$  if interest rates keep jumping from 12% to either 10% or 14%?
- Consider now an active style of management, whereby your horizon is one year only.

In conclusion, which bond would you choose?



# 9

## THE YIELD CURVE AND THE TERM STRUCTURE OF INTEREST RATES

WOLSEY. Your envious courses . . . no doubt in time will find their fit rewards.

*Henry VIII*

### Chapter outline

<b>9.1</b>	<b>Spot rates</b>	<b>185</b>
<b>9.2</b>	<b>The yield curve and the derivation of the spot rate curve</b>	<b>187</b>
<b>9.3</b>	<b>Derivation of the implicit forward rates from the spot rate structure</b>	<b>192</b>
<b>9.4</b>	<b>A first approach: individuals are risk-neutral; the pure expectations theory</b>	<b>195</b>
<b>9.5</b>	<b>A more refined approach: determination of expected spot rates when agents are risk-averse</b>	<b>201</b>
9.5.1	The behavior of investors	201
9.5.1.1	Investors with a short horizon	201
9.5.1.2	Investors with a longer horizon	202
9.5.2	The borrowers' behavior	203

### Overview

Interest rates vary according to the maturities of the debt issues they correspond to. It is a well-known fact that, in the long run, the term structure of interest rates often increases with maturity; in other words, short-term paper usually carries smaller interest rates than long-term bonds. Our purpose in this chapter is twofold: we first explain the generally increasing structure of spot rates; we then turn to the more difficult question of whether knowledge of the term structure enables us to infer anything about the future spot rates expected by the market.

We start by reminding the reader of the definition of a yield curve: a yield curve depicts the yield to maturity of bonds with various maturities. This curve should not be confused with the term structure of interest rates, defined by the spot rate curve. The yield to maturity of a bond is the discount rate which, when applied to each of the bond's cash flows, would make their total present value equal to the cost of the bond. As such, it is the horizon return on the bond for the given maturity if all cash flows of the bond (all coupons) can be reinvested systematically at a rate equal to

the yield to maturity—and we know that there is no reason for this to be the case. Thus, the concept of yield to maturity carries the same drawbacks as the concept of the internal rate of return of an investment project.

We then move on to show how the spot rate (term) structure of interest rates can be calculated from the prices of coupon-bearing bonds of different maturities. We then explain why, generally, this structure is increasing with maturity.

From this spot rate structure, we may infer an implicit forward structure, that is, the structure of interest rates that would apply to contracts that start in the future and have different maturities. Indeed, it is possible to show that any combination of two spot rates (for two different maturities) always implies a certain forward rate for a span of time equal to the difference between these maturities. Both an investor and a borrower can always lock in a forward rate by using spot rates for different maturities. For instance, suppose a one-year spot rate and a two-year spot rate. An investor who wants to borrow in one year for one year can always do two mutually offsetting things today, implying no net cash disbursement for him, which will give him what he wants. If today he borrows a given amount for two years and at the same time he lends that same amount for one year, he will not disburse anything today. On the other hand, he will receive some cash in one year and will disburse another amount in two years. This is tantamount to borrowing in one year, with a one-year contract. We can easily determine the forward rate implied by such an operation; indeed, such a forward rate is determined solely by the very existence of spot rates for various maturities.

From that point we will try to answer a much more difficult question: what are the reasons behind a given structure of interest rates? For instance, suppose that a given structure is increasing with maturity at one point in time. What are the reasons for this? Does it imply that the market believes future spot rates will be increasing? As the reader may well surmise, the answer to that question is far from obvious. Using reasonable assumptions, we will show that a downward sloping term structure always implies that the market expects future spot rates to fall. On the other hand, an increasing structure does not necessarily mean that the market expects future spot rates to rise.

Among the central concepts in bond analysis and management is the term structure of interest rates and its implications with regard to possible

future interest rates. Admittedly, this subject is not especially easy, and, unfortunately, its difficulty is compounded by a confusion often made between the yield curve and the term structure of interest rates (the spot rate curve). This is the reason we have chosen to present in this chapter these two concepts, which are quite distinct. Our first task will be to define them with precision. We will then show how the spot rate curve can be derived from the yield curve. This will enable us to derive from the spot rate curve the implied forward rate curve, and we will then ask whether implied forward rates have anything to tell us regarding expected future interest rates.

## 9.1 Spot rates

The spot rate is the interest that prevails today for a loan made today with a maturity of  $t$  years. It will be denoted  $i_{0,t}$ . The first index (0 here) refers to the time at which the loan starts; the second index ( $t$ ) designates the number of years during which the loan will be running. We should warn the reader that it is not unusual to encounter other notations for interest rates bearing three indexes rather than our two. Sometimes interest rates are referred to as  $i_{0,0,t}$ ; the first index (0) refers to the time at which the decision to make the loan is made; the second index (0) refers to the start-up time of the loan; and the third index ( $t$ ) may be either the date at which the loan is due or the number of years during which the loan will be running. In this text, since we will practically always deal with interest rate contracts that are decided today that come into existence either immediately or at some future date, for simplicity we have decided to suppress the first of the above three indexes.

The spot rate is, in effect, the yield to maturity of a zero-coupon bond with maturity  $t$ , because  $i_{0,t}$  verifies the equation

$$B_t = B_0(1 + i_{0,t})^t \quad (1)$$

where  $B_0$  is the value today of a zero-coupon bond,  $t$  is its maturity, and  $B_t$  is its par (or reimbursement) value. We could also say, equivalently, that  $i_{0,t}$  is the horizon rate of return of a zero-coupon bond bought today at price  $B_0$  and having an expected—and certain—value of  $B_t$  at time  $t$ . From (1), we can extract the horizon rate of return, as we have done in earlier chapters, and obtain

$$i_{0,t} = \left( \frac{B_t}{B_0} \right)^{1/t} - 1 \quad (2)$$

So the concept of spot rate is entirely equivalent to the concept of horizon rate of return.

Dividing both sides of (1) by  $(1 + i_{0,t})^t$ , we can write that  $i_{0,t}$  is such that

$$B_0 = \frac{B_t}{(1 + i_{0,t})^t} \quad (3)$$

We deduce from (3) that the spot rate is the internal rate of return of the investment, derived from buying a bond  $B_0$  and receiving  $B_t$   $t$  years later. Alternatively, we can see that the spot rate  $i_{0,t}$  is the rate that makes the net present value of that project equal to zero:

$$NPV = -B_0 + \frac{B_t}{(1 + i_{0,t})^t} = 0 \quad (4)$$

We conclude that the spot rate is the internal rate of the investment project derived from buying the zero-coupon bond, or equivalently the yield to maturity of the zero-coupon bond. The following table summarizes these properties of the spot rate.

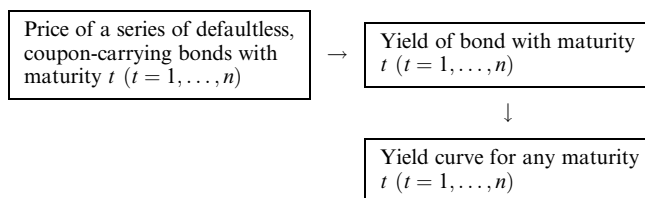
Equivalent Definitions	
Spot rate $i_{0,t}$ for a loan starting at time 0 and in effect for $t$ years	<ul style="list-style-type: none"> <li>• Horizon rate of return of an investment transforming an investment <math>B_0</math> into <math>B_t</math> <math>t</math> years later: <math display="block">B_t = B_0(1 + i_{0,t})^t</math> or <math>i_{0,t} = \left( \frac{B_t}{B_0} \right)^{1/t} - 1</math> </li> <li>• Yield to maturity <math>t</math> of a zero-coupon bond <math>B_0 = \frac{B_t}{(1+i_{0,t})^t}</math></li> <li>• Internal rate of return of a project where <math>B_0</math> is invested today and <math>B_t</math> is recouped in <math>t</math> years: <math display="block">NPV = -B_0 + \frac{B_t}{(1 + i_{0,t})^t} = 0</math> </li> </ul>

*The term structure of interest rates is the structure of spot rates  $i_{0,t}$  for all maturities  $t$ .* It is important *not* to confuse this with the yield curve, which we will define in section 9.2. How can we know the spot rate curve  $i_{0,t}$ ? The easiest way to determine this information would be to calculate the horizon rates of return for a whole series of defaultless zero-coupon bonds; this would give us our result immediately. Suppose that  $B_{0,t}$  designates the price today of a zero-coupon bond that will be reimbursed at  $B_t$  in  $t$  years. We would simply have to calculate  $i_{0,t} = (B_t/B_{0,t})^{1/t} - 1$  for the whole series of observed prices  $B_{0,t}$ , and we would be done. Unfortunately, markets do not carry many zero-coupon bonds, especially in the range of medium to long maturities. Therefore, we have to infer, or calculate, the spot rates by relying on the values of coupon-bearing bonds. This is what we will do in the following section.

## 9.2 The yield curve and the derivation of the spot rate curve

Let us first give a precise definition of the yield curve: it is the curve representing the relationship between the maturity of defaultless *coupon-bearing* bonds and their yield to maturity. In other words, it is the geometrical depiction of the various yields to maturity corresponding to the maturities of coupon-bearing bonds. This yield curve is easy to determine and widely published, because it is necessary simply to consider a set of coupon-bearing bonds with various maturities and to calculate their yields to maturity. However, this is *not* the term structure of interest rates, that is, the structure which is important to us and from which we want to infer the rates with which we can discount future cash flows or from which we can infer what the market thinks of future interest rates.

The following scheme summarizes the definitional and informational content of the yield curve:



It is clear that the yield curve is very different from the term structure of interest rates, that is, the spot rate curve. The yield curve does not convey

more information than its contents, which is the yield to maturity, and we know that the drawbacks of this concept are entirely similar to the internal rate of return of an investment. This return is a horizon return only if all reinvestments are made at a constant rate equal itself to the internal rate of return—which has reason not to be true.

So we must try to deduce the spot rate curve from the prices of the bonds with different maturities, rather than simply limiting ourselves to the yield curve. We now present a method that enables us to do this.

Consider first the one-year spot rate,  $i_{0,1}$ . We will be able to calculate it immediately from the price of a zero-coupon bond maturing in one year or from a coupon-bearing bond also maturing in one year. Suppose that our bond bears a coupon of 10 and that the bond's price is \$99. The spot rate  $i_{0,1}$  must then be such that

$$99(1 + i_{0,1}) = 110$$

and thus

$$i_{0,1} = \frac{110}{99} - 1 = 0.111 = 11.1\%$$

Let us now consider the price of a coupon-bearing bond with a maturity equal to 2 years,  $B_{0,2}$ . Its price is equal to the present value of all cash flows it will bring to its holder, discounted by the spot rates. One of them,  $i_{0,1}$ , is already known; the second one,  $i_{0,2}$ , is the unknown, which we can determine from the following equation:

$$B_{0,2} = \frac{10}{1 + i_{0,1}} + \frac{110}{(1 + i_{0,2})^2} \quad (5)$$

where, of course, the one-year spot rate  $i_{0,1}$  is known to be equal to 11.1%. Suppose, for instance, that  $c = 10$  and  $B_{0,2} = 97.5$ . Then  $i_{0,2}$  is determined by

$$97.5 = \frac{10}{1.11} + \frac{110}{(1 + i_{0,2})^2}$$

and so

$$i_{0,2} = 11.5\%$$

Suppose now that a third bond has a maturity of three years. Its coupon may very well be different from 10, but in our example, we will keep a

coupon of 10. The corresponding spot rate  $i_{0,3}$  will be determined by the following equation:

$$96 = \frac{10}{1.11} + \frac{10}{(1.115)^2} + \frac{110}{(1 + i_{0,3})^3}$$

which leads to

$$i_{0,3} = 11.7\%$$

This process can be repeated for all other spot rates, if we have the necessary information about a defaultless bond's price with the relevant maturity. If we want to calculate the spot rate  $i_{0,n}$ , knowing all spot rates  $i_{0,1}, \dots, i_{0,n-1}$  and the bond's value  $B_{0,n}$ , then  $i_{0,n}$  must verify the relationship

$$B_{0,n} = \frac{c}{1 + i_{0,1}} + \dots + \frac{c}{(1 + i_{0,n-1})^{n-1}} + \frac{c + B_{n,0}}{(1 + i_{0,n})^n} \quad (6)$$

where  $B_{n,0}$  denotes the value, at year  $n$ , of a bond whose maturity is 0 (that is, the par value of the bond). This relationship thus implies that

$$i_{0,n} = \left\{ \frac{c + B_{n,0}}{B_{0,n} - c \left[ \frac{1}{1 + i_{0,1}} + \dots + \frac{1}{(1 + i_{0,n-1})^{n-1}} \right]} \right\}^{1/n} - 1 \quad (7)$$

There is, of course, a more elegant way of presenting this derivation of the spot rate structure. If we use the concept of a discount rate as a *price*, we simply have to solve a system of linear equations.

Suppose indeed that we have  $n$  bonds at our disposal, each with maturity  $T = 1, \dots, n$ . Denote by  $B^{(t)}$  the value of the bond with the  $t$ th maturity and  $c_1^{(t)}, \dots, c_t^{(t)}$  the cash flows corresponding to bond  $t$ ;  $c_t^{(t)}$  is always the par value of the  $t$ th bond plus the last coupon. Also, denote, as we did in chapter 2, the discount factor  $1/(1 + i_{0,t})^t$  as the *price*  $b(0, t)$  of one dollar received at time  $t$  in terms of dollars of year zero. We then have the system of  $n$  equations in  $n$  unknowns  $b(0, 1) \dots b(0, n)$ :

$$B^{(1)} = c_1^{(1)} b(0, 1)$$

$$B^{(2)} = c_1^{(2)} b(0, 1) + c_2^{(2)} b(0, 2)$$

$$\begin{aligned}
B^{(3)} &= c_1^{(3)}b(0,1) + c_2^{(3)}b(0,2) + c_3^{(3)}b(0,3) \\
&\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\
B^{(n)} &= c_1^{(n)}b(0,1) + c_2^{(n)}b(0,2) + c_3^{(n)}b(0,3) + \cdots + c_n^{(n)}b(0,n)
\end{aligned}$$

Using the following notation:

$\mathbf{B} \equiv$  the vector of the bonds' values

$\mathbf{C} \equiv$  the (diagonal) matrix of the bonds' cash flows

$\mathbf{b} \equiv$  the (unknown) discount factors vector

we have

$$\mathbf{B} = \mathbf{C}\mathbf{b}$$

which generally can be solved in  $\mathbf{b}$  as

$$\boxed{\mathbf{b} = \mathbf{C}^{-1}\mathbf{B}} \quad (8)$$

For instance, in our former simple case, matrix  $\mathbf{C}$  and vector  $\mathbf{B}$  would be

$$\mathbf{C} = \begin{bmatrix} 110 & 0 & 0 \\ 10 & 110 & 0 \\ 10 & 10 & 110 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 99 \\ 97.5 \\ 96 \end{bmatrix}$$

Vector  $\mathbf{b}$  is then

$$\mathbf{b} = \begin{bmatrix} 0.9 \\ 0.8045 \\ 0.7178 \end{bmatrix}$$

Remember now that  $i_{0,t}$  is obtained through

$$i_{0,t} = \left[ \frac{1}{b(0,t)} \right]^{1/t} - 1$$

We then have the spot rate structure

$$i_{0,1} = 11.1\%$$

$$i_{0,2} = 11.5\%$$

$$i_{0,3} = 11.7\%$$



**Table 9.1**  
**Maturity, coupon, and equilibrium price for a set of bonds**

Maturity $t$ (years)	Coupon	Equilibrium price vector $\mathbf{B}$
1	5	100.478
2	5.5	101.412
3	6.25	103.591
4	6	103.275
5	5.75	102.501
6	5.5	101.199
7	4	92.220
8	4.5	94.247
9	6.5	107.434
10	6	104.213

**Table 9.2**  
**Resulting discount factors (zero-coupon prices) and spot rate structure**

Maturity $t$ (years)	Discount factor $b(0, t)$	Spot rate interest term structure $i_{0,t}$
1	0.95693	0.045
2	0.91136	0.0475
3	0.86507	0.0495
4	0.81957	0.051
5	0.77609	0.052
6	0.73355	0.053
7	0.69202	0.054
8	0.65408	0.0545
9	0.61763	0.055
10	0.58543	0.055

Let us now consider a more realistic example, where we try to determine a 10-year term structure, that is, the set of all spot rates spanning horizons from one to 10 years. Table 9.1 summarizes the information we have about the maturity, the coupon, and the equilibrium prices of a series of 10 bonds paying their coupons on an annual basis. Each of these bonds is considered as having a \$100 par value.

Table 9.1 provides all the ingredients needed to construct the diagonal matrix  $\mathbf{C}$  whose inverse has to be multiplied by vector  $\mathbf{B}$  of the bonds' prices to obtain the zero-coupon price vector  $\mathbf{b}$ . From this the spot rates can be deduced through  $i_{0,t} = [b(0, t)]^{-1/t} - 1$ . The results are in table 9.2. The spot rate interest structure we get is exactly the one we presented as an example in chapter 2 (section 2.4, table 2.5).

Simple and appealing as this method may look, it is unfortunately difficult to apply directly. Indeed, observations may be more than complete in the sense that many coupon bonds mature on each coupon date. In the past 30 years, researchers have tried to cope with this problem with more and more refined econometric methods. One could even say that they have displayed remarkable ingenuity in their pursuit. However, we will have to wait until chapter 11, where we deal with continuous spot and forward rates of return, to present the latest research in this field.

### 9.3 Derivation of the implicit forward rates from the spot rate structure

It is now natural to ask the following question: is it possible to exploit our knowledge of the term structure of interest rates (the spot rate curve) to calculate the implicit forward rates, and then to try to deduce from those the future spot rates that are expected by the market? For instance, suppose the current structure is increasing, with the long rates being higher than the short rates. Does this reveal any information about future spot rates that are expected by the market? In this example, does it mean that the market believes that the future short rates will be higher than the current short rates? As we shall see, the answer to such a question is far from obvious. We will proceed in two steps: first, we will suppose that individuals are risk-neutral; then we will suppress this rather stringent hypothesis and suppose that the majority of agents are risk-averse.

We should first recognize that if, on any given market, there exists a term structure of interest rates, this implies the existence of a forward structure. The following example will make this clear. Suppose we know the two spot rates corresponding to the one-year and two-year contracts, denoted as  $i_{0,1}$  and  $i_{0,2}$ . From this structure, any individual can construct today a one-year forward loan starting in one year; in other words, if an individual wished to borrow a certain amount in one year, he can lock in today a rate of interest that will be prevailing with certainty in one year, by doing the following:

- He borrows \$1 today for two years and repays  $(1 + i_{0,2})^2$  in two years.
- He lends \$1 today for one year; he will therefore receive  $1 + i_{0,1}$  in one year. Hence, since he receives  $1 + i_{0,1}$  in one year and will have to reimburse  $(1 + i_{0,2})^2$  a year later, in effect he will have paid an (implicit) forward rate, applicable in one year for a one-year time span. This is denoted  $f_{1,1}$ , such that it transforms  $1 + i_{0,1}$  into  $(1 + i_{0,2})^2$  a year later.

Thus, the implied forward rate  $f_{1,1}$  must be such that

$$(1 + i_{0,1})(1 + f_{1,1}) = (1 + i_{0,2})^2$$

$$1 + f_{1,1} = \frac{(1 + i_{0,2})^2}{1 + i_{0,1}} \quad (9)$$

which is equivalent to

$$f_{1,1} = \frac{(1 + i_{0,2})^2}{1 + i_{0,1}} - 1 \quad (10)$$

Of course, the process is perfectly symmetrical from the point of view of a lender. Suppose someone wishes to lock in a forward rate for a loan he wants to make in one year for one year. He can easily lend \$1 today for two years at rate  $i_{0,2}$ , and he can borrow \$1 for one year at  $i_{0,1}$ . In one year, he will repay his one-year loan  $1 + i_{0,1}$ , and after two years he will receive  $(1 + i_{0,2})^2$ . This implies that he will have made a loan in one year such that the forward rate he will get will be equal to  $f_{1,1} = \frac{(1+i_{0,2})^2}{1+i_{0,1}} - 1$ , as before.

These conclusions can be generalized to any forward rate, given a whole spot rate structure. For instance, consider the implicit forward interest rate starting at  $t$  and valid for the  $n$  following years,  $f_{t,n}$ . A borrower can lock in this rate, for instance, by borrowing \$1 today at rate  $i_{0,t+n}$  and lending the same \$1 today for a period of  $t$  years at rate  $i_{0,t}$ . In  $t$  years, our individual will pay back  $(1 + i_{0,t})^t$  dollars, and  $n$  years later, he will receive  $(1 + i_{0,t+n})^{t+n}$  dollars. This amounts to transforming, at time  $t$ ,  $(1 + i_{0,t})^t$  into  $(1 + i_{0,t+n})^{t+n}$   $n$  years later. Thus the implied forward rate,  $f_{t,n}$ , is such that

$$(1 + i_{0,t})^t (1 + f_{t,n})^n = (1 + i_{0,t+n})^{t+n} \quad (11)$$

which is equivalent to

$$(1 + f_{t,n})^n = \frac{(1 + i_{0,t+n})^{t+n}}{(1 + i_{0,t})^t}$$

or

$$f_{t,n} = \frac{(1 + i_{0,t+n})^{(t+n)/n}}{(1 + i_{0,t})^{t/n}} - 1 \quad (12)$$

Let us call 1 plus any interest rate the *capitalization rate*, which is therefore the ratio of a capital one year hence over the capital invested today, when a given rate of interest transforms the latter into the former. (This capitalization rate is also called the *dollar rate of return*.)

The reader will notice from (11) that the long-term spot rate  $i_{0,t+n}$  is related to the shorter-term spot rate  $i_{0,t}$ , as well as the implicit forward rate  $f_{t,n}$ , by a time-weighted geometric average of their respective capitalization rates. From (11) we can write

$$1 + i_{0,t+n} = (1 + i_{0,t})^{\frac{t}{t+n}} (1 + f_{t,n})^{\frac{n}{t+n}} \quad (13)$$

which is indeed a geometric average of the short-term capitalization rate  $1 + i_{0,t}$  and the implicit forward capitalization rate, the respective weights being the shares  $\frac{t}{t+n}$  and  $\frac{n}{t+n}$  in the total time span considered  $t + n$ .

From one given spot rate structure, we have calculated all implicit forward rate structures. First, we considered an increasing spot structure (table 9.3; the same results were also shown in figure 9.1). We then supposed that the term structure was “humped,” increasing in a first phase and then decreasing. These results are summarized in table 9.4 and figure 9.2.

The reader should notice that when we consider *one* spot structure, this implies the existence of a *series* of implicit forward structures, which can be classified in several ways. For instance, we have all forward rates that start in one year, for loans with maturities ranging from one to nine years (in our example). This is the second line of table 9.3. Next, the implicit forward rates start in two years, for time spans between one and eight years (the third line in our table). Finally, we have the implicit forward rate that starts in nine years and ends one year later (the last line and last figure of our table). Figure 9.1 represents the first six lines of table 9.3, and the same has been done in figure 9.2 for table 9.4. But we could also visualize implicit forward rates under yet another classification. If we consider all forward rates with a one-year maturity, starting at different dates, we will see that these are all the rates located in the main diagonal of our table. All forward rates starting at various years and corresponding to maturities of  $t$  years would be those in the diagonal, which is located  $t - 1$  spaces further to the right of the main diagonal. As an example, we have shown in the table all forward rates with a maturity of four years, which are in the diagonal located three spaces to the right of the main diagonal.

**Table 9.3**  
**Implicit forward rates derived from a given time structure of interest rates**  
**(increasing structure)**

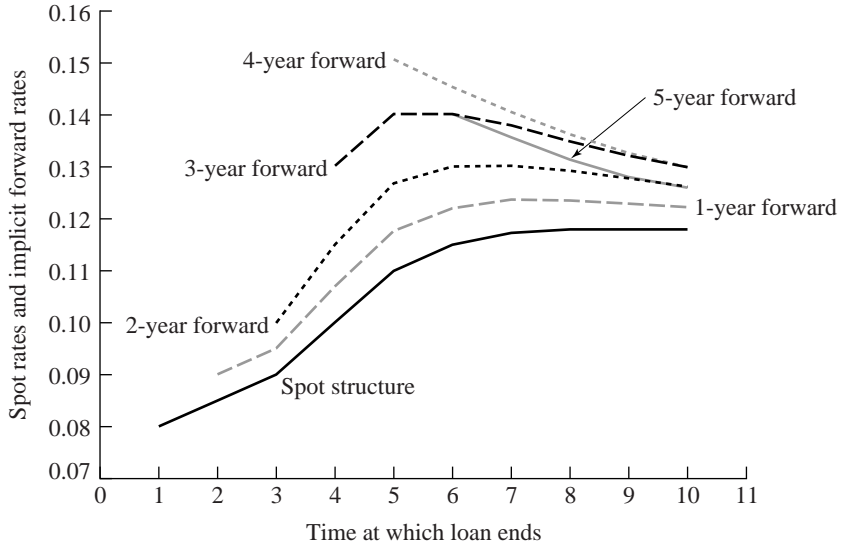
Time at which loan starts ( $t$ )	Time at which loan ends ( $t+n$ )									
	1	2	3	4	5	6	7	8	9	10
0	8.0	8.5	9.0	10.0	11.0	11.5	11.73	11.8	11.8	11.8 ← Spot rate structure
1		9.0	9.5	10.7	11.8	12.2	12.4	12.4	12.3	12.2
2			10.0	11.5	12.7	13.0	13.0	12.9	12.8	12.6
3				13.1	14.1	14.1	13.8	13.5	13.2	13.0
4					15.1	14.6	14.1	13.6	13.3	13.0
5						14.0	13.6	13.1	12.8	12.6 ← Implicit four-year maturity forward contracts
6							13.1	12.7	12.4	12.3
7								12.0	12.0	11.9
8									11.8	11.8
9										11.8 ← Implicit one-year maturity forward contracts

#### 9.4 A first approach: individuals are risk-neutral; the pure expectations theory

Let us consider two periods. Our problem is to try to know what the market expects the future one-year spot rate to be in one year, that is,  $i_{1,1}$ , relying on our knowledge of the present spot rates, which we call the short rate ( $i_{0,1}$ ) and the long rate ( $i_{0,2}$ ). We will suppose that

- Lenders try to get the highest return for their loans.
- Borrowers try to pay the smallest interest rate.

Two possibilities are offered to lenders as well as to borrowers. Either lenders can invest on the two-year market or they can perform two suc-



**Figure 9.1**  
Implicit forward rates derived from a given spot rates structure (increasing structure).

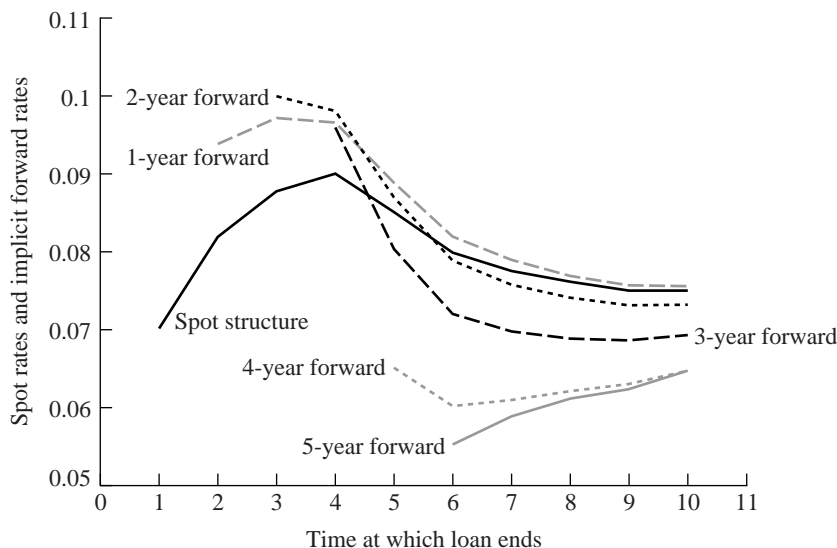
cessive operations: the first on the one-year spot market and the second in one year on the one-year spot market. In a symmetrical way, borrowers, on a two-year period, can borrow on the two-year market, or alternatively, they can borrow twice for a one-year period. In the first case they will sell a two-year maturity bond; in the second case they will sell two successive one-year bonds.

We will show that if lenders and borrowers are indifferent with respect to those two strategies, then the expected one-year rate for year 1,  $E[i_{1,1}]$ , must be linked to the spot rates  $i_{0,1}$  and  $i_{0,2}$  in the following manner:

$$(1 + i_{0,2})^2 = (1 + i_{0,1})(1 + E[i_{1,1}]) \quad (14)$$

In other words,  $E[i_{1,1}]$  must be the implied forward rate  $f_{1,1}$  we introduced previously. Analogous to what we showed earlier, the two-year capitalization rate  $1 + i_{0,2}$  is the geometric average between  $1 + i_{0,1}$  and the *expected* capitalization rate  $1 + E[i_{1,1}]$ ; (13) implies indeed

$$1 + i_{0,2} = \{(1 + i_{0,1})(1 + E[i_{1,1}])\}^{1/2} \quad (15)$$



**Figure 9.2**  
Implicit forward rates derived from a given spot rates structure (hump-shaped spot structure).

Let us now show that (13) is valid under the risk-neutrality hypothesis we have made. Suppose that (13) does not hold and that the expected rate  $E[i_{1,1}]$  is such that it seems less profitable for the investor to invest on the two-year market than to lend twice consecutively on the one-year market. This hypothesis implies that it will be less costly for the borrower to borrow on the two-year market than to try to raise funds twice on the shorter term markets. Suppose our data are the following:

- One-year spot rate =  $i_{0,1} = 6\%$
- Expected future spot rate, in one year =  $E[i_{1,1}] = 7\%$

The geometric average of  $(1 + i_{0,1})$  and  $(1 + E[i_{1,1}])$  is  $[(1.06)(1.07)]^{1/2} = 1.064988 \approx 1.065$ ; the two-year spot rate that would make both lenders and borrowers indifferent between the two strategies would be  $i_{0,2} = 1.065 - 1 = 6.5\%$ . Note that for two fairly close figures, namely 1.06 and 1.07, the simple geometric mean is extremely close to the arithmetic mean.) Let us then suppose that the two-year spot rate is significantly smaller than 6.5%, being equal to 6.2%. We would then have

**Table 9.4**  
**Implicit forward rates derived from a given time structure of interest rates**  
**("humped" shape)**

Time at which loan starts ( $t$ )	Time at which loan ends ( $t + n$ )										
	1	2	3	4	5	6	7	8	9	10	
0	7.0	8.2	8.8	9.0	8.5	8.0	7.75	7.6	7.5	7.5	← Spot rate structure
1		9.4	9.7	9.7	8.9	8.2	7.9	7.7	7.6	7.6	
2			10.0	9.8	8.7	7.9	7.6	7.4	7.3	7.3	
3				9.6	8.0	7.2	7.0	6.9	6.9	6.9	
4					6.5	6.0	6.1	6.2	6.3	6.5	
5						5.5	5.9	6.1	6.2	6.5	
6							6.3	6.4	6.5	6.8	← Implicit four-year maturity forward contracts
7								6.6	6.6	6.9	
8									6.7	7.1	
9										7.5	← Implicit one-year maturity forward contracts

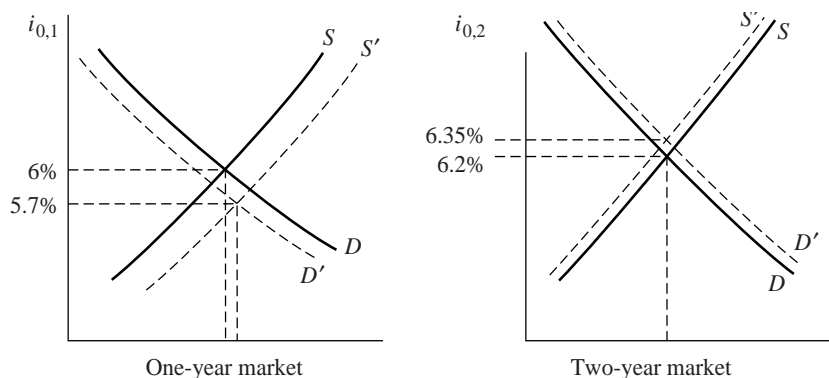
$$(1.062)^2 < (1.06)(1.07)$$

and therefore

$$(1 + i_{0,2})^2 < (1 + i_{0,1})(1 + E[i_{1,1}])$$

Under these conditions, borrowers will want to enter the two-year market, and they will leave the one-year market. Symmetrically, investors will do just the opposite: they will certainly want to quit the two-year market to invest in the one-year market. Here is a possible scenario, which we have summarized in figure 9.3: the initial supply and demand of loanable funds on the one-year and two-year markets are depicted in full lines. The equilibrium rates are 6% and 6.2%, respectively, as we supposed earlier.



**Figure 9.3**

If 6% and 6.2% are equilibrium rates, respectively, and if the expected spot rate  $E[i_{1,1}]$  is 7%, we have the following inequality:

$$(1.062)^2 < (1.06)(1.07)$$

which entails arbitrage opportunities. It also becomes:

- less expensive for borrowers to go to the long market; they will enter the two-year market and exit the one-year market;
- less rewarding for investors to be on market 2 than on market 1; they will want to exit market 2 and enter market 1.

A new equilibrium can be reached with 5.7% on market 1 and 6.35% on market 2. We will then have

$$1.0635^2 = (1.057)(1.07)$$

which does not allow arbitrage opportunities any more.

Since borrowers prefer to choose the two-year market rather than rolling over a one-year loan, they will increase their demand for loanable funds on the two-year market and will reduce their demand on the one-year market. On the former market, their demand will move to the right, and it will move to the left on the latter. As for investors, they will want to stay away from the long rates and will start looking for the short rates. Their supply of loanable funds will move to the left on the two-year market and to the right on the one-year market.

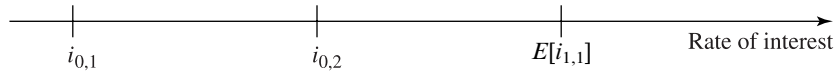
We can readily see the consequences of these modifications in the behavior of borrowers and lenders: the two-year spot rate will increase and the one-year spot rate will diminish, under the conjugate effects of an excess demand for longer loans and an excess supply for shorter ones.

Note here that we do not know which new equilibrium will be obtained. (We have only one equation (13) in two unknowns  $(i_{0,1}$  and  $i_{0,2})$ ,

supposing that the market's forecast of the future spot rate  $E[i_{1,1}]$  is not affected by the movements in the spot rates we have just described.) The only thing we can be sure of in this model where investors and borrowers are risk-neutral is that  $i_{0,1}$ ,  $i_{0,2}$ , and  $E[i_{1,1}]$  must be related by (13). For instance,  $i_{0,2}$  could increase from 6.2% to 6.35%, and  $i_{0,1}$  could drop from 6% to 5.7%, as indicated in our graph (dotted lines). This would indeed lead to a new equilibrium, since

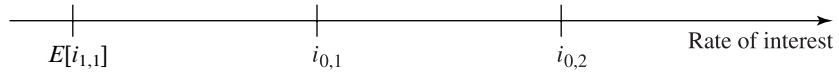
$$(1.0635)^2 \approx (1.057)(1.07)$$

Let us now remind ourselves of the property according to which the two-year ratio is the geometric mean of the one-year capitalization ratio and the one-year expected capitalization ratio. This implies that spot rates and the expected future spot rate are those indicated in the diagram below, if the two-year rate is higher than the one-year rate.



This implies that  $E[i_{1,1}]$  is higher than  $i_{0,1}$  and thus that the market believes that the short rate will increase. Consequently, when agents are risk-neutral, an increasing term structure of interest rates indicates that the market forecasts an increase in short rates.

Symmetrically, suppose that the term structure of interest rates is decreasing. We then have the following situation:



Since  $E[i_{1,1}]$  is necessarily lower than  $i_{0,2}$ , it is lower than  $i_{0,1}$ . A decreasing structure implies a forecast of falling short rates. Finally, if the term structure is flat (if  $i_{0,1} = i_{0,2}$ ), this means that agents throughout the market believe that short rates will stay constant.

We have shown how, in equilibrium and in a universe where agents are risk-neutral, the expected one-year spot rate can be determined simply by applying equation (13), or, equivalently, by writing

$$E[i_{1,1}] = \frac{(1 + i_{0,2})^2}{1 + i_{0,1}} - 1 \quad (16)$$

which is exactly the implicit forward rate defined in (10).

An obvious and serious problem arises, however, if we consider long periods for lending or borrowing. We have supposed that all agents were risk-neutral and that they accepted the arbitrage equations (12) and (15) as if the forecast of future rates,  $i_{1,1}$  and  $i_{t,n}$ , respectively, did not convey any sort of risk worth worrying about. In real life, this is certainly not the case: those forecasts are highly error prone; they do carry some element of risk; and the vast majority of agents are risk-averse. The time has now come to abandon the risk-neutrality hypothesis and to evaluate the consequences associated with risk-averse agents.

## 9.5 A more refined approach: determination of expected spot rates when agents are risk-averse

We want to answer the following question: are lenders and borrowers indifferent between investing (borrowing) on the two-year market and undertaking similar operations twice on the one-year market, the first time at a rate known with certainty ( $i_{0,1}$ ) and the second time at an expected rate  $E[i_{1,1}]$ ? In other words, does an arbitrage relationship such as

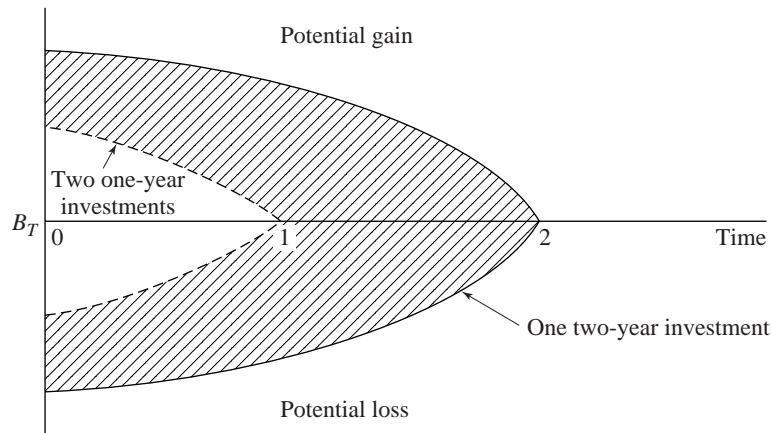
$$(1 + i_{0,2})^2 = (1 + i_{0,1})(1 + E[i_{1,1}]) \quad (17)$$

really hold, for investors as well as for borrowers? Earlier, we answered yes to this question, because we supposed that whatever risk was involved in using the estimate  $E[i_{1,1}]$ , agents considered that bad surprises were weighted exactly evenly against good surprises, which is what made the agents neutral to risk. But this is an oversimplification of reality. We will first consider the case of investors. In this category, we will distinguish those with a short horizon from those with a long horizon. We will then turn to the case of the borrowers and examine the same type of distinction.

### 9.5.1 The behavior of investors

#### 9.5.1.1 Investors with a short horizon

The short route (consisting of making two one-year loans) is not risky; however, the two-year route involves much more risk, because if investors want to cash in their investment, they are exposed to a potential capital loss—although it is also possible that they will benefit from a capital gain if in the meantime interest rates have dropped.



**Figure 9.4**

Value of a single two-year investment and two one-year investments if the rate of interest is modified immediately after the first investment, at time 0. The two-year investment entails additional profits and losses compared to the one-year investment. It is thus riskier. Those potential losses, as well as the potential gains, are represented by the hatched area.

To make matters simple, suppose that both spot rates  $i_{0,1}$  and  $i_{0,2}$  are equal (for instance, both are equal to 9%). An investor can lend his money for one year at 9%. Suppose that he lends at 9%, buying a bond that might be at par, and that immediately afterward the interest rates move, either up or down. If they move up, the value of this bond becomes the dotted line below the horizontal (at height  $B_T$ ) in figure 9.4. After one year, however, he can recoup his original capital ( $B_T$ ). If he then reinvests his money, and if interest rates do not move in any significant way, his capital will not be affected. This is what we have described by the horizontal dotted line between  $t = 1$  and  $t = 2$ . Note, of course, the fact that our investor will be able to roll over his loan at a higher rate, since we have supposed that rates would not fluctuate after their initial rise.

If rates go down, the short investor benefits from a capital gain during the first year but none whatsoever in the second year, and his reinvestment rate after one year will be supposedly lower than in the first year.

#### 9.5.1.2 Investors with a longer horizon

For those investors who choose the long two-year route, the possible capital loss following a rise in interest rates is indicated by the continuous line in figure 9.4, below  $B_T$ . As the reader can immediately see, these

**Table 9.5**  
**Potential gains and losses of investors, corresponding to the long and short strategies**

Strategy	Capital gain or loss		Reinvestment gain or loss	
	$i$ increases	$i$ decreases	$i$ increases	$i$ decreases
Short investment and roll-over	Small loss	Small gain	Reinvestment at a higher rate	Reinvestment at a lower rate
Long investment	Large loss	Large gain	No reinvestment; same coupon	

potential losses last for the whole two-year time span, and their magnitude is always higher than those for the investor who chose the short route. Admittedly, the potential gains (indicated by the hatched area above the horizontal line  $B_T$ ) are higher, but the risk is also considerably higher.

The advantages and the drawbacks of the short investments compared to those of the long investment are summarized in table 9.5. From the point of view of capital gains, the two-year investment is much riskier than rolling over a one-year loan; but, of course, if we consider reinvestment risk, the double loan is riskier, since our investor is liable to see his coupon rate decrease if interest rates go down. It is thus clear that each strategy carries a specific risk: the long investor fears a capital loss in case of an increase in interest rates, while the short strategy involves the risk of rolling over at a lower interest rate. It is obvious that the capital loss risk is more substantial than the reinvestment risk. This will have important consequences.

Indeed, we may conclude from this analysis that investors will be more interested in looking for short deals rather than long ones; therefore, there will be a natural tendency for supply to be relatively abundant on short markets and relatively scarce on long ones. It remains for us now to see whether the behavior of borrowers will reinforce these consequences or not.

### 9.5.2 The borrowers' behavior

Borrowers with a short horizon will have a natural tendency to borrow on short-term markets. They could of course borrow on the two-year market, but they would have to redeem their bonds after one year, and we know that capital gains or losses are at stake. Consequently, short-term borrowers will try to minimize their risk by borrowing on the short market.

Borrowers with longer horizons will preferably choose the long markets. Of course, they could roll over a series of short loans, but they would

thus take the risk of an interest rate increase, while long markets do not carry such risk. Thus, their behavior will tend to counter the consequences of the short-term borrowers' action.

As before, we can ask the question about which tendency will prevail. The answer depends on the magnitude of the loanable funds' demands on each market. Generally speaking, we can say that borrowers have typically long horizons, and they will usually accept paying a risk premium. Since their behavior will tend to increase the long-term rates compared to the short-term ones, we can see that the consequences generated by the lenders' behavior will be reinforced. In sum, on the long markets we will generally find a relatively scarce supply of funds and a relatively abundant demand for funds. Symmetrically, short-term markets will usually witness an ample supply and a short demand for funds, leading to long interest rates that will often be higher than short ones.

The conclusion we can draw from this analysis is this: generally, *both* suppliers and borrowers of funds will agree that a liquidity premium must be paid on long loans. It will then be quite acceptable to admit that the long-term capitalization rate is not a weighted average of the short-term rate and the expected short-term rate but that it is somewhat higher than that. Thus we have an inequality of the following type:

$$(1 + i_{0,2})^2 > (1 + i_{0,1})(1 + E[i_{1,1}]) \quad (18)$$

We can therefore justify the existence of a “risk premium” to be received by investors and paid by borrowers in the following way. It is simply equal to the additional amount that has to be added to the expected spot rate in order to make the square of the long capitalization rate equal to the product of the two one-year capitalization rates. Thus, it will always be such that

$$(1 + i_{0,2})^2 = (1 + i_{0,1})(1 + E(i_{1,1}) + \pi) \quad (19)$$

We can now draw some conclusions from the general shape of the interest rates term structure. Suppose first that this structure is flat, that is,  $i_{0,2} = i_{0,1}$ . Does this observation imply that the market forecasts that the future spot rates will remain constant? We just showed that both lenders and borrowers agreed that long investments should be rewarded by a risk premium. Consequently, we can conclude from this that future spot rates are expected by the market to go down. If  $i_{0,2} = i_{0,1}$ , equation (19) permits us to write

$$1 + i_{0,1} = 1 + E[i_{1,1}] + \pi \quad (20)$$

and thus

$$E[i_{1,1}] = i_{0,1} - \pi \quad (21)$$

The market therefore expects a decrease in the spot rate when the spot rate structure is constant.

Let us suppose now that we face a *decreasing* structure ( $i_{0,2} < i_{0,1}$ ). We have, for instance,  $1 + i_{0,2} = \theta(1 + i_{0,1})$ ,  $0 < \theta < 1$ . Inequality (18) can then be written, after simplifying by  $(1 + i_{0,1})$ ,

$$\theta^2(1 + i_{0,1}) > 1 + E[i_{1,1}] \quad (22)$$

and we have

$$\frac{1 + E[i_{1,1}]}{1 + i_{0,1}} < \theta^2 < 1 \quad (23)$$

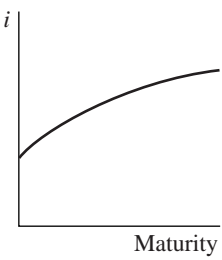
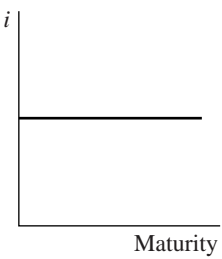
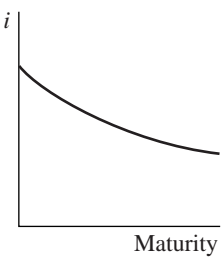
Consequently,  $E[i_{1,1}] < i_{0,1}$ ; the market expects the future spot rate to be lower than today.

Let us examine finally what conclusions can be drawn if we face an increasing structure of interest rates. Since long rates integrate a liquidity premium, we are not in a situation to know, a priori, whether the structure that would not incorporate such a premium would be increasing or not. As a result, we cannot deduce from such an increasing structure that the market expects spot rates to rise. What we *are* able to say is that an increasing structure corresponding to *constant* expected spot rates definitely exists—but we do *not* know where exactly it is located, since we do not know the exact magnitudes of all risk premiums. If the observed structure is above this unknown error, then the market definitely expects spot rates to rise. If it is under it, spot rates are expected to decrease.

We have summarized in table 9.6 the conclusions that can be drawn from a set of observations of the term structure of interest rates when we try to make our reasoning under two different types of hypotheses: the pure expectations hypothesis, implying risk-neutrality of agents (a hypothesis we are not very willing to accept); and the liquidity premium hypothesis, corresponding to risk aversion of economic actors, a hypothesis that is generally accepted. We need to stress, however, that even if we can infer from an interest rate structure what the market *thinks* the future interest rates will be, this is a far cry from being able to predict what will

**Table 9.6**

**Relationship between observed term structures of interest rates and the implied forecasts of future spot rates, according to the hypotheses of risk neutrality and risk aversion of economic agents**

Observed time structure of interest rates	Implied forecast of future spot rates	
	Pure expectations theory (risk-neutrality hypothesis)	Liquidity premium theory (risk-aversion hypothesis)
	Interest rates are expected to rise	Indeterminacy of expected spot rates
	Interest rates are expected to remain constant	Interest rates are expected to fall
	Interest rates are expected to fall	Interest rates are expected to fall



actually happen in the future. Hence the importance of immunization, for which we will demonstrate a general theorem in our next chapter.

### Key concepts

- Yield curve: the relationship between the maturity of defaultless coupon-bearing bonds and their yield to maturity.
- Term structure of interest rates: the spot rate curve, or the relationship between the maturity of defaultless zero-coupon bonds and their yield to maturity.
- Forward interest rates: interest rates set up today for loans that will be made at a future date for a well-determined time span.
- Implied forward interest rates: the forward rates that are implied by the current spot rate curve.

### Main formulas

- Spot rate (for rates compounded once per year):

$$i_{0,n} = \left\{ \frac{c + B_{n,0}}{B_{0,n} - c \left[ \frac{1}{1+i_{0,1}} + \cdots + \frac{1}{(1+i_{0,n-1})^{n-1}} \right]} \right\}^{1/n} - 1 \quad (7)$$

- Discount factor:

$$b_{0,t} = \frac{1}{(1 + i_{0,t})^t}$$

- Vector of bonds values (**B**):

$$\mathbf{B} = \mathbf{C}\mathbf{b}$$

where

**C** = (diagonal) matrix of the bonds' cash flows

**b** = discount factors vector

- Discount factors vector:

$$\mathbf{b} = \mathbf{C}^{-1}\mathbf{B} \quad (8)$$

- Forward rate decided upon today for contract starting at  $t$  for a trading period of  $n$  years:

$$f_{t,n} = \frac{(1 + i_{0,t+n})^{(t+n)/n}}{(1 + i_{0,t})^{t/n}} - 1 \quad (12)$$

### Questions

9.1. Consider a spot rate  $i_{0,t}$  for a loan starting at time 0 and in effect for  $t$  years. Can you give three other equivalent concepts of this spot rate in terms of horizon rate of return, yield to maturity, and internal rate of an investment project, respectively?

The next three questions can be considered as projects.

9.2. Using as input the data on the set of bonds given in table 9.1 (section 9.2), derive the discount factors and the spot rate interest term structure  $i_{0,t}$  as contained in table 9.2 of the same section.

9.3. Determine the implicit forward rates derived from an increasing structure of spot interest rates of your choosing.

9.4. Determine the implicit forward rates derived from a hump-shaped structure of spot interest rates of your choosing.

# 10 IMMUNIZING BOND PORTFOLIOS AGAINST PARALLEL MOVES OF THE SPOT RATE STRUCTURE

EARL of SALISBURY. . . . much more general than  
these lines import . . .

*King John*

## Chapter outline

10.1	Value of a bond in continuous time for a given term structure	211
10.2	A fundamental property of duration	212
10.3	Value of a bond at horizon $H$	212
10.4	A first-order condition for immunization	213
10.5	A second-order condition for immunization	214
10.6	The difficulty of immunizing a bond portfolio when the term structure of interest rates is subject to any kind of variation	215

## Overview

We will now generalize the result we obtained in chapter 5, according to which a fixed income investment may well, in certain circumstances, be immunized against a change in the structure of interest rates. We had supposed that this structure was horizontal and that it would receive a parallel shift. We are now in a position to suppose that the initial structure has any kind of shape (not necessarily horizontal); however, we will still suppose that it undergoes a parallel shift. We will of course have to redefine duration within this somewhat broader framework. For that purpose, we will use the concept introduced initially by Fisher and Weil.<sup>1</sup> The difference between their concept and that of Macaulay is that the weights of the times of payment now make use of the *spot rates* pertaining to each term, while Macaulay relied on the bond's yield to maturity to discount the cash flows.

In order to facilitate our calculations, we will work in continuous time. Let us designate any time structure of spot interest rates as

$$i(0, 1), i(0, 2), \dots, i(0, t), \dots, i(0, T)$$

where the first index (0) designates the present time and the second index the term we consider. To simplify notation, let  $i(z)$  designate the *instantaneous* rate of interest prevailing at time 0 for a loan starting at time  $z$  with

<sup>1</sup>L. Fisher and R. Weil, "Coping with the Risk of Interest-rate Fluctuations: Return to Bondholders from Naive and Optimal Strategies." *The Journal of Business*, 44, no. 4 (October 1971).

an infinitely small maturity. With this notation, we can easily derive a definite relationship between the spot interest rates  $i(0, t)$ —the interest rate prevailing today for a loan maturing at time  $t$ —and the implicit instantaneous forward rates  $i(z)$ . We must have the arbitrage-driven relationship

$$e^{\int_0^t i(z) dz} = e^{i(0, t)t} \quad (1)$$

because \$1 invested today at the spot rate  $i(0, t)$  must yield the same amount as \$1 rolled over at all infinitesimal forward rates  $i(z)$ . Consequently, taking the log of (1), we get

$$i(0, t) = \frac{\int_0^t i(z) dz}{t} \quad (2)$$

We can see that the spot rate  $i(0, t)$  is none other than the average of all implicit forward rates. Should the structure of interest rates receive a shock  $\alpha$ , identical for all terms, we can conclude that this would be entirely equivalent to each implicit forward rate receiving the same increase  $\alpha$ . In fact, from (2) we may write

$$i(0, t)t = \int_0^t i(z) dz \quad (3)$$

This equality remains unchanged if we add  $\alpha t$  to each side:

$$i(0, t)t + \alpha t = \int_0^t i(z) dz + \alpha t$$

which may also be written as

$$[i(0, t) + \alpha]t = \int_0^t i(z) dz + \int_0^t \alpha dz = \int_0^t [i(z) + \alpha] dz \quad (4)$$

We can see that increasing each rate  $i(0, t)$  by  $\alpha$  is equivalent to displacing vertically by  $\alpha$  the implicit forward rate structure.

We shall denote the present value of a cash flow  $c(t)$  received at time  $t$  (per infinitesimal period  $dt$ )<sup>2</sup> as

$$c(t)e^{-\int_0^t i(z) dz}$$

<sup>2</sup>Here we assume a continuously paid coupon; so, for instance, in nominal terms the cash flow corresponding to the first year is  $\int_0^1 c(t) dt = \int_0^1 c dt = c$ . This means that within any infinitesimal time interval  $dt$ , however small, the bond owner receives  $c dt$ . Also in nominal terms, the cash flow for the last year is  $\int_{T-1}^T c(t) dt = \int_{T-1}^T (c + B_T) dt = c + B_T$ .

or, in an entirely equivalent manner, as

$$c(t)e^{-i(0,t)t}$$

and we shall keep in mind that any increase given to  $i(z)$  is equivalent to an identical increase given to  $i(0, t)$ .

Let us now define a bond's value and its duration when we consider a term structure of interest rates denoted as  $i(0, t)$ .

### 10.1 Value of a bond in continuous time for a given term structure

Consider a term structure  $i(0, t)$ , which we will denote as  $\vec{i}$  for simplicity. The value of a bond when the structure is  $\vec{i}$  will be equal to

$$B(\vec{i}) = \int_0^T c(t)e^{-i(0,t)t} dt \quad (5)$$

The reader who is familiar with the calculus of variations will immediately recognize in (5) a functional, that is, a relationship between a function ( $\vec{i}$ ) and a number ( $B(\vec{i})$ ).

Let us now suppose that the whole structure receives an increase  $\alpha$  (in the language of the calculus of variations, we would say that  $\vec{i}$  receives a variation  $\alpha$ ). We consider that this variation takes place immediately after the bond has been bought. The structure then becomes  $\vec{i} + \alpha$ , and the bond's value is

$$B(\vec{i} + \alpha) = \int_0^T c(t)e^{-[i(0,t)+\alpha]t} dt \quad (6)$$

As a consequence, for any given initial term structure of interest rates, our bond becomes a function of the single variable  $\alpha$ . (This increase in the function  $i(0, t)$  is merely a particular case of the more general variation defined by functions such as  $\alpha\eta(t)$  in the calculus of variations; here we have supposed that  $\eta(t) = 1$ .)

As the reader may well surmise, duration of the bond with the initial term structure will be defined as

$$D(\vec{i}) = \frac{1}{B(\vec{i})} \int_0^T tc(t)e^{-i(0,t)t} dt \quad (7)$$

As before, duration is the weighted average of the bond's times of payment, the weights being the cash flows in present value. The only differ-

ence is that the cash flows are discounted by a nonconstant structure of interest rates, and that we are in continuous analysis. We can also see that, should the structure  $\vec{i}$  receive a variation  $\alpha$ , duration will take a new value, denoted as  $D(\vec{i} + \alpha)$ , and equal to

$$D(\vec{i} + \alpha) = \frac{1}{B(\vec{i} + \alpha)} \int_0^T tc(t) e^{-[i(0,t) + \alpha]t} dt \quad (7a)$$

We will soon make use of this last observation.

## 10.2 A fundamental property of duration

We can show immediately that the logarithmic derivative of the bond's value with respect to  $\alpha$ , with a minus sign, is exactly equal to the bond's duration. From (6), we can write

$$-\frac{1}{B(\vec{i} + \alpha)} \frac{dB(\vec{i} + \alpha)}{d\alpha} = -\frac{1}{B(\vec{i} + \alpha)} \int_0^T -t \cdot c(t) e^{-[i(0,t) + \alpha]t} dt = D(\vec{i} + \alpha) \quad (8)$$

because of (7a). Also, this relative increase, when  $\alpha = 0$ , is equal to the bond's duration before any change in the structure. Setting  $\alpha = 0$ , in (8), we can write

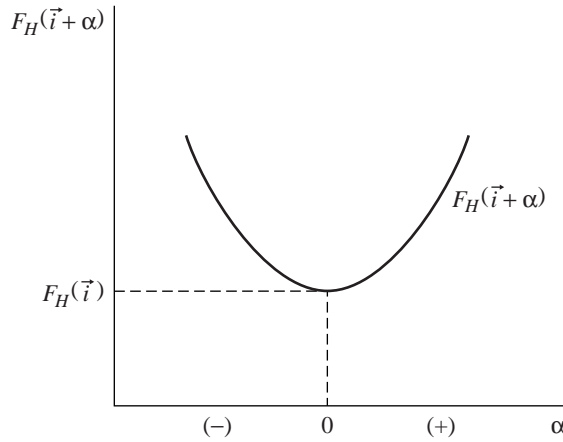
$$-\frac{1}{B(\vec{i})} \frac{dB(\vec{i})}{d\alpha} = \frac{1}{B(\vec{i})} \int_0^T tc(t) e^{-i(0,t)t} dt = D \quad (9)$$

because of (7) or (7a).

Consequently, we can see that duration is exactly equal to minus the relative rate of increase of the bond with respect to a parallel shift in the structure of interest rates. We will soon make use of this important property. (The reader will notice that here, in continuous analysis, duration is no longer divided by a quantity reminding us of  $1 + i$ , which was the coefficient by which we had divided duration to get, in *discrete* time,  $-\frac{1}{B} \frac{dB}{di}$ ; we called  $D/(1 + i)$  the *modified duration*.)

## 10.3 Value of a bond at horizon $H$

As we did before, we will now determine the bond's value at horizon  $H$  in order to immunize this future value as well as the rate of return at horizon

**Figure 10.1**

Value of the bond at horizon  $H$ , when the term structure of interest rates is  $\vec{i} + \alpha$ .

$H$ . Let us designate by  $F_H(\vec{i})$  this future value when the term structure of interest rates is  $\vec{i}$ ; we have

$$F_H(\vec{i}) = B(\vec{i}) \cdot e^{[i(0,H)]H} \quad (10)$$

More generally, of course, when the structure becomes  $\vec{i} + \alpha$ , we can write

$$F_H(\vec{i} + \alpha) = B(\vec{i} + \alpha) \cdot e^{[i(0,H) + \alpha]H} \quad (11)$$

Our purpose now is to determine a horizon  $H$  such that this future value is equal, at a minimum, to the value it would have if the structure did not change, that is, if  $\alpha = 0$ . This minimal value is the future value of the bond as calculated today, before the structure undergoes any change; it is equal to  $F_H(\vec{i})$ . We must therefore find  $H$  such that  $F_H(\vec{i} + \alpha)$  goes through a minimum in space  $(\alpha, F_H)$ , the coordinates of this minimum being  $(\alpha = 0, F_H = F_H(\vec{i}))$ , as indicated in figure 10.1.

#### 10.4 A first-order condition for immunization

Minimizing  $F_H(\vec{i} + \alpha)$  is of course equivalent to minimizing  $\ln F_H(\vec{i} + \alpha)$ ; this is what we will do in order to simplify calculations. With

$$\ln F_H(\vec{i} + \alpha) = \ln B(\vec{i} + \alpha) + i(0, H)H + \alpha H \quad (12)$$

we can write

$$\left. \frac{d \ln F_H(\vec{i} + \alpha)}{d\alpha} \right|_{\alpha=0} = \left. \frac{d \ln B(\vec{i} + \alpha)}{d\alpha} \right|_{\alpha=0} + H = 0 \quad (13)$$

The last equation implies

$$H = -\frac{1}{B(\vec{i})} \frac{dB(\vec{i})}{d\alpha} = D(\vec{i}) \quad (14)$$

Our immunizing horizon must be equal to duration, a result that generalizes the one we obtained previously. Of course, the reader will notice that duration is calculated with respect to the initial *structure* of interest rates  $\vec{i}$  (while before duration was calculated simply with respect to the horizontal structure summarized by the number  $i$ ).

### 10.5 A second-order condition for immunization

We now have to make sure that the function whose derivative we have just set equal to zero is convex with respect to  $\alpha$ , because in such a case we would have indeed reached a minimum.

Let us calculate the second derivative of  $\ln F_H(\vec{i} + \alpha)$ . From (12) we have

$$\frac{d^2 \ln F_H}{d\alpha^2} = \frac{d}{d\alpha} \left[ \frac{1}{B(\vec{i} + \alpha)} \frac{dB(\vec{i} + \alpha)}{d\alpha} \right] = \frac{d}{d\alpha} [-D(\vec{i} + \alpha)] \quad (15)$$

(The last equation can be written because of (8).)

The derivative of  $D(\vec{i} + \alpha)$  with respect to  $\alpha$  is equal to

$$\begin{aligned} \frac{dD(\vec{i} + \alpha)}{d\alpha} &= \frac{d}{d\alpha} \left\{ \frac{1}{B(\vec{i} + \alpha)} \int_0^T tc(t) e^{-[i(0,t)+\alpha]t} dt \right\} \\ &= \frac{1}{[B(\vec{i} + \alpha)]^2} \left\{ \int_0^T -t^2 c(t) e^{-[i(0,t)+\alpha]t} dt \cdot B(\vec{i} + \alpha) \right. \\ &\quad \left. - \int_0^T tc(t) e^{-[i(0,t)+\alpha]t} dt \cdot B'(\vec{i} + \alpha) \right\} \\ &= - \left\{ \frac{\int_0^T t^2 c(t) e^{-[i(0,t)+\alpha]t} dt}{B(\vec{i} + \alpha)} + \frac{\int_0^T tc(t) e^{-[i(0,t)+\alpha]t} dt}{B(\vec{i} + \alpha)} \frac{B'(\vec{i} + \alpha)}{B(\vec{i} + \alpha)} \right\} \end{aligned} \quad (16)$$



We recognize in the second term of the last brace the expression  $-D^2(\vec{i} + \alpha)$ . Simplifying the notation, we may thus write

$$w(t) \equiv \frac{c(t)e^{-[i(0,t)+\alpha]t}}{B(\vec{i} + \alpha)}, \quad \text{with } \int_0^T w(t) dt = 1,$$

and  $D(\vec{i} + \alpha) \equiv D(\alpha)$ .

$$\begin{aligned} \frac{dD}{d\alpha} &= - \left\{ \int_0^T t^2 w(t) dt - D^2 \right\} \\ &= - \left\{ \int_0^T t^2 w(t) dt - 2D \int_0^T t w(t) dt + D^2 \int_0^T w(t) dt \right\} \\ &= - \int_0^T w(t) [t^2 - 2tD + D^2] dt = - \int_0^T w(t) [t - D]^2 dt \quad (17) \end{aligned}$$

As a happy surprise, the last expression is none other than minus the dispersion, or variance, of the terms of the bond,<sup>3</sup> and such a variance is of course always positive. Denoting the bond's variance as  $S(\vec{i}, \alpha)$ , we may write

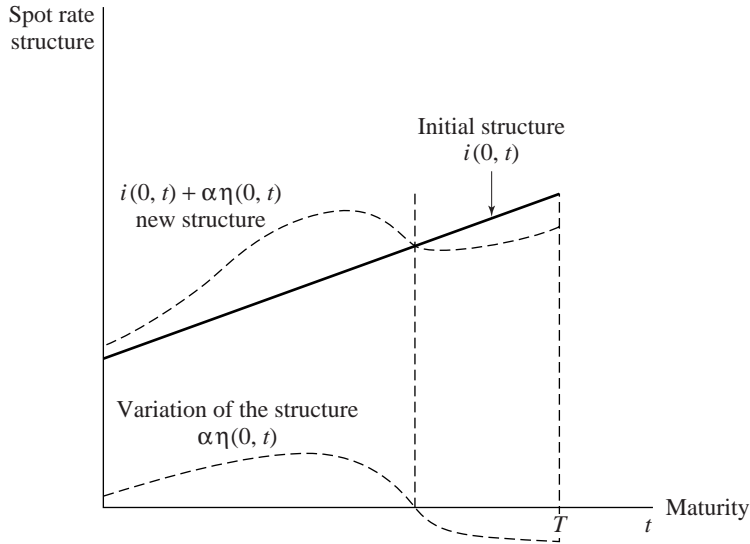
$$\frac{d^2 \ln F_H}{d\alpha^2} = - \frac{d}{d\alpha} [D] = S(\vec{i}, \alpha) > 0$$

Consequently, we may conclude that  $\ln F_H$  is indeed convex. Together with the first-order condition indicated in section 10.4, this condition is then sufficient for  $\ln F_H$  to go through a global minimum at point  $\alpha = 0$  (that is, for the initial term structure of interest rates). Our bond (or bond portfolio) is thus immunized for a short span of time, whatever parallel displacement this structure may receive immediately after the bond (or bond portfolio) has been purchased.

## 10.6 The difficulty of immunizing a bond portfolio when the term structure of interest rates is subject to any kind of variation

We can now consider immunization in the case of a variable term structure, when this structure undergoes any kind of variation. We will show

<sup>3</sup>This result could also have been obtained by observing that the first part of (17),  $-\{\int_0^T t^2 w(t) dt - D^2\}$ , itself defines minus the dispersion, or variance, of the terms of the bond.



**Figure 10.2**  
Initial term structure of interest rates  $i(0, t)$ , variation of the structure  $\alpha \eta(0, t)$ , and new structure  $i(0, t) + \alpha \eta(0, t)$ .

that we cannot be sure we can immunize our portfolio, contrary to the result we obtained previously, when we had considered a parallel shock to the structure.

Let us define a variation in the term structure in the following way. Let  $i(0, t)$  be the initial structure, existing at the time we buy our bond portfolio. Consider now  $\eta(0, t)$ , another term structure of interest rates, and  $\alpha$ —a number that, as before, may be positive, negative, or equal to zero. We will call  $\alpha \eta(0, t)$  a *variation* of the structure; this variation is supposed to occur immediately after the purchase of the bond portfolio. The new structure is thus  $i(0, t) + \alpha \eta(0, t)$ . Recapitulating, we have

- Initial term structure of interest rates:  $i(0, t)$
- Variation of the structure:  $\alpha \eta(0, t)$
- New structure:  $i(0, t) + \alpha \eta(0, t)$

Figure 10.2 illustrates an example of each of these functions.

To each structure  $i(0, t) + \alpha \eta(0, t)$  corresponds a value of the bond (or of the portfolio) equal to

$$B[i(0, t) + \alpha\eta(0, t)] = \int_0^T c(t)e^{-[i(0, t) + \alpha\eta(0, t)]t} dt \quad (18)$$

Clearly, the bond's value is a functional, that is, a relationship between a function of  $t$ —in our case the whole structure  $i(0, t) + \alpha\eta(0, t)$ —and a number (the bond's value  $B$ ). But the functions  $i(0, t)$  and  $\eta(0, t)$  can be considered as fixed. (At the time of the bond's purchase,  $i(0, t)$  is fixed; immediately after this purchase, the structure  $i(0, t)$  undergoes a variation  $\alpha\eta(0, t)$ , where  $\eta(0, t)$  can be considered as fixed, although the investor was unaware of this at the time he bought the bond.) As a consequence,  $B$  depends on  $\alpha$  only. We have thus transformed our functional into a function of the sole variable  $\alpha$ . We will therefore denote this function as  $B(\alpha)$ .

By taking the derivative of  $B(\alpha)$ , we can see that duration is no longer equal to the logarithmic derivative of  $B$  with respect to  $\alpha$ , with a minus sign. Indeed, we can calculate that

$$-\frac{1}{B} \frac{dB}{d\alpha} \Big|_{\alpha=0} = \frac{1}{B} \int_0^T t\eta(0, t)c(t)e^{-i(0, t)t} dt$$

This expression is not equal to the duration before the shock to the structure, because the expression of duration would be the right-hand side of the above equation without the variation  $\eta(0, t)$ , which was unknown at the time of the bond's purchase.

The value of the bond at horizon  $H$ , after the shock to the structure has happened, is equal to

$$F_H = B(\alpha)e^{[i(0, H) + \alpha\eta(0, H)]H} \quad (19)$$

In order to minimize this value with respect to  $\alpha$ , it would be sufficient, as before, to minimize  $\ln F_H$ :

$$\ln F_H = \ln B(\alpha) + [i(0, H) + \alpha\eta(0, H)]H \quad (20)$$

Taking the derivative of (20) with respect to  $\alpha$ , we get

$$\frac{d \ln F_H}{d\alpha} = \frac{d \ln B(\alpha)}{d\alpha} + \eta(0, H)H$$

Unfortunately, it is not possible in this case to choose a horizon  $H^*$  that would minimize  $F_H$ ; indeed, we would have to be able to choose  $H^*$  such that

$$H^* = - \left. \frac{\frac{d \ln B(\alpha)}{d\alpha}}{\eta(0, H)} \right|_{\alpha=0}$$

and this is impossible since we do not know the variation  $\eta(0, t)$ , although this was equal to 1 in the preceding case of a parallel displacement of the term structure. Hence it is generally not possible to immunize our bond portfolio if the structure is subject to any kind of variation.

There remains the question regarding the kind of immunization process we want to undertake: shall we aim at protecting our investor against a parallel shift in the term structure, supposing that initially it is flat or that it has an initial nonflat shape? In the first case we would consider the Macaulay duration, and in the second we would have to calculate the Fisher-Weil duration, which implies calculating first the term structure. Fortunately, many tests have been performed, and the second strategy does not seem to yield significantly better results than the first one. Possibly, the reason for this is twofold: first, the term structure is relatively flat where it counts most (that is, for long maturities); and second, it is quite possible that in the long run the nonparallel displacements of the term structure have a tendency to compensate each other in their effects, since over long periods the shape of the structure generally increases smoothly, as we have explained in this text.

A hopefully satisfactory answer to immunization of any kind of change in the spot rate structure will have to wait until chapter 15. There we will propose a new method of immunizing a portfolio and will give quite a number of applications, where we will consider dramatic changes in the spot rate structure.

### Main formulas

- Value of a bond in continuous time, with  $\vec{i} \equiv i(0, t)$  being the term structure or interest rates:

$$B(\vec{i}) = \int_0^T c(t) e^{-i(0, t)t} dt \quad (5)$$

- Duration of the bond:

$$D(\vec{i}) = \frac{1}{B(\vec{i})} \int_0^T t c(t) e^{-i(0, t)t} dt \quad (7)$$

- Duration of the bond when  $\vec{i}$  receives a (constant) variation  $\alpha$ :

$$D(\vec{i} + \alpha) = \frac{1}{B(\vec{i} + \alpha)} \int_0^T tc(t) e^{-[i(0,t) + \alpha]t} dt \quad (7a)$$

- Fundamental property of duration:

$$-\frac{1}{B(\vec{i})} \frac{dB(\vec{i})}{d\alpha} = \frac{1}{B(\vec{i})} \int_0^T tc(t) e^{-i(0,t)t} dt = D \quad (9)$$

- First-order condition for immunization:

$$H = -\frac{1}{B(\vec{i})} \frac{dB(\vec{i})}{d\alpha} = D(\vec{i}) \quad (14)$$

- Second-order condition for immunization:

$$\frac{d^2 \ln F_H}{d\alpha^2} > 0 \Rightarrow -\frac{d}{d\alpha} [D(\vec{i} + \alpha)] = S(\vec{i}, \alpha) > 0$$

## Questions

10.1. How do you derive equation (8) in section 10.2 of this chapter?

10.2. This question could be considered as a project. You have noticed that in section 10.3, we sought to protect the investor against a parallel displacement of the term structure for interest rates by seeking a horizon such that the future value of the bond (or the bond portfolio),  $F_H$ , goes through a minimum for the initial term structure, that is, for  $\alpha = 0$  in  $(F_H, \alpha)$  space. How would you have obtained the ensuing immunization theorem if you had considered finding a minimum for the investor's horizon rate of return? (Hint: First define carefully the investor's horizon rate of return by extending the concept you encountered in section 6.2 of Chapter 6 to the concepts introduced in this chapter.)



# 11 CONTINUOUS SPOT AND FORWARD RATES OF RETURN, WITH TWO IMPORTANT APPLICATIONS

HOTSPUR. Are there not some of them set forward already?

*Henry IV*

## Chapter outline

11.1	The continuously compounded rate of return	222
11.2	Continuously compounded yield	226
11.3	Forward rates for instantaneous lending and borrowing, and spot rates	228
11.4	Relationships between the forward rates, the spot rates, and a zero-coupon bond price	232

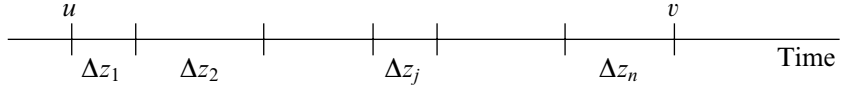
## Overview

Until now we have calculated rates of return over finite periods of time (for instance, one year). There are, however, considerable advantages to considering rates of return over infinitesimally small amounts of time. A first reward is to simplify many computations. For example, the reader will recall that, in chapter 7, with numbers of years  $n_1 < n_2 < n_3$ , the  $n_2$  spot rate was a weighted geometric mean of 1 plus the  $n_1$  spot rate and 1 plus the  $n_3$  spot rate minus 1. This is a rather cumbersome expression, whose properties are not easy to analyze. When considering continuously compounded rates, we will be able to replace affine transformations of geometric averages with simple arithmetic averages. But by far the most important advantage of such a procedure is that it enables us to apply the central limit theorem and thus derive from that major result of statistics a theory about the probabilistic nature of both dollar annual returns and asset prices. We will thus be able to justify the lognormal distribution of the dollar returns on assets, as well as the lognormal distribution of the asset prices.

This chapter will be organized as follows: we will first introduce the concept of a continuously compounded rate of return on an arbitrary time interval. We will then move on to describe in the same context yields, spot rates, and forward rates, and their relationships to zero-coupon bond prices. We will finally apply the central limit theorem in order to have a theory about the probability distribution of dollar returns and asset prices.

### 11.1 The continuously compounded rate of return

Consider, on the time axis, a length of time  $[u, v]$  of size  $v - u$  and an *arbitrary* partition of this length into  $n$  intervals denoted  $\Delta z_1, \Delta z_2, \dots, \Delta z_n$ .



These intervals, *not necessarily of the same length*, are measured in years; their sum,  $\sum_{j=1}^n \Delta z_j$ , is equal to  $v - u$ . Consider, without any loss in generality, the first interval,  $\Delta z_1$ . Suppose that we are at time  $u$  (at the beginning of this interval) and that we have agreed to lend capital  $C_u$  at that time at *annual* rate  $i_1$  during the amount of time  $\Delta z_1$ . The contract specifies that the interest will be computed and paid at the end of that period (at time  $u + \Delta z_1$ ) and will be in the amount of  $i_1 \Delta z_1$ . Indeed,  $\Delta z_1$  is simply the fraction of year corresponding to this contract. For our purposes, we could consider that  $\Delta z_1$  is shorter than one year. In order to stress the fact that the contract stipulates that the interest is paid once per period, we write it  $i_1^{(1)}$ , where the superscript denotes the number of times the annual rate of interest is paid within  $\Delta z_1$ ;  $C_u i_1^{(1)} \Delta z_1$  is then the interest paid at  $u + \Delta z_1$ . Including principal, you would receive  $C_u + C_u i_1^{(1)} \Delta z_1 = C_u(1 + i_1^{(1)} \Delta z_1)$ .

Consider now another contract. This time, the contract stipulates that the interest will be paid twice per period  $\Delta z_1$ . The amount received as interest in the middle of period  $\Delta z_1$ , at time  $u + \frac{1}{2} \Delta z_1$ , would be  $C_u i_1^{(2)} \frac{\Delta z_1}{2}$ ; the total amount received at time  $u + \Delta z_1$  would therefore be  $C_u \left(1 + \frac{i_1^{(2)}}{2} \Delta z_1\right)^2$ .

The generalization to a contract paying interest  $m$  times, at equal distances within the interval  $\Delta z_1$ , is immediate. It would pay the amount

$$C_{u+\Delta z_1} = C_u \left(1 + \frac{i_1^{(m)}}{m} \Delta z_1\right)^m \quad (1)$$

at time  $u + \Delta z_1$ . Of course, this amount is different from the amount corresponding to a contract with an interest rate  $i_1^{(1)}$  paid once per period. In fact, we can show that it is always superior to it; with  $i_1^{(1)} = i_1^{(m)}$ ,  $1 + i_1^{(1)} \Delta z_1$  is simply the sum of the first two terms of the binomial development of  $(1 + i_1^{(m)} \Delta z_1 / m)^m$  and is therefore always strictly inferior to



it. Thus, the  $i_1^{(m)}$  contract is definitely better for the lender—and more expensive for the borrower—than the  $i_1^{(1)}$  contract. For both contracts to produce the same amount,  $i_1^{(m)}$  should be related to  $i_1^{(1)}$  by the following relationship:

$$\left(1 + \frac{i_1^{(m)}}{m} \Delta z_1\right)^m = 1 + i_1^{(1)} \Delta z_1 \quad (2)$$

and therefore  $i_1^{(m)}$  should be set equal to

$$i_1^{(m)} = \frac{m}{\Delta z_1} [(1 + i_1^{(1)} \Delta z_1)^{1/m} - 1] \quad (3)$$

For example, suppose  $i_1^{(1)} = 8\%$  per year;  $\Delta z_1 = 3$  months = 0.25 year;  $m = 10$ . From (3),  $i_1^{(m)}$  should be equal to 7.9289% per year for both contracts to be equivalent, both producing 1.02  $C_u$  at time  $u + \Delta z_1$ .

Consider now what happens when  $m$ , the number of times the interest is paid throughout the interval  $\Delta z_1$ , increases to infinity. Setting in (1)  $\frac{i_1^{(m)} \Delta z_1}{m} = \frac{1}{k}$ , with  $m = ki_1^{(m)} \Delta z_1$ , we have

$$C_{u+\Delta z_1} = C_u \left(1 + \frac{1}{k}\right)^{ki_1^{(m)} \Delta z_1} \quad (4)$$

In the limit when  $m \rightarrow \infty$  we have also  $k \rightarrow \infty$  and therefore

$$\lim_{k \rightarrow \infty} C_{u+\Delta z_1} = C_u e^{i_1^{(\infty)} \Delta z_1} \quad (5)$$

an all-important relationship, where  $i_1^{(\infty)}$  is called the *continuously compounded* rate of interest prevailing throughout interval  $\Delta z_1$ . For the  $i_1^{(\infty)}$  contract to yield the same amount as the  $i_1^{(1)}$  contract, we should have

$$e^{i_1^{(\infty)} \Delta z_1} = 1 + i_1^{(1)} \Delta z_1 \quad (6)$$

or, equivalently,

$$i_1^{(\infty)} = \frac{\log(1 + i_1^{(1)} \Delta z_1)}{\Delta z_1} \quad (7)$$

Coming back to our former example,  $i_1^{(\infty)}$  should be equal to  $\frac{\log(1+0.02)}{0.25} = 7.921\%$  for both contracts to yield 1.02  $C_u$  at time  $u + \Delta z_1$ , that is, after three months.

Notice an important special case for (7): if  $\Delta z_1 =$  one year, the equivalence between the continuously compounded interest rate and the once-

**Table 11.1**

**Convergence of the equivalent continuously compounded interest rate  $i^{(\infty)}$  toward the simple interest rate  $i^{(1)}$  when  $i^{(1)}$  tends toward zero (in percentage per year)**

Simple interest rate $i^{(1)}$	Equivalent continuously compounded interest rate $i^{(\infty)} = \log(1 + i^{(1)})$	Difference $i^{(1)} - i^{(\infty)}$
0	0	0
2	1.98	0.02
4	3.92	0.08
6	5.83	0.17
8	7.70	0.30
10	9.53	0.47

per-year compounded (or simple) interest rate is, suppressing the “1” subscript for simplicity,

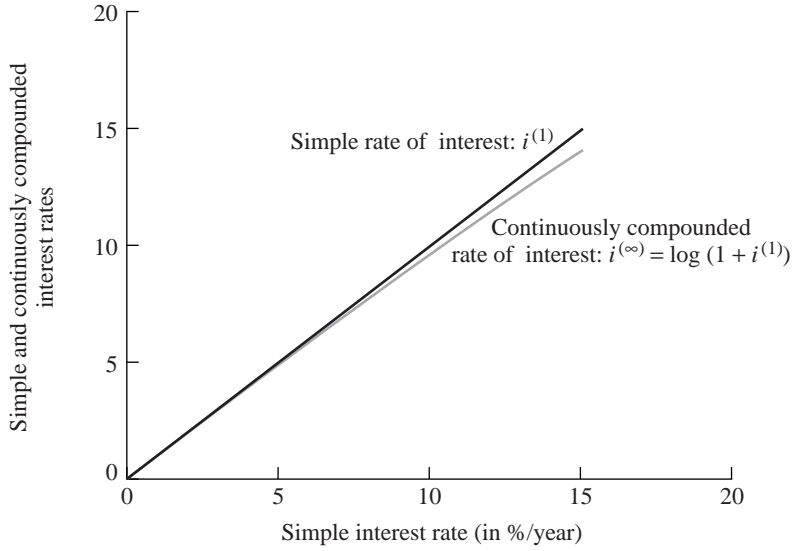
$$i^{(\infty)} = \log(1 + i^{(1)}) \quad (8)$$

Is the continuously compounded annual interest rate  $i^{(\infty)} = \log(1 + i^{(1)})$  close in some sense to the simple interest rate  $i^{(1)}$ ? The answer is yes. It turns out that  $i^{(1)}$  is nothing else than the linear approximation of  $i^{(\infty)} = \log(1 + i^{(1)})$ . Indeed, the first-order (or linear) Taylor development of  $\log x$  at  $x = 1$  is  $x - 1$ ; thus the same development of  $\log(1 + i_1^{(1)})$  at  $i_1^{(1)} = 0$  is simply  $i^{(1)}$ . (From a geometric point of view, the ray  $i^1$  is tangent to the curve  $i^{(\infty)} = \log(1 + i^{(1)})$  at point  $i^{(1)} = 0$ .) Thus, the smaller the value of  $i^{(1)}$ , the better the approximation and the closer the continuously compounded rate  $i^{(\infty)}$  is to the simple rate  $i^{(1)}$ . For instance, table 11.1 gives  $i^{(\infty)}$  as a function of  $i^{(1)}$  and illustrates the convergence of these rates when  $i^{(1)} \rightarrow 0$ .

Figure 11.1 represents the continuously compounded interest rate  $i^{(\infty)}$  as a function of the simple rate. The “straight line” character of  $i^{(\infty)}$  is a surprise. It comes from the fact that even when  $i^{(1)} = 15\%$ , we are still very close to zero and therefore the linear approximation is still very good.

We now point to a final property of the continuously compounded interest rate: that rate is the only rate that makes an investment’s value come back to its initial value when the rate has had a given value in one period (one year, for instance) and the opposite value in a second period of equal length. Indeed, consider three consecutive *years*,  $j - 1$ ,  $j$ , and  $j + 1$ , and let  $i^{(\infty)}$  denote for simplicity the continuously compounded interest rate; we have

$$S_{j+1} = S_{j-1} e^{i^{(\infty)}} e^{-i^{(\infty)}} = S_{j-1}$$



**Figure 11.1**

The proximity of the equivalent continuously compounded interest rate and the simple interest rate. The latter is the linear approximation of the former at  $i^{(1)} = 0$ .

Therefore,  $S$  does not change value. This would not be true with an interest rate compounded a number  $m$  ( $1 \leq m < \infty$ ) times per period; if  $i^{(m)}$  denotes such a rate of return, a negative difference between  $S_{j+1}$  and  $S_{j-1}$  would always appear:

$$S_{j+1} = S_{j-1} \left(1 + \frac{i^{(m)}}{m}\right)^m \left(1 - \frac{i^{(m)}}{m}\right)^m = S_{j-1} \left[1 - \left(\frac{i^{(m)}}{m}\right)^2\right]^m$$

The coefficient multiplying  $S_{j-1}$  is clearly smaller than 1, and therefore  $S_{j+1} < S_{j-1}$ . This inequality becomes an equality if and only if  $m$  tends toward infinity, in other words, if and only if the rate of return is compounded an infinite number of times per period. Indeed, we then get

$$\begin{aligned} \lim_{m \rightarrow \infty} S_{j+1} &= \lim_{m \rightarrow \infty} S_{j-1} \left(1 + \frac{i^{(m)}}{m}\right)^m \left(1 - \frac{i^{(m)}}{m}\right)^m \\ &= S_{j-1} e^{i^{(\infty)}} e^{-i^{(\infty)}} = S_{j-1} \end{aligned} \quad (9)$$

Let us now come back to our partition of the time span  $[u, v]$ . Consider that contracts always provide for continuously compounded interest rates  $i_j^{(\infty)}$ ,  $j = 1, \dots, n$ .

At time  $u + \Delta z_1$ ,  $C_u$  has become  $C_u e^{i_1^{(\infty)} \Delta z_1}$ . By the same reasoning, at time  $u + \Delta z_1 + \Delta z_2$  (at the end of interval  $\Delta z_2$ ),  $C_u$  becomes  $C_u e^{i_1^{(\infty)} \Delta z_1} \cdot e^{i_2^{(\infty)} \Delta z_2} = C_u e^{i_1^{(\infty)} \Delta z_1 + i_2^{(\infty)} \Delta z_2}$ , and of course at time  $v$  (at the end of the last interval  $\Delta z_n$ )  $C_u$  becomes

$$C_v = C_u e^{\sum_{j=1}^n i_j^{(\infty)} \Delta z_j} \quad (10)$$

Until now the rate of interest was considered a discontinuous function of time. Indeed, since the  $i_j^{(\infty)}$  are not necessarily equal,  $i_j^{(\infty)}$  is a function of time defined by a succession of constant values over each interval  $\Delta z_j$ . Since these values are not necessarily equal to each other, we may have a series of  $n - 1$  successive jumps for the interest rate between time  $u$  and time  $v$ . Consider now what happens to our expression (10) if  $i_j^{(\infty)}$  becomes a continuous function of time, defined over infinitesimally small time intervals. (Note that we could still consider a discontinuous function, but it is not necessary to do so for our purposes here.) To that effect, transform the partition of  $[u, v]$  in such a way that the number of intervals  $n$  tends toward  $\infty$ , and the largest interval  $\Delta z_j$  tends toward zero. If for any partition the sum  $\sum_{j=1}^n i_j^{(\infty)} \Delta z_j$  tends toward a limit, that limit is the definite integral of  $i_j^{(\infty)}$  between  $u$  and  $v$ . We thus have

$$\lim_{\substack{n \rightarrow \infty \\ \max \Delta z_j \rightarrow 0}} \sum_{j=1}^n i_j^{(\infty)} \Delta z_j = \int_u^v i(z) dz \quad (11)$$

where  $i(z)$  now denotes the continuously compounded interest rate as a function of time, and therefore  $C_v$  can be written as

$$C_v = C_u e^{\int_u^v i(z) dz} \quad (12)$$

## 11.2 Continuously compounded yield

Consider an investment  $C_u$  that is transformed into  $C_v$  after a time span  $v - u$ . We define the yearly continuously compounded yield as the (con-

stant) continuously compounded interest rate  $R(u, v)$  that transforms  $C_u$  into  $C_v$  in time  $v - u$ . Therefore, applying what we have done before at the level of any time interval, we can define  $R(u, v)$  implicitly as

$$C_u e^{R(u, v)(v-u)} = C_v \quad (13)$$

and thus  $R(u, v)$  is equal to

$$R(u, v) = \frac{\log(C_v/C_u)}{v-u} \quad (14)$$

There is an important relationship between the continuously compounded return and the yield. From (12) and (13) we may write

$$C_u e^{\int_u^v i(z) dz} = C_u e^{R(u, v)(v-u)} \quad (15)$$

implying

$$R(u, v) = \frac{\int_u^v i(z) dz}{v-u} \quad (16)$$

and therefore the continuously compounded yield is equal to the *average value of the continuously compounded interest rate*. These relations provide a nice interpretation of the Euler number. We use equations (13) and (16) to write

$$C_v = C_u \exp \left[ \frac{\int_u^v i(z) dz}{v-u} (v-u) \right] = C_u e^{R(u, v)(v-u)} \quad (17)$$

Consider an investment of  $C_u = \$1$  and a yield over  $[u, v]$  equal to  $\frac{1}{v-u}$ . Then from (17),  $C_v = e = 2.71828 \dots$ . We can state the following:

The Euler number  $e$  is what \$1 becomes after a time span  $v - u$  years when the continuously compounded yield (the average of the continuously compounded instantaneous interest rates) is  $\frac{1}{v-u}$ .

Indeed, we would have  $C_v = 1 \cdot e^{\frac{1}{v-u}(v-u)} = e$ . For instance, suppose that  $v - u$  is 20 years; then, with  $R(u, v) = 1/(v - u) = 5\%/year$ , \$1 becomes

\$2.71828 after 20 years. Of course, this holds for fractional years just as well: if  $(v - u) = 20.202$  years and  $R(u, v) = \frac{1}{20.202 \text{ years}} = 4.95\%/year$ , \$1 becomes  $e$  dollars 20.202 years later.

Notice the generality of this interpretation. It does not suppose that *all* interest rates are necessarily positive; it would apply to the *real* value of \$1 at time  $v$  if, during some subintervals of  $[u, v]$ , *real* interest rates were negative, that is, nominal interest rates were lower than instantaneous inflation rates.

### 11.3 Forward rates for instantaneous lending and borrowing, and spot rates

Until now, we have supposed that when  $n$  was finite, there was a finite number of contracts that were signed at the various dates  $u$ ,  $u + \Delta z_1$ ,  $u + \Delta z_1 + \Delta z_2, \dots$ , that is, at the beginning of each interval  $\Delta z_j$ . But we could of course simplify our scenario and imagine that at date  $u$  there exist  $n$  *forward* markets, each with a time span  $\Delta z_j$ , and that one can enter all those contracts at initial time  $u$ . Therefore, our interest rates pertaining to any interval  $j$ , denoted  $i_j$ , would be replaced by *forward* rates. Now consider what happens when the number of periods goes to infinity and the duration of the loan goes to zero. We can define an instantaneously compounded forward rate for infinitely short borrowing as  $f(u, z)$ , where  $u$  is the contract inception time and  $z$  ( $z > u$ ) is the time at which the loan is made, for an infinitesimally small period. We thus have

$f(u, z)$  = instantaneously compounded forward rate for  
a loan contracted at time  $u$ , starting at time  $z$ ,  
for an infinitesimal period

Let us now define the *spot rate* at time  $u$  as the continuously compounded return that transforms  $C_u$  into  $C_v$  after a time span  $v - u$ . Applying what we saw in section 11.2, this is exactly the continuously compounded yield, which we can therefore denote as  $R(u, v)$ , such that  $C_v = C_u e^{R(u, v)(v-u)}$ . If agents have costless access to all forward markets in their capacity both as lenders and as borrowers, arbitrage ensures that \$1 invested in all forward markets between  $u$  and  $v$  will be transformed into the same amount as \$1 invested at the continuously compounded spot rate  $R(u, v)$ . We must have

$$e^{\int_u^v f(u, z) dz} = e^{R(u)du} \quad (18)$$

and therefore

$$R(u, v) = \frac{\int_u^v f(u, z) dz}{v - u} \quad (19)$$

Equation (19) is important. It tells us that the spot rate  $R(u, v)$  is equal to the arithmetic average of the forward rates  $f(u, z)$ ,  $z \in [u, v]$ . Note that this relationship, valid when all rates are continuously compounded, is much simpler than the relationship between the spot rate and the forward rates when these are compounded a finite number of times. (Indeed, we showed that when the compounding took place once a year, *one plus* the spot rate was a *geometric* average of *one plus* the forward rates—a much more cumbersome relationship.) Using the arithmetic average property expressed in (19), or the arbitrage equation (18), we can write

$$\int_u^v f(u, z) dz = R(u, v)(v - u) \quad (20)$$

and take the derivative of (20) with respect to  $v$  to obtain a direct relationship between any forward rate and the spot rate:

$$f(u, v) = R(u, v) + (v - u) \frac{dR(u, v)}{dv} \quad (21)$$

Equation (21) has a nice economic interpretation. Suppose that the spot rate  $R(u, v)$  is increasing. This means that the *average* rate of return increases. What does this increase imply in terms of the forward (the “marginal”) rate? In other words, given an increase in the spot rate, what is the *implied* value of the forward rate in relation to both the value of the initial spot rate and the increase this spot rate has received? The answer is quite simple and follows the rules relating average values to marginal values: the forward rate (the marginal rate of return) must be equal to the spot rate (the average rate of return) *plus* the rate of increase of the spot rate multiplied by the sum of all infinitesimally small time increases between  $u$  and  $v$  (this sum is equal to  $(v - u)$ ). Thus the forward rate *must* be equal to  $R(u, v) + (v - u) \frac{dR(u, v)}{dv}$  as indicated in (21). Of course, the same reasoning would have applied had we considered a decreasing spot rate.

The relationship between the forward rate and the spot rate, as expressed in (21) or in its integral forms ((19) or (20)), entails useful properties of the forward rate. We can see immediately that:

1. the forward rate is higher than the spot rate if and only if the spot rate is increasing;
2. the forward rate is lower than the spot rate if and only if the spot rate is decreasing; and
3. the forward rate is equal to the spot rate if and only if the spot rate is constant.

One caveat is in order here: an increasing spot rate curve does *not* necessarily entail an increasing forward rate curve. Indeed, an increasing average implies only a marginal value that is above it (and *not* necessarily an increasing marginal value). In fact, the forward rate can be increasing, decreasing, or constant, while the spot rate is always increasing. To illustrate this, consider the well-known, common case where the spot rate slowly increases and then reaches a constant plateau. We can show that in this case if the spot rate is a concave function of maturity, the forward rate will *always be decreasing* in an interval to the left of the point where the spot rate reaches its maximum.

Let  $[0, T]$  denote our interval  $[u, v]$ . We then write

$$R(0, T) = \frac{\int_0^T f(0, z) dz}{T} \quad (22)$$

Suppose that  $R(0, T)$  is an increasing, concave function of  $T$ , reaching a maximum at  $\hat{T}$  with a zero slope at  $\hat{T}$ . This translates as

$$\frac{dR(0, \hat{T})}{dT} = 0 \quad (23)$$

and

$$\frac{d^2 R(0, \hat{T})}{dT^2} < 0 \quad (24)$$

Therefore, (23) implies, from (21),

$$f(0, \hat{T}) = R(0, \hat{T}) \quad (25)$$



On the other hand, (24) and (25) imply

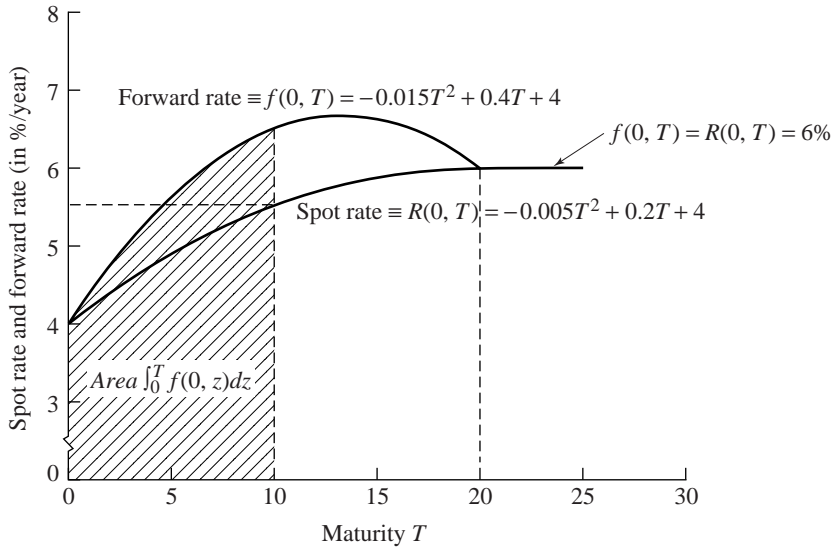
$$\frac{d^2 R(0, \hat{T})}{d\hat{T}^2} = \frac{1}{\hat{T}} \left[ \frac{df(0, \hat{T})}{d\hat{T}} - \frac{dR(0, \hat{T})}{d\hat{T}} \right] < 0$$

and therefore, since  $\frac{dR(0, \hat{T})}{d\hat{T}} = 0$ ,  $\frac{df(0, \hat{T})}{d\hat{T}} < 0$ . Thus the forward rate must be *decreasing* when the spot rate reaches its plateau.

The following example illustrates what we have just shown. Suppose that the spot rate  $R(0, T)$ , expressed in percent per year, is the following function of maturity:

$$R(0, T) = \begin{cases} -0.005T^2 + 0.2T + 4 & (\text{for maturities } 0 \text{ to } 20 \text{ years}) \\ 6 & (\text{for maturities beyond } 20 \text{ years}) \end{cases}$$

Applying the properties of the forward rate, we know that it will be equal to the spot rate (6% per year) for maturities beyond 20 years. To recover the forward rate for maturities between 0 and 20 years, we could apply (21), replacing  $u$  and  $v$  by 0 and  $T$ , respectively. But perhaps a more elegant way to go about it would be to come back to the arbitrage rela-



**Figure 11.2**

An example of the correspondence between the forward rate  $f(0, T)$  for instantaneous borrowing and the spot rate  $R(0, T)$ .  $R(0, T)$  is the average of all forward rates between 0 and  $T$ . Hence  $\text{area } R(0, T) \cdot T = \int_0^T f(0, z) dz$ .

tionship between spot and forward rates, which implies

$$R(0, T) \cdot T = \int_0^T f(0, z) dz \quad (26)$$

In our example, this yields

$$-0.005T^3 + 0.2T^2 + 4T = \int_0^T f(0, z) dz \quad (27)$$

Taking the derivative of (27) with respect to  $T$  yields the forward curve between  $T = 0$  and  $T = 20$ :

$$f(0, T) = -0.015T^2 + 0.4T + 4$$

Both the spot rate curve and the forward rate curve are shown in figure 11.2. Indeed, it can be verified that there is a whole range of maturities where the forward curve is *decreasing* while the spot rate is increasing; that range is between maturities 13.3 years (corresponding to the maximum of the forward rate curve) and 20 years.

#### 11.4 Relationships between the forward rates, the spot rates, and a zero-coupon bond price

Consider a defaultless zero-coupon bond at time  $t$ , which matures at time  $T$  with \$1 face value. Denote the price of that bond as  $B(t, T)$ . Suppose also that today is time 0; so, with  $t > 0$ ,  $B(t, T)$  is a random variable. Nevertheless, it can be related in a very precise way to the future forward rates  $f(t, u)$  that will apply at time  $t$  and to the future spot rate  $R(t, T)$  that will apply at time  $t$ . Applying the definition of forward rates, we can first write

$$B(t, T)e^{\int_t^T f(t, u) du} = 1 \quad (28)$$

from which we deduce

$$B(t, T) = e^{-\int_t^T f(t, u) du} \quad (29)$$

In (29), the bond's value  $B(t, T)$ , a random value, is a function of all values  $f(t, u)$  that the forward rate will take at time  $t$  for horizons from  $u = t$  to  $u = T$ . For each horizon  $u$  (now fixing  $u$ ), the value  $f(t, u)$  may

be considered a random process starting at a known value (today)  $f(0, u)$  and having properties that we will describe in later chapters. For the time being, it is important to note that the zero-coupon bond's value  $B(t, T)$  is a function of an infinitely large number of outcomes of random variables (the forward rates  $f(t, u)$ , where  $u$  ranges between  $t$  and  $T$ ), *each* of them being the outcome of a random process taking place between 0 and  $t$ .

Alternatively, we could express the future bond's value in terms of the future spot rate  $R(t, T)$ . We can write

$$B(t, T)e^{R(t, T)(T-t)} = 1 \quad (30)$$

and, therefore,

$$B(t, T) = e^{-R(t, T)(T-t)} \quad (31)$$

This time  $B(t, T)$  is a function of the *single* random variable  $R(t, T)$ , which may also be considered the result of a random process started at time 0. In any case, we can see that we could model as well the evolution of *each* forward rate  $f(t, u)$  (for each  $u$ ) *or* the evolution of the single spot rate  $R(t, T)$  as a random process starting at  $t$ . Note that a remarkably comprehensive model—that of Heath, Jarrow, and Morton (1992)—follows the first route and models the forward rates. (This is the model we shall describe and use from chapter 17 onward.)

### Main formulas

- Equivalence relationship between the once-per-year compounded (or simple) interest rate  $i^{(1)}$  and the continuously compounded interest rate  $i^{(\infty)}$ :

$$i^{(\infty)} = \log(1 + i^{(1)}) \quad (8)$$

or

$$i^{(1)} = e^{i^{(\infty)}} - 1$$

- Over a time span  $[u, v]$  partitioned into  $n$  intervals  $\Delta z_j$ ,  $j = 1, \dots, n$ , a sum  $C_u$  becomes

$$C_v = C_u e^{\sum_{j=1}^n i_j^{(\infty)} \Delta z_j} \quad (10)$$

- If the following limit exists:

$$\lim_{\substack{n \rightarrow \infty \\ \max \Delta z_j \rightarrow 0}} \sum_{j=1}^n i_j^{(\infty)} \Delta z_j = \int_u^v i(z) dz \quad (11)$$

$$C_v = C_u e^{\int_u^v i(z) dz} \quad (12)$$

- Continuously compounded yield = continuously compounded spot rate =  $R(u, v)$  such that

$$C_u e^{R(u, v)(v-u)} = C_v \quad (13)$$

or

$$R(u, v) = \frac{\log(C_v/C_u)}{v-u} \quad (14)$$

- Relationship between continuously compounded rate of return over period  $[u, v]$ ,  $R(u, v)$ , and instantaneous rate of interest  $i(z)$ :

$$R(u, v) = \frac{\int_u^v i(z) dz}{v-u} \quad (16)$$

The continuously compounded rate of return is the average of the instantaneous rates of interest  $i(z)$  over period  $[u, v]$ .

- Economic interpretation of the Euler number:

The Euler number  $e$  is what \$1 becomes after a time span  $v - u$  years when the continuously compounded yield (the average of the continuously compounded instantaneous interest rates) is  $\frac{1}{v-u}$ .

- Definition:  $f(u, z)$  = instantaneously compounded forward rate for a loan contracted at time  $u$ , starting at time  $z$ , for an infinitesimal period.
- Relationship between forward rates  $f(u, z)$  and continuously compounded spot rate  $R(u, v)$ :

$$R(u, v) = \frac{\int_u^v f(u, z) dz}{v-u} \quad (19)$$

equivalent to

$$\int_u^v f(u, z) dz = R(u, v)(v-u) \quad (20)$$

and to

$$f(u, v) = R(u, v) + (v - u) \frac{dR(u, v)}{dv} \quad (21)$$

## Questions

11.1. Consider a contract over a period of six months, with a yearly rate of interest of 7%. Suppose that the interest is paid only once at the end of that period. What should the rate of interest be for a contract yielding the same return after six months if the interest is to be paid after each month?

11.2. Using the same data as in question 11.1, what should the continuously compounded rate of interest be for the contracts to yield the same return?

11.3. In section 11.3 we showed that the continuously compounded spot rate over a period  $v - u$  ( $R(u, v)$ , or  $R(0, T)$  if we consider a time interval  $T$ ) was the average of all forward rates for infinitesimal trading periods over this interval (see equation (19)). This in turn implies that the product of the time interval ( $v - u$ , or  $T$ ) by the spot rate is equal to the sum (the integral in this case) of all forward rates from  $u$  to  $v$  (or from 0 to  $T$ ), such that  $\int_u^v f(u, z) dz$  or  $\int_0^T f(0, z) dz$ —see equation (20). The aim of this exercise is to verify this result by considering the data in figure 11.2.

a. Consider, as we have done, a time interval of 10 years. First, verify that the spot rate  $R(0, 10)$  is 5.5% per year according to the hypothesis we have chosen for the interest rate structure.

b. Verify that the product of this spot rate by the time interval (the area of the rectangle with height equal to the spot rate and width equal to the time interval) is indeed equal to the definite integral of the forward rate between 0 and 10.

c. What are the measurement units of this rectangle?

d. Give an economic, or financial, interpretation of the area of this rectangle or of the definite integral whose value is equal to that area.

e. Does your answer to (d) lead you to think that you could have dispensed with performing the calculation of the definite integral corresponding to (b)?



# 12 TWO IMPORTANT APPLICATIONS

MARIA. . . . all the power thereof it doth apply . . .  
*Love's Labour's Lost*

## Chapter outline

<b>12.1</b>	<b>A theoretical justification of the lognormal distribution of asset prices</b>	<b>238</b>
<b>12.2</b>	<b>In search of the term structure of interest rates, or to spline or not to spline—and how?</b>	<b>240</b>
12.2.1	Parametric methods	244
12.2.1.1	The Nelson-Siegel model	244
12.2.1.2	The Svensson model	247
12.2.2	Spline methods: the Fisher-Nychka-Zervos, the Waggoner, and the Anderson-Sleath models	248
12.2.2.1	What is a spline?	248
12.2.2.2	Some strange—and nice—consequences	249
12.2.2.3	Determining the order of the optimal polynomial spline	251
12.2.2.4	The Fisher-Nychka-Zervos model	253
12.2.2.5	An extension of the Fisher-Nychka-Zervos model: the Waggoner model	253
12.2.2.6	The Anderson-Sleath model	256
<b>Appendix 12.A.1</b>	<b>A proof of the central limit theorem and a determination of the probability distribution of the dollar return on an asset</b>	<b>257</b>
<b>Appendix 12.A.2</b>	<b>Deriving the Euler-Poisson equation from economic reasoning</b>	<b>259</b>

## Overview

We will now apply the concepts of continuous spot and forward rates to two central subjects of finance. The first is the theoretical justification of the lognormal distribution of asset prices. It is a hypothesis that has been at the core of the evaluation of assets and their derivatives for the past several decades. The second is the search for the term structure of interest rates, an extraordinarily elusive object but a highly interesting one.

Indeed, for the last 30 years or so, researchers have displayed tremendous ingenuity in attempting to extract the term structure of interest rates from bond prices. We will describe recent advances in this field.

### 12.1 A theoretical justification of the lognormal distribution of asset prices

The hypothesis of lognormally distributed asset prices is among the most widespread assumptions in financial economics. We will now show how it can be justified through the central limit theorem; we will also show that alternatively, it can be founded on a Wiener process.

Let a unit time span (one year, for instance) be divided into  $n$  equal intervals (one interval is, for instance, one day). Let  $j$  denote any of these intervals;  $j = 1, \dots, n$ . By convention  $j$  will indicate the end of the time interval, and  $j - 1$  its beginning. An asset is priced at  $S_{j-1}$  at the beginning of the  $j$ th interval and at  $S_j$  at its end. Let  $r_{j-1,j}$  denote the *daily* continuously compounded rate of return on  $S$  within one interval, thus equal to  $\log(S_j/S_{j-1})$ .

Let us now express  $S_n$ , the asset's value after  $n$  periods, equivalently, after one year, as follows:

$$S_n = S_0 \cdot \frac{S_1}{S_0} \cdots \frac{S_j}{S_{j-1}} \cdots \frac{S_n}{S_{n-1}} = S_0 e^{r_{0,1}} \cdots e^{r_{j-1,j}} \cdots e^{r_{n-1,n}} = S_0 e^{\sum_{j=1}^n r_{j-1,j}} \quad (1)$$

Thus  $S_n$  is expressed as a simple exponential whose exponent is a sum of the  $n$  random variables  $r_{j-1,j}$ .

Suppose now that these random variables are independent, identically distributed with mean  $m$  and (finite) variance  $s^2$ . We can now appeal to the central limit theorem: the sum  $\sum_{j=1}^n r_{j-1,j}$  will be normally distributed with mean  $\sum_{j=1}^n m = nm$  and variance  $\sum_{j=1}^n s^2 = ns^2$  when  $n$  is large. (A proof of the theorem can be found in appendix 12.A.1). Call the mean and variance  $\mu$  and  $\sigma^2$ , respectively; let  $\sum_{j=1}^n r_{j-1,j} \equiv r_{0,n}$ . We then have

$$S_1 = S_0 e^{r_{0,1}}, \quad r_{0,1} \sim N(\mu, \sigma^2) \quad (2)$$

if time is expressed in years; thus  $S_1/S_0$  is a lognormal variable with parameters  $(\mu, \sigma^2)$ .



It can be seen that the lognormal property of an asset's value is independent from the peculiar distribution of the continuously compounded rate of return of the asset over an interval, so long as the conditions of the central limit theorem are met. The strength of the theorem stems precisely from the fact that these conditions are quite unrestrictive: the continuously compounded daily rates of return,  $r_{j-1,j}$ , are simply supposed to be independent and identically distributed. The remarkable property of the lognormally distributed variable  $S_1/S_0$  stems from the fact that it will apply *whatever* the type of distribution followed by the  $r_{j-1,j}$  variable is, so long as it has finite variance and the  $r_{j-1,j}$ 's are independent.

There is another way of justifying *directly* the lognormality of asset prices, which rests on the assumption that the continuously compounded rate of return between any time 0 and time  $t$ ,  $\log(S_t/S_0)$ , follows a general Wiener process. Define  $W_t$  as a Wiener process such that  $W_0 = 0$ ,  $W_t \sim N(0, t)$ , with  $t_1 < t_2$ ,  $\Delta W_t = W_{t_2} - W_{t_1} \sim N(0, t_2 - t_1) \equiv N(0, \Delta t)$ . Suppose that

$$\log(S_t/S_0) = \mu t + \sigma W_t = \mu t + \sigma \sqrt{t} \varepsilon_t, \quad \varepsilon_t \sim N(0, 1) \quad (3)$$

or

$$\log(S_t/S_0) \sim N(\mu t, \sigma^2 t)$$

Over a finite interval  $\Delta t$ , the continuously compounded rate of return,  $\log S_{t+\Delta t}/S_t$ , is

$$\begin{aligned} \log\left(\frac{S_{t+\Delta t}}{S_t}\right) &= \log\left(\frac{S_{t+\Delta t}}{S_0}\right) - \log\left(\frac{S_t}{S_0}\right) \\ &= \mu \cdot (t + \Delta t) + \sigma W_{t+\Delta t} - \mu t - \sigma W_t \\ &= \mu \Delta t + \sigma(W_{t+\Delta t} - W_t) = \mu \Delta t + \sigma \Delta W_t \\ &= \mu \Delta t + \sigma \sqrt{\Delta t} \varepsilon_t, \quad \varepsilon_t \sim N(0, 1) \end{aligned}$$

Hence,  $S_{t+\Delta t}/S_t$  is also lognormal and can be written

$$\frac{S_{t+\Delta t}}{S_t} = e^{\mu \Delta t + \sigma \sqrt{\Delta t} \varepsilon_t}, \quad \varepsilon_t \sim N(0, 1)$$

Two special cases are important:  $\Delta t = 1$  year and  $\Delta t = p$  years. We have, respectively,

$$\frac{S_1}{S_0} = e^{\mu + \sigma \varepsilon_t}$$

which implies  $\log(S_1/S_0) = \mu + \sigma \varepsilon_t \sim N(\mu, \sigma^2)$  as before, and

$$\frac{S_p}{S_0} = e^{\mu p + \sigma \sqrt{p} \varepsilon_t}$$

entailing  $\log(S_p/S_0) \sim N(\mu p, \sigma^2 p)$ .

Thus the same results are obtained using two different probabilistic assumptions. The first is that over very short intervals the continuously compounded rates of return are independent and identically distributed, and we are then free to apply the central limit theorem. The second is a bit more restrictive because it assumes that the continuously compounded return over the same small interval follows a process with very precise properties—the Wiener process.

## 12.2 In search of the term structure of interest rates, or to spline or not to spline—and how?

Before dealing with the important flow of current research in this area, let us remember how the following three concepts are defined and related to each other (in continuous time):

- the price today of a series of default-free zero-coupon bonds with various maturities,
- the spot rate, or zero-coupon yield curve, and
- the forward rate curve.

We will take advantage of this quick refresher to simplify the notation somewhat in order to make it identical or immediately comparable to that used in recent research on the subject.

First, in section 11.4 we denoted the price at time  $t$  of a default-free zero-coupon bond paying one monetary unit at maturity  $T$  as  $B(t, T)$ . We will now simplify this notation by supposing that time today is 0. So one zero-coupon bond price can be written as  $B(0, T)$ , or, more com-

pactly, as  $B(T)$ , omitting the first index. This function of  $T$  is often called the *discount function*.

The *spot rate* is the yield to maturity of that zero-coupon bond, denoted  $R(0, T)$  in section 11.4. We can now do away with the first index and write simply  $R(T)$ . It is equal to the yield-to-maturity of the zero-coupon. Hence its relationship to the zero-coupon's price is

$$B(T)e^{R(T)T} = 1 \quad (4)$$

From (4), we can see that  $B(T)$  can be expressed as a function of  $R(T)$ , and *conversely*. Indeed, we can write

$$B(T) = e^{-R(T)T} \quad (5)$$

and

$$R(T) = -\frac{\log B(T)}{T} \quad (6)$$

Those relationships, (5) and (6),<sup>1</sup> are shown in the right-hand side of figure 12.1.

Consider now the forward rate. In section 11.4, we denoted the forward rate decided upon at time  $t$ , for a loan starting at time  $T$  for an infinitesimal trading period, as  $f(t, T)$ . Since today is time 0, we can denote it as  $f(0, T)$  or more simply as  $f(T)$ . The relationship between the discount function  $B(T)$  and the forward rate is given by equation (28) in chapter 11, where  $t$  is now replaced by 0. With our simplified notation, it thus reads as

$$B(T)e^{\int_0^T f(u) du} = 1 \quad (7)$$

from which  $B(T)$  can be immediately deduced:

$$B(T) = e^{-\int_0^T f(u) du} \quad (8)$$

<sup>1</sup> There is of course no surprise in seeing the *minus* sign in (6); the surprise would be *not* to see one. Indeed,  $B(T)$  being smaller than 1, its natural logarithm (as the logarithm  $B(T)$  with *any* base larger than 1 for that matter) is *always* a negative number. Since  $R(T)$  is positive, a minus sign is warranted in front of  $\log B(T)/T$ .

To determine the forward rate  $f(T)$  from  $B(T)$ , first take the logarithm of (8):

$$\log B(T) = - \int_0^T f(u) du$$

and differentiate  $\log B(T)$  with respect to  $T$ :

$$\frac{d}{dt} \log B(T) = -f(T)$$

The forward rate is thus

$$f(T) = - \frac{d}{dt} \log B(T) \quad (9)$$

The relationships between the discount function and the forward rate, (8) and (9), are shown in the left-hand side of figure 12.1.

Finally, what about any relationship between the forward rate and the spot rate? From (5) and (8), we can write

$$e^{-\int_0^T f(u) du} = e^{-R(T)T} \quad (10)$$

Hence the spot rate  $R(T)$  is simply the average of the forward rates between 0 and  $T$ :

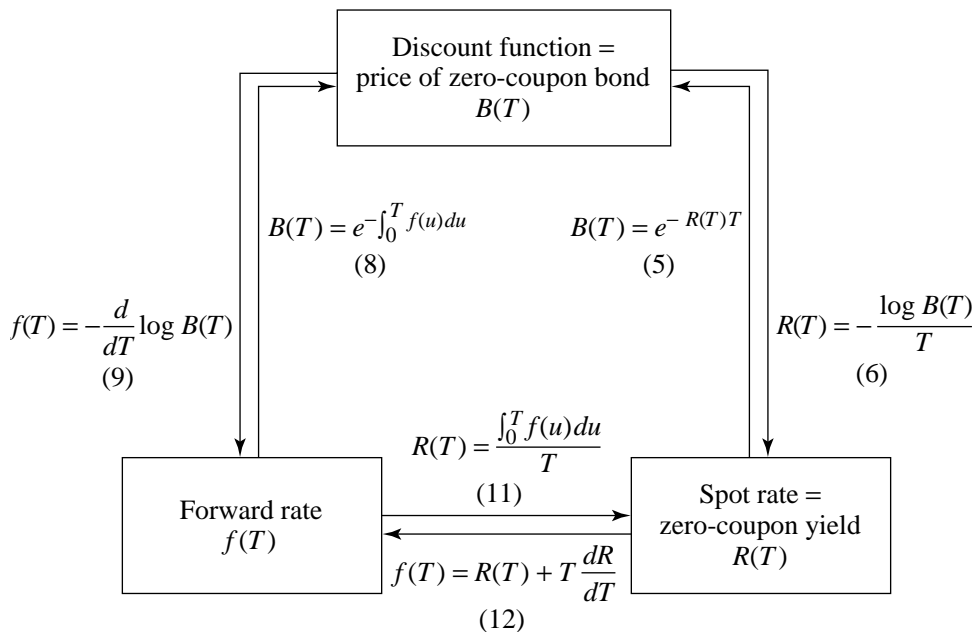
$$R(T) = \frac{\int_0^T f(u) du}{T} \quad (11)$$

From (10) or (11) we can write:

$$\int_0^T f(u) du = T \cdot R(T) \quad (12)$$

Taking the derivative of (12) with respect to  $T$  yields:

$$f(T) = R(T) + T \frac{dR}{dt} \quad (13)$$

**Figure 12.1**

The one-to-one relationship between the discount function, the spot rate, and the forward rate.

The relationships between the forward rate and the spot rate, (11) and (12), are shown at the bottom of figure 12.1. Thus there is indeed a one-to-one relationship between each of the concepts of discount function, spot rate, and forward rate. The moral: once you know one, you know the other two, and there are always *two* ways of going from one to another—a direct way and an indirect one.

Two schools of thought have had a friendly battle these last 30 years in the quest for the Holy Grail. We can first distinguish the so-called “parametric” methods; their promoters’ aim is to model, for instance, the forward curve by a parametric function. The other contender are the “spline” methods (to be defined soon). We will now present those two schools of thought, focusing on models that are relatively recent and that are widely used by practitioners, especially by central banks. As the reader might surmise, the most recent development in this field, quite interestingly, draws from both schools of thought.

### 12.2.1 Parametric methods

We will first examine the Nelson-Siegel (1987) model, and then turn to its extension by Svensson (1994, 1995).

#### 12.2.1.1 The Nelson-Siegel model

A very interesting model for the forward rate curve was proposed by Charles Nelson and Andrew Siegel (1987).<sup>2</sup> As we know, modeling the forward rate curve is equivalent to modeling the spot rate curve (through averaging the forward rate curve). The authors consider for the forward rate quite an interesting function, which can give rise to practically all shapes of spot curves observed on the markets. This is a Laguerre function<sup>3</sup> plus a constant, the formula of which is

$$f(T) = \beta_0 + \beta_1 e^{-T/\tau_1} + \frac{\beta_2}{\tau_1} T e^{-T/\tau_1} \quad (14)$$

where  $T$  is the variable, and  $\beta_0$ ,  $\beta_1$ ,  $\beta_2$ , and  $\tau_1$  are parameters to be estimated.

This expression implies the following equation for the spot rate. (See equation (11) and also the bottom of figure 12.1.) Taking the average value of the forward rates, we get

$$R(T) = \frac{\int_0^T f(u) du}{T} = \beta_0 + (\beta_1 + \beta_2) \frac{\tau_1}{T} (1 - e^{-T/\tau_1}) - \beta_2 e^{-T/\tau_1} \quad (15)$$

Let us evaluate limiting values of the spot rate and the forward rate when  $T$  goes either to infinity or to zero.

Consider first the case where  $T \rightarrow \infty$ ; from (15),  $R(T)$  tends toward the constant  $\beta_0$ . From (14), the same conclusion applies to the forward rate:  $\lim_{T \rightarrow \infty} f(T) = \beta_0$ .<sup>4</sup>

Suppose now that the maturity  $T$  goes to zero. From what we know of the spot rate as an average of forward rates, any limit when  $T \rightarrow 0$  must

<sup>2</sup> C. Nelson and A. Siegel, "Parsimonious Modeling of Yield Curve," *Journal of Business*, 60, no. 4 (1987): 473–489.

<sup>3</sup> For a description of Laguerre functions, see for instance E. Courant and D. Hilbert, *Methods of Mathematical Physics* (New York: Wiley, 1953), pp. 93–97.

<sup>4</sup> Notice that from  $\lim_{T \rightarrow \infty} f(T) = \beta_0$ ,  $\lim_{T \rightarrow \infty} R(T) = \beta_0$ , because  $R(T)$  is the average of the  $f(u)$ 's. However, the converse would have been much harder to prove.

be the same for either concept. Indeed, from (14) we have

$$\lim_{T \rightarrow 0} f(T) = \beta_0 + \beta_1$$

and the same is true for  $R(T)$ ; applying L'Hôpital's rule to the middle term of the right-hand side of equation (15), we get

$$\lim_{T \rightarrow 0} R(T) = \beta_0 + \beta_1$$

We are now ready to examine the shapes that the *average* of a Laguerre function can take, and we will make sure they fit a great deal of observed forms of the spot rate structure. We will try not to lose any generality in the function (15) by setting  $\beta_0 = 4.5$  (expressed in percent per year) and  $\tau_1 = 1$ . Also, we will set  $\beta_1 = -1$ . Denoting by  $a$  the only parameter left, we get from (15)

$$R(T) = 4.5 + (-1 + a) \frac{1 - e^{-T}}{T} - ae^{-T} \quad (16)$$

We can check that  $\lim R(T)_{T \rightarrow \infty} = 4.5$  ( $\beta_0 = 4.5$ ), and that  $\lim R(T)_{T \rightarrow 0} = 3.5(\beta_0 + \beta_1 = 3.5)$ .<sup>5</sup> We have now given to parameter  $a$  the values  $-2, -1, 0, 1, 3, 5$ . The resulting curves are depicted in figure 12.2. Indeed, we obtain the most familiar shapes of the spot rate structure. One feature, however, is worth mentioning here because it is a little surprising: the relatively slow convergence of the spot rates to their common value ( $\beta_0 = 4.5$ ) when  $T$  increases indefinitely. Table 12.1 gives the values we obtain with our values (in percent per year) for the Nelson-Siegel model.

This slowness of convergence is even more apparent in the original Nelson-Siegel model, where the authors considered a wider range of values for parameter  $a$  (from  $-6$  to  $+12$ ). (We have chosen to set their  $\beta_0$  coefficient to  $+3$  in order to conform with their graph on page 476.) (Note that table 12.2 does not carry the values for  $a = 1$ , for which maximum convergence speed can be observed.)

<sup>5</sup>There is probably a small typo on page 476 of the Nelson and Siegel article. To be consistent with figure 1 of the paper, the first coefficient of the equation on the same page should be  $+3$  (instead of  $+1$ ). Or, if the equation is correct, then the first node of the curves should be at the origin, and all curves should be translated down by two.

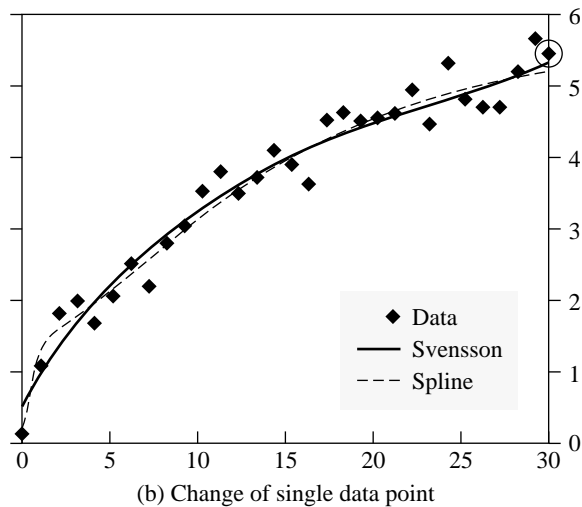
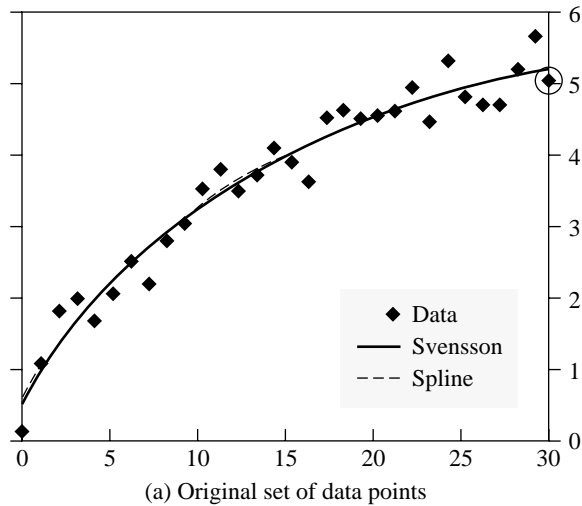


Fig. 12.2

Comparison between the sensitivity of a spline regression and that of a parametric regression to a change of one data point. (Reproduced from Nicola Anderson and John Sleath, "New Estimates of the UK Real and Nominal Yield Curves," *Bank of England Quarterly Bulletin* (November 1999): 386, with the kind permission of the authors and the Bank of England, Monetary Instruments and Markets Division.)



**Table 12.1**  
**Spot rate values for long maturities;  $\beta_0 = 4.5$ ;  $\beta_1 = -1$**

Maturity (years)	$a = -2$	$a = -1$	$a = 0$	$a = 1$	$a = 3$	$a = 5$
25	4.38	4.42	4.46	4.5	4.58	4.66
100	4.47	4.48	4.49	4.5	4.52	4.54
1000	4.497	4.498	4.499	4.5	4.502	4.504

**Table 12.2**  
**Spot rates for long maturities;  $\beta_0 = +3$ ;  $\beta_1 = -1$ ; original Nelson-Siegel values for  $a$**

Maturity (years)	$a = -6$	$a = -3$	$a = 0$	$a = 3$	$a = 6$	$a = 12$
25	2.72	2.84	2.96	3.08	3.2	3.32
100	2.93	2.96	2.99	3.02	3.05	3.08
1000	2.993	2.996	2.999	3.002	3.005	3.008

### 12.2.1.2 The Svensson model

The Svensson model adds flexibility to the Nelson-Siegel formulation by adding a potential extra hump in the forward curve.<sup>6</sup> Its basic equation reads:

$$f(T) = \beta_0 + \beta_1 e^{-T/\tau_1} + \beta_2 \frac{T}{\tau_1} e^{-T/\tau_1} + \beta_3 \frac{T}{\tau_1} e^{-T/\tau_2} \quad (17)$$

which implies that six coefficients ( $\beta_0$ ,  $\beta_1$ ,  $\beta_2$ ,  $\beta_3$ ,  $\tau_1$ , and  $\tau_2$ ) have to be estimated. This model had considerable impact in the practitioners' world: it was used between 1995 and the end of the twentieth century by many central banks, most notably the Bank of England.

The highly innovative paper by Nicola Anderson and John Sleath<sup>7</sup> (of the Bank of England's Monetary Instruments and Markets Division) explains very clearly why these authors looked for a new approach that

<sup>6</sup> L. Svensson, "Estimating and Interpreting Forward Interest Rates: Sweden 1992–94," *IMF Working Paper*, No. 114 (1994); L. Svensson, "Estimating Forward Interest Rates with the Extended Nelson and Siegel Method," *Sveriges Riksbank Quarterly Review*, 3 (1995): 13.

<sup>7</sup> N. Anderson and J. Sleath, "New Estimates of the UK Real and Nominal Yield Curves," *Bank of England Quarterly Bulletin* (November 1999): 384–392.

would improve on the estimates of the Nelson-Siegel and Svensson approaches, and why this approach is now in use at the Bank of England. Apart from the appearance of additional information from the gilt market,<sup>8</sup> the main impetus for changing the in-house method of estimating the yield curve was the appearance of a new model developed by Daniel Waggoner (1997), of the Federal Reserve Bank of Atlanta.<sup>9</sup> Waggoner's work belongs to the family of "spline" methods that we will now present.

### 12.2.2 Spline methods: the Fisher-Nychka-Zervos, the Waggoner, and the Anderson-Sleath models

#### 12.2.2.1 What is a spline?

From a mathematical standpoint, a *spline* is a piecewise polynomial, which is a function made up of individual polynomial segments joined at so-called "knot points." At those knot points the function and its first derivative are continuous; note that the latter condition means the corresponding curve is smooth in the sense that its curve does not make any angle at any of its points. To the eye, a curve corresponding to a spline will always look as if it represented one polynomial, albeit perhaps of a high order.<sup>10</sup>

In fact, as stated in most finance texts that deal with the subject, most splines are made up with piecewise *cubic* polynomials, and the reader might ask why this is so and why at least some of the segments are not second-order or fifth-order polynomials, for instance. The answer to this question is far from obvious and quite interesting. It certainly requires going back to the origin of the word "spline." We will then discover some new and fascinating territory.

<sup>8</sup> The gilt-edged market is the UK Government bond market. "Its origins," says the *New Palgrave Dictionary of Money and Finance* (New York: Macmillan, 1992), p. 240 "go back to 1694 when the Bank of England was founded to help the Government raise money to fight the French" (!) (Let us not forget, on the other hand, that the Banque de France was founded by Napoleon to fight the rest of the world, and for a number of other no more commendable purposes.)

<sup>9</sup> D. Waggoner, "Spline Methods for Extracting Interest Rate Curves from Coupon Bond Prices," *Working Paper*, No. 97-10 (1997), Federal Reserve Bank of Atlanta.

<sup>10</sup> Technical references to splines are from C. de Boor, *A Practical Guide to Splines* (New York: Springer-Verlag, 1978), and G. Wahba, *Spline Models for Observational Data* (Philadelphia: SIAM, 1990).

More than a century ago, the word “spline” had a number of different meanings; among them is the following, which we extracted from the *Oxford English Dictionary* (Vol. XVI, p. 283): “A flexible strip of wood or hard rubber used by draftsmen in laying out broad sweeping curves, especially in railroad work.” When draftsmen designed in such a way curves that had to pass through fixed points, knowingly or not, they achieved two results at the same time. First, they minimized the curvature of the spline. Second, the lath took a form that minimized its deformation energy.

### 12.2.2.2 Some strange—and nice—consequences

Closing your favorite dictionary, it dawns on you that your younger sister is majoring in physics (which of course makes you a little jealous, but you know you can always count on her unless she is watching *Beverly Hills*). So you ask her what the deformation energy is for a thin, homogeneous lath whose function  $f(x)$  is defined over an interval  $[x_0, x_n]$  going through  $n + 1$  pivotal point  $(x_j, y_j)$ ,  $j = 1, \dots, n$ . A little surprised by the simplicity of your question, she answers that the deformation energy is, disregarding a few constants, basically represented by the integral

$$I = \int_{x_0}^{x_n} [f''(x)]^2 dx \quad (18)$$

Luck is still on your side. It turns out that you have been fortunate enough to take a course on growth and investment.<sup>11</sup> One of the instructor’s pet hobbies was the calculus of variations—the calculus by which you look for extremals of functional relationships. These are relationships between a function and a number equal to the integral of an expression involving  $x$ ,  $f(x)$  and its successive derivatives, for instance. Such a functional may be written as

$$v[y(x)] = \int_a^b F[x, f(x), f'(x), f''(x), \dots, f^{(n)}(x)] dx \quad (19)$$

So you recognize what your little sister has quickly scribbled on the palm of your hand as nothing else than a particular case of (19), where all

<sup>11</sup> Was it by any chance given by Olivier de La Grandville? That is a possibility.

arguments of  $F(\cdot)$  are missing, except the fourth ( $f''(x)$ ). Furthermore,  $F(\cdot)$  in your case is  $(f''(x))^2$ . Now you remember that your instructor highly praised an excellent, very readable text by L. Elsgolc<sup>12</sup> that gives the solution to the problem in the form of an Euler-Poisson equation: if  $y(x)$  minimizes  $v[y(x)]$ , it has to be a solution of the following Euler-Poisson equation:

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \cdots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0 \quad (20)$$

where  $F_y$  denotes the partial derivative of the integrand  $F$  with respect to  $y$ , and  $F_{y^{(j)}}$  the partial derivative of  $F$  with respect to the derivative of order  $j$  of  $y$ .

The Euler-Poisson equation is a differential equation of order  $2n$ . It generalizes the solution discovered by Euler in 1744 to the following problem: find a curve  $y(x)$  that would be an extremum of

$$u[y(x)] = \int_a^b F[x, f(x), f'(x)] dx \quad (21)$$

Euler found that if  $y(x)$  was such an extremal, it had to be the solution of the second-order differential equation

$$F_y - \frac{d}{dx} F_{y'} = 0 \quad (22)$$

Now why is the Euler-Poisson a differential equation of order  $2n$ ? Since *Beverly Hills* has not yet started, you turn again to your sister for some sibling support. This time, deservedly, she gives you a bad rap. “You should be ashamed of yourself,” she says. “I do remember seeing in your class notes that your instructor (what was his funny name again?) had

<sup>12</sup>L. Elsgolc, *Calculus of Variations*, International Series of Monographs in Pure and Applied Mathematics (Reading, Mass.: Pergamon Press, 1962). Another excellent source is by the same author (with a different spelling: L. Elsgolts), *Differential Equations and the Calculus of Variations* (Moscow: MIR Publishers, 1970).

taken the trouble to write in full the Euler equation (22), which reads

$$F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0$$

or

$$\frac{\partial F}{\partial y}(x, y, y') - \left[ \frac{\partial^2 F}{\partial x \partial y'}(x, y, y') + \frac{\partial^2 F}{\partial y \partial y'}(x, y, y')y' + \frac{\partial^2 F}{\partial y'^2}(x, y, y')y'' \right] = 0 \quad (23)$$

“The last term of (23),” your sister goes on, “contains a function of  $x$ ,  $y$ , and  $y'$  multiplying the second derivative  $y''$ ; hence the Euler equation is a nonlinear *second-order* differential equation with nonconstant coefficients, generally depending not only on  $x$  but on  $y$  and  $y'$  as well. If you do the same with the Euler-Poisson equation (20), you notice that you have to take the  $n$ th-order derivative of an expression depending on the  $n$ th-order derivative of  $y$ —hence you are facing a  $2n$ -order differential equation.

“What’s more,” she adds, “your instructor gave you a way of obtaining the Euler-Poisson equation through simple economic reasoning. To that effect, he handed you an article of his. He could guarantee the quality of the paper, he said, because you could use it to hold those greasy french fries the next time you went to the ballpark. You thought it very funny to do just that the last time you went to see the Giants take care of the Dodgers, and now you pay the price.” With that she slammed her door.<sup>13</sup>

### 12.2.2.3 Determining the order of the optimal polynomial spline

Ashamed of yourself as you may be, you are nevertheless ready to go back to the functional minimizing the curvature as well as the deformation energy of the spline (18):

$$I[y(x)] = \int_{x_0}^{x_n} [f''(x)]^2 dx$$

Applying the Euler-Poisson equation (20), the spline should be a solution

<sup>13</sup> We are grateful to your sister for having taken a few notes on this paper before you headed for Pacific Bell Park. These can be found in appendix 12.A.2.

of the fourth-order differential equation

$$\frac{d^2}{dx^2} F_{y''} = 0 \quad (24)$$

In this case,  $F = (y'')^2 \equiv [f''(x)]^2$ ; therefore  $F_{y''} = 2f''(x)$ . The Euler-Poisson is thus

$$\frac{d^2}{dx^2} [2f''(x)] = 2f^{(4)}(x) = 0, \quad (25)$$

which is equivalent to

$$f^{(4)}(x) = 0 \quad (26)$$

The most general solution of this simple differential equation is obtained by four successive integrations, which indeed lead to the third-order polynomial

$$ax^3 + bx^2 + cx + d = 0 \quad (27)$$

A minimal convexity, minimal deformation energy curve will thus correspond to a third-order polynomial.

To the best of our knowledge, J. H. McCulloch (1971 and 1975)<sup>14</sup> was the first author to propose using regression cubic splines to determine the discount function. He was followed by a number of authors who have either improved the estimation techniques or set out to analyze potential heteroskedasticity in the residuals. (The interested reader should consult, in particular, Oldrich Vasicek and Gifford Fong (1982);<sup>15</sup> G. Shea (1984);<sup>16</sup> D. Chambers, W. Carleton, and D. Waldman (1992);<sup>17</sup> and also T. S. Coleman, L. Fisher, and R. Ibbotson (1992).<sup>18</sup> See also

<sup>14</sup> J. H. McCulloch, "Measuring the Term Structure of Interest Rates," *Journal of Business*, 44 (1971): 19–31; J. H. McCulloch, "The Tax-Adjusted Yield Curve," *Journal of Finance*, 30 (1975): 811–830.

<sup>15</sup> O. Vasicek and G. Fong, "Term Structure Estimation Using Exponential Splines," *Journal of Finance*, 38 (1982): 339–348.

<sup>16</sup> G. Shea, "Pitfalls in Smoothing Interest Rate Term Structure Data: Equilibrium Models and Spline Approximations," *Journal of Financial and Quantitative Studies*, 19, no. 3 (1984): 253–269.

<sup>17</sup> D. Chambers, W. Carleton, and D. Waldman, "A New Approach to Estimation of Terms Structure of Interest Rates," *Journal of Financial and Quantitative Studies*, 19, no. 3 (1984): 233–252.

<sup>18</sup> T. S. Coleman, L. Fisher, and R. Ibbotson, "Estimating the Term Structure of Interest from Data That Include the Prices of Coupon Bonds," *Journal of Fixed Income* (September 1992): 85–116.

the very innovative work of K. J. Adams and D. R. Van Deventer (1994).<sup>19</sup>)

There was, however, one drawback to that approach. It implied undue oscillations of the forward rate curve. In recent years, we have witnessed three major successive breakthroughs to circumvent that problem. Each of them displays remarkable ingenuity. These are the Fisher-Nychka-Zervos (1995),<sup>20</sup> the Waggoner (1997),<sup>21</sup> and the Anderson-Sleath (1999)<sup>22</sup> models. We will describe them in order.

#### **12.2.2.4 The Fisher-Nychka-Zervos model**

The main objective of this powerful paper was twofold: first, to price bonds accurately and, second, to produce a relatively stable forward curve. The originality of their contribution stems from the fact that the spline is aimed directly at the forward rate curve. Furthermore, a “roughness penalty” is added in the process of minimizing the residual sums of squares. (We will soon present this subject in more detail when we describe the Waggoner model.) The results obtained by the authors for *daily* (!) data from December 1987 through September 1994 are quite impressive.

In an extremely detailed study, however, Robert Bliss (1997)<sup>23</sup> observed a slight problem with the Fisher-Nychka-Zervos method, in the sense that it tended to misprice very short or short bonds. To try to correct that slight defect, Daniel Waggoner (1997) suggested the following method.

#### **12.2.2.5 An extension of the Fisher-Nychka-Zervos model: the Waggoner model**

Basically, Waggoner introduced a “variable roughness penalty” (VRP), which we now describe.

<sup>19</sup> K. J. Adams and D. R. Van Deventer, “Fitting Yield Curves and Forward Rate Curves with Maximum Smoothness,” *Journal of Fixed Income* (June 1994): 52–62.

<sup>20</sup> M. Fisher, D. Nychka, and D. Zervos, “Fitting the Term Structure of Interest Rates with Smoothing Splines,” *Working Paper*, No. 95-1 (1995), Finance and Economics Discussion Series, Federal Reserve Board, 1995.

<sup>21</sup> D. Waggoner, “Spline Methods for Extracting Interest Rate Curves from Coupon Bond Prices,” *Working Paper*, No. 97-10 (1997). Federal Reserve Bank of Atlanta.

<sup>22</sup> N. Anderson and J. Sleath, “New Estimates of the UK Real and Nominal Yield Curves,” *Bank of England Quarterly Bulletin* (November 1999): 384–392.

<sup>23</sup> R. Bliss, “Testing Term Structure Estimation Methods,” *Advances in Futures and Options Research*, 9 (1997): 197–231.

Pure arbitrage, as we presented it in previous chapters, is a simplified description of what can happen in financial markets, in the sense that a number of imperfections, or biases, are introduced into the picture. Those biases mainly take the form of taxes (we could even say tax systems) and various transaction costs.

Following Waggoner—where the author denotes the spot rate  $R(T)$  as  $y(t)$  and the discount function  $B(T)$  as  $\delta(t)$ —suppose that the cashflows of a given coupon-bearing bond are paid on times  $t_j$ ,  $j = 1, \dots, K$ . The theoretical value of a bond  $i$ ,  $P_i$ , is therefore given equivalently by any of the following three expressions:

$$P_i = \sum_{j=1}^K c_j \delta(t_j) = \sum_{j=1}^K c_j e^{-t_j y(t_j)} = \sum_{j=1}^K c_j e^{-\int_0^{t_j} f(u) du} \quad (28)$$

and the pricing equation, where  $\hat{P}_i$  denotes the observed value of the  $i$ th bond<sup>24</sup> and  $\varepsilon_i$  the error term, is

$$P_i = \hat{P}_i + \varepsilon_i, \quad i = 1, \dots, n \quad (29)$$

(supposing we consider a sample of  $n$  bonds).

As Waggoner explains clearly, the oscillation of a spline is often an increasing function of the number of its nodes. This is disturbing, since there are very few reasons for the forward curve not to be a horizontal line for very long maturities.

These oscillations can be controlled by imposing a roughness penalty (denoted  $\lambda$ ) in the regression process. The idea, originally introduced by Fisher, Nychka, and Zervos (1995), is to choose an optimal spot rate function  $R(T)$ —denoted by Waggoner as  $\psi(t)$ —that minimizes the sum of the following two elements:

- the sum of the square of error terms

$$\varepsilon_i^2 = \sum_{i=1}^n \{P_i - \hat{P}_i[\psi(t)]\}^2 \quad (30)$$

<sup>24</sup> As many authors do, Waggoner uses as observed price the average between the bid and ask quotes for the bond.



- the roughness penalty value  $\lambda$  times the curvature (or, as we have seen, a measure of the lath deformation energy) of the interest rate spline,<sup>25</sup>  $\lambda \int_0^{t_K} [\psi''(t)]^2 dt$ .

The size of  $\lambda$  acts as a trade-off between minimizing estimation errors and minimizing curvature (or, equivalently, between maximizing accuracy of fit and minimizing smoothness). So we have to solve the following problem:

$$\min_{\psi(t)} \left[ \sum_{i=1}^K \{P_i - \hat{P}_i[\psi(t)]\}^2 + \lambda \int_0^{t_K} [\psi''(t)]^2 dt \right] \quad (31)$$

This is what Fisher, Nychka, and Zervos did using a generalized cross-validation method. Using the Waggoner notation, this method consists of minimizing the expression

$$\gamma(\lambda) = \frac{RSS(\lambda)}{[n - \theta e_p(\lambda)]^2}$$

where

$n \equiv$  number of bonds

$RSS(\lambda) \equiv$  sum of residual sum of squares (first part of (31))

$e_p(\lambda) \equiv$  effective number of parameters

$\theta \equiv$  a “cost” or “tuning” parameter, set by Fisher, Nychka, Zervos, and Waggoner to 2

The contribution of Waggoner was to introduce a *variable* roughness penalty (a function depending on time,  $\lambda(t)$ ), so that one is now looking for a function  $\psi(t)$  such that

$$\min_{\psi(t)} \left[ \sum_{i=1}^K \{P_i - \hat{P}_i[\psi(t)]\}^2 + \lambda \int_0^{t_K} \lambda(t) [\psi''(t)]^2 dt \right] \quad (32)$$

<sup>25</sup> Independently, C. H. Reinsch (1971) and O. Vasicek, in the appendix of the K. Adams and D. Van Deventer (1994) paper, showed, through variational methods, that directly attacking the forward curve with a spline necessarily implied looking for a *quartic* (and not a cubic) spline. (See C. H. Reinsch “Smoothing by Spline Functions II,” *Numerische Mathematik*, 16 (1971): 451–454.) We may then be facing a contradiction between what theory dictates and what practitioners do.

This was a fine idea: introducing an ad hoc roughness penalty enabled Waggoner to retain the much needed flexibility at the short end, and to damp oscillations for long maturities. The  $\lambda(t)$  function retained by Waggoner was the following:

$$\lambda(t) = \begin{cases} 0.1 & 0 \leq t \leq 1 \\ 100 & 1 \leq t \leq 10 \\ 100,000 & t \geq 10 \end{cases}$$

The first question the reader could ask is: how sensitive is the optimal function  $\psi(t)$  to the above ad hoc choice of  $\lambda(t)$ ? Quite surprisingly, Waggoner found that the optimal  $\psi(t)$  was not very sensitive to quite different choices of the  $\lambda(t)$  function (the author chose step functions for  $\lambda(t)$  as well as continuous functions), with results that seem to improve somewhat on those obtained in the Fisher-Nychka-Zervos model.

#### 12.2.2.6 The Anderson-Sleath model

In their 1999 contribution, Nicola Anderson and John Sleath first asked a very interesting question: what is the main advantage of spline methods over parametric methods? The answer is not so obvious. The principal advantage of the former methods over the latter is that individual segments of the spline can move almost independently of one another, contrary to a parametric curve. Anderson and Sleath showed quite persuasively that whenever a shock affects the data at one end of the curve, the spline is (quite naturally) modified at that end, while the curve obtained through a parametric method can be severely affected, with no good reason, at the other end. They give an excellent example to demonstrate this, which they were kind enough to let us reproduce here.

In figure 12.2(a), a set of points (made up from a power function with noise added) is regressed using both a cubic spline and a Svensson two-hump functional form (equation (17)). The closeness between the two curves is such that it is almost impossible to distinguish between them. We now change the last point of the regression: while retaining the same abscissa, its ordinate moves up from 5 to 5.5 (see the circled point in both figures). Not only does the Svensson curve move practically on its whole length, but it receives a hump at its short end, certainly a surprising and unwarranted consequence of such an insignificant change in the data.

Anderson and Sleath modified the Waggoner model in important ways. First, they weighed differences between any bond and its theoretical value

with the inverse of its modified duration (in order to penalize pricing errors on bonds that are more sensitive to interest rate changes than others). Second, they chose to estimate an optimal function of maturity  $\lambda(m)$  that would be such that

$$\log \lambda(m) = L - (L - S)e^{-m/\mu}$$

where  $L, S$ , and  $\mu$  are parameters to be estimated. Finally, they made prices direct functions of the forward rates.

So they chose to minimize the expression

$$X_s = \sum_{i=1}^N \left[ \frac{P_i - \pi_i(\mathbf{c})}{D_i} \right]^2 + \int_0^M \lambda_t(m) [f''(m)]^2 dm \quad (33)$$

where

$D_i \equiv$  modified duration of bond  $i$

$\mathbf{c} \equiv$  parameter vector of polynomial spline to be estimated

$M \equiv$  maturity of the longest bond.<sup>26</sup>

Their results definitely seem to improve upon those obtained in the earlier models. As the reader will notice, Anderson and Sleath draw upon both schools of thought—splining *and* parameter estimation.

A final word is in order. While the Fisher, Nychka, and Zervos method postulates that  $\lambda$  is constant accross maturities but variable from day to day, the Waggoner and the Anderson and Sleath models allow  $\lambda$  to vary across maturities. One could think perhaps of a roughness penalty that would share both properties.

### Appendix 12.A.1 A proof of the central limit theorem and a determination of the probability distribution of the dollar return on an asset

There are many ways to prove the central limit theorem. Here is one of the most direct (inspired by S. Ross<sup>27</sup> and adapted to the concepts and

<sup>26</sup> We have taken the liberty of modifying very slightly their notation.

<sup>27</sup> S. Ross, *A First Course in Probability* (New York: Macmillan, 1987), p. 346.

notation of this chapter). As in many cases, proving the theorem amounts to showing that when  $n$  becomes large, the standardized variable of  $r_{0,n}$  has a moment-generating function that converges toward  $e^{\lambda^2/2}$ , that is, the moment-generating function of the unit normal variable.

Let  $r_{0,n} \equiv r$  for short. We have

$$r = \sum_{t=1}^n r_{t-1,t}, \quad \text{with } r_{t-1,t} = \log(S_t/S_{t-1})$$

By hypothesis, the  $r_{t-1,t}$ 's are independent and identically distributed; moreover,  $E(r_{t-1,t}) = m$  and  $VAR(r_{t-1,t}) = s^2 < \infty$ ; hence, from independence of the  $r_{t-1,t}$ 's, we have

$$E(r) = nm \equiv \mu$$

$$VAR(r) = ns^2 \equiv \sigma^2$$

$$\sigma(r) = \sqrt{ns} \equiv \sigma$$

The standardized variable of the sum is

$$\rho = \frac{r - nm}{\sqrt{ns}} = \frac{\sum_{t=1}^n r_{t-1,t} - nm}{\sqrt{ns}} = \sum_{t=1}^n \left( \frac{r_{t-1,t} - m}{\sqrt{ns}} \right) = \sum_{t=1}^n \frac{Y_t}{\sqrt{n}}$$

where  $Y_t = \frac{r_{t-1,t} - m}{s}$  is the standardized variable of  $r_{t-1,t}$ , with  $E(Y_t) = 0$  and  $VAR(Y_t) = 1$ .

The moment-generating function of  $\rho$  is the  $n$ th power of the moment-generating function of  $\frac{Y_t}{\sqrt{n}}$ , itself  $\varphi_{Y_t/\sqrt{n}}(\lambda)$ , equal to  $\varphi_{Y_t}\left(\frac{\lambda}{\sqrt{n}}\right)$ .<sup>28</sup> Thus we have

$$\varphi_\rho(\lambda) = \left[ \varphi_{Y_t}\left(\frac{\lambda}{\sqrt{n}}\right) \right]^n$$

What we want to prove is that  $\lim_{n \rightarrow \infty} \varphi_\rho(\lambda) = e^{\lambda^2/2}$  or, equivalently, that  $\lim_{n \rightarrow \infty} \log \varphi_\rho(\lambda) = \lambda^2/2$ . We have

$$\log \varphi_\rho(\lambda) = n \log \varphi_{Y_t}\left(\frac{\lambda}{\sqrt{n}}\right) = \frac{\log \varphi_{Y_t}\left(\frac{\lambda}{\sqrt{n}}\right)}{n^{-1}} \equiv \frac{L\left(\frac{\lambda}{\sqrt{n}}\right)}{n^{-1}}$$

<sup>28</sup> In this simple proof, we suppose that the moment-generating function for  $Y_t$  exists.

where  $L(x) = \log \varphi_{Y_t}(x)$ . We have

$$L(0) = \log \varphi_{Y_t}(0) = \log 1 = 0$$

$$L'(0) = \frac{\varphi'_{Y_t}(0)}{\varphi_{Y_t}(0)} = 0 \quad (\text{since } \varphi'(0) = 0)$$

$$L''(0) = \frac{\varphi''_{Y_t}(0)\varphi_{Y_t}(0) - [\varphi'_{Y_t}(0)]^2}{\varphi_{Y_t}^2(0)} = 1 \quad (\text{since } \varphi''(0) = 1)$$

Let us apply L'Hôpital's rule to (34):

$$\lim_{n \rightarrow \infty} \log \varphi_p(\lambda) = \lim_{n \rightarrow \infty} \frac{L'(\frac{\lambda}{\sqrt{n}}) \cdot \lambda n^{-3/2}}{2n^{-2}} = \lim_{n \rightarrow \infty} \frac{\lambda L'(\frac{\lambda}{\sqrt{n}})}{2n^{-1/2}}$$

Applying L'Hôpital's rule to (35), we have

$$\lim_{n \rightarrow \infty} \log \varphi_p(\lambda) = \lim_{n \rightarrow \infty} \frac{\lambda^2}{2} L''\left(\frac{\lambda}{\sqrt{n}}\right) = \frac{\lambda^2}{2}$$

This shows that  $\rho$ , the standardized variable of  $r$ , converges in distribution toward the unit normal distribution  $N(0, 1)$ .

## Appendix 12.A.2 Deriving the Euler-Poisson equation from economic reasoning

In a highly insightful paper, Robert Dorfman (1969)<sup>29</sup> showed how the Pontryagin maximum principle (the fundamental theorem of optimal control theory<sup>30</sup>) could be derived from pure economic reasoning, more precisely by using capital theory. To that effect, he introduced the concept of a modified Hamiltonian. Differentiating this Hamiltonian with respect to both the state variable and the control variable led him to the maximum principle. In this appendix, we will show how the Euler-Poisson

<sup>29</sup>R. Dorfman, "An Economic Interpretation of Optimal Control Theory," *The American Economic Review* (1969): 817–831.

<sup>30</sup>The original, technically demanding, work of Pontryaguine is in L. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E. L. Mishchenko, *The Mathematical Theory of Optimal Processes* (New York: Interscience Publishers, 1962). A deep (and difficult) discussion of the relationship between the calculus of variations and optimal control theory is given in Pontryaguine et al. (1962, chapter 5). A simpler presentation may be found in I. M. Gelfand and S. V. Fomin, *Calculus of Variations* (Englewood Cliffs, N.J.: Prentice Hall, 1963) (see Appendix II).

equation of the calculus of variations can be obtained by extending his method to new Hamiltonians.

First, recall briefly a simple version of Pontryagin's principle. Let  $k(t)$  denote a *state* variable (for instance, the capital stock of a company or in a country) and  $x(t)$  a *control* variable (that is, a variable that reflects decisions taken at time  $t$ ). Suppose one wants to maximize (or minimize) a functional such that  $\int_a^b u(k_t, x_t, t) dt$ , where  $k_t$  and  $x_t$  stand for  $k(t)$  and  $x(t)$  subject to the constraint

$$\dot{k} = f(k_t, x_t, t)$$

Note that the control variable and the state variable have some effect both on the integrand of the functional and on the rate of increase of the state variable. Furthermore, these relationships may change in time. Pontryagin's principle states that if one defines a function  $\lambda(t)$  and a Hamiltonian<sup>31</sup>

$$H = u(k_t, x_t, t) + \lambda(t)f(k_t, x_t, t)$$

then the optimal control time path  $x^*(t)$  must be such that it solves the following system of equations:

$$\frac{\partial H}{\partial x_t} = \frac{\partial u}{\partial x_t}(k_t, x_t, t) + \lambda(t) \frac{\partial f}{\partial x_t}(k_t, x_t, t) \quad (1)$$

$$\frac{\partial H}{\partial k_t} = \frac{\partial u}{\partial k_t}(k_t, x_t, t) + \lambda(t) \frac{\partial f}{\partial k_t}(k_t, x_t, t) = -\dot{\lambda}(t) \quad (2)$$

$$\frac{\partial H}{\partial \lambda_t} = f(k_t, x_t, t) = \dot{k}_t \quad (3)$$

For anyone familiar with differential calculus and classical optimization of functions only, there are at least three surprises in Pontryagin's principle. The first two come from the definition of the Hamiltonian, which has little to do with a Lagrangian. First, the constraint  $\dot{k} = f(k_t, x_t, t)$  does not appear in the form it has in a traditional Lagrangian. Second, the familiar, constant Lagrange multiplier is here replaced by a function of time to be determined. Finally, equation (2) may seem quite astonishing  $\left(\frac{\partial H}{\partial k_t} = -\dot{\lambda}(t)\right)$ .

<sup>31</sup> There is no relationship between the  $\lambda$  used in this appendix and the  $\lambda$  used in the preceding chapter.

To explain Pontryagin's principle from an economic point of view, Dorfman had the stroke of genius to define a new Hamiltonian that had profound economic meaning, where the state variable and the control variable would play symmetrical roles. Dorfman defined the total value of the system at any point of time as

$$H^* = u(k_t, x_t, t) + \frac{d}{dt}[\lambda(t)k(t)] \quad (4)$$

In this expression, he was adding to the integrand of the functional the rate of increase of the state variable's value, itself equal to the product of the *amount* of the state variable  $k(t)$  and its *price*  $\lambda(t)$ .  $H^*$  could be written as

$$H^* = u(k_t, x_t, t) + \dot{\lambda}_t k_t + \lambda_t \dot{k}_t \quad (5)$$

It now makes a lot of sense to take the two partial derivatives of  $H^*$  with respect to  $k_t$  and  $x_t$ , respectively, and equate each of them to zero to obtain

$$\frac{\partial H^*}{\partial x_t} = \frac{\partial u}{\partial x_t}(k_t, x_t, t) + \lambda(t) \frac{\partial f}{\partial x_t}(k_t, x_t, t) = 0 \quad (6)$$

$$\frac{\partial H^*}{\partial k_t} = \frac{\partial u}{\partial k_t}(k_t, x_t, t) + \dot{\lambda}(t) + \lambda_t \frac{\partial f}{\partial k_t}(k_t, x_t, t) = 0 \quad (7)$$

It is immediately apparent that equations (6) and (7) are equivalent to Pontryagin's equations (1) and (2). Furthermore,  $\frac{\partial H^*}{\partial \lambda_t} = \dot{k}_t$  as in equation (3).

Let us illustrate the power of Dorfman's (economic) Hamiltonian. Suppose we want to address the classic, first-order, variational problem set by Jean Bernoulli in 1696<sup>32</sup> and solved in a general way by Euler half

<sup>32</sup> The origin of the calculus of variations can be traced to 1696, when Jean Bernoulli organized for his fellow mathematicians the following competition: let  $A$  and  $B$  be two fixed points in a vertical plane. Find the curve joining  $A$  and  $B$  such that a particle sliding along the curve reaches  $B$  in minimum time. The solution is a cycloid (a curve generated by the point of a circle rolling on a horizontal axis), and was found by Jean Bernoulli, his brother Jacob, Newton, and the Marquis de L'Hôpital. Newton did not sign his solution (perhaps fearing that H. M. Treasury, for which he was working at the time, would discover that he was still dabbling in mathematics?). In any case, upon reading his solution, Jean Bernoulli supposedly said, in awe: "I recognized the lion's paw."

a century later (in 1744): find  $k_t$  (or  $\dot{k}_t$ ) such that the following functional is minimized:

$$\min_{k_t} \int_a^b u(k_t, \dot{k}_t, t) dt$$

In this original variational problem, there are no constraints. One can consider it a special case of an optimal control problem where the constraint  $\dot{k}_t = f(k_t, x_t, t)$  is  $\dot{k}_t = x_t$ . Form then the modified Hamiltonian:

$$H^* = u(k_t, \dot{k}_t, t) + \dot{\lambda}_t k_t + \lambda_t \dot{k}_t \quad (8)$$

Now take the derivatives of  $H^*$  with respect to  $k_t$  and  $\dot{k}_t$ , and set both equal to zero. We get

$$\frac{\partial H^*}{\partial k_t} = \frac{\partial u}{\partial k_t}(k_t, k_t, t) + \dot{\lambda}_t = 0 \quad (9)$$

$$\frac{\partial H^*}{\partial \dot{k}_t} = \frac{\partial u}{\partial \dot{k}_t}(k_t, \dot{k}_t, t) + \lambda_t = 0 \quad (10)$$

Differentiate (9) with respect to time, and replace  $\dot{\lambda}_t$  in (8). The Euler equation immediately appears:

$$\frac{\partial u}{\partial k_t}(k_t, \dot{k}_t, t) - \frac{d}{dt} \frac{\partial u}{\partial \dot{k}_t}(k_t, \dot{k}_t, t) = 0 \quad (11)$$

So this famous equation, which took so long to be found, can be derived in three lines, thanks to Dorfman's modified Hamiltonian.

We should not stop at this point. It is only natural to ask whether this method can be extended to obtain higher-order equations of the calculus of variations. In particular, we should wonder whether we could obtain the Euler-Poisson equation that we needed in this chapter to determine the function corresponding to the optimal polynomial spline, that is, minimizing both its curvature and the lath's deformation energy. The problem amounted to finding  $y(t)$  such that the integral  $\int_a^b [y''(t)]^2 dt$  was minimized.

So let us define  $\lambda_1(t)$  as the price of  $y_t$  and  $\lambda_2(t)$  as the price of  $y'_t$  and a second-order Hamiltonian as



$$\begin{aligned}
H^{(2)} &= u(x, y, y', y'') + \frac{d}{dt}[\lambda_1(t)y_t + \lambda_2(t)y'_t] \\
&= u(x, y, y', y'') + \lambda'_1(t)y_t + \lambda_1(t)y'_t + \lambda'_2(t)y'_t + \lambda_2(t)y''_t \quad (12)
\end{aligned}$$

Take the partial derivative of  $H^{(2)}$  with respect to  $y_t, y'_t$ , and  $y''_t$ , and set them equal to zero:

$$\frac{\partial H^{(2)}}{\partial y} = \frac{\partial u}{\partial y}(x, y, y', y'') + \lambda'_1(t) = 0 \quad (13)$$

$$\frac{\partial H^{(2)}}{\partial y'} = \frac{\partial u}{\partial y'}(x, y, y', y'') + \lambda_1(t) + \lambda'_2(t) = 0 \quad (14)$$

$$\frac{\partial H^{(2)}}{\partial y''} = \frac{\partial u}{\partial y''}(x, y, y', y'') + \lambda_2(t) = 0 \quad (15)$$

Now take once the derivative of (14) with respect to time and twice the derivative of (15). We get, respectively, with simplified notation,

$$\frac{d}{dt} \frac{\partial u}{\partial y'} + \lambda'_1(t) + \lambda''_2(t) = 0 \quad (16)$$

and

$$\frac{d^2}{dt^2} \frac{\partial u}{\partial y''} + \lambda''_2(t) = 0 \quad (17)$$

Substituting  $\lambda''_2(t)$  from (17) into (16), we get

$$\frac{d}{dt} \frac{\partial u}{\partial y'} + \lambda'_1(t) - \frac{d^2}{dt^2} \frac{\partial u}{\partial y''} = 0 \quad (18)$$

Finally, substituting  $\lambda'_1(t)$  from (18) into (13), we obtain

$$\frac{\partial u}{\partial y} - \frac{d}{dt} \frac{\partial u}{\partial y'} + \frac{d^2}{dt^2} \frac{\partial u}{\partial y''} = 0 \quad (19)$$

which is nothing else than the Euler-Poisson equation, where the derivatives of the function  $y = f(x)$  appear until the second order in the integrand of the functional. Since minimizing the spline curvature or the lath's deformation energy amounted to minimizing an integral containing only the second derivative  $y''$ , the Euler-Poisson equation (19) boiled down to  $\frac{d^2}{dt^2} \frac{\partial u}{\partial y''} = 0$ , which was equation (24) in our text.

This procedure can be generalized to any kind of variational problem. Functionals involving all derivatives of  $y = f(x)$  up to the  $n$ th order can be tackled in exactly this way. Also, this reasoning can be extended to problems involving multiple integrals, where the integrand depends on  $2n + 1$  arguments:

- (a)  $n$  variables  $(x_1, \dots, x_n)$ ,
- (b) a function of these  $n$  variables  $z = f(x_1, \dots, x_n)$ , and
- (c) the  $n$  partial derivatives  $\frac{\partial f}{\partial x_j}$ ,  $j = 1, \dots, n$ .

The functional would thus be

$$\int \dots \int_R F \left[ x_1, \dots, x_n; z; \frac{\partial z}{\partial x_1}, \dots, \frac{\partial z}{\partial x_n} \right] dx_1 \dots dx_n$$

One would be led to the Ostrogradski equation, from which the Laplace and Schrödinger equations, so central to physics, are derived. Finally, we could apply this method to deal with functionals involving functions of  $n$  variables and their partial derivatives up to the  $m$ th order. We would then obtain generalizations both of the Euler-Poisson and the Ostrogradski equations.<sup>33</sup>

### Main formulas

- Forward rate in the Nelson-Siegel model:

$$f(T) = \beta_0 + \beta_1 e^{-T/\tau_1} + \frac{\beta_2}{\tau_1} T e^{-T/\tau_1} \quad (14)$$

- Spot rate in the Nelson-Siegel model:

$$R(T) = \frac{\int_0^T f(u) du}{T} = \beta_0 + (\beta_1 + \beta_2) \frac{\tau_1}{T} (1 - e^{-T/\tau_1}) - \beta_2 e^{-T/\tau_1} \quad (15)$$

- Forward rate in the Svensson model:

$$R(T) = \beta_0 + \beta_1 e^{-T/\tau_1} + \beta_2 \frac{T}{\tau_1} e^{-T/\tau_1} + \beta_3 \frac{T}{\tau_1} e^{-T/\tau_2} \quad (17)$$

<sup>33</sup> We take the liberty of referring the interested reader to our paper, “New Hamiltonians for Higher-Order Equations of the Calculus of Variations: A Generalization of the Dorfman Approach,” *Archives des Sciences*, 45, no. 1 (May 1992): 51–58.

- Curvature or deformation of spline:

$$I = \int_{x_0}^{x_n} [f''(x)]^2 dx \quad (18)$$

- Euler-Poisson equation: to minimize

$$v[y(x)] = \int_a^b F[x, f(x), f'(x), f''(x), \dots, f^{(n)}(x)] dx \quad (19)$$

a first-order condition is that  $y(x)$  is a solution of

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \dots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0 \quad (20)$$

which leads to a piecewise third-order polynomial minimizing  $I$ .

- The Fisher-Nychka-Zervos model: choose optimal spot rate function  $\psi(t)$  such that

$$\min_{\psi(t)} \left[ \sum_{i=1}^K \{P_i - \hat{P}_i[\psi(t)]\}^2 + \lambda \int_0^{t_K} [\psi''(t)]^2 dt \right] \quad (31)$$

where  $P_i$  are coupon-bearing prices and  $\lambda$  is a roughness penalty value, determined by generalized cross-validation method:

$$\min_{\lambda} \gamma(\lambda) = \frac{RSS(\lambda)}{[n - \theta e_p(\lambda)]^2}$$

where

$n \equiv$  number of bonds

$RSS(\lambda) \equiv$  sum of residual sum of squares (first part of (31))

$e_p(\lambda) \equiv$  effective number of parameters

$\theta \equiv$  a “cost” or “tuning” parameter, set by Fisher, Nychka, Zervos, and Waggoner to two

- The Waggoner model:

$$\min_{\psi(t)} \left[ \sum_{i=1}^K \{P_i - \hat{P}_i[\psi(t)]\}^2 + \int_0^{t_K} \lambda(t) [\psi''(t)]^2 dt \right] \quad (32)$$

- The Anderson-Sleath model:

$$\min \sum_{i=1}^N \left[ \frac{P_i - \pi_i(\mathbf{c})}{D_i} \right]^2 + \int_0^M \lambda_t(m) [f''(m)]^2 dm$$

where

$D_i \equiv$  modified duration of bond  $i$

$\mathbf{c} \equiv$  parameter vector of polynomial spline to be estimated

$M \equiv$  maturity of the longest bond

and

$$\log \lambda(m) = L - (L - S)e^{-m/\mu}$$

## Questions

12.1. Make sure that, in the Nelson-Siegel model, you can deduce the spot rate structure (equation (15)) from the forward rate curve (equation (14)).

12.2. You can find the demonstration of the general Euler-Poisson equation in the texts on the calculus of variations, indicated in this chapter. Make sure, however, that you can deduce, from an economic reasoning point of view, the Euler-Poisson equation corresponding to

$$\min_{y(x)} \int_{x_0}^{x_1} F(x, y, y', y'') dx$$

12.3. Make sure that you understand why

$$\min_{y(x)} \int_{x_0}^{x_1} [f''(x)]^2 dx$$

leads to a third-order polynomial.

# 13 ESTIMATING THE LONG-TERM EXPECTED RATE OF RETURN, ITS VARIANCE, AND PROBABILITY DISTRIBUTION

The key issue in investments is estimating expected return.

*Fisher Black (1993)*

In Memoriam (1941–1995)

## Chapter outline

<b>13.1</b>	<b>Notation</b>	<b>268</b>
<b>13.2</b>	<b>Estimating the expected value of the one-year return, its variance, and its probability distribution</b>	<b>269</b>
13.2.1	Determining $E(\log X_{t-1,t})$ and $VAR(\log X_{t-1,t})$	269
13.2.2	What would be the consequences of errors in parameters' estimates?	270
13.2.3	Recovering the expected value and variance of the yearly rate of return, $E(R_{t-1,t})$ and $VAR(R_{t-1,t})$	271
13.2.4	The probability distribution of the one-year return	272
13.2.4.1	Determination of the lognormal density function and its relationship to the normal density	276
13.2.4.2	Fundamental properties of the yearly rate of return and their justification	277
<b>13.3</b>	<b>The expected <math>n</math>-year horizon rate of return, its variance, and its probability distribution</b>	<b>279</b>
<b>13.4</b>	<b>Direct formulas of the expected <math>n</math>-horizon return and its variance in terms of the yearly return's mean value and variance</b>	<b>281</b>
<b>13.5</b>	<b>The probability distribution of the <math>n</math>-year horizon rate of return</b>	<b>283</b>
<b>13.6</b>	<b>Some concrete examples</b>	<b>284</b>
13.6.1	Example 1	284
13.6.2	Example 2	285
<b>13.7</b>	<b>The central limit theorem at work for you again</b>	<b>285</b>
<b>13.8</b>	<b>Beyond lognormality</b>	<b>288</b>
<b>Appendix 13.A.1</b>	<b>Estimating the mean and variance of the rate of return from a sample</b>	<b>289</b>
<b>Appendix 13.A.2</b>	<b>Determining the expected long-term return when the continuously compounded yearly return is uniform</b>	<b>291</b>

## Overview

We all wish we could know more about the future rate of return on any kind of investment, be it an investment in stocks or one in bonds. Know-

ing more means having a precise idea about the expected value, its variance, and its probability distribution. The subject is important and comes with a wealth of surprises. Suppose, for instance, that you set out to protect yourself against inflation by buying a bond portfolio with a yearly expected return of 5%, and the expected inflation rate is also 5%; your expected *real* return is zero. Suppose also that the annual real returns are lognormally distributed, with a variance of 2.25%. It will probably surprise you that your 30-year expected real return is *minus* 1.07% and that the probability of having a *negative* 30-year real return is about 2/3!

In this chapter we will explain how we can determine the expected rate of return on bond portfolios over a long horizon, the rate's variance, and its probability distribution.<sup>1</sup>

### 13.1 Notation

We will use the following notation:

$S_t$  = an asset's value at end of year  $t$

$R_{t-1,t}$  = the yearly rate of return, compounded once per year;

$$R_{t-1,t} = (S_t - S_{t-1})/S_{t-1}$$

$X_{t-1,t} = S_t/S_{t-1} = 1 + R_{t-1,t}$  = the yearly growth *factor*, or one plus the yearly rate of return; in financial circles it is called the yearly “dollar” return

$\log X_{t-1,t} = \log(S_t/S_{t-1})$  = the yearly continuously compounded rate of return

$S_j$  = the asset's value at the end of a trading day

$R_{j-1,j}$  = the daily rate of return, compounded once a day;

$$R_{j-1,j} = (S_j - S_{j-1})/S_{j-1}$$

$X_{j-1,j} = S_j/S_{j-1} = 1 + R_{j-1,j}$  = the daily growth *factor*; equivalently, the daily “dollar” return, or one plus the daily return, compounded once a day

<sup>1</sup>Some of the material in this chapter has been drawn from our paper, “The Long-Term Expected Rate of Return: Setting It Right,” *Financial Analysts Journal* (November/December 1998): 75–80, with kind permission of the *Journal*. Copyright 1998, Association for Investment Management and Research, Charlottesville, Virginia. All rights reserved.

$\log X_{j-1,j} = \log(S_j/S_{j-1})$  = the daily continuously compounded rate of return

$R_{0,n}$  = the  $n$ -year horizon rate of return;  $n$ , not necessarily an integer, is expressed in years.  $R_{0,n}$  solves  $S_n = S_0(1 + R_{0,n})^n$ ;  $R_{0,n} = (S_n/S_0)^{1/n} - 1$ .

$X_{0,n} \equiv S_n/S_0$  = the  $n$ -year horizon dollar return over a period of  $n$  years

$X_{0,n}^{1/n} \equiv (S_n/S_0)^{1/n}$  = the  $n$ -year horizon dollar return per year;  
 $R_{0,n} = X_{0,n}^{1/n} - 1$

$Y(\lambda) = E(e^{\lambda Y})$  = the moment-generating function of  $Y$ , where  $Y$  is a continuous random variable

### 13.2 Estimating the expected value of the one-year return, its variance, and its probability distribution

Our first aim is to estimate the probability distribution of  $1 + R_{t-1,t} = X_{t-1,t}$  and its first two moments. A first step may be to estimate the probability distribution of  $\log X_{t-1,t}$  and  $E(\log X_{t-1,t})$ ,  $VAR(\log X_{t-1,t})$ . We will be able to do this by having recourse to the central limit theorem. We will then recover  $E(R_{t-1,t})$ ,  $VAR(R_{t-1,t})$  and its probability distribution. Of course, we all know how little value or credence can be attributed to an estimation of  $E(R_{t-1,t})$  based on past data. However, we should not forget that the procedure we describe is widely used for estimating variances and covariances; we therefore keep it for the record and for reasons of symmetry. Furthermore, the practitioner may well have his own method of estimating the expected value of the continuously compounded return and its variance; but there will still remain the question of recovering from those estimates the expected value and variance of the yearly rate of return.

#### 13.2.1 Determining $E(\log X_{t-1,t})$ and $VAR(\log X_{t-1,t})$

Suppose we know the daily trading prices  $S_j$  for a long period. We can therefore determine over such a long period  $\log(S_j/S_{j-1})$  the daily continuously compounded rates of return. For the time being, suppose we have no indication as to the nature of the probability distribution of the  $\log(S_j/S_{j-1})$  variables. Assume, however, that these variables are independent of one another and that they have finite variance. It is possible

to estimate both  $E[\log(S_j/S_{j-1})] \equiv m$  and  $VAR[\log(S_j/S_{j-1})] = s^2$  from our sample, in the usual way.<sup>2</sup> Consider now that there are  $v$  trading days in a year (for instance,  $v = 250$ ). We have

$$\log(S_t/S_{t-1}) = \sum_{j=1}^v \log(S_j/S_{j-1}) \quad (1)$$

The central limit theorem ensures that  $\log(S_t/S_{t-1})$  converges toward a normal variable when  $v$  becomes large (250 is very large in this instance), with mean  $vm \equiv$  and variance  $vs^2 \equiv \sigma^2$ . The important point to realize here is that this applies *whatever* the probability distribution of  $\log(S_j/S_{j-1})$  may be. For the central limit theorem to apply, we need only independence of the  $\log(S_j/S_{j-1})$  and a variance for each that is finite.

We thus have

$$\log(S_t/S_{t-1}) = \log(X_{t,t-1}) \quad N(vm, vs^2) \equiv N(\quad, \sigma^2)$$

### 13.2.2 What would be the consequences of errors in parameters' estimates?

As the reader may surmise, errors in the estimation of the mean will be of greater consequence than errors in estimating variances (or covariances in the case of portfolios). As Vijay Chopra and William Ziemba put it very well in their 1993 paper,<sup>3</sup> the following question arises: “How much worse off is the investor if the distribution of returns is estimated with an error? . . . Investors rely on limited data to estimate the *parameters* of the distribution, and estimation errors are unavoidable” (p. 55; our italics). The authors answer this crucial question by showing that the “percentage cash equivalent loss” for the investor (see their exact definition of this concept) due to errors in means is about 11 times that for errors in vari-

<sup>2</sup> In appendix 13.A.1 we recall why an unbiased estimate of a variance from a sample necessitates adjusting the measured variance of the sample.

<sup>3</sup> V. Chopra and W. Ziemba, “The Effect of Errors in Means, Variances and Covariances on Optimal Portfolio Choice,” *Journal of Portfolio Management* (Winter 1993): 6–11; reprinted in W. T. Ziemba and J. Mulvey, *Worldwide Asset and Liability Modelling* (Cambridge: Cambridge University Press, Publications of the Newton Institute, 1998).



ances and over 20 times that for errors in covariances. These results confirm those obtained earlier by J. Kallberg and W. Ziemba (1984), who found that consequences of errors in means were approximately 10 times more important than errors in variances and covariances, considered together as a set of parameters.<sup>4</sup> Also, the statistically minded reader will find interest in applications of mean estimation techniques in a series of papers by R. R. Grauer and N. H. Hakansson.<sup>5</sup>

### 13.2.3 Recovering the expected value and variance of the yearly rate of return, $E(R_{t-1,t})$ and $VAR(R_{t-1,t})$

We now want to recover the implied expected value and variance of the yearly rate of return  $R_{t-1,t} = (S_t - S_{t-1})/S_{t-1} = X_{t-1,t} - 1$ .

Start with  $E(X_{t-1,t})$ , equal to  $1 + E(R_{t-1,t})$ . We can write

$$X_{t-1,t} = e^{\log X_{t-1,t}} \quad \log X_{t-1,t} \sim N(\mu, \sigma^2)$$

Denoting by  $\phi_{\log X_{t-1,t}}(\lambda)$  the moment-generating function of  $\log X_{t-1,t}$  at point  $\lambda$ , we can write

$$E(X_{t-1,t}) = E(e^{\log X_{t-1,t}}) = \phi_{\log X_{t-1,t}}(1) \quad (2)$$

$E(X_{t-1,t})$  can thus be seen to be the moment-generating function of  $\log X_{t-1,t}$  at point 1.<sup>6</sup> Since we have

<sup>4</sup>J. G. Kallberg and W. T. Ziemba, "Mis-specification in Portfolio Selection Problems," in G. Bamberg and K. Spremann (eds.), *Risk and Capital: Lecture Notes in Econometrics and Mathematical Systems* (New York: Springer-Verlag, 1989).

<sup>5</sup>On these estimation techniques, see in particular, R. R. Grauer and N. H. Hakansson, "Higher Return, Lower Risk: Historical Returns on Long-Run, Actively Managed Portfolios of Stocks, Bonds and Bills, 1936–1978," *Financial Analysts Journal*, 38 (March–April 1982): 39–53; "Returns of Levered, Actively Managed Long-Run Portfolios of Stocks, Bonds and Bills, 1934–1984," *Financial Analysts Journal*, 41 (September–October 1985): 24–43; "Gains from International Diversification: 1968–85 Returns on Portfolios of Stocks and Bonds," *Journal of Finance*, 42 (July 1987): 721–739; and "Stein and CAPM Estimators of the Means in Asset Allocation," *International Review of Financial Analysts*, 4 (1995): 35–66.

<sup>6</sup>There would have been two alternate ways of determining  $E(X_{t-1,t})$ . The first would have been to take the integral  $\int_0^\infty xf(x)dx$ , where  $f(x)$  is the lognormal density; the second one would have been to calculate  $\int_{-\infty}^\infty \exp(u)g(u)du$ , where  $u = \log x$  and  $g(u)$  is the normal density. The method we offer here, however, does not involve any calculations once we know that the moment-generating function of a normal  $U = \log X_{t-1,t} \sim N(\mu, \sigma^2)$  is  $\phi_U(\lambda) = E(e^{\lambda U}) = \exp(\lambda\mu + \lambda^2\sigma^2/2)$ .

$$E(e^{\lambda \log X_{t-1,t}}) = {}_{\log X_{t-1,t}}(\lambda) = e^{\lambda + \frac{\lambda^2 \sigma^2}{2}} \quad (3)$$

we get immediately

$$E(X_{t-1,t}) = {}_{\log X_{t-1,t}}(1) = e^{+\frac{\sigma^2}{2}} \quad (4)$$

and

$$E(R_{t-1,t}) = e^{+\frac{\sigma^2}{2}} - 1 \quad (5)$$

The variance of  $R_{t-1,t}$  can be determined in an analogous way:

$$VAR(R_{t-1,t}) = VAR(X_{t-1,t}) = E(X_{t-1,t}^2) - [E(X_{t-1,t})]^2 \quad (6)$$

We have

$$E(X_{t-1,t}^2) = E(e^{2 \log X_{t-1,t}}) = {}_{\log X_{t-1,t}}(2) = e^{2 + 2\sigma^2} \quad (7)$$

Therefore,

$$VAR(R_{t-1,t}) = e^{2 + 2\sigma^2} - e^{2 + \sigma^2} = e^{2 + \sigma^2} [e^{\sigma^2} - 1] \quad (8)$$

and

$$\sigma(R_{t-1,t}) = e^{+\frac{\sigma^2}{2}} [e^{\sigma^2} - 1]^{1/2} \quad (9)$$

As an example, consider the case where estimates of  $\mu$  and  $\sigma^2$  are 0.07905 and 0.03252, respectively. Applying (5), (8), and (9), we get  $E(R_{t-1,t}) = 10\%$ ,  $VAR(R_{t-1,t}) = 4\%$ , and  $\sigma(R_{t-1,t}) = 20\%$ , respectively.

### 13.2.4 The probability distribution of the one-year return

Before rushing to the analysis of the  $n$ -year horizon rate of return, it will be useful to consider carefully the probability distribution of the one-year return. First, remember that, from the hypotheses we made on the daily rate of return, the central limit theorem led us to conclude that the logarithm of  $X_{t-1,t}$  was normally distributed  $N(\mu, \sigma^2) \equiv N(\mu, \sigma^2)$ .  $X_{t-1,t}$  thus

being lognormally distributed, suppose we wish to determine the probability that  $R_{t-1,t}$  is between two values  $a$  and  $b$ . Keeping in mind that  $\log X_{t-1,t}$  can be written as  $\mu + \sigma Z$ ,<sup>7</sup>  $Z \sim N(0, 1)$ , we can write

$$\begin{aligned} \text{Prob}(a < R_{t-1,t} < b) &= \text{Prob}(1 + a < X_{t-1,t} < 1 + b) \\ &= \text{Prob}[\log(1 + a) < \log X_{t-1,t} < \log(1 + b)] \\ &= \text{Prob}[\log(1 + a) < \mu + \sigma Z < \log(1 + b)] \\ &= \text{Prob}\left[\frac{\log(1 + a) - \mu}{\sigma} < Z < \frac{\log(1 + b) - \mu}{\sigma}\right] \end{aligned}$$

So, finally, we have

$$\text{Prob}(a < R_{t-1,t} < b) = \Phi\left(\frac{\log(1+b)-\mu}{\sigma}\right) - \Phi\left(\frac{\log(1+a)-\mu}{\sigma}\right) \quad (10)$$

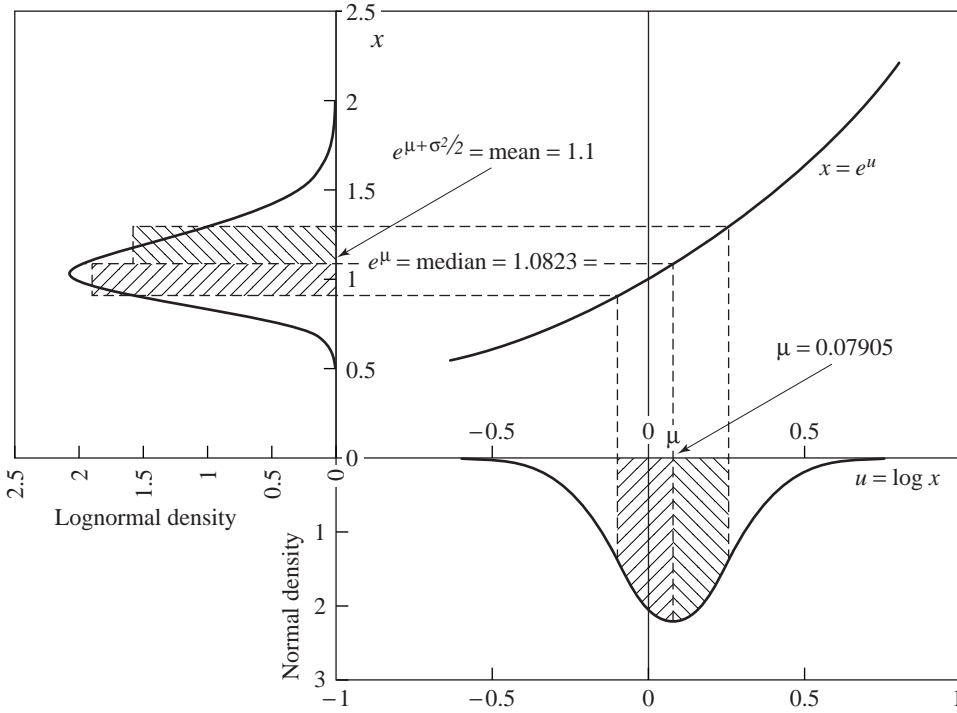
where  $\Phi(\cdot)$  denotes the cumulative probability distribution of the unit normal variable.

As a first example of an application, let us consider our former case where  $\mu = 0.07905$ ,  $\sigma^2 = 0.03252$  ( $\sigma = 0.18033$ ), implying  $E(R_{t-1,t}) = 10\%$ , and  $\text{VAR}(R_{t-1,t}) = 4\%$ , respectively. An apparently innocent question is the following: what is the probability that the yearly rate of return  $R_{t-1,t}$  is below its expected value, equal to 10%? This is akin to determining the probability that  $R_{t-1,t}$  is between  $a = -1$  (−100%, the minimum it can reach even if the asset becomes worthless) and  $b = 0.1$  (10%). Applying (10), where  $a = -1$  and  $b = 0.1$ , we get

$$\begin{aligned} \text{Prob}(-1 < R_{0,1} < 0.1) &= \text{Prob}[R_{0,1} < E(R_{0,1})] \\ &= \Phi\left[\frac{\log 1.1 - 0.07905}{0.03252}\right] - \Phi(-\infty) \\ &= \Phi(0.0902) = 53.6\% \end{aligned}$$

So there is a slightly higher chance that the yearly return will be less than its expected value (10%). This result, a direct consequence of the skewness

<sup>7</sup> For notational convenience, we omit the subscript  $t - 1, t$  and write  $Z_{t-1,t} \equiv Z$ .



**Figure 13.1**

A geometrical interpretation of three fundamental properties of the lognormal distribution: (a) the expected value of  $X_{t-1,t} = e^U = e^{\mu + \sigma Z}$  ( $Z \sim N(0, 1)$ ),  $E(X) = e^{\mu + \sigma^2/2}$  is above the median  $e^\mu$ ; (b) the probability that  $X_{t-1,t}$  is below its expected value is above 50%; and (c) the expected value of  $X_{t-1,t}$  is an increasing function of the variance of  $U$ ,  $\sigma^2$ . In this example,  $E(\log X) = E(U) = 0.07905$ ;  $\sigma(\log X) = \sigma_U = 0.18033$ .

of the lognormal distribution, is important; it will be of consequence when we consider longer horizons, and therefore deserves explanation.

The fact that it is more likely that the yearly return will be lower than its expected value (10%) has to be clearly understood. In our opinion, a good starting point is to have a clear picture of the links between the normal distribution and the lognormal distribution. (Remember that the normal distribution governs  $U \equiv \log X_{t-1,t} = \log(S_t/S_{t-1})$  and that the lognormal distribution is that of  $e^{\log X_{t-1,t}} = X_{t-1,t}$ .) In figure 13.1 we depict on the horizontal axis the probability distribution of  $U = \log X_{t-1,t}$  which is normal:  $U \sim N(\mu, \sigma^2)$ . In our case,  $\mu = 0.07905$  and  $\sigma^2 =$

0.03252, or  $\sigma = 0.18033$ . Now the exponential of  $U$ ,  $e^U$  or  $e^{\log X_{t-1,t}} = X_{t-1,t}$  is (by definition) lognormally distributed, and our purpose is to discover the fundamental properties of the probability density of the lognormal distribution of our variable of interest,  $X_{t-1,t}$ , the *yearly* dollar return on our investment.

Let us also draw in figure 13.1 the exponential  $x = e^u$ , the possible outcome values of the random variable  $X_{t-1,t} = e^U = e^{\log X_{t-1,t}}$ . Now consider the average of the normal  $U \sim N(\mu, \sigma^2)$ ,  $\mu$ , which also is its median. The exponential of  $\mu$ ,  $e^\mu$ , will therefore be the *median* of the lognormal, and we can already mark this value  $e^\mu = e^{0.07905} = 1.08226$  on the vertical axis  $x$ . On the other hand, we have determined through (4) that the *average* of the lognormal was  $e^{\mu + \sigma^2/2}$ , and therefore that the average of the lognormal was higher than its median. For any continuous random variable, its cumulative probability density to the left of its median is 50%. Now if the expected value of  $X_{t-1,t}$  is above the median, it means that the probability for  $X_{t-1,t}$  to be lower than its expected value is 50% *plus* the cumulative density between the median and the average. This is precisely what is happening here: there is a 53.6% probability that  $R(0, 1)$  is lower than 10% (or that  $X_{t-1,t} = 1 + R(0, 1)$  is lower than 1.1); 3.6% is exactly the cumulative density of the lognormal distribution between the median of  $X_{t-1,t}$  ( $e^\mu = 1.0822$ ) and the mean of  $X_{t-1,t}$  ( $e^{\mu + \sigma^2/2} = 1.1$ ). Let us verify this by applying what we have just seen; suppressing the lower indexes  $t - 1$ ,  $t$  for convenience, we have

Prob(median of  $X < X < \text{mean of } X$ )

$$\begin{aligned} &= \text{Prob}(e^\mu < X < e^{\mu + \sigma^2/2}) = \text{Prob}(\mu < \log X < \mu + \sigma^2/2) \\ &= \text{Prob}(\mu < \mu + \sigma Z < \mu + \sigma^2/2) = \text{Prob}(0 < \sigma Z < \sigma^2/2) \\ &= \text{Prob}(0 < Z < \sigma/2) \end{aligned}$$

In our case,  $\sigma = 0.18033$ ; therefore,

Prob(median of  $X < X < \text{mean of } X$ )

$$\begin{aligned} &= \Phi(0.09017) - \Phi(0) = 0.536 - 0.5 = 0.036 \\ &= 3.6\% \text{ as we wanted to ascertain.} \end{aligned}$$

Our purpose is now to explain and generalize these properties.

### 13.2.4.1 Determination of the lognormal density function and its relationship to the normal density

It is easy to determine the lognormal density function once we know the density of the normal function. Suppose that  $U = \log X_{t-1,t}$  is normally distributed  $N(\mu, \sigma^2)$ . In figure 13.1 we have represented this density in the example chosen above, that is, with  $\mu = 0.07905$  and  $\sigma = 0.18033$  or  $\sigma^2 = 0.03252$ . Let  $f(u)$  denote the normal density with average  $\mu$  and standard deviation  $\sigma$ . We have

$$f(u) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}$$

Now the lognormal distribution is the distribution of the exponential of  $U \sim N(\mu, \sigma^2)$ , that is,<sup>8</sup>

$$X = e^U$$

From a realization of events point of view, each realized value  $u$  is related to  $x$  by  $x = e^u$ , or  $x(u) = e^u$ .

To determine the density function of  $X$ , which we choose to denote as  $g(x)$ , we simply have to make sure that to each infinitesimal probability  $f(u) du$  corresponds an equal infinitesimal probability  $g(x) dx$ . So we must have

$$g(x) dx = f(u) du$$

which implies in turn<sup>9</sup>

$$g(x) = f(u) \left| \frac{dx}{du}(u) \right|$$

where  $u$  is replaced, in the right-hand side, by  $u = \log x$  and  $dx/du = d(e^u)/du = e^u = x$ . Performing those substitutions, we have

$$g(x) = f(\log x)/e^{\log x} = f(\log x)/x$$

<sup>8</sup> Notice how very unfortunate the standard terminology turns out to be: if  $U$  is normal, it would have been logical to call  $e^U$  the *exponormal* variable and not the *lognormal* variable (i.e., the variable whose *logarithm* is normal, leaving the reader with the exercise of finding that the lognormal is  $e^U$ ).

<sup>9</sup> In order to ensure that the probability densities  $g(x)$  are always positive, we have to take the absolute value  $|\frac{dx}{du}|$ . In our case, however, since  $x = e^u$ ,  $dx/du = x > 0$  so we do not need here the absolute value of the derivative.

and therefore we get the lognormal density function

$$g(x) = \frac{1}{x\sigma \sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\log x - \mu}{\sigma}\right]^2}$$

Both densities (the normal and the lognormal) are represented in figure 13.1, the normal on the horizontal axis  $u$  and the lognormal on the vertical  $x$ . The relationship between  $x$  and  $u$ ,  $x = e^u$  has been drawn as well. This diagram will be useful in understanding the important properties of the lognormal.

#### 13.2.4.2 Fundamental properties of the yearly rate of return and their justification

Our purpose is now threefold. We want to show why

1. the expected value of the yearly dollar return,  $X_{t-1,t} = e^U = e^{\log X_{t-1,t}}$  is higher than  $e^{E(\log X_{t-1,t})} = e$  ;
2. the probability of having a yearly dollar return *smaller* than its expected value is *higher* than 50%; and
3. the expected value of the yearly dollar return is an increasing function of the variance of the instantaneously compounded rate of return.

• **The expected value of the yearly dollar return is higher than its median  $e$ .** We had observed earlier that (equation (4))

$$E(X_{t-1,t}) = e^{+\sigma^2/2}$$

This is clearly higher than  $e$  , and it is a direct consequence of Jensen's inequality, according to which, if  $h(U)$  is a convex transformation of  $U$  (this is the case with  $X = h(U) = e^U$ ), then we always have

$$E(X) = E[h(U)] > h[E(U)]$$

the proof of which can be found in most statistics books. But there is an easy explanation of this inequality in the case where  $U$  has a symmetrical distribution (our case, since  $U = \log X_{t-1,t}$  is normal). Consider two intervals of equal, arbitrary length  $I = \sigma$  around on the horizontal  $u$  axis (figure 13.1). To each interval  $I$  corresponds a given equal mass probability under the normal unit curve; on the vertical  $x$  axis we can read

the value  $x = e$ , as well as the intervals corresponding to the intervals  $I$  defined previously on the abscissa. We notice immediately that the intervals on the  $x$  axis are *not* equal, due to the convexity of the exponential function  $x = e^u$ . So the probability masses corresponding to the  $I$  intervals, if represented by rectangles of equal areas (as they are on the left-hand side of the diagram), will be of different heights, the height of the upper rectangle being smaller than the height of the lower rectangle.

Where will the center of gravity of the mass of the two rectangles be located on the  $x$  axis? Obviously to the right of the point corresponding to the median,  $e$ . Repeat this process for a new interval of the same length, to the right and to the left of  $+I$  and  $-I$ . You will add, on the left-hand side of the picture, two additional rectangles of equal area, the upper one with a basis larger than the basis of the lower one. Therefore, your center of gravity will be displaced to the right. Notice that each time you repeat this process, you are adding intervals with mass probabilities that decrease and tend toward zero. Henceforth, you may well surmise that each time your center of gravity moves up on the  $x$  axis, it does so by diminishing increments, tending toward a limit that turns out to be equal, thanks to our previous calculations, to  $e^{+\sigma^2/2}$ .

• **The probability that the yearly dollar return is lower than its expected value is higher than 50%.** This is an immediate corollary of our last property. Since the average of the lognormal  $X = e^U$ ,  $e^{+\sigma^2/2}$  is larger than its median  $e$ , it is clear that the probability of having an  $X$  value smaller than  $E(X)$  is higher than 50%. In fact, the probability of  $X$  being lower than  $E(X)$  is

$$\begin{aligned} \text{Prob}(X < E(X) = e^{+\sigma^2/2}) &= \text{Prob}(\log X < +\sigma^2/2) \\ &= \text{Prob}(+\sigma Z < +\sigma^2/2) \\ &= \text{Prob}(Z < \sigma/2) \end{aligned}$$

(A verification of our former calculations can be given here: with  $\sigma^2 = 0.18033$ ,  $\text{prob}(Z < \sigma/2 = 0.09017) = 53.6\%$ , a result matching the one we reached before.)

Note the simplicity of the result: *the probability for the yearly return to be lower than its expected value is simply the probability of the normal unit variable to be lower than half of the standard deviation of the continuously compounded return.*



This result has a nice geometrical interpretation and can be obtained without recourse to any calculation. Refer again to figure 13.1, and watch out for the values of the median and the mean of the lognormal, equal to  $e^{-\sigma^2/2}$  and  $e^{+\sigma^2/2}$ , respectively. The probability that  $X$  will be *below*  $E(X)$  is the probability that  $\log X$  will be below  $E(\log X) = +\sigma^2/2$ . This probability can be recovered by going back to the normal curve through the log transformation on the diagram (the reciprocal of the exponential). This is just the area spanned by the normal curve below  $+\sigma^2/2$ , or the area below  $Z = (+\sigma^2/2 - 0)/\sigma = \sigma/2$ .

We will now explain why this probability is an increasing function of the standard deviation of the continuously compounded rate of return.

• **The expected value of the yearly dollar return is an increasing function of the variance of the continuously compounded return.** From equation (4), we notice a positive dependence between the variance  $\sigma^2$  of the continuously compounded rate of return and the expected yearly return—and a powerful dependence at that. Our purpose is now to explain why this is so.

Let us come back once more to figure 13.1. We can see immediately that if  $U = \log X_{t-1,t}$  has a larger variance than before, and if we consider the same mass probability as the one that corresponded to the  $I$  intervals, our  $I$  intervals this time will have to be broader, entailing a larger discrepancy in the bases of the two rectangles in the left part of the diagram. Hence the center of gravity will be farther away from  $e^{-\sigma^2/2}$  than it was when the variance of  $U$  was lower. The same argument carries to additional intervals; hence the center of gravity (the expected value of  $X_{t-1,t}$ ) is definitely increasing with the variance of  $U$ .

### 13.3 The expected $n$ -year horizon rate of return, its variance, and its probability distribution

Suppose that  $n$  is the length of our horizon, expressed in years;  $n$  is *not necessarily* an integer number. For instance, if one year has 250 trading days, as in our example before,  $n = 2.2$  years means two years plus  $(0.2)(250) = 50$  trading days. Our  $n$ -year horizon yearly rate of return, denoted  $R_{0,n}$ , is such that

$$S_0(1 + R_{0,n})^n = S_n \quad (11)$$

and is equal to

$$R_{0,n} = \left( \frac{S_n}{S_0} \right)^{1/n} - 1 \equiv X_{0,n}^{1/n} - 1 \quad (12)$$

where  $X_{0,n} = S_n/S_0$ .

We have

$$\begin{aligned} \log X_{0,n} &= \log S_n - \log S_0 = \sum_{j=1}^{n.250} \log(S_j/S_{j-1}) \quad N(n.250m \quad n.250s^2) \\ &\equiv N(n \quad n\sigma^2) = n \quad + \quad n\sigma Z, \quad Z \quad N(0, 1) \end{aligned} \quad (13)$$

Indeed, from our hypotheses the random variable  $\log X_{0,n}$  turns out to be normally distributed with mean  $n.250m \equiv n$  and variance  $n.250s^2 \equiv n\sigma^2$ .

We can then write

$$E(X_{0,n}^{1/n}) = E[(e^{\log X_{0,n}})^{1/n}] = E(e^{\frac{1}{n} \log X_{0,n}}) \quad (14)$$

The term on the far right of (14) is simply the moment-generating function of  $\log X_{0,n}$  at point  $1/n$ . We therefore have

$$E(X_{0,n}^{1/n}) = \log X_{0,n} \left( \frac{1}{n} \right) = e^{\frac{1}{n}n + \frac{1}{2} \frac{1}{n^2} n\sigma^2} = e^{\frac{n}{n} + \frac{\sigma^2}{2n}} \quad (15)$$

and

$$E(R_{0,n}) = e^{\frac{n}{n} + \frac{\sigma^2}{2n}} - 1 \quad (16)$$

Consider, as before, our estimates of  $\mu = 7.905\%$  per year and  $\sigma^2 = 3.252\%/year^2$ , that is,  $\sigma = 18.003\%$  per year. We have seen that they entailed a yearly expected return  $E(R_{0,1})$  equal to  $10\%$ , and a standard deviation of  $R_{0,1}$  equal to  $20\%$ . Should our horizon be  $2.2$  years, we would have

$$E(R_{0,2.2}) = e^{0.07905 + \frac{0.03252}{4.4}} - 1 = 9.03\% \text{ per year}$$

Let us generalize this method to the determination of the  $k$ th moment of  $X_{0,n}$ ; we will use this result to determine the variance of  $R_{0,n}$  in a straightforward way.

The  $k$ th moment of  $X_{0,n}^{1/n}$  can be written as

$$E(X_{0,n}^{k/n}) = E(e^{\frac{k}{n} \log X_{0,n}}) = \log X_{0,n}(k/n) \quad (17)$$

and is thus equal to the moment-generating function of  $\log X_{0,n}$  at point  $k/n$ ; it is therefore equal to

$$E(X_{0,n}^{k/n}) = e^{\frac{k}{n} \mu + \frac{1}{2} \frac{k^2}{n^2} n \sigma^2} = e^k + \frac{k^2}{2n} \sigma^2 \quad (18)$$

As an application, let us determine the variance of the  $n$ -year horizon expected rate of return.

$$VAR(R_{0,n}) = VAR(X_{0,n}^{1/n} - 1) = VAR(X_{0,n}^{1/n}) = E(X_{0,n}^{2/n}) - [E(X_{0,n}^{1/n})]^2 \quad (19)$$

Using (18), we have

$$E(X_{0,n}^{2/n}) = e^2 + \frac{2\sigma^2}{n} \quad (20)$$

As a consequence

$$VAR(R_{0,n}) = e^2 + \frac{2\sigma^2}{n} - e^2 = \frac{\sigma^2}{n} \quad (21)$$

and therefore

$$VAR(R_{0,n}) = e^2 + \frac{\sigma^2}{n} (e^{\frac{\sigma^2}{n}} - 1)$$

The standard deviation of  $R_{0,n}$  is

$$\sigma(R_{0,n}) = e^{\frac{\sigma^2}{2n}} (e^{\frac{\sigma^2}{n}} - 1)^{1/2} \quad (22)$$

Pursuing our previous example, we have

$$\sigma(R_{0,2.2}) = e^{0.07905 + \frac{0.03252}{4.4}} (e^{0.07905} - 1) = 8.96\%$$

### 13.4 Direct formulas of the expected $n$ -horizon return and its variance in terms of the yearly return's mean value and variance

It would, of course, be very convenient to express both the expected value and the variance of the  $n$ -horizon return *directly* as functions of the one-year expected return and variance. Let  $E(X_{0,1}) = E(R_{0,1} + 1) \equiv E$  and  $VAR(X_{0,1}) = VAR(R_{0,1} + 1) = VAR(R_{0,1}) \equiv V$ . We would like to express formulas (16) and (21) directly in terms of  $E$  and  $V$ , which pertain to the

yearly return. Recall that we have

$$E \equiv E(X_{0,1}) = e^{+\frac{\sigma^2}{2}} \quad (4)$$

and

$$V \equiv V(X_{0,1}) = e^{2+\sigma^2}(e^{\sigma^2} - 1) = E^2 \cdot (e^{\sigma^2} - 1) \quad (8)$$

From system (4) and (8) we can easily determine  $\sigma^2$ :

$$= \log E - \frac{1}{2} \log \left( 1 + \frac{V}{E^2} \right) \quad (23)$$

$$\sigma^2 = \log \left( 1 + \frac{V}{E^2} \right) \quad (24)$$

Plugging (23) and (24) into (16), (21), and (22), we get

$$E(R_{0,n}) = E \cdot (1 + V/E^2)^{\frac{1}{2}(\frac{1}{n}-1)} - 1 \quad (25)$$

$$VAR(R_{0,n}) = E^2 \cdot (1 + V/E^2)^{(\frac{1}{n}-1)} [(1 + V/E^2)^{\frac{1}{n}} - 1] \quad (26)$$

which also leads to

$$\sigma(R_{0,n}) = E \cdot (1 + V/E^2)^{\frac{1}{2}(\frac{1}{n}-1)} [(1 + V/E^2)^{\frac{1}{n}} - 1]^{1/2} \quad (27)$$

Using (25), (27) can be expressed simply as

$$\sigma(R_{0,n}) = [1 + E(R_{0,n})][(1 + V/E^2)^{\frac{1}{n}} - 1]^{1/2} \quad (28)$$

As a check, note that with  $n = 1$ ,  $E(R_{0,1}) = E - 1$ ,  $VAR(R_{0,1}) = VAR(X_{0,1}) = V$ , and that with  $n \rightarrow \infty$ ,  $VAR(R_{0,\infty}) = 0$ , as they should.

In the limit, the infinite horizon expected return is

$$E(R_{0,\infty}) = e^{-1} = E \cdot \left(1 + \frac{V}{E^2}\right)^{-1/2} - 1 = \frac{E^2}{(E^2 + V)^{1/2}} - 1 \quad (29)$$

For instance, with  $E = 1.1$  and  $V = 0.04$ , the infinite horizon expected return is  $E(R_{0,\infty}) = 8.23\%$ . It is well known that the sample geometric mean of a series of independent random variables converges asymptotically, by decreasing values, toward the geometric mean of the random variable (see Hines<sup>1</sup>). Here  $E(X_{0,\infty}^{1/n}) = E(R_{0,\infty}) + 1 = E^2(E^2 + V)^{-1/2}$  is precisely the geometric mean of the probability distribution of the yearly dollar return  $X_{t-1,t}$ . So the infinite horizon expected rate of return (29) is simply the geometric mean of the lognormal minus one.

### 13.5 The probability distribution of the $n$ -year horizon rate of return

Coming back to one of our basic formulas, (13), we can see that *one plus* the  $n$ -horizon rate of return is a lognormal distribution with parameters  $\log(1 + R_{0,n})$  and  $\sigma^2/n$ ; that is, the logarithm of the  $n$ -year horizon *dollar* return,  $\log(1 + R_{0,n})$ , is normally distributed  $N(\log(1 + R_{0,n}), \sigma^2/n)$ . This implies that  $\frac{\log(1 + R_{0,n}) - \log(1 + a)}{\sigma/\sqrt{n}}$  is  $N(0, 1)$ ; therefore, the probability that  $R_{0,n}$  is between any two values  $a$  and  $b$  is  $\Phi\left(\frac{\log(1+b) - \log(1+a)}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{\log(1+a) - \log(1+a)}{\sigma/\sqrt{n}}\right)$ , where  $\Phi$  is the cumulative probability distribution of the unit normal variable.

More formally, keeping in mind that  $X_{0,n}^{1/n} = 1 + R_{0,n}$  and that, from (13),  $\log X_{0,n}^{1/n} = \log(1 + R_{0,n})$  can be written as  $\log(1 + R_{0,n}) = \log(1 + a) + \frac{\sigma}{n}Z$ ,  $Z \sim N(0, 1)$ , we have

$$\begin{aligned} \text{Prob}(a < R_{0,n} < b) &= \text{Prob}(1 + a < 1 + R_{0,n} < 1 + b) \\ &= \text{Prob}[\log(1 + a) < \log(1 + R_{0,n}) < \log(1 + b)] \\ &= \text{Prob}\left[\log(1 + a) < \log(1 + a) + \frac{\sigma}{n}Z < \log(1 + b)\right] \\ &= \text{Prob}\left[\frac{\log(1 + a) - \log(1 + a)}{\sigma/\sqrt{n}} < Z < \frac{\log(1 + b) - \log(1 + a)}{\sigma/\sqrt{n}}\right] \end{aligned}$$

<sup>1</sup> W. G. S. Hines, "Geometric Mean," in S. Kotz and N. L. Johnson (eds.), *Encyclopedia of Statistical Science*, vol. 3 (New York: Wiley, 1983), pp. 397–450.

Therefore, we have

$$\text{Prob}(a < R_{0,n} < b) = \Phi \left[ \frac{\log(1+b) - \log(1+a)}{\sigma/\sqrt{n}} \right] - \Phi \left[ \frac{\log(1+a) - \log(1+a)}{\sigma/\sqrt{n}} \right] \quad (30)$$

Conversely, we can easily determine an interval for  $R_{0,n}$  spanning a given probability interval, for instance 95%. We can write

$$\text{Prob}(-1.96\sigma/\sqrt{n} < \log(1+R_{0,n}) < +1.96\sigma/\sqrt{n}) = 0.95 \quad (31)$$

or, equivalently,

$$\text{Prob}(e^{-1.96\sigma/\sqrt{n}} - 1 < R_{0,n} < e^{+1.96\sigma/\sqrt{n}} - 1) = 0.95 \quad (32)$$

Thus, a 95% probability interval for the  $n$ -horizon rate of return is given by  $(e^{-1.96\sigma/\sqrt{n}} - 1, e^{+1.96\sigma/\sqrt{n}} - 1)$ ; in turn, and if needed, this interval can be expressed directly in terms of  $E$  and  $V$ , using (23) and (24). We will now give examples of applications.

## 13.6 Some concrete examples

### 13.6.1 Example 1

As a first example, let us verify the exactness of the figures we have given in our introduction. By hypothesis, the one-year expected real return on the portfolio is 0%, and its variance is 2.25%. So we can plug  $E = 1 + 0\% = 1$ ,  $V = 0.0225$ , and  $n = 30$  into formula (25), to yield  $E(R_{0,30}) = -1.07\%$ . The probability that  $R_{0,n}$  is negative is given by

$$\text{Prob}(R_{0,n} < 0) = \text{Prob}(1 + R_{0,n} < 1)$$

Since  $\log(1 + R_{0,n})$  is normal  $N(\log E, \sigma^2/n)$ , we first have to determine  $\sigma^2$  from (23) and (24). We get

$$= \log E - \frac{1}{2} \log \left( 1 + \frac{V}{E^2} \right) = -0.011125$$

and

$$\sigma^2 = \log \left( 1 + \frac{V}{E^2} \right) = 0.0222506 \quad \sigma = 0.149166$$

Now we have

$$\begin{aligned}
 \text{Prob}(1 + R_{0,n} < 1) &= \text{Prob}[\log(1 + R_{0,n}) < 0] \\
 &= \text{Prob}\left[\frac{\log(1 + R_{0,n}) - \bar{\mu}}{\sigma/\sqrt{n}} < \frac{-\bar{\mu}}{\sigma/\sqrt{n}}\right] \\
 &= \text{Prob}\left(Z < \frac{-\bar{\mu}}{\sigma/\sqrt{n}} = 0.4085\right), \quad Z \sim N(0, 1) \\
 &= \Phi(0.4085) = 0.658
 \end{aligned}$$

as stated in the introduction to this chapter.

### 13.6.2 Example 2

Let us now consider the case  $E(R_{0,1}) = 10\%$ ;  $\sigma(R_{0,1}) = 20\%$ . We then have  $E \equiv E(X_{t-1,t}) = 1.1$ . Using (25),  $E(R_{0,10})$  turns out to be 8.4%; similarly the standard deviation of  $R_{0,10}$  can be calculated, using (28), as 6.2%. Suppose now that we want to know the probability for the 10-year horizon to be negative. Referring to our results in section 13.4, we first have to translate  $E(= 1.1)$  and  $V(= 0.04)$  in terms of  $\bar{\mu}$  and  $\sigma$ . Using (23) and (24) respectively, we get  $\bar{\mu} = 7.90\%$ ,  $\sigma^2 = 3.25\%$ ,  $\sigma = 18.03\%$ . The probability for  $R_{0,10}$  to be lower than  $b = 0$  is equal to  $\Phi\left(\frac{\log(1+b)-\bar{\mu}}{\sigma/\sqrt{n}}\right)\Big|_{b=0} = \Phi\left(\frac{-7.90\%}{18.03\%/\sqrt{10}}\right) = \Phi(-1.38) \approx 8\%$ . Similarly, the 95% prediction interval for  $R_{0,10}$  is  $(e^{-1.96\sigma/\sqrt{n}} - 1, e^{+1.96\sigma/\sqrt{n}} - 1) = (-3.2\%, +21.0\%)$ . On the other hand,  $R_{0,10}$  has a 68% probability of being between 2.2% and 14.6%.

### 13.7 The central limit theorem at work for you again

Suppose you are not quite sure about the exact nature of the probability distribution of the yearly continuously compounded rate of return,  $\log X_{t-1,t}$ . In particular, you are not certain it is normal, which implies that you are not sure about the lognormality of  $X_{t-1,t}$ . Nevertheless, you go ahead and perform the calculations of the long-term expected return  $E(R_{0,n})$  as indicated before, assuming that the  $X_{t-1,t}$ 's are lognormal. Suppose you want to know the type of error you have been making by relying on a distribution (the lognormal) that is not necessarily the true one. In order to have an idea of the error made, let us conduct the

**Table 13.1**

**Comparison of  $E(R_{0,n})$  when  $\log X_{t-1,t}$  is either normal or uniform, with  $E(\log X_{t-1,t}) = 10\%$  and  $\sigma(\log X_{t-1,t}) = 20\%$**

Horizon $n$ (years)	$E(R_{0,n})$ with $\log X_{t-1,t}$ normal	$E(R_{0,n})$ with $\log X_{t-1,t}$ uniform
2	0.116278	0.116267
4	0.110711	0.110709
6	0.108861	0.108861
8	0.107937	0.107937
10	0.107383	0.107383
20	0.106277	0.106277
30	0.105908	0.105908
100	0.105392	0.105392

following experiment. Assume that the true probability distribution of  $\log X_{t-1,t}$  was uniform. As shown in appendix 13.A.2, this implies that the underlying probability density of  $X_{t-1,t}$  is hyperbolic, indeed quite a departure from the lognormal, and that the expected value of  $1 + R_{0,n}$  (denoted  $Y_{0,n}$ ) is

$$\begin{aligned}
 E(Y_{0,n})|_{\log X_{t-1,t} \sim \text{uniform}[a,b]} &= \left[ \frac{n}{2-3\sigma} (e^{(+3\sigma)/n} - e^{(-3\sigma)/n}) \right]^n \\
 &= \left[ \frac{n}{b-a} (e^{b/n} - e^{a/n}) \right]^n
 \end{aligned} \tag{33}$$

where  $a = -3\sigma$  and  $b = +3\sigma$ .

This formula looks exceedingly different from the result corresponding to the normal distribution for  $\log X_{t-1,t}$ :

$$E(Y_{0,n})|_{\log X_{t-1,t} \sim N(\mu, \sigma^2)} = e^{\mu + \frac{\sigma^2}{2n}} = e^{\frac{b+a}{2} + \frac{(b-a)^2}{24n}} \tag{34}$$

which applies in the case of the normal distribution. Now try it in the following example: let  $E(\log X_{t-1,t}) = 10\%$  and  $\sigma(\log X_{t-1,t}) = \sigma = 20\%$ , and apply (14) and (15). The surprise comes when it turns out that the first *four* digits of the long-term expected return  $E(R_{0,n}) = E(Y_{0,n}) - 1$  are the *same* in each case when the horizon is as low as two years! (See table 13.1.) Six digits are caught with five years. The explanation of this remarkable result lies of course in the strength of the central limit theorem. Indeed,  $Y_{0,n}$ , the geometric average of the dollar yearly returns, can



be written as

$$Y_{0,n} = \left[ \prod_{t=1}^n X_{t-1,t}^{1/n} \right] = e^{\frac{1}{n} \sum_{t=1}^n \log X_{t-1,t}} \quad (35)$$

and the central limit theorem can be applied to the power of  $e$ . We thus have

$$Y_{0,n} \approx e^{E(\log X_{t-1,t}) + \frac{\sigma(\log X_{t-1,t})}{n} Z}, \quad Z \sim N(0, 1) \quad (36)$$

and the expected value of  $Y_{0,n}$  is<sup>11</sup>

$$E(Y_{0,n}) \approx e^{E(\log X_{t-1,t}) + \frac{\sigma^2(\log X_{t-1,t})}{2n}} \quad (37)$$

It turns out that in most cases (and ours is such a case) the average in the power of  $e$  converges extremely quickly. (For a very good illustration of the power of the central limit theorem, see the spectacular diagrams in J. Pitman (1993).<sup>12</sup>) Hence, the result we have here: very early the distribution of  $Y_{0,n}$  becomes lognormal, even if the continuously compounded return  $\log X_{t-1,t}$  is far from normal (a uniform distribution in our case). We should add that the stability of the mean reflects the small annual variance, as well as the central limit theorem.

In section 13.5 we promised to determine the limit of the long-term dollar return when the horizon tends to infinity; this is now easy to do, thanks again to the central limit theorem. Indeed, turning to (36), it is immediately clear that when  $n \rightarrow \infty$ , the limit is  $e^{E(\log X_{t-1,t})}$ , which is none other than the geometric mean of the distribution of  $X_{t-1,t}$ . (On this, see Hines (1983).) In our example, the distribution of  $X_{t-1,t}$  is hyperbolic if  $\log X_{t-1,t}$  is uniform, lognormal if  $\log X_{t-1,t}$  is normal. It turns out that this property can be extended to *all* moments of  $Y_{0,n}$ : each  $k$ th moment of  $Y_{0,n}$  tends toward the  $k$ th geometric moment of the  $X_{t-1,t}$  distribution. (For a definition and properties of geometric moments of order  $k$ , see O. de La Grandville (2000).<sup>13</sup>)

We can conclude that what matters most when forecasting expected long-term returns are the first two moments of the yearly returns, *not* their

<sup>11</sup> To show this, use again the fact that  $E(Y_{0,n})$  is a linear transformation of the moment-generating function of  $Z$ , where  $\frac{\sigma(\log X_{t-1,t})}{n}$  plays the role of the parameter.

<sup>12</sup> J. Pitman, *Probability* (New York: Springer-Verlag, 1993), pp. 199–202.

<sup>13</sup> O. de La Grandville, “Introducing the Geometric Moments of Continuous Random Variables, with Applications to Long-Term Growth Rates,” Research Report, University of Geneva, Geneva, Switzerland, 2000.

probability distribution. Whenever you assume yearly returns to be independent and to have finite variance, do not worry about their probability densities, and let the central limit theorem do the work for you.

### 13.8 Beyond lognormality

The lognormality hypothesis is indeed very convenient mathematically. We must stress, however, that the lognormality assumption for asset prices rests on the central limit theorem, which itself requires two sufficient (but not necessary) conditions<sup>14</sup>: first, independence of the continuously compounded returns; second, identical distributions for those returns. On the other hand, all market participants are well aware that the market systematically exaggerates up and down movements when it experiences huge swings; therefore, this independence condition is not always met.

Researchers have been conscious of this fact and have tried to model the market's behavior by using so-called "fat-tailed distributions" (for instance, Pareto-Lévy distributions). Those distributions have in common the fact that they do not possess finite moments (the corresponding integrals do not converge), and they are known by their characteristic functions only. Therefore, it becomes much more difficult to work with them than with the convenient lognormal distribution. The first person to consider these possibilities was the mathematician Benoît Mandelbrot, in his paper "The Variation of Certain Speculative Prices".<sup>15</sup> His further work, as well as other important contributions in this area, are well referenced in his recent book *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk*.<sup>16</sup>

No doubt advances will still be made in this subject. Although we have not applied these distributions in this chapter, we think that if it had been the case, the long-term expected return would still have been smaller than the one-year expected return, with a difference of a magnitude comparable to that obtained with the lognormal distribution. An interesting ques-

<sup>14</sup> Note that the central limit theorem is valid under much wider conditions: changing distributions are allowed under certain conditions and under some degree of dependence between the log-returns, but not too much so.

<sup>15</sup> B. Mandelbrot, "The Variation of Certain Speculative Prices," *Journal of Business*, 36 (1993): 394–419.

<sup>16</sup> B. Mandelbrot, *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk* (New York: Springer, 1997).

tion, of course, would be to know how much the skewness introduced in these distributions influences this difference.

### Appendix 13.A.1 Estimating the mean and variance of the rate of return from a sample

Let  $X$  designate a random variable with unknown mean and unknown variance  $\sigma^2$ . All possible values of  $X$  are supposed to be independent from each other. Let  $(x_1, \dots, x_{n_s})$  be a sample of size  $n_s$  of this random variable. For instance,  $X$  may be the continuously compounded daily return on an asset ( $X = \ln(S_j/S_{j-1})$  when  $S_j$  is the asset's value at end of day  $j$ ;  $x_j$  may be its sample value measured at end of day  $j$ :  $x_j = \ln(S_j/S_{j-1})$ ). We now consider two statistics corresponding to this sample: its mean,  $\bar{x}$ , and its variance,  $s^2$ . Its mean,  $\bar{x} = \frac{1}{n_s} \sum_{i=1}^{n_s} x_i$ , can be considered as *one* sample value of a random variable that is the average of  $n_s$  random variables  $X_i$ :

$$\bar{X} = \frac{1}{n_s} \sum_{i=1}^{n_s} X_i$$

The expected value of  $\bar{X}$  is  $E(\bar{X}) = \frac{1}{n_s} \sum_{i=1}^{n_s} E(X_i) = \frac{1}{n_s} n_s E(X) = E(X) = \mu$ . This means that we can consider the sample's average as representative of the population's average. Let us now consider the variance of the random variable  $\bar{X}$ :

$$VAR(\bar{X}) = \frac{1}{n_s^2} \sum_{i=1}^{n_s} VAR(X_i)$$

(since the  $X_i$  variables are independent)

$$= \frac{1}{n_s^2} n_s VAR(X_i) = \frac{VAR(X)}{n_s} = \frac{\sigma^2}{n_s}$$

(since all variances of the  $X_i$ 's are common and equal to the (unknown)  $VAR(X) = \sigma^2$ ).

So the statistic  $\bar{X}$  has an expected value that is the mean of the population; its variance is  $\sigma^2/n_s$ , which converges to zero when  $n_s \rightarrow \infty$ . The larger the sample, the more confident we can be that the sample's average is close to the true population's mean.

Consider now another statistic: the variance of the sample,

$$s^2 = \frac{1}{n_s} \sum_{i=1}^{n_s} (x_i - \bar{x})^2$$

This is just one outcome value of a random variable that may be written as

$$S^2 = \frac{1}{n_s} \sum_{i=1}^{n_s} (X_i - \bar{X})^2$$

Let us now consider the expected value of  $S^2$ ,  $E(S^2)$ :

$$\begin{aligned} E(S^2) &= \frac{1}{n_s} E \left[ \sum_{i=1}^{n_s} (X_i^2 - 2X_i\bar{X} + \bar{X}^2) \right] = \frac{1}{n_s} E \left[ \sum_{i=1}^{n_s} (X_i^2) - n_s \bar{X}^2 \right] \\ &= \frac{1}{n_s} \left[ \sum_{i=1}^{n_s} E(X_i^2) - n_s E(\bar{X}^2) \right] = E(X_i^2) - E(\bar{X}^2) = \sigma^2 + \frac{\sigma^2}{n_s} - E(\bar{X}^2) \end{aligned}$$

We can write

$$\begin{aligned} E(\bar{X}^2) &= \frac{1}{n_s^2} E(X_1 + \cdots + X_{n_s} - \frac{1}{n_s} \sum_{i=1}^{n_s} X_i)^2 \\ &= \frac{1}{n_s^2} \{ E[X_1 + \cdots + X_{n_s} - \frac{1}{n_s} \sum_{i=1}^{n_s} X_i]^2 \\ &\quad + E[2n_s (X_1 - \frac{1}{n_s} \sum_{i=1}^{n_s} X_i)] + E(n_s^2 \frac{\sigma^2}{n_s^2}) \} \end{aligned}$$

In the above expression, the first expectation is the expected value of a quadratic form, which is equal first to the  $n$  variances of  $X_1, \dots, X_{n_s}$  (hence to  $n_s \sigma^2$ ), to which one has to add  $n_s(n_s - 1)$  covariances, all equal to zero, since the  $X_i$ 's are independent. The second expected value is zero, since  $E(X_i - \frac{1}{n_s} \sum_{i=1}^{n_s} X_i) = 0$ ,  $i = 1, \dots, n_s$ ; the third one is just  $n_s^2 \frac{\sigma^2}{n_s^2}$ .

We have

$$E(\bar{X}^2) = \frac{1}{n_s^2} (n_s \sigma^2 + n_s^2 \frac{\sigma^2}{n_s^2}) = \frac{\sigma^2}{n_s} + \frac{\sigma^2}{n_s}$$

(Note that this last expression does make sense: since  $\bar{X}$  approaches  $\mu$  when  $n \rightarrow \infty$ , it is reasonable to think that the expectation of  $\bar{X}^2$  should approach  $\mu^2$  when  $n \rightarrow \infty$ . From the above expression, we can see that this indeed is the case.)

Finally, for  $E(S^2)$  we get

$$E(S^2) = \sigma^2 + \frac{\sigma^2}{n_s} - \frac{\sigma^2}{n_s} = \frac{n_s - 1}{n_s} \sigma^2$$

In order to get an estimate of the true variance, we will therefore consider the statistic  $\frac{n_s}{n_s - 1} S^2$ , denoted  $S^*$ , whose expected value will be  $\sigma^2$ :

$$E(S^*) = E\left(\frac{n_s}{n_s - 1} S^2\right) = \frac{n_s}{n_s - 1} \frac{n_s - 1}{n_s} \sigma^2 = \sigma^2$$

### Appendix 13.A.2 Determining the expected long-term return when the continuously compounded yearly return is uniform

If  $\log X_{t-1,t}$  is uniform on  $[a, b]$ , we have the following relationships:

$$E(\log X_{t-1,t}) \equiv \frac{a + b}{2} \quad (1)$$

$$\sigma(\log X_{t-1,t}) \equiv \sigma = \frac{b - a}{2\sqrt{3}} \quad (2)$$

Conversely,  $a$  and  $b$  are related to  $E$  and  $\sigma$  by

$$a = E - \sqrt{3}\sigma \quad (3)$$

$$b = E + \sqrt{3}\sigma \quad (4)$$

Denote for simplicity  $U = \log X_{t-1,t}$ ; if  $h(u)$  is the density function of  $U$  and  $f(x)$  is the density function of  $X_{t-1,t}$ , we have the well-known relationship  $f(x) = h(u) \left| \frac{du}{dx} \right| = \frac{1}{b-a} \frac{1}{x}$ . Hence the density function of  $X_{t-1,t}$  is a hyperbola defined between  $e^a$  and  $e^b$ .

$E(Y_{0,n})$  is equal to

$$E(Y_{0,n}) = E\left(\prod_{t=1}^n X_{t-1,t}^{1/n}\right) = \prod_{t=1}^n E(X_{t-1,t}^{1/n}) = [E(X_{t-1,t}^{1/n})]^n$$

(from the independence of the  $X_{t-1,t}$ 's). Evaluate now each  $E(X_{t-1,t}^{1/n})$  in this case:

$$\begin{aligned} E(X_{t-1,t}^{1/n}) &= E(e^{\frac{1}{n} \log X_{t-1,t}}) = E(e^{\frac{1}{n} U}) \\ &= \int_a^b e^{\frac{1}{n} u} \cdot \frac{1}{b-a} du = \frac{n}{b-a} (e^{b/n} - e^{a/n}) \end{aligned} \quad (5)$$

Finally,

$$E(Y_{0,n}) = \left[ \frac{n}{b-a} (e^{b/n} - e^{a/n}) \right]^n \quad (6)$$

which is the second part of equation (14) in our text; the first part of (14) is obtained by plugging equations (3) and (4) into (6).

### Main formulas

- Definition:  $X_{t-1,t} = S_t/S_{t-1} = 1 + R_{t-1,t} \equiv$  the yearly “dollar” return, or one plus the yearly rate of return.

- Hypothesis:  $X_{t-1,t}$  lognormal because  $\log X_{t-1,t} \sim N(\mu, \sigma^2)$

#### 1. Properties of the yearly rate of return

- Expected value, variance, and standard deviation of the yearly rate of return  $R_{t-1,t}$ :

$$E(R_{t-1,t}) = e^{+\frac{\sigma^2}{2}} - 1 \quad (5)$$

$$VAR(R_{t-1,t}) = VAR(X_{t-1,t}) = E(X_{t-1,t}^2) - [E(X_{t-1,t})]^2 \quad (6)$$

$$\sigma(R_{t-1,t}) = e^{+\frac{\sigma^2}{2}} [e^{\sigma^2} - 1]^{1/2} \quad (9)$$

- Probability for  $R_{t-1,t}$  to be between  $a$  and  $b$ :

$$\text{Prob}(a < R_{t-1,t} < b) = \Phi\left(\frac{\log(1+b) - \mu}{\sigma}\right) - \Phi\left(\frac{\log(1+a) - \mu}{\sigma}\right) \quad (10)$$

- Particular case:

$$\text{Prob}(R_{t-1,t} < E(R_{t-1,t})) = \text{Prob}(Z < \sigma/2), \quad Z \sim N(0, 1)$$

#### 2. Properties of the horizon- $n$ rate of return $R_{0,n}$

$$R_{0,n} = \left( \frac{S_n}{S_0} \right)^{1/n} - 1$$

- Expected value, variance, and standard deviation of the horizon rate of return  $R_{0,n}$ :

$$E(R_{0,n}) = e^{+\frac{\sigma^2}{2n}} - 1 \quad (16)$$

$$VAR(R_{0,n}) = e^{2+\frac{\sigma^2}{n}}(e^{\frac{\sigma^2}{n}} - 1) \quad (21)$$

$$\sigma(R_{0,n}) = e^{+\frac{\sigma^2}{2n}}(e^{\frac{\sigma^2}{n}} - 1)^{1/2} \quad (22)$$

• Formulas of  $\sigma^2$  in terms of  $E \equiv E(X_{0,1}) \equiv 1 + R_{0,n}$  and  $V \equiv VAR(X_{0,1}) = VAR(R_{0,1})$ :

$$= \log E - \frac{1}{2} \log \left( 1 + \frac{V}{E^2} \right) \quad (23)$$

$$\sigma^2 = \log \left( 1 + \frac{V}{E^2} \right) \quad (24)$$

$$E(R_{0,n}) = E \cdot (1 + V/E^2)^{\frac{1}{2}(1/n - 1)} - 1 \quad (25)$$

$$VAR(R_{0,n}) = E^2 \cdot (1 + V/E^2)^{\frac{1}{2}(1/n - 1)} [(1 + V/E^2)^{\frac{1}{2}} - 1] \quad (26)$$

$$\sigma(R_{0,n}) = [1 + E(R_{0,n})][(1 + V/E^2)^{\frac{1}{2}} - 1]^{1/2} \quad (28)$$

• Probability for  $R_{0,n}$  to be between  $a$  and  $b$ :

$$\text{Prob}(a < R_{0,n} < b) = \Phi \left[ \frac{\log(1+b) - \log E}{\sigma / \sqrt{n}} \right] - \Phi \left[ \frac{\log(1+a) - \log E}{\sigma / \sqrt{n}} \right] \quad (30)$$

## Questions

13.1. This first exercise is for the reader who is not absolutely sure that the moment-generating function of a normal variable  $\log X_{t-1,t} \equiv U \equiv N(\mu, \sigma^2)$  is  $U(\lambda) = e^{\lambda\mu + \frac{1}{2}\lambda^2\sigma^2}$ , a result of considerable importance in this chapter and the subsequent ones. You can either turn to an introductory probability text or do the following exercise. First write the normal variable  $N(\mu, \sigma^2)$  as  $U = \mu + \sigma Z$ , where  $Z$  is the unit normal variable. Then apply the definition of the moment-generating function and write

$$U(\lambda) = E(e^{\lambda U}) = E(e^{\lambda\mu + \lambda\sigma Z}) = e^{\lambda\mu} E(e^{\lambda\sigma Z})$$

You can see that calculating the moment-generating function of  $U \sim N(\mu, \sigma^2)$  amounts to calculating the moment-generating function of the unit normal variable  $Z$  where  $\lambda\sigma$  plays the role of the parameter. This is what you should do in this exercise. (Hint: Set  $\lambda\sigma$  as  $p$ , for instance, and

calculate  ${}_Z(p) = E(e^{pz}) = \int_{-\infty}^{+\infty} e^{pz} g(z) dz$ , where  $g(z)$  is the unit normal density.)

13.2. Suppose that you estimate the expected yearly continuously compounded return to be  $\mu = 8\%$  and its variance  $\sigma^2 = 4\%$ . You also suppose that these yearly returns are normally distributed. What are the expected yearly return (compounded once a year) and its standard deviation over a one-year horizon?

13.3. Supposing that  $\log(S_t/S_{t-1})$  is normal  $N(\mu, \sigma^2)$ , what is the probability that the yearly return  $R_{t-1,t}$  (compounded once a year) will be lower than its expected value  $E(R_{t-1,t})$ ? (The result you will get is quite striking in its simplicity.)

13.4. Use the result you obtained under 13.3 with the data of 13.2 to calculate the probability that  $R_{t-1,t}$  will be smaller than  $E(R_{t-1,t})$ .

13.5. What is the shape of the curve depicting the probability of  $R_{t-1,t}$  being smaller than  $E(R_{t-1,t})$ ? Draw this curve in  $[\text{prob}(R_{t-1,t} < E(R_{t-1,t})), \sigma]$  space; use values of  $\sigma$  from 5% to 35% in steps of 5%.

13.6. Consider again the hypotheses and data of 13.2. What are the 10-year horizon expected return and standard deviation, supposing that the continuously compounded yearly returns are independent?

13.7. Assuming that the continuously compounded annual returns are independent and normally distributed  $N(\mu, \sigma^2)$ :

a. Determine the probability that the  $n$ -year horizon return will be smaller than its expected value. As in exercise 13.3, you will be surprised by the simplicity of the result you obtain.

b. Verify your result by setting  $n = 1$ ; it should yield the result you got in 13.3.

c. What is the limit of this result when  $n \rightarrow \infty$ ?

13.8. Consider the hypotheses and data of exercise 13.2 ( $\mu = 8\%$  and  $\sigma^2 = 4\%$ ). Use two methods to find the infinite horizon expected return on your investment.



# 14 INTRODUCING THE CONCEPT OF DIRECTIONAL DURATION

SHYLOCK. Give him direction for this merry bond.  
*The Merchant of Venice*

## Chapter outline

<b>14.1</b>	<b>A brief reminder on the concept of directional derivative</b>	<b>295</b>
<b>14.2</b>	<b>A generalization of the Macaulay and Fisher-Weil concepts of duration to directional duration</b>	<b>296</b>
<b>14.3</b>	<b>Applying directional duration</b>	<b>299</b>
14.3.1	The case of a parallel displacement	302
14.3.2	The case of a nonparallel displacement	303

## Overview

In the immunization problem treated in chapter 10, we showed that it was not possible to systematically protect the investor when the term structure was subject to *any* kind of variation. However, we pointed out that if we made a forecast of the possible variation of the term structure, we could still immunize the investor. In this chapter we will show that the concept of duration can be generalized to any kind of variation of the term structure, and we will call this new duration concept *directional duration*. It stems from the concept of directional derivative.

We will show how this may help generalize the concepts of the Macaulay or Fisher-Weil durations. First, we begin with a reminder of a simple way to define the directional derivative.

### 14.1 A brief reminder on the concept of directional derivative

Suppose that  $y = f(x_1, \dots, x_j, \dots, x_n)$  is a differentiable function of  $n$  variables. Its total differential is

$$dy = \sum_{j=1}^n \frac{f}{x_j}(x_1, \dots, x_n) dx_j \quad (1)$$

Now suppose that each differential  $dx_j$  is written in the following form:

$$dx_j = a_j d\lambda \quad (j = 1, \dots, n) \quad (2)$$

where  $d\lambda$  is a given arbitrary small increment different from zero. The  $n$   $a_j$  elements form an  $n$ -vector, which we may call a directional vector, written as  $\mathbf{a}$ . The directional derivative may be defined by substituting (2) into (1) and dividing the result by  $d\lambda$ :

$$\frac{dy}{d\lambda} = \sum_{j=1}^n \frac{f}{x_j}(x_1, \dots, x_j, \dots, x_n) a_j = \nabla \cdot \mathbf{a} \quad (3)$$

where  $\nabla$  designates the gradient vector of  $f$  at point  $(x_1, \dots, x_n)$ . Thus the directional derivative is the scalar product of the function's gradient  $\nabla$  and the directional vector  $\mathbf{a}$ . From a geometric point of view, the directional derivative is the slope of the tangent plane (if  $n = 2$ ) or the hyperplane (if  $n > 2$ ) in the direction of the vector  $\mathbf{a}$  at point  $(x_1, \dots, x_n)$  in  $(x_1, \dots, x_n, y)$  space.

## 14.2 A generalization of the Macaulay and Fisher-Weil concepts of duration to directional duration

In order to keep notation simple, we will call the *continuously compounded spot interest* pertaining to period  $[0, t]$   $r_{0,t}$  (previously denoted  $R(0, t)$ );  $r_{0,t}^0$  ( $t = 1, \dots, n$ ) will then designate the continuously compounded *initial* interest rate structure (before this structure has received any shock); and  $r_{0,t}^1$  will be the new interest rate structure (the structure after the shock).

Using this continuously compounded spot rate of interest, a bond paying  $T$  cash flows  $c_t$  ( $t = 1, \dots, T$ ) has the following arbitrage-driven equilibrium values before and after the shock, respectively:

$$B_0^0 = B(r_{0,1}^0, \dots, r_{0,t}^0, \dots, r_{0,T}^0) = \sum_{t=1}^T c_t e^{-r_{0,t}^0 t} \quad (4)$$

$$B_0^1 = B(r_{0,1}^1, \dots, r_{0,t}^1, \dots, r_{0,T}^1) = \sum_{t=1}^T c_t e^{-r_{0,t}^1 t}$$

Its total differential is, at initial point  $(r_{0,1}^0, \dots, r_{0,T}^0)$ ,

$$dB = \sum_{t=1}^T c_t(-t)e^{-r_{0,t}^0} dr_{0,t} \quad (5)$$

Let the increase received by each spot rate  $r_{0,t}^0$  ( $t = 1, \dots, T$ ) be written as

$$dr_{0,t}^0 = a_t d\lambda, \quad t = 1, \dots, T \quad (6)$$

where  $d\lambda$  is a small, normalized increment (for instance, 1% per year or, in financial parlance, 10 basis points per year) and  $a_t$  is any given arbitrary number (which can be negative, positive, or zero). The  $a_t$ 's form a directional vector  $\mathbf{a} \equiv a_1, \dots, a_t, \dots, a_T$ . We then have  $dr_{0,t} = a_t d\lambda$ , which we can substitute in (5);  $dB$  thus becomes equal to

$$dB = \sum_{t=1}^T c_t(-t)e^{-r_{0,t}^0} a_t d\lambda \quad (7)$$

Dividing both sides of (7) by  $B_0^0 d\lambda$ , we get

$$\frac{1}{B_0^0} \frac{dB}{d\lambda} = - \sum_{t=1}^T ta_t c_t e^{-r_{0,t}^0} / B = - \sum_{t=1}^T ta_t w_t \quad (8)$$

where  $w_t \equiv c_t e^{-r_{0,t}^0} / B$  is the share of each cash flow in the bond's value.

We will now introduce the concept of *directional duration*, denoted  $D_{\mathbf{a}}$ . Let each weight  $w_t$ , defined as

$$w_t = c_t e^{-r_{0,t}^0} / B_0^0$$

be multiplied by the directional coefficient  $a_t$  pertaining to the change in the spot rate  $i_t$ ,  $a_t w_t$  called *directional weight* and denoted  $w_t^a \equiv a_t w_t$ . These  $n$  weights form a  $T$ -element directional vector denoted  $\mathbf{w}^a$ . The directional duration denoted  $D_{\mathbf{a}}$  is the weighted *sum* of the times of payment, the weights being  $w_t^a$ ,  $t = 1, \dots, T$ :

$$D_{\mathbf{a}} = \sum_{t=1}^T ta_t c_t e^{-r_{0,t}^0} / B_0^0 = \sum_{t=1}^T t w_t^a = \mathbf{t} \cdot \mathbf{w}^a \quad (9)$$

Thus the directional duration of a bond (or a bond portfolio) is just the scalar product of the time payment vector and the directional weight vector.

Notice that, generally, this directional duration is not a weighted average, since the sum  $\sum_{t=1}^T a_t w_t$  will not usually be unitary. But this possibility should not be excluded, because one could still observe that there is an infinity of directional vectors  $\mathbf{a}$  such that  $\sum_{t=1}^T a_t w_t = 1$ , or  $\mathbf{a} \cdot \mathbf{w} = 1$ . One such case would be such that all directional coefficients  $a_t$  ( $t = 1, \dots, T$ ) are equal to one ( $\mathbf{a}$  would be the unit vector);  $D_{\mathbf{a}}$  would then be the Fisher-Weil duration concept, corresponding to a parallel displacement of the term structure curve.

We notice that the concept of a weighted average can be kept if, instead of transforming the weights  $w_t$  by multiplying them by the directional coefficient  $a_t$ , we transform the times of payments by considering the  $T$  products  $a_t t$  rather than  $t$ . It makes sense of course, since we can choose to weight the times of payments *or* the cash flows to obtain the same results. For that matter, we can see the directional coefficient as equivalently multiplying by  $a_t$  ( $t = 1, \dots, T$ ) *any one* of the following:

1. the times of cash flow payments  $t$  ( $t = 1, \dots, T$ );
2. the cash flows in current value  $c_t$  ( $t = 1, \dots, T$ );
3. the cash flows in present value  $c_t e^{-r_{0,t}^0}$  ( $t = 1, \dots, T$ ); or
4. the discount factors  $e^{-r_{0,t}^0}$  ( $t = 1, \dots, T$ ).

Thus we have the simple relationship measuring the bond's sensitivity to *any* change in the term structure of interest rates:

$$\frac{1}{B_0^0} \frac{dB}{d\lambda} \equiv -D_{\mathbf{a}} \quad (10)$$

or, equivalently,

$$\boxed{\frac{dB}{B_0^0} = -D_{\mathbf{a}} d\lambda} \quad (11)$$

Equation (10) may be read as follows: in linear approximation, the relative increase of the bond's value divided by the increase  $d\lambda$  (which

could be called the bond's *relative* directional derivative) is simply *minus* the directional duration.

We can verify that directional duration boils down to the Fisher-Weil concept by setting  $a_t = 1$  ( $t = 1, \dots, n$ ). We get  $\frac{1}{B} \frac{dB}{d\lambda} = -\sum_{i=1}^n t c_t e^{-r_{0,t}^0} / B_0^0$ , as usual, or the Macaulay duration by setting  $r_{0,t}^0 = i^0$  (a constant) in the last formula.

Note also that the simplicity of these formulas stems from the fact that we have used *continuously* compounded rates of interest (and, of course, discount rates) instead of the once-per-year compounded rates (or the rates compounded a finite number of times per year), which makes formulas more cumbersome.

We could say that the concept of directional duration generalizes well the classical Macaulay and the Fisher-Weil duration concepts. Its only drawback reinforces what we have said earlier: when attempting to protect the investor against *any* change in the interest rate structure, one has to make a prediction about the exact changes in all of the spot interest rates—and this is of course where difficulties begin.

### 14.3 Applying directional duration

Let us now show how directional duration can be applied to the measurement of the sensitivity of a bond's value when the term structure of interest rates does not receive a parallel shift. In order to give such an example, let us come back to the original example of a nonhorizontal spot rate structure as it appeared in chapter 2, section 2.3, table 2.5 (or in chapter 9, section 9.2), using the same bond (coupon: 7%; par value: 1000; maturity: 10 years).

Our first task is to convert these rates  $i_t$ , which are compounded once per year, into equivalent continuously compounded rates  $r_{0,t} = \ln(1 + i_t)$  (because  $(1 + i_t)^t = e^{\ln(1+i_t)t} \equiv e^{r_{0,t}^0 t}$ ). This is done in the third column of table 14.1a. In our example, we have then considered  $d\lambda = 0.1\%$  (or +10 basis points). In the first part of the table (part a), we take up the case of a parallel displacement: all directional coefficients  $a_t$  are equal to one, so  $dr_{0,t}^0 = d\lambda = 0.1\%$  (or 10 basis points;  $t = 1, \dots, 10$ ). The initial value of the bond is denoted as  $B_0^0$  and is equal to

$$B_0^0 = \sum_{t=1}^T c_t e^{-r_{0,t}^0 t} = 1118.254$$

**Table 14.1**  
**Parallel and nonparallel displacements of the continuously compounded spot rate curve: an application of the concept of directional duration  $D_a$**

(a) Parallel displacement of the continuously compounded spot rate curve								
Term of payment $t$	Cash flow $C_t$	Initial spot rate (compounded once a year) $i_{0,t}$	Initial spot rate (continuously compounded) $\ln(1 + i_{0,t}) = r_{0,t}^0$	Initial discounted cash flow $c_t e^{-r_{0,t}^0 t}$	Calculation of Fisher-Weil duration terms $t c_t e^{-r_{0,t}^0 t} / B^0$	Normalized continuously compounded spot rate increase $d\lambda$	New continuously compounded spot rate $r_{0,t}^1 = r_{0,t}^0 + d\lambda$	New discounted cash flow $c_t e^{-r_{0,t}^1 t}$
1	70	0.045	0.044017	66.986	0.0599	0.001	0.045017	66.919
2	70	0.0475	0.046406	63.795	0.1141	0.001	0.047406	63.668
3	70	0.0495	0.048314	60.555	0.1625	0.001	0.049314	60.374
4	70	0.051	0.049742	57.370	0.2052	0.001	0.050742	57.141
5	70	0.052	0.050693	54.327	0.2429	0.001	0.051693	54.056
6	70	0.053	0.051643	51.349	0.2755	0.001	0.052643	54.041
7	70	0.054	0.052592	48.441	0.3032	0.001	0.053592	48.103
8	70	0.0545	0.053067	45.785	0.3275	0.001	0.054067	45.420
9	70	0.055	0.053541	43.234	0.3480	0.001	0.054541	42.847
10	1070	0.055	0.053541	626.411	5.6017	0.001	0.054541	620.178
				$B^0 = 1118.254$	Fisher-Weil duration: $D_{FW} = 7.641$ years	$B^1 = 1109.748$		

(b) Nonparallel displacement of the continuously compounded spot rate curve

Term of payment $t$	Cash flow $C_t$	Initial spot rate (continuously compounded) $\ln(1 + i_{0,t}) = r_{0,t}^0$	Initial discounted cash flow $C_t e^{-r_{0,t}^0 t}$	Directional coefficients vector $\vec{a}$ $a_t$	Calculation of directional terms $ta_t C_t e^{-r_{0,t}^0 t} / B_0^0$	Increase in continuously compounded spot rate $a_t d\lambda$	New continuously compounded spot rate: $r_{0,t}^1 = r_{0,t}^0 + a_t d\lambda$	New discounted cash flows $C_t e^{-r_{0,t}^1 t}$
1	70	0.044017	66.986	1	0.0599	0.001	0.045017	66.919
2	70	0.046406	63.795	0.9	0.1027	0.0009	0.047336	63.681
3	70	0.048314	60.555	0.8	0.1300	0.0008	0.049114	60.410
4	70	0.049742	57.370	0.75	0.1539	0.00075	0.050492	57.198
5	70	0.050693	54.327	0.7	0.1700	0.0007	0.051393	54.138
6	70	0.051643	51.349	0.65	0.1791	0.00065	0.052293	51.149
7	70	0.052592	48.441	0.6	0.1819	0.0006	0.053192	48.238
8	70	0.053067	45.785	0.58	0.1900	0.00058	0.053647	45.573
9	70	0.053541	43.234	0.55	0.1914	0.00055	0.054091	43.021
10	1070	0.053541	626.411	0.55	3.0809	0.00055	0.054091	622.975
								$B_0^1 = 1113.301$
								Directional duration: $D_a = 4.4398$ years

(This of course is the same value as the one we had obtained in chapter 2, section 2.3, because we have used continuously compounded spot rates that are *equivalent* to the former spot rates applied once a year.)

The Fisher-Weil duration, denoted as  $D_{FW}$ , is equal to

$$D_{FW} = \sum_{t=1}^T t c_t e^{-r_{0,t}t} / B^0 = 7.641 \text{ years} \quad (12)$$

Let us now turn to directional duration. In table 14.1b, column 5 displays directional coefficients  $a_t$ , which play the role of giving to short-term continuously compounded spot rates higher increases than to long ones: notice that these directional coefficients decline from  $a_1 = 1$  to  $a_{10} = 0.55$ . Directional duration, denoted  $D_a$ , is calculated in column 5 and is equal to

$$D_a = \sum_{t=1}^T a_t c_t e^{-r_{0,t}^0 t} / B_0^0 = 4.4398 \text{ years} \quad (13)$$

The next column shows the actual increases in each continuously compounded spot rate,  $a_t d\lambda$  ( $t = 1, \dots, T$ ), which give rise to the new spot rates  $r_{0,t}^1 = r_{0,t}^0 + a_t d\lambda$ . When applied as discount rates to the bond's cash flows, the new spot rates  $r_{0,t}^1$  yield the new value of the bond ( $B_0^1$ ) when spot rates have sustained a nonparallel displacement (last column of table 14.1b). We have

$$B_0^1 = \sum_{t=1}^T c_t e^{-r_{0,t}^1 t} = 1113.301$$

Let us now see how directional duration  $D_a$  fares when we want to evaluate the sensitivity of the bond to a change in the spot rate structure.

### 14.3.1 The case of a parallel displacement

First, consider the parallel displacement of the structure. From table 14.1, the bond's relative change in value is

$$\frac{\Delta B}{B} = \frac{B_0^1 - B_0^0}{B_0^0} = \frac{1109.748 - 1118.254}{1118.254} = -0.761\%$$



In linear approximation, this relative change is given by *minus* the Fisher-Weil duration (or the directional duration with unit directional coefficients), multiplied by  $d\lambda = 0.1\%$  per year:

$$\frac{dB}{B} = -D_{FW} d\lambda = -7.6405 \text{ years } (0.1\%/year) = -0.764\%$$

This shows a relative error, entailed by the linear approximation, of

$$\text{Relative error} = \frac{(dB/B) - (\Delta B/B)}{\Delta B/B} = 0.45\%$$

a very small error indeed, because of the small increase in the spot term structure (10 basis points).

### 14.3.2 The case of a nonparallel displacement

From table 14.1b, the bond's relative change in value is

$$\frac{\Delta B}{B} = \frac{B_0^1 - B_0^0}{B_0^0} = \frac{1113.301 - 1118.254}{1118.254} = -0.443\%$$

In linear approximation, the relative change entailed by this nonparallel displacement of the spot structure is given by *minus* the directional duration multiplied by  $d\lambda = 0.1\%$  per year. We have

$$\frac{dB}{B_0^0} = -D_a d\lambda = -4.439806 \text{ years } (0.1\%/year) = -0.444\%$$

This time, the relative error entailed by the linear approximation is

$$\text{Relative error} = \frac{(dB/B) - (\Delta B/B)}{\Delta B/B} = 0.25\%$$

—about two times smaller than previously. This comes, of course, from the fact that the nonparallel displacement involved increases in the spot rates, each of which was smaller than in the parallel displacement case, except for the first term. (Our directional coefficients started with 1 for the first term, and all other coefficients were smaller than 1—see the fifth column of table 14.1b.)

So there is no real problem in measuring a bond portfolio's sensitivity to any interest structure change, but as we have already said in chapter 10, the problems arise when it comes to *protecting* the investor against

such changes. Fortunately, there are some ways out, and this is what we will turn to in the next chapters.

### Main formulas

- Directional derivative:

$$\frac{dy}{d\lambda} = \sum_{j=1}^n \frac{f}{x_j}(x_1, \dots, x_j, \dots, x_n) a_j = \nabla \cdot \mathbf{a} \quad (3)$$

- Directional duration: with  $w_t = c_t e^{-r_{0,t}} / B_0^0$ :

$$D_{\mathbf{a}} = \sum_{t=1}^T t a_t c_t e^{-r_{0,t}} / B_0^0 = \sum_{t=1}^T t w_t^a = \mathbf{t} \cdot \mathbf{w}^a \quad (9)$$

- Bond's sensitivity to any change in the term structure of interest rates:

$$\frac{1}{B_0^0} \frac{dB}{d\lambda} \equiv -D_{\mathbf{a}} \quad (10)$$

or

$$\frac{dB}{B_0^0} = -D_{\mathbf{a}} d\lambda \quad (11)$$

### Questions

14.1. When we suppose—as in table 14.1a, column 4—that the five-year horizon continuously compounded spot rate is, initially,  $r_{0,5}^0 = 5.069\%$ , what does it imply in terms of the forward rates for infinitesimal trading periods  $f(0, \cdot)$ , for  $\cdot = 0$  to  $\cdot = 5$ ?

14.2. If we suppose, as we have done in this chapter, that the whole spot structure receives an increase (either parallel or nonparallel), does it imply an increase in forward rates?

14.3. What is the dimension (the measurement units) of the directional coefficients  $a_t$  ( $t = 1, \dots, T$ ) we have used in the definition of the directional duration?

14.4. This exercise can be considered as a project. Consider again a bond with the same properties and the ones we retained in section 14.3 and the

same initial structure. However, consider now the following directional vector  $\mathbf{a}$ :

Term $t$	Directional coefficient $a_t$
1	1
2	0.95
3	0.9
4	0.85
5	0.8
6	0.75
7	0.7
8	0.68
9	0.65
10	0.65

Use also the same normalized increment  $d\lambda = 0.001$  (+10 basis points).

Determine in this case:

- the exact relative change in the bond's price;
- the bond's directional duration  $D_{\mathbf{a}}$ ;
- the relative change in the bond's price in linear approximation; and
- the relative error you are introducing by using this linear approximation.

Without doing any calculations, could you have predicted the magnitude of the results you have obtained under (a), (b), (c), and (d)? (This of course should give you a good way to check that the results you obtained have a good chance of being correct.)



# 15 A GENERAL IMMUNIZATION THEOREM, AND APPLICATIONS

AARON. . . . so must you resolve,  
That what you cannot as you would achieve,  
You must perforce accomplish as you may.  
*Titus Andronicus*

SPEED. But tell me true, will't be a match?

LAUNCE. Ask my dog: if he say aye, it will! if he say no, it will;  
if he shake his tail and say nothing, it will.  
*The Two Gentlemen of Verona*

## Chapter outline

15.1	Some brief methodological remarks	307
15.2	A general immunization theorem	308
15.3	Applications	312
15.4	Epilogue: a new approach to the calculus of variations	324

## Overview

In the remainder of this book we will be concerned with protecting the investor against nonparallel shifts in the term structure. This chapter will be in the spirit of what we did in chapters 6 and 10. Here, however, we will deal with both any initial term structure and *any* kind of shock received by this structure. We will show how our previous results can be generalized in a theorem. We will then put our theorem to the test by giving a number of immunization applications that the reader will hopefully find quite forceful.

In the next chapters we will consider a stochastic approach, dealing with the justly celebrated Heath-Jarrow-Morton model. The necessary methodology, the basic model, and applications will be presented in chapters 16, 17, and 18, respectively.

### 15.1 Some brief methodological remarks

As we recalled in chapter 12, at the end of the seventeenth century and during the eighteenth century, mathematicians were confronted with novel problems that could not be tackled immediately with differential or integral calculus: those problems required analyzing relationships between

a *function* and a number, for instance. In order to distinguish those objects from the more usual concept of functions, these relationships were given the name of functionals. Here we have a problem very much like one of those tricky questions posed by Jean Bernoulli to his brother and other colleagues for their enjoyment and enlightenment. As we have seen however, both an extension of duration to directional duration (chapter 14) and a variational approach (section 10.6 of chapter 10) seem to indicate that it is not possible to tackle directly the question of immunizing an investor against *any* kind of interest structure variation. However, we will now suggest a method that will achieve what we are looking for with as much precision as we need, and which also rests on transforming a functional into a function, this time a function of several variables.<sup>1</sup>

We will state our theorem and demonstrate it in section 15.2. In section 15.3 we will apply it in a number of very diverse—we could even call them extreme—situations. We hope, finally, that the reader will find a nice surprise in the epilogue of this chapter.

## 15.2 A general immunization theorem

We first have to recall a concept that will play a central role. In chapters 11 and 12 we defined the continuously compounded rate of return *over a period*  $T$  as  $R(T)T$ , equal to  $\int_0^T f(u) du$ , where  $R(T)$  is the continuously compounded rate of return *per year* and  $f(u)$  is the yearly forward rate agreed upon at time 0 for an infinitesimal trading period starting at  $u$ . We will give the name  $G(T)$  to this function. We have

$$G(T) = R(T)T = \int_0^T f(u) du \quad (1)$$

**Definition.** We now define the *moment of order  $k$  of a bond portfolio as the weighted average of the  $k$ th power of its times of payment, the weights being the shares of the portfolio's cash flows in the initial portfolio value.*

<sup>1</sup> At the time of completion of this book, we were unfortunately unaware of the work by Donald R. Chambers and Willard T. Carleton, "A Generalized Approach to Duration," *Research in Finance*, 7 (1988): 163–181, where the authors introduced and developed a "multiple factor duration model" along similar lines, albeit with some differences. It is to be noted that D. R. Chambers, W. T. Carleton, and R. W. McEnally, using real data, have tested their model for immunization purposes with impressive success. See their paper "Immunizing Default-Free Bond Portfolios with a Duration Vector," *Journal of Financial and Quantitative Analysis*, 23, no. 1 (March 1988): 89–104.

Such a moment can therefore be written as

$$\sum_{t=1}^N t^k c_t e^{-\int_0^t f(u) du} / B_0 = \sum_{t=1}^N t^k c_t e^{-R(t)t} / B_0 = \sum_{t=1}^N t^k c_t e^{-G(t)} / B_0$$

Suppose that an investor has a horizon  $H$ , and that when he buys a bond portfolio the spot structure (or the forward structure) is given. However, immediately after his purchase, this structure undergoes a shock—or equivalently, receives a variation. We are now ready to state our theorem.

**Theorem.** Suppose that the continuously compounded rate of return over a period  $t$ ,  $G(t) = R(t) \cdot t = \int_0^t f(u) du$  is developable in a Taylor series  $\sum_{j=0}^m A_j t^j + \frac{1}{(m+1)!} G^{(m+1)}(\theta t) t^{m+1}$ ,  $0 < \theta < 1$ , where  $A_j = (1/j!) G^{(j)}(0)$ ,  $j = 0, 1, \dots, m$ . Let  $B_H$  designate the horizon- $H$  bond portfolio's value after a change in the spot rate structure. This change intervenes immediately after the portfolio is bought. For an investor with horizon  $H$  to be immunized against any such change in the spot interest structure, a sufficient condition is that at the time of the purchase

- (a) any moment of order  $j$  ( $j = 0, 1, \dots, m$ ) of the bond portfolio is equal to  $H^j$ ,
- (b) the Hessian matrix of the second partial derivatives ( $\partial^2 B_H / \partial A_j^2$ ) is positive-definite.

**Proof.** Suppose we can write the McLaurin series:

$$\begin{aligned} G(t) = & G(0) + G'(0)t + \frac{1}{2!} G''(0)t^2 + \dots + \frac{1}{m!} G^{(m)}(0)t^m \\ & + \frac{1}{(m+1)!} G^{(m+1)}(\theta t)t^{m+1}, \quad 0 < \theta < 1 \end{aligned} \quad (2)$$

This implies that  $G(t)$  can be approximated with any desired accuracy by a polynomial of the form

$$G(t) \approx A_0 + A_1 t + A_2 t^2 + \dots + A_m t^m = \sum_{j=0}^m A_j t^j \quad (3)$$

where  $A_j = (1/j!) G^{(j)}(0)$ ,  $j = 0, 1, \dots, m$ .

Consider a time span  $[1, N]$ , where  $N$  designates the last time a bond or a bond portfolio used for immunization will pay a cash flow. Let  $R(t)$  designate the continuously compounded rate of return function and  $f(u)$

the forward rate function for instantaneous lending, agreed on *at the time* (0) *the portfolio is bought*. Using (3), the value of the bond—or the bond portfolio—can be written

$$B_0 = \sum_{t=1}^N c_t e^{-\int_0^t f(u) du} = \sum_{t=1}^N c_t e^{-R(t)t} \approx \sum_{t=1}^N c_t e^{-(A_0 + A_1 t + \dots + A_m t^m)} \quad (4)$$

Note that we have transformed a *functional*  $B_0$ , depending on function  $f(u)$ , into a *function* of  $m + 1$  variables  $(A_0, \dots, A_m)$ . This will be the key to our demonstration.

We know already from chapter 6 that to immunize a portfolio we have to guarantee a future value of the investment at a given horizon  $H$  ( $H < N$ ). Let us then express the portfolio's future value at horizon  $H$ :

$$\begin{aligned} B_H &= \left[ \sum_{t=1}^N c_t e^{-\int_0^t f(u) du} \right] e^{\int_0^H f(u) du} = \left[ \sum_{t=1}^N c_t e^{-R(t)t} \right] e^{R(H)H} \\ &= \left[ \sum_{t=1}^N c_t e^{-G(t)} \right] e^{G(H)} \end{aligned} \quad (5)$$

By the hypothesis of our theorem, we can then write

$$B_H \approx \left[ \sum_{t=1}^N c_t e^{-(A_0 + A_1 t + \dots + A_j t^j + \dots + A_m t^m)} \right] e^{(A_0 + A_1 H + \dots + A_j H^j + \dots + A_m H^m)} \quad (6)$$

Given  $H$ , this future value of the portfolio expressed at horizon  $H$  *remains* a function of the  $m + 1$  variables  $A_0, A_1, \dots, A_m$ . We know that to immunize our portfolio this future value should pass through a minimum in the  $m + 1$  dimension space  $(B_H, A_0, \dots, A_m)$ . Following what we did already in chapter 6, minimizing  $B_H$  is tantamount to minimizing  $\log B_H$ . We have

$$\begin{aligned} \log B_H &\approx \log \left[ \sum_{t=1}^N c_t e^{-(A_0 + A_1 t + \dots + A_j t^j + \dots + A_m t^m)} \right] \\ &\quad + A_0 + A_1 H + \dots + A_j H^j + \dots + A_m H^m \end{aligned} \quad (7)$$

A first-order condition for minimizing  $\log B_H$  is that the gradient of  $\log B_H$  be zero at the initial point  $A_0, \dots, A_m$ . This implies the  $m + 1$  following equations:

$$\frac{\log B_H}{A_0} = 0$$



$$\begin{aligned}
\frac{\log B_H}{A_1} &= 0 \\
&\vdots \\
\frac{\log B_H}{A_m} &= 0
\end{aligned} \tag{8}$$

Written more compactly, this first-order condition is

$$\frac{\log B_H}{A_j} = 0, \quad j = 0, 1, \dots, m \tag{9}$$

Let us now see what conditions (8) or (9) imply. Using (7), the first of these  $m + 1$  equations implies

$$\frac{\log B_H}{A_0} = \frac{\sum_{t=1}^N c_t e^{-(A_0 + A_1 t + \dots + A_j t^j + \dots + A_m t^m)}}{B_0} (-1) + 1 = 0 \tag{10}$$

which leads to

$$\frac{\sum_{t=1}^N c_t e^{-(A_0 + A_1 t + \dots + A_j t^j + \dots + A_m t^m)}}{B_0} = 1 \tag{11}$$

Equation (11) is an accounting identity. It says that the shares of the present value cash flows of the immunization portfolio add up to 1. But the innocuous-looking equation (11) also carries another message: it has a nice mathematical interpretation, which the reader will discover very soon.

Consider now the second equation of system (8). It leads to

$$\frac{\log B_H}{A_1} = \frac{\sum_{t=1}^N c_t e^{-(A_0 + A_1 t + \dots + A_j t^j + \dots + A_m t^m)}}{B_0} (-t) + H = 0 \tag{12}$$

or

$$H = \frac{\sum_{t=1}^N t c_t e^{-(A_0 + A_1 t + \dots + A_j t^j + \dots + A_m t^m)}}{B_0} \tag{13}$$

which amounts to equalizing horizon  $H$  to the Fisher-Weil duration. Equivalently, it requires equalizing horizon  $H$  to the *first moment* of the bond.

Consider now the generic term of (9). It leads to

$$\frac{\log B_H}{A_j} = \frac{\sum_{t=1}^N c_t e^{-(A_0 + A_1 t + \dots + A_j t^j + \dots + A_m t^m)}}{B_0} (-t^j) + H^j = 0$$

or

$$H^j = \frac{\sum_{t=1}^N t^j c_t e^{-(A_0 + A_1 t + \dots + A_j t^j + \dots + A_m t^m)}}{B_0} \quad (14)$$

Equation (14) is central to our theorem: it tells us that the  $j$ th moment of the bond portfolio must be equal to the  $j$ th power of horizon  $H$ . Thus conditions (9) are equivalent to

$$H^j = \sum_{t=1}^N t^j c_t e^{-(A_0 + A_1 t + \dots + A_j t^j + \dots + A_m t^m)} / B_0, \quad j = 0, 1, \dots, m \quad (15)$$

This is part (a) of our theorem; part (b) results from usual second-order conditions for a nonconstrained minimum of a function of several variables.

Let us now come back for an instant to equation (11), resulting from equalizing to zero our first partial derivative of  $\log B_H$  with respect to  $A_0$ . We now recognize in (11) not only a necessary accounting condition but the equality between  $H^0 (= 1)$  and the *moment of order zero* of the bond portfolio (necessarily equal to 1, as you remember your statistics teacher telling you when he introduced the moment-generating function). So system (15) is perfectly general and applies to *all* moments of the bond, starting with the *moment of order zero*.

Observe also that system (15) involves  $m + 1$  equations: if the order of the Taylor expansion you choose is  $m$ , you will want to build up an immunization portfolio of  $m + 1$  bonds.

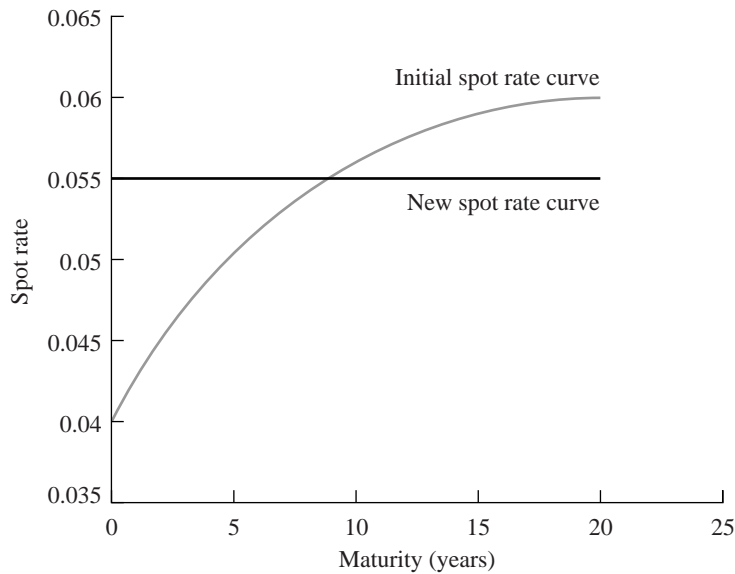
### 15.3 Applications

We will now illustrate the power of immunization generated by this method. Suppose that practitioners have determined, either through spline or parametric methods, that the spot rate curve is given by

$$R(t) = 0.04 + 0.00248t - 10^{-4}t^2 + (1.5)10^{-6}t^3 \quad (16)$$

This is our initial spot rate curve, represented in Figure 15.1. It implies that the continuously compounded return over period  $t$  is

$$G(t) = tR(t) = 0.04t + 0.00248t^2 - 10^{-4}t^3 + (1.5)10^{-6}t^4 \quad (17)$$



**Figure 15.1**  
Initial and new spot rate curves.

**Table 15.1**  
Main features of bonds *a*, *b*, *c*, and *d*

	Coupon	Maturity (years)	Par value
Bond <i>a</i>	4	7	100
Bond <i>b</i>	4.75	8	100
Bond <i>c</i>	7	15	100
Bond <i>d</i>	8	20	100

So our function  $G(t)$  already has the form of a Taylor series expansion. Alternatively,  $G(t)$  could be considered to be formed from a Taylor expansion of an estimated spot rate curve.

The initial term structure  $R(t)$  corresponding to this  $G(t)$  function is extremely close to that depicted as a simple illustration in figure 11.2 of chapter 11, where it was denoted  $R(0, T)$ . In fact, it can hardly be visually distinguished from  $R(0, T)$  because of the smallness of the third-order term.

Consider now a portfolio made up of four bonds *a*, *b*, *c*, and *d*. Their characteristics are summarized in table 15.1. For simplicity we assume they pay annual coupons.

Suppose that you want to invest today a value of \$100 for a length of time of seven years. Your investment horizon is seven years, but unfortunately you cannot find some piece of an AAA zero-coupon, seven-year maturity bond that would guarantee you an  $R(7) = 5.28\%$  yield per year (continuously compounded), equivalent to a terminal value of  $100e^{(0.0528)7} = 144.72$  in seven years. On the other hand, could you be guaranteed this seven-year return, or the corresponding terminal value, by choosing an appropriate portfolio using bonds  $a$ ,  $b$ ,  $c$ , and  $d$ ?

In other words, what should the numbers  $n_a$ ,  $n_b$ ,  $n_c$ , and  $n_d$  of these bonds be for you to purchase them such that your \$100 would be transformed into that terminal value even if, just after you have made your investment, the term structure shifts in a dramatic way? By dramatic way we mean, for example, that the steep spot curve we have chosen (increasing from 4% to 6%, by 200 basis points, over 20 years of maturity) turns instantly clockwise to become horizontal at a given level, for instance 5.5%; this implies that the shortest spot rate moves up by 150 basis points and that the longest spot rate moves down by 50 points immediately after you have made your purchase. How should you constitute your bond portfolio in order to secure a value extremely close to \$144.72 in seven years?

Let us compute the moments  $m_0$ ,  $m_1$ ,  $m_2$ , and  $m_3$  for each bond. First, table 15.2 presents the *values* of each bond as determined by the initial spot rate structure. The calculations of moments  $m_0$ ,  $m_1$ ,  $m_2$ , and  $m_3$  under the same structure are to be found in tables 15.3, 15.4, 15.5, and 15.6, respectively.

We now have to build a portfolio of  $n_a$ ,  $n_b$ ,  $n_c$ , and  $n_d$  bonds such that its first four moments are equal to 1,  $H$ ,  $H^2$ , and  $H^3$  (respectively to  $D^0 = 1$ ,  $D$ ,  $D^2$ , and  $D^3$ ). Since we have chosen a horizon of seven years, those numbers are 1, 7, 49, and 343.

So we are led to solve the following system of four equations in the four unknowns  $n_a$ ,  $n_b$ ,  $n_c$ ,  $n_d$ :

$$\begin{aligned}
 \frac{n_a B_0^a}{B_0} m_0^a + \frac{n_b B_0^b}{B_0} m_0^b + \frac{n_c B_0^c}{B_0} m_0^c + \frac{n_d B_0^d}{B_0} m_0^d &= H^0 \\
 \frac{n_a B_0^a}{B_0} m_1^a + \frac{n_b B_0^b}{B_0} m_1^b + \frac{n_c B_0^c}{B_0} m_1^c + \frac{n_d B_0^d}{B_0} m_1^d &= H^1 \\
 \frac{n_a B_0^a}{B_0} m_2^a + \frac{n_b B_0^b}{B_0} m_2^b + \frac{n_c B_0^c}{B_0} m_2^c + \frac{n_d B_0^d}{B_0} m_2^d &= H^2 \\
 \frac{n_a B_0^a}{B_0} m_3^a + \frac{n_b B_0^b}{B_0} m_3^b + \frac{n_c B_0^c}{B_0} m_3^c + \frac{n_d B_0^d}{B_0} m_3^d &= H^3
 \end{aligned} \tag{18}$$

**Table 15.2**

**Valuation of bonds  $a$ ,  $b$ ,  $c$ , and  $d$  under the initial spot rate structure  $R(t) = 0.04 + 0.00248t - 10^{-4}t^2 + (1.5)10^{-6}t^3$**

Maturity (years) $t$	Initial spot rate structure $R(t)$	Discounted cash flows			
		Bond $a$ $c_t^a e^{-R(t)t}$	Bond $b$ $c_t^b e^{-R(t)t}$	Bond $c$ $c_t^c e^{-R(t)t}$	Bond $d$ $c_t^d e^{-R(t)t}$
0	0.04				
1	0.042378	3.83403	4.55291	6.709552	7.668059
2	0.044558	3.658956	4.34501	6.403173	7.317912
3	0.046549	3.478656	4.130904	6.087648	6.957312
4	0.04836	3.296471	3.91456	5.768825	6.592943
5	0.05	3.115203	3.699304	5.451605	6.230406
6	0.051477	2.93713	3.487842	5.139977	5.874259
7	0.0528	71.86511	3.282301	4.837075	5.528085
8	0.053978		68.01664	4.545265	5.194588
9	0.05502			4.266237	4.875699
10	0.055934			4.001101	4.572686
11	0.05673			3.75048	4.286263
12	0.057415			3.514599	4.016685
13	0.058			3.293366	3.763847
14	0.058492			3.086439	3.527358
15	0.058901			44.22598	3.306615
16	0.059235				3.100858
17	0.059503				2.909221
18	0.059714				2.730772
19	0.059877				2.564542
20	0.06				32.52897
Initial bond's value		92.18556	95.42947	111.0813	123.5471

For simplicity, let us call

$\mathbf{n} \equiv$  the (row) vector of unknowns  $(n^a, n^b, n^c, n^d)$

$\mathbf{m} \equiv$  the square matrix of terms  $(1/B_0)B_0^l m_k^l$ ,  $k = 0, 1, 2, 3$   
 $l = a, b, c, d$

Each of these terms is the moment of various orders of each of the bonds entering the immunizing portfolio, weighted by the ratio of each bond's value to the initial investment cost.<sup>2</sup> Matrix  $\mathbf{m}$  is

$$\mathbf{m} = \begin{bmatrix} 0.921856 & 0.954295 & 1.110813 & 1.235471 \\ 5.710284 & 6.478267 & 11.00425 & 13.96598 \\ 38.07528 & 48.53687 & 138.4661 & 216.5303 \\ 260.1156 & 375.6751 & 1895.132 & 3779.024 \end{bmatrix}$$

<sup>2</sup> These weights are just the first line of matrix  $m$ .

**Table 15.3**  
**Moments of order 0 of bonds  $a$ ,  $b$ ,  $c$ , and  $d$**

(Maturity) <sup>0</sup> $t^0$	Initial spot rate structure $R(t)$	Share of discounted cash flows in each bond's value			
		$c_t^a e^{-R(t)t} / B_0^a$	$c_t^b e^{-R(t)t} / B_0^b$	$c_t^c e^{-R(t)t} / B_0^c$	$c_t^d e^{-R(t)t} / B_0^d$
1	0.042378	0.04159	0.04771	0.060402	0.062066
1	0.044558	0.039691	0.045531	0.057644	0.059232
1	0.046549	0.037735	0.043288	0.054804	0.056313
1	0.04836	0.035759	0.04102	0.051933	0.053364
1	0.05	0.033793	0.038765	0.049078	0.050429
1	0.051477	0.031861	0.036549	0.046272	0.047547
1	0.0528	0.77957	0.034395	0.043545	0.044745
1	0.053978		0.712743	0.040918	0.042045
1	0.05502			0.038406	0.039464
1	0.055934			0.03602	0.037012
1	0.05673			0.033763	0.034693
1	0.057415			0.03164	0.032511
1	0.058			0.029648	0.030465
1	0.058492			0.027785	0.028551
1	0.058901			0.398141	0.026764
1	0.059235				0.025099
1	0.059503				0.023547
1	0.059714				0.022103
1	0.059877				0.020758
1	0.06				0.263292
Moments of order 0 (pure numbers)		1	1	1	1

$\mathbf{h} \equiv$  the (column) vector of horizon (or duration) to the powers 0, 1, 2, 3

$$\mathbf{h} = (0, 7, 49, 343)$$

System (18) can now be written

$$\mathbf{nm} = \mathbf{h} \quad (19)$$

which can easily solved as

$$\mathbf{n} = \mathbf{m}^{-1}\mathbf{h}$$

where matrix  $\mathbf{m}^{-1}$  is

$$\mathbf{m}^{-1} = \begin{bmatrix} 43.62181 & -11.4693 & 0.818443 & -0.01877 \\ -48.7515 & 13.49682 & -1.00556 & 0.023675 \\ 10.24403 & -3.30703 & 0.307824 & -0.00877 \\ -3.29338 & 1.106155 & -0.11074 & 0.003599 \end{bmatrix}$$

**Table 15.4**  
**Moments of order 1 (Fisher-Weil durations) of bonds  $a$ ,  $b$ ,  $c$ , and  $d$**

Maturity $t$	Initial spot rate structure $R(t)$	Maturity times shares of discounted cash flows in each bond's value			
		$tc_t^a e^{-R(t)t} / B_0^a$	$tc_t^b e^{-R(t)t} / B_0^b$	$tc_t^c e^{-R(t)t} / B_0^c$	$tc_t^d e^{-R(t)t} / B_0^d$
0	0.04				
1	0.042378	0.04159	0.04771	0.060402	0.062066
2	0.044558	0.079382	0.091062	0.115288	0.118464
3	0.046549	0.113206	0.129863	0.164411	0.168939
4	0.04836	0.143036	0.164082	0.207733	0.213455
5	0.05	0.168964	0.193824	0.245388	0.252147
6	0.051477	0.191166	0.219293	0.277633	0.28528
7	0.0528	5.456991	0.240765	0.304817	0.313213
8	0.053978		5.70194	0.327347	0.336363
9	0.05502			0.345658	0.355179
10	0.055934			0.360196	0.370117
11	0.05673			0.371397	0.381627
12	0.057415			0.379679	0.390136
13	0.058			0.385427	0.396043
14	0.058492			0.388996	0.39971
15	0.058901			5.972108	0.40146
16	0.059235				0.401578
17	0.059503				0.400307
18	0.059714				0.397856
19	0.059877				0.394395
20	0.06				5.265842
Moments of order 1 (Fisher-Weil durations; in years)		6.194337	6.788539	9.90648	11.30418

The solution  $\mathbf{n}$  is then

$$n^a = -2.99784$$

$$n^b = 4.57423$$

$$n^c = -0.82821$$

$$n^d = 0.257722$$

Suppose now that at time  $t$ , immediately after the purchase of portfolio  $(n^a, n^b, n^c, n^d)$ , the initial spot structure that was steeply increasing turns abruptly clockwise to become horizontal, at level 5.5% per year (continuously compounded for every maturity from  $t = 0$  to  $t = 20$  years).

**Table 15.5**  
**Moments of order 2 of bonds  $a$ ,  $b$ ,  $c$ , and  $d$**

(Maturity) <sup>0</sup> $t^0$	Initial spot rate structure $R(t)$	(Maturity) <sup>2</sup> times shares of discounted cash flows in each bond's value			
		$t^2 c_t^a e^{-R(t)t} / B_0^a$	$t^2 c_t^b e^{-R(t)t} / B_0^b$	$t^2 c_t^c e^{-R(t)t} / B_0^c$	$t^2 c_t^d e^{-R(t)t} / B_0^d$
0					
1	0.042378	0.04159	0.04771	0.060402	0.062066
4	0.044558	0.158765	0.182124	0.230576	0.236927
9	0.046549	0.339618	0.389588	0.493232	0.506817
16	0.04836	0.572145	0.656327	0.830934	0.853821
25	0.05	0.844819	0.96912	1.22694	1.260735
36	0.051477	1.146998	1.31576	1.665799	1.711682
49	0.0528	38.19894	1.685357	2.133722	2.192493
64	0.053978		45.61552	2.618775	2.690906
81	0.05502			3.110921	3.196608
100	0.055934			3.601956	3.701169
121	0.05673			4.085368	4.197896
144	0.057415			4.556142	4.681637
169	0.058			5.010553	5.148564
196	0.058492			5.445938	5.595941
225	0.058901			89.58162	6.021902
256	0.059235				6.42524
289	0.059503				6.805219
324	0.059714				7.1614
361	0.059877				7.493497
400	0.06				105.3168
Moments of order 2 (in years <sup>2</sup> )		41.30287	50.86151	124.6529	175.2614

Let us call  $R^*(t)$  the new spot structure (see figure 15.1). The new values of each bond making up our portfolio are shown in table 15.7.

The total values of each bond and of the portfolio under the initial and the new structure are summarized in table 15.8. From \$100 at time 0, the portfolio's value diminishes almost instantly (at time 0) to \$98.63686. As to the value at horizon  $H = 7$  years of the portfolio under the initial structure, it was  $100e^{R(7)7} = 100e^{(0.0528)7} = 144.72$ . In order to judge the efficiency of our immunization strategy, let us calculate the portfolio's value at the same horizon. We obtain  $98.63686e^{R^*(7)7} = 98.63686e^{(0.055)7} = 144.96$ .

This good result prods us to submit our strategy to even more radical changes in the spot rate structure: from the initial change, let us make it turn clockwise to twelve horizontal structures, from 1% to 12%. The



**Table 15.6**  
**Moments of order 3 of bonds  $a$ ,  $b$ ,  $c$ , and  $d$**

(Maturity) <sup>0</sup> $t^0$	Initial spot rate structure $R(t)$	(Maturity) <sup>3</sup> times shares of discounted cash flows in each bond's value			
		$t^3 c_t^a e^{-R(t)t} / B_0^a$	$t^3 c_t^b e^{-R(t)t} / B_0^b$	$t^3 c_t^c e^{-R(t)t} / B_0^c$	$t^3 c_t^d e^{-R(t)t} / B_0^d$
0	0.04				
1	0.042378	0.04159	0.04771	0.060402	0.062066
8	0.044558	0.31753	0.364249	0.461152	0.473854
27	0.046549	1.018855	1.168763	1.479695	1.520452
64	0.04836	2.288582	2.625309	3.323734	3.415284
125	0.05	4.224093	4.845599	6.134701	6.303676
216	0.051477	6.881989	7.894561	9.994795	10.27009
343	0.0528	267.3926	11.7975	14.93605	15.34745
512	0.053978		364.9242	20.9502	21.52725
729	0.05502			27.99829	28.76947
1000	0.055934			36.01956	37.01169
1331	0.05673			44.93905	46.17685
1728	0.057415			54.67371	56.17965
2197	0.058			65.13719	66.93134
2744	0.058492			76.24313	78.34318
3375	0.058901			1343.724	90.32852
4096	0.059235				102.8038
4913	0.059503				115.6887
5832	0.059714				128.9052
6859	0.059877				142.3764
8000	0.06				2106.337
Moments of order 3 (in years <sup>3</sup> )		282.1652	393.6679	1706.076	3058.772

results are given in table 15.9. In each case the new portfolio's value at horizon  $H$  is greater than the portfolio's value under the initial spot rate curve.

Suppose now that the initial spot structure shifts in the following way: from very steep, it turns less steep, but all spot rates increase (except the last one). In other words, it turns around its end point, conserving some concavity. The new short rate ( $R(0)$ ) increases by 80 basis points, from 4% to 4.8%. Figure 15.2 describes this move. The new spot structure is now given by

$$\hat{R}(t) = 0.048 + 0.00146 - (5.5)10^{-5}t + (6)10^{-7}t^3 \quad (20)$$

In that case, performing the necessary calculations as we have done before, the immunization process gives us a seven-year horizon value of \$144.88 (instead of 144.72 before the spot structure change).

**Table 15.7**  
**New bond values at time  $\varepsilon$  (after spot structure shift) under new structure  $R^*(t)$**

Maturity	New term structure	Discounted cash flows			
		Bond $a$ $c_t^a e^{-R^*(t)t}$	Bond $b$ $c_t^b e^{-R^*(t)t}$	Bond $c$ $c_t^c e^{-R^*(t)t}$	Bond $d$ $c_t^d e^{-R^*(t)t}$
0					
1	0.055	3.785941	4.495804	6.625396	7.571881
2	0.055	3.583337	4.255212	6.270839	7.166673
3	0.055	3.391575	4.027495	5.935256	6.78315
4	0.055	3.210075	3.811964	5.617632	6.42015
5	0.055	3.038288	3.607968	5.317005	6.076577
6	0.055	2.875695	3.414888	5.032466	5.75139
7	0.055	70.76687	3.232141	4.763154	5.443605
8	0.055		67.46282	4.508255	5.152291
9	0.055			4.266996	4.876567
10	0.055			4.038649	4.615598
11	0.055			3.822521	4.368595
12	0.055			3.617959	4.134811
13	0.055			3.424345	3.913537
14	0.055			3.241091	3.704105
15	0.055			46.89114	3.50588
16	0.055				3.318263
17	0.055				3.140687
18	0.055				2.972614
19	0.055				2.813535
20	0.055				35.95008
New bond values at time		90.65178	94.30829	113.3727	127.68

Let us now consider the following change in the initial structure: it turns around its first point; it still increases, but becomes flatter, conserving some concavity. The end point (for  $t = 20$  years) decreases from 6% to 5.5% (by 50 basis points). Figure 15.3 describes this move. The new term structure is given by

$$R^{**}(t) = 0.04 + 0.001712t - (6.7)10^{-5}t^2 + (9.5)10^{-7}t^3 \quad (21)$$

and is pictured in figure 15.3. Under this new spot structure, the new seven-year horizon of our portfolio is \$144.80.

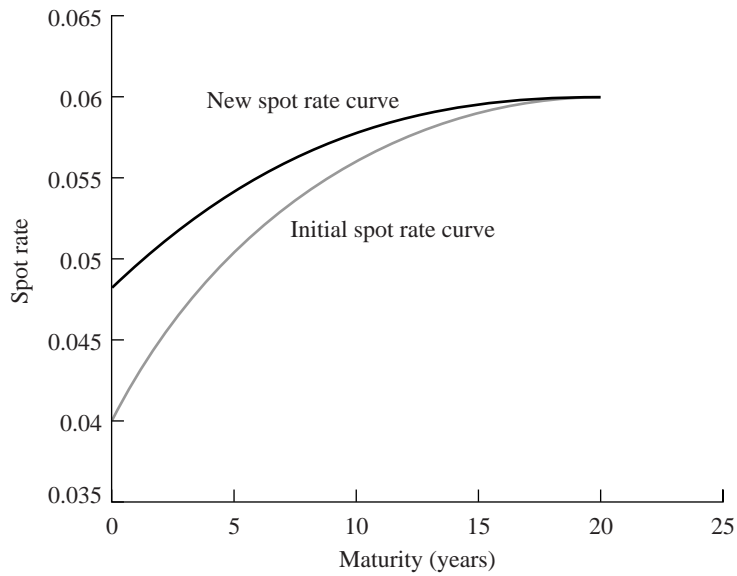
Let us try even more dramatic changes in the spot structure: we will suppose that the new structures become either  $\hat{R}(t)$  plus 50 basis points or  $R^{**}$  minus 50 points. Those new structures are pictured in figures 15.4 and 15.5, respectively. In each case, the new seven-year horizon portfolio

**Table 15.8****Value of bond portfolio under initial and new spot rate structure, at times 0,  $\varepsilon$ , and  $H$** 

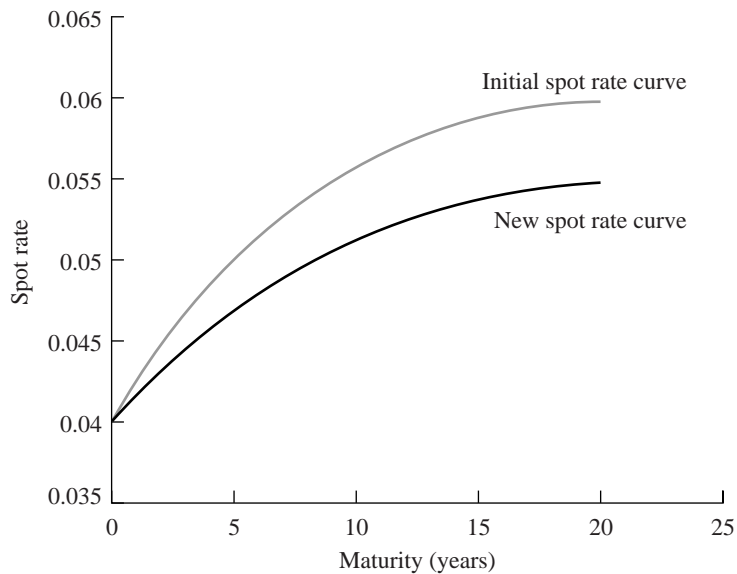
	$n_a$	$n_b$	$n_c$	$n_d$
Optimal number of bonds	-2.99784	4.57423	-0.82821	0.257722
Value of each bond at time 0, under initial structure	92.18556	95.42947	111.0813	123.5471
Value of each bond's share in portfolio	-276.358	436.5163	-91.999	31.8408
<b>Total = 100</b>				
Value of each bond at time $\varepsilon$ , under new structure	90.65178	94.30829	113.3727	127.68
Value of each bond's share in portfolio	-271.76	431.3878	-93.8969	32.90593
<b>Total = 98.63686</b>				
Value of bond portfolio at horizon $H = 7$ years				
(a) under initial term structure: $100e^{(0.0528)7} = 144.72$				
(b) under new term structure: $98.63686e^{(0.055)7} = 144.96$				

**Table 15.9****Horizon  $H = 7$  years portfolio value under new spot rate structures.****Initial portfolio value before spot structure change: \$144.7156**

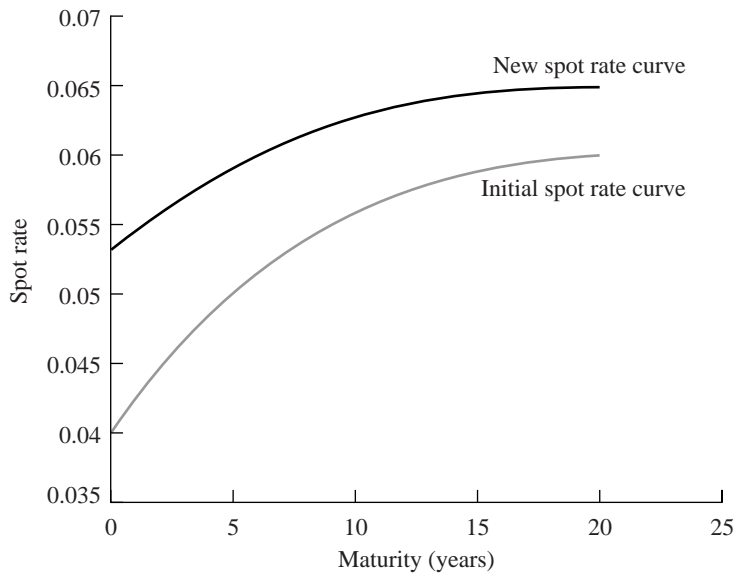
New spot rate structure	New bond portfolio's value at horizon $H = 7$ years (in \$)
0.01	144.7241
0.02	144.7448
0.03	144.791
0.04	144.8524
0.05	145.9218
0.06	145.9952
0.07	145.0708
0.08	145.149
0.09	145.2321
0.10	145.3237
0.11	145.4287
0.12	145.5533



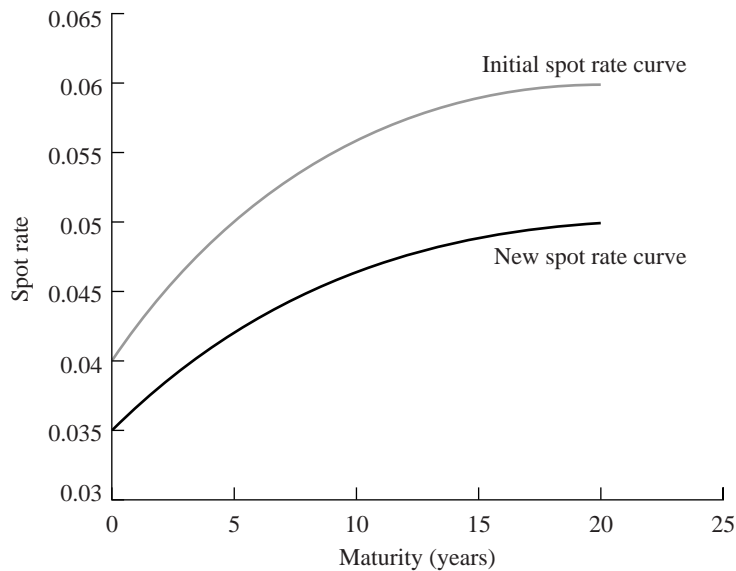
**Figure 15.2**  
Rotation of initial spot rate around its terminal point.



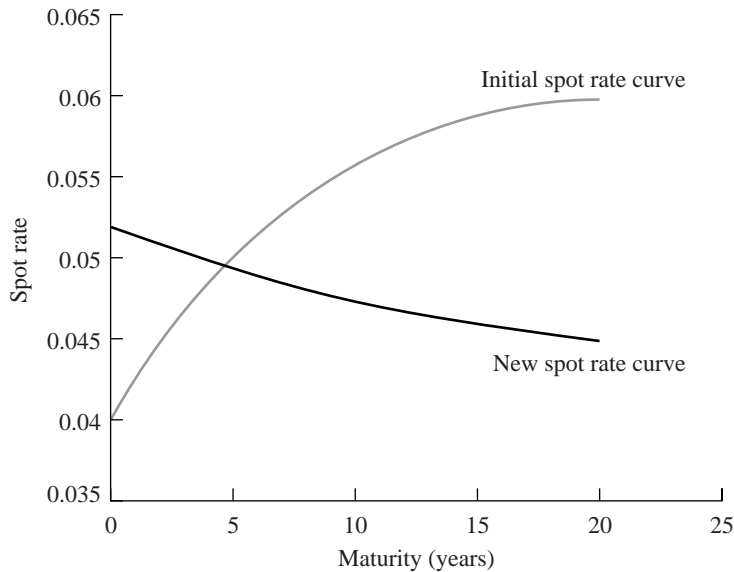
**Figure 15.3**  
Rotation of spot rate curve around its initial point.



**Figure 15.4**  
Nonparallel increase of spot rate curve.



**Figure 15.5**  
Nonparallel decrease of spot rate curve.



**Figure 15.6**  
Change from an increasing spot curve to a decreasing one.

value becomes \$144.89 and \$144.79, respectively. Once more the investor is well protected.

At this point, the reader may wonder whether we will dare to make the bold move to go from a steeply increasing curve to a *decreasing* one. Before proceeding, we looked for reassurance from Launce's dog. Since he was not only wagging his tail but also saying "Yes, sir!" we moved ahead. The new spot structure, denoted  $R^-(t)$ , is

$$R^-(t) = 0.052 - 0.0004t - (1.7)10^{-6}t^2 + (2)10^{-7}t^3 \quad (22)$$

This time, the short rate moves *up* by 120 basis points, and the 20-year rate moves *down* by 150 points! (See figure 15.6.) The new seven-year horizon value in this case is 144.86. Even Launce might be surprised that in each case we get even *more* than a match.

#### 15.4 Epilogue: a new approach to the calculus of variations

We may wonder whether the method we proposed in this chapter does not open new perspectives on the calculus of variations. Those who have been

studying even briefly this beautiful part of calculus know that it is only in very simple cases that variational problems possess analytical solutions. The difficulty stems from the usual complexity of the differential equations resulting from the Euler-Lagrange equation or from its extensions (the Euler-Poisson equation and the Ostrogradski equation).

One way to solve such problems might be to suppose that the optimal function we are looking for can be developed in a Taylor series. The integral representing the functional (such as  $\int_a^b F(x, y, y', \dots, y^{(n)}) dx$  for instance) would become a *function* of our  $m + 1$  variables  $A_0, \dots, A_m$ . This function would not be hard to determine, thanks especially to the integration software that is available today. Finding the optimal vector  $(A_0^*, \dots, A_m^*)$  would then be an easy task.

### Main formulas

**Definition.** *The moment of order  $k$  of a bond portfolio as the weighted average of the  $k$ th power of its times of payment, the weights being the shares of the portfolio's cash flows in the initial portfolio value.*

Such a moment can therefore be written as

$$\sum_{t=1}^N t^k c_t e^{-\int_0^t f(u) du} / B_0 = \sum_{t=1}^N t^k c_t e^{-R(t)t} / B_0 = \sum_{t=1}^N t^k c_t e^{-G(t)} / B_0$$

**Theorem.** Suppose that the continuously compounded rate of return over a period  $t$ ,  $G(t) = R(t) \cdot t = \int_0^t f(u) du$  is developable in a Taylor series  $\sum_{j=0}^m A_j t^j + \frac{1}{(m+1)!} G^{(m+1)}(\theta t) t^{m+1}$ ,  $0 < \theta < 1$ , where  $A_j = (1/j!) G^{(j)}(0)$ ,  $j = 0, 1, \dots, m$ . Let  $B_H$  designate the horizon- $H$  bond portfolio's value after a change in the spot rate structure. This change intervenes immediately after the portfolio is bought. For an investor with horizon  $H$  to be immunized against any such change in the spot interest structure, a sufficient condition is that at the time of the purchase

- (a) any moment of order  $j$  ( $j = 0, 1, \dots, m$ ) of the bond portfolio is equal to  $H^j$ ,
- (b) the Hessian matrix of the second partial derivatives ( ${}^2B_H / A_j^2$ ) is positive-definite.

This implies

$$H^j = \sum_{t=1}^N t^j c_t e^{-(A_0 + A_1 t + \dots + A_j t^j + \dots + A_m t^m)} / B_0, \quad j = 0, 1, \dots, m \quad (15)$$

Let

$\mathbf{n} \equiv$  the (row) vector of the optimal *amounts* of each bond in the immunizing portfolio

$\mathbf{m} \equiv$  the square matrix of terms  $(1/B_0)B_0^l m_k^l$ ,  $k = 0, 1, 2, 3$   
 $l = a, b, c, d$

$\mathbf{h} \equiv$  the (column) vector of horizon  $H$  to the powers  $0, 1, \dots, m-1$  if there are  $m$  bonds in the optimal portfolio

$\mathbf{n}$  is then the solution of

$$\mathbf{n}\mathbf{m} = \mathbf{h}$$

or

$$\mathbf{n} = \mathbf{m}^{-1}\mathbf{h}$$

## Questions

These exercises could be considered as a project.

Consider the initial spot rate curve as given by  $R(t)$  in this chapter. Suppose that the spot curve is initially increasing, then takes an even more dramatic counterclockwise movement than the one we considered. The spot rate, represented by a third-order polynomial, decreases from 6% for  $t = 0$  to 4% for  $t = 20$ . Furthermore, its curve goes through points (5; 0.051) and (13; 0.043).

15.1. Determine the coefficients of the third-order polynomial  $R^*(t)$  corresponding to the new spot structure.

15.2. Determine the optimal portfolio immunizing the investor under the initial spot structure.

15.3. What is the seven-year horizon portfolio value corresponding to the new structure? How does it compare to the same horizon portfolio value under the initial structure?



## 16

ARCITE.                                Let the event,  
That never erring arbitrator, tell us  
When we knew all ourselves, and let us follow  
The backing of our chance.

*The Two Noble Kinsmen*

### *The Two Noble Kinsmen*

## Chapter outline

<b>16.1</b>	<b>The right price in a one-step model: a geometrical approach</b>	<b>328</b>
<b>16.2</b>	<b>Arbitrage at work for you</b>	<b>331</b>
16.2.1	The case of an undervalued derivative ( $P_0 < V_0$ )	331
16.2.2	The case of an overvalued derivative ( $P_0 > V_0$ )	333
<b>16.3</b>	<b>Some important properties of the derivative's value</b>	<b>336</b>
<b>16.4</b>	<b>Two fundamental properties of the <math>Q</math>-probability measure</b>	<b>338</b>
<b>16.5</b>	<b>A two-period evaluation model and extensions</b>	<b>339</b>
<b>16.6</b>	<b>The concept of probability measure change</b>	<b>344</b>
<b>16.7</b>	<b>Extension of arbitrage pricing to continuous-time processes</b>	<b>347</b>
16.7.1	A statement of the Cameron-Martin-Girsanov theorem	347
16.7.2	An intuitive proof of the Cameron-Martin-Girsanov theorem	348
<b>16.8</b>	<b>An application: the Baxter-Rennie martingale evaluation of a European call</b>	<b>352</b>

## Overview

The law of one price tells us that in a market without frictions a given commodity cannot have various prices at one time, because otherwise arbitrageurs would step in and reap the absolute value of the difference between the arbitrage price and any other price. Our purpose in this chapter is to show how a derivative's arbitrage price can easily be established geometrically in the so-called binomial configuration.

We suppose that at time 0 a random variable (the price of a stock, for instance) is known; call it  $S_0$ . At time  $\Delta t$ , there are two states of the nature: an up-state and a down-state, where the stock can take values  $S_u$  and  $S_d$ , respectively. We suppose that at that time a derivative value corresponds to each of those values, denoted  $f(S_u)$  and  $f(S_d)$ . There are probabilities  $p$  and  $1 - p$  that the down- and up-states are reached, respectively. We suppose also that a bond with maturity  $\Delta t$  and par

$B_{\Delta t} = 1$  can be either bought or sold at price  $B_0 = e^{-i\Delta t}$  at time 0. (We can verify that the implicit, continuously compounded interest over that time interval is  $i(0, \Delta t) = \Delta t^{-1} \log(B_{\Delta t}/B_0) = -(\log B_0)\Delta t^{-1} = i$ .<sup>1</sup>) Our goal is now to determine the arbitrage value of the derivative at time 0.

The approach usually taken here is to surmise that building a portfolio of a number  $a$  of stocks and a number  $b$  of bonds can replicate at time  $\Delta t$  the values the derivative can take. Before rushing to the corresponding system of two equations in two unknowns, however, let us consider a geometric approach. Not only will we be able to get the derivative's value without any calculations, but we will be in a position to understand easily all comparative statics<sup>2</sup> of our results. There is an additional advantage in the geometric approach: since all the results become natural, they are easy to remember.

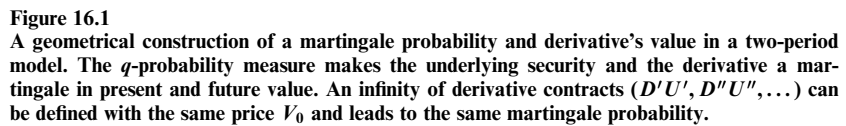
### 16.1 The right price in a one-step model: a geometrical approach

Figure 16.1 summarizes our situation at time  $\Delta t$ . Today,  $S_0 = 5$ . After  $\Delta t$ , we can be either at point  $U$  or  $D$ . (To each value  $S_u = 8$  or  $S_d = 4$  corresponds the derivative's value  $f(S_u) = 9$  or  $f(S_d) = 3$ , respectively.) Let us consider how we could form a portfolio such that its value at time  $\Delta t$  would be exactly  $f(S_u)$  or  $f(S_d)$  if we are in the up- or the down-state.

Consider first a portfolio made up of  $a$  units of the stock only;  $a$  can be positive or negative, an integer or a fraction. We understand of course that we could then replicate *all* derivatives of the stock that would be located on the ray from the origin  $f(S) = aS$ . It turns out that we have chosen our points  $U$  and  $D$  definitely not on such a ray from the origin. However, we notice that they are on a *displaced* ray from the origin. Hence, the portfolio's value at  $\Delta t$  must be of the form  $f(S) = aS + \beta$ , where  $\beta$  is the ordinate at the origin of the line  $UD$ , equal to the amount that must come in addition to  $aS$  to replicate both  $U$  and  $D$ . Therefore, if  $b$  denotes the number of bonds held at time 1 and if  $B_1$  is the value of each bond at time 1, then  $\beta = bB_1$ . The coefficient  $a$  is simply the

<sup>1</sup> Notice that we can always go from prices to yields, or vice versa. Here, if  $i = \Delta t^{-1} \log(B_{\Delta t}/B_0)$ , with  $B_{\Delta t} = 1$  this last equality implies  $i = \Delta t^{-1}(-\log B_0)$ ; therefore,  $\log B_0 = -i\Delta t$  and  $B_0 = e^{-i\Delta t}$ .

<sup>2</sup> "Comparative statics" is an expression widely used in economics, defining the analysis of changes in equilibrium values entailed by changes in parameters of the underlying model.



ratio  $[f(S_u) - f(S_d)]/(S_u - S_d)$ ; to determine  $b$ , we know that it must be such that  $f(S_u) = aS_u + bB_1$  or  $f(S_d) = aS_d + bB_1$ . Hence,  $b = B_1^{-1}[f(S_u) - aS_u]$ , or  $B_1^{-1}[f(S_d) - aS_d]$  and our results: at time 0, form a portfolio of a number  $a$  of stocks, and *lend* an amount  $bB_0 = be^{-i\Delta t}$ , which will become  $\beta = bB_1 = b$  a period  $\Delta t$  later. You could, presumably, make that loan directly, but you could also proceed as follows. Remember that there are on the market zero-coupon bonds that pay \$1 at maturity  $\Delta t$ ; each of them is worth  $e^{-i\Delta t}$  today. Therefore, you should buy a number  $b$  of those at time 0; at time  $\Delta t$ , you will sell them at par for  $\beta = bB_1 = b$ . At time 0, your portfolio is therefore  $aS_0 + bB_0$ , and this is the value of the derivative at time 0, which we may call  $V_0$ . We have

$$V_0 = aS_0 + bB_0 \quad (1)$$

where  $a$ , the number of stocks in the replicating strategy, is

$$a = \frac{f(S_u) - f(S_d)}{S_u - S_d} \quad (2)$$

and  $b$ , the number of bonds in the same strategy, is

$$b = B_1^{-1}[f(S_u) - aS_u] = B_1^{-1}[f(S_d) - aS_d] \quad (3)$$

Innocuous as equations (2) and (3) may seem, they are of great importance; they mean that *whatever* state of nature at  $\Delta t$ , there is *one and only one* value for  $a$  and  $b$  that guarantees the portfolio to replicate the derivative. It means that a *unique* set of values  $a$  and  $b$ , *known at time 0*, corresponds to both states of nature at  $\Delta t$ . Notice that the number of bonds to be purchased,  $b$ , can be expressed solely as a function of variables that are *known at time 0*:

$$b = B_0^{-1}(V_0 - aS_0) \quad (3a)$$

In our example, suppose  $B_0 = 0.9$ . (This implies that if  $\Delta t$  is one year, the continuously compounded rate of return—or rate of interest—over that period is  $-\log 0.9 = 10.54\%$ ; equivalently, the implied rate of interest compounded once a year would be  $11.11\%$ .) Suppose as before that  $S_0 = 5$ ;  $S_u = 8$ ;  $S_d = 4$ ;  $f(S_u) = 9$ ;  $f(S_d) = 3$ . We should then buy  $a = 1.5$  shares and  $b = 9 - (1.5)8 = 3 - (1.5)4 = -3$  bonds; in other words, we should borrow the worth of 3 bonds. The cost of the portfolio is thus  $(1.5)5 - 3(0.9) = 4.8$ , and that is  $V_0$ , the price of our derivative. At time  $\Delta t$ , in the up-state, the portfolio is worth  $(1.5)8 - 3 = 9$ ; in the

down-state, it is worth  $(1.5)4 - 3 = 3$ , that is, it replicates exactly the derivative's possible values at time  $\Delta t$ .

Of course, we could also say that (2) and (3) are the solutions of the system

$$aS_u + bB_1 = f(S_u) \quad (4)$$

$$aS_d + bB_1 = f(S_d) \quad (5)$$

but the step of solving the system in  $(a, b)$  can be skipped altogether because the result is obvious in our geometrical approach. Before discussing important properties of results (1) to (3), we have to make sure that  $V_0$  is indeed an arbitrage price.

## 16.2 Arbitrage at work for you

Let us now verify that this price is arbitrage determined; in other words, any other price  $P_0 \neq V_0$  would enable arbitrageurs (like you, of course) to step in and profit from that situation.

### 16.2.1 The case of an undervalued derivative ( $P_0 < V_0$ )

An individual A may very well make the following (erroneous) reasoning about the value of the derivative at time 0. Suppose that the probabilities of  $S_d$  and  $S_u$  are  $p = 0.95$  and  $1 - p = 0.05$  and that everybody, including A and yourself, agrees that those are the true probabilities corresponding to the event  $S(\Delta t)$ . So the expected value of the derivative at  $\Delta t$  is  $E[f(S)] = pf(S_d) + (1 - p)f(S_u) = 0.95(3) + 0.05(9) = 3.30$ . (Notice also that the expected value of the stock is  $0.95(4) + 0.05(8) = 4.2$ ; it is expected to *fall* from 5 by 16%.)

Individual A continues his reasoning as follows: the expected value of the derivative at maturity is, in present value,  $3.30(0.9) = 2.97$ . From his old business school days, A remembers that uncertain cash flows should be discounted with a risk-free interest rate to which an appropriate risk premium should be added. More eager to make some quick money than to flip through the thousand pages of his old investments text, A decides to inflict a hefty risk premium to that kind of investment (25%). The discount rate jumps to  $11.11\% + 25\% = 36.11\%$ , and A is then confident that the true value of the derivative at time 0 is  $3.30/1.3611 = 2.42$ . (If you are interested in making a classroom experiment about the partic-

ipants' evaluations of the derivative at  $t = 0$ , this is typically the order of magnitude you will get. We did so with 21 professionals and received an average estimate of 2.2 for the derivative's value.)

So A thinks the derivative's value is 2.42. You come along and offer to buy this contract for 4, a price way above his estimate and even well above the expected value of the derivative's payoffs *in future value* (3.30). So A eagerly accepts. He now has the obligation of paying you 3 with probability 95%, and 9 with probability 5%, random outcomes with an expected value of 3.30. For his effort, he receives 4 today, which he can of course transform into  $4(1.11) = 4.44$  in one year with certainty. Of course, he has difficulty containing his satisfaction at having met such a gullible counterpart.

A has, however, committed two fatal mistakes: first, he has fallen into the trap of considering variables that turn out to be useless (the probabilities of the up- and down-states); second, he has totally neglected a variable of central importance: today's stock value  $S_0$ . Here now is how you will proceed to gain 0.8 immediately—the difference between the true value of the derivative (4.8) and what you have paid (4)—with all your future obligations being faithfully met.

Pay A the sum of 4 for entering the contract; you are then entitled to the payoffs of the contract at its maturity. On the other hand, at time 0 you short the portfolio whose value is  $V_0$ . (Remember it is made of  $a$  stocks, and *borrowing* a future value of  $b$ .) So short-selling the portfolio means that you take a negative position with respect to the portfolio: you short-sell  $a$  shares of the stock and you *lend* a future value of  $b$ . This amounts to receiving  $aS_0$  and paying  $bB_0$  (the present value of one bond times  $b$  bonds with one-par value). In our example, you receive  $(1.5)5 - 3(0.9) = 4.8$ . Your net cash flow is  $4.8 - 4 = 0.8$  at time 0.

Consider now what happens at time  $\Delta t$ . There are two states to consider. We will show that in either of those states you can exactly meet your obligations thanks to the contract you signed with A and to the lending you did.

- In the up-state, you receive a payoff from the derivative equal to 9; you also receive the par of the bonds you have bought, which takes you to +12. Now you have to buy back the stocks you borrowed and remit them to their owners. This costs you exactly  $(1.5)8 = 12$ , and you are therefore just even.

- In the down-state, your payoff from the contract is +3 only, to which you can add +3 from the bonds that come to maturity. This gives you +6. You have to buy and remit the stocks, which costs you in the down-state  $(1.5)4 = 6$ . So your net payoff is exactly 0, as in the up-state. Meanwhile, of course, you have made at time 0 a certain net profit of 0.8 (which you can transform into 0.888 one year later). Table 16.1 summarizes these operations and their related cash flows.

What will happen, of course, is that A will soon realize his mistake: you make a *sure* profit of 0.8. His evaluation of the contract was erroneous: he could have sold it to you for nearly 20% more than the price he received (4.8 instead of 4). He will thus discover that, far from being overvalued, the price he had agreed on was grossly undervalued. The price will thus be bid up until it reaches at time 0 the arbitrage price  $V_0 = aS_0 + bB_0 = 4.8$ .

### 16.2.2 The case of an overvalued derivative ( $P_0 > V_0$ )

We now have to show that if the price of the derivative is *overvalued*, arbitrage will force it down to its  $V_0$  level. Suppose that the  $p$  probabilities are inverted and that there is a 95% probability that up-state happens and a 5% probability for the down-state. If B, like anybody else, takes the expected value route, he will determine that in probability the future value of the derivative is  $(0.95)9 + (0.05)3 = 8.7$ . Not willing to take undue risks, he also applies the hefty discount rate of 36.11% and gets a “fair value” of  $8.7/1.3611 = 6.39$ . You come along and offer to sell him this contract for an amount of 5.5; B is thus offered to buy a contract for an amount that is way below his estimate of the true value of the contract, and of course he accepts.

Let us now see what you will do. You know that the derivative is *overvalued* (5.5 against an arbitrage value of 4.8). So the portfolio reflecting the derivative is undervalued: you will then buy it. You buy  $a$  units of the stock and  $b$  bonds, for a cost  $V_0 = aS_0 + bB_0 = 4.8$ . You sell the contract for 5.5 and profit 0.7.

At maturity, consider first the up-state. You have to pay 9 to B and reimburse 3, for a total of 12. On the other hand, your stocks are worth  $(1.5)8 = 12$ . It clicks again. In the down-state, you have to pay 3 to B and reimburse 3; but you have  $(1.5)4 = 6$  worth of stocks to cover exactly those costs. Table 16.2 summarizes these operations and the corresponding cash flows. So you can again profit from a price not driven by arbi-

**Table 16.1**  
Arbitrage resulting from an undervalued derivative ( $P_0 < V_0$ )

Time 0		Time $\Delta t$	
Operation	Cash flow	Operation	Cash flow
Buy derivative at $P_0 = 4$ Short-sell portfolio of $a$ shares and $b$ bonds $aS_0 + bB_0 = (1.5)5 - 3(0.9) = 4.8$	$-P_0 = -4$ $+V_0 = aS_0 + bB_0 = 4.8$	<b>1. In up-state:</b> a. Receive payoff of derivative: $f(S_u) = +9$ b. Receive par for $-b$ bonds ( $b < 0$ ): $-b = +3$ c. Buy and remit $a$ shares for $aS_u = (1.5)8 = 12$	$f(S_u) = +9$ $-b = +3$ $-aS_u = -12$
		Net cash flow	$f(S_u) - b - aS_u = 0$
		<b>2. In down-state:</b> a. Receive payoff of derivative: $f(S_d) = +3$ b. Receive par for $-b$ bonds: $-b = +3$ c. Buy and remit $a$ shares for $aS_d(1.5)4 = 6$	$f(S_d) = +3$ $-b = +3$ $-aS_d = -6$
Net cash flow	$V_0 - P_0 = 0.8$	Net cash flow	$f(S_d) - b - aS_d = 0$



**Table 16.2**  
Arbitrage resulting from overvalued derivative ( $P_0 > V_0$ )

Time 0		Time $\Delta t$	
Operation	Cash flow	Operation	Cash flow
Sell derivative at $P_0 = 5.5$ Buy portfolio of $a$ shares and $b$ bonds (borrow the present value of $b$ bonds); pay $aS_0 + bB_0$	$+P_0 = +5.5$ $-aS_0 - bB_0 = -4.8$	<b>1. In up-state:</b>	
		a. Pay out derivative's liability: $f(S_u) = 9$	$-f(S_u) = -9$
		b. Pay back loan: $-b = 3$	$b = -3$
		c. Sell $a$ shares for $aS_u = (1.5)8 = 12$	$+aS_u = +12$
		Net cash flow	$aS_u - f(S_u) + b = 0$
Net cash flow	$P_0 - V_0 = 0.8$	<b>2. In down-state:</b>	
		a. Pay out derivative's liability: $f(S_d) = 3$	$-f(S_d) = -3$
		b. Reimburse loan: $-b = 3$	$b = -3$
		c. Sell shares for $aS_d = (1.5)4 = 6$	$+aS_d = +6$
		Net cash flow	$aS_d - f(S_d) + b = 0$

trage considerations. The market will certainly take notice and bid down the price of that contract until it reaches a level that now may be duly called an arbitrage equilibrium level:  $V_0 = aS_0 + bB_0$  (4.8 in our case).

### 16.3 Some important properties of the derivative's value

Making use of equations (1), (2), and (3), the arbitrage portfolio's (or the derivative's) value turns out to be

$$V_0 = \frac{f(S_u) - f(S_d)}{S_u - S_d} S_0 + B_1^{-1} \left[ f(S_u) - S_u \frac{f(S_u) - f(S_d)}{S_u - S_d} \right] B_0 \quad (6)$$

Since  $B_1 = B_0 e^{i\Delta t}$ , it is also equal to

$$V_0 = \frac{f(S_u) - f(S_d)}{S_u - S_d} S_0 + e^{-i\Delta t} \left[ f(S_u) - S_u \frac{f(S_u) - f(S_d)}{S_u - S_d} \right] \quad (6a)$$

Notice that this expression makes a lot of sense: the derivative's value is the present value of the stocks in the portfolio, plus the present value of the additional amount you need at maturity to replicate the derivative's payoffs (that is, the present value of the bonds you need to buy or sell).

Let us rewrite  $V_0$  by factoring out  $e^{-i\Delta t}$ :

$$V_0 = e^{-i\Delta t} \left\{ f(S_u) \left[ \frac{e^{i\Delta t} S_0 - S_d}{S_u - S_d} \right] + f(S_d) \left[ \frac{S_u - e^{i\Delta t} S_0}{S_u - S_d} \right] \right\} \quad (7)$$

Hence  $V_0$  is a weighted average of the derivative's payoffs in present value. The weights have no relationship to the initial  $p$  and  $1 - p$  probabilities; any such relationship would be mere coincidence. However, these weights can be considered probabilities in their own right and called  $q$  and  $1 - q$  respectively. (In our example,  $q = 0.38$  and  $1 - q = 0.61$ .)

To ensure that the  $q$ 's qualify as probabilities, we need only the condition  $S_d < e^{i\Delta t} S_0 < S_u$ , which results from arbitrage considerations. Suppose that the first inequality is not met and that  $S_d > e^{i\Delta t} S_0$ . You would borrow  $S_0$  and buy the stock;  $\Delta t$  later, you would end up with at least  $S_d$ , which is larger than the sum you have to reimburse. On the other hand, suppose that  $S_0 e^{i\Delta t} > S_u$ . The initial stock's value this time is definitely

overvalued: you could short-sell the stock, earn  $S_0 e^{i\Delta t}$  on the risk-free interest market, and buy back the security at a lower price, thus earning unlimited arbitrage profits as before.

These probabilities are made up solely of the different values of the stock ( $S_0$ ,  $S_u$ , and  $S_d$ ) and of the bond's present value. Alternatively, we could say that besides the stock's values, the sole determinant value of the  $q$  is the rate of interest for the time span between time 0 and time  $\Delta t$ .

We thus have an important first result: the derivative's price is the present value of the expected value of the derivative's payoffs *under a probability measure  $q$*  that is independent of the actual probabilities and dependent on all the values, actual and potential, of the stock, as well as the present value of the bond.

Note that the result here of the derivative's value as a weighted average is perfectly general; it has a nice geometric interpretation. In figure 16.1, denote on the horizontal axis the future value of the stock  $S_0 e^{i\Delta t}$ ; it necessarily lies between  $S_d$  and  $S_u$  as we have just shown. We can use the  $(S_d, S_u)$  segment to form an axis  $(0, 1)$  measuring probability  $q = (e^{i\Delta t} S_0 - S_d) / (S_u - S_d)$ , as shown in figure 16.1. This probability  $q$  can be used to determine point  $Q$  on the  $DU$  line. The height of  $Q$  is thus the weighted average of  $f(S_u)$  and  $f(S_d)$  with weights  $q$  and  $1 - q$ , respectively, or  $qf(S_u) + (1 - q)f(S_d)$ . Indeed, that height is equal to  $f(S_d) + [\text{slope of } DU \text{ in } (f(S), q) \text{ space}] \cdot q = f(S_d) + [f(S_u) - f(S_d)]q = qf(S_u) + (1 - q)f(S_d)$ . Now in the left part of the diagram, express the  $\Delta t$  values in present value through the inverse of the linear transformation  $e^{i\Delta t} V$ . You thus get  $V_0 = e^{-i\Delta t} [qf(S_u) + (1 - q)f(S_d)]$ , which is the derivative's value at time 0.

From the diagram we can quickly deduce important properties, which make a lot of sense.

First, we can immediately see that in order to construct a portfolio equivalent to the derivative's value, you will always

- buy shares when  $f(S_u) > f(S_d)$ ;
- short-sell shares when  $f(S_u) < f(S_d)$ ; and
- take no position in shares if  $f(S_u) = f(S_d)$ ; in that case your derivative reduces to a bond whose fair value is  $B_0 f(S_u) = B_0 f(S_d)$ .

In order to know whether you will borrow (as in our first example), consider all positions of the payoff line  $UD$  such that it intersects the ordinate below the origin. For a positive slope of line, for instance, bor-

rowing will always be the rule if the payoffs are such that  $\frac{f(S_u)}{S_u} > \frac{f(S_d)}{S_d}$ . On the contrary, lending will be the rule if the equality is reversed; there will be no borrowing or lending, as we saw earlier, if the derivative's payoffs are linearly related to the underlying security (if  $f(S_u)/S_u = f(S_d)/S_d$ , as we noted earlier).

It is interesting to realize how, for a given set of  $i$ ,  $S_0$ ,  $S_u$ , and  $S_d$ , we can define an infinity of derivatives that have exactly the same arbitrage value (this results from (7)), and we can construct those geometrically. Pivot the straight line around point  $Q$ . You get an infinity of lines with positive, negative, and zero slopes, intersecting verticals with abscissas  $S_d$  and  $S_u$  at points marked  $(D', U')$ ,  $(D'', U'')$ , and so on, which are the payoffs for the new derivatives whose unique value remains fixed at  $V_0$ . Note that it may very well be the case that one of the payoffs is negative; in our diagram,  $(D'', U'')$  corresponds to such a case.

#### 16.4 Two fundamental properties of the $Q$ -probability measure<sup>3</sup>

We should now stress two central properties of the  $q$ -probabilities that have been defined here.

1. The  $q$ -probabilities have been defined in such a way as to make the *stock's process* a martingale, when that process is expressed *either* in present value or in future value. This is easy to see. Indeed, with  $q = \frac{S_0 e^{i\Delta t} - S_d}{S_u - S_d}$ , we have

$$S_0 = E_Q(S_{\Delta t} e^{-i\Delta t}) = q S_u e^{-i\Delta t} + (1 - q) S_d e^{-i\Delta t} \quad (8)$$

and the second part of our statement comes from multiplying both sides of (8) by  $e^{i\Delta t}$ .

2. The  $q$ -probabilities make the *derivative's payoff* process, measured in terms of *either* time 0 *or* time  $\Delta t$ , a martingale. Indeed, consider (7):  $V_0$ , the present value of the derivative, is the  $Q$ -measure expected value of the derivative's payoffs in present value. Also, multiplying (7) by  $e^{i\Delta t}$ , we find that the derivative's value today, expressed in future value, is the  $Q$ -measure expected value of the derivative's payoffs.

<sup>3</sup> A simple way to define a probability measure is the following: it is the collection of probabilities on the set of all possible outcomes (see M. Baxter and A. Rennie, *An Introduction to Derivative Pricing* (Cambridge, U.K.: Cambridge University Press, 1998), p. 223).

We thus have a preview of a fundamental result, which still needs to be proven for multiple period processes or continuous processes: in order to determine a derivative's arbitrage-free value, determine a  $Q$ -probability measure that makes the underlying asset process a martingale<sup>4</sup> in present value (or any period's value for that matter) and determine today's derivative value as the expected value of its payoffs in present value, under that  $Q$ -measure. In other words, the key issue in evaluating a derivative will be to transform random variables under a  $P$ -probability measure into random variables under a  $Q$ -probability measure.

## 16.5 A two-period evaluation model and extensions

What we have shown in a one-period model can be extended in a straightforward way to two-period models and then further to  $n$ -period models. This is what we will do in this section. We will first develop a two-period model, keeping the geometric approach we developed in the one-period model. The extension to  $n$ -period models will then come as a natural extension.

We will first offer an example based on the following possible values of the underlying security and its derivative, as illustrated in figure 16.2. Also, we suppose that between time 1 and time 2, the simple interest is now 10%. (This means that the simple *forward* interest for the second period, as determined at period 0 and applying from time 1 to time 2, is 10%. A continuously compounded interest rate corresponds to this simple interest—denoted for simplicity  $i_1$ —equal to  $\log(1.1) = 9.53\%$ .) What is known at time 0 are

1. all values of  $S$  on the tree;
2. the derivative's values *only at time 2*; and
3. the interest rates  $i_0$  and  $i_1$  (11.1% and 10%, respectively).

Our task is to construct the derivative's value along the tree. We can proceed to evaluate the derivative on each branch geometrically, as we have done in the one-period model. Let us draw our data corresponding

<sup>4</sup> We recall that if  $Z_t$  is a stochastic process such that the expected value of  $|Z_t|$  is finite for all  $t$ , and if the expected value of  $Z_t$  conditional on its time path until time  $s$  ( $s \leq t$ ) is  $Z_s$ , then  $Z_t$  is defined as a martingale.

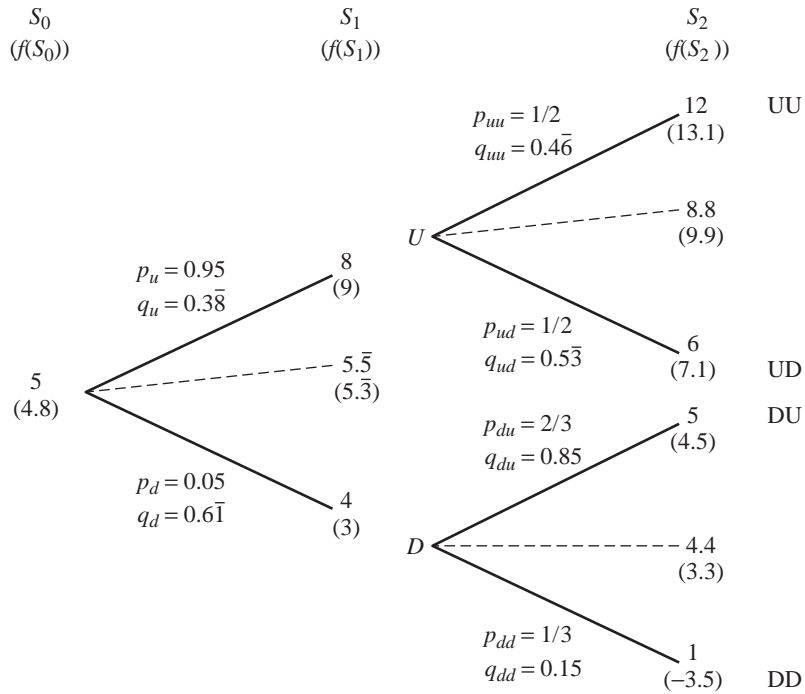


Figure 16.2

A three-period tree carrying the underlying asset's values and the corresponding derivatives values. For each branch, the asset's and derivative's values are expressed both in current and present value; those values are linked by the dashed lines. For the first period, the simple interest rate is 11.1% per period; for the second period, the interest rate is 10% per period.

to time 2 (the possible stock prices and their corresponding derivative values) in quadrant I<sup>5</sup> of figure 16.3. There are four possible points:  $UU$ ,  $UD$ ,  $DU$ ,  $DD$ . (Notice that one of the coordinates of  $DD$  is negative, due to the fact that if  $S_2$  is equal to 1, the derivative's value  $f(S_1)$  is negative—here  $-3.5$ .) Consider the first pair of points,  $UU$  and  $UD$ . We can immediately see that to replicate the derivative's value we will have to buy a number of stocks  $a_u$  and a number of bonds  $b_u$ , which means that we will be *lending* some money at time 1. (Indeed, the slope of line  $UDUU$  is positive and its ordinate at the origin, the amount we will have to receive at time 1, is also positive.) Applying (2), the exact amount

<sup>5</sup> We designate by “quadrant I” the quadrant in the upper right part of figure 16.3. Quadrants II, III, and IV run counterclockwise in the same figure.

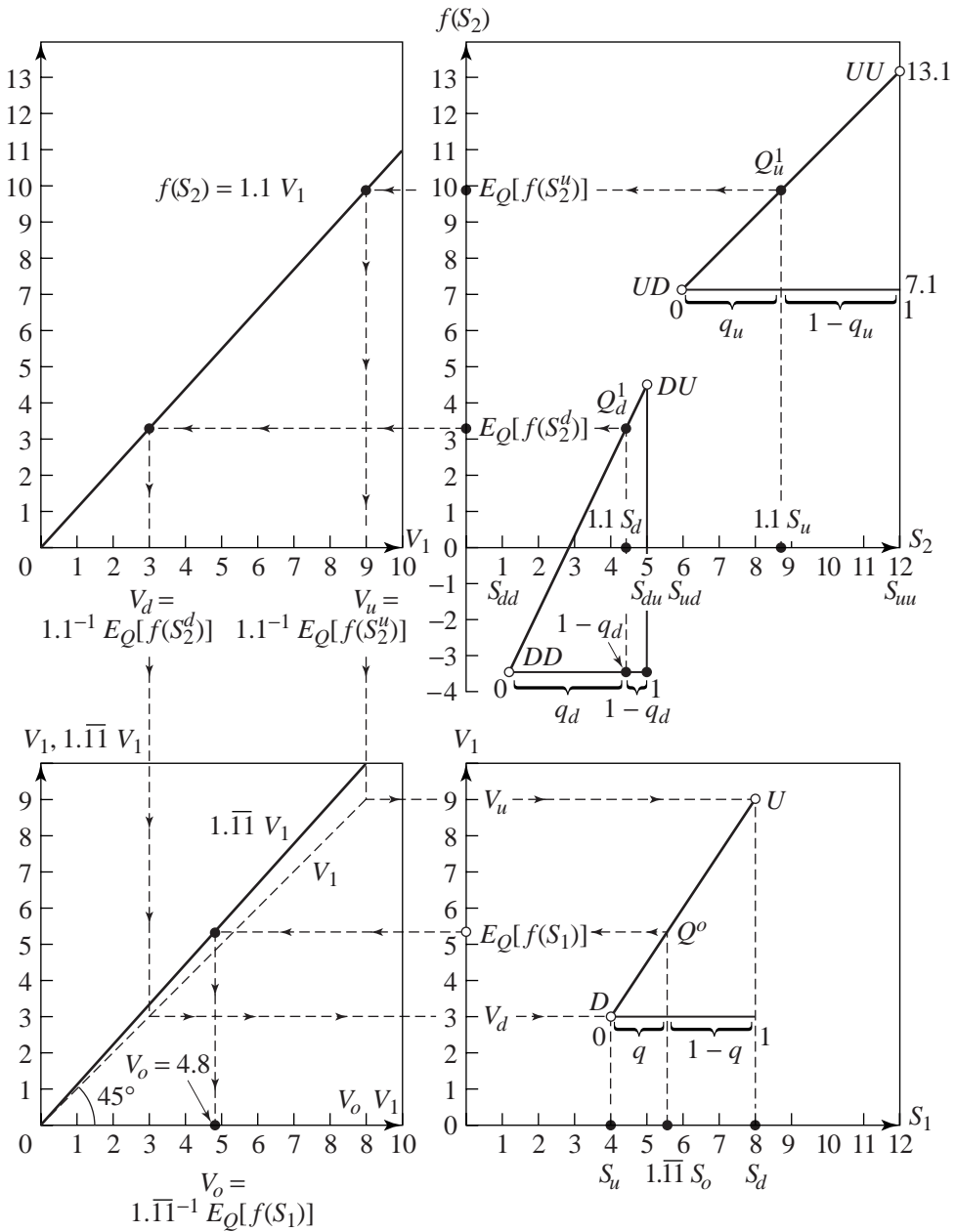


Figure 16.3

A geometrical construction of a  $Q$ -probability measure and a derivative's valuation ( $V_0 = 4.8$ ) over a two-period time span.

of  $a_u$  is

$$a_u = \frac{13.1 - 7.1}{12 - 6} = 1$$

Now the amount you want to receive at time 2 is the ordinate at the origin; it is equal to  $7.1 - (1)6 = 13.1 - (1)12 = 1.1$ . Since the forward rate of interest from time 1 to time 2 is 10%, you want to lend one at time 1; the bond's price at time 1 being one, you then buy one bond at time 1. So your replicating portfolio is made of one stock and one bond; therefore, its *arbitrage-enforced* value at time 1 is

$$V_u = (1)8 + (1)1 = 9$$

Relying on what we showed before, we could have calculated this value just as well by determining the  $q$ -probability pertaining to that branch of the tree; constructing that  $q$ -probability, denoted here  $q_{uu}$ , as

$$q_{uu} = \frac{S_u(1.1) - S_{ud}}{S_{uu} - S_{ud}} = \frac{8.8 - 6}{12 - 6} = 0.4\bar{6}$$

and applying directly the appropriate formula, transposed from (7) to our case, which yields

$$V_u = (1.1)^{-1}[0.4\bar{6}(13.1) + 0.5\bar{3}(7.1)] = 9$$

The construction of the value  $V_u$  is indicated in quadrant II of figure 16.3. First, the point  $Q_u^1$  of line  $UDUU$  is obtained by the abscissa of  $S_ue^{i\Delta t} = 8(1.1) = 8.8$  on the  $S$  axis. As before, that point splits the line  $UDUU$  in such a way that the ordinate of  $Q_u^1$  becomes the  $Q$ -measure expected value of the derivative, conditional to the fact that we are in the up-state at time 1. We thus get  $q_uf(S_{uu}) + (1 - q_u)f(S_{ud})$ , of which we have to take the present value at time 1. We do this in quadrant II by using the reciprocal of  $Ve^{i\Delta t} = V(1.1)$ , and thus get  $V_u$  on the horizontal axis. From there this value is taken (through the  $45^\circ$  line in quadrant III) to  $U$ 's ordinate (9) in quadrant IV.

Of course, the same procedure may be applied to the down branch, and the reader will not be surprised that it yields a value of the derivative at time 1 that is  $V_d = 3$  (since our data for time 2 have been determined precisely to yield the data at time 1 we used in our preceding case). In that case, the reader can see from quadrant I that the replicating portfolio will be made of *buying* 2 stocks and a *debt* whose value at time 2 must be 5.5.



If 5 bonds are sold at time 1, the portfolio at that time has a value of  $2(4) - 5(1) = 3$ . We are, of course, back at our initial situation, with the value of the derivative equal to either 9 or 3 (in the up- or down-state), and the replicating portfolio of those values is 4.8 at time 0.

The latter calculation is illustrated by the construction of point  $Q_d$  on line  $DDDU$  in quadrant I, from which point  $D$  in quadrant IV can be deduced as before. We thus get the  $DU$  line in quadrant IV that corresponds exactly to our figure 16.1, from which the value of the derivative at time 0 ( $V_0 = 4.8$ ) had been calculated and which is indicated again on the  $V$  axis of quadrant III.

We conclude that in this two-period model the arbitrage price of a derivative defined on a claim to be received at time  $\Delta t$  has a  $q$ -probability expected value of two expected values, *each* expected value being discounted by one period.

We have

$$V_0 = e^{-i_0 \Delta t} E_Q(V_1) = e^{-i_0 \Delta t} [qV_u + (1 - q)V_d]$$

with  $V_u$  and  $V_d$  being equal, respectively, to

$$V_u = e^{-i_1 \Delta t} [q_u V_{uu} + (1 - q_u) V_{ud}]$$

$$V_d = e^{-i_1 \Delta t} [q_d V_{du} + (1 - q_d) V_{dd}]$$

Therefore, we can write

$$\begin{aligned} V_0 &= e^{-(i_0+i_1)\Delta t} [qq_u V_{uu} + q(1 - q_u) V_{ud} + (1 - q)q_d V_{du} + (1 - q)(1 - q_d) V_{dd}] \\ &= e^{-(i_0+i_1)\Delta t} E_Q(V_2) \end{aligned}$$

**Basic properties.** The value of the derivative fully qualifies as a discounted expected value of the claim at time  $t$ . This, of course, can be generalized to an  $n$ -period model. The essential feature of that expected value is that it is determined under a  $Q$ -probability measure that is completely independent from the  $P$ -probability measure. The  $Q$ -measure is that which makes both the *discounted* underlying security and the discounted corresponding derivative a martingale. Also, as we observed in the one-step model, we are not surprised that the  $Q$ -measure makes the underlying security and its derivative martingales when the security and the derivative are expressed in *any* year's value (so there is in fact nothing special about the present value). For instance,  $4.8(1.\bar{1})(1.1) = 5.8\bar{6}$  is the

value of the derivative at time 0 expressed in terms of the second period, and that is just the  $q$ -measure expected value of the derivative's possible values in the second period.

That all these properties extend to  $n$ -period models will not be a surprise either. The real problem comes from certain technical restrictions that may apply if one wants to go from an  $n$ -period model to a continuous time model. (For these, see the excellent text by Baxter and Rennie (1998).) But the general ideas described in this chapter will carry over to continuous time, the framework we will be using from chapter 17 onward.

## 16.6 The concept of probability measure change

Consider as before a nonrecombining binomial event tree. By “nonrecombining” we mean a tree that has the following general property: at any node, one step up followed by one step down will not necessarily lead to the same value as the one that would correspond to a first step down followed by one step up. This implies that at time 1 there are two possible values, at time 2,  $2^2 = 4$  possible values (instead of 3 in a recombining tree). Our tree is defined over a number of steps in time equal to  $T$ . At  $t = 0$ , our process has a nonrandom initial value; at time  $t$ , it has  $2^t$  possible values defined at  $2^t$  nodes. In all, there are  $1 + 2 + 2^2 + \cdots + 2^t + \cdots + 2^T = 2^{T+1} - 1$  nodes on the tree. To each of these nodes corresponds one path, and a probability that is simply the product of all probabilities from the initial node to the node considered at time  $t$ . All these  $2^{T+1} - 1$  probabilities define a  $P$ -measure. In the same way as we showed before, we can define on the same set of nodes a set of  $2^{T+1} - 1$  risk-neutral probabilities defining a  $Q$ -measure. We know that we have to evaluate a derivative received at time  $T$  as the expectation of the derivative in present value under the  $Q$ -measure, that is, using the  $Q$ -probabilities.

Now let us ask the following question: would there be a way to obtain our result by still using the event  $P$ -probabilities or the  $P$ -measure? Our intuition here can provide an answer: it should be possible, provided that either the relevant random variable or the  $P$ -probability undergoes a change. Define, on the same tree, a process in the following way: at each node, it is the ratio of the  $Q$ -probability to the  $P$ -probability to reach that node. This process is well defined if the following condition is met: if the event corresponding to any node has positive  $P$ -probability, it also has

positive  $Q$ -probability, and the converse is true. If such a condition is met, both probability measures are said to be equivalent.

Our process, which we can denote  $\xi_t$ , is such that at any time  $t$  its value at each node is the ratio of the  $Q$ -probabilities to the  $P$ -probabilities for that node to be reached. This process is called a discrete-time Radon-Nikodym process, to which probabilities can be attached. Figure 16.4 illustrates such a construction. For instance, to node  $DU$  corresponds a value  $\frac{q_d q_{du}}{p_d p_{du}}$  for the outcome of the Radon-Nikodym derivative; under the  $P$ -measure, the probability for  $\xi_2$  to take that value is  $p_d p_{du}$ ; and under measure  $Q$ , that probability is  $q_d q_{du}$ .

For horizon  $T$ , the possible values of the Radon-Nikodym process constitute a random variable denoted  $dQ/dP$ . In our  $T = 2$  example, we have the following random variable with its associated  $P$ -measure probabilities. (The actual values of our example are in parentheses.)

It can be seen immediately that multiplying  $X_T$  (or the claims at time  $T$ ) by the Radon-Nikodym variable at time  $T$  enables us to revert to the  $P$ -probabilities in order to calculate the  $Q$ -measure expected value of the claim  $X$  at  $T$ . Indeed, call  $X_{uu}$ ,  $X_{ud}$ ,  $X_{du}$ , and  $X_{dd}$  the possible values of the claim at  $T$ . The claim's expected value at time 0 under  $Q$ -measure is

$$E_Q(X_T) = q_u q_{uu} X_{uu} + q_u q_{ud} X_{ud} + q_d q_{du} X_{du} + q_d q_{dd} X_{dd}$$

and this is equal to the expected value of  $\frac{dQ}{dP} X_T$  under  $P$ -measure:

$$\begin{aligned} E_P\left(\frac{dQ}{dP} X_T\right) &= p_u p_{uu} \frac{q_u q_{uu}}{p_u p_{uu}} X_{uu} + p_u p_{ud} \frac{q_u q_{ud}}{p_u p_{ud}} X_{ud} \\ &\quad + p_d p_{du} \frac{q_d q_{du}}{p_d p_{du}} X_{du} + p_d p_{dd} \frac{q_d q_{dd}}{p_d p_{dd}} X_{dd} \end{aligned}$$

We can also see immediately from our construction that the Radon-Nikodym process is a  $P$ -martingale. At any point of time  $s$  ( $s < t$ ), its value is equal to the  $P$ -measure expected value at horizon  $t$ , conditional on the path taken up to  $s$  (which we can call the *filtration* of the process  $\xi$  at time  $s$ , denoted  $F_s$ ). We thus have

$$\xi_s = E_P(\xi_t | F_s)$$

Consider first the expected value of  $\xi_1$  at time 0. We have  $E_P(\xi_1 | F_0)$   
 $= p_u \frac{q_u}{p_u} + p_d \frac{q_d}{p_d} = q_u + q_d = 1$ , as it should. The expected value of  $\xi_2$  at time 0 is  $E_P(\xi_2 | F_0) = p_u p_{uu} \frac{q_u q_{uu}}{p_u p_{uu}} + p_u p_{ud} \frac{q_u q_{ud}}{p_u p_{ud}} + p_d p_{du} \frac{q_d q_{du}}{p_d p_{du}} + p_d p_{dd} \frac{q_d q_{dd}}{p_d p_{dd}}$   
 $= q_u(q_{uu} + q_{ud}) + q_d(q_{du} + q_{dd}) = q_u + q_d = 1$ , without surprise. In the

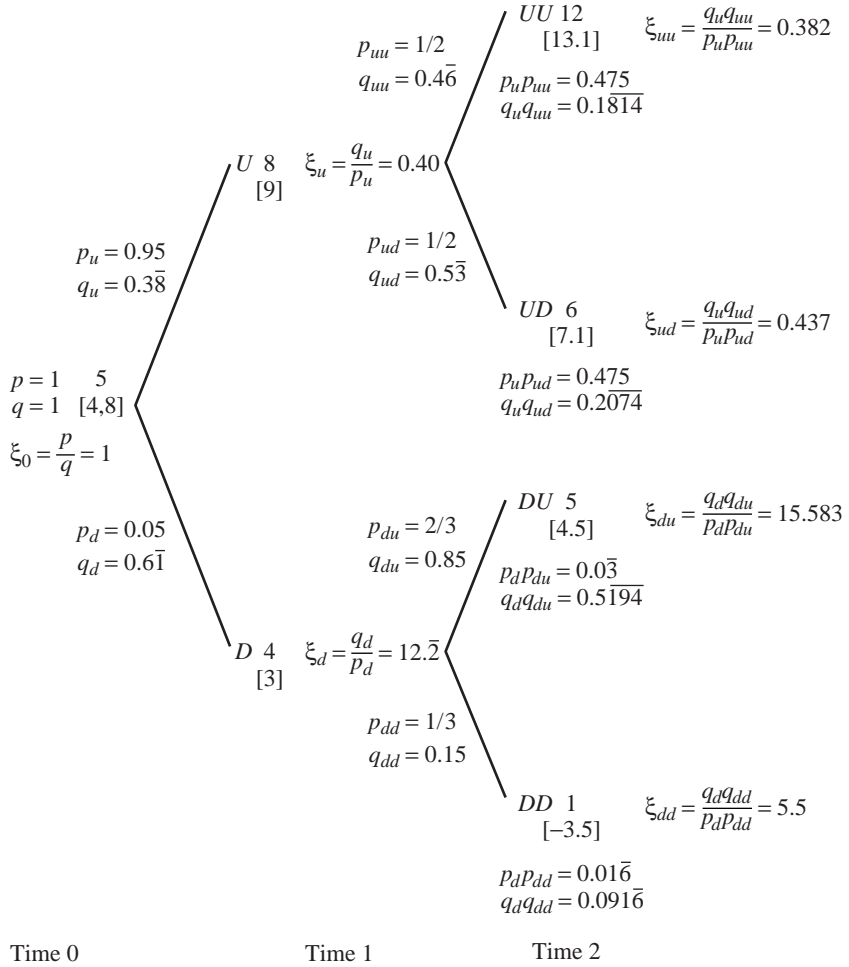


Figure 16.4

Construction of the Radon-Nikodym process  $\xi_t$  for  $t = 0, 1, 2$ . At each node ( $U, D, UU$ , etc.) the first number indicates the asset value; immediately under it, the figure in brackets is the derivative's value. For instance, at node  $D$ , 4 is the stock's value and 3 is the derivative's value. At each node the value taken by the Radon-Nikodym random process is indicated. (For instance, at node  $D$  the value of the Radon-Nikodym process is 12.2.)

**Table 16.3****A derivation of  $Q$ -measure probabilities using the Radon-Nikodym derivative**

	State of nature (nodes) at $T$			
	$UU$	$UD$	$DU$	$DD$
Radon-Nikodym derivative values	$\frac{q_u q_{uu}}{p_u p_{uu}}$ (0.382)	$\frac{q_u q_{ud}}{p_u p_{ud}}$ (0.437)	$\frac{q_d q_{du}}{p_d p_{du}}$ (15.583)	$\frac{q_d q_{dd}}{p_d p_{dd}}$ (5.5)
Probabilities under $P$ -measure	$p_u p_{uu}$ (0.475)	$p_u p_{ud}$ (0.475)	$p_d p_{du}$ (0.03)	$p_d p_{dd}$ (0.016)
Probabilities under $Q$ -measure	$q_u q_{uu}$ (0.1814)	$q_u q_{ud}$ (0.2074)	$q_d q_{du}$ (0.5194)	$q_d q_{dd}$ (0.0916)

same way, we can determine  $\xi_1$  as the expected values of  $\xi_2$  subject to the filtration  $F_1$ ; thus  $E_P(\xi_2|U) = p_{uu} \frac{q_u q_{uu}}{p_u p_{uu}} + p_{ud} \frac{q_u q_{ud}}{p_u p_{ud}} = \frac{q_u}{p_u} (q_{uu} + q_{ud}) = \frac{q_u}{p_u}$ . The conditional expectation of  $\xi_2$  at  $D$  would of course lead to  $q_d/p_d$ .

## 16.7 Extension of arbitrage pricing to continuous-time processes

The risk-neutral measure (the martingale  $Q$ -measure) transforms the initial random process into a driftless martingale process. So in order to find a risk-neutral measure we will have to *remove* any drift from the original process. When, in continuous time, we are given a random process with drift, our first task will be to remove the drift by changing the measure. We saw in the discontinuous case that the Radon-Nikodym derivative was a random process that could be considered a multiplier of either the initial random variable or the initial set of probabilities. Suppose now that we want to transform a continuous Wiener process with drift into a martingale.

Will changing the drift of the initial process be like multiplying either the initial probability measure or the random variable by a random variable? The answer is positive and is provided by the Cameron-Martin-Girsanov theorem.

### 16.7.1 A statement of the Cameron-Martin-Girsanov theorem

The Cameron-Martin-Girsanov theorem can be stated in numerous ways with varied levels of complexity, depending in particular on the number of Brownian motions involved. It was first formulated by R. H. Cameron

and W. T. Martin in 1944<sup>6</sup> in a rather difficult setting; it was then rediscovered, apparently independently, by I. V. Girsanov in 1960.<sup>7</sup> Its access is somewhat easier in the Girsanov version, although it can still take a variety of forms. For our purposes we will use only the single-Brownian motion version as expressed in the Baxter and Rennie text<sup>8</sup>:

If  $W_t$  is a  $P$ -Brownian motion and  $\varphi_t$  is an  $F$ -previsible process satisfying the boundedness condition  $E_P[\exp(\frac{1}{2} \int_0^T \varphi_t^2 dt)] < \infty$ , then there exists a measure  $Q$  such that

- (a)  $Q$  is equivalent to  $P$ ,
- (b)  $\frac{dQ}{dP} = \exp(-\int_0^T \varphi_t dW_t - \frac{1}{2} \int_0^T \varphi_t^2 dt)$ , and
- (c)  $\tilde{W}_t = W_t + \int_0^t \varphi_s ds$  is a  $Q$ -Brownian motion; equivalently,  $W_t$  has drift  $-\varphi_t$  under  $Q$ .

### 16.7.2 An intuitive proof of the Cameron-Martin-Girsanov theorem

We will now give an intuitive proof of the theorem.<sup>9</sup> Let  $(t_1, \dots, t_n)$  denote a series of times belonging to the time interval  $[0, T]$ . Let  $W_t$  be a Brownian motion defined on  $[0, T]$ , with  $W_0 = 0$ . The probability for  $W_t$  to be at time  $t_1$  within a very short vicinity  $dx_1$  of  $x_1$  is given by

$$P(W_{t_1} \in dx_1) = \frac{1}{\sqrt{2\pi t_1}} e^{-\frac{x_1^2}{2t_1}} dx_1 \quad (9)$$

The probability that the trajectory governed by  $W_t$  goes through the vicinity of  $x_1$  at time  $t_1$  and through the vicinity of  $x_2$  at time  $t_2$  ( $t_2 > t_1$ ) is given by the product of the probability of getting to the vicinity of  $x_1$  and the probability of  $W_{t_1}$  to receive an increase  $\Delta W_{t_1} = W_{t_2} - W_{t_1}$ . It is thus

<sup>6</sup> R. H. Cameron and W. T. Martin, "Transformations of Wiener Integrals under Translations," *Annals of Mathematics*, 45, no. 2 (April 1944): 386–396.

<sup>7</sup> I. V. Girsanov, "On Transforming a Certain Class of Stochastic Processes by Absolutely Continuous Substitution of Measures," *Theory of Probability and Its Applications*, 5, no. 3 (1960): 285–301. I am grateful to Wolfgang Stummer for mentioning to me that the theorem had another precedent; see G. Murayama, "On the Transition Probability Functions of the Markov Processes," *Nat. Sci. Rep. Ochanomizu Univ.* 5 (1954): 10–20, and "Continuous Markov Processes and Stochastic Equations," *Rend. Circ. Mat. Palermo Ser. 4* (1955): 48–90.

<sup>8</sup> Baxter and Rennie, op. cit., p. 74.

<sup>9</sup> We owe this proof to an excellent colleague of ours, who unfortunately insisted upon anonymity.

equal to

$$P(W_{t_1} \in dx_1, W_{t_2} \in dx_2) = \frac{1}{2\pi t_1} e^{-\frac{1}{2} \frac{x_1^2}{t_1}} \frac{1}{\sqrt{2\pi(t_2 - t_1)}} e^{-\frac{1}{2} \frac{(x_2 - x_1)^2}{t_2 - t_1}} dx_1 dx_2 \quad (10)$$

More generally, consider the probability for  $W_t$  to go through the vicinity of  $x_1, x_2, \dots, x_n$  at times  $t_1, t_2, \dots, t_n$ . We have

$$\begin{aligned} P(W_{t_1} \in dx_1, W_{t_2} \in dx_2, \dots, W_{t_n} \in dx_n) \\ = \frac{1}{2\pi t_1} e^{-\frac{1}{2} \frac{x_1^2}{t_1}} \frac{1}{\sqrt{2\pi(t_2 - t_1)}} e^{-\frac{1}{2} \frac{(x_2 - x_1)^2}{t_2 - t_1}} \dots \\ \dots \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} e^{-\frac{1}{2} \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}} dx_1 dx_2 \dots dx_n \quad (11) \end{aligned}$$

Consider now another probabilistic model (called  $Q$ ) in which the trajectory is governed by a Brownian motion with drift  $-t$ , where  $t$  is a constant. (Later we will see what happens when  $t$  is allowed to be a deterministic function of time.) The probability  $Q$  for  $\tilde{W}_t$  to go through the vicinity of  $x_1, \dots, x_n$  at times  $t_1, \dots, t_n$  is given by

$$\begin{aligned} Q(\tilde{W}_{t_1} \in dx_1, \tilde{W}_{t_2} \in dx_2, \dots, \tilde{W}_{t_n} \in dx_n) \\ = \frac{1}{2\pi t_1} \cdot \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \dots \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} e^{-\frac{1}{2} \frac{(x_1 + t_1)^2}{t_1}} \\ \cdot e^{-\frac{1}{2} \frac{[(x_2 + t_2) - (x_1 + t_1)]^2}{t_2 - t_1}} \dots e^{-\frac{1}{2} \frac{[(x_n + t_n) - (x_{n-1} + t_{n-1})]^2}{t_n - t_{n-1}}} dx_1 dx_2 \dots dx_n \quad (12) \end{aligned}$$

Let us now evaluate the exponent in the exponential part of (12). It can be written as

$$\begin{aligned} -\frac{1}{2} \left[ \frac{(x_1 + t_1)^2}{t_1} + \frac{[(x_2 + t_2) - (x_1 + t_1)]^2}{t_2 - t_1} \right. \\ \left. + \dots \frac{[(x_n + t_n) - (x_{n-1} + t_{n-1})]^2}{t_n - t_{n-1}} \right] \\ = -\frac{1}{2} \left[ \frac{x_1^2 + 2x_1 t_1 + t_1^2}{t_1} \right. \\ \left. + \frac{(x_2 - x_1)^2 + 2(x_2 - x_1)(t_2 - t_1) + t_2^2 - t_1^2}{t_2 - t_1} \right] \end{aligned}$$

$$\begin{aligned}
& + \dots \frac{(x_n - x_{n-1})^2 + 2(x_n - x_{n-1})(t_n - t_{n-1}) + (t_n - t_{n-1})^2}{t_n - t_{n-1}} \Big] \\
& = -\frac{1}{2} \left[ \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \dots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}} \right] - x_1 - (x_2 - x_1) \\
& \quad - \dots - (x_n - x_{n-1}) - \frac{1}{2} [ (t_1)^2 + (t_2 - t_1)^2 + \dots + (t_n - t_{n-1})^2 ] \\
& \hspace{15em} (13)
\end{aligned}$$

The first part of the right-hand side of (13) corresponds to the  $P$ -probability measure (see equation (11)); it is followed by a series of terms that all cancel out except  $-x_n$ . Finally, observe that the last bracketed term of (13) contains only  $(t_n)^2$ . Hence we may write

$$Q(\tilde{W}_1 \in dx_1, \dots, \tilde{W}_n \in dx_n) = P(W_1 \in dx_1, \dots, W_n \in dx_n) e^{-x_n - \frac{1}{2} t_n^2} \quad (14)$$

We have here a formula that changes a  $P$ -probability into a  $Q$ -probability through a quantity that depends on the *realization* of a random variable  $(x_n)$ , the  $t_n$  variable as well as terminal time  $t_n$ . Suppose now that we consider *all* possible values of time on the interval  $[0, T]$ . We will take up similarly all possible values of  $x$  between  $x_0 = 0$  and  $x_T$ . Exactly the same analysis will apply, and we can say that probability  $Q$  to have a given trajectory under a Brownian motion with drift  $-x$  is the probability of getting the same trajectory under a Brownian motion with no drift, multiplied by  $e^{-W_T - \frac{1}{2} T^2}$ .

Let us now see how our analysis is modified if we want to determine the  $Q$ -probability when changing the  $P$ -model by a *function of time*, which we denote  $(t)$ . In equation (12) we would change all the constants into variables  $(t_1), (t_2), \dots, (t_n)$ . The exponent of the exponential would thus become

$$\begin{aligned}
& -\frac{1}{2} \left[ \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \dots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}} \right] \\
& \quad - [ (t_1)(x_1 - x_0) + (t_2)(x_2 - x_1) + \dots + (t_n)(x_n - x_{n-1}) ] \\
& \quad - \frac{1}{2} [ (t_1)^2(t_1 - t_0) + (t_2)^2(t_2 - t_1) + \dots + (t_n)^2(t_n - t_{n-1}) ] \quad (15)
\end{aligned}$$

(with  $t_0 = 0$  and  $x_0 = 0$ ).



Suppose finally that we consider all possible instants in the interval  $[0, T]$  and, similarly, all possible values our Brownian motion can take. Then the second bracket in (15) is the realization value of the *stochastic integral* of  $(t)dW_t$  from  $t = 0$  to  $t = T$ , and the third bracket is the simple integral of  $^2(t)$ . Thus, if  $\omega$  denotes any trajectory of the Brownian motion, we can write

$$Q(\omega) = P(\omega) \exp \left[ - \int_0^T (t) dW_t - \frac{1}{2} \int_0^T ^2(t) dt \right] \quad (16)$$

We now have a way of translating one probability measure into another. Inasmuch as in the discrete case we can consider ratios of probabilities to convert one model into another, here we have a ratio that could be expressed in its limited form as

$$\frac{Q(\omega)}{P(\omega)} = \exp \left[ - \int_0^T (t) dW_t - \frac{1}{2} \int_0^T ^2(t) dt \right]$$

This expression is also denoted  $\frac{dQ}{dP}(\omega)$ , which is the Radon-Nikodym derivative. The reason for this notation comes from the way expected values of a continuous random variable under probability density functions  $f(x)$  and  $g(x)$  are calculated. Suppose that  $f(x)$  corresponds to the  $P$  model and  $g(x)$  to the  $Q$  model;  $D$  is the domain of definition of  $X$ ; call  $P$  and  $Q$  the cumulative distributions of  $f(x)$  and  $g(x)$ . We have

$$\begin{aligned} E_P(X) &= \int_D x f(x) dx = \int_D x dP \\ E_Q(X) &= \int_D x g(x) dx = \int_D x dQ \\ &= \int_D x \frac{g(x)}{f(x)} f(x) dx \equiv \int_D x \xi(x) f(x) dx \end{aligned}$$

where  $\xi(x) \equiv \frac{g(x)}{f(x)} = \frac{dQ}{dx} / \frac{dP}{dx} = \frac{dQ}{dP}$ .

These concepts will now be put to work in describing a model of central importance today, which we believe to be the most recent major advance in finance, the Heath-Jarrow-Morton model of forward interest rates, bond prices, and their derivatives. This will be the subject of our next chapter.

### 16.8 An application: the Baxter-Rennie martingale evaluation of a European call

We now illustrate the considerable importance of the martingale probability measure by showing how a central result of finance (the Black-Scholes formula) can be obtained in a straightforward way under that framework.

The famous Black-Scholes formula for the evaluation of a European call was a major advance for both economics and finance. It was obtained in 1973 through a rather difficult resolution of a second-order partial differential equation. (In fact, Fisher Black recalled that when he saw this equation for the first time, he did not recognize a transformation of the propagation of heat equation, although he had studied the latter quite extensively.)

Applying ingeniously the concept of expectation under a  $Q$ -measure of a martingale process, Baxter and Rennie (1998) obtained what turns out to be, in our opinion, the simplest and at the same time the most elegant way of finding the Black-Scholes formula. We will present it here. At some point we will even take the liberty of simplifying their method, choosing a shortcut to obtain the limiting mean and variance of the continuously compounded rate of return process.

The reader will recall that the basic hypothesis of Black and Scholes is the lognormal distribution of the underlying asset:  $\log S_t$  follows a Brownian motion  $\log S_0 + t + \sigma W_t$ , where  $W_t \sim N(0, t)$ . In other words, the stock's value can be written as

$$S_t = S_0 e^{t + \sigma W_t} = S_0 e^{t + \sigma \sqrt{t} Z_t}, \quad Z_t \sim N(0, 1) \quad (17)$$

Baxter and Rennie took the following route in order to evaluate a claim depending on the terminal value  $S_T$  of the stock (equal to  $X = \max(0, S_T - K)$ , where  $K$  is the exercise price of the call). They first looked for a discontinuous (discrete) binomial process whose continuous limit was (17). This gave them the up or down step that  $S_t$  follows. In turn, they determined the  $Q$ -probability measure pertaining to those steps and the continuous limit of the process under the  $Q$  measure, which translated simply as another Brownian motion. Finally, they calculated the call's value today as the present value of the expected value of the derivative under the  $Q$  measure.

Consider the *discontinuous* process with  $P$ -measure  $\frac{1}{2}$ :

$$S_u = S_0 e^{\Delta t + \sigma \sqrt{\Delta t} Z_t} \quad (18)$$

$$S_d = S_0 e^{(\Delta t - \sigma \sqrt{\Delta t})} \quad (19)$$

For any branch,  $\log(S_u/S_0)$  (the continuously compounded return over  $\Delta t$ ) follows the binomial process

$$\log(S_u/S_0) = \Delta t + \sigma \sqrt{\Delta t} \quad (20)$$

$$\log(S_d/S_0) = \Delta t - \sigma \sqrt{\Delta t} \quad (21)$$

with  $E[\log(S_{\Delta t}/S_0)] = \Delta t$  and  $VAR[\log(S_{\Delta t}/S_0)] = \sigma^2 \Delta t$  under the  $P$ -measure. Split a time period, from 0 to  $t$ , into  $n$  equal intervals,  $t = n\Delta t$ , and consider the distribution of  $S_t$ :

$$S_t = S_0 e^{B_1 + \dots + B_n} \quad (22)$$

where  $B_i (i = 1, \dots, n)$  is the binomial process ( $\log S_{\Delta t}/S_0$ ). We then have  $\log(S_t/S_0) = \sum_{i=1}^n B_i$ . Following the central limit theorem, when  $\Delta t \rightarrow 0$  (and consequently, when  $n \rightarrow \infty$ ),  $\sum_{i=1}^n B_i$  tends in law toward a normal distribution with mean  $nE(B_i) = n \Delta t = t$  and variance  $nVAR(B_i) = n\sigma^2 \Delta t = \sigma^2 t$ . Hence, we have

$$\log(S_t/S_0) \approx N(t, \sigma^2 t) \quad (23)$$

or, equivalently,

$$\begin{aligned} S_t &= S_0 e^{\log(S_t/S_0)} \approx S_0 e^{N(t, \sigma^2 t)} \\ &= S_0 e^{t + \sigma \sqrt{t} Z_t}, \quad Z_t \sim N(0, 1) \end{aligned} \quad (24)$$

Baxter and Rennie now set out to determine the  $Q$ -martingale probability measure. We know that it is independent of the original  $P$ -measure and that it is simply given by

$$q = \frac{S_0 e^{r\Delta t} - S_d}{S_u - S_d} = \frac{e^{r\Delta t} - e^{(\Delta t - \sigma \sqrt{\Delta t})}}{e^{(\Delta t + \sigma \sqrt{\Delta t})} - e^{(\Delta t - \sigma \sqrt{\Delta t})}} \quad (25)$$

The only hurdle in Baxter and Rennie's demonstration lies here: what is the limit of  $q$  when  $\Delta t$  goes to zero? For notational simplicity, let us denote  $\Delta t \equiv h$ . Our purpose is to show that

$$\lim_{h \rightarrow 0} q = \lim_{h \rightarrow 0} \frac{e^{rh^2} - e^{h^2 - \sigma h}}{e^{h^2 + \sigma h} - e^{h^2 - \sigma h}} \approx \frac{1}{2} \left[ 1 - h \left( \frac{+\frac{\sigma^2}{2} - r}{\sigma} \right) \right]$$

Let us call  $N(h)$  and  $D(h)$  the numerator and the denominator of  $q$ . Developing  $N(h)$  and  $D(h)$  in a Taylor series and making some cancella-

tions, we end up with

$$N(h) = \sigma h + \left(r - \frac{\sigma^2}{2}\right)h^2 + \text{terms of order } h^3 \text{ or higher}$$

$$D(h) = 2\sigma h + \text{terms of order } h^3 \text{ or higher}$$

Dividing  $N$  and  $D$  by  $2\sigma h$ , the ratio  $N/D$  can be written as

$$\frac{N}{D} = \frac{\frac{1}{2} - \frac{+\frac{\sigma^2}{2}-r}{2\sigma}h + \text{terms of order } h^2 \text{ or higher}}{1 + \text{terms of order } h^2 \text{ or higher}}$$

Neglecting the terms of order  $h^2$  and above, we get the result  $\lim_{h \rightarrow 0} q \approx \frac{1}{2}(1 - \frac{+\frac{\sigma^2}{2}-r}{\sigma}h)$ .

We can now determine the  $Q$ -measure expected value and variance of the binomial  $B_i$ . Denoting  $a \equiv \frac{1}{\sigma}(r + \sigma^2/2 - r)$  and therefore, coming back to the initial notation  $h = \Delta t$ ,  $q \equiv \frac{1}{2}(1 - a \Delta t)$ , we then have

$$\begin{aligned} E_Q(B_i) &= (\Delta t + \sigma \Delta t) \left[ \frac{1}{2}(1 - a \Delta t) \right] + (\Delta t - \sigma \Delta t) \left[ \frac{1}{2}(1 + a \Delta t) \right] \\ &= \Delta t(r - \sigma^2/2) = (r - \sigma^2/2) \frac{t}{n} \end{aligned} \quad (26)$$

(Note that  $\sigma$  has disappeared from this expected value *under the  $Q$ -measure*.)

Similarly,  $VAR(B_i)$  can be shown to converge toward  $\sigma^2 \Delta t = \sigma^2 \frac{t}{n}$ . Therefore, the sum  $\sum_{i=1}^n B_i$  converges in law toward a normal distribution with mean  $(r - \sigma^2/2)t$  and variance  $\sigma^2 t$ . We can write

$$\sum_{i=1}^n B_i \approx N[(r - \sigma^2/2)t, \sigma^2 t] = (r - \sigma^2/2)t + \sigma \sqrt{t} Z_t, \quad Z_t \sim N(0, 1) \quad (27)$$

Hence  $S_t$  under the  $Q$ -martingale measure converges toward a lognormal process that can be written as

$$S_t = S_0 e^{(r - \sigma^2/2)t + \sigma \sqrt{t} Z_t}, \quad Z_t \sim N(0, 1) \quad (28)$$

We can note here that, under this  $Q$ -martingale measure, the present value of  $S_t$ ,  $S_t e^{-rt}$  is indeed a martingale. Let us determine the expected value of  $S_t$ 's present value:

$$E_Q(e^{-rt} S_t) = e^{-rt} E_Q[S_0 e^{(r - \sigma^2/2)t + \sigma \sqrt{t} Z_t}] \quad (29)$$

Note that the expectation on the right-hand side of (29) is simply a linear transformation of the moment-generating function of the unit normal distribution where  $\sigma \sqrt{t}$  plays the role of the parameter. We have

$$e^{-rt} S_0 e^{(r-\sigma^2/2)t} E_Q[e^{\sigma \sqrt{t} Z}] = e^{-rt} S_0 e^{(r-\sigma^2/2)t} \cdot e^{\sigma^2 t/2} = S_0 \quad (30)$$

and therefore  $E_Q(e^{-rt} S_t) = S_0$ ; so  $S_t$  is indeed a  $Q$ -martingale; the limiting process we have taken from binomials to a normal has preserved its martingale character.

Applying what we know from derivative evaluation, we can now calculate today's call value ( $C_0$ ) on  $S_T$  as the present value of the claim at  $T$  under the  $Q$ -martingale measure. If this claim at  $T$  is  $\max(S_T - K)$ , where  $K$  is the exercise price of the call, we simply have to calculate

$$\begin{aligned} C_0 &= e^{-rT} E_Q[\max(0, S_T - K)] \\ &= e^{-rT} E_Q[\max(0, S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma \sqrt{T} Z_T} - K)] \end{aligned} \quad (31)$$

$S_T$  is a strictly increasing function of the realization value of  $Z, z$ . Let us determine the value of  $z$ , denoted  $\bar{z}$ , above which  $\max(0, S_T - K) = S_T - K$ . The equation

$$S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma \sqrt{T} \bar{z}} = K \quad (32)$$

implies

$$\bar{z} = \frac{\log(K/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}} \quad (33)$$

We then have

$$\begin{aligned} C_0 &= e^{-rT} \int_{\bar{z}}^{+\infty} [S_T(z) - K] f(z) dz \\ &= e^{-rT} \left[ \int_{\bar{z}}^{+\infty} S_T(z) f(z) dz - K \int_{\bar{z}}^{+\infty} f(z) dz \right] \equiv e^{-rT} (I_1 - KI_2) \end{aligned} \quad (34)$$

where  $I_1$  and  $I_2$  designate each of the above integrals, respectively.

$$I_1 = \frac{S_0}{2\pi} \int_{\bar{z}}^{+\infty} e^{(r-\sigma^2/2)T+\sigma \sqrt{T} z - z^2/2} dz \quad (35)$$

In the exponent of  $e$ , the terms in  $z$  are part of the development of  $\frac{1}{2}(z - \sigma \sqrt{T})^2$ ; therefore, this exponent can be written as  $-\frac{1}{2}(z - \sigma \sqrt{T})^2 +$

$rT$ . The integration is now straightforward:

$$I_1 = S_0 e^{rT} \frac{1}{2\pi} \int_{\bar{z}}^{\infty} e^{-\frac{1}{2}(z - \sigma - T)^2} dz \quad (36)$$

Let us make the change of variable  $y = -z + \sigma - T$ ; to  $\bar{z} = \frac{\log(K/S_0) - (r - \frac{\sigma^2}{2})T}{\sigma - T}$  corresponds  $\bar{y} = \frac{\log(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma - T}$  and

$$\begin{aligned} I_1 &= S_0 e^{rT} \frac{1}{2\pi} \int_{\bar{y}}^{-\infty} e^{-y^2/2} (-dy) = S_0 e^{rT} \int_{-\infty}^{\bar{y}} \frac{1}{2\pi} e^{-y^2/2} dy \\ &= S_0 e^{rT} \Phi(\bar{y}) \end{aligned} \quad (37)$$

where  $\Phi(\bar{y})$  is the cumulative unit normal distribution corresponding to  $z = \bar{y}$ , that is,  $\Phi(\bar{y}) = \text{prob}(Z < \bar{y})$ . The second integral is

$$I_2 = \int_{\bar{z}}^{\infty} f(z) dz = \int_{-\infty}^{-\bar{z}} f(z) dz = \Phi(-\bar{z}) = \Phi(\bar{y} - \sigma - T) \quad (38)$$

and therefore the call's value is

$$C_0 = S_0 \Phi(\bar{y}) - e^{-rT} K \Phi(\bar{y} - \sigma - T) \quad (39)$$

with  $\bar{y} = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma - T}$ , which is the Black-Scholes formula, obtained through a method that involved only the calculation of a simple definite integral.

### Main formulas

By far the most important formulas of this chapter are contained in the single-Brownian version of the Cameron-Martin-Girsanov theorem, as expressed in Baxter and Rennie<sup>1</sup> :

If  $W_t$  is a  $P$ -Brownian motion and  $\varphi_t$  is an  $F$ -previsible process satisfying the boundedness condition  $E_P[\exp(\frac{1}{2} \int_0^T \varphi_t^2 dt)] < \infty$ , then there exists a measure  $Q$  such that

- (a)  $Q$  is equivalent to  $P$ ,
- (b)  $\frac{dQ}{dP} = \exp(-\int_0^T \varphi_t dW_t - \frac{1}{2} \int_0^T \varphi_t^2 dt)$ , and

<sup>1</sup> Baxter and Rennie, op. cit., p. 74.

(c)  $\tilde{W}_t = W_t + \int_0^t \lambda_s ds$  is a  $Q$ -Brownian motion; equivalently,  $W_t$  has drift  $-\lambda_t$  under  $Q$ .

### Questions

16.1. In section 16.3, verify the steps leading from equation (6a) to equation (7).

16.2. What is the algebraic explanation of the fact that, for given  $i$ ,  $S_0$ ,  $S_u$ , and  $S_d$  values, there is an infinity of derivative values  $f(S_d)$ ,  $f(S_u)$  at time  $\Delta t$  that lead to the same derivative value at time 0,  $V_0$ ?

16.3. Using the same  $i$ ,  $S_0$ ,  $S_u$ , and  $S_d$  values as the ones in the text, and assuming that  $f(S_u) = -4.6$  and  $\Delta t = 1$  year, give the value of  $f(S_d)$  that would lead to the same derivative value at time  $V_0$ . (You can check your answer with the height of point  $D''$  in figure 16.1.)

16.4. Suppose that, in the example of the determination of a Radon-Nikodym process given in this chapter you want to add a third period and therefore eight realization values to the Radon-Nikodym process. What information would you need to do just that?

16.5. Confirm that under the  $Q$ -measure the expected value of the binomial  $B_i (i = 1, \dots, n)$  is indeed  $(r - \sigma^2/2)t/n$  (equation (26)).





# 17 THE HEATH-JARROW-MORTON MODEL OF FORWARD INTEREST RATES, BOND PRICES, AND DERIVATIVES

COUNTESS. It must be an answer of most monstrous size, that must fill all demands.

*All's Well That Ends Well*

## Chapter outline

<b>17.1</b>	<b>The one-factor model</b>	<b>360</b>
17.1.1	The forward rate as a Wiener process	361
17.1.2	The current short rate as a Wiener process	361
17.1.3	The cash account as a Wiener geometric process	362
17.1.4	The zero-coupon bond in current and present values as Wiener geometric processes	363
17.1.4.1	The zero-coupon bond in current value	363
17.1.4.2	The zero-coupon bond in present value	363
17.1.5	Finding a change of probability measure	363
17.1.6	The no-arbitrage condition	365
<b>17.2</b>	<b>An important application: the case of the Vasicek volatility reduction factor</b>	<b>368</b>
<b>Appendix 17.A.1</b>	<b>Determining the cash account's value <math>B(t)</math></b>	<b>372</b>
<b>Appendix 17.A.2</b>	<b>Derivation of Itô's formula, with an application to the solution of two important stochastic differential equations</b>	<b>374</b>

## Overview

Coupon-bearing bonds are portfolios of zero-coupon bonds. Because of the random nature of interest rates (or yields), these zero-coupon bonds, whose prices fluctuate in the opposite direction of interest rates, also qualify as random variables; inasmuch as interest rates follow random processes, zero-coupon bonds are transforms of such random processes. The difficulty in modeling the behavior of those zero-coupons is twofold: first, one should agree on a stochastic process for interest rates; second—and this is even more serious and profound—one has to recognize that if, for instance, instantaneous forward rates are driven by a Wiener process,

then the (time-dependent) drift coefficient of the process has to depend on the volatility coefficient in order to prevent arbitrage opportunities. This last property constitutes a major discovery in finance. It was made in 1989 by David Heath, Robert Jarrow, and Andrew Morton, and published in 1992 in *Econometrica*.

In its most general form, the Heath-Jarrow-Morton model is quite complex since it is an  $n$ -factor model. For our purposes we will need the one-factor version only, which we present here in some detail. As the reader will see, it constitutes a neat, powerful, and compelling means of analysis.

### 17.1 The one-factor model<sup>1</sup>

In this chapter and in the next one, we will always use the concept of the “forward rate” as introduced in section 11.3 of chapter 11 under the heading “Forward rates for instantaneous lending and borrowing.” We repeat the definition of those forward rates, where  $u$  is the contract inception time and  $z$  ( $z > u$ ) is the time at which the loan is made for an *infinitesimal* period (the time at which the borrowing starts for an infinitesimal period):

$f(u, z)$  = instantaneously compounded forward rate for a loan contracted at time  $u$ , and starting at time  $z$ , for an infinitesimal period.

We will first suppose that the *forward interest* rate follows a general Wiener process. As a particular case of that forward rate, we will then consider the *current rate*<sup>2</sup> (the forward rate when starting trading time tends toward the current time). We will define and evaluate the current account (the value that \$1 becomes continuously when rolled over at the current rate). These are the basic building blocks of the model. We

<sup>1</sup> This first section draws heavily on chapter 5, section 3, of M. Baxter and A. Rennie, *An Introduction to Derivative Pricing* (Cambridge, U.K.: Cambridge University Press, 1998), pp. 142–145.

<sup>2</sup> We prefer using here the term “current rate” rather than the term “spot rate” used in the original HJM (1992) article, since we have reserved the term “spot rate” for the rate agreed upon today for a loan *starting today* and *maturing* at  $T$ , continuously compounded. We designated that spot rate as  $R(0, T)$ , and we saw that it was equal to the average of all forward rates from 0 to  $T$  with infinitesimal trading periods;  $R(0, T) = T^{-1} \int_0^T f(0, t) dt$ .

then turn toward the zero-coupon bond's *current value* under the initial probability measure (i.e., under the initial Wiener process governing the forward rate) and to the zero-coupon's *present value* under the same probability measure. We will then be led to the two crucial steps of the Heath-Jarrow-Morton models. The first step is to find a *change in probability measure* such that the process describing the zero-coupon bond's present value becomes a martingale. In order to do that, using Itô's lemma, we will derive from the present value's process a differential equation; in that equation, we will make a change of variable such that the process becomes driftless. We will then take the second step, which will amount to showing that to prevent arbitrage there must be a definite relationship between the volatility coefficient and the drift rate; we will show the nature of that relationship. We will then be ready to use our valuation model, which we will apply in an all-important special case.

### 17.1.1 The forward rate as a Wiener process

Let  $t$  belong to the time interval  $[0, T]$ . On this interval we can consider the evolution of the forward interest rate  $f(t, T)$  either in differential form:

$$df(t, T) = \sigma(t, T) dW_t + \alpha(t, T) dt \quad (1)$$

or, equivalently, in integral form:

$$f(t, T) = f(0, T) + \int_0^t \sigma(s, T) dW_s + \int_0^t \alpha(s, T) ds \quad (2)$$

where  $W_t$  stands for a standard Brownian motion.

It is important to realize that both the drift of the forward process  $\alpha(t, T)$  and its volatility  $\sigma(t, T)$  are functions of the two variables  $t$  and  $T$ . Note also that for all forward rates  $f(t, T)$  there is only one source of uncertainty (the Brownian motion); thus, whatever their maturity, all forward rates will be perfectly correlated. Allowing multiple sources of uncertainty would allow for a less than perfect correlation among the rates. This is the case in the general Heath-Jarrow-Morton model.

### 17.1.2 The current short rate as a Wiener process

As a particular case of equation (2) giving the stochastic evolution of the forward rate  $f(t, T)$ , let us consider the forward rate *at time*  $t$  for a

trading time  $T$  tending toward  $t$  (when  $T \rightarrow t$ ), that is, when the trading horizon is itself  $t$  (of course when the trading *period* remains infinitesimally small). This forward rate thus becomes the current short rate at  $t$ , which we can denote as  $\lim_{T \rightarrow t} f(t, T) = f(t, t) \equiv r(t)$ . We then get

$$r(t) = f(0, t) + \int_0^t \sigma(s, t) dW_s + \int_0^t \alpha(s, t) ds \quad (3)$$

### 17.1.3 The cash account as a Wiener geometric process

Let  $B(t, t) \equiv B(t)$  denote the amount on the cash account at time  $t$ , opened at time 0, with \$1:  $B(0, 0) = 1$ . Suppose it earns the short rate  $r(t)$ . A time  $t$ , the value of that account is the random variable equal to the geometric Wiener process:

$$B(t) = e^{\int_0^t r(s) ds} \quad (4)$$

(Heath, Jarrow, and Morton call this cash account an accumulation factor or a money market account rolling over at  $r(t)$ .)

Substituting (3) into (4), this process is

$$B(t) = \exp \left[ \int_0^t f(0, s) ds + \int_0^t \int_0^s \sigma(u, s) dW_u ds + \int_0^t \int_0^s \alpha(u, s) du ds \right] \quad (5)$$

Because it will soon simplify computations, we will write the two double integrals in (5) in a different form. In appendix 17.A.1 we show that their sum can be written equivalently as

$$\int_0^t \int_s^t \sigma(s, u) du dW_s + \int_0^t \int_s^t \alpha(s, u) du ds$$

and therefore the value of the cash account, continuously reinvested at the stochastic rate  $r(t)$ , is equal to

$$B(t) = \exp \left[ \int_0^t f(0, u) du + \int_0^t \int_s^t \sigma(s, u) du dW_s + \int_0^t \int_s^t \alpha(s, u) du ds \right] \quad (6)$$

The inverse of (6),  $B^{-1}(t) = e^{-\int_0^t r(u) du}$ , will be used to determine the present value of the bond  $B(t, T)$ . Of course,  $B^{-1}(t)$  is nothing else than the present value (at time 0) of \$1 to be received at time  $t$ .

### 17.1.4 The zero-coupon bond in current and present values as Wiener geometric processes

#### 17.1.4.1 The zero-coupon bond in current value

Consider now the price at  $t$  of the bond maturing at  $T$ ,  $B(t, T)$ . It is equal to

$$B(t, T) = e^{-\int_t^T f(t, u) du} \quad (7)$$

or, making use of the forward rate given by (2),

$$B(t, T) = \exp \left[ - \int_t^T f(0, u) du - \int_0^t \int_t^T \sigma(s, u) du dW_s - \int_0^t \int_t^T \alpha(s, u) du ds \right] \quad (8)$$

(inverting the order of integration in the last two double integrals).

#### 17.1.4.2 The zero-coupon bond in present value

The present value (at time 0) of  $B(t, T)$  is  $e^{-\int_0^t r(u) du} B(t, T) = B^{-1}(t) \cdot B(t, T)$ . Dividing (8) by (6) and denoting by  $Z(t, T)$  the resulting present value of  $B(t, T)$ , we have

$$Z(t, T) = \exp \left[ - \int_0^T f(0, u) du - \int_0^t \int_s^T \sigma(s, u) du dW_s - \int_0^t \int_s^T \alpha(s, u) du ds \right] \quad (9)$$

(As the reader can notice, the calculation for the double integrals is straightforward, thanks to the change in notation just made in (5) and explained in appendix 17.A.1.)

### 17.1.5 Finding a change of probability measure

When valuing derivatives, we had seen previously that we needed a probability measure that would make the underlying security a martingale in present value. The last hurdle is thus to find a change in probability measure such that  $Z(t, T)$  becomes a martingale. One way to perform this is to take the differential of the stochastic process (using Itô's lemma—see appendix 17.A.2, part 1) and from there look for a change in the Brownian differential  $dW$  such that there is no drift term left.

Let us first determine the differential of  $Z(t, T)$ . An easy way to apply Itô's lemma is to consider that  $Z(t, T)$  is written in the form  $Z(t, T) =$

$e^{-X_t}$ , where  $X_t$  is the following Wiener process:

$$X_t = \int_0^T f(0, u) du + \int_0^t \int_s^T \sigma(s, u) du dW_s + \int_0^t \int_s^T \alpha(s, u) du dt \quad (10)$$

We can then write the differential of  $X_t$  as

$$dX_t = \int_t^T \sigma(t, u) du dW_t + \int_t^T \alpha(t, u) du dt \equiv v(t, T) dW_t + \int_t^T \alpha(t, u) du dt \quad (11)$$

where  $v(t, T) \equiv \int_t^T \sigma(t, u) du$  denotes the volatility of the  $X_t$  process. Applying Itô's lemma to  $Z(t, T) = e^{-X_t}$ , we get

$$dZ(t, T) = Z(t, T) \left[ -v(t, T) dW_t - \int_t^T \alpha(t, u) du dt + \frac{v^2}{2}(t, T) dt \right] \quad (12)$$

In order to find a new probability measure that makes  $Z(t, T)$  a martingale, we should make a change in  $dW_t$  such that equation (12) can be written in the form

$$dZ(t, T) = -Z(t, T) v(t, T) d\tilde{W}_t \quad (13)$$

Indeed, it is easy to show that in those circumstances, the solution of (13) is of the form

$$Z(t) = Z_0 \exp \left( \int_0^t v(\cdot, T) d\tilde{W} - \frac{1}{2} \int_0^t v^2(\cdot, T) dt \right) \quad (14)$$

which turns out to be a martingale (see 17.A.2, part 2.2). So our first task is to make the necessary measure change so that equation (12) is transformed into a driftless geometric Wiener motion.

What we want is a relationship between  $d\tilde{W}$  and  $dW_t$  such that

$$-v(t, T) dW_t - \int_t^T \alpha(t, u) du dt + \frac{v^2}{2}(t, T) dt = -v(t, T) d\tilde{W}_t \quad (15)$$

We are thus led to

$$d\tilde{W}_t = dW_t + \frac{1}{v(t, T)} \int_t^T \alpha(t, u) du dt - \frac{v(t, T)}{2} dt \equiv dW_t + \beta_t dt$$

and hence the change of measure  $\mathbb{Q}_t$  is

$$\mathbb{Q}_t = \frac{1}{v(t, T)} \int_t^T \alpha(t, u) du - \frac{v(t, T)}{2} \quad (16)$$

### 17.1.6 The no-arbitrage condition

The fundamental contribution of Heath, Jarrow, and Morton was to show that in order to avoid arbitrage, the volatility function had to be related to the original drift term through a constraint. This important result comes from the following reasoning: in order to prevent arbitrage opportunities, *any* bond with shorter maturity than  $T$  must have the *same* change of measure  $\mathbb{Q}_t$ . This implies that  $\mathbb{Q}_t$  *must* be independent from  $T$ . Multiply then (16) by  $v(t, T)$  and differentiate with respect to  $T$ ; we get

$$\alpha(t, T) = \frac{v(t, T)}{T} [v(t, T) + \mathbb{Q}_t] = \sigma(t, T)[v(t, T) + \mathbb{Q}_t] \quad (17)$$

Equations (16) and (17) are the two major constraints of the single-factor Heath-Jarrow-Morton model. Equation (16) stems from the change of drift necessitated by the transformation of  $Z(t, T)$  into a martingale; (17) results from the need to suppress arbitrage opportunities.

We thus arrive at the fundamental result of the model: the determination of the value for the drift coefficient of  $f(t, T)$  *under the  $\tilde{W}_t$  Brownian motion*. Recall that, from the basic property of the forward rate,

$$df(t, T) = \sigma(t, T) dW_t + \alpha(t, T) dt \quad (1)$$

In this equation, replace  $dW_t$  with  $d\tilde{W}_t - \mathbb{Q}_t dt$  and  $\alpha(t, T)$  with constraint (17). We get

$$\begin{aligned} df(t, T) &= \sigma(t, T)(d\tilde{W}_t - \mathbb{Q}_t dt) + \sigma(t, T)[v(t, T) + \mathbb{Q}_t] dt \\ &= \sigma(t, T) d\tilde{W}_t + \sigma(t, T)v(t, T) dt \end{aligned} \quad (18)$$

Thus in the single-factor Heath-Jarrow-Morton model, under the martingale measure, we *must* have a drift coefficient equal to

$$\sigma(t, T)v(t, T) = \sigma(t, T) \int_t^T \sigma(t, u) du$$

The following two-page “guide” summarizes the Heath-Jarrow-Morton one-factor stochastic model of bond evaluation. We will now apply the model in an important example.

### A Guide to the Heath-Jarrow-Morton One-Factor Model: Four Essential Steps<sup>3</sup>

#### 1. Hypotheses

- Evolution of forward rate  $f(t, T)$ :

$$df(t, T) = \sigma(t, T) dW_t + \alpha(t, T) dt \quad (1)$$

or

$$f(t, T) = f(0, T) + \int_0^t \sigma(s, T) dW_s + \int_0^t \alpha(s, T) ds \quad (2)$$

- Particular case: evolution of current short rate:

$$r(t) \equiv f(t, t) = f(0, t) + \int_0^t \sigma(s, t) dW_s + \int_0^t \alpha(s, t) ds \quad (3)$$

#### 2. Valuations

- Cash account:

$$\begin{aligned} B(t) = e^{\int_0^t r(s) ds} &= \exp \left[ \int_0^t f(0, u) du + \int_0^t \int_s^t \sigma(s, u) du dW_s \right. \\ &\quad \left. + \int_0^t \int_s^t \alpha(s, u) du ds \right] \end{aligned} \quad (6)$$

- Zero-coupon bond in current value:

$$\begin{aligned} B(t, T) = e^{-\int_t^T f(t, u) du} &= \exp \left[ -\int_t^T f(0, u) du - \int_0^t \int_t^T \sigma(s, u) du dW_s \right. \\ &\quad \left. - \int_0^t \int_t^T \alpha(s, u) du ds \right] \end{aligned} \quad (8)$$

<sup>3</sup> The numbering of the equations in this summary is that used in the text. We suppose throughout this “guide” that a number of technical conditions on  $\sigma(t, T)$  and  $\alpha(t, T)$ , as expressed in Baxter and Rennie (op. cit., p. 143), are met. Also, for the justification for the interchanging of limits in the double integrals involving stochastic differentials, see D. Heath, R. Jarrow, and A. Morton, “Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation,” *Econometrica*, 60, no. 1 (January 1992), appendix.



- Zero-coupon bond in present value:

$$\begin{aligned} Z(t, T) &= B^{-1}(t)B(t, T) \\ &= \exp \left[ - \int_0^T f(0, u) du - \int_0^t \int_s^T \sigma(s, u) du dW_s \right. \\ &\quad \left. - \int_0^t \int_s^T \alpha(s, u) du ds \right] \end{aligned} \quad (9)$$

### 3. Finding a change in probability measure that makes $Z(t, T)$ a martingale

- Apply Itô's lemma to  $Z(t, T)$ , set  $v(t, T) = \int_t^T \sigma(t, u) du$ :

$$dZ(t, T) = Z(t, T) \left[ -v(t, T) dW_t - \int_t^T \alpha(t, u) du dt + \frac{v^2}{2}(t, T) dt \right] \quad (12)$$

- Set  $d\tilde{W}_t$  equal to

$$d\tilde{W}_t = dW_t + \frac{1}{v(t, T)} \int_t^T \alpha(t, u) du dt - \frac{v(t, T)}{2} dt \equiv dW_t + \gamma_t dt$$

with the change of measure  $\gamma_t$ :

$$\gamma_t = \frac{1}{v(t, T)} \int_t^T \alpha(t, u) du - \frac{v(t, T)}{2} \quad (16)$$

### 4. The fundamental no-arbitrage condition: a restriction on the change of measure $\gamma_t$

- The change of measure  $\gamma_t$  must be independent from  $T$ ; set the differential of  $\gamma_t$  with respect to  $T$  equal to zero to obtain

$$\alpha(t, T) = \sigma(t, T)[v(t, T) + \gamma_t] \quad (17)$$

- In equation (1) replace  $dW_t$  with  $d\tilde{W}_t - \gamma_t dt$  and  $\alpha(t, T)$  with the right-hand side of (17).
- Then

$$df_t = \sigma(t, T) d\tilde{W}_t + \sigma(t, T)v(t, T) dt \quad (18)$$

Thus the drift coefficient under the martingale measure must be  $\sigma(t, T)v(t, T) = \sigma(t, T) \int_t^T \sigma(t, u) du$ .

### 17.2 An important application: the case of the Vasicek volatility reduction factor

In a pioneering study of the stochastic term structure, Oldrich Vasicek<sup>4</sup> introduced a volatility  $\sigma(t, T)$  coefficient in the form of  $\sigma \exp[-\lambda(T - t)]$ , where  $\sigma$  and  $\lambda$  are positive constants. We will make use of that hypothesis in the Heath-Jarrow-Morton model. Thus we can first calculate the value of the drift coefficient  $\alpha(t, T)$  under the martingale measure, with no arbitrage opportunities. Applying (17), we obtain

$$\alpha(t, T) = \sigma e^{-\lambda(T-t)} \int_t^T \sigma e^{-\lambda(u-t)} du = \sigma^2 \left[ \frac{e^{-\lambda(T-t)} - e^{-2\lambda(T-t)}}{\lambda} \right] \quad (19)$$

The forward rate process, under those circumstances, is thus

$$df(t, T) = \sigma e^{-\lambda(T-t)} d\tilde{W}_t + \frac{\sigma^2}{\lambda} [e^{-\lambda(T-t)} - e^{-2\lambda(T-t)}] dt \quad (20)$$

Integrating (20), we can write the process governing the forward rate as

$$\begin{aligned} f(t, T) &= f(0, T) + \sigma \int_0^t e^{-\lambda(T-v)} d\tilde{W}_v + \frac{\sigma^2}{\lambda} \int_0^t [e^{-\lambda(T-u)} - e^{-2\lambda(T-u)}] du \\ &= f(0, T) + \sigma \int_0^t e^{-\lambda(T-v)} d\tilde{W}_v \\ &\quad - \frac{\sigma^2}{2\lambda^2} [e^{-2\lambda T}(e^{2\lambda t} - 1) - 2e^{-\lambda T}(e^{\lambda t} - 1)] \end{aligned} \quad (21)$$

From (21), it is now possible to determine both  $B(t, T)$  and  $r(t)$ . First, calculate the bond's current value  $B(t, T)$ . Note that now the integrating variable is the second argument of  $f(\cdot)$ .

$$\begin{aligned} B(t, T) &= \exp \left[ - \int_t^T f(t, u) du \right] \\ &= \exp \left\{ - \int_t^T f(0, u) du + \frac{\sigma^2}{2\lambda^2} \int_t^T [e^{-2\lambda u}(e^{2\lambda t} - 1) - 2e^{-\lambda u}(e^{\lambda t} - 1)] du \right. \\ &\quad \left. - \sigma \int_t^T \int_0^t e^{-\lambda(u-v)} d\tilde{W}_v du \right\} \end{aligned}$$

<sup>4</sup> O. Vasicek, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5 (November 1977): 177–188.

$$= \exp \left\{ - \int_t^T f(0, u) du + \frac{\sigma^2}{4\lambda^3} [(1 - e^{2\lambda t})(e^{-2\lambda T} - e^{2\lambda t}) + 4(e^{\lambda t} - 1)(e^{-\lambda T} - e^{-\lambda t})] - \sigma \int_t^T \int_0^t e^{-\lambda(u-v)} d\tilde{W}_v du \right\} \quad (22)$$

This is the value of  $B(t, T)$ 's bond, viewed as a geometric Wiener process. There is unfortunately no way to express the stochastic integral  $\int_0^t e^{-\lambda(u-v)} d\tilde{W}_v$  in closed form; however, there is a nice way out, which was first presented by Heath, Jarrow, and Morton in their classic 1992 paper, in the special case of  $\lambda = 0$  (that is,  $\sigma(t, T) = \sigma$ , a constant). Here let us first derive  $r(t)$  by writing in (21)  $f(t, t) = r(t)$ :

$$r(t) = f(0, t) + \sigma \int_0^t e^{-\lambda(t-v)} d\tilde{W}_v - \frac{\sigma^2}{2\lambda^2} (2e^{-\lambda t} - e^{-2\lambda t} - 1) \quad (23)$$

(It can be checked that if  $\lambda = 0$ , then  $r(t) = f(0, t) + \sigma\tilde{W}_t + \frac{\sigma^2 t^2}{2}$ —as in the 1992 paper—by applying L'Hospital's rule to the last term of (23) twice.) Consider now the exponential part of the last term of (22). It can be written as

$$\begin{aligned} -\sigma \int_t^T \int_0^t e^{-\lambda(u-v)} d\tilde{W}_v du &= -\sigma \int_t^T \int_0^t e^{-\lambda(u-t+t-v)} d\tilde{W}_v du \\ &= -\sigma \int_t^T \int_0^t e^{-\lambda(u-t)} e^{-\lambda(t-v)} d\tilde{W}_v du \\ &= \sigma \int_0^t e^{-\lambda(t-v)} d\tilde{W}_v \cdot \left[ \frac{e^{-\lambda(T-t)} - 1}{\lambda} \right] \end{aligned} \quad (24)$$

Using equation (23), the stochastic integral  $\sigma \int_0^t e^{-\lambda(t-v)} d\tilde{W}_v$  can be expressed as the difference between the Wiener process  $r(t)$  and  $f(0, t)$  plus a deterministic function of time:

$$\sigma \int_0^t e^{-\lambda(t-v)} d\tilde{W}_v = r(t) - f(0, t) + \frac{\sigma^2}{2\lambda^2} (2e^{-\lambda t} - e^{-2\lambda t} - 1)$$

and thus the term containing the double integral in (22) can be written as

$$\begin{aligned} -\sigma \int_t^T \int_0^t e^{-\lambda(u-v)} d\tilde{W}_v du &= \left[ r(t) - f(0, t) + \frac{\sigma^2}{2\lambda^2} (2e^{-\lambda t} - e^{-2\lambda t} - 1) \right] \left[ \frac{e^{-\lambda(T-t)} - 1}{\lambda} \right] \end{aligned} \quad (25)$$

Using (25), we can now express the double integral in (22) in terms of the random variable  $r(t)$ . The bond's value thus becomes

$$\begin{aligned}
 B(t, T) = \exp \Bigg\{ & - \int_t^T f(0, u) du + \frac{\sigma^2}{4\lambda^3} [(1 - e^{2\lambda t})(e^{-2\lambda T} - e^{2\lambda t}) \\
 & + 4(e^{\lambda t} - 1)(e^{-\lambda T} - e^{-\lambda t})] \\
 & + \left[ r(t) - f(0, t) + \frac{\sigma^2}{2\lambda^2} (2e^{-\lambda t} - e^{-2\lambda t} - 1) \right] \\
 & \cdot \left[ \frac{e^{-\lambda(T-t)} - 1}{\lambda} \right] \Bigg\} \quad (26)
 \end{aligned}$$

After performing the appropriate calculations and cancellations of the deterministic terms in (26), we can write it as

$$\begin{aligned}
 B(t, T) = \exp \Bigg\{ & - \int_t^T f(0, u) du + [r(t) - f(0, t)] \left[ \frac{e^{-\lambda(T-t)} - 1}{\lambda} \right] \\
 & - \frac{\sigma^2}{4\lambda} \left[ \frac{1 + e^{-2\lambda(T-t)} - 2e^{-\lambda(T-t)}}{\lambda^2} \right. \\
 & \left. - e^{-2\lambda t} \left( \frac{1 + e^{-2\lambda(T-t)} - 2e^{-\lambda(T-t)}}{\lambda^2} \right) \right] \Bigg\} \quad (27)
 \end{aligned}$$

We now introduce the following simplifying notation, which will play an important role:

$$X(t, T) \equiv \frac{1 - e^{-\lambda(T-t)}}{\lambda} \quad (28)$$

Using this notation and observing that  $1 + e^{-2\lambda(T-t)} - 2e^{-\lambda(T-t)} = (1 - e^{-\lambda(T-t)})^2$ , we can write  $B(t, T)$  as

$$\begin{aligned}
 B(t, T) = \exp \Bigg\{ & - \int_t^T f(0, u) du - [r(t) - f(0, t)] X(t, T) \\
 & - \frac{\sigma^2}{4\lambda} X^2(t, T) (1 - e^{-2\lambda t}) \Bigg\} \quad (29)
 \end{aligned}$$

One last simplification is in order. The first exponential term on the right-hand side of (29) is

$$\begin{aligned}
e^{-\int_0^T f(0,u) du} &= e^{-[\int_0^T f(0,u) du - \int_0^t f(0,u) du]} \\
&= e^{-\int_0^T f(0,u) du} \cdot e^{\int_0^t f(0,u) du} = \frac{B(0, T)}{B(0, t)} \quad (30)
\end{aligned}$$

The bond's value can thus be expressed as a function of the random short rate  $r(t)$ :

$$B(t, T) = \frac{B(0, T)}{B(0, t)} \exp \left\{ -[r(t) - f(0, t)]X(t, T) - \frac{\sigma^2}{4\lambda} X^2(t, T)(1 - e^{-2\lambda t}) \right\} \quad (31)$$

This is the important formula given by Robert Jarrow and Stuart Turnbull in their excellent book *Derivative Securities*,<sup>5</sup> where they use the following notation:

$$r(t) - f(0, t) \equiv I(t)$$

and

$$-\frac{\sigma^2}{4\lambda} X^2(t, T)(1 - e^{-2\lambda t}) \equiv a(t, T)$$

At any time  $t$ , a bond with maturity  $T$  has the following value: it is the product of a nonrandom number (the ratio of the prices at time 0 of two bonds with maturities  $T$  and  $t$ ), which is perfectly well known, and a random variable, the exponential part of (31), which we will call  $J(t, T)$ . We thus have

$$B(t, T) = \frac{B(0, T)}{B(0, t)} J(t, T) \quad (31a)$$

We will want to verify that this last (random) number collapses to one if we take uncertainty out of the model by setting  $\sigma(t, T) = 0$ . But first we should check that  $B(t, T)$  takes the right values when  $t = 0$  and  $t = T$ .

• At  $t = 0$ ,  $r(0) = f(0, 0)$   $I = r(0) - f(0, 0) = 0$ ; also,  $a = 0$ . Therefore  $B(0, T) = \frac{B(0, T)}{B(0, 0)} = B(0, T)$ , as it should.

<sup>5</sup> R. Jarrow and S. Turnbull, *Derivative Securities* (Cincinnati, Ohio: South-Western, 1996).

- At  $t = T$ ,  $X(T, T) = 0$ ;  $a(T, T) = 0$ ; thus  $B(T, T) = 1$ .
- Suppose now that we remove all uncertainty by setting  $\sigma = 0$ . Taking into account the nonarbitrage restriction, the current short rate at  $t$ ,  $r(t)$ , is always equal to the forward rate for trading at  $t$ ,  $f(0, t)$  and is a constant, which we can denote by  $r$ . In those conditions of certainty,  $I = 0$  and  $a = 0$ .<sup>6</sup> Thus  $J(t, T) = 1$  and

$$B(t, T) = \frac{B(0, T)}{B(0, t)} = \frac{e^{-rT}}{e^{-rt}} = e^{-r(T-t)}$$

as it should.

Before going further, let us examine the exact nature of the random variable  $r(t)$ , since that variable carries all the randomness of the bond's value. From (23) the current, short rate can be written as

$$r(t) = f(0, t) + \frac{\sigma^2}{2} \left( \frac{1 - e^{-\lambda t}}{\lambda} \right)^2 + \sigma e^{-\lambda t} \int_0^t e^{\lambda v} d\tilde{W}_v \quad (32)$$

(We can verify that when  $t = 0$ ,  $r(0) = f(0, 0)$ , as it should.)

The stochastic integral  $\int_0^t e^{\lambda v} d\tilde{W}_v$  does not have a closed form; however, we know<sup>7</sup> that it is a normal variable with mean 0 and variance  $\int_0^t e^{2\lambda v} dv = \frac{1}{2\lambda} (e^{2\lambda t} - 1)$ . Thus the mean of  $r(t)$ , under the martingale measure, is  $f(0, t) + \frac{\sigma^2}{2} \left( \frac{1 - e^{-\lambda t}}{\lambda} \right)^2$  and its variance is  $\sigma^2 e^{-2\lambda t} \frac{1}{2\lambda} (e^{2\lambda t} - 1) = \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda t})$ .

We will now give an example of application of this one-factor Heath-Jarrow-Morton model by considering the problem of bond immunization. This will be the subject of our next chapter.

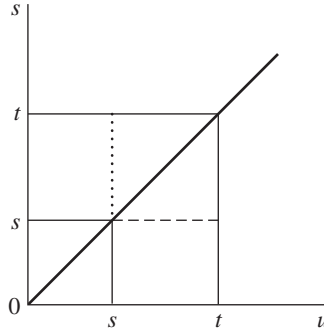
### Appendix 17.A.1 Determining the cash account's value $B(t)$

We explain here how to change the writing of the double integrals in equation (5), which gives the cash account's value. We start with

<sup>6</sup> See the definition of  $a(t, T)$  above—before equation (31a)—which contains  $\sigma$  in multiplication form.

<sup>7</sup> An intuitive proof for this is the following:  $dW_v$  has variance  $dv$ ;  $e^{\lambda v} dW_v$  has variance  $(e^{\lambda v})^2 dv = e^{2\lambda v} dv$ ; finally the sum of all independent variables  $e^{\lambda v} dW_v$  has a variance equal to the sum of the variances, which is  $\int_0^t e^{2\lambda v} dv$ .

$\int_0^t \int_0^s \alpha(u, s) du ds$ . This double integral is taken over the upper triangle of the following diagram:



For a given value of  $s$ —for a given height  $s$  on the ordinate—we integrate all elements  $\alpha(u, s) du$  for  $u$  from 0 to  $s$ ; this is  $\int_0^s \alpha(u, s) du$ . Each of these slices is a function of  $s$  for the following two reasons: parameters enter the function  $\alpha(u, s)$ , and the upper bound of the integral is  $s$ . Each of these slices is then integrated with respect to  $s$  from 0 to  $t$  to give  $\int_0^t \int_0^s \alpha(u, s) du ds$ . This is our initial double integral.

There are alternatives to this procedure. First, instead of adding horizontal slices as we did, we could as well add (horizontally) vertical slices  $\int_s^t \alpha(u, s) ds$ , corresponding, for instance, to the vertical dotted line. This amounts to calculating the integral  $\int_0^t \int_s^t \alpha(u, s) ds du$ ; this, however, involves an interchange in the order of integration. Second, an alternative method preserves the initial order of integration: consider the horizontal dashed line (symmetric to the vertical dotted line and of equal length) and integrate a slice along this line, with a proviso: change along that line any point with coordinates  $(u, s)$  into its symmetric point  $(s, u)$ . Our slice corresponding to the dashed line will thus be  $\int_s^t \alpha(s, u) du$ , equal to  $\int_s^t \alpha(u, s) ds$ ; our double integral is equal to  $\int_0^t \int_s^t \alpha(s, u) du ds$ . We can write the equality:  $\int_0^t \int_0^s \alpha(u, s) du ds = \int_0^t \int_s^t \alpha(s, u) du ds$ .

For the same reason, and some additional technical reasons related to its random character, the first double integral in equation (5) can be written as  $\int_0^t \int_s^t \sigma(s, u) du dW_s$ . (The precise justification for this may be found in the appendix of the Heath, Jarrow, and Morton 1992 paper, p. 99.) Equation (5) may thus be written in the form of equation (6).

## Appendix 17.A.2 Derivation of Itô's formula, with an application to the solution of two important stochastic differential equations

### 1. Itô's formula

We shall give here a heuristic derivation of Itô's formula. Suppose that  $X_t$  is a stochastic process governed by

$$dX_t = \mu_t dt + \sigma_t dW_t \quad (1)$$

where  $dW_t$  is the infinitesimal increment of Brownian motion  $W_t$ , equal to  $Z_t \sqrt{dt}$ ,  $Z_t \sim N(0, 1)$ . Consider  $Y_t = f(X_t, t)$ , where  $f$  is twice differentiable. We want to determine  $dY_t$ . Expanding  $Y_t$  in Taylor series gives

$$dY_t = \frac{f}{X_t} dX_t + \frac{f}{t} dt + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial X_t^2} dX_t^2 + 2 \frac{\partial^2 f}{\partial X_t \partial t} dX_t dt + \frac{\partial^2 f}{\partial t^2} dt^2 \right] + \text{higher order terms in } dX_t, dt. \quad (2)$$

Squaring  $dX_t$ , we get

$$dX_t^2 = \sigma_t^2 dW_t^2 + 2\sigma_t \mu_t dW_t dt + \mu_t^2 dt^2 \quad (3)$$

The last two terms in (3) are of order higher than  $dt$  and can be ignored when  $dt$  is sufficiently small. However, consider the  $(dW_t)^2$  term, which is equal to

$$(dW_t)^2 = (Z \sqrt{dt})^2 = dt Z^2, \quad Z \sim N(0, 1)$$

where  $Z$  denotes  $Z_t$ .

The expected value of  $(dW_t)^2$  is  $E(dt Z^2) = dt E(Z^2) = dt$  since  $E(Z^2) = 1$ . Its variance is equal to  $\text{VAR}(dt Z^2) = dt^2 \text{VAR}(Z^2) = 2dt^2$ , because  $\text{VAR}(Z^2)$  is equal to 2; indeed, we can write

$$\text{VAR}(Z^2) = E(Z^4) - [E(Z^2)]^2$$

The fourth moment of  $Z$  can be calculated as the fourth derivative of the moment-generating function of  $Z$ ,  $Z(\lambda) = e^{\lambda^2/2}$ , at point  $\lambda = 0$ ; we get  $E(Z^4) = 3$ , and therefore  $\text{VAR}(Z^2) = 2$ .

From the above, we conclude that the variance of  $(dW_t)^2$  tends toward zero when  $dt$  becomes extremely small; therefore, it ceases to be a random



variable and tends toward the constant  $E(Z^2 dt) = dt$ . Alternatively, we could make use of Chebyshev's inequality. We can write

$$\text{prob}\{|(dW_t)^2 - dt| < \} > 1 - \frac{\text{VAR}(dW)^2}{}$$

and, equivalently,

$$\text{prob}\{|(dW_t)^2 - dt| < \} > 1 - \frac{2dt^2}{}$$

which shows that however small a value we may choose for  $\epsilon$ , we can make  $dW_t^2$  converge toward  $dt$  by making  $dt$  sufficiently small. We then have

$$dX_t^2 \Rightarrow \sigma_t^2 dW_t^2 \Rightarrow \sigma_t^2 dt$$

Now  $dY_t$  has a part that tends toward  $\frac{1}{2} \frac{2f}{X_t^2} \sigma_t^2 dt$  when  $dt$  becomes very small. Being of order  $dt$ , it cannot be neglected, like the two other higher order terms  $\frac{2f}{X_t} dX dt$  and  $\frac{1}{2} \frac{2f}{X_t^2} dt^2$ . Therefore, the first-order differential of  $Y_t$  is equal to

$$dY_t = \frac{f}{X_t} dX_t + \frac{f}{t} dt + \frac{1}{2} \sigma_t^2 \frac{2f}{X_t^2} dt \quad (4)$$

Plugging  $dX_t$  from (1) into (4), we get

$$dY_t = \left( \frac{f}{X_t} + \frac{f}{t} + \frac{1}{2} \sigma_t^2 \frac{2f}{X_t^2} \right) dt + \sigma_t \frac{f}{X_t} dW_t \quad (5)$$

This is Itô's lemma. We can verify immediately that from any process

$$X_t = X_0 + \int_0^t v dv + \int_0^t \sigma_s dW_s \quad (6)$$

we can deduce  $dX_t = v dt + \sigma_t dW_t$ . Indeed, let us write  $f(X_t) = X_t$ ; then  $\frac{f}{X_t} = 1$ ;  $\frac{2f}{X_t^2} = 0$ ;  $\frac{f}{t} = 0$  and Itô's formula then leads to

$$dX_t = v dt + \sigma_t dW_t \quad (7)$$

as it should. Thus (6) is the solution, or the diffusion, of the stochastic differential equation (7).

## 2. The solution of the two important stochastic differential equations

### 2.1 The Doléans equation

Suppose we are given a stochastic differential equation (SDE) of the following form:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad (8)$$

This is called the Doléans SDE, and it is frequently used in finance. In order to find its solution, as is often the case in integration, we need to use trial and error. We first try a solution of the form

$$S_t = S_0 e^{mt + sW_t} \quad (9)$$

and then use Itô's formula to derive the differential of  $S$ ; we next see whether we can identify  $m$  and  $s$  such that, if used in (9), the differential of  $S$  would exactly match (8).

Let us apply Itô's formula to (9). We have, with  $X_t = mt + sW_t$  and  $dX_t = m dt + s dW_t$ ,  $S_t = S_0 e^{X_t}$ . Note here that  $\frac{d}{dt} = 0$  because  $S_t$  depends on time solely through Brownian motion  $X_t$  and does not exhibit additional dependence upon time. Therefore,

$$dS_t = (mS_0 e^{X_t} + \frac{1}{2}s^2 S_0 e^{X_t}) dt + sS_0 e^{X_t} dW_t \quad (10)$$

or, equivalently, with  $S_t = S_0 e^{X_t}$ ,

$$\frac{dS_t}{S_t} = (m + \frac{1}{2}s^2) dt + s dW_t \quad (11)$$

We can now identify the coefficients of  $dt$  and  $dW_t$  in (11) with those of (8) to obtain the values of  $m$  and  $s$ . We can write  $m + \frac{1}{2}s^2 = \mu$  and  $s = \sigma$ . The identification of  $s$  as  $\sigma$  is immediate, and  $m$  is given by  $\mu - \frac{\sigma^2}{2}$ . The solution is then

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \quad (12)$$

or, equivalently,

$$\log S_t = \log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t \quad (13)$$

This implies that

$$\log(S_t/S_0) = \left( -\frac{\sigma^2}{2} \right) t + \sigma W_t \quad (14)$$

Thus, the solution of the SDE (8) implies that the continuously compounded rate of return of  $S_t$  over period  $[0, t]$  is a Brownian motion with drift  $-\frac{\sigma^2}{2}$  and variance  $\sigma^2$ .

## 2.2 The solution of $dY_t = \sigma(t)Y_t dW_t$

In our last example, we saw that Itô's lemma led, in an exponential Wiener process, to subtracting in the exponent the term  $\frac{1}{2}\sigma^2$  from . So we can imagine that a solution to the above equation would be given by

$$Y_t = Y_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds\right) \quad (15)$$

Indeed, applying Itô's lemma to  $Y_t$  will yield the above differential equation.

We will now show that a necessary condition for  $Y_t$  to be a martingale is met. We must have  $E(Y_t) = Y_0$ . The expectation of  $Y_t$  is

$$E(Y_t) = Y_0 \exp\left(-\frac{1}{2} \int_0^t \sigma_s^2 ds\right) E\left[\exp\left(\int_0^t \sigma_s dW_s\right)\right] \quad (16)$$

The expectation term on the right-hand side of (16) can be written in the following way: let  $\Delta s_j$ ,  $j = 1, \dots, n$  be an equally spaced partition of the interval  $[0, t]$ , and  $\int_0^t \sigma(s) dW_s = \lim_{n \rightarrow \infty} \sum_{j=1}^n \sigma_j \Delta W_j$ , where all  $\Delta W_j$  variables are independent and normal  $N(0, \Delta s_j)$ . We can write

$$E\left\{e^{\int_0^t \sigma(s) dW_s}\right\} = E\left\{e^{\lim_{n \rightarrow \infty} \sum_{j=1}^n \sigma_j \Delta W_j}\right\} = \lim_{n \rightarrow \infty} E\left\{e^{\sum_{j=1}^n \sigma_j \Delta W_j}\right\}$$

Each expectation  $E(e^{\sigma_j \Delta W_j})$  is the moment-generating function of  $\Delta W_j \sim N(0, \Delta s_j)$ , where  $\sigma_j$  plays the role of the parameter; it is equal to  $e^{\frac{1}{2}\sigma_j^2 \Delta s_j}$ . Thus we have

$$\lim_{n \rightarrow \infty} E\left\{e^{\sum_{j=1}^n \sigma_j \Delta W_j}\right\} = \lim_{n \rightarrow \infty} e^{\frac{1}{2} \sum_{j=1}^n \sigma_j^2 \Delta s_j} = e^{\frac{1}{2} \int_0^t \sigma^2(s) ds}$$

and, finally,

$$E(Y_t) = Y_0 \exp\left(-\frac{1}{2} \int_0^t \sigma^2(s) ds\right) \exp\left(\frac{1}{2} \int_0^t \sigma^2(s) ds\right) = Y_0$$

Thus a necessary condition to be met for  $Y(t)$  to be a martingale.

### Main formulas

#### 1. A Guide to the Heath-Jarrow-Morton One-Factor Model: Four Essential Steps<sup>8</sup>

##### Hypotheses

- Evolution of forward rate  $f(t, T)$ :

$$df(t, T) = \sigma(t, T) dW_t + \alpha(t, T) dt \quad (1)$$

or

$$f(t, T) = f(0, T) + \int_0^t \sigma(s, T) dW_s + \int_0^t \alpha(s, T) ds \quad (2)$$

- Particular case: evolution of current, short rate:

$$r(t) \equiv f(t, t) = f(0, t) + \int_0^t \sigma(s, t) dW_s + \int_0^t \alpha(s, t) ds \quad (3)$$

##### Valuations

- Cash account:

$$\begin{aligned} B(t) &= e^{\int_0^t r(s) ds} \\ &= \exp \left[ \int_0^t f(0, u) du + \int_0^t \int_s^t \sigma(s, u) du dW_s + \int_0^t \int_s^t \alpha(s, u) du ds \right] \end{aligned} \quad (6)$$

- Zero-coupon bond in current value:

<sup>8</sup> The numbering of the equations in this summary is that used in the text. We suppose throughout this “guide” that a number of technical conditions on  $\sigma(t, T)$  and  $\alpha(t, T)$ , as expressed in M. Baxter and A. Rennie (op. cit., p. 143), are met. Also, for the justification for interchanging of limits in the double integrals involving stochastic differentials, see the appendix of Heath, Jarrow, and Morton (op. cit.).

$$\begin{aligned}
B(t, T) &= e^{-\int_t^T f(t, u) du} \\
&= \exp \left[ -\int_t^T f(0, u) du - \int_0^t \int_t^T \sigma(s, u) du dW_s - \int_0^t \int_t^T \alpha(s, u) du ds \right]
\end{aligned} \tag{8}$$

- Zero-coupon bond in present value:

$$\begin{aligned}
Z(t, T) &= B^{-1}(t) B(t, T) \\
&= \exp \left[ -\int_0^T f(0, u) du - \int_0^t \int_s^T \sigma(s, u) du dW_s - \int_0^t \int_s^T \alpha(s, u) du ds \right]
\end{aligned} \tag{9}$$

**Finding a change in probability measure that makes  $Z(t, T)$  a martingale**

- Apply Itô's lemma to  $Z(t, T)$ , set  $v(t, T) = \int_t^T \sigma(t, u) du$ :

$$dZ(t, T) = Z(t, T) \left[ -v(t, T) dW_t - \int_t^T \alpha(t, u) du dt + \frac{v^2}{2}(t, T) dt \right] \tag{12}$$

- Set  $d\tilde{W}_t$  equal to

$$d\tilde{W}_t = dW_t + \frac{1}{v(t, T)} \int_t^T \alpha(t, u) du dt - \frac{v(t, T)}{2} dt \equiv dW_t + \gamma_t dt$$

with the change of measure  $\gamma_t$ :

$$\gamma_t = \frac{1}{v(t, T)} \int_t^T \alpha(t, u) du - \frac{v(t, T)}{2} \tag{16}$$

**The fundamental no-arbitrage condition, a restriction on the change of measure  $\gamma_t$**

- The change of measure  $\gamma_t$  must be independent from  $T$ ; set the differential of  $\gamma_t$  with respect to  $T$  equal to zero to obtain

$$\alpha(t, T) = \sigma(t, T)[v(t, T) + \gamma_t] \tag{17}$$

- In equation (1) replace  $dW_t$  with  $d\tilde{W}_t - \gamma_t dt$  and  $\alpha(t, T)$  with the right-hand side of (17).

- Then

$$df_t = \sigma(t, T) d\tilde{W}_t + \sigma(t, T) v(t, T) dt \tag{18}$$

Thus the drift coefficient under the martingale measure must be  $\sigma(t, T)v(t, T) = \sigma(t, T) \int_t^T \sigma(t, u) du$ .

## 2. Application: the case of the Vasicek volatility reduction factor

### Drift coefficient

$$\alpha(t, T) = \sigma e^{-\lambda(T-t)} \int_t^T \sigma e^{-\lambda(u-t)} du = \sigma^2 \left[ \frac{e^{-\lambda(T-t)} - e^{-2\lambda(T-t)}}{\lambda} \right] \quad (19)$$

### Differential of forward rate process

$$df(t, T) = \sigma e^{-\lambda(T-t)} d\tilde{W}_t + \frac{\sigma^2}{\lambda} \left[ e^{-\lambda(T-t)} - e^{-2\lambda(T-t)} \right] dt \quad (20)$$

### Forward process

$$f(t, T) = f(0, T) + \sigma \int_0^t e^{-\lambda(T-v)} d\tilde{W}_v - \frac{\sigma^2}{2\lambda^2} \left[ e^{-2\lambda T} (e^{2\lambda t} - 1) - 2e^{-\lambda T} (e^{\lambda t} - 1) \right] \quad (21)$$

### Bond's current value

$$B(t, T) = \exp \left\{ - \int_t^T f(0, u) du + \frac{\sigma^2}{4\lambda^3} [(1 - e^{2\lambda t})(e^{-2\lambda T} - e^{2\lambda t}) + 4(e^{\lambda t} - 1)(e^{-\lambda T} - e^{-\lambda t})] - \sigma \int_t^T \int_0^t e^{-\lambda(u-v)} d\tilde{W}_v du \right\} \quad (22)$$

which is shown to be equal to

$$B(t, T) = \frac{B(0, T)}{B(0, t)} \exp \left\{ -[r(t) - f(0, t)]X(t, T) - \frac{\sigma^2}{4\lambda} X^2(t, T)(1 - e^{-2\lambda t}) \right\} \quad (31)$$

with

$$X(t, T) \equiv \frac{1 - e^{-\lambda(T-t)}}{\lambda} \quad (28)$$

## Questions

17.1. Make sure you can derive the value of the cash account  $B(t)$ , opened at time 0 with an initial value of \$1, as given by equation (5).

17.2. Make sure you can see why the double integral  $\int_0^t \int_0^s \alpha(u, s) du ds$  can be written as  $\int_0^t \int_s^t \alpha(s, u) du ds$  in the expression yielding the current account's value  $B(t)$  (equation (6)).

17.3. If you are not already familiar with Itô's lemma, read section 1 of appendix 17.A.2 and try to do the intuitive demonstration on your own.

17.4. Using Itô's lemma, obtain the expression of the differential  $dZ$  (equation (12)).

17.5. Derive the value of the drift term as expressed in equation (19).

17.6. Show that obtaining the bond's value using Vasicek's volatility coefficient  $\sigma(t, T)$  in the Heath-Jarrow-Morton framework implies three successive definite integrations, and carry them out.





# 18 THE HEATH-JARROW-MORTON MODEL AT WORK: APPLICATIONS TO BOND IMMUNIZATION

All losses are restor'd, and sorrows end.

*Sonnets*

## Chapter outline

<b>18.1</b>	<b>The immunization process</b>	<b>383</b>
18.1.1	The case of a zero-coupon bond	384
18.1.2	Comparison of longer-term hedging portfolios	389
<b>18.2</b>	<b>Immunizing a bond under the Heath-Jarrow-Morton model</b>	<b>391</b>
<b>18.3</b>	<b>Other applications of the Heath-Jarrow-Morton model</b>	<b>401</b>

## Overview

In the preceding chapter we described how to obtain, from Wiener processes governing the forward rate, a closed form for the price of a zero-coupon bond at time  $t$ , valued as a function of the difference between the short rate at time  $t$  and the forward rate that had been set by the market at time 0 for that date. We did so using the Heath-Jarrow-Morton one-factor model, where only one Brownian motion drove the forward rate. We will now give an application in the form of immunization. From the first part of this book, we remember that replicating a zero-coupon bond required us to construct another portfolio with a matching duration. Here we will not be surprised to find that the hedging portfolio will exhibit a somewhat different structure, and an important question will be, how different is a Heath-Jarrow-Morton portfolio from a classical portfolio, constructed on the premises that the structure of interest rates receives a parallel shift? In the case of nonparallel moves of the yield curve, how different are the portfolios, and how different are their durations? Important conclusions can be derived from such a comparison.

## 18.1 The immunization process

The sensitivity of the bond to any shock received by the random variable  $I(t) = r(t) - f(0, t)$ , at any point in time, can be analyzed as follows: let  $I_0$  denote an *initial* value of  $I$  at any time  $t$ ;  $I_1$  denotes a new value that  $I$

can take a very short instant after. Developing  $B(I_1)$  in Taylor series at point  $I_0$  up to the second order,

$$B(I_1) = B(I_0) + B'(I_0)(I_1 - I_0) + \frac{1}{2} B''(I_0)(I_1 - I_0)^2 \\ + \text{terms of higher order in } I_1 - I_0 \quad (1)$$

and thus, setting  $\frac{\Delta B}{B} \equiv \frac{B(I_1) - B(I_0)}{B(I_0)}$  and  $I_1 - I_0 \equiv \Delta I$ , a second-order approximation of the shock received by  $B$  is

$$\frac{\Delta B}{B} \approx \frac{B'(I_0)}{B(I_0)} \Delta I + \frac{1}{2} \frac{B''(I_0)}{B(I_0)} (\Delta I)^2 \quad (2)$$

In the one-factor Heath-Jarrow-Morton model, the expressions  $\frac{B'(I_0)}{B(I_0)}$  and  $\frac{B''(I_0)}{B(I_0)}$  are obtained by differentiating equation (31) of our chapter 17. Because of the exponential nature of (31), these turn out to be simply

$$\frac{B'(I_0)}{B(I_0)} = -X(t, T) \quad (3)$$

and

$$\frac{B''(I_0)}{B(I_0)} = X^2(t, T) \quad (4)$$

Thus the relative increase incurred by  $B$  is, in second-order approximation,

$$\frac{\Delta B}{B} \approx -X(t, T) \Delta I + \frac{1}{2} X^2(t, T) (\Delta I)^2 \quad (5)$$

This relationship enables us to define easily a hedging policy for any zero-coupon bond or any coupon-bearing bond or any portfolio of bonds for that matter. We will first consider protecting a single zero-coupon bond and will then turn toward the immunization of a coupon-bearing bond.

### 18.1.1 The case of a zero-coupon bond

Suppose we want to protect a bond<sup>1</sup> against a shock in  $r(t)$ , or equivalently, against a shock in  $I(t)$ . What we want is to constitute a portfolio

<sup>1</sup>In this section the term “bond” will always mean a zero-coupon bond.

replicating the bond to be protected with other bonds (three of them, for reasons that will appear soon) such that its initial value is zero and its new value (in the case of a change in  $I(t)$ ) is also zero. Let  $V(I_0)$  be the value of that portfolio and  $n_a, n_b, n_c$  the number of bonds  $a, b, c$  with values  $B_a(I), B_b(I)$ , and  $B_c(I)$  to be sold (or bought). We have, by definition,

$$V(I_0) = B(I_0) + n_a B_a(I_0) + n_b B_b(I_0) + n_c B_c(I_0) = 0 \quad (6)$$

When  $I$  moves from  $I_0$  to  $I_1$ ,  $V$  becomes, in quadratic approximation

$$V(I_1) \approx V(I_0) + V'(I_0)\Delta I + \frac{1}{2} V''(I_0)(\Delta I)^2 \quad (7)$$

For  $V(I_1)$  to remain equal to  $V(I_0) = 0$  whatever the change  $\Delta I$ , a set of necessary and sufficient conditions is that all the coefficients of the polynomial in the right-hand side of (7) are zero:

- i)  $V(I_0) = 0$
- ii)  $V'(I_0) = 0$
- iii)  $V''(I_0) = 0$

Equation (i) translates as

$$B(I_0) + n_a B_a(I_0) + n_b B_b(I_0) + n_c B_c(I_0) = 0 \quad (8)$$

For simplicity, denote by  $X$  the coefficient  $X(t, T)$  corresponding to the initial bond and by  $X_a, X_b, X_c$  the coefficients corresponding to bonds  $a, b$ , and  $c$ . Use the fact that  $B'(I_0) = -XB(I_0)$ ; equation (ii) implies, after simplification,

$$XB(I_0) + n_a X_a B_a(I_0) + n_b X_b B_b(I_0) + n_c X_c B_c(I_0) = 0 \quad (9)$$

Finally, use the relationship  $B''(I_0) = X^2 B(I_0)$ ; equation (iii) implies

$$X^2 B(I_0) + n_a X_a^2 B_a(I_0) + n_b X_b^2 B_b(I_0) + n_c X_c^2 B_c(I_0) = 0 \quad (10)$$

(The reader can now see why we have to find three zero-coupons to complete our portfolio. Developing the portfolio's value  $V(I)$  into a second-order Taylor series entails a second-order polynomial whose three coefficients have to vanish—hence a system of three equations, which in turn implies at least three unknowns—the numbers  $n_a, n_b$ , and  $n_c$  of additional bonds to make up the portfolio.)

For sake of convenience, we have rewritten and regrouped equations (8) to (10) in a slightly simplified form, which enables us to see immediately the simple matrix structure of the system (in particular, we have not included the dependencies of the values of the zero-coupons on  $I$ ):

$$B_a n_a + B_b n_b + B_c n_c = -B \quad (8a)$$

$$X_a B_a n_a + X_b B_b n_b + X_c B_c n_c = -XB \quad (9a)$$

$$X_a^2 B_a n_a + X_b^2 B_b n_b + X_c^2 B_c n_c = -X^2 B \quad (10a)$$

Solving this system yields the number of bonds  $n_a$ ,  $n_b$ , and  $n_c$  to be sold or purchased. Note that the  $B$ 's are positive and that, for  $\lambda > 0$ , the  $X$ 's are always positive; this implies that at least one of the additional bonds will be held short in the portfolio. An example will illustrate the construction of such a portfolio.

Before we proceed with this example, we should reflect on the particular significance of the  $\lambda$  coefficient in our model and on the central role played by the  $X(t, T)$  variable. Remember that  $\sigma(t, T) = \sigma e^{-\lambda(T-t)}$ . Thus  $\lambda = -\frac{1}{\sigma} \frac{d\sigma}{dT}$ : it is the *instantaneous* rate of decrease of the forward rate's volatility coefficient with respect to the trading horizon  $T - t$ , longer horizons having a lower volatility than shorter ones.

When the trading horizon increases by one year, the volatility's relative increase is, in *exact value*,

$$\frac{\Delta\sigma(t, T)}{\sigma(t, T)} = \frac{e^{-\lambda(T+1-t)} - e^{-\lambda(T-t)}}{e^{-\lambda(T-t)}} = \frac{e^{-\lambda(T+1-t)}}{e^{-\lambda(T-t)}} - 1 = e^{-\lambda} - 1$$

The volatility increase turns out to be  $e^{-\lambda} - 1$ . In linear approximation (developing  $e^{-\lambda} - 1$  in a Taylor series limited to its term of order one in  $\lambda$ ), it is approximately equal to  $-\lambda$ ; indeed,  $e^{-\lambda} - 1 \approx -\lambda$ . For instance, suppose that  $\lambda = 10\%$ . This implies that the *exact* relative increase in the volatility is  $e^{-0.1} - 1 = -0.0952 = -9.52\%$ ; thus the volatility coefficient decreases by 9.52% per additional year in the trading period, while it decreases by 10% in linear approximation.

Let us now turn to the all-important coefficient

$$X(t, T) = \frac{1 - e^{-\lambda(T-t)}}{\lambda}$$

As we noticed before, the exponential expression for the zero-coupon bond's value—equation (31) of chapter 17—reveals that  $X(t, T)$  is simply

minus the sensitivity coefficient of  $B(t, T)$  with respect to the variable  $I(t) = r(t) - f(0, t)$  and that  $X^2(t, T)$  is the convexity coefficient of  $B(t, T)$  with respect to the same variable  $I(t)$ . Those properties enable us to answer easily the question of the units in which  $X(t, T)$  is expressed. Since  $X$  is  $-\frac{1}{B(t, T)} \frac{dB(t, T)}{dI}$  and since  $dI$  is expressed in 1/year, it is immediate that  $X(t, T)$  is expressed in years.

From the hypotheses of the model, we know that  $\lambda = 0$  implies that  $\sigma(t, T) = \sigma$ , a constant. Whatever shock interest rates will receive, the shock will be independent from the time horizon, which means that the forward rate curve will receive a parallel displacement. Hence, a zero-coupon bond with maturity (or duration)  $T - t$  will have a sensitivity equal to its duration  $T - t$ . This can be verified by taking the limit of  $X(t, T)$  when  $\lambda \rightarrow 0$ ; indeed, applying L'Hospital's rule, we have

$$\lim_{\lambda \rightarrow 0} X(t, T) = \lim_{\lambda \rightarrow 0} \frac{1 - e^{-\lambda(T-t)}}{\lambda} = T - t$$

We now take up again the question we alluded to in this chapter's introduction: consider the general single-factor Heath-Jarrow-Morton model where  $\lambda$  is not equal to zero ( $\lambda > 0$ ). Suppose we want to immunize a zero-coupon bond against shocks in interest rates. How does a hedging policy based upon the Heath-Jarrow-Morton model differ from a policy resulting from the classical results about duration? The question makes a lot of sense, since we may well imagine that a model like the one developed by Heath, Jarrow, and Morton might yield quite different results compared to those obtained with the classical model.

For easy reference, and in order to have comparable results, the example that will serve as our basis for work is the one considered in Jarrow and Turnbull.<sup>2</sup> In this base case,  $\sigma = 0.01$  and  $\lambda = 0.1$ ; a one-year zero-coupon is to be hedged with three zero-coupon bonds, with maturities equal to 0.5, 1.5, and 2 years, respectively. The  $B$ ,  $BX$ , and  $BX^2$  values are given in table 18.1.

Let us now apply these data to systems (8) through (10) and solve it in  $n_a$ ,  $n_b$ ,  $n_c$ . We get, as Jarrow and Turnbull did:  $n_a = -0.3093$ ,  $n_b = -1.0776$ , and  $n_c = 0.3872$ . Now there are some interesting questions to ask: how far is this hedging portfolio from a "classical" portfolio that

<sup>2</sup>R. Jarrow and S. Turnbull, *Directive Securities* (Cincinnati, Ohio: South-Western, 1996), p. 494.

**Table 18.1**  
**Base case:  $\sigma = 0.01$ ;  $\lambda = 0.1$**

Maturity (years)	Discount rate (per year)	Price $B$	$X$ (years)	$XB$	$X^2B$
1	0.0458	95.42	0.951626	90.80414	86.41156
0.5	0.0454	97.73	0.487706	47.66348	23.24576
1.5	0.0461	93.085	1.39292	129.66	180.606
2	0.0464	90.72	1.812692	164.4475	298.0927

would be built on the premise that the whole structure would undergo a parallel shift? And how different is the duration of the hedging portfolio (which will be referred to henceforth as the HJM portfolio) from the duration of the classical portfolio (which, of course, is equal to one)?

To answer the first question, let us recalculate our values  $X$ ,  $XB$ , and  $X^2B$  when  $\lambda = 0$  (or, equivalently, when  $X = T - t$ ) and solve systems (8) through (10). We obtain  $n_a = -0.325$ ,  $n_b = -1.025$ , and  $n_c = 0.351$ . That “classical portfolio” is rather close to the HJM one. Now calculate the duration of the initial portfolio. Denoting by  $\alpha_a$ ,  $\alpha_b$ , and  $\alpha_c$  the shares of each bond in the portfolio, those shares are

$$\alpha_a = n_a \frac{B_a}{B_p} = 0.31681$$

$$\alpha_b = n_b \frac{B_b}{B_p} = 1.051271$$

$$\alpha_c = n_c \frac{B_c}{B_p} = -0.36808$$

On the other hand, the duration of the portfolio is equal to

$$D_p = \alpha_a D_a + \alpha_b D_b + \alpha_c D_c$$

(that is, it is equal to the average of the durations, weighted by the respective shares of each bond in the portfolio). We thus get

$$D_p = (0.31681)0.5 + (1.051271)1.5 + (-0.36808)2 = 0.9992$$

The duration of the HJM portfolio is practically the same as the duration of the classical portfolio. This is an interesting property, which is somewhat at odds with what Jarrow and Turnbull state. In their remarkable text (p. 495), the authors started by hedging the one-year bond with

the first-order sensitivity coefficient only, thereby using two equations in two unknowns. They had (quite correctly, of course) obtained the portfolio  $n_a = -0.4760$  and  $n_b = -0.5254$ . Unfortunately, they were led to calculate (by some unfortunate typos) the duration of the resulting portfolio as

$$0.4760(0.5) + 0.5254(1.5) = 1.0261$$

In this equation, the left-hand side adds two quantities expressed in different units:  $(0.4760)(0.5)$  is in (units of bond  $a$ ) times years;  $0.5254(1.5)$  is in (units of bond  $b$ ) times years. These cannot be added, and the result is not in years, as it should be. The correct calculation should be the weighted average of the durations, where the weights are the shares of each zero-coupon in the balancing portfolio. Here those shares are  $-0.476 \cdot \frac{97.73}{-95.42} = 0.4875$  and  $-0.525 \cdot \frac{93.085}{-95.42} = 0.5125$ , and therefore the duration is

$$0.4875(0.5) + 0.5125(1.5) = 1.0124 \text{ years}$$

which yields a duration significantly closer to one—quite a remarkable result considering that only a linear approximation had been used in the hedging process. This invites us to examine whether the kind of (true) proximity we have observed between the duration of an HJM portfolio and a classical portfolio stands the trial of larger  $\lambda$  values, as well as longer maturities for the bond to be hedged and the bonds of the hedging portfolio.

We started by considering the same set of zero-coupon bonds as above and varying the volatility reduction coefficient from  $\lambda = 0.05$  to 0.2. The results in terms of the hedging portfolios, and their durations, are found in table 18.2.

Note that even if the volatility reduction factor is as high as 20%, the duration of the hedging portfolio differs from one by less than 1% (3.5 per thousand means that the order of magnitude of the difference in durations between the classical portfolio and the HJM portfolio is of the order of one day—a negligible difference).

### 18.1.2 Comparison of longer-term hedging portfolios

Let us now run the test of longer zero-coupon bonds. We have considered a two-year zero-coupon bond to be hedged with the help of bonds  $a$ ,  $b$ ,

**Table 18.2**  
**Hedging portfolios and their durations. Same bonds;  $\lambda = 0$  to 0.2.**

Number of bonds in hedging portfolio	Classical portfolio $\lambda = 0$	HJM portfolios ( $\lambda > 0$ )			
		$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.15$	$\lambda = 0.2$
$n_a$	-0.325	-0.317	-0.309	-0.301	-0.284
$n_b$	-1.025	-1.051	-1.078	-1.105	-1.133
$n_c$	0.351	0.369	0.387	0.407	0.427
Portfolio's duration	1	0.9998	0.9991	0.998	0.9965

**Table 18.3**  
**Data for longer maturities**

Maturity	Discount	Price
2	0.0458	90.84
1	0.0454	95.46
3	0.0461	86.17
4	0.0464	81.44

and  $c$ , whose maturities are twice those considered earlier (1, 3, and 4 years, respectively). The term structure as defined by discount rates and the corresponding data are shown in table 18.3.

The resulting hedging portfolios and their durations, corresponding to volatility reduction factor values from  $\lambda = 0$  to  $\lambda = 2$ , are represented in table 18.4.

We can see from these results that for a usual value of the volatility reduction factor ( $\lambda = 0.1$ ) the duration of the hedging HJM portfolio is still less than 1% of a year different from the duration of the classical hedging portfolio (1.993 years as opposed to two years). We have to double the volatility reduction factor ( $\lambda = 0.2$ ) to have a difference of 3% of a year between those durations.

It is our view that such small differences stem from a remarkable property of the Heath-Jarrow-Morton model: it provides the classical immunization results in the special case of  $\lambda = 0$ ; and when  $\lambda$  goes away from zero, the immunization process *generally* starts to differ from the duration that a classical process would call for, but it does so in a smooth and regular way, rather than abruptly. Also, it is perfectly logical that there is an infinite number of cases in which, even if the term structure displacement is not parallel, the durations will match *exactly* in the clas-



**Table 18.4**

**Hedging portfolios and their durations bond  $a$ : 1 year; bond  $b$ : 3 years; bond  $c$ : 4 years;  $\lambda = 0$  to 2.0**

Number of bonds in hedging portfolio	Classical portfolio $\lambda = 0$	HJM portfolios ( $\lambda > 0$ )			
		$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.15$	$\lambda = 0.2$
$n_a$	-0.317	-0.301	-0.285	-0.271	-0.256
$n_b$	-1.054	-1.108	-1.165	-1.225	-1.288
$n_c$	0.372	0.411	0.351	0.498	0.547
Portfolio's duration (years)	2	1.998	1.993	1.984	1.971

sical and the HJM cases. This is, after all, what we would expect from a model that provides nonparallel shifts through a reduction factor in volatility of the forward rate process.

Our task now is to see whether the same conclusions apply in the immunization of coupon-bearing bonds.

## 18.2 Immunizing a bond under the Heath-Jarrow-Morton model

In order to avoid possible confusion between the price of a zero-coupon bond and the price of a coupon-bearing bond, we will use the following notation:

$P(t, T_n \mid I)$  = value of a bond at time  $t$ , with maturity at date  $T_n$ ; its dependency on  $I = r(t) - f(0, t)$  is underlined; the letter  $P$  stands both for the *price* of the bond and for a *portfolio* of  $n$  zero-coupon bonds: the  $n$  payments, or cash flows paid out by the issuer of the bond to its owner; the first  $n - 1$  are equal to the coupons; the last one,  $c_n$ , is the last coupon *plus* the principal.

For short,  $P(t, T_n \mid I)$  will be abbreviated to  $P(I)$  or even  $P$ .

$c_i$  = number of dollars paid by the issuer at time  $T_i$  ( $i = 1, \dots, n$ ).

$B(t, T_i \mid I)$  = value at time  $t$  of a zero-coupon paying one dollar at time  $T_i$ .

$c_i B(t, T_i \mid I)$  = value at time  $t$  of  $c_i$  dollars paid out by the issuer at time  $T_i$ .

We thus have

$$P(t, T_n \mid I) = \sum_{i=1}^n c_i B(t, T_i \mid I) \quad (11)$$

Under the Heath-Jarrow-Morton one-factor model, the bond's theoretical value is thus the portfolio of the  $n$  zero-coupons, \$1 principal bonds  $B(t, T_i)$ , each of which can be evaluated through formula (31) of chapter 17.

In order to immunize this coupon-bearing bond, we will proceed in the same way as before in the case of zero-coupon bonds. To simplify notation, our variables  $P(t, T_n | I)$  and  $B(t, T_i | I)$  will be written  $P(I)$  and  $B_i(I)$ ; thus  $P(I) = \sum_{i=1}^n c_i B_i(I)$ .

First, develop  $P(I)$  in a second-order Taylor series around  $I_0$ . We have

$$P(I_1) \approx P(I_0) + P'(I_0)(I_1 - I_0) + \frac{1}{2} P''(I_0)(I_1 - I_0)^2 \quad (12)$$

Setting  $\frac{\Delta P}{P} \equiv \frac{P(I_1) - P(I_0)}{P(I_0)}$  and  $I_1 - I_0 \equiv \Delta I$ , the relative increase in the bond's value after a shock  $\Delta I$  is, with a second-order approximation

$$\frac{\Delta P}{P} \approx \frac{P'(I_0)}{P(I_0)} \Delta I + \frac{1}{2} \frac{P''(I_0)}{P(I_0)} (\Delta I)^2 \quad (13)$$

Let us now determine the two coefficients of this second-order polynomial. First, from (11) we have

$$\frac{P'(I_0)}{P(I_0)} = \sum_{i=1}^n \frac{c_i B'_i(I_0)}{P(I_0)} = \sum_{i=1}^n \frac{c_i B_i(I_0)}{P(I_0)} \cdot \frac{B'_i(I_0)}{B_i(I_0)} \quad (14)$$

Replacing  $B'(I_0)/B_i(I_0)$  with  $-X(t, T_i) \equiv -X_i$  and the shares of each cash flow in the bond  $\frac{c_i B_i(I_0)}{P(I_0)}$  with  $\alpha_i$  ( $\sum_{i=1}^n \alpha_i = 1$ ), we have

$$\frac{P'(I_0)}{P(I_0)} = - \sum_{i=1}^n \alpha_i X_i \equiv -\bar{X} \quad (15)$$

Thus the relative change in the coupon-bearing bond is simply *minus* the weighted average of the  $X_i$ 's, the weights being the share of each of the cash flows in the bond's value.

Similarly, the second-order term is

$$\frac{P''(I_0)}{P(I_0)} = \sum_{i=1}^n \frac{c_i B''_i(I_0)}{P(I_0)} = \sum_{i=1}^n \frac{c_i B_i(I_0)}{P(I_0)} \cdot \frac{B''_i(I_0)}{B_i(I_0)} \quad (16)$$

We know that each variable  $B''_i(I_0)/B_i(I_0)$  is simply  $X^2(t, T_i) \equiv X_i^2$ . Thus

$$\frac{P''(I_0)}{P(I_0)} = \sum_{i=1}^n \alpha_i X_i^2 \equiv \bar{X}^2 \quad (17)$$

The second-order coefficient is the weighted average of the squares of the  $X_i$ 's. The relative change in the bond's value as expressed in (13) can be written as

$$\frac{\Delta P}{P} \approx -\bar{X}\Delta I + \frac{1}{2}\bar{X}^2 P(\Delta I)^2 \quad (18)$$

and the increase in the bond's value is approximately equal to

$$\Delta P = -\bar{X}P\Delta I + \frac{1}{2}\bar{X}^2 P(\Delta I)^2 \quad (19)$$

This relationship applies directly to any bond, with corresponding  $\bar{X}$  and  $\bar{X}^2$  coefficients.

Suppose we now want to protect our coupon-bearing bond whose value is  $P(I)$  against any shock, at time  $t$ , received by  $I$ . We can think of building a hedging portfolio by adding positions (positive or negative) in three bonds,  $A$ ,  $B$ , and  $C$ , to our initial bond. Let  $N_A$ ,  $N_B$ , and  $N_C$  be the number of bonds to be determined; the prices of the bonds are  $P_A$ ,  $P_B$ , and  $P_C$ , respectively, and the value of the hedging portfolio is  $V_h$ . Thus, if our hedging portfolio is to have initial zero value, a first equation to be satisfied by the unknowns  $N_A$ ,  $N_B$ ,  $N_C$  must be

$$V_h(I_0) = P(I_0) + N_A P_A(I_0) + N_B P_B(I_0) + N_C P_C(I_0) = 0 \quad (20)$$

When  $I$  undergoes a shock from  $I_0$  to  $I_1$ , the hedging portfolio takes a new value equal to (in second-order approximation)

$$V_h(I_1) \approx V_h(I_0) + V_h'(I_0)\Delta I + \frac{1}{2}V_h''(I_0)(\Delta I)^2 \quad (21)$$

For the hedging portfolio to remain fixed at  $V_h(I_0) = 0$  whatever the values of  $\Delta I$  and  $(\Delta I)^2$  may be, a first condition is that (20) is met. In addition, the coefficients  $V_h'(I_0)$  and  $V_h''(I_0)$  in (21) must be equal to zero.  $V_h'(I_0) = 0$  implies that

$$V_h'(I_0) = P'(I_0) + N_A P_A'(I_0) + N_B P_B'(I_0) + N_C P_C'(I_0) = 0 \quad (22)$$

From (15), we know that  $P'(I_0) = -\bar{X}P(I_0)$ . As we mentioned earlier, this relation extends of course to bonds  $A$ ,  $B$ , and  $C$ , so we have  $P_A'(I_0) = -\bar{X}_A P_A(I_0)$ , and so forth. The above relation can then be written as

$$V'_h(I_0) = -\bar{X}P(I_0) - N_A\bar{X}_AP_A(I_0) - N_B\bar{X}_BP_B(I_0) - N_C\bar{X}_CP_C(I_0) = 0 \quad (23)$$

which implies

$$\bar{X}P(I_0) + N_A\bar{X}_AP_A(I_0) + N_B\bar{X}_BP_B(I_0) + N_C\bar{X}_CP_C(I_0) = 0 \quad (24)$$

This is the second equation of our system in  $N_A$ ,  $N_B$ ,  $N_C$ . The third equation is provided by  $V''_h(I_0) = 0$ :

$$V''_h(I_0) = P''(I_0) + N_AP''_A(I_0) + N_BP''_B(I_0) + N_CP''_C(I_0) = 0 \quad (25)$$

From (17), we know that  $P''(I_0) = \bar{X}^2P(I_0)$ , and as for the first-order derivative, this relationship extends to any other bond. The preceding equation may therefore be written as

$$\bar{X}^2P(I_0) + N_A\bar{X}_A^2P_A(I_0) + N_B\bar{X}_B^2P_B(I_0) + N_C\bar{X}_C^2P_C(I_0) = 0 \quad (26)$$

The system to be solved corresponds to the following, written in simplified form:

$$P_AN_A + P_BN_B + P_CN_C = -P \quad (27)$$

$$\bar{X}_AP_AN_A + \bar{X}_BP_BN_B + \bar{X}_CP_CN_C = -\bar{X}P \quad (28)$$

$$\bar{X}_A^2P_AN_A + \bar{X}_B^2P_BN_B + \bar{X}_C^2P_CN_C = -\bar{X}^2P \quad (29)$$

Thus, the linearity of the bonds in their components has provided us with a system of equations (27) through (29), which is basically the same as our previous system (8a) through (10a), corresponding to hedging a zero-coupon. The prices of the zero-coupons (the  $B$ 's) are replaced by the prices of the bonds (the  $P$ 's); the numbers of each individual zero-coupon (the  $n$ 's) are replaced by the numbers of each bond; each individual zero-coupon's  $X$  is replaced by  $\bar{X}$ , the weighted average of the  $X$ 's of each bond; and finally the square of  $X$  is replaced by the weighted average of all squares  $X^2$  pertaining to the cash flows of each bond.

We first experimented with the following bond:  $P$ , the bond to be immunized, has a 10-year maturity, and a coupon of 5; the initial term structure is given by the polynomial

$$R(0, T) = (-0.005T^2 + 0.2T + 4)/100$$

(It corresponds to the first part of our term structure described in chapter 11.)

The bond's price, its  $\bar{X} = -B'/B$  sensitivity coefficient, as well as its  $-B\bar{X}$  and  $B\bar{X}^2$  values and the initial term structure, are given in table 18.5; tables 18.6, 18.7, and 18.8 provide the same information for the hedging bonds whose maturities are 7, 8, and 15 years, with coupons of 4, 4.75, and 7, respectively. The common volatility reduction factor ranges from 0 to 0.2.

The results are presented in table 18.9, in the form of the classical portfolio (corresponding to  $\lambda = 0$ ) and the HJM portfolio ( $\lambda > 0$ ). For each portfolio, the Fisher-Weil duration is also indicated.

Now watch carefully on the last line of table 18.9, the evolution of the duration of the HJM portfolio as  $\lambda$  increases from  $\lambda = 0$  to  $\lambda = 0.1$ . Contrary to what our intuition would suggest, it does not diverge more and more from the duration of the classical portfolio: first, it increases, and then not only does it decrease, but it falls *below* the level of the classical duration. This prompts us to think that, since  $D_P$  is a continuous function of  $\lambda$ , there *must* be at least one value of  $\lambda$  different from zero for which both the classical portfolio and the HJM portfolio have the same value. Indeed, for a value slightly above  $\lambda = 0.09$  (in fact,  $\lambda = 0.09357$ ), the difference between the durations falls to  $10^{-6}$ .

This shows that there is not necessarily a difference between the duration of an HJM portfolio and that of a classical portfolio, although the *portfolios* themselves will be different, if ever so slightly. (In the case just mentioned, the HJM portfolio would be  $N_A = 1.444$   $N_B = -2.265$ ; and  $N_C = -0.109$ .) Note also that the nonmonotonicity of the duration is mirrored in that of the HJM portfolios: between  $\lambda = 0$  and  $\lambda = 0.1$ ,  $N_A$  first decreases and then increases; the negative position in bond  $B$  first decreases and then increases; and the reverse is true for bond  $C$ .

We may then well surmise that if, for a given set of bonds, there exists a value  $\lambda$  ( $\neq 0$ ) for which the classical and the HJM durations are identical, the converse must be true: for a given value of  $\lambda$ , there must be one (and probably an infinitely high) number of coupon sets for which the durations are the same, if coupons have the same continuity as  $\lambda$ . Indeed, for  $\lambda = 0.1$ , for instance, it suffices to consider for bond  $C$  a coupon of 9.25 for both durations to be separated by less than  $4 \cdot 10^{-4}$ . The same reasoning would apply if we considered varying the bonds' maturities, keeping  $\lambda$  and the coupons as given. Naturally, we could now envision the

**Table 18.5**  
**Bond to be immunized\***

T (years)	R(0,T)	c <sub>t</sub>	c <sub>T</sub> * = c <sub>T</sub> e <sup>-R(0,T)T</sup>	s <sub>T</sub> * ≡ c <sub>T</sub> */P	-B'/B = X <sub>P</sub> (T) (years)	-s*B'/B = s*X <sub>p</sub> (T) (years)	s*B''/B = s*X <sub>p'</sub> '(T) (years <sup>2</sup> )	s*T (years)
1	0.04195	5	4.795	0.05	0.952	0.048	0.045	0.050
2	0.0438	5	4.581	0.048	1.813	0.087	0.157	0.096
3	0.04555	5	4.361	0.045	2.592	0.118	0.305	0.136
4	0.0472	5	4.140	0.043	3.297	0.142	0.469	0.173
5	0.04875	5	3.918	0.041	3.935	0.161	0.632	0.204
6	0.0502	5	3.700	0.039	4.512	0.174	0.785	0.231
7	0.05155	5	3.485	0.036	5.034	0.183	0.921	0.254
8	0.0528	5	3.277	0.034	5.507	0.188	1.036	0.273
9	0.05395	5	3.077	0.032	5.934	0.190	1.130	0.289
10	0.055	105	60.580	0.632	6.311	3.993	25.238	6.316
<hr/>								
P = 95.914					X̄ = 5.283	D = 8.023		
					X̄² P = 506.717	X̄² = 30.719		
						X̄² P = 2946.35		

\*Except for  $R(0, T)$ , all numbers in this table and in the following ones have been rounded to the third decimal.

**Table 18.6**  
**Bond  $A$  used for immunization**

$T$ (years)	$R(0, T)$	$c_T^*$	$c_T^* = c_T e^{-R(0, T)T}$	$s_T^* \equiv c_T^*/P_A$	$-B'/B = X_A(T)$ (years)	$-s^* B'/B = s^* X_A(T)$ (years)	$s^* B''/B = s^* X_A^2(T)$ (years <sup>2</sup> )	$s^* T$ (years)
1	0.04195	4	3.836	0.041	0.952	0.039	0.041	0.050
2	0.0438	4	3.665	0.039	1.813	0.072	0.079	0.096
3	0.04555	4	3.489	0.038	2.591	0.097	0.113	0.136
4	0.0472	4	3.312	0.036	3.297	0.118	0.143	0.173
5	0.04875	4	3.135	0.034	3.935	0.133	0.169	0.204
6	0.0502	4	2.960	0.032	4.512	0.144	0.191	0.231
7	0.05155	104	72.497	0.78	5.034	3.929	5.463	0.254
$P_A = 92.892$								
						$\bar{X}_A = 4.531$	$\bar{X}_A^2 = 21.756$	$D_A = 6.198$
						$\bar{X}_A P_A = 420.901$	$\bar{X}_A^2 P_A = 2020.99$	

**Table 18.7**  
**Bond  $B$  used for immunization**

$T$ (years)	$R(0, T)$	$c_T$	$c_t^* = c_T e^{-R(0, T)T}$	$s_T = c_T / P_B$	$-B' / B = X_B(T)$ (years)	$-s^* B' / B = s^* X_B(T)$ (years)	$s^* B'' / B = s X_B^2(T)$ (years <sup>2</sup> )	$s^* T$ (years)
1	0.04195	4.75	4.555	0.047	0.952	0.045	0.043	0.047
2	0.0438	4.75	4.351	0.045	1.813	0.082	0.149	0.090
3	0.04555	4.75	4.143	0.043	2.592	0.112	0.289	0.129
4	0.0472	4.75	3.933	0.041	3.297	0.135	0.444	0.164
5	0.04875	4.75	3.723	0.039	3.935	0.152	0.599	0.193
6	0.0502	4.75	3.515	0.037	4.512	0.165	0.744	0.219
7	0.05155	4.75	3.311	0.034	5.034	0.173	0.872	0.241
8	0.0528	104.75	68.661	0.714	5.507	3.931	21.645	5.710
<hr/>								
$P_B = 96.192$				$\bar{X}_B = 4.795$	$\bar{X}_B^2 = 24.786$	$D_B = 6.795$		
				$\bar{X}_B P_B = 461.195$	$\bar{X}_B^2 P_B = 2384.15$			

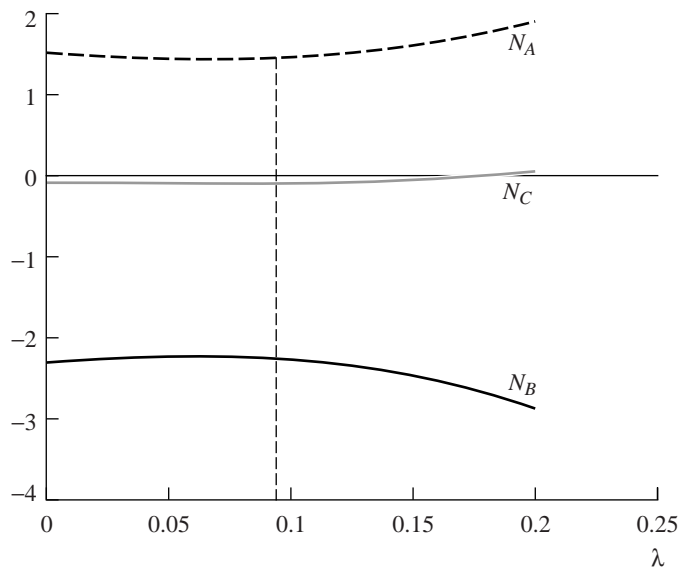
**Table 18.8**  
**Bond  $C$  used for immunization**

$T$ (years)	$R(0, T)$	$c_T$	$c_T^* = c_T e^{-R(0, T)T}$	$s_T^* = c_T^* / P_C$	$-B' / B = X_C(T)$ (years)	$-s^* B' / B = s^* X_C(T)$ (years)	$s^* B'' / B = s X_C^2(T)$ (years <sup>2</sup> )	$s^* T$ (years)
1	0.04195	7	6.712	0.060	0.952	0.057	0.054	0.060
2	0.0438	7	6.413	0.057	1.813	0.104	0.189	0.115
3	0.04555	7	6.106	0.055	2.592	0.142	0.368	0.164
4	0.0472	7	5.796	0.052	3.297	0.171	0.565	0.208
5	0.04875	7	5.486	0.049	3.935	0.193	0.761	0.246
6	0.0502	7	5.180	0.046	4.512	0.209	0.945	0.279
7	0.05155	7	4.880	0.044	5.034	0.220	1.108	0.306
8	0.0528	7	4.588	0.041	5.507	0.226	1.247	0.329
9	0.05395	7	4.308	0.039	5.934	0.229	1.360	0.347
10	0.055	7	4.039	0.036	6.321	0.229	1.446	0.362
11	0.05595	7	3.783	0.034	6.671	0.226	1.509	0.373
12	0.0568	7	3.541	0.032	6.986	0.221	1.550	0.380
13	0.05755	7	3.313	0.030	7.275	0.216	1.571	0.386
14	0.0582	7	3.099	0.028	7.534	0.209	1.577	0.389
15	0.05875	107	44.326	0.40	7.769	3.087	23.978	5.960
$P_C = 111.568$								
					$\bar{X}_C = 5.742$	$\bar{X}_C^2 = 38.228$	$D_C = 9.904$	
					$\bar{X}_C B = 640.605$	$\bar{X}_C^2 B = 4265.08$		

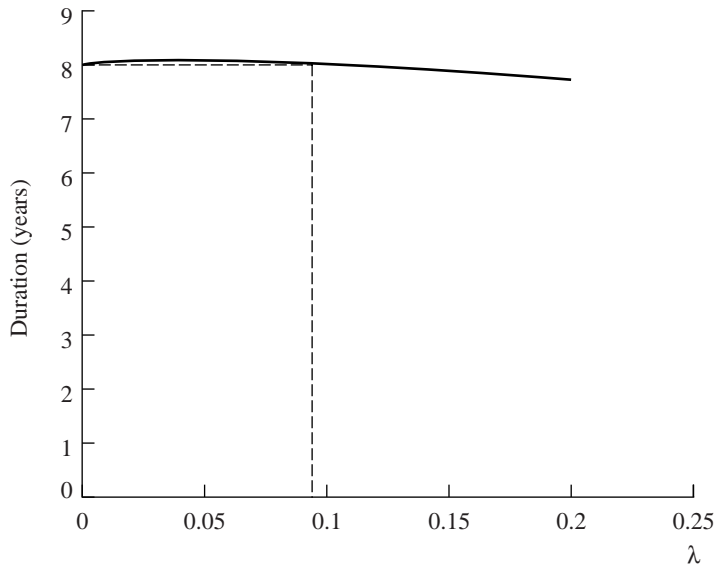


**Table 18.9**  
**Immunization results**

Number of bonds	Classical portfolio $\lambda = 0$	HJM portfolios ( $\lambda > 0$ )			
		$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.15$	$\lambda = 0.2$
$N_a$	1.487	1.428	1.453	1.601	1.896
$N_b$	-2.314	-2.245	-2.277	-2.469	-2.878
$N_c$	-0.102	-0.113	-0.107	-0.064	0.043
Duration (years)	8.023	8.027	8.020	7.952	7.733



**Figure 18.1**  
 Classical ( $\lambda = 0$ ) and Heath-Jarrow-Morton ( $\lambda > 0$ ) hedging portfolios. The portfolio on the dotted line corresponds to  $\lambda = 0.09357$ , and, although somewhat different, the classical portfolio ( $\lambda = 0$ ) has exactly the same duration.



**Figure 18.2**

**Duration of classical ( $\lambda = 0$ ) and Heath-Jarrow-Morton ( $\lambda = 0$ ) hedging portfolios as a function of  $\lambda$ . The HJM portfolio indicated by the dashed line (for  $\lambda = 0.09357$ ) is slightly different from the classical portfolio but has the same duration.**

whole spectrum of the  $\lambda$ , coupons, and maturities. From our experience with numerical examples, we can conclude that

- *in general* there will be a difference in duration and composition between the HJM portfolio and the classical portfolio;
- these differences in composition and in duration are not monotonic functions of  $\lambda$ , and convergence may be the case even if  $\lambda$  increases;
- there is an infinite number of combinations of  $\lambda$ , coupons, and maturities of the immunizing bonds for which there will be no difference between durations of the classical portfolio and the HJM portfolio.

We do not consider the fact that the durations may be the same as a sign of weakness in the HJM model. On the contrary, this kind of similarity in one aspect of the immunizing portfolios is exactly what we may expect from a model that smoothly generalizes the classical one, allowing for a reduction in the volatility factor of the forward rate with respect to

maturity (at any fixed time) or, equivalently, allowing an increase in that volatility factor with respect to time (for any fixed maturity). Indeed, one might conjecture that in some circumstances classical immunization will produce the same beneficial effects as the HJM immunization, and we have shown some of them precisely.

### 18.3 Other applications of the Heath-Jarrow-Morton model

One of the great advantages of the Heath-Jarrow-Morton model is that it provides a unified framework under which it is possible to evaluate interest rate derivatives. Indeed, since the evolution of bond prices is described under their martingale probabilities, it suffices to consider, for the valuation of those derivatives, the expected value of the derivative's payoff *under* those martingale probabilities. The following derivatives can be priced in this way: futures on Treasury bills and Treasury bonds; options on Treasury bills, on Treasury bonds, on Treasury bill futures, and on Treasury bond futures; caps, floor, collars, and swaptions. The reader is referred to chapters 16 and 17 in Jarrow and Turnbull (1995), which give all the formulas for these derivatives and furthermore provide software to perform the necessary calculations.

#### Main formulas

$$\frac{B'(I_0)}{B(I_0)} = -X(t, T) \quad (3)$$

$$\frac{B''(I_0)}{B(I_0)} = X^2(t, T) \quad (4)$$

with

$$I(t) \equiv r(t) - f(0, t)$$

$$X(t, T) \equiv \frac{1 - e^{-\lambda(T-t)}}{\lambda}$$

$$\frac{\Delta B}{B} \approx -X(t, T)\Delta I + \frac{1}{2}X^2(t, T)(\Delta I)^2 \quad (5)$$

Using the following notation:

$P(t, T_n \mid I)$  = value of a bond at time  $t$ , with maturity at date  $T_n$ ; its dependency on  $I = r(t) - f(0, t)$  is underlined; the letter  $P$  stands both for the *price* of the bond, and for a *portfolio* of  $n$  zero-coupon bonds: the  $n$  payments, or cash flows paid out by the issuer of the bond to its owner; the first  $n - 1$  are equal to the coupons; the last one,  $c_n$ , is the last coupon *plus* the principal. For short,  $P(t, T_n \mid I)$  will be abbreviated to  $P(I)$  or even  $P$ .

$c_i$  = number of dollars paid by the issuer at time  $T_i$  ( $i = 1, \dots, n$ ).

$B(t, T_i \mid I)$  = value at time  $t$  of a zero-coupon paying one dollar at time  $T_i$ .

$c_i B(t, T_i \mid I)$  = value at time  $t$  of  $c_i$  dollars paid out by the issuer at time  $T_i$ .

$$P(t, T_n \mid I) = \sum_{i=1}^n c_i B(t, T_i \mid I) \quad (11)$$

$$\frac{\Delta P}{P} \approx \frac{P'(I_0)}{P(I_0)} \Delta I + \frac{1}{2} \frac{P''(I_0)}{P(I_0)} (\Delta I)^2 \quad (13)$$

With  $\alpha_i = \frac{c_i B_i(I_0)}{P(I_0)}$ ,  $\sum_{i=1}^n \alpha_i = 1$ :

$$\frac{P'(I_0)}{P(I_0)} = - \sum_{i=1}^n \alpha_i X_i \equiv -\bar{X} \quad (15)$$

$$\frac{\Delta P}{P} \approx -\bar{X} \Delta I + \frac{1}{2} \bar{X}^2 (\Delta I)^2 \quad (18)$$

A hedging portfolio comprising  $N_A$ ,  $N_B$ , and  $N_C$  bonds of types A, B, and C must solve the system:

$$P_A N_A + P_B N_B + P_C N_C = -P \quad (27)$$

$$\bar{X}_A P_A N_A + \bar{X}_B P_B N_B + \bar{X}_C P_C N_C = -\bar{X} P \quad (28)$$

$$\bar{X}_A^2 P_A N_A + \bar{X}_B^2 P_B N_B + \bar{X}_C^2 P_C N_C = -\bar{X}^2 P \quad (29)$$

## Questions

18.1. Using the same bonds and discount rates as in Jarrow and Turnbull (1995, p. 494; see table 18.1 in this chapter), determine the hedging Heath-

Jarrow-Morton portfolios in the cases  $\lambda = 0.075$ ,  $\lambda = 0.125$ ,  $\lambda = 0.175$ ,  $\lambda = 0.225$ , and in the classical case ( $\lambda = 0$ ). Determine also the duration of those portfolios.

18.2. Suppose you want to immunize a bond with maturity 10 years, par value 100, bearing a 5% coupon. You want to replicate this bond with three bonds ( $a$ ,  $b$ , and  $c$ ) with the following characteristics:

	Maturity	Coupon rate	Par value
Bond $a$	7 years	4.25%	100
Bond $b$	8 years	4.5%	100
Bond $c$	15 years	9%	100

Suppose also that the initial term structure is given by

$$R(0, T) = (-0.005T^2 + 0.2T + 4)/100$$

for  $T = 1, \dots, 15$ . (This is the term structure we used in section 11.3 of chapter 11.)

What would be the replicating classical portfolio and the Heath-Jarrow-Morton portfolios for  $\lambda = 0.1$ ,  $\lambda = 0.15$ ,  $\lambda = 0.2$ ,  $\lambda = 0.25$ , and  $\lambda = 0.3$ ? Determine the durations of the respective portfolios. What do you observe? Do your results, for instance, suggest that there might exist a nonzero value for  $\lambda$  and therefore a Heath-Jarrow-Morton portfolio the duration of which would be the same as the classical portfolio's duration?



## BY WAY OF CONCLUSION: SOME FURTHER STEPS

QUEEN. Having thus far proceeded  
... is't not meet  
That I did amplify my judgment in  
Other conclusions?

*Cymbeline*

In the past thirty years, research in finance has essentially been founded on Gaussian statistical processes, which have their roots in Bachelier's modeling of Brownian motion at the beginning of the twentieth century.<sup>1</sup> It is now recognized more and more by academics and practitioners alike that Wiener geometric processes—and generally the lognormality hypotheses—have been used so extensively mainly for two reasons. First, they rest on one of the most powerful results of statistics obtained in these past two centuries: the central limit theorem. Second, they are extremely convenient from a mathematical point of view. Unfortunately, these processes do not describe well the uncertainty of financial markets, and they are poorly suited to representing the large swings that have come to be systematically observed in the value of either individual assets or whole market indices.

Today theorists and practitioners have concluded that more sophisticated tools are needed to deal with what is observed in the markets: discontinuities—or jumps—in the assets' value, more frequent observations of very small (either positive or negative) returns than assumed by the normal distribution, and more frequent steep downfalls or upward moves than warranted by the norm. However, for nearly four decades, research spawned from earlier work by Benoît Mandelbrot<sup>2</sup> has called for the use of statistical distributions that would better conform to the observed

<sup>1</sup>L. Bachelier, "Théorie de la spéculation," *Annales scientifiques de l'Ecole normale supérieure*, 17 (1900): 21–86.

<sup>2</sup>See, in particular, B. Mandelbrot, "The Variation of Certain Speculative Prices," *Journal of Business*, 36 (1963): 394–419; B. Mandelbrot, *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk* (New York: Springer, 1997); E. Fama, "Mandelbrot and the Stable Paretian Hypothesis," in *The Random Character of Stock Market Prices*, ed. P. Cootner (Cambridge, Mass.: MIT Press, 1964); E. Fama, "The Behavior of Stock Market Prices," *Journal of Business*, 38 (1965); E. Fama, "Portfolio Analysis in a Stable Paretian Market," *Management Science*, 11 (1965); and P. A. Samuelson, "Efficient Portfolio Selection for Pareto-Levy Investments," *Journal of Financial and Quantitative Analysis*, June (1967).

behavior of markets. These are so-called *fractal* distributions. In addition to meeting the above-mentioned properties, they share the property of being time-homothetic in the sense that, over different time scales, their statistical characteristics remain the same. They are, however, very much at odds with our intuition, not to mention our wishes: in particular, they generate continuous functions that are nowhere differentiable.<sup>3</sup> This is also true of a Brownian motion generated by a normal distribution. But with regard to the difference of the normal distribution, their density functions are not known; only their characteristic functions are, which makes them much more difficult to deal with.

On the other hand, as we saw earlier, the distribution of a sum, or an average of random variables, converges toward a normal distribution (the central limit theorem would apply) only if, among other conditions, the individual probability distributions are independent. Independence was, and still is, very difficult to justify. A lot of research has investigated autoregressive processes, giving rise to frequency distributions that also have the necessary characteristics called for (fat tails, slim center). These are the ARCH (for autoregressive conditional heteroscedasticity) processes and their multiple descendants. A good synthesis of these strands of thought, as well as a strong call for uniting them, can be found in Edgar Peters.<sup>4</sup> Regarding the current extreme volatility of financial markets, the reader should refer to the very convincing book by Robert Shiller.<sup>5</sup> As an example of recent research in fractal theory, we refer the reader to the work of Jean-Philippe Bouchaud and Marc Potters<sup>6</sup>, who use a truncated

<sup>3</sup>The profile of a mountain, or of a mountain chain, constitutes a very good example of a fractal and how fractal objects can have a deceiving appearance. From far away, we could think of mountain chains as sets of triangles and estimate the difficulty to climb them by the steepness of their “slope” as seen from such distance. Unfortunately for amateur alpinists, there is no such thing as the slope of a fractal. After six unsuccessful attempts to climb the Matterhorn from its Italian side, which looked “easier,” the British draftsman turned climber Edward Whymper decided to circle the whole mountain, carefully drawing its fractal structure from all edges. He noticed two things: the fractal structure was basically the same on both the Swiss and the Italian sides; but the strata of the *whole* mountain had, for some geological reason, been tilted in such a way that the short, nearly vertical steps were more pronounced on the Italian side (sometimes leading to overhangs), although the *average* slope of some Italian edges looked, and was, lower than on the Swiss side. He then decided to try the Swiss route and was the first to reach the summit in 1865.

<sup>4</sup>E. Peters, *Fractal Market Analysis* (New York: Wiley, 1994).

<sup>5</sup>R. Shiller, *Irrational Exuberance* (Princeton, N.J.: Princeton University Press, 2000).

<sup>6</sup>J.-Ph. Bouchaud and M. Potters, *Théorie des risques financiers* (Paris: Aléa Saclay, 1997), pp. 134–135.



Lévy law. This law was itself proposed in the form of a general characteristic function by I. Koponen.<sup>7</sup>

Why was the fractal approach not more in use? The reasons probably go back to the very origins of our attempts to comprehend the laws of nature. Quite understandably, we started trying to solve complex problems by making simple assumptions—but these were unfortunately somewhat flawed. In his path-breaking essay on growth theory, Robert Solow wrote: “All theory makes assumptions which are not quite true. This is what makes it theory.” He added: “The art of successful theorizing is to make the inevitable simplifying assumption in such a way that the final results are not very sensitive. . . . When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect.”<sup>8</sup>

Here we are encountering a case in which the results are so sensitive to assumptions that a revision of the hypotheses is needed. Since such a revision has been felt necessary for many years, a legitimate question to ask is why it was not already undertaken and generalized long ago. Possibly a number of reasons are at play, and we will try to delineate some of them. Quite a profound one, in our opinion, has been suggested by E. Peters.<sup>9</sup> His reasoning makes us go back to the very origin of scientific thinking in Western civilization. Through their philosophers, mathematicians, and geometers, ancient Greeks formed the idea of a perfect universe, corresponding to simple (and we may add, often harmonious in an ultimate sense) constructs. In doing so, they were also led by practical considerations: if the area of a field had to be measured, the easiest route was to assimilate its shape to a simple geometric figure, or to a set of those, whose areas could be easily calculated. They knew that nature did not conform to their models, but as E. Peters put it very well, they turned the conclusions around: “the problem was not with the geometry, but with the world itself.”<sup>1</sup>

A good illustration of the problems we inherited from the ancient Greeks is the way mathematics has been taught since the Renaissance: most functions (whether dependent on one or several variables) are smooth in the sense that they are differentiable. One property of those

<sup>7</sup>I. Koponen, *Physical Review, E*, 52 (1995): 1197.

<sup>8</sup>R. M. Solow, “A Contribution to the Theory of Economic Growth,” *The Quarterly Journal of Economics*, 70 (1956): 439–469.

<sup>9</sup>Peters, op. cit.

<sup>1</sup> Peters, op. cit., p. 3

functions is that if you constantly magnify a given portion of their corresponding curves, you ultimately obtain a straight line (or, equivalently, a plane). Unfortunately, nature does not possess that property: a natural object retains its complexity at every scale, and physicists tell us that its complexity can even increase and not necessarily diminish as science has suggested—and perhaps hoped—since Pericles’s century.

In finance, and especially in the subject area of this book, simplicity was certainly an important factor for the central limit theorem to be used so extensively. Also, the theorem served as a powerful vehicle for the so-called efficient market hypothesis, according to which information was continuously reflected by agents in price formation, with new information reaching us in a random fashion. In fact, it seems that one should recognize three things: information is not spread out uniformly; even if information is the same for all actors in the market, it is not interpreted by all in the same way; and, finally, even if this information conveys the same message to everybody, it will have different consequences on agents according to their *horizon*. (For instance, facing a given set of news, short-term traders may behave quite differently than pension-fund managers.) These differences in information, analysis, and horizon, far from entailing disequilibrium, on the contrary work together toward *establishing equilibrium* prices. In this sense we have a reconciliation between determinism and randomness. The two can coexist, leading to a market equilibrium that is disrupted precisely when everybody thinks in the same way, thus giving rise to abrupt changes and even discontinuities that can be of major proportions, until diversity restores some degree of stability.

We take this opportunity to indulge in a conjecture. A primary goal of social scientists is to describe and explain the progress of societies. At an even more general level, a perennial question has been to discover the causes of the rise and eventual decline of civilizations. To the best of our knowledge, the first thinker who asked the question and came up with an answer was the great Arab historian Ibn Khaldun (1376)—see his masterly work.<sup>11</sup>

<sup>11</sup>Ibn Khaldun’s work was translated into English for the first time by Franz Rosenthal under the title *The Muqaddimah: An Introduction to History* (New York: Bollingen, 1958). A revised edition has been published by Princeton University Press (2nd edition, 3 volumes, 1980). For a discussion of Ibn Khaldun’s contribution to the questions we alluded to, we take the liberty of referring the reader to our work (O. de La Grandville, *Théorie de la croissance économique* (Paris, Masson, 1977).)

It is quite possible that some day historians, sociologists, political scientists, and—yes—even economists will communicate with each other. They may then discover that, like natural phenomena and human behavior, the evolution of civilizations conforms to fractal patterns. (Would Ibn Khaldun be surprised? Maybe not.) We could then witness the beginning of a reconciliation between long-term determinism due to the will of the human being, and short-term randomness. Will that be good news or bad news? Probably good news, because we will be better prepared to face what we have come to call “accidents of history.”



# ANSWERS TO QUESTIONS

DON ADRIAN DE ARMADO. I do say you are quick in answers.  
*Love's Labour's Lost*

## Chapter 1

- 1.1. Applying equation (1) in the text we get 9.28% per year.
- 1.2. From (1) or (2), you can determine  $B_t = B_T/[1 + i(T - t)] = \$97.087$ .
- 1.3. From (4),  $i = 5.396\%$  per year.
- 1.4. From (3), we can determine  $B_t$  as  $B_T(1 + i)^{-(T-t)} = \$82.989$ .
- 1.5. Applying (7) with  $t = 2$  years and  $T = 5$  years, we get:  $f(0, 2, 5) = 6.67\%$  per year. The dollar five-year forward rate, 1.06, is the geometrical average between the dollar two-year spot rate (1.05) and the dollar forward rate (1.0667), with weights  $\frac{2}{5}$  and  $\frac{3}{5}$ , respectively. We have indeed  $1.06 = (1.05)^{2/5}(1.0667)^{3/5}$ .
- 1.6. Transforming (6) or (7), we have

$$i(0, t) = \frac{[1 + i(0, T)]^{T/t}}{1 + f(0, t, T)^{(T-t)/t}} - 1 = 6.75\%$$

## Chapter 2

- 2.1. a. *Any* bond, whatever its par value, coupon rate, or (nonzero) maturity, will increase in value whenever interest rates decrease, because, fundamentally, it pays fixed amounts in a world in which interest rates are—in our hypothesis—decreasing. We also know that  $B(i)/B_T$  is the weighted average of  $\frac{c/B_T}{i}$  and 1. So when the interest rate is equal to the coupon rate, the bond's value is at par, that is, \$1000. This result is independent of the bond's coupon rate and maturity.
- b. The bond's value will be farthest from its par value when  $i = 6\%$  because of the convexity of the curve  $B = B(i)$ . (A bond's value is the sum of convex functions—see the open-form equation (11) or figure 2.3).
- c. The bond's value in each case  $i = 12\%$ ,  $i = 9\%$ , and  $i = 6\%$  (those for  $i = 12\%$  and  $i = 6\%$  are calculated with the closed-form (15)) are \$830.5,

\$1000, and \$1220.8, respectively, thus confirming what we could infer in (a) and (b). They would then be directly proportional to the coupon.

2.2. The price of bond  $B$  will be lower than the price of bond  $A$  by *less* than 20% because the bonds' prices are given by

$$B = \frac{c}{i} [1 - (1 + i)^{-T}] + B_T(1 + i)^{-T}$$

which is not a linear function of coupon  $c$  but an affine function of  $c$  with a positive, additive term  $B_T(1 + i)^{-T}$ , which is independent of  $c$ . Cases in which the price of  $B$  is inferior to the price of  $A$  by exactly 20% are the cases in which either  $B_T = 0$  or  $T \rightarrow \infty$ . (In the latter case, the bond would be called a *consol*.)

2.3. In this riskless world, the equilibrium expected return is  $i = 6\%$  per year. The current price of the asset is, either from equation (17) (setting  $\pi = 0$ ) or from figure 2.4,  $p_0 = E(q + p_1)/(1 + i) = \$105,000/1.06 = \$99,056$ .

2.4. From (17) or figure 2.4,  $p_0$  would be worth an amount equal to:  $p_0 = E(q + p_1)/(1 + i + \pi) = \$105,000/1.16 = 90,517$ .

2.5. From (17) or figure 2.4, the consequences are the following:

- a.  $E(R)$  increases
- b.  $E(R)$  increases
- c.  $E(R)$  decreases

### Chapter 3

3.1. The reason for this can best be seen by transforming the definitional equation of  $i^*$ , (1), into (2). Then, the left-hand side of (2) is a polynomial in  $i^*$  of order  $T$ , whose complete expression is given by a Newton binomial expansion of order  $T$ . Its right-hand side is the sum of  $T$  polynomials of order  $T - 1$ ,  $T - 2, \dots, 0$  and therefore a polynomial of order  $T - 1$ . Consequently, equation (2) amounts to a polynomial of order  $T$ . This should be solved in order to obtain  $i^*$ .

3.2. If  $i_1 = i_0$ ,  $B_1 = B_0$  and from (5)  $r_H = i_0$  for any horizon  $H$ .

3.3. When the bond was bought, its price was  $B(0.08) = \$1112.6$ . One day later, this value increases to the par value ( $B(0.09) = \$1000$ ). The horizon rates of return are given by

$$R_H = \left[ \frac{B(i)}{B_0} \right]^{1/H} (1 + i) - 1$$

and, respectively, by

- a.  $R_{H=1} = \left( \frac{1000}{1112.6} \right) (1.09) - 1 = -2.0\%$
- b.  $R_{H=4} = \left( \frac{1000}{1112.6} \right)^{1/4} (1.09) - 1 = 6.1\%$
- c.  $R_{H=10} = \left( \frac{1000}{1112.6} \right)^{1/10} (1.09) - 1 = 7.8\%$

3.4. From equation (5),  $\lim_{H \rightarrow \infty} r_H = 1 + i_1 - 1 = i_1$ . The infinite horizon rate of return turns out to be the new interest rate  $i_1$ . This has an intuitive explanation: whatever a finite maturity  $T$  for a bond, any change in the value of the bond entailed by a change of interest from  $i_0$  to  $i_1$  is of no consequence to the par value of the bond, which will be reinvested for an infinite period at rate  $i_1$ . Consequently, the infinite horizon rate of return is simply  $i_1$ .

3.5. Calculate  $B_1$ , using a closed-form formula such as (13) in chapter 2, and then  $r_H$  using equation (5) of chapter 3.

3.6. Such a horizon may indeed exist if the capital loss due to an increase in the rates of interests is exactly compensated by gains on the reinvestment of coupons and, symmetrically, if the capital gain due to a decrease in the interest rates is exactly compensated by losses on the reinvestment of coupons. This is what happens with a horizon equal to 7.7 years. Intuitively, we can understand that compared to the maturity of the bond, this horizon cannot be too long—otherwise the effect of the reinvestment of coupons would dominate the change of price effect—and it cannot be too short—otherwise the price effect would be stronger than the reinvestment effect.

## Chapter 4

4.1. In the left-hand side of equation (6), the units of  $B$  cancel out; we are left with units of  $1/i$ , which are  $1/(1/\text{year}) = \text{years}$ . In the right-hand side of (6), the units of  $c_t$  cancel out with those of  $B$ ;  $1 + i$  is without units (because in it  $i$  has been multiplied by 1 year). Therefore, the right-hand side of (6) is also in years, as it should be.

4.2. The units of modified duration (see the first equation of section 4.4) are years because  $1 + i$  is unitless.

4.3. In the equation of the same section giving the value of  $\frac{1}{B} \frac{dB}{di}$ , multiply by  $-(1+i)$  to get  $D$ . Then, in the resulting equation, factor out  $(1+i)^{-T}$  in both the numerator and the denominator of the right-hand side.

4.4. Durations are best calculated using the closed-form equation (11).

a. For  $i = 9\%$ ,  $D = 6.995$  years (the case of this chapter).

b. For  $i = 8\%$ ,  $D = 7.098$  years.

From the negatively sloped relationship between interest rates and duration demonstrated in section 4.5, one could infer that duration would be somewhat larger when  $i$  gets smaller.

4.5. The durations of bonds  $A$  and  $B$  are, respectively,

$$D_A = 7.888 \text{ years}$$

$$D_B = 5.268 \text{ years}$$

One can infer from these figures that the relative decrease in value of bond  $A$  will be higher than that of bond  $B$ . Indeed, we first get the following relative changes in exact value:

	Value of bond for		Relative change in bond's value, in exact value
	$i_0 = 12\%$	$i_1 = 14\%$	
Bond $A$	\$863.8	\$754.3	$\Delta B_A/B_A = -12.7\%$
Bond $B$	\$908.7	\$828.5	$\Delta B_B/B_B = -8.8\%$

In linear approximation, those relative changes in value are

$$\frac{dB_A}{B_A} = -\frac{1}{1+i_0} D_A di = -\frac{1}{1.12} (7.888)(2\%) = -14.1\%$$

and

$$\frac{dB_B}{B_B} = -\frac{1}{1+i_1} D_B di = -\frac{1}{1.12} (5.268)(2\%) = -9.4\%$$

From the convexity of  $B(i)$ , one can infer that the relative changes following an *increase* in interest rates are *less* pronounced in (absolute) exact value than in linear approximation, which is confirmed by our results.



4.6. The durations are the following:  $D_A = 8.36$  years and  $D_B = 8.94$  years. Since bond  $A$ 's coupon rate is higher than bond  $B$ 's,  $A$  has a smaller duration than  $B$ 's duration and will react less than  $B$  to a change in interest rates. This is confirmed in the following results (in exact value):

	Value of bond for		Relative change in bond's value, in exact value
	$i = 12\%$	$i = 14\%$	
$B_A$	\$1000	\$867.5	$\Delta B_A/B_A = -13.3\%$
$B_B$	\$701.2	\$602.6	$\Delta B_B/B_B = -14.1\%$

4.7. The durations of bonds  $A$  and  $B$  are the following:  $D_A = 8.602$  years and  $D_B = 4.981$ . Bond  $A$ 's duration is considerably higher than bond  $B$ 's, for the following two reasons: its maturity is longer (without reaching the levels such that the corresponding duration decreases) and the coupon rate is smaller. Therefore  $A$  will be more sensitive to an increase in interest rates, as confirmed by the results in the following table:

	Value of bond for		Relative change in bond's value, in exact value
	$i = 12\%$	$i = 14\%$	
$B_A(i)$	\$850.6	\$735.1	$\Delta B_A/B_A = -13.6\%$
$B_B(i)$	\$1091.3	\$1000	$\Delta B_B/B_B = -8.4\%$

4.8. Compared to the situation described in question 4.5., increasing the maturity of bond  $B$  from 7 to 11 years and reducing its coupon rate from 10% to 5% has a powerful increasing effect on the bond's duration in such a way that the bonds now have approximately the same duration:  $D_A = 7.888$  years and  $D_B = 7.898$  years. One can infer that their sensitivity to interest rate changes will be approximately the same.

	Value of bond for		Relative change in bond's value, in exact value
	$i = 12\%$	$i = 14\%$	
$B_A(i)$	\$863.8	\$754.3	$\Delta B_A/B_A = -12.7\%$
$B_B(i)$	\$584.4	\$509.3	$\Delta B_B/B_B = -12.9\%$

Although both bonds are of a very different nature, their level of interest rate riskiness is indeed about the same.

4.9. We obtain here a result that is quite contrary to our intuition: bond  $B$ , although having half of  $A$ 's maturity (20 years as opposed to 40 years), has a *higher* duration (12.34 years against 10.8 years) and therefore exhibits more risk.

	Value of bond for		Relative change in bond's value, in exact value
	$i = 12\%$	$i = 14\%$	
$B_A(i)$	\$175.6	\$147.4	$\Delta B_A/B_A = -16.1\%$
$B_B(i)$	\$253.1	\$205.22	$\Delta B_B/B_B = -18.9\%$

This is due of course to the nonmonotonicity property of duration with maturity for under-par (low-coupon) bonds when maturity is very high, which we explained in section 4.5.3.

4.10.  $dV/V$  is a random variable that is itself a weighted sum of two random variables  $\tilde{A}$  and  $\tilde{B}$ ,  $\tilde{A} \equiv di$  and  $\tilde{B} \equiv de/e$ . The weights are  $\alpha \equiv -\frac{D}{1+i}$  and  $\beta = 1$ . We thus have

$$dV/V \equiv \alpha \tilde{A} + \beta \tilde{B}$$

Applying the formula for the variance of a weighted sum of dependent random variables,  $\tilde{A}$  and  $\tilde{B}$ , we have

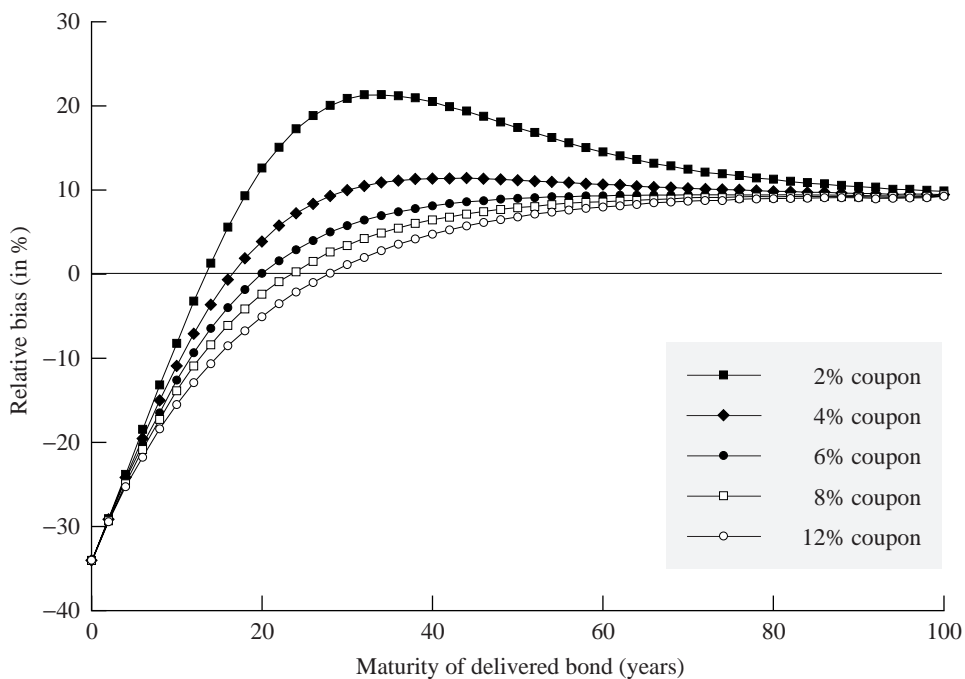
$$VAR(dV/V) = \alpha^2 VAR(\tilde{A}) + \beta^2 VAR(\tilde{B}) + 2\alpha\beta COV(\tilde{A}, \tilde{B})$$

which establishes equation (18).

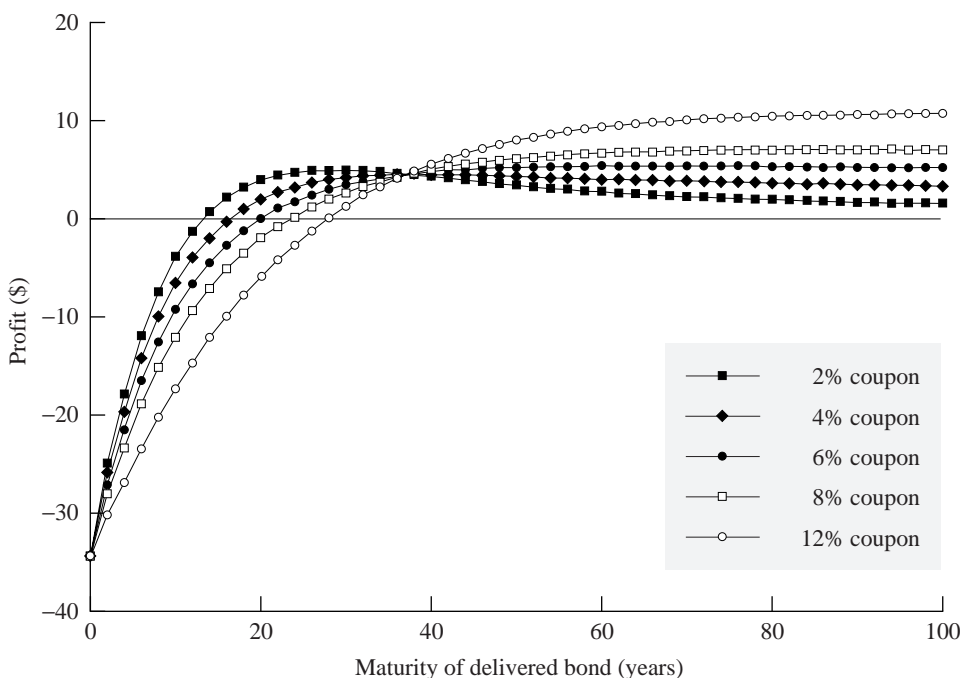
## Chapter 5

5.1. As a check, you should obtain, for a 4% coupon and 20-year maturity bond, a relative bias of 3.14% when rates of interest are at 10%. For the same bond, the profit at delivery is \$1.525. The graphs of all relative biases and profits are given in figures 1a and 1b.

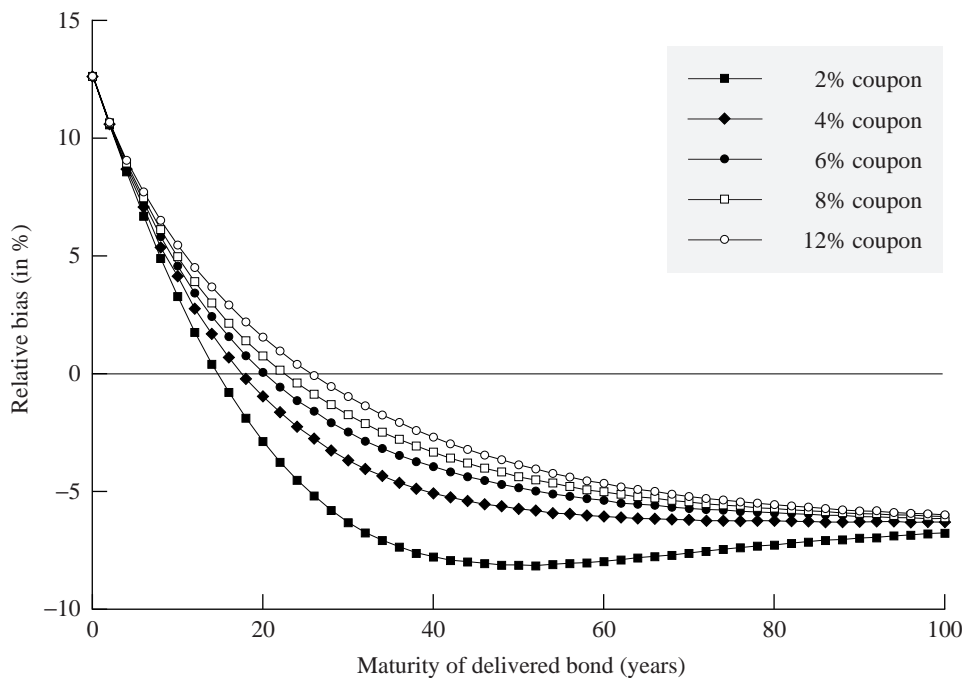
5.2. As a check, when rates of interest are at 6%, for a 4% coupon and 20-year maturity bond, the relative bias is  $-3.26\%$ . For the same bond, the profit at delivery is  $-\$2.506$ . The graphs of all relative biases and profits are given in figures 2a and 2b.



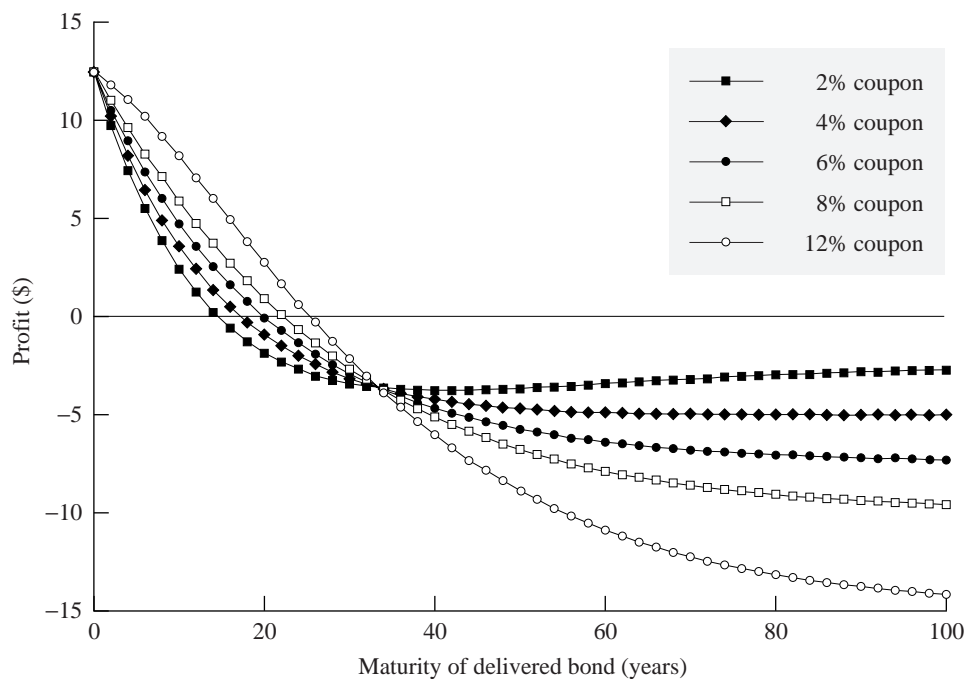
**Figure 1a**  
Relative bias of conversion factor in  $T$ -bond futures contract, with  $i = 10\%$



**Figure 1b**  
Profit from delivery with  $T$ -bond futures contract;  $F(T) = 82.84$ ;  $i = 10\%$



**Figure 2a**  
Relative bias of conversion factor in *T*-bond futures contract, with  $i = 6\%$



**Figure 2b**  
Profit from delivery with *T*-bond futures contract;  $F(T) = 123.12$ ;  $i = 6\%$

## Chapter 6

6.1. Using equation (5) in chapter 3, we can replace  $i_1$  with  $i_0$ . We then get  $B_1 = B_0$ , and therefore  $r_H = i_0$  for any  $H$ . Thus,  $(i_0, i_0)$  is a fixed point for all curves  $r_H(i)$ .

6.2. Indeed, that kind of problem could not be tackled simply by the intuitive reasoning we used in section 6.1 since it requires more refined analysis.

What we want to show is that for *any* bond, either under- or above-par, the graph pictured in figure 6.1 is correct. This amounts to showing the following relationship:

$$\frac{dr_H(i_0)}{di} \cong 0 \text{ if and only if } H \cong D$$

In other words, the horizon rate of return has a positive (zero, negative) slope at fixed point  $(i_0, i_0)$  if and only if the horizon  $H$  is larger than (equal to, smaller than) duration.

Recall our formula (5) of chapter 3 for  $r_H$ :

$$r_H = \left[ \frac{B(i)}{B_0} \right]^{1/H} (1+i) - 1$$

Let us now study, in linear approximation, the *relative* rate of increase of the dollar horizon rate of return  $1 + r_H$ , that is,  $\frac{1}{1+r_H} \frac{d \log(1+r_H)}{di}$ . We have

$$1 + r_H = \left[ \frac{B(i)}{B_0} \right]^{1/H} (1+i) = (1/B_0)^{1/H} [B(i)]^{1/H} (1+i) \quad (1)$$

Denoting the constant  $\frac{1}{H} \log(1/B_0)$  by  $k$ , we have

$$\log(1 + r_H) = k + \frac{1}{H} \log B(i) + \log(1 + i) \quad (2)$$

Furthermore, the derivative of  $\log(1 + r_H)$  with respect to  $i$  can be written as

$$\frac{d \log(1 + r_H)}{di} = \frac{1}{1 + r_H} \frac{d(1 + r_H)}{di} = \frac{1}{1 + r_H} \frac{dr_H}{di} \quad (3)$$

and is equal, at point  $i_0$ , to

$$\frac{1}{1 + r_H(i_0)} \frac{dr_H(i_0)}{di} = \frac{1}{H} \frac{1}{B(i_0)} \frac{dB(i_0)}{di} + \frac{1}{1 + i_0} \quad (4)$$

Since  $1 + r_H(i_0)$  can be assumed to be always positive (because  $r_H(i_0) > -1$ ),  $\frac{dr_H(i_0)}{di}$  and the above expression (4) are always positive (equal to zero, negative) if and only if

$$\frac{1}{H} \frac{B'(i_0)}{B(i_0)} \cong -\frac{1}{1 + i_0} \quad (5)$$

or

$$-\frac{(1 + i_0)}{B(i_0)} \frac{B'(i_0)}{B(i_0)} \cong H \quad (6)$$

(Note the changes in the inequality signs.) In the left-hand side of the set of inequalities (6) we recognize the duration of the bond whatever its coupon rate (and therefore its initial price) may be. Therefore, we can write

$$\frac{dr_H(i_0)}{di} \cong 0 \text{ if and only if } H \cong D$$

Thus the intuitive reasoning that is valid for a bond at par can be extended to above- or below-par bonds through analytical reasoning.

6.3. The “variance of the bond” means the *dispersion* of the bond’s times of payments around its duration and is measured by  $S = \sum_{t=1}^T (t - D)^2 [c_t(1 + i)^{-t}/B(i)]$ , as we defined it in section 4.5.2.

6.4. Consider first the left-hand side of (9) in this chapter. We have  $\frac{d^2 \ln F_H}{di^2} = \frac{d}{di} \left[ \frac{d \ln F_H}{di} \right] = \frac{d}{di} \left[ \frac{1}{F_H} \frac{dF_H}{di} \right]$ ; in the bracketed term, the units of  $F_H$  cancel out, and therefore the remaining units in this term are those of  $1/i$ , which are years. Therefore,  $\frac{d}{di} \left[ \frac{1}{F_H} \frac{dF_H}{di} \right]$  is in years/(1/year) = years<sup>2</sup>.

This property is confirmed both by the right-hand side of (9), which is in years/(1/year) = years<sup>2</sup> (since  $(1 + i)$  is unitless) and by the last equation of section 6.2, which shows that  $d^2 \ln F_H / di^2 = S / (1 + i)^2$ , where  $S$  is expressed in years<sup>2</sup>.

## Chapter 7

7.1. There are at least two ways of calculating convexity; they rely on either formula (2) or formula (19). Both formulas are open-form. A spreadsheet is quite appropriate for such calculations, and it may be advisable, for checking purposes, to use both equations (2) and (19). The result is 62.79 (years<sup>2</sup>).

7.2. The explanation of this apparently nonintuitive result is as follows: the bond's value curve is a sum of curves of the hyperbola type, which all tend toward 0 when  $i$  increases indefinitely, and therefore this curve has an asymptote which is the horizontal  $i$  axis. On the other hand, the quadratic approximation is a decreasing parabola; it will go through a minimum and then start increasing toward infinity. It is therefore quite normal that at some point it will dominate both the bond's value curve and its tangent.

7.3. The key assumption we make when calculating the bond portfolio's convexity (or duration, or dispersion, for that matter) is that for each bond the rate of return to maturity is the same.

7.4. After equating (17) to  $-(1+i)S$ , where  $S$  is the dispersion of the bond, remember that  $-(1+i)B'/B$  is equal to duration  $D$  and replace that expression in the bracketed term of (17). Then factor out  $1/B$  from the bracketed term and multiply throughout by  $-1$  to obtain (18).

## Chapter 8

8.1. There are, in fact, many ways to perform the calculations of table 8.1. The most natural one would be to determine closed-form formulas both for duration and convexity. For duration, this closed form is given in equation (11) of section 4.5.1. However, the calculation of the second derivative of  $B(i)$ , necessary for the determination of convexity, turns out to be a very cumbersome exercise (its expression is spread out on three lines), and writing a program to do this is no less of a chore. This is why we hinted in this exercise to choose another route, founded upon the approximations

$$D = -\frac{1+i}{B(i)} \frac{dB}{di} \approx -\frac{1+i}{B(i)} \frac{\Delta B}{\Delta i} \quad (7)$$

and

$$C = \frac{1}{B(i)} \frac{d^2 B}{di^2} \approx \frac{1}{B(i)} \frac{\Delta}{\Delta i} \left[ \frac{\Delta B}{\Delta i} \right] \quad (8)$$

which can be rendered as close to the exact value as we choose, simply by considering sufficiently small, finite increases in  $i$ ,  $\Delta i$ .

It turns out that there are various ways of performing such calculations. Quite an appropriate one would be to choose a first increase of  $i$  equal to

$\Delta i = 10^{-5} = 0.00001$ , for instance. However, we can obtain better precision in the calculations by choosing the following method. Call

$$i_0 = 0.09$$

$$i_1 = 0.09 + 10^{-5} = 0.09001$$

$$i_2 = 0.09 - 10^{-5} = 0.08999$$

To each of these values of  $i$  correspond the following values of  $B(i)$  for a bond at par (with coupon 9%) and maturity 10 years (easily determined through the closed-form formula (13) of chapter 2):

$i$	$B(i)$
$i_0 = 0.09$	$B(i_0) = 100$
$i_1 = 0.09001$	$B(i_1) = 99.99358$
$i_2 = 0.08999$	$B(i_2) = 100.0064$

Thus, the first derivative can very well be approximated, at point  $i_0$ , by

$$B'(i_0) = \frac{dB}{di}(i_0) \approx \frac{B(i_1) - B(i_2)}{i_1 - i_2}$$

(instead of considering simply  $\frac{B(i_1) - B(i_0)}{i_1 - i_0}$  or  $\frac{B(i_0) - B(i_2)}{i_0 - i_2}$ , both of which would have entailed more error than the first formula; a simple diagram can show nicely why this will generally be so).

The second derivative—to be used in the calculation of convexity—can easily be determined through the following approximation. Denoting  $i_1 - i_0 = i_0 - i_2 = \Delta i$ , we have

$$\begin{aligned} B''(i_0) &= \frac{d^2 B}{di^2}(i_0) \approx \frac{\Delta}{\Delta i} \left[ \frac{\Delta B}{\Delta i}(i_0) \right] \\ &= \frac{[B(i_1) - B(i_0)] - [B(i_0) - B(i_2)]}{(i_1 - i_0)^2} \end{aligned}$$

These expressions are very handy to program and yield excellent results, provided  $\Delta i$  is sufficiently small. (Note that this is precisely the way some financial services determine duration and convexity for practi-



tioners.) It is quite remarkable that the results

$$D = -\frac{1+i_0}{B(i_0)} B'(i_0) = 6.9952 \text{ years}$$

and

$$C = \frac{B''(i_0)}{B'(i_0)} = 56.4962 \text{ (years}^2\text{)}$$

can be obtained with *more* than four decimals (five, in fact) by using an increase of one basis point for the rate of interest ( $\Delta i = 0.0001$ ), which corresponds to two basis points for  $i_1 - i_2$ .

The figures in table 8.3 (horizon rates of return) do not give rise to any special problem and are straightforward to establish.

8.2. The results contained in tables 8.7–8.10 are quite straightforward to establish using the corresponding closed-form formulas.

a. Although the durations of both bonds are very close (about 7.9 years), their convexities are significantly different:

$$\frac{1}{B_A} \frac{dB_A^2}{di^2} = C_A = 76.9 \text{ (years}^2\text{)}$$

$$\frac{1}{B_B} \frac{dB_B^2}{di^2} = C_B = 67.2 \text{ (years}^2\text{)}$$

We now show why the higher convexity of  $B$  makes it preferable to  $A$ .

b. The rates of return at horizon  $H = D$  years are the following, if  $i$  moves from  $i = 12\%$  either to  $10\%$  or to  $14\%$ :

	$i = 10\%$	$i = 12\%$	$i = 14\%$
$r_{H=D}^A$	12.06%	12%	12.06%
$r_{H=D}^B$	12.03%	12%	12.03%

In either case, the higher convexity of  $A$  makes  $A$  more attractive than  $B$  in a defensive management style.

c. The rate of return at horizon  $H = 1$  year:

	$i = 10\%$	$i = 12\%$	$i = 14\%$
$r_{H=1}^A$	27.3%	12%	-0.4%
$r_{H=D}$	27.1%	12%	-0.7%

The results under (b) and (c) both confirm that bond  $A$  is to be preferred to bond  $B$ .

## Chapter 9

9.1. See the table at the end of section 9.1.

9.2. To check your answer, replace the increasing spot interest rate structure you have chosen with the first line of table 9.1.

9.3. To check your answer, replace your hump-shaped spot interest rate structure with the first line of table 9.2.

## Chapter 10

10.1. You have to use Leibniz's formula of the derivative of an integral with respect to a parameter. Let  $\alpha$  be such a parameter, and a definite integral  $I = I(\alpha)$ , depending on  $\alpha$  for these reasons: the function of  $x$  to be integrated,  $f$ , depends on  $x$  and  $\alpha$  and is written as  $f = f(x, \alpha)$ ; furthermore, both integration bounds,  $a$  and  $b$ , depend on  $\alpha$ . We thus have

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) dx$$

The Leibniz formula is the following:

$$I'(\alpha) \equiv \frac{dI}{d\alpha}(\alpha) = \int_{a(\alpha)}^{b(\alpha)} \frac{f(x, \alpha)}{\alpha} dx + \frac{db(\alpha)}{d\alpha} f[b(\alpha), \alpha] - \frac{da(\alpha)}{d\alpha} f[a(\alpha), \alpha]$$

(The demonstration of this formula can be found in any analysis text; however, you should convince yourself that the formula makes sense by drawing a simple diagram showing the exact increase in the definitive integral  $\Delta I$  entailed by a finite increase  $\Delta\alpha$  of parameter  $\alpha$  and then considering the limit  $\lim_{\Delta\alpha \rightarrow 0} \frac{\Delta I}{\Delta\alpha} = I'(\alpha)$ .)

In equation (6) of section 10.1, we have a special case of such an integral: only the function to be integrated depends on parameter  $\alpha$ ; the integration bounds are independent of  $\alpha$ . Thus only the first part of Leibniz's

formula applies to our problem. Taking the partial derivative of the function to be integrated in (6) with respect to  $\alpha$  and the definite integral of this partial derivative from 0 to  $T$  yields the right-hand side of (8).

10.2. Our first task is to define in continuous time the horizon rate of return for our investor when the spot rate structure  $\vec{i}$  receives a parallel displacement  $\alpha$ . In a way that extends what we did in section 6.2 (chapter 6), we can define this horizon rate of return  $r_H$  such that it transforms the bond's initial value  $B(\vec{i})$  into a future value  $F_H$  when the structure  $\vec{i}$  has undergone a shock  $\alpha$  immediately after the bond's purchase. This future value  $F_H(\vec{i} + \alpha)$  is given by equation (11) in section 10.3. Thus,  $r_H$  is defined by the following relationship:

$$B(\vec{i})e^{r_H H} = F_H = B(\vec{i} + \alpha)e^{[i(0, H) + \alpha]H} \quad (9)$$

Dividing equation (9) by  $B(\vec{i})$ , taking the logarithm on both sides, and finally dividing by  $H$ , yields

$$r_H = \frac{1}{H} \log \left[ \frac{B(\vec{i} + \alpha)}{B(\vec{i})} \right] + i(0, H) + \alpha \quad (10)$$

We can verify that in the case where no shock takes place ( $\alpha = 0$ ), the horizon rate of return is equal to  $i(0, H)$ , as it should be.

We have now a function  $r_H = r_H(\alpha)$  in  $(r_H, \alpha)$  space that we can try to minimize, in a way similar to what we did in section 6.2 (chapter 6). Consider equation (10) and perform in reverse order the operations described in the paragraph preceding it. Minimizing the horizon rate of return is thus equivalent to minimizing the future value of the investment, and we can then continue our demonstration of the immunization theorem with sections 10.4 and 10.5.

## Chapter 11

11.1. Applying formula (3) of section 11.1 with  $m = 6$ ,  $\Delta z_1 = 0.5$  years,  $i_1^{(1)} = 6\%$  per year, we get  $i_1^{(m)} = \frac{6}{0.5} \{ [1 + 0.06(1/2)]^{1/6} - 1 \} = 5.926\%$  per year, which is smaller than  $6\%$ , as it should be.

11.2. Applying (7), we have  $i_1^{(\infty)} = \frac{\log[1+0.06(1/2)]}{1/2} = 5.912\%$  per year, which is the smallest interest rate that would yield the same return to the lender (or cost the same amount to the borrower).

11.3. a. From  $R(0, T) = -0.005T^2 + 0.2T + 4$  (expressed in percent per year), the spot rate is indeed  $5.5\%$  per year for a horizon of 10 years.

b. The area of the rectangle is  $(5.5\% \text{ per year} \cdot 10 \text{ years}) = 0.55$  (a pure number). This area can be verified to be the integral (the sum) of the infinitely large number of forward rates from 0 to 10,  $\int_0^{10} f(0, T) dT = \int_0^{10} (-0.015T^2 + 0.4T + 4) dT = [-0.005T^3 + 0.2T^2 + 4T]_0^{10} = 55\% = 0.55$ .

c. Note that this must be a *pure* number (as observed above) because in the definite integral you are considering a sum of infinitesimal elements  $f(0, T) dT$ , whose dimension is indeed  $(1/\text{time}) \cdot \text{time}$ , a pure number as it should be.

d. The economic (or financial) interpretation of this area—expressed as the pure number 0.55, or 55%—is the continuously compounded rate of interest *per 10-year period* that applies to a loan made at time 0 and reimbursed 10 years later (or, equivalently, 5.5% *per year*). Indeed, we have

$$C_{10} = C_0 e^{R(0,10) \cdot 10} = C_0 e^{(0.055)10} = C_0 e^{0.55}$$

and therefore

$$\ln(C_{10}/C_0) = 0.55 = 55\%$$

e. In fact, you could have dispensed with any calculation:  $\ln(C_{10}/C_0) = 0.055 \times 10$ , the continuously compounded rate of return *over a 10-year period*, *must* be equal to the definite integral of the forward rates  $\int_0^{10} f(0, T) dT$  you have determined, both  $R(0, 10) = 5.5\% \text{ per year}$  and  $T = 10 \text{ years}$  having been given to you in (a). But it is definitely a smart idea to verify at least once those important properties.

## Chapter 12

12.1. You have to perform the following integration:  $R(T) = \frac{1}{T} [\int_0^T f(u) du]$ ;  $f(u)$  is obtained through equation (14). You just have to be careful that in (14),  $T$  is now transformed into an integration variable. Therefore, you have to calculate

$$\begin{aligned} R(T) &= \frac{1}{T} \left[ \int_0^T f(u) du \right] \\ &= \frac{1}{T} \left[ \int_0^T \left( \beta_0 + \beta_1 e^{-u/1} + \frac{\beta_2}{1} u e^{-u/1} \right) du \right] \end{aligned}$$

which does not present any difficulty. Integrating once by parts the last term yields (15).

12.2. First, write down a modified Hamiltonian corresponding to (14). Here  $x$  replaces  $t$ . The rest of the demonstration is given in Appendix 12.A.2.

12.3. A first integration of  $y^{(4)}(x)$  leads to

$$y'''(x) = c_1$$

where  $c_1$  is a constant. A second integration enables us to write

$$y''(x) = c_1x + c_2$$

and so forth:

$$y' = \frac{c_1}{2}x^2 + c_2x + c_3$$

$$y = \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3x + c_4$$

which amounts to a third-degree polynomial.

## Chapter 13

13.1. Using the fact that the normal density of the unit normal variable  $Z$  is  $(2\pi)^{-1/2}e^{-(1/2)z^2}$ , the moment-generating function (with parameter  $p$ ) of the unit normal is

$$z(p) = E(e^{pZ}) = \int_{-\infty}^{+\infty} e^{pz} \frac{1}{2\pi} e^{-z^2/2} dz = \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-\frac{z^2}{2} + pz} dz \quad (11)$$

Observe, in the right-hand side of (11), that the exponent of  $e$  can be written as  $-\frac{z^2}{2} + pz = -\frac{1}{2}(z^2 - 2pz + p^2) + p^2/2 = -\frac{1}{2}(z - p)^2 + p^2/2$ . (This operation is called “completing the square” because you observe that the exponent of  $e$  is “nearly” the perfect square of an affine function of  $z$ ,  $z - p$ .) Making now the change of variable  $z - p/2 = y$ , the moment-generating function of  $Z$  can be written as

$$z(p) = e^{p^2/2} \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-\frac{1}{2}y^2} dy \quad (12)$$

The integral in the right-hand side of (12) is just 1 (the integral of the unit normal density from  $-\infty$  to  $+\infty$ ). Hence the moment-generating function

of  $Z$  is simply

$$Z(p) = e^{p^2/2} \quad (13)$$

The moment-generating function of the normal  $U = +\sigma Z$   $N(0, \sigma^2)$  is

$$U(\lambda) = E(e^{\lambda U}) = E(e^{\lambda + \lambda \sigma Z}) = e^{\lambda} E(e^{\lambda \sigma Z})$$

Now in order to evaluate  $E(e^{\lambda \sigma Z})$ , it suffices to observe that this expression is none other than the moment-generating function of  $Z$ , where  $\lambda \sigma$  now plays the role of the parameter  $p$ . We then have

$$E(e^{\lambda \sigma Z}) = Z(\lambda \sigma) = e^{\lambda^2 \sigma^2 / 2}$$

and hence the moment-generating function of  $U \sim N(0, \sigma^2)$  is

$$U(\lambda) = E(e^{\lambda u}) = E(e^{\lambda + \lambda \sigma Z}) = e^{\lambda + \lambda^2 \sigma^2 / 2}$$

This result will be of great importance in this and subsequent chapters.

13.2. Applying equations (5), (8), and (9) of chapter 13, we get

$$E(R_{t-1,t}) = e^{+\frac{\sigma^2}{2}} - 1 = 10.52\%$$

$$VAR(R_{t-1,t}) = e^{2 + \sigma^2} [e^{\sigma^2} - 1] = 4.98\%$$

$$\sigma(R_{t-1,t}) = e^{+\frac{\sigma^2}{2}} [e^{\sigma^2} - 1]^{1/2} = 22.33\%$$

13.3. We have to determine

$$\text{Prob}(R_{t-1,t} < E(R_{t-1,t}))$$

This probability is equal to

$$\begin{aligned} & \text{Prob}(1 + R_{t-1,t} = X_{t-1,t} < 1 + E(R_{t-1,t}) = E(X_{t-1,t})) \\ &= \text{Prob}(X_{t-1,t} < e^{+\frac{\sigma^2}{2}}) = \text{Prob}\left(\log X_{t-1,t} < +\frac{\sigma^2}{2}\right) \end{aligned}$$

Since  $\log X_{t-1,t} \sim N(0, \sigma^2)$ ,  $\log X_{t-1,t} = +\sigma Z$ ,  $Z \sim N(0, 1)$ , where  $Z$  denotes  $Z_{t-1,t}$  for short. Hence this probability is equal to

$$\text{Prob}\left(+\sigma Z < +\frac{\sigma^2}{2}\right) = \text{Prob}\left(Z < \frac{\sigma}{2}\right)$$

This result is quite striking in its simplicity.

13.4. Denote  $f(z)$  as the probability density of the unit normal variable. With  $\sigma^2 = 4\%$ , using the result of question 13.3, the probability that  $R_{t-1,t}$  is smaller than  $E(R_{t-1,t})$  is

$$\text{Prob}(R_{t-1,t} < E(R_{t-1,t})) = \text{Prob}(Z < 0.01) = \int_{-\infty}^{0.1} f(z) dz = 0.5398$$

13.5.  $\text{Prob}(R_{t-1,t} < E(R_{t-1,t}))$  as a function of  $\sigma$  is as follows:

$\sigma$	$\text{Prob}(R_{t-1,t} < E(R_{t-1,t})) = \int_{-\infty}^{\sigma/2} f(z) dz$
0.05	0.51
0.1	0.52
0.15	0.53
0.2	0.54
0.25	0.55
0.3	0.56
0.35	0.57

This probability, within this range of  $\sigma$  values, is pretty much an affine function of  $\sigma$ .

13.6. A first method would be to apply equations (16) and (22); we would get

$$E(R_{0,n}) = e^{+\frac{\sigma^2}{2n}} - 1 = 8.55\%$$

$$\sigma(R_{0,n}) = e^{+\frac{\sigma^2}{2n}(e^{\frac{\sigma^2}{n}} - 1)} - 1 = 6.87\%$$

A second method would be to apply equations (25) and (28), using  $E \equiv 1 + E(R_{t-1,t})$  and  $V \equiv \text{VAR}(R_{t-1,t})$  in them, as obtained in 13.2. (Note, however, that *these* are the data that are more commonly used by practitioners.) Applying (4) and (8), respectively,  $E$  and  $V$  turn out to be  $E = 1.10517$  and  $V = 0.04985$ . We get

$$E(R_{0,n}) = E \cdot (1 + V/E^2)^{\frac{1}{2}(1 - \frac{1}{n})} - 1 = 8.55\%$$

$$\sigma(R_{0,n}) = [1 + E(R_{0,n})][(1 + V/E^2)^{1/n} - 1]^{1/2} = 6.87\%$$

as they should.

13.7. a. We want to determine

$$\text{Prob}(R_{0,n} < E(R_{0,n}))$$

From equation (12) of this chapter, we know that

$$R_{0,n} = X_{0,n}^{1/n} - 1$$

Therefore,  $1 + R_{0,n} = X_{0,n}^{1/n}$ , and  $\text{prob}(R_{0,n} < E(R_{0,n}))$  is equal to

$$\begin{aligned} \text{Prob}(1 + R_{0,n} < 1 + E(R_{0,n})) &= \text{Prob}(X_{0,n}^{1/n} < E(X_{0,n}^{1/n})) \\ &= \text{Prob}\left(\frac{1}{n} \log X_{0,n} < \log E(X_{0,n}^{1/n})\right) \end{aligned}$$

From (13), we know that  $\log X_{0,n} = n^{-1/2} + n\sigma Z$ ,  $Z \sim N(0, 1)$  and from (15),  $E(X_{0,n}^{1/n}) = e^{+\frac{\sigma^2}{2n}}$ . Therefore, this probability is equal to

$$\text{Prob}\left(-\frac{\sigma}{n} Z < \frac{\sigma^2}{2n}\right)$$

As in question 13.3,  $n$  cancels out. We are left with the simple, and important, result:

$$\text{Prob}(R_{0,n} < E(R_{0,n})) = \text{Prob}\left(Z < \frac{\sigma}{2}\right)$$

b. As a check, if we plug  $n = 1$  into this result, we get the same result obtained in exercise 13.3:  $\text{prob}(R_{0,1} < E(R_{0,1})) = \text{prob } Z < \sigma/2$ .

c. When  $n$  tends toward  $\infty$ , we have

$$\text{Prob}(R_{0,\infty} < E(R_{0,\infty})) = \text{Prob}(Z < 0) = 50\%$$

13.8. You can work either in terms of  $\sigma$ ,  $\sigma^2$  or in terms of  $E$ ,  $V$ .

a. In terms of  $\sigma = 0.08$  and  $\sigma^2 = 0.04$ , apply the left-hand side of equation (29):

$$E(R_{0,\infty}) = e^{-\sigma^2/2} - 1 = 8.33\%$$

b. In terms of  $E$ ,  $V = 1.1052$ , 4.98% as obtained in question 13.2, apply the right-hand side of equation (29):

$$E(R_{0,\infty}) = \frac{E^2}{(E^2 + V)^{1/2}} - 1 = 8.33\%$$



## Chapter 14

14.1. It implies that the *average* of all forward rates  $f(0, \cdot)$  for  $\cdot = 0$  to  $\cdot = 5$ ,  $\int_0^5 f(0, \cdot) d\cdot / 5$ , is equal to 5.069%.

14.2. No, because it implies only an increase in the *average* of forward rates, not necessarily an increase in *all* forward rates.

14.3. These directional coefficients are dimensionless—they are pure numbers. There are at least two ways of seeing this.

First, consider equation (7). Its left-hand side,  $dB$ , is in dollars; so must be its right-hand side.  $d\lambda$  is in (1/time) units;  $e^{-r_{0,t}^0}$  is unitless; so  $c_t(-t)e^{-r_{0,t}^0} d\lambda$  is in dollars. Hence  $a_t$  *must* be dimensionless.

Second, consider equation (8). Its left-hand side is in  $1/(1/\text{years}) = \text{years}$ . On the other hand, in the right-hand side of (8),  $c_t e^{-r_{0,t}^0}/B$  is in  $(\$/\$)$ , or a pure number. Since  $t$  is in years,  $a_t$  must be a pure dimensionless number.

14.4. a. The exact relative change in the bond's price is

$$\frac{\Delta B}{B} = \frac{B_0^1 - B_0^0}{B_0^0} = \frac{1112.464 - 1118.254}{1118.254} = -0.518\%$$

b. The directional duration of the bond is equal to  $D_a = 5.192$  years.

c. The relative change in the bond's price in linear approximation is given by

$$\frac{dB}{B_0^0} = -D_a d\lambda = -5.192163 \text{ years } (0.1\%/\text{year}) = -0.519\%$$

d. Using 5 decimals for  $dB/B$  and  $\Delta B/B$ , the relative error you are introducing by using this linear approximation is

$$\text{relative error} = \frac{(dB/B) - (\Delta B/B)}{\Delta B/B} = 0.29\%$$

Indeed, you could have figured out what the orders of magnitude of the answers to all questions (a), (b), (c), and (d) would be by reasoning in the following way.

Consider the directional vector  $\mathbf{a}$  in this case. It has the following properties: it lies between the directional vector used in the example treated in section 14.3, and the unit vector corresponding to a parallel

shift of the continuously compounded term structure, but it is definitely closer (in many senses) to the directional vector leading to a nonparallel displacement. We can conclude that

- a. the exact relative change in the bond's price will be slightly larger in absolute value than in the nonparallel displacement case we treated in this chapter (in algebraic value,  $-0.518\%$  against  $-0.444\%$ );
- b. the directional duration of the bond is slightly higher (5.192 years against 4.440 years);
- c. the relative change in the bond's price in linear approximation should be slightly higher in absolute value (in algebraic value:  $-0.519\%$  instead of  $-0.444\%$ );
- d. the relative error made by using the linear approximation should prove to be slightly higher ( $0.29\%$  against  $0.25\%$ ).

Table 14a summarizes the results of this project.

## Chapter 15

15.1. The polynomial corresponding to the new spot rate curve is given by

$$R^*(t) = 0.06 - 0.00218t + 0.000083t^2 - (1.2)10^{-6}t^3$$

15.2. The optimal (immunizing) portfolio is the one given in this chapter.

15.3. The seven-year horizon portfolio value corresponding to the new structure is  $103.3544e^{(0.04836)7} = \$144.99$ . It is even higher than the initial portfolio's value (\$144.72).

## Chapter 16

16.1. In (6a), factor out  $e^{-i\Delta t}$  to get

$$V_0 = e^{-i\Delta t} \left[ \frac{f(S_u) - f(S_d)}{S_u - S_d} S_0 e^{i\Delta t} + f(S_u) - S_u \frac{f(S_u) - f(S_d)}{S_u - S_d} \right] \quad (6a')$$

In the bracketed term of (6a'), factor out  $f(S_u)$  and  $f(S_d)$ :

$$V_0 = e^{-i\Delta t} \left\{ f(S_u) \left[ \frac{e^{i\Delta t} S_0}{S_u - S_d} + 1 - \frac{S_u}{S_u - S_d} \right] + f(S_d) \left[ \frac{S_u - S_0 e^{i\Delta t}}{S_u - S_d} \right] \right\} \quad (6a'')$$

**Table 14a**  
Nonparallel displacement of the continuously compounded spot rate curve

Term of payment $t$	Cash flow $C_t$	Initial spot rate (continuously compounded) $\ln(1 + i_{0,t}) = r_{0,t}^0$	Initial discounted cash flow $c_t e^{-r_{0,t}^0, t}$	Directional coefficient vector $\mathbf{a}$ $a_t$	Calculation of directional terms $ta_t c_t e^{-r_{0,t}^0, t} / B_0^0$	Increase in continuously compounded spot rate $a_t d\lambda$	New continuously compounded spot rate $r_{0,t}^1 = r_{0,t}^0 + a_t d\lambda$	New discounted cash flows $c_t e^{-r_{0,t}^1, t}$
1	70	0.044017	66.986	1	0.0599	0.001	0.045017	66.919
2	70	0.046406	63.795	0.95	0.1084	0.00095	0.047356	63.674
3	70	0.048314	60.555	0.9	0.1462	0.0009	0.049214	60.392
4	70	0.049742	57.370	0.85	0.1744	0.00085	0.050592	57.176
5	70	0.050693	54.327	0.8	0.1943	0.0008	0.051493	54.181
6	70	0.051643	51.349	0.75	0.2066	0.00075	0.052393	51.118
7	70	0.052592	48.441	0.7	0.2123	0.0007	0.053292	48.204
8	70	0.053067	45.785	0.68	0.2227	0.00068	0.053747	45.537
9	70	0.053541	43.234	0.65	0.2262	0.00065	0.054191	42.982
10	1070	0.053541	626.411	0.65	3.6411	0.00065	0.054191	622.975
$B_0^0 = 1118.254$					Directional duration: $D_a = 5.1922$	$B_0^1 = 1112.464$		

Simplifying now the first bracketed term of  $(6a'')$ , we get equation (7) in the text.

16.2. That an infinity of derivative values  $f(S_u)$  and  $f(S_d)$  leading to the same value  $V_0$  corresponds to one set of values  $i, S_0, S_u$ , and  $S_d$  comes from the fact that  $V_0$  is determined by the *single* equation (7), whose right-hand side is a function of the *two* variables  $f(S_u)$  and  $f(S_d)$ , given  $i, S_0, S_u$ , and  $S_d$ .

16.3. Solve equation (7) in  $f(S_d)$  with the following parameters:

$$4.8 = e^{-0.11}[-4.6(0.38) + f(S_d)0.61]$$

You get  $f(S_d) = 11.7$ , which can be checked with the height of point  $D''$  in figure 16.1.

16.4. You need the following information: first, the interest rate pertaining to the time span from the second to the third period and all possible values of the underlying asset at each node, which will enable you to determine the martingale probabilities  $Q$ ; second, the conditional probabilities of reaching each node of period 3.

16.5. You simply have to develop the first part of equation (26), not forgetting that  $a \equiv \frac{1}{\sigma}(\frac{1}{2}\sigma^2 + \sigma^2/2 - r)$ .

## Chapter 17

17.1. You first have to derive the current rate  $r(t)$  from the forward rate process given in integral form by equation (2), and then evaluate  $\exp[\int_0^t r(s) ds]$ .

17.2. See the detailed required steps in appendix 17.A.1.

17.3. You can retrace the various steps described in part 1 of appendix 17.A.2. Note the crucial point that when  $dt$  becomes infinitesimally small,  $(dW_t)^2$  ceases to become a random variable and becomes equal to  $dt$ .

17.4. This is a direct application of Itô's lemma (see appendix 17.A.2.).

17.5. You just need to calculate  $\sigma e^{-\lambda(T-t)} \int_t^T \sigma e^{-\lambda(u-t)} du$ ; after the necessary simplifications, you will obtain the drift term  $\alpha(t)$ .

17.6. The following three definite integrations are needed:

- the one you have performed in question 17.5,
- the integration of the differential  $df$  (equation (20)), and

c. the integration of  $\exp[-\int_t^T f(t, u) du]$ , which leads to the value of  $B(t, T)$  (equation (22)). The integrations are fortunately straightforward for the nonrandom terms.

## Chapter 18

18.1. The HJM portfolios and the classical portfolio can be easily calculated by inverting the matrix  $\mathbf{M}$ :

$$\mathbf{M} = \begin{pmatrix} B_a & B_b & B_c \\ X_a B_a & X_b B_b & X_c B_c \\ X_a^2 B_a & X_b^2 B_b & X_c^2 B_c \end{pmatrix}$$

and multiplying the inverted matrix  $\mathbf{M}^{-1}$  by the vector  $(-B - XB - X^2 B)'$ . The durations are straightforward to determine, and the results are summarized in the table 18a.

**Table 18a**  
Hedging portfolios and their durations\*

Number of bonds in hedging portfolio	Classical portfolio $\lambda = 0$	HJM portfolios ( $\lambda > 0$ )			
		$\lambda = 0.075$	$\lambda = 0.125$	$\lambda = 0.175$	$\lambda = 0.225$
$n_a$	-0.325	-0.313	-0.305	-0.297	-0.290
$n_b$	-1.025	-1.064	-1.091	-1.119	-1.147
$n_c$	0.351	0.378	0.397	0.417	0.437
Portfolio's duration	1	1	0.999	0.997	0.996

\*Rounded to the third decimal place

18.2. The immunization portfolios are given in table 18b.

**Table 18b**  
Immunization results (summary)

Number of bonds	Classical portfolio $\lambda = 0$	Heath-Jarrow-Morton portfolios				
		$\lambda = 0.1$	$\lambda = 0.15$	$\lambda = 0.2$	$\lambda = 0.25$	$\lambda = 0.3$
$N_a$	1.148	0.977	0.957	0.966	0.976	0.967
$N_b$	-1.996	-1.766	-1.733	-1.746	-1.776	-1.791
$N_c$	-0.117	-0.160	-0.169	-0.166	-0.152	-0.135
Duration (years)	8.023	8.061	8.083	8.076	8.034	7.965

Those results indicate first that the components of the vector  $(N_A, N_B, N_C)$ —the components of the replicating portfolio—are not monotonic functions of the reduction volatility factor  $\lambda$ . Also, we notice that when  $\lambda$  increases from  $\lambda = 0$  to  $\lambda = 0.3$ , the duration of the immunizing portfolio first increases above the classical portfolio's duration (8.023), then decreases to reach a value *below* that duration. This suggests that there must exist at least one  $\lambda$  value for which both durations are the same. This proves to be true, as table 18c shows.

**Table 18c**

**Immunization results for  $\lambda$  values between  $\lambda = 0.258$  and  $\lambda = 0.26$ , and for classical portfolio ( $\lambda = 0$ )**

Number of bonds	Classical portfolio $\lambda = 0$	Heath-Jarrow-Morton portfolios		
		$\lambda = 0.258$	$\lambda = 0.259$	$\lambda = 0.26$
$N_a$	1.148	0.976	0.976	0.976
$N_b$	-1.996	-1.780	-1.780	-1.781
$N_c$	-0.117	-0.150	-0.149	-0.149
Duration (years)	8.023	8.024	8.023	8.022

For  $\lambda = 0.259$ , the corresponding Heath-Jarrow-Morton portfolio, although somewhat different from the classical portfolio, exhibits the same duration (8.023 years).

## FURTHER READING

FERDINAND, KING OF NAVARRE. How well he's read,  
to reason against reading!  
*Love's Labour's Lost*

A complete bibliography on bonds would take more than one volume, since nearly every professional financial journal has at least one paper on the subject in each of its issues. For example, the remarkable book by Marek Musiela and Marek Rutkowski, *Martingale Methods in Financial Modelling* (Cambridge, U.K.: Cambridge University Press, 1998)—written from a methodological point of view—lists more than one thousand recent references rather closely related to our subject, and many are of a highly technical nature.

We will therefore limit ourselves here to a number of books and articles that we find essential, either for the new methodology they offer or for the historical development of the subject. Also, some books contain a lot of institutional information, such as auctions, special kinds of bonds, and so forth.

Many of the issues discussed in this book have been treated in a very lucid way by David Luenberger in his text *Investment Science* (Oxford: Oxford University Press, 1998). At about the same level of technicality as Luenberger's text is the now classic *Options, Futures and Derivative Securities* by John Hull (Englewood Cliffs, N.J.: Prentice Hall, 1993). A detailed descriptive and institutional point of view is taken both in the recent book by Suresh Sundaresan, *Fixed Income Markets and Their Derivatives* (Cincinnati, Ohio: South-Western, 1997), and in Frank Fabozzi's book *The Handbook of Treasury Securities* (Probus, 1986). Interest rate derivatives are thoroughly covered in *Derivative Securities* by Robert Jarrow and Stuart Turnbull (Cincinnati, Ohio: South-Western, 1996). A deep account of the implications of a given spot interest structure will be found in the thorough study by Robert Shiller, "The Term Structure of Interest Rates," in Benjamin Friedman and Frank Hahn (eds.), *Handbook of Monetary Economics* (North Holland, 1989).

From a methodological point of view, we feel that the recent book by Martin Baxter and Andrew Rennie, *An Introduction to Derivative Pricing*, (Cambridge, U.K.: Cambridge University Press, 1998) is a must for anyone not familiar with concepts such as probability measure change and the Cameron-Martin-Girsanov theorem. We have made ample use of it in section 13.1 (chapter 13) of this book. It is remarkable in its clarity. At

an advanced level, the book by Marek Musiela and Marek Rutkowski cited above offers the reader very current insights.

In these past twenty years, research has organized itself first in continuous time models, which were followed by discrete time analysis, and returned finally to continuous models. We now describe briefly some important contributions in this area.

Research began with arbitrage models in continuous time with one or more state variables. In general, this type of model is based on the specification of diffusion processes governing interest rates and the definition of risk premia. The first model in this vein was due to Vasicek ("An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5 (November 1977): 177–188), who developed a stochastic model using one state variable (the short-term rate). Two-state variable models were introduced by S. Richard ("An Arbitrage Model of the Term Structure of Interest Rates," *Journal of Financial Economics*, 6 (1978) and Cox, Ingersoll, and Ross ("An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 53, no. 2 (1985): 363–384), where the second-state variable is the inflation rate.

Nelson and Schaefer ("The Dynamics of the Term Structure and Alternative Portfolio Immunization Strategies," in G. O. Bierwag, G. Kaufman, and A. Toevs, eds., *Innovation in Bond Portfolio Management: Duration Analysis and Immunization* (Greenwich, Conn.: JAI Press 1983)) and Schaefer and Schwartz ("A Two-Factor Model of the Term Structure: An Approximate Solution," *Journal of Financial and Quantitative Analysis*, no. 4 (1984)) considered as state variables the spread difference between the long-term rate and the short-term rate. The contribution found in Cox, Ingersoll, and Ross ("A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, no. 2 (1985): 385–407) is a general equilibrium model of considerable breadth.

In recent years, the tendency has been to develop the modeling of the whole-term structure, following a suggestion made by T. S. Y. Ho and S. B. Lee in their article "Term Structure Movements and Pricing Interest Rate Contingent Claims" (*Journal of Finance*, December (1986): 1011–1029). This model, which rests also on arbitrage pricing, was recently extended by Hull and White in "One-Factor Interest Rate Models and the Valuation of Interest Rate Derivative Securities" (*Journal of Financial and Quantitative Analysis*, June (1993): 235–254). Another approach consists of defining a process for the short rate; this was used by Black, Derman, and Toy in "A One-Factor Model of Interest Rates and Its



Application to Treasury Bond Options” (*Financial Analysis Journal*, January (1990): 33–39) and Hull and White in “Pricing Interest Rate Derivative Securities” (*Review of Financial Studies*, 4, (1990): 573–592).

To the best of our knowledge, the idea of developing the price of a discount bond in a Taylor series is due to Donald R. Chambers and Willard T. Carleton, “A Generalized Approach to Duration,” *Research in Finance*, 7 (1988): 163–181. For immunization purposes, the model was very successfully tested with real data by D. R. Chambers, W. T. Carleton, and R. W. McEnally, “Immunizing Default-Free Bond Portfolios with a Duration Vector,” *Journal of Financial and Quantitative Analysis*, 23, no. 1 (March 1988): 89–104.

As we have stressed in the introduction as well as in the last two chapters of this book, a general and innovative approach to the problem of modeling the term structure from a stochastic point of view was suggested by David Heath, Robert Jarrow, and Andrew Morton in their now classic paper, “Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation” (*Econometrica*, 60, no. 1 (January 1992): 77–105). At an advanced level, the reader will certainly be interested in the recent paper by Martin Baxter, “General Interest Rate Models and the Universality of HJM” (in *Mathematics of Derivative Securities*, M. A. H. Dempster and S. R. Pliska, eds., pp. 315–335) (Cambridge, U.K.: Cambridge University Press, 1997).

Recent advances in stochastic optimization of bond portfolios have been carried out by a number of researchers, among them Marida Bertocchi and Vittorio Moriggia (of the Department of Mathematics at the University of Bergamo), Jitka Dupačová (Department of Probability and Mathematical Statistics, University of Prague), and Stavros Zenios (University of Cyprus, and Decision Sciences Department, The Wharton School, University of Pennsylvania). In particular, the reader should have a close look at two papers published in W. Ziemba and J. Mulvey, *Worldwide Asset and Liability Modelling* (Cambridge, U.K.: Cambridge University Press, 1998): J. Dupačová, M. Bertocchi, and V. Moriggia, “Postoptimality for Scenario Based Financial Planning Models with an Application to Bond Portfolio Management” (pp. 263–285), and S. Zenios, “Asset and Liability Management under Uncertainty for Fixed Income Securities” (pp. 537–560).

Also in the same vein, and of great interest, are the following studies: M. Bertocchi, J. Dupačová, and V. Moriggia, “Security of Bond Portfolios Behavior with Respect to Random Movements in Yield Curve: A Simula-

tion Study,” to appear in *Annals of Operations Research* (1999); J. Dupačová and M. Bertocchi, “Management of Bond Portfolios via Stochastic Programming—Postoptimality and Sensitivity Analysis” (in J. Doležal and J. Fidler, eds., *System Modelling and Optimization*, New York: Chapman and Hall, 1995, 574–582); and J. Dupačová and M. Bertocchi, “From Data to Model and Back to Data: A Bond Portfolio Management Problem” (to appear in the *European Journal of Operations Research*). Finally, the reader should also consult the following recent contributions: A. Beltratti, A. Consiglio, and S. Zenios, “Scenario Modelling for the Management of International Bond Portfolios” (*Annals of Operations Research*, 85 (1999): 227–247), and A. Consiglio and S. Zenios, “A Model for Designing Callable Bonds and Its Solution Using Tabu Research” (*Journal of Economic Dynamics and Control*, 21 (1997): 1445–1470). All of these papers contain numerous references to related research.

The reader should also be aware of an important area of research that deals with the optimization of portfolios carrying complex securities, such as callable or putable bonds, mortgage-backed securities, and insurance products. In order to tackle the problems raised, in particular, by cash infusion and withdrawal and transaction costs for certain types of securities, the development of multiperiod dynamic models for portfolio management was called for. At an expository level, we refer the reader to R. S. Hiller and C. Schaak, “A Classification of Structured Bond Portfolio Modeling Techniques” (*Journal of Portfolio Management* (1990): 37–48). For the problem of managing portfolios of mortgage-backed securities, see S. A. Zenios, “A Model of Portfolio Management with Mortgage-Backed Securities” (*Annals of Operations Research*, 43 (1993): 337–356), as well as D. R. Carion et al., “The Russel-Yasuda Kasai Model: An Asset/Liability Model for a Japanese Insurance Company Using Multistage Stochastic Programming” (*Interfaces* 24(1) (1994): 29–49). A successful application to money management can be found in B. Golub, M. Holmer, R. MacKendall, and S. Zenios, “Stochastic Programming Models for Money Management” (*European Journal of Operational Research*, 85 (1995): 282–296). At an advanced level, the reader will find a textbook treatment in Y. Censor and S. Zenios, *Parallel Optimization: Theory, Algorithms and Applications* (New York: Oxford University Press, 1997). An interesting new approach for various currency risk management strategies was developed by A. Beltratti, A. Laurent, and S. Zenios in “Scenario Modeling of Selective Hedging Strategies” (The Wharton School, University of Pennsylvania, Working Paper 99-15, 1999).

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From whence doth spring the true Promethean fire.  
*Loves' Labour's Lost*

PETRUCHIO. A herald, Kate? O put me in thy books!  
*The Taming of the Shrew*

These references should be considered of course as nothing but a small sample of the true Promethean fire.

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# INDEX

- Adams, K. J., 253
- Anderson, N., 247
- Anderson-Sleath model, 256–257
- Arak, M., 127
- Arbitrage, x, xi, xii, xiv, 6–9
  - bond valuation, 25–55
    - coupon-bearing bonds, 29–34, 49–52
    - discount factors as prices, 34–36
    - opened- and closed-form expressions, 39–44
    - process, 36–39
    - zero-coupon bonds, 26–29, 35–36
  - defined, 7
  - derivatives pricing, 327–356
    - Baxter-Rennie martingale evaluation, 351–356
    - continuous-time processes, 347–351
    - derivative value, properties, 336–338
    - one-step model, 328–331
    - overvalued derivative, 333, 336
    - probability measure change, 344–347
    - $Q$ -probability measure, 338–339
    - two-period evaluation model and extensions, 339–344
    - undervalued derivative, 331–334
  - of forward and spot rates, 228–229, 231–232
  - futures pricing, 111–125
    - basis. *See* Basis
    - with carry costs, 117–123
    - with interest only carry costs, 116–117
    - potential errors, 123–125
  - on interest rates
    - bond prices and, 44–52, 54
    - forward rates, 201
- Arbitrageurs, 101
- ARCH processes, 406
- Asked price *versus* bid price, 17
- Asset prices, lognormality, 238–240, 257–259, 288
- Autoregressive conditional heteroscedasticity processes, 406
- Bachelier, L., 405
- Basis
  - convergence to zero at maturity, 112
  - defined, 112
  - properties, before maturity, 112–115
  - sign of, 124–125
- Baxter, M., 338, 348, 352, 360, 378, 437, 439
- Baxter-Rennie martingale evaluation, 352–356
- Beltratti, A., 440
- Bernoulli, J., 261
- Bertocchi, M., 439
- Bid price *versus* asked price, 17
- Bierwag, G., 73
- Black, F., 352, 439
- Black-Scholes formula, xii, 351–356
- Bliss, R., 253
- Bond valuation, xiii, 37, 39
  - arbitrage, 25–55. *See also* Arbitrage
  - convexity and, 157–160
  - duration and, 76–78, 211–212
  - at given horizon, 212–213
  - at given term structure, 211–212
  - immunization and. *See* Immunization
  - at variable term structure, 215–218
- Bonds
  - classification, 13, 14
  - convertible, 19, 21–24
  - convexity, 153–155, 162–163
  - corporate, 18–19, 20, 162
  - evaluation. *See* Bond valuation
  - foreign, 87–90, 162
  - perpetual, 40–41
  - portfolio. *See* Portfolios
  - risk ratings, 19, 20. *See also* Riskiness
  - Treasury bills, 13–17
  - Treasury bonds, 18
  - Treasury notes, 17–18
- Bouchaud, J.-Ph., 406
- Brennan, M. J., 22, 24
- Brownian differential, 363–364
- Brownian motions
  - Cameron-Martin-Girsanov theorem, 347–351
  - HJM model, 365
  - Itô's formula, 374, 376–377
- Cameron, R. H., 347
- Cameron-Martin-Girsanov theorem, 347–351
- Capital asset pricing model, xii
- Capital gain or loss, horizon rate of return and, 64
- Capitalization rate, 194, 204
- Carion, D. R., 440
- Carleton, W. T., 252, 308, 439
- Carrying costs
  - defined, 113
  - futures pricing, 117–123
- Cash account
  - valuation, 372–373
  - Wiener process, 362
- Cash and carry, reverse, 117, 118, 122–123
- Cash settlement, futures contract, 109–110
- CBOT. *See* Chicago Board of Trade
- Censor, Y., 440
- Central limit theorem, xiv, 408
  - estimating value of yearly return, 269–270, 285–288

- Central limit theorem (cont.)
  - lognormality, 238–239, 240, 257–259, 288, 405
- Chambers, D. R., 252, 308, 309
- Chicago Board of Trade, conversion factor, 140
  - calculation, 139, 141
  - delivery option, 109, 126–127
  - as a price, 127–128
  - relative bias, 129–134
- Choice of the bond, 125, 126–127
- Chopra, V., 270
- Clearing houses, 102–106
- Closed-form expressions
  - of duration, 79–81
  - open-form expressions and, 39–44
- Closing out, futures position, 107–111
- Coleman, T. S., 252
- Compound rate of return. *See*
  - Continuously compounded rate of return; Rate of return
- Consiglio, A., 440
- Consol bond, 40
- Continuously compounded rate of return, 222–226
  - central limit theorem and, 285–288
  - directional duration and, 295
    - application, 299, 300–304
    - Macaulay and Fisher-Weil durations and, 296–299, 302, 303
  - mean and variance, estimating, 289–291
  - n*-horizon rate
    - uniform continuous compounding and, 291–292
  - variance, probability distribution and, 279–285
  - one-year return, value, variance and probability distribution, 269–279, 284–285
- Continuously compounded yield, 226–228
- Contracts. *See* Forward contracts; Futures contracts
- Convenience return, 118–119
- Conversion factor
  - CBOT. *See* Chicago Board of Trade
  - delivery of grades, 108–109
  - “true” or “theoretical,” 128–129
- Convertible bonds, xiii, 19, 21–24
- Convexity, xiv, 43, 153–169
  - bond management and, 171–172
  - coupon-bearing bonds, 172–175
  - portfolio development, 176–180
  - risk measurement, 157–160
  - zero-coupon *versus* coupon-bearing bonds, 174–176
  - calculation, 155–157
  - defined, 155
  - duration and, 77–78, 163–168
  - factors, 153–155, 160–163
  - immunization and, 168–169, 173–174, 214–215
- Cornering the market, 108
- Corporate bonds, 18–19
  - convexity and, 162
  - Moody’s ratings, 20
- Coupon, 1–2
- Coupon-bearing bonds
  - arbitrage-enforced valuation, 29–34
  - formulations, 40–44
  - interest rates and price relativity, 49–52
  - overvalued bonds, 32–34
  - undervalued bonds, 30–32
  - convexity, management and, 171–172
  - two bonds, 172–175
  - zero-coupon bonds *versus*, 174–176
  - duration and maturity, 83–85
  - immunization, HJM model, 391–399, 400–401
  - rate of return
    - coupon reinvestment, 63–64, 145–148
    - horizon. *See* Horizon rate of return
    - term structure derived from, 187–192
    - yield curve and, 187
    - yield to maturity, 61–62, 66–68
- Coupon rate, 2, 79–81
- Coupon reinvestment, 63–64, 145–148
- Cox, 438
- Cross-hedging, 101
- Daily settlements, futures markets, 102–106
- Debt ratings, Standard and Poor’s, 19
- Declare his intent, 125, 126
- Default consequences, futures markets, 111
- Delivery day, 126
- Delivery, futures contract, 107–109
  - T-bonds, 134–139
  - options, 109, 125–127
  - profit, 134–139
- Dempster, M. A. H., 439
- Density function, 276–277, 286
- Derivatives
  - arbitrage, 327–356. *See also* Arbitrage
  - directional, 295–296
  - HJM model. *See* Heath-Jarrow-Morton model
- Derman, 439
- Direct yield, 49–50
- Directional derivative, 295–296
- Directional duration, xiv, 295
  - application, 299, 300–302

- nonparallel displacement, 303–304
- parallel displacement, 302–303
- Macaulay and Fisher-Weil durations and, 296–299, 302, 303
- Discount factors, as prices
  - in bond valuation, 34–36
  - term rate structure and, 189–190
- Discount function, 241–243, 252
- Distributions
  - ARCH processes and, 406
  - fat-tailed, 288, 406
  - fractal, 406–407
  - lognormal. *See* Lognormal distributions
  - probability. *See* Probability distributions
- Doléans SDE, 376–377
- Dollar rate of return, 194
  - defined, 35–36
  - n*-year horizon rate, 279–285, 288
  - notation, 268–269
  - one-year return, 269–279, 285–287
- Dorfman, R., 259, 261–262
- Dupačová, J., 439
- Duration, xiii, 65–66, 71–91
  - bond portfolio, 85–87
  - calculation, 73–74
  - CBOT T-bond contract, 132–134
  - as center of gravity, 75
  - conceptual importance, 75–76
  - convexity and, 163–168
    - in bond management, 172–176
    - in portfolio development, 176–180
  - zero-coupon bonds, 175–176
  - defined, 72
  - directional. *See* Directional duration
  - Fisher-Weil. *See* Fisher-Weil duration
  - immunization and, 75–76
    - coupon-bearing bonds, 395–399, 400–401
    - immunizing horizon in, 143–145, 214
    - zero-coupon bonds, 387–389, 390–391
- Macaulay's. *See* Macaulay's duration
- modified duration, 78–79
- properties
  - bond value, 211–212
  - coupon rate, 80–81
  - maturity, 83–85
  - yield to maturity, 76–78, 81–82
- EEP agreement, 110–111
- Elsrgolc, L., 250
- End-of-the-month option, 126
- Equilibrium prices, x, xi, 55, 191, 408
- Equilibrium rate of return, x, 11–12, 198–200
- Euler equation, xi, 250–251
- Euler-Lagrange equation, xi, 325
- Euler number, 227–228, 234
- Euler-Poisson equation, 250–252
  - calculus of variations and, 325
  - derivation, xi, 259–264
- Exchange for physicals, 110–111
- Fabozzi, F., 437
- Face value, 1
- Fat-tailed distributions, 288, 406
- Financial institutions, investment protection, 163–169
- Fisher, I., xi, 46
- Fisher, L., 73, 252, 254
- Fisher equation, xi, xii, 44–45, 46–47
- Fisher-Nychka-Zervos model, 253–256, 257
- Fisher-Weil duration, xiv, 209, 218
  - classical *versus* HJM portfolios, 395, 399
  - directional duration and, 296–299, 302, 303
- Fixed exchange rates, ix
- Fixed rate of return, ix, 12, 13
- Fluctuation
  - rate of return
    - bond pricing and, 52, 53, 54
    - financial institutions and, 163–169
  - yield, 2–3
- Fong, G., 252
- Foreign bonds
  - convexity, 162
  - riskiness, 87–90
- Forward contracts
  - arbitrage, 112–113
  - characteristics and deficiencies, 96–99
  - closing out, 110
  - defined, 4
  - synthetic, 9–10
- Forward rate curve, splines and
  - Anderson-Sleath model, 256–257
  - Fisher-Nychka-Zervos model, 253
  - Waggoner model, 253–256
- Forward rate of return, xiv
  - arbitrage and, xiii, 6–9
  - derived from term structure, 192–198
    - risk averse approach, 201–207
    - risk neutral approach, 195–201, 206
  - HJM model. *See* Heath-Jarrow-Morton model
  - in search for term structure, 240–243
    - Anderson-Sleath model, 256–257
    - Fisher-Nychka-Zervos model, 253
    - Nelson-Siegel model, 244–247
    - parametric methods, 244–248
    - spline methods, 248–257
    - Svensson model, 247–248
    - Waggoner model, 253–256

- Forward rate of return (cont.)
    - spot rates and, 238–240
    - in instantaneous lending and borrowing, 228–232
    - lognormality, 238–240
    - in search for term structure, 240–257
    - zero-coupon bond price and, 232–233
  - Wiener process, 361
  - Fractal theory, 406–407
  - Fractional numbers, 5–6
  - Friedman, B., 437
  - Functionals, 308
  - Functions
    - density, 276–277, 286
    - discount, 241–243, 252
    - fractal distributions, 406
    - functionals and, 308
    - Laguerre function, 244, 245
    - pedagogy and, 407–408
  - Future value. *See also* Horizon rate of return; Long horizon rate of return
    - horizon rate of return, 65
    - one-year return, estimating, 269–279, 284–285
    - zero-coupon bond, forward and spot rates and, 232–233
  - Futures contracts
    - arbitrage, 111–125
      - basis. *See* Basis
      - with carry costs, 117–123
      - with interest only carry costs, 116–117
    - potential errors, 123–125
  - closing out, 107
    - cash settlement, 109–110
    - default consequences, 111
    - delivery, 107–109
    - exchange for physicals, 110–111
    - reversing trades, 110
  - HJM applications, 401
  - immunization with, 168–169
  - riskiness, 106–107
  - standardization, 102
  - T-bond. *See* Treasury bonds
- Futures markets
- contracts. *See* Futures contracts
  - daily settlements, 102–106
  - financial instruments, 100
  - margins, 102–106
  - operators, 100–101
  - safeguards, 102
  - trading categories, 100
- Geometric processes. *See* Wiener process
- Girsanov, I. V., 348
- Gold reserves, ix
- Golub, B., 440
- Goodman, L. S., 127
- Grades, delivery, 107–109
- Grandville, O. de La, 287
- Grauer, R. R., 271
- Gross return, 35–36
- Growth theory, variational problems in, xi
- Hahn, F., 437
- Hakansson, N. H., 271
- Hamiltonians, Euler-Poisson equation and, 259–264
- Harrison, M., xii
- Hawawini, G., 85
- Heath, D., xii, 360, 439
- Heath-Jarrow-Morton model, x, xiv, 359–402
  - cash account, valuation, 372–373
  - derivatives pricing, 401
  - immunization
    - coupon-bearing bonds, 395–399, 400–401
    - zero-coupon bonds, 387–389, 390–391
  - Itô's formula, with applications, 374–378
  - one-factor model, 360–361
    - cash account, 362, 372–373
    - current short rate, 361–362
    - forward rate, 361
    - no-arbitrage condition, 365–367
    - probability measure change, 363–365
    - zero-coupon bonds, 363
  - Vasicek volatility reduction factor, 368–372
  - zero-coupon bond, 388
- Hedgers, 100–101
- Hedging. *See also* Immunization
  - coupon-bearing bonds, 391–399, 400, 401
  - cross-hedging, 101
  - forward contract, 96–99
  - zero-coupon bonds, 384–392
- Hicks, J., 73
- Hiller, R. S., 440
- HJM model. *See* Heath-Jarrow-Morton model
- Ho, T. S. Y., 438
- Holmer, R., 440
- Holding period rate of return. *See* Horizon rate of return
- Horizon, 4, 408
- Horizon rate of return, xiii
  - bond valuation at, 212–213
  - coupon-bearing bonds, 65
  - coupon reinvestment, 63–64, 145–148
  - forward rates, derivation
    - borrowers and, 203–207
    - investors and, 201–203

- fractional numbers, 5–6
- future value, 59, 62–66
  - central limit theorem and, 287–288
  - $n$ -year, estimating, 279–285
  - one year or less, 4–5
- immunization. *See* Immunization
- integer numbers, 5
- long return. *See* Long horizon rate of return
- short return, 145, 146
  - borrowers and, 203
  - investors and, 201–202
  - one year or less, 4–5
- spot rate and, 185–187
- Hull, J., 438, 439
- Ibbotson, R., 252
- Immunization, xiii–xiv, 143–150, 209–218, 383–401
  - bond value and
    - at given term structure, 211–212
    - at horizon  $H$ , 212–213
    - at variable term structure, 215–218
  - conditions, 213–215
  - convexity and, 168–169, 214–215
  - management strategy, 173–174
  - coupon-bearing bonds, 392–401
  - defined, 144
  - duration and, 75–76
    - coupon-bearing bonds, 395, 396–399, 400–401
    - immunizing horizon in, 143–145, 214
    - zero-coupon bonds, 387–389, 390–391
- HJM applications. *See* Heath-Jarrow-Morton model
- likelihood, 145–148
- term structure variation and, 216
  - defensive management, 173–176
  - difficulties with, 215–218
  - general theorem, with applications, 308–324
  - zero-coupon bonds, 384–391
- Indices, cash settlement and, 109–110
- Inflation, interest rates and, x, 10–11
- Ingersoll, J. E., 22, 438
- Initial margin, 104
- Integer numbers, horizons and, 5
- Intention day, notice of, 125
- Interest rate. *See* Rate of return
- Interest rate futures, 100
- Internal rate of return
  - ask yield and, 18
  - Macaulay's duration, 73–74
  - spot rate and, 186
  - yield to maturity and, 60
- International bonds
  - convexity, 162
  - riskiness, 87–90
- IRR. *See* Internal rate of return
- Itô's formula
  - applications, 376–378
  - derivation, 374–375
- Itô's lemma, 363–364, 375, 377–378
- Jarrow, R., xii, 360, 372, 387, 437, 439
- Kallberg, J., 271
- Kaufman, G., 73
- Keynes, J. M., 55, 123–125
- Khaldun, I., 408
- Killcollin, T., 127
- Kolb, R., 126
- Koponen, I., 407
- Labuszewski, J., 127
- Laguerre function, 244, 245
- Last trading day, 126
- Laurent, A., 440
- Lee, S. B., 438
- L'Hôpital's rule, 245, 259, 369
- Lintner, J., xii
- Lognormal distribution, xiv, 238–240
  - central limit theorem and, 257–259, 285–288
  - general use, 405
  - one-year return, 272–277
- Long forward contract, 98, 99
- Long horizon rate of return, xiv, 267–292
  - central limit theorem and, 269–270, 285–288
  - forward rates, derivation
    - borrowers and, 203–205
    - investors, 202, 203
  - immunization, 145–150
  - lognormality and, 285–289
  - mean and variance, estimating, 289–291
  - $n$ -year horizon return
    - probability distribution, 283–285
    - variance, 279–283
  - one-year return, 274
    - probability distribution, 272–279
    - variance, 267–272
  - uniform continuous compounding and, 291–292
- Luenberger, D., 437
- Macaulay, F., 73, 163
- Macaulay's duration, xiv, 73–74
  - directional duration and, 296–299

- Macaulay's duration (cont.)
  - immunization and, 209, 218
  - riskiness measurement and, 166
- MacKendall, R., 440
- MacLaurin approximation, 68
- MacLaurin series, 309
- Maintenance margin, 103–104
- Management style
  - active, 171, 174, 176
  - coupon-bearing bonds, 174
  - defensive, 171, 176
- Mandelbrot, B., 288, 405
- Margins, 102–106
- Markovitz, H., xii
- Martin, W. T., 348
- Martingale probability, xii, 329
  - Baxter-Rennie evaluation, 351–356
  - HJM model, 363–365, 367–368, 401
  - Itô's lemma and, 377–378
  - $P$ -martingale, 345
  - $Q$ -probability and, 338–339, 347
- Maturity
  - in bond valuation, 41–44, 54
  - convergence of basis to zero at, 112
  - defined, 1
  - duration and, 82–85
  - profit from delivery and, 138
  - properties of basis before, 112–115
  - yield to maturity, 59–62
- McCulloch, J. H., 252
- McEnally, R. W., 308, 439
- Meisner, J., 127
- Merton, R., xii
- Modified duration, 78–79
- Moriggia, V., 439
- Moody's corporate bond ratings, 20
- Morton, A., xii, 360, 439
- Mulvey, J., 439
- Musiela, M., 437
- Nelson, C., 244, 438
- Nelson-Siegel model, 244–245, 246, 247
- Notice of intention day, 125
- Numbers
  - Euler number, 227–228, 234
  - fractional, 5–6
  - integer, 5
- Nychka, D., 253
- Offsetting positions, 110
- Oil shocks of 1973 and 1979, x
- One-year return, estimating value, variance and probability distribution, 269–279, 284–285
- Open-form expressions
  - closed-form expressions and, 39–44
  - of duration, 81–82
- Opportunity cost, 113
- Ostrogradski equation, 264, 325
- $P$ -martingale, 345
- $P$ -measure, 344, 345, 347, 352
- $P$ -probability measure, 344–345
- Par value, 1
- Parametric methodology, 246, 247
  - Nelson-Siegel model, 244–245, 246, 247
  - versus* spline methodology, 256–257
  - Svensson model, 247–248
- Perpetual bond, 40–41
- Peters, E., 406, 407
- Physicals, exchange for, 110–111
- Pliska, S. R., xii, 439
- Polynomials. *See* Spline methodology
- Pontryagin's principle, 260–261
- Portfolios
  - convexity, 153–155, 160–163
  - in portfolio development, 176–180
  - duration, 85–87, 143–144
  - HJM model. *See* Heath-Jarrow-Morton model
  - immunization, 143–144, 160–163
  - coupon-bearing bonds, 395, 396–399, 400–401
  - variable term structure and, 215–218
  - zero-coupon bonds, 387–389, 390–391
- Position day, 125
- Potters, M., 406
- Price
  - arbitrage. *See* Arbitrage
  - bid *versus* asked, 17
  - Capital asset pricing model, xii
  - CBOT conversion factor as, 127–128
  - determination, 54–55
  - discount factors as
    - term rate structure and, 189–190
    - in valuation, 34–36
  - equilibrium, 55, 191, 408
  - fluctuating rate of return and, 52, 53, 54
  - HJM model. *See* Heath-Jarrow-Morton model
  - lognormality, 238–240, 257–259, 288
  - sensitivity, duration and, 76–78
- Principal, 1
- Probability distribution
  - central limit theorem and, 257–259, 285–288
  - independence, 406

- n*-year horizon return, 283–285
- one-year return, 272–279
- Probability measures, xiv
  - change in, 344–347, 363–365
  - Martingale, 329, 338–339
  - Q*-probability measure. *See Q*-probability measure
- Profit from delivery, 134–139
- Pure-discount security, 2
- Q*-martingale process, 353–355
- Q*-measure
  - Black-Scholes formula and, 352–356
  - martingale, 345, 347
- Q*-probability measure, 338–339
  - Black-Scholes formula and, 352–356
  - Cameron-Martin-Girsanov theorem, 347–351
  - probability measure change, 344–347
  - in two-period evaluation model and extensions, 342–345
- Quality option, 125, 126–127
- Radon-Nikodym process, 345, 346, 347
- Rate of return, ix, xiii, 3–24. *See also* Yield
  - arbitrage, 6–9
  - bond prices and, 44–52, 53, 54
  - forward rates, 201
  - in bond valuation, 41–44
  - compound. *See* Continuously compounded rate of return
  - convexity. *See* Convexity
  - dollar rate, 194
  - defined, 35–36
  - n*-year horizon rate, 279–285, 288
  - notation, 268–269
  - one-year return, 269–279, 285–287
  - equilibrium rate, 11–12, 198–200
  - fixed rate, 12, 13
  - forward rate. *See* Forward rate of return
  - HJM applications, 401
  - horizon rate. *See* Horizon rate of return; Long horizon rate of return
  - immunization. *See* Immunization
  - internal rate
    - ask yield and, 18
    - Macaulay's duration, 73–74
    - spot rate and, 186
    - yield to maturity and, 60
  - long-term. *See* Long horizon rate of return
  - rationales for, 10–13
  - spot rate, 4, 185–187
  - term structure. *See* Term structure
  - volatility of, xiii
- Redington, F. M., 73, 76, 163
- Redington conditions, 164–168
- Redington property, 164
- Reinvestment, of coupons, 63–64, 145–148
- Reinvestment risk, 3
- Rennie, A., 338, 348, 352, 360, 378, 437
- Reverse cash and carry, 117, 122–123
- Reversing trades, 110
- Richard, S., 438
- Risk premium, 47–48, 204
- Riskiness
  - foreign bonds, 87–90
  - forward contracts, 96–99
  - futures trading, 106–107
  - long horizon forward rates
    - borrowers, 203–207
    - investors, 202–203
  - measures, xii
    - convexity and, 157–160
    - duration, 77–79
    - Moody's corporate ratings, 20
    - S&P debt ratings, 19
  - risk premium, xii, 47–48, 204
  - short horizon forward rates
    - borrowers, 203
    - investors, 201–202
- Ross, S., 127, 257, 438
- Roughness penalty, 253–256
- Rutkowski, M., 437
- Samuelson, P., 73
- Savings and loan association failures, 165
- Scalpers, 101
- Schaak, C., 440
- Schaefer, S., 438
- Schwartz, E. S., 22, 24, 438
- SDE, Doléans, 376–377
- Securities, pure-discount, 2
- Settlement price, 125–126
- Sharpe, W., xii, 48
- Shea, G., 252
- Shiller, R., 406, 437
- Short (the), 97
- Short forward contract, 97–98
- Short horizon rate of return, 145, 146
  - borrowers and, 203
  - investors and, 201–202
- Wiener process, 361–362
- Siegel, A., 244
- Simple interest rate, continuous
  - compounding and, 222–225
- Sleath, J., 247
- Solow, R., xi, 407
- S&P. *See* Standard and Poor's

- Speculators, 101
- Spline, 248–249
- Spline methodology, 248–253
  - Anderson-Sleath model, 256–257
  - Fisher-Nychka-Zervos model, 253
  - versus* parametric methodology, 256–257
- Waggoner model, 253–256
- Spot price
  - forward contracts, 97–98
  - futures trading
    - arbitrage, 101, 111–125. *See also* Arbitrage
    - speculators, 101
- Spot rate curve. *See* Term structure
- Spot rate of return
  - arbitrage, xiv, 6–9
  - defined, 4, 185–187
  - forward rates and. *See* Forward rate of return
  - immunization and. *See* Immunization structure. *See* Term structure
- Standard and Poor's, 19, 109–110
- Stocks
  - arbitrage pricing. *See* Arbitrage, derivatives pricing
  - convertible bonds and, 19, 21–22
- Sundaresan, S., 437
- Svensson model, 247–248
  
- Taylor approximation, 68
- Taylor development, 224
- Taylor expansion, 158, 312
- Taylor series
  - convexity calculations, 157
  - immunization calculations, 309, 312, 325
  - coupon-bearing bonds, 392
  - shock sensitivity, 384
- Term structure, xii–xiii, xiv
  - in bond valuation, 37, 55
  - given, 211–212
  - at horizon  $H$ , 212–213
  - variable, 215–218
- calculation, 187
- defined, 187
- forward rates derived from, 192–198
- risk averse approach, 201–207
- risk neutral approach, 195–201, 206
- parallel changes in, xiv
- of rates, search for, 240–243
  - Anderson-Sleath model, 256–257
  - Fisher-Nychka-Zervos model, 253
  - Nelson-Siegel model, 244–245, 246, 247
  - parametric methods, 244–248
  - spline methods, 248–257
  - Svensson model, 247–248
  - Waggoner model, 253–256
- variation, immunization and. *See also* Directional duration
- defensive management, 173–174, 175, 176
- difficulties with, 215–218
- general theorem, with applications, 308–324
- yield curve and, 187–192
- Theorems
  - Cameron-Martin-Girsanov, 347–351
  - central limit. *See* Central limit theorem
  - immunization, with applications, 308–324
- Time to deliver option, 125–126
- Toevis, A., 73
- Toy, 439
- Trade balance, ix
- Trading Act of 1921, 108
- Trading day, last, 126
- Trading period, 4, 408
- Treasury bills, 13–17, 401
- Treasury bonds, 14, 18
- Treasury bonds futures contract, 125–139
- conversion factor
  - CBOT. *See* Chicago Board of Trade, conversion factor
  - true or theoretical, 128–129
- delivery options, 109, 125–127
- delivery profit, 134–139
- HJM applications, 401
- Treasury notes, 14, 17–18
- Turgot de L'Aulne, A., x, xi, xii
- Turgot-Fisher equation, xi, 47
- Turnbull, S., 371, 387, 437
  
- Valuation. *See* Bond valuation
- Van Deventer, D. R., 253
- Variable roughness penalty, 253–256
- Variance
  - mean and, estimating, 289–291
  - $n$ -year horizon return, 279–283
  - one-year return, 267–272
- Vasicek, O., 252, 368, 438
- Vasicek volatility reduction factor, 368–372
- Volatility reduction factor
  - classical *versus* HJM portfolios, 389–391, 395, 401
  - Vasicek's, 368–372
- Volatility term, xiii
- VRP, 253–256



- Waggoner, D., 248, 253
- Waggoner model, 253–257
- Waldman, D., 252
- Weil, R., 73
- White, A., 438, 439
- Wiener process, xiii, 405
  - cash account as, 362
  - forward rate as, 361
  - lognormality, 239–240
  - probability measure change, 364–365
  - short rate as, 361–362
  - Vasicek volatility reduction factor, 369
  - zero-coupon bond as, 363
- Wild card option, 125–127
  
- Yield
  - continuously compounded, 226–228
  - direct, 49–50
  - fluctuation, 2–3
- Yield curve, 183, 187–192
- Yield to maturity, xiii, 59–62
  - ask yield, 17–18, 67–68
  - defined, 183–184
  - duration and, 76–78, 81–82
  - as performance measure, 62
  - semi-annual coupons, 61–62, 66–68
  - yield curve and, 187
  
- Zenios, S., 439, 440
- Zero-coupon bonds
  - defined, 2
  - duration and maturity, 83
  - forward and spot rates and, 232–233
  - HJM model. *See also* Heath-Jarrow-Morton model
  - immunization, 387–389, 390–391
  - one-factor model, 363
  - protection, 384–387
  - management, convexity in, 174–176
  - term structure derived from, 188, 191, 191.
    - See also* Term structure
  - valuation, 26–29, 35–36, 38–39
  - overvalued bonds, 28–29
  - undervalued bonds, 27–28
  - Wiener processes, 363, 364
- Zervos, D., 253
- Ziemba, W., 270, 271, 439