Optimal Credit Allocation for Buy-and-Hold Investors

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- Buy-and-hold investors face the tradeoff between risk and return from a very different viewpoint than their total return counterparts. Especially in credit, the longer horizon carries with it a much more asymmetric return distribution in which the maximum return is the yield earned and the maximum loss is represented by default. Diversification can help mitigate the exposure to default risk, but issuer correlations reduce the benefit of diversification and create a systematic component of un-diversifiable default risk.

- The standard tools used by total return portfolio managers for top-down asset allocation (e.g., mean-variance optimization) and bottom-up security selection (e.g., multi-factor risk models) do not fill the needs of buy-and-hold investors. We present an approach to help such investors deal with these issues.

- The biggest single decision for many buy-and-hold investors is the allocation between A-rated and Baa-rated credit. How should this allocation be set? Given current spreads, expected default rates and correlations, and loss tolerance, we derive an optimal long-term allocations among segments of the credit market.

- We offer a simple approach to translate portfolio default rates into horizon returns. It allows any model of portfolio defaults to be combined with current yield and spread information. The porfolio optimization problem is then cast in terms of expected return versus risk. The model for portfolio defaults incorporates the impact of issuer correlations.

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1. INTRODUCTION

Credit by its very nature offers an asymmetric return profile. A corporate bond rewards investors with a small advantage over Treasuries (spread) over the course of its lifetime in compensation for bearing the risk of a large loss (default) with a small probability. To a certain extent, default risk is an issuer-specific, or non-systematic risk, and can be diminished via diversification. However, correlations among issuers make it impossible to entirely eliminate default risk through diversification. The common dependence of all issuers on general economic conditions, and the common exposures of all issuers within a given industry, give default risk a systematic component which cannot be diversified away.

This extremely asymmetric view of credit investing corresponds most closely to the considerations of a long-term investor who intends to buy bonds and hold them to maturity. In this case, the maximum upside is just the yield or spread earned, while the maximum loss is potentially the entire investment. Investors with a much shorter time frame may perceive a very different, and less asymmetric, risk/return profile. For a total return manager evaluating his investments on a monthly horizon, the dominant risks of investment-grade credit are the exposure to spread widening and the possible loss of liquidity. Yet spreads are just as likely to tighten as to widen, offering some upside to partially offset this risk. Also, credit degradation for investment-grade debt usually involves a sequence of downgrade events rather than direct default, so that even this component of risk is seen by total return investors primarily as downgrade risk rather than default risk[1].

This difference in risk horizon has two main implications for buy-and-hold investors. First, the spreads at which credit trades in the market are set by the interaction between investors of all different types. High estimates of short-term spread volatility or liquidity risk on the part of total return investors can sometimes drive spreads up beyond the level justified by long-term default risk alone. For long-term credit investors, who are unaffected by these short-term risks[2] , these high spreads represent a buying opportunity. The ability to identify and exploit such opportunities is the key to their success.

Second, the asymmetric nature of the risk/return profile for long-term investors must be considered in the asset allocation process. The most common approach to asset allocation is mean-variance optimization, in which the key measure of risk is the standard deviation of asset return (or of outperformance). This approach may be suitable for total return managers, who can model the means, standard deviations, and correlations of monthly excess returns among various asset classes. However, for very asymmetric return profiles, standard deviation is not a good measure of risk. In fact, it is safe to say that no single measure of risk is universally appropriate for dealing with the extreme events at the "tail" of a probability distribution. The treatment of this "tail risk" is very subjective, and must be tailored to the needs and considerations of each investor. Different approaches have been taken: downside risk measures (also known as lower

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[1] For a study of downgrade risk in investment grade credit, and the portfolio structuring implications for total return managers, see *Sufficient Diversification in Credit Portfolios*, Lehman Brothers, May 2002.

[2] In reality, very few investors are entirely immune to short-term risks. Certain book value investors, for example, may be subject to downgrade risk (e.g., insurance companies with risk-based capital requirements). For the purposes of this paper, however, we will continue with the simplifying assumption that a buy-and-hold investor is concerned only with default risk. This point of view may correspond to that of a CDO manager.

partial moments) characterize the portion of the return distribution that is below some target, which can be viewed as the minimum required return. Alternatively, utility functions that incorporate risk aversion can be used to penalize negative returns more than we reward positive returns when comparing two return distributions. Asset allocation optimizations can be carried out using either of these approaches. However, all of these approaches require an explicit distribution of asset class returns. A simple characterization by mean and standard deviation is not sufficient.

As a result of these basic differences in investment objectives and risk horizons, quantitative decision support tools for buy-and-hold managers need to analyze portfolio risk and return at a different level than those used by total return managers. Nevertheless, the management decisions in both settings can be grouped into the same two broad categories: top-down allocation among the various segments of a given market, and bottom-up selection of the specific securities used to implement a desired allocation.

For total return managers, the Lehman Brothers Global Risk Model[3] provides a complete analysis of both systematic and non-systematic risks over a one-month horizon based on the security-level composition of a fixed-income portfolio and its benchmark. For macro-level asset allocation, we have developed a risk budgeting framework that helps translate manager views into an optimal allocation subject to various types of constraints.

For bottom-up analysis of buy-and-hold portfolios, the Lehman Brothers' Quantitative Credit Research group has developed a proprietary application known as COMPASS (Credit OptiMised Portfolio Asset Selection System)[4]. COMPASS finds the detailed security-level composition of a portfolio that will minimize the expected shortfall due to defaults for a given average spread. Originally designed for valuation of complex credit derivatives, COMPASS uses a Monte Carlo approach with a rich set of options for modeling default correlations and tail dependence.

In this paper, we use a similar (but much simplified[5]) model to address the task of asset allocation among various subsets of the credit market from the viewpoint of a buy-and-hold investor. The goal is to find the optimal trade-off between the long-term payoff corresponding to current spread and the long-term risk of "unacceptably large" default losses, subjectively defined. We set out to answer the following types of questions faced by a buy-and-hold credit investor:

- How do we evaluate the tradeoff between current credit spreads and expected horizon defaults? When is credit "cheap" from a buy-and-hold perspective?

[3] A thorough discussion of multi-factor risk models is contained in *The Lehman Brothers Multi-Factor Risk Model*, Lehman Brothers, 1999. Our current model incorporates many improvements, including extension to multiple currencies and the addition of a model of default risk for high yield debt. The modules of the new model devoted to investment-grade and high yield credit are described in "The New Lehman Brothers Credit Risk Model", *Quantitative Credit Research Quarterly*, May 2002, and "The New Lehman Brothers High Yield Risk Model", *Quantitative Credit Research Quarterly*, November 2003.

[4] While the details of this application have not been made public, some of the underlying models and assumptions are discussed in D. O'Kane and L. Schloegl, "Tail Dependence and Portfolio Risk", *Quantitative Credit Research Quarterly*, September 2002.

[5] Later in the paper, we will investigate the loss of accuracy entailed in these model simplifications.

- How many issuers should a portfolio contain to project a certain confidence of outperforming Treasuries over the horizon?
- How do issuer correlations affect the answers to both of the above questions?
- What is the optimal allocation between single-A and Baa credits in a portfolio for a given loss tolerance level?

This paper does not provide definitive numerical answers to each of these questions, but rather outlines an approach to addressing them. The result of our analysis is not a single one-size-fits-all optimal allocation, but a methodology for achieving a customized solution given each investor's individual situation: the types of assets used and their spreads, views on expected default probabilities and correlations, and the precise formulation of the constraint on default risk.

The model underlying our analysis is a well-known firm-value model originally due to Vasicek[6]. In its simplest form, this model treats credit markets as a homogeneous set of issuers all characterized by the same set of parameters. Correlations among issuer returns are represented by imposing identical correlations between each issuer and a single central asset return variable. In our variation, different credit asset classes are viewed as homogeneous sub-populations. Within each group, all issuer firms are characterized by the same set of parameters. The parameter values change from one group to another, but the asset returns of issuers in all groups are driven by the single common central asset return variable. The default parameters and the spread assumptions are combined to form a return distribution for any allocation. This approach can help investors tailor their allocation within credit to their appetite for default risk.

While the approach is broadly applicable to the task of long-term asset allocation among credit asset classes, we motivate the discussion by considering a more specific problem often faced by insurance companies. A typical strategy is to fund a set of projected liabilities with a higher-yielding portfolio of corporate bonds. For example, an Aa-rated insurer that purchases a portfolio of Baa-rated debt might expect to earn the spread between typical Aa and Baa yields, minus a certain allowance for default losses. Assuming the risk of default losses in such a strategy is considered too great, we instead seek the blend of A and Baa debt that finds the optimum tradeoff between spread pickup and default risk.

We proceed as follows:

In Section 2, we present a very simple model of a buy-and-hold portfolio. We consider an equally weighted portfolio of *n* bonds, and present a simple approximation for the portfolio return as a function of the number of bonds defaulting over the period. Using this approximation, any distribution of the number of portfolio defaults can be transformed into a distribution of portfolio return. The simplest such distribution is the binomial distribution, which assumes that each issuer is equally likely to default and that what happens to one issuer is independent of what happens to any other issuer. The

[6] Vasicek, Oldrich, "Probability of Loss on Loan Portfolio", KMV corporation, February 1987.

default probability is assumed to be a constant, provided as an input parameter. Infinitely large portfolios will always realize this default rate exactly, and thus earn a constant return. For small portfolios, this model analyzes the random component of return due to uncertainty in the realized portfolio default rate. We show how this model can be used to relate the assumed default probability, the spread, and the number of bonds in the portfolio.

The major shortcoming of the binomial model is that it assumes a constant default rate. In reality, observed overall corporate bond default rates can vary significantly over time. This gives rise to correlations between the default probabilities of different issuers[7]. In Section 3, we present a model that includes this correlation effect. Because defaults are relatively rare events, it is difficult to work directly with default correlations. The model therefore begins by modeling the root cause of default based on the value of an issuer's assets relative to its liabilities, and then models correlations among the asset returns of the various issuers. As shown by Vasicek, assuming a constant asset return correlation among all pairs of issuers is equivalent to assuming correlations with a single market variable. The model turns out to be equivalent to using the binomial model, but with the default probability itself modeled as a random variable instead of being specified as a constant. We explore the distribution of the default probabilities in this model, and how it depends on the correlation assumption. We find that as the assumed correlation increases, the shape of this distribution becomes the main driver of portfolio performance, and the number of securities in the portfolio plays a smaller role. In the limit when the portfolio contains a large number of bonds (i.e., n is large), the realized portfolio default rate exactly follows the outcome of the random market default probability. The large homogeneous portfolio (LHP) approximation, based on this assumption, allows us to broadly characterize the risk and return of a credit asset class.

In Section 4, we extend this model to cover two (or more) distinct groups of credits, which could correspond to different quality ratings. Each group of issuers is homogeneous, and all issuers are linked to the same central asset return variable, but each group can have different spread, expected default probability and a different correlation. Under this set of assumptions, the LHP approximation gives us a very simple one-dimensional characterization of the return distribution of a portfolio defined as a weighted blend of these asset classes.

Given the ability to project the entire distribution of long-term returns for a given set of asset weights, we can offer several different approaches to finding the optimal allocation for a given set of risk tolerances. One can maximize expected return given a specific limit on some measure of tail risk. Tail risk can be measured by lower partial moments: shortfall probability, expected shortfall, or target semivariance. Alternatively, a utility function incorporating risk aversion can be used to evaluate a given distribution as a whole.

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[7] To understand the connection between time-varying default rates and default correlations, consider the effect of the overall health of the economy. In a recession, default rates will increase, and the default probabilities will tend to increase for all issuers. When the default probabilities for two issuers tend to rise and fall together, the two default events are correlated, and the probability of both defaulting within a given time period is higher than would be calculated under an assumption of independence.

We apply this model to the example problem of allocation between A and Baa debt, and show some numerical examples detailing the optimal allocation to Baa for different assumptions about spreads, default probabilities, correlations, and risk limits.

In Section 5, we take a critical look at some of the simplifying assumptions used in this analysis. The COMPASS system is used to illustrate the magnitude of the performance differences that might be expected under some more realistic assumptions about asset return distributions.

2. THE BINOMIAL MODEL – UNCORRELATED ANALYSIS OF SPREAD VS. DEFAULT RISK

Our analysis of the long-term risk and return of a corporate bond portfolio begins with the following simple interpretation of the buy-and-hold assumption. We choose a fixed time horizon – say 10 years – and model the possibility of default as a single-period problem: each bond either defaults during the next ten years or survives to maturity. Bonds that do not default are assumed to earn an annualized total return equal to their yield; bonds that default do not contribute anything to the cumulative performance beyond their recovery value. We ignore any coupon payments that might have been made before a bond defaults as well as any reinvestment, essentially assuming that all defaults occur immediately at the start of the period. This makes our analysis more conservative.

Using this model, we can compare the returns on portfolios of non-callable 10-year credits to those of 10-year Treasuries. Given the current 10-year Treasury yield, we can easily calculate the terminal value V_T of the portfolio for each dollar invested in Treasuries and the annualized return r_T by

$$(1)\quad V_T = (1 + y_T)^{10} = (1 + r_T)^{10}.$$

For riskless bonds held to maturity, the total return according to our assumptions is deterministic and equal to the yield.[8] For credit portfolios, we add an element of uncertainty – the realized portfolio default rate D.

If we let s denote the average portfolio spread over Treasuries, and R the assumed recovery rate on defaulted bonds[9], then our simple model for the terminal value and return of the credit portfolio over 10 years is

$$(2)\quad V_C = (1 - D)(1 + y_T + s)^{10} + D \cdot R = (1 + r_C)^{10}.$$

[8] We have chosen to ignore the effects of reinvestment and inflation for simplicity. These factors would affect both Treasury and credit portfolios.

[9] In reality, both the default rate and the recovery rate should be considered as random variables. For simplicity, we assume a constant recovery rate; we will deal with the uncertainty in recovery rates by investigating the effect of different recovery assumptions.

A comparison between equations (1) and (2) emphasizes the fundamental aspects of credit: the risk of default loss is offset by the additional return due to the spread. Figure 1 illustrates the breakeven point between these two effects. Assuming a Treasury yield of 4% and a recovery rate of 20%, we show the maximum realized default rate that will allow the credit portfolio to at least break even with Treasuries for a given level of spread. While the figure is based on a Treasury yield of 4%, the results will change only slightly with changes in Treasury yield. At this level, using simple annual compounding, a 10-year Treasury investment of $1 will have a terminal value of $1.48, while a credit investment with a spread of 200 bp will have a terminal value of $1.79 (if it does not default). Even assuming a very conservative 20% recovery rate[3] , equation (2) tells us that a realized portfolio default rate of 19.5% would make the return on the credit portfolio equal to the Treasury return. This breakeven default rate demonstrates just how much cushion can be generated by credit spreads – with a spread of 200 bp, we can experience 9 defaults in a 50-bond portfolio and still outperform Treasuries!

The key to understanding the risk/return tradeoff of credit investing is to model the likelihood of credit losses. The model given in (2) provides a simple translation of a realized portfolio default rate to a realized portfolio return. As we proceed through different approaches to modeling the distribution of default losses, we will continue to use this simple transformation to obtain corresponding distributions of portfolio return.

The first model we will consider for the portfolio default rate is the binomial model. We assume that the portfolio is an equally weighted blend of *n* bonds with equal weights, that each bond has the same known probability of default *p*, and that the outcomes for all bonds are independent. The probability distribution of the number of defaulted bonds, $n_{default}$, is given by

$$(3) \quad P(n_{default} = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

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[10] According to Moody's, while historical recovery rates for defaulted bonds span the range from 0% to 100%, the average historical recovery rate was 41%, with a standard deviation of 28%; the median recovery rate was 35%.

Figure 1. **Break-even Cumulative 10-year Portfolio Default Rates, 20% Recovery**

Corporate Spread (bp)	**Corporate Yield** (%)	**Corporate Terminal Value**	**Breakeven Realized Defaults**
100	5.00	1.63	10.4%
125	5.25	1.67	12.8%
150	5.50	1.71	15.1%
175	5.75	1.75	17.4%
200	6.00	1.79	19.5%
225	6.25	1.83	21.6%
250	6.50	1.88	23.7%
275	6.75	1.92	25.6%
300	7.00	1.97	27.6%
325	7.25	2.01	29.4%
350	7.50	2.06	31.2%
375	7.75	2.11	33.0%
400	8.00	2.16	34.6%
Treasury yield			4%
Treasury terminal value			1.48
Recovery Rate			20%

Figure 2. **Distribution of Number of Defaulted Bonds in a 20-Bond Portfolio, Binomial Model, 5% Cumulative Default Probability**

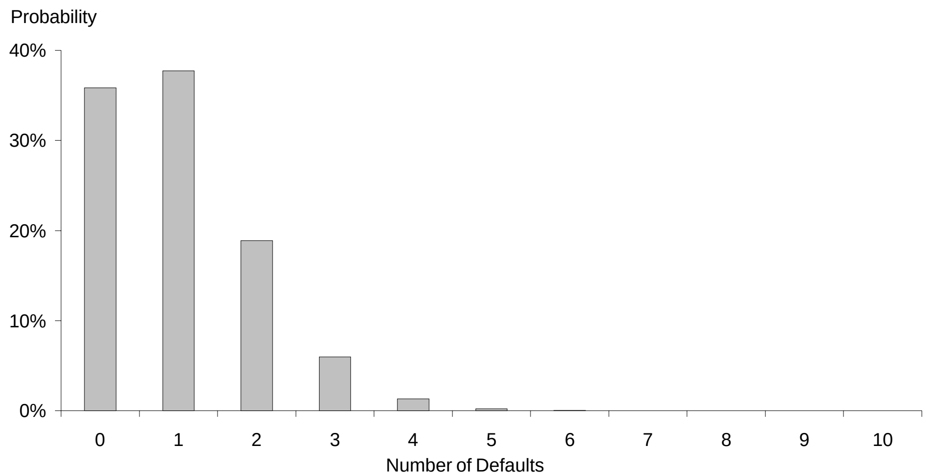
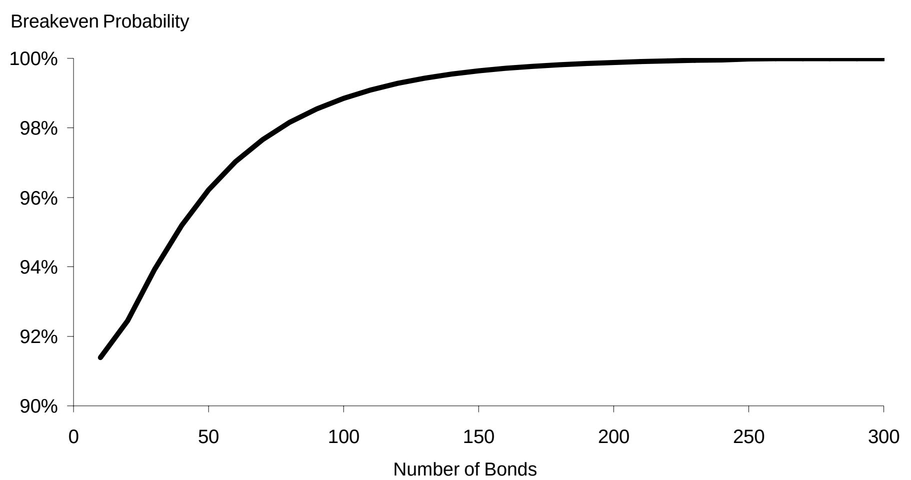


Figure 3. **Breakeven Probability as a Function of the Number of Bonds in the Portfolio, Binomial Model, 5% Default Probability, Spread 100 bp**



This distribution is illustrated in Figure 2 for a 20-bond portfolio with a probability of default 5%. This can be easily mapped into a return distribution by substituting $D = n_{default} / n$ in equation (2). For example, if we assume a spread of 150 bp and a recovery rate of 20%, Figure 1 shows us that as long as realized defaults are 15.1% or less, the portfolio will outperform Treasuries. The distribution in Figure 2 shows that for a 20-bond portfolio, where we can tolerate up to 3 defaults, the probability of outperformance is over 98%. If the spread is only 100 bp, then realized defaults must be under 10.4% over our 10-year horizon, so only 2 defaults out of 20 can be absorbed. Assuming that the distribution of Figure 2 still applies (i.e. the same 5% default probability is assumed despite the lower spread), the probability of break-even in this case is only 92.4%.

The binomial distribution is often used to examine the role of portfolio diversification in reducing default risk. As *n* grows, the tails of the distribution grow smaller, and the distribution tends to converge around its mean. For the same set of parameters used in Figure 2, we can vary the number of bonds in the portfolio, observe the new distribution, and recalculate the probability of outperformance. This dependence on the number of bonds is shown in Figure 3. We see that by increasing the number of bonds in the portfolio, we can achieve an arbitrarily high level of confidence that we will outperform Treasuries. For a 200-bond portfolio, for example, the probability of realized defaults over 10.4% is almost nil.

It is very important to exercise care in applying the binomial model in this way and interpreting the results. The model is based on the assumptions that the default probability for each issuer is a known constant, and that each issuer has an independent chance of defaulting over the horizon period. The result of this combination of assumptions, as we have seen, is that as the portfolio grows, the default losses over the horizon period converge to a known deterministic amount. This clearly does not correctly reflect the reality of owning a credit portfolio. In fact, we do not know what the next 10 years hold in store for the credit markets, and no amount of diversification can guarantee achieving a particular default rate.

In our interpretation, the default probability p that appears in equation (3) is the realized market-wide cumulative default rate. This is the proportion of bonds in the marketplace that will default over the next 10 years, or the cohort default rate. This quantity is not yet known, and must itself be treated as a random variable. The binomial model can then be used to draw conclusions about the portfolio default rate conditional on the cohort default rate. By using an appropriately pessimistic "worst-case" value for p, one can gain a high level of confidence that the portfolio is sufficiently protected from default risk. To establish these worst-case market default rates, we obtained 10-year cumulative default rates from Moody's.[11] Figure 4a shows such rates for two investment-grade rating categories, A and Baa, issued from 1970 through 1994 (so that the last observed 10-year time frame runs from January 1994 through December 2003). The highest 10-year default rates were observed in the period spanning the recession of the early 90's, with peaks of 9.51% (1982 cohort) for Baa and 4.67% (1985 cohort) for A. The long-term average cumulative default rates are relatively modest at 1.56% for single-A and 4.84% for Baa. The recent data points in this series have risen in response to the credit events of 2000-2002, but they are still well below the long-term average, thanks to the benefit of the placid mid-90's experience of these cohorts.

To get a better idea of the relative magnitude of the recent credit crisis, we looked at 3-year cumulative default rates as well. These are shown in Figure 4b. We find that the recent rise in default rates, for Baa bonds in particular, have recently approached their historical peaks, but remain below them. Nonetheless, the rapid rise in the 3-year rates leaves room to imagine that the "worst case" may exceed even the historical maximum default rate. In the next section, the effect of correlations is incorporated to help quantify the probability of such events.

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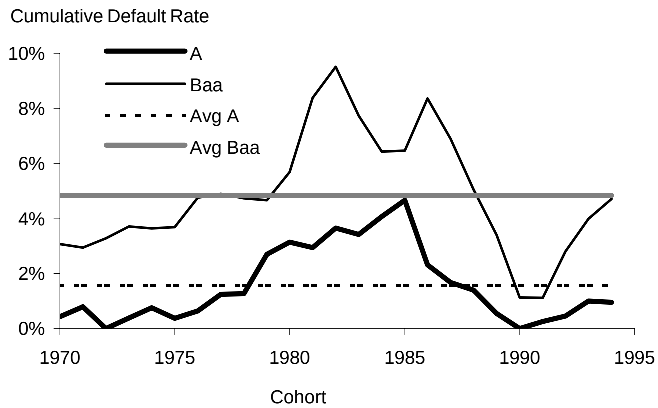
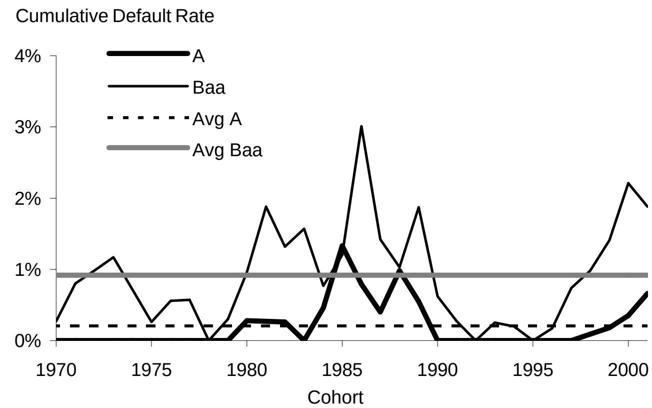
[11] "Default & Recovery Rates of Corporate Bond Issuers," Moody's Investors Service, January 2004.

Figure 4. **Moody's 10-Year and 3-Year Cumulative Default Rates by Annual Cohort**

4a. 10-Year cumulative default rates



4b. 3-Year cumulative default rates



3. INCORPORATING ISSUER CORRELATIONS

The big risk for buyers of a diversified corporate bond portfolio versus Treasuries is that difficult economic conditions could produce a wave of defaults throughout the sector[12]. We showed in the previous section that the binomial model can be used to bound the portfolio default rate D under a worst case assumption for the market default rate p. However, we did not offer a very rigorous process for setting this worst case assumption. The fact that Moody's historical data over the last thirty years show a maximum 10-year cumulative default rate of under 10% certainly does not guarantee that the next ten years will not be even worse. How can we estimate the likelihood of such an event? Clearly, if we are to consider the cohort default rate p as a random variable, we would like to have a model for its distribution. In this section, we present such a model, based on the correlated evolution of the asset values of issuing firms.

The model we will use is a default-mode CreditMetrics™ portfolio credit model discussed by our Quantitative Credit Research colleagues in an earlier publication[13]. An issuer is represented by the total value of its assets and liabilities. Liabilities are assumed to be constant, but asset values are subject to random fluctuations. If changes in the asset values ever bring the net worth below zero, the issuer goes into default. The key determinant of the likelihood of default is thus the relationship between the volatility of asset returns and the current net asset value of the firm.

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[12] Note that a systematic increase in default rates may not be a very big concern for the manager of a corporate bond portfolio benchmarked to a corporate bond index. Even abysmal absolute returns can be excused when the entire asset class suffers together. The risk of high overall defaults, and thus the risk of correlated defaults, is much more harmful to a portfolio measured against a benchmark that does not share the same level of exposure to default risk. Many buy-and-hold portfolios are benchmarked against a set of liabilities that must be assumed to be default-free, and so the risk of high overall defaults poses a very real threat.

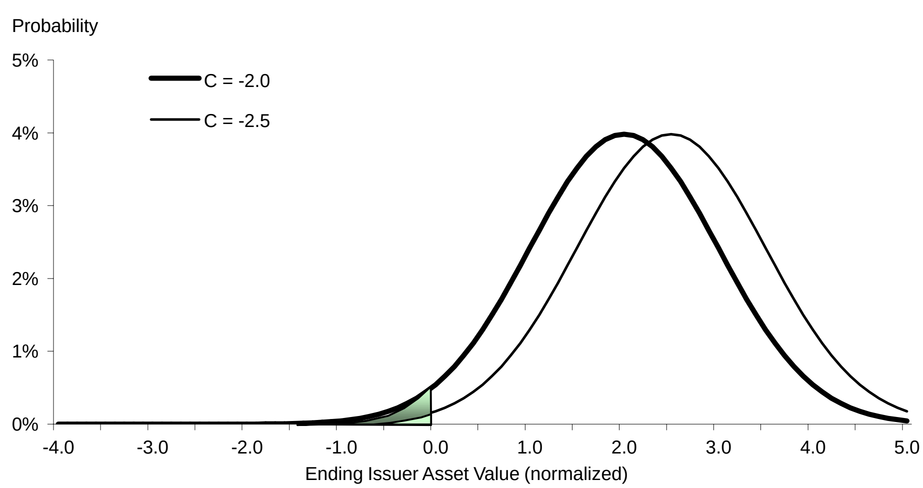
[13] For a detailed description of this model, see "Modelling Credit: Theory and Practice" by Dominic O'Kane and Lutz Schloegl, Lehman Brothers, February 2001, pages 35-37.

To keep things very simple, we will work with a one-period problem. That is, rather than look at the evolution of asset value over time, we just choose a horizon (say 10 years) and use a single random variable $A(i)$ to represent the cumulative asset return of issuer i over that period. We assume that this variable follows the standard normal distribution. $A(i)$ can be interpreted as a rescaling of the asset return in terms of the volatility. For example, say that an issuer has a current net worth of $20 billion, and assume that the change in issuer asset value over the next 10 years is normally distributed with mean zero and a standard deviation of $10 billion. In this case the event $A(i) = -1$ means that the issuer suffers a one-standard-deviation loss over the period, and ends with a net asset value of $10 billion. If, $A(i) < -2$ the net asset value will become negative, pushing the issuer into default. We define $C(i)$ as the return threshold which, if crossed, will result in default. In our example, $C(i) = -2$. The probability of default is then given very simply by the cumulative standard normal distribution, $N(.)$,

(4) $$p(i) = P(A(i) < C(i)) = N(C(i))\ .$$

Figure 5 gives a graphical depiction of this calculation, and shows the effect of changing the threshold from -2.0 to -2.5. For our issuer that begins with $20 billion in assets (2.0 times the standard deviation of return), the ending issuer asset value is assumed to follow a normal distribution centered on this mean. This distribution is shown normalized by the $10 billion standard deviation, so that an ending value of 4.0, for example, would represent the outcome in which the issuer's asset value grows to $40 billion over the 10-year horizon. A default is triggered if the asset value becomes negative, which happens when the normalized asset return is below -2. This occurs with probability 2.275%. If the issuer instead begins with $25 billion in assets, the whole distribution is shifted to the right, and a normalized return below $C(i) = -2.5$ is required to trigger a default. As seen in the figure, the shaded area under this curve is much smaller, and the default probability is reduced to 0.621%. Note that the input data describing the issuer's condition is represented by a single parameter $C(i)$, which is the negative of the number of standard deviations away from default over our selected time horizon.

Figure 5. **The Probability of Negative Asset Values, Which Trigger Default, Depend on the Return Threshold C**



In practice, when looking at asset classes such as sets of bonds with similar ratings, we do not really have a good way to determine the net asset value of a firm, or the volatility of its asset values. However, we can use historical rating agency data to estimate the default probabilities, and work backwards from there. For example, as shown in Figure 4, the long-term average cumulative 10-year default rates reported by Moody's for A and Baa issuers are approximately 2% and 5%, respectively. Using the inverse of the standard normal distribution, we can obtain the values of $C(i)$ that correspond to these default probabilities. We find that $C(i)$ is -2.054 for A-rated issuers and -1.645 for Baa-rated ones. That is, a typical A-rated issuer is more than 2 standard deviations away from default (over a 10-year horizon), while a Baa-rated issuer is substantially closer to a default condition.

What happens when we apply this model to a homogeneous portfolio of n bonds? We assume that all the bonds are from firms carrying the same quality rating, and therefore have the same expected default probability p, and the same implied threshold C. If we further assume that the outcomes of the asset return variables $A(i)$ for all of the issuers are independent, then the default processes are independent as well, and the distribution of the number of realized defaults is given by the binomial distribution of equation (3) just as in the previous section.

The usefulness of this model becomes apparent with the addition of issuer correlations. It is very difficult to work directly with default correlations[14], due to the fact that defaults from investment grade are rare events. In the firm value model, the correlations in the default processes are results of correlations in asset returns, which are more easily observable. The homogeneous portfolio model assumes that any two asset return variables $A(i)$ are correlated to each other with the same correlation coefficient. Vasicek showed that this set of n correlated variables can be decomposed into a model with $n+1$ independent variables, as follows:

$$(5)\quad A(i) = \beta\, Z + \sqrt{1-\beta^2}\, Z(i)$$

The asset return of each issuer is a weighted sum of two terms: one due to a systematic market return Z, and one due to an issuer-specific return variable $Z(i)$ which is assumed to be independent of both the market return and the issuer-specific returns of all other issuers. Both Z and all of the $Z(i)$ are assumed to follow the standard normal distribution.[15] The correlations among the overall asset return variables $A(i)$ are thus due entirely to the common exposure to the market return variable. It can be easily shown from equation (5) that each $A(i)$ has a correlation of β with the market return variable Z and a correlation of β^2 with the asset return $A(j)$ of any other issuer j.[16]

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[14] The default correlation between issuers A and B relates the individual issuer default probabilities p_A and p_B to the joint default probability p_{AB}, the probability that both issuers will default over the period. It is particularly difficult to estimate these joint default probabilities.

[15] The coefficients of the two terms have been set such that if Z and $Z(i)$ are independent standard normal variables (with a mean of zero and a standard deviation of one) then $A(i)$ is a standard normal variable as well, and it has a correlation of β to the market variable Z.

[16] Throughout the numerical examples in this paper, we will refer to the correlations among the issuer asset returns. Thus, when we discuss a correlation of 20% between any two issuers, the underlying assumption is that each issuer's asset return $A(i)$ has a correlation of $\beta = \sqrt{0.2} = 0.447$ with the market variable Z.

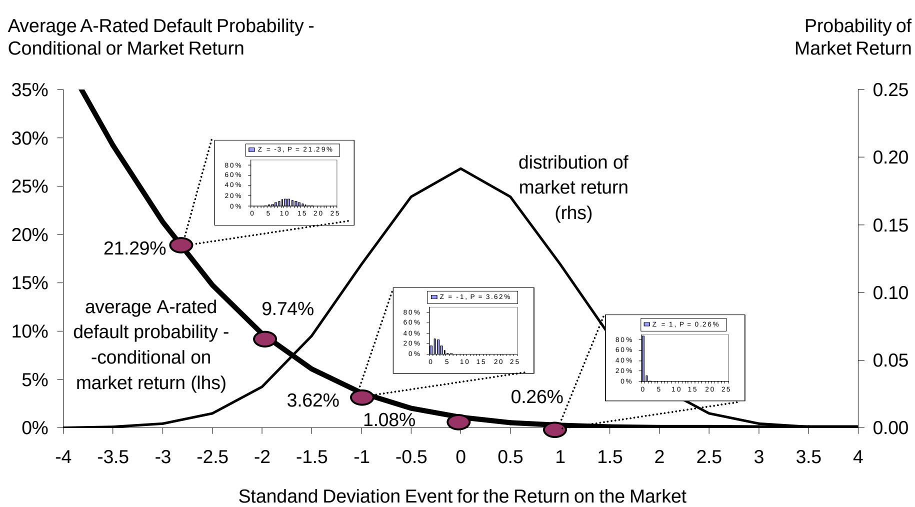
Returning to our homogeneous portfolio of n bonds, we find that this formulation, including issuer correlations, retains the form of the binomial distribution if we condition on the outcome of the market variable Z. That is, we analyze the probabilities of what might happen to the portfolio in two stages. In the first stage, we project the possible outcomes of the systematic variable Z. In the second stage, we consider the possibility of default for each issuer. The outcome of the market variable Z will determine the level of idiosyncratic asset return $Z(i)$ that will result in default. To find the conditional default probabilities, we re-express the default condition $A(i) \le C$ in terms of the idiosyncratic asset returns, to obtain

(6) $$p(i \mid Z) = P\big(A(i) \le C \mid Z\big) = P\left(Z(i) \le \frac{C - \beta Z}{\sqrt{1-\beta^2}} \right) = N\left(\frac{C - \beta Z}{\sqrt{1-\beta^2}} \right)$$

The quantity in the parentheses at the right of equation (6) is the value of $Z(i)$ that will trigger a default of issuer i, conditioned on the market return Z. Comparing this with equation (4), we see that the introduction of correlations can be viewed as adjusting the default threshold C in two ways. The main adjustment, in the numerator, reflects the effect of the market return. A negative market return will make the default threshold less negative, and increase the probability of default for all issuers. The second adjustment, in the denominator, scales up the magnitude of the default threshold based on the correlation. The greater the correlation with the market, the less the role of the idiosyncratic return in determining whether an issuer will default.

For example, assume that the correlation between any pair of assets is given by $\beta^2 = 20\%$ and the realization of the market return is $Z = -1$. A –1 standard deviation event in the market will bring down the net asset value of every firm by 0.447

Figure 6. **Building a Distribution of the Number of Portfolio Defaults in Two Steps: Any Realization of the Market Return Variable Z Gives an Average Default Probability P for the Market; the Number of Portfolio Defaults Follows a Binomial Distribution Conditional on This P.**



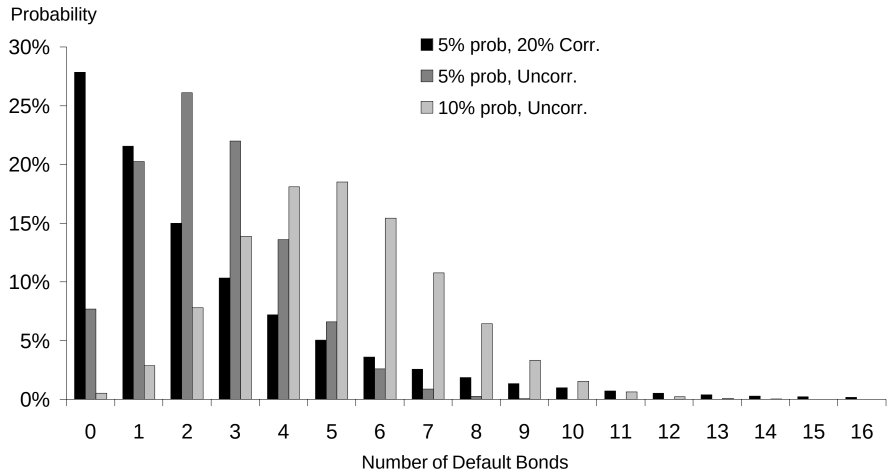
standard deviations. Clearly, the systematic depression of asset values will increase the default probability for every issuer, as there is now a smaller cushion to protect the firms from negative returns on $Z(i)$.

Conditioned on the market return, the number of portfolio defaults follows the binomial distribution with the probability of default given by equation (6). If the market return is very negative, and β is positive, then the probability of default is increased for all issuers simultaneously. If the market return is positive, then all issuers will have smaller default probabilities.

This two-step construction of the portfolio default distribution is illustrated in Figure 6 for a 50-bond portfolio of A-rated bonds with 20% correlation. For any possible realization of the market return Z, we compute the conditional default probability using equation (6). Although we start our analysis with the assumption that the ex ante 10-year cumulative default probability for A-rated debt is 2%, this can be decomposed into an average of very different default rates in different market conditions. A positive market return of Z=1 will result in a very low default probability of 0.26%, while negative market returns can result in much higher default rates: 3.62% if Z=-1, and 21.29% if Z=-3. In each of these cases, the number of defaults in a particular 50-bond portfolio will vary around this market-wide default rate *p*, and can be modeled using a binomial distribution parametrized by *p*. The binomial distributions for these three values of Z are shown in the figure, and illustrate how this distribution changes with Z. For Z=1, due to the low market default rate, the most likely portfolio outcome by far is zero defaults (87.87% probability); there is a much smaller likelihood (11.38%) of one default, and less then a 1% chance of two or more defaults. The dominance of the zero default outcome is characteristic of the entire right-hand side of this graph. As we move over to the left, we find that the distribution of portfolio defaults moves to the right and widens. In the unlikely event of Z=-3, the market default rate is just over 20%, and so the number of defaults in a 50-bond portfolio is centered around 10, with the bulk of the distribution falling between 5 and 15 defaults.

While we have drawn the conditional binomial distributions for 3 values of Z, there are actually an infinite number of them across a continuous distribution. Assuming a standard normal distribution for the market return Z, and integrating numerically over these conditional binomial distributions for all possible outcomes, we can obtain the unconditional distribution of the number of portfolio defaults. In Figure 7, we plot this distribution for a 50-bond Baa portfolio with an expected market default rate of 5% and a correlation assumption of 20%. We compare this distribution with those produced by the uncorrelated case (the plain binomial distribution) using market default rates of 5% and 10%. First let us compare the correlated and uncorrelated cases using the same 5% value for the expected default rate. In this case, for a 50-bond portfolio, the expected number of defaults is 2.5 for both the correlated and uncorrelated cases. The binomial distribution with no correlations has its peak near this value, and a relatively short tail. In the correlated case, the distribution shows a decreased probability of realizing the average default rate, and increased probabilities of either extremely high or extremely low defaults.

Figure 7. **Distribution of Number of Defaulted Bonds in a 50-bond Portfolio: 5% Expected Default Probability with 20% Correlation, Uncorrelated Model With Market Default Rates of 5% and 10%.**



If we increase the market default rate to 10% in the uncorrelated binomial distribution, the whole distribution shifts to the right, and the tail of the distribution includes high probabilities that 8, 9, or 10 bonds may default over the period. This comes much closer to the tail of the correlated distribution with a 5% expected default rate. However, even in this case, the probability of 11 or more bonds defaulting is higher in the correlated model.

Figure 7 illustrates two methods for identifying "worst case" Baa portfolio default rates. Historical data on 10-year cumulative defaults on Baa securities indicate a long-term average default rate of about 5%, with the worst observed cohort experiencing a default rate just under 10%. Using a simple binomial model, a worst case assumption for realized portfolio defaults can be obtained by using the tails of the binomial distribution with the highest observed default rate of 10%. In the correlated model, we use an expected default rate of 5%, and the fat tails of the distribution are generated by the 20% correlation assumption.

It is also very interesting to examine the unconditional distribution of the market default rate. As shown in Figure 6, the realization of the market return variable Z drives the market default rate over the next ten years according to equation (6). By integrating this function over all values of Z, we can obtain the unconditional (*ex ante*) distribution of the market default rate. This distribution is shown in Figure 8 for an expected default probability of 5% and a correlation of 20%. Note that while the mean of the distribution shown in Figure 8 is 5%, the distribution is very asymmetric. The bulk of the distribution lies below the mean, but there is a large positive tail showing small chances of much higher default rates – as high as 25%! The higher the assumed correlation, the greater the asymmetry, and the larger the probabilities of very high market default rates. In the limit as the number of bonds in the portfolio grows infinitely large, the realized portfolio default rate converges to the market default rate. Note that the correlation model considers the possibility of market default rates as high as 25%. According to this assumption, the overall probability of a market default rate worse than 10% is 13.6%.

When we used the binomial model with a "worst case" market default rate of 10%, the tail of the portfolio default distribution was entirely due to the portfolio underperforming the market due to poor security selection in a small portfolio. We see now that the reason for the increased tail probabilities shown in Figure 7 is that the correlation model considers the possibility of much higher market default rates as well. This is a systematic risk that cannot be diversified away.

In Figure 9, we compare the worst case realized portfolio default rates at 95% and 99% confidence levels using two different assumptions. The first is the worst case assumption that we used in the uncorrelated case, with the market default rate assumed to take on its worst observed historical value but with no correlations. The second assumes asset correlations of 20%, with the expected default rate set to the long-term historical average. The results are shown for portfolios of 20 and 50 bonds, using default probabilities characteristic of A and Baa ratings. We find that the two sets of assumptions give quite similar results, particularly at the 95% confidence level. The most striking difference is that the assumption of 20% correlation reduces the advantage of increasing the portfolio size from 20 to 50 bonds. For a portfolio of Baa bonds, the worst case realized portfolio default rate at a 99% confidence level improves from 30% to 20% as we go from 20 to 50 bonds under the uncorrelated assumption. With the 20% correlation assumption, even a 50-bond portfolio may have a 28% default rate.

Figure 10 summarizes the risk/return characteristics of a 50-bond Baa portfolio using the correlation model with expected default probability assumptions of 5%, 7.5% and 10%, and correlation assumptions of 20% and 30%. A recovery rate of 20% is assumed throughout. In addition to the mean and standard deviation of the distribution of outperformance, we look at various measures of the risk in the negative tail of the distribution. The probability of outperforming Treasuries is quite high under all parameter sets considered, but the key question is my just how much we might underperform in a crisis. We use two additional measures of tail risk, based on a specific level of confidence: the worst case outperformance, and the expected shortfall of outperformance, which is the average outperformance conditioned on being in the tail.

Figure 8. **The Probability of Negative Asset Values, Which Trigger Default, Depend on the Return Threshold C**

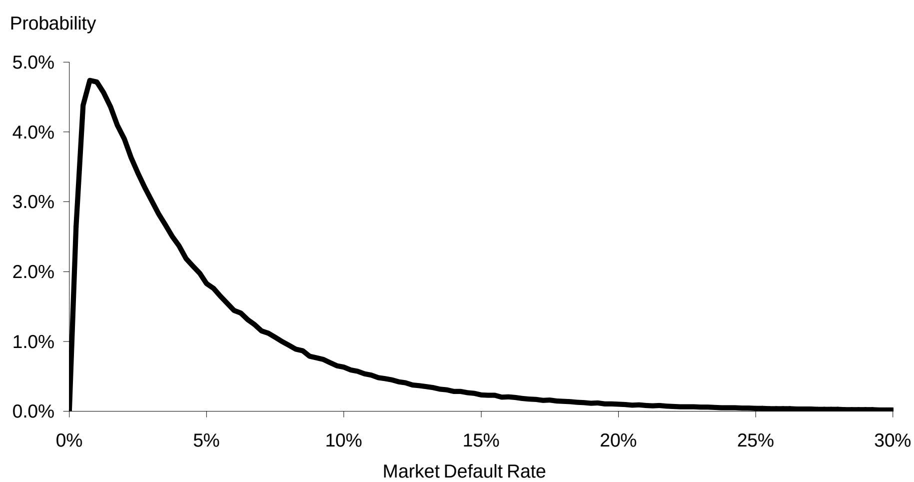


Figure 9. **Comparing Worst Case Realized Portfolio Default Rates Using Historical Average 10-year Default Rates With 20% Correlation and Worst Case Historical Default Rates With 0% Correlation**

		95% Confidence Number of bonds			99% Confidence Number of bonds		
A-rated portfolios	**Correlation**	**20**	**50**	**100**	**20**	**50**	**100**
Historic Worst Case: *p* = 5%	0%	15%	10%	9%	20%	**14%**	11%
Historic Mean: *E[p]* = 2	20%	10	8	8	20	16	**14**
		95% Confidence Number of bonds			**99% Confidence Number of bonds**		
BBB-rated portfolios	**Correlation**	**20**	**50**	**100**	**20**	**50**	**100**
Historic Worst Case: *p* = 10%	0%	20%	18%	15%	30%	**20%**	18%
Historic Mean: *E[p]* = 5%	20%	20	18	16	30	28	**26**

For example, under the assumption of 10% defaults with no correlations, there is a 95% probability of outperforming Treasuries by 0.17% or more. Over the 5% of cases in which outperformance is below this value, the expected outperformance is 0.02. Under the assumption of 5% expected default probability and 20% correlation, we obtain the same worst case underperformance of 0.17%, but an expected shortfall of -0.48%, reflecting a worse degradation of performance beyond this point. This can be seen as well in the worst case outperformance at the 99% level.

With different assumptions, the correlation model allows for even more extreme predictions of portfolio default rates, and shows that in the worst case corporates can underperform by a substantial amount. The most pessimistic assumptions shown combine a 10-year expected default probability of 10% with a firm value correlation of 30%. According to these assumptions, there is a 1% chance that more than half the portfolio will default. With our assumption of only 20% recovery, the resulting underperformance can be -4.37% per year or worse. Yet even according to these most pessimistic assumptions, there is compensation for taking these risks. The probability of outperformance over Treasuries is 83.2%, and the mean outperformance is 0.95% per year. Under more benevolent assumptions, the information ratio can be greater than 2. If we use the historical average default rate of 5% as the expected value of the market default rate (keeping in mind that this reflects the possibility of much higher cohort default rates, as shown in Figure 7), then even under an assumption of 30% correlation, the portfolio will outperform Treasuries with almost 95% confidence.

The correlation model used here takes advantage of the simplifying assumption that any two issuers are related by the same correlation coefficient. In reality, the correlations among different issuers reflect two types of factors: general macro-economic trends that affect all issuers, and industry-specific circumstances that can affect a particular sector of the market. A generally accepted market practice is to assume 30% correlation among issuers within the same industry, and 15% correlation between issuers from different industries. As the model uses just a single coefficient, 20% seems like a reasonable value. While our model cannot account for industry-specific correlations[17], these can be in large part avoided by diversification of industry exposures in the portfolio. If lack of

[17] The model can compute portfolio loss distributions assuming a different beta for each asset. However, this would complicate the analysis without necessarily changing any of the main results.

Figure 10. **Risk/Return Characteristics of a 50-bond Baa Portfolio Under Various Assumptions For Expected Default Rate and Correlation. Additional Assumptions: Corporate Yield 6.00%, Treasury Yield 4.00%, Recovery Rate 20%.**

Prob. of Default	**10%**	**5%**	**7.5%**	**10%**	**5%**	**7.5%**	**10%**
Correlation	**0%**	**20%**	**20%**	**20%**	**30%**	**30%**	**30%**
Mean Outperformance (%/yr)	1.01%	1.50%	1.24%	0.97%	1.49%	1.22%	0.95%
Stdev of Outperformance (%/yr)	0.44%	0.63%	0.85%	1.05%	0.81%	1.09%	1.35%
Information Ratio	2.31	2.38	1.46	0.93	1.84	1.12	0.70
Probability of Outperformance	97.5%	96.3%	91.4%	85.0%	94.4%	89.2%	83.2%
Worst Case Num. Defaults, 95% conf.	9	9	12	14	10	14	17
Worst Case Outperformance, 95% conf.	0.17%	0.17%	-0.51%	-0.99%	-0.05%	-0.99%	-1.74%
Expected shortfall of Outperf., 95% conf.	0.02%	-0.48%	-1.34%	-1.96%	-1.08%	-2.28%	-3.24%
Worst Case Num. Defaults, 99% conf.	10	14	18	21	17	22	26
Worst Case Outperformance, 99% conf.	-0.05%	-0.99%	-2.01%	-2.84%	-1.74%	-3.13%	-4.37%
Expected shortfall of Outperf., 99% conf.	-0.18%	-1.70%	-2.87%	-3.82%	-2.91%	-4.50%	-5.89%

liquidity in the market makes such diversification impossible, our break-even default rates would have to be adjusted upwards for industry correlations. Nevertheless, we believe it is feasible under most market conditions to construct a corporate portfolio of twenty or fifty names well-diversified across industries[18].

4. FINDING THE OPTIMAL ALLOCATION TO TWO CREDIT QUALITIES

For a population of homogeneous issuers, we have seen that our model can be used to generate a distribution for the number of defaults over a given time horizon – and hence, the outperformance over Treasuries – for a portfolio of n issuers. In addition, in the limit as n gets very large, the proportion of portfolio defaults converges to the conditional probability of default given in equation (6), and the overall distribution of the default rate is obtained by combining this with the standard normal distribution for Z, as illustrated in Figure 8.

This large homogenous portfolio (LHP) approximation provides a characterization of a particular group of credits as an asset class that is very well-suited to the task of asset allocation among the different parts of the credit market. From the point of view of the buy-and-hold investor, the distribution of returns over the holding period is the essential piece of information needed to evaluate risk and return, and to determine how much of a given asset class he should hold.

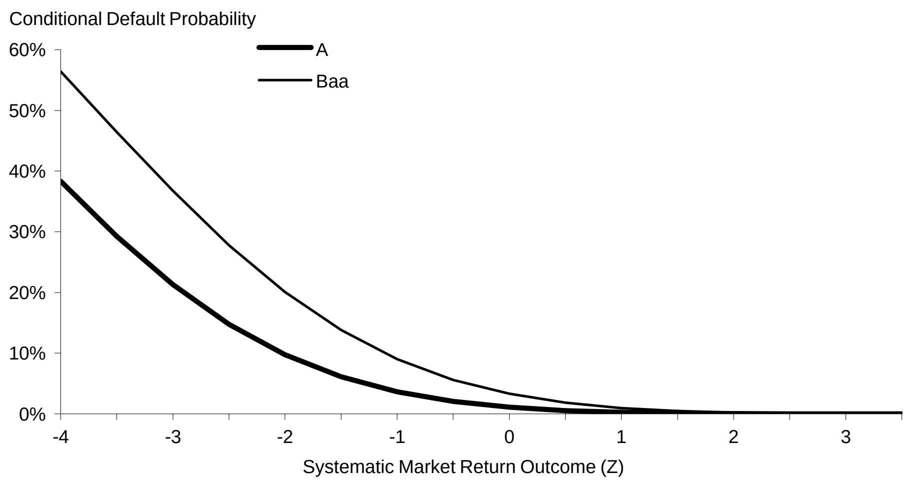
Consider an investor who plans to invest in a credit portfolio on a buy-and-hold basis over a ten-year horizon. He intends to reduce non-systematic risk to a minimum by diversifying his positions among many issuers within each market sector selected. He assumes, broadly speaking, that securities rated Aa or better will not be able to meet his yield targets, but he is restricted to use only investment-grade credits. If we can overlook the finer distinctions within the credit market (quality tiers, industry, etc.), the asset allocation decision essentially boils down to an allocation between A-rated and Baa-rated bonds. We show how the LHP model can be used in this allocation process.

[18] While testing our risk model for total return portfolios, we found that a proxy portfolio of 20 bonds could track the Corporate Bond Index with a projected tracking error of 44 bp/year, and that this number could be reduced to 29 bp/year in a 50-bond portfolio. For details, see *The Lehman Brothers Multi-factor Risk Model*, July 1999, p. 39.

In our two-quality version of the model, we assume that instead of a single homogeneous population of issuers, there are two distinct homogeneous groups. The two groups are tied together by sharing a common exposure to the same systematic market variable *Z*, which we can assume relates to the overall condition of the economy. All issuers within each group are assumed to have the same correlation with the market variable and the same default threshold. The default threshold *C* is set to two different values for the two groups, to reflect a higher probability of default for the lower-rated credits; the correlation assumptions for the two groups can be the same or different. For any outcome of *Z*, we can calculate the conditional default probabilities of the two qualities as:

$$(7) \qquad p_A(Z) = N\left(\frac{C_A - \beta_A Z}{\sqrt{1-\beta_A^2}}\right), \qquad p_{Baa}(Z) = N\left(\frac{C_{Baa} - \beta_{Baa} Z}{\sqrt{1-\beta_{Baa}^2}}\right)$$

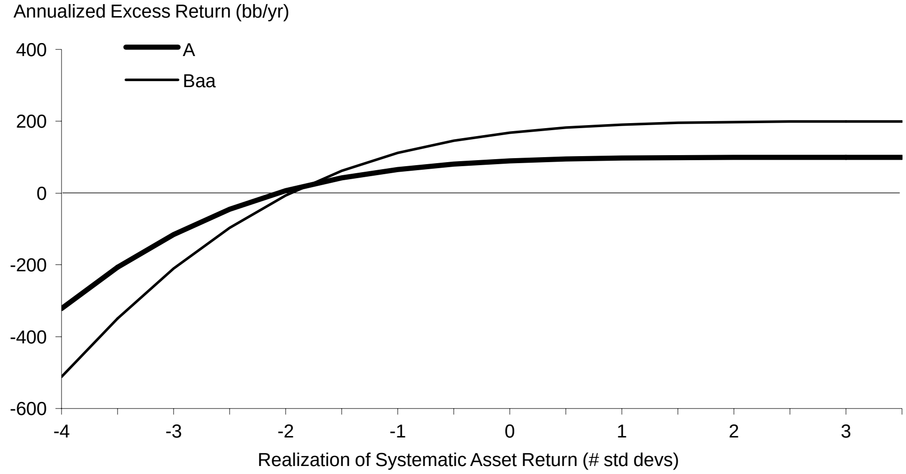
Figure 11. **Conditional Default Probabilities For Different Qualities, and Their Dependence on The Market Variable. Assumptions For A-Rated Bonds: (2% Expected Cumulative Default Probability, 20% Correlation); For Baa-Rated Bonds: (5%, 20%)**



Systematic Variable			A-rated Bonds		Baa-rated Bonds	
Z	p(Z)	Cumulative Probability	Threshold	Conditional Default Probability	Threshold	Conditional Default Probability
-4.00	0.008%	0.008%	-0.296	38.36%	0.161	56.40%
-3.50	0.049	0.057	-0.546	29.25	-0.089	46.45
-3.00	0.240	0.297	-0.796	21.30	-0.339	36.73
-2.50	0.924	1.221	-1.046	14.77	-0.589	27.79
-2.00	2.783	4.005	-1.296	9.75	-0.839	20.07
-1.50	6.559	10.564	-1.546	6.10	-1.089	13.81
-1.00	12.098	22.662	-1.796	3.62	-1.339	9.03
-0.50	17.467	40.128	-2.046	2.04	-1.589	5.60
0.00	19.741	59.870	-2.296	1.08	-1.839	3.30
0.50	17.467	77.336	-2.546	0.54	-2.089	1.84
1.00	12.098	89.434	-2.796	0.26	-2.339	0.97
1.50	6.559	95.993	-3.046	0.12	-2.589	0.48
2.00	2.783	98.776	-3.296	0.05	-2.839	0.23
2.50	0.924	99.701	-3.546	0.02	-3.089	0.10
3.00	0.240	99.941	-3.796	0.01	-3.339	0.04
3.50	0.049	99.990	-4.046	0.00	-3.589	0.02

Figure 11 shows these probabilities, as a function of *Z*, for parameters corresponding to A-rated and Baa-rated bonds. Based on the historical data shown in Figure 4, we have assumed expected 10-year cumulative default probabilities of 2% for A and 5% for Baa, with correlations of 20% for both. Figure 11 shows a very coarse discrete representation of the distribution of *Z*, and is used strictly for illustration. To calculate statistics of the various distributions, we use a much smaller step size. The "threshold" shown in Figure 11 is the quantity in parentheses in equation (7). For $Z = 0$, the threshold for a single-A bond is –2.296. That is, if there is no systematic market return, a typical A-rated issuer would not default unless its idiosyncratic asset return was 2.296 standard deviations below its mean. The probability of this event, from the standard normal cumulative distribution function, is just 1.08%. However, if there is a systematic downturn of one standard deviation in the market ($Z = -1.0$), the default threshold will be reduced in magnitude for all issuers, and now a move of only –1.796 standard deviations will result in default. This carries a probability of 3.62%. It is interesting to compare how bonds

Figure 12. **Annualized Excess Returns Realized By Bonds of Different Qualities, Conditioned on the Market Variable**



Systematic Variable		Cumulative	Bond Portfolio Performance (Annualized Excess Returns, bp) Conditioned on Different Outcomes of Systematic Variable Z		
Z	**p(Z)**	**Probability**	**A**	**Baa**	**50/50 Blend**
-4.00	0.008%	0.008%	-322	-512	-417
-3.50	0.049	0.057	-207	-349	-278
-3.00	0.240	0.297	-115	-210	-163
-2.50	0.924	1.221	-45	-96	-71
-2.00	2.783	4.005	7	-6	0
-1.50	6.559	10.564	42	62	52
-1.00	12.098	22.662	66	112	89
-0.50	17.467	40.128	81	146	114
0.00	19.741	59.870	90	169	129
0.50	17.467	77.336	95	183	139
1.00	12.098	89.434	98	191	144
1.50	6.559	95.993	99	195	147
2.00	2.783	98.776	100	198	149
2.50	0.924	99.701	100	199	149
3.00	0.240	99.941	100	200	150
3.50	0.049	99.990	100	200	150

of different credit qualities are affected. The same one-standard-deviation systematic event that drove the single-A default probability from 1.08% to 3.62% raises the Baa default probability from 3.30% to 9.03%. While this systematic event increases the default probabilities for both credit qualities by approximately a factor of three, clearly the absolute change in default losses is much more severe for Baa credits.

These distributions of conditional default rates can be transformed into conditional distributions of portfolio excess return via the approximation in equation (2). In addition to the default rate assumptions shown in Figure 11, we must provide assumptions for the spread over Treasuries and the recovery rate for each asset class. Figure 12 shows the conditional excess return distributions for A-rated credits, Baa-rated credits, and a 50/50 blend of the two. We see that in all three cases, a diversified credit portfolio approximately will underperform Treasuries when the systematic variable takes a value of about –2.0 or worse, which we expect to happen with a probability of about 4%. In these times of credit distress, Baa investments will underperform their single-A counterparts; at all other times, they will outperform. Yet it is not at all clear from Figure 12 how to choose which of these return distributions is better – this will depend on a particular investor's goals and risk appetite.

Once we have plotted out the complete return distributions as illustrated in Figure 12 (but of course, at a much finer resolution), we can calculate various types of summary statistics that could drive investment decisions. Figure 13 compares the return distributions obtained for various blends of A and Baa securities. In addition to the expected excess return over Treasuries, we report its standard deviation, as well as several measures of tail risk.

Two types of statistics are reported on tail risk for a given confidence level. The interpretation of the value at risk (VaR) is: you can be 95% confident that the result you will obtain will be the VaR or better. The interpretation of the expected shortfall (ES) is the average of all the possible outcomes that go beyond the VaR. (One could imagine two different distributions with the same VaR but with one having a much worse ES than the other.)

Figure 13. **Characteristics of Corporate Bond Portfolios Using Different Blends of A and Baa**

	A Weight	Baa Weight	Mean Ann ExcRet	Stdev Ann ExcRet	95% VaR	95% ES	99% VaR	99% ES	Breakeven Prob	IR
(All A)	100%	0%	81	26	33	-4	-25	-70	98.10%	3.15
	90	10	88	29	34	-6	-29	-76	98.01	3.08
	80	20	95	31	36	-7	-33	-83	97.93	3.02
	70	30	102	34	37	-9	-36	-89	97.86	2.97
	60	40	109	37	38	-11	-40	-96	97.79	2.93
	50	50	116	40	39	-13	-44	-103	97.73	2.89
	40	60	123	43	40	-15	-48	-109	97.68	2.86
	30	70	130	46	41	-17	-51	-116	97.63	2.83
	20	80	137	49	42	-19	-55	-123	97.58	2.80
	10	90	144	52	43	-21	-59	-129	97.54	2.78
(All Baa)	0	100	151	55	44	-23	-63	-136	97.50	2.76

In the example calculation shown, we use a Treasury yield of 4%, and spreads of 100 bp for A and 200 bp for Baa. The achieved excess return over the 10-year horizon for A is found to have an average of 81 bp/year[19] with a standard deviation of 26, but has a 1% probability of underperforming Treasuries (99% VaR) by -25 bp/year or more. For Baa, the mean outperformance is 151 with a standard deviation of 55, with a 99% VaR of -63 bp/year. Assuming that the portfolio is composed of a linear blend of the two qualities gives a linear blend of the excess return numbers for every value of Z, and thus gives a linear blend of the results for all of the performance measures shown (except IR).

One way to use this analysis to set the allocation is to seek to maximize the expected return subject to a specified risk limit. For example, using the data shown in Figure 13, an investor who requires 95% confidence that portfolio excess returns will be at least 40 bp/year would choose an allocation of 60% to Baa. As long as the maximum amount of risk that can be tolerated is known, then this method can be used to back out the blend of A and Baa bonds that will achieve that level of risk, whether specified by VaR or expected shortfall, at any confidence level.

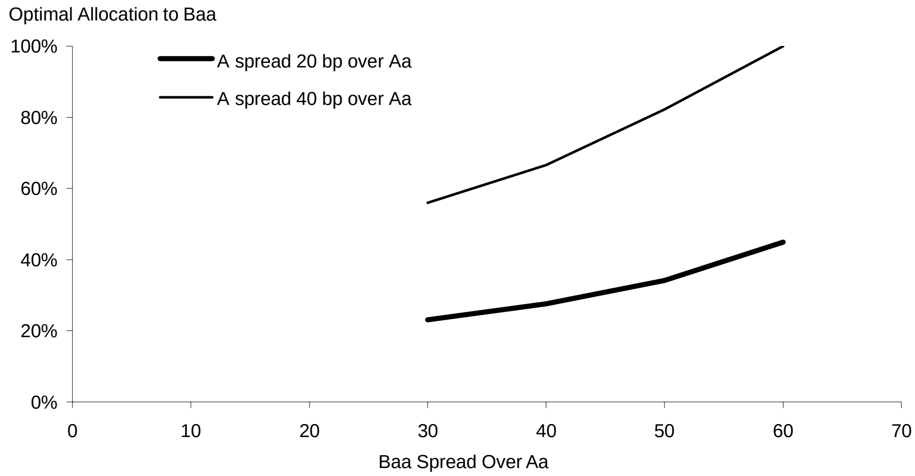
The analysis can be modified in various ways to fit various types of long-term investment objectives. At many institutions, for example, formal risk constraints are defined in terms of VaR limits on the dollar amounts of portfolio default losses, not in terms of excess returns as illustrated above. In this case, the risk limit will dictate a maximum exposure to Baa that will not change with fluctuating spreads. This can give rise to a two-tiered approach. First, an analysis of the portfolio loss distribution based on models of default rates, correlations and recovery rates is used to establish a ceiling on the Baa exposure that is acceptable under the loss constraint. However, it may not always be desirable to take on this maximum exposure to Baa. A second analysis may be carried out to establish the tactical allocation, including the effect of spread levels on outperformance. At this level, the distribution of excess returns is used to compute an optimal allocation to Baa that will tend to increase when spreads have widened enough to justify the additional risk. The limit on default losses places a fixed upper bound on how high this allocation is allowed to go.

[19] In the absence of any defaults, the portfolio would outperform by 100 bp/year. The expected default rate of 2% over the horizon reduces this amount. From equation (2), the horizon value per dollar of the corporate portfolio net default losses is $98\% \cdot 1.05^{10} + 2\% \cdot 0.20 = 1.60$, which gives an annualized return of 4.81%, outperforming the assumed 4.00%/year Treasury return by 81 bp/year.

Figure 14. **Risk and return of Blends of A and Baa assets vs. Aa liabilities**

	A Weight	Baa Weight	Mean Ann ExcRet	Stdev Ann ExcRet	95% VaR	95% ES	99% VaR	99% ES	Breakeven Prob	IR
(All A)	100%	0%	4	22	-37	-68	-86	-123	75.14%	0.18
	90	10	6	25	-40	-76	-96	-137	77.00	0.26
	80	20	9	28	-44	-83	-106	-151	78.20	0.31
	70	30	11	31	-48	-91	-117	-166	79.07	0.36
	60	40	14	34	-52	-99	-127	-180	79.79	0.39
	50	50	16	38	-56	-107	-137	-194	80.21	0.43
	40	60	18	41	-60	-115	-147	-208	80.76	0.45
	30	70	21	44	-64	-123	-158	-222	81.03	0.47
	20	80	23	47	-68	-130	-168	-237	81.43	0.49
	10	90	26	50	-72	-138	-178	-251	81.70	0.51
(All Baa)	0	100	28	54	-76	-146	-189	-265	81.83	0.52

Figure 15. **Optimal Allocation to Baa as a Function of Spread Differentials**



An Insurance Industry Example

Consider the following credit allocation problem, typical for an insurance company portfolio. Assume that the company has an Aa rating, and can fund at LIBOR. It collects premiums up front against an estimated liability stream. Essentially, over the long term, the portfolio invested against these liabilities needs to outperform a default-free investment at Aa rates; and it must ensure that the chance of underperforming is minimal.

In this case, we can modify the above analysis to view the credit portfolio versus the firm's liabilities, rather than versus Treasuries. We repeat the analysis shown in Figure 11 through Figure 13, but calculate the distribution of excess returns over Aa, rather than over Treasuries. Such an analysis is carried out in Figure 14 using the following assumptions: Treasury yield 4%; spreads of 60, 80, and 130 bp for Aa, A and Baa, respectively; single-A expected default probability 2% with 20% issuer correlation; Baa default probability 5% with correlation 25%; and 40% recovery rates throughout. We see that due to the small spread differential, A-rated securities offer an expected return advantage over the Aa liabilities of only 4 bp/year, with a standard deviation of 22 bp. The breakeven probability is only 75%. For Baa securities, the larger spread cushion more than makes up for the higher expected default rate, and thus we obtain a higher expected return as well as a greater probability of breakeven. However, the Baa distribution has more risk in the tails. If we seek to maximize the expected return subject to a maximum underperformance of 50 bp/year at 95% confidence, then we can allocate between 30% and 40% of the portfolio to Baa.

These results are very sensitive to the spread assumptions, and in particular the spread differentials from Aa to A and from A to Baa. In Figure 14 we assumed that A spreads were 20 bp over Aa, and that Baa spreads were 50 bp wider still; our loss constraint of 95% VaR = -50 bp led to a Baa allocation of 34%. Figure 15 shows how this optimal allocation would change as we vary these two spread differentials. Naturally, an increase in the spread advantage of Baa over A (without any adjustment of the expected default rates) increases the optimal allocation to Baa. When this advantage goes below a certain level (here shown to be about 30 bp) the expected return is higher for A, and there is no longer any incentive to take on Baa risk.

An even larger effect can be seen as we increase the spread differential between A and Aa. This increases the spread cushion on which the strategy rests, improving the mean excess return and the breakeven probabilities for both A and Baa assets. This allows us to take much more risk before challenging the VaR limit, and hence permits much larger contributions to Baa.

5. ISSUER LEVEL OPTIMIZATION

The macro-level analysis presented in this paper is based on many simplifying assumptions. These include:

- uniform correlations (*i.e.*, homogeneous assets);
- asset returns assumed to be distributed normally;
- constant recovery rate; and
- issues that default are assumed to do so immediately.

When selecting specific assets for a portfolio, however, many of these simplifying assumptions should be relaxed. For example, asset returns are typically modeled as having a "fat-tailed" distribution, implying a greater probability of large positive and negative returns compared to a Normal distribution. This also implies that the joint probability of default is underestimated when a Normal distribution is assumed (so-called "tail-dependence"). Also, issuer correlations are unlikely to be uniform. Issuers belonging to the same sector are more likely to have a higher default correlation with each other than with issuers belonging to different sectors. Recovery rates have been shown to be correlated with default rates, further skewing the overall loss distribution. A more realistic, non-uniform, correlation matrix might produce a more realistic joint probability distribution of default losses. Unfortunately, relaxing the assumptions of the "macro" model greatly increases the complexity of the model.

How might the risks of various A-Baa allocations change if we were to make more realistic assumptions regarding asset return distributions? We can investigate such questions, under a much broader range of assumptions, using COMPASS, Lehman's portfolio simulation and optimization tool for buy-and-hold credit investors[20]. An

Figure 16. **Comparison of Macro Model Results With COMPASS Results, In Terms of Overall Default Loss Statistics, Under Various Assumptions. (50/50 Blend of A and Baa Credits; 10-year Horizon; Default Rates 2% for A, 5% for Baa; Spreads 100 and 200 bp For A and Baa, Constant 20% Recovery)**

Modeling Assumptions	Expected Loss	VaR (99%)	Expected Shortfall (1%)
Macro model	2.9%	15.1%	19.3%
Gaussian; uniform correlation = 0.2	2.7	16.2	19.3
Student-t dist, (12 degrees of freedom); uniform correlation = 0.2	2.6	20.8	25.1
Gaussian; sector-based correlation matrix	2.7	13.0	14.8
Student-t dist (12 degrees of freedom); sector-based correlation matrix	2.7	17.4	20.6

investor can use COMPASS to construct an optimized credit portfolio that minimizes expected shortfall while satisfying various constraints, including an expected return target. It is instructive to compare the results from our "macro" allocation model presented above with results obtained using COMPASS, which works at the issuer-level. As we show below, optimal allocation results using COMPASS are similar to the results from the "macro" allocation model. This supports an approach for investors of first determining their macro allocation to the A and Baa sectors, and then using an optimizer such as COMPASS to select individual names for the portfolio.

COMPASS works as follows. Individual issuer default rates are mapped to historical default rates based on the issuer's credit rating. COMPASS then uses historical equity return correlations to estimate joint default correlations between issuers depending on their respective sectors. Using this information and assuming that asset return distributions are "fat-tailed," COMPASS generates a joint default probability distribution for all assets in a portfolio. Using this default distribution and applying a model of recovery rates, COMPASS can generate a portfolio's loss distribution. For a given level of expected return, COMPASS generates various possible portfolio loss distributions using as inputs various combinations of the available assets. COMPASS then finds the single portfolio with the smallest expected shortfall given the level of expected return.

In Figure 16, we use COMPASS to examine the risks of various "macro" allocations when the assets come from heterogeneous sectors. We first set up all of the asset characteristics to match as closely as possible those assumed in this paper. A-rated and Baa-rated assets are assumed to have a spread over Treasuries of 100 bp and 200 bp, respectively. In addition, a fixed recovery rate of 20% is assumed in both asset classes. For the first COMPASS run, we match all of the distributional assumptions of the macro model as well: a Normal distribution, uniform correlations of 20%, and fixed 20% recovery rates. We find an expected loss of 2.69%, expressed as the total cumulative losses due to defaults, net recoveries, as a percentage of starting value. The VaR of such losses at the 99% level is 16.17%, and the expected shortfall is 19.31%. Note that performance is measured here in a somewhat different way than throughout this paper. For the purpose of comparison, we backed out similar numbers from the results of our macro model shown in Figure 13. The results agree quite closely with the COMPASS results for Normal returns and uniform correlations; this is expected because the underlying models are the same.

We then investigate the effect of introducing some more complex assumptions. The introduction of a more fat-tailed asset return distribution (the Student-t distribution) is found to increase both VaR and ES by a substantial amount. However, the introduction of a non-uniform correlation matrix based on equity market sector correlations leads to a decrease in both VaR and ES in this example. (This is due to the combination of two factors: the 20% correlation assumption is high relative to many of the correlations found in the sector-based model, and the portfolio used in this example was well-diversified in its sector exposures.) It is interesting that when we include both of these effects (Student-t distribution and sector-based correlations), they tend to cancel each other out, leaving the VaR and ES only mildly higher than in the macro model. Overall, the fact that the results from COMPASS are similar to those from the "macro" model support the use of the "macro" model for A – Baa allocation, while using COMPASS for portfolio issue selection.

[20] See footnote 4 above.

Even within the confines of our macro framework, there is a lot of flexibililty for changing various assumptions. For example, the return approximation of equation (2) assumes that defaults occur immediately, ignoring the coupon income that such bonds generate up to the default event. This conservative assumption underestimates the return advantage offered by high spreads. This is desirable when the goal is to demonstrate that credit asset classes are attractive relative to Treasuries even under the most conservative assumptions. However, in the allocation between A and Baa credits, this assumption may provide somewhat of a bias against Baa. A modified return approximation that phases in defaults over time could certainly be investigated in future research.

Similarly, throughout this paper we have assumed nominal 10-year default rates of 2% for A and 5% for Baa, based on Moody's long term averages. Are these the best estimates of forward-looking default rates, regardless of current spread levels or the economic climate? Possibly not. Our goal has to present a framework for analyzing credit allocations at the macro level; the selected horizon, expected default rates, correlation levels and spreads are inputs that can be modified to suit an investor's views.

Furthermore, the desire to include the effect of tail dependence does not require a move to a full issuer-based optimization. It has been shown[21] that tail dependence can be incorporated into the large homogeneous portfolio (LHP) approximation without requiring simulation. Even without introducing this extra level of complexity, the tail risk of an asset class can be increased within our framework by increasing the correlation assumption.

Several notes are in order concerning the portfolio optimization criteria as well. It is widely accepted that the standard mean/variance optimization framework is not appropriate for asset classes with asymmetric return distributions, and that some adjustment needs to be made to reflect risk aversion. In the literature on downside risk[22] , the efficient frontier is redefined using alternative measures of risk, such as target shortfall (the mean of the distribution conditioned on it being below a specified minimum, or target, return) or target semivariance (the variance of this portion of the distribution, also called conditional variance, or CVAR). The expected shortfall measure used here is similar in nature to target shortfall, but is defined in terms of a confidence level rather than a fixed threshold.

An alternative approach to such optimization problems departs completely from the notion of the efficient frontier. Rather than to maximize the mean of the distribution subject to a risk constraint on some other property of the distribution, a utility function[23] is used to evaluate the value to the investor of achieving a certain return. This function is characterized by a risk aversion parameter, which ensures that the penalty for a very negative return is greater than the reward for a positive return. (The greater the risk aversion parameter, the greater the difference in magnitudes.) Optimization in

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[21] D. O'Kane and L. Schloegl, "An Analytical Portfolio Credit Model with Tail Dependence", *Lehman Brothers Quantitative Credit Research Quarterly*, January 2003.

[22] See, for example, W.V. Harlow, "Asset Allocation in a Downside-Risk Framework", *Financial Analysts Journal*, September-October 1991.

[23] J. Ingersoll, *Theory of Financial Decision Making*, Rowman and Littlefield Publishers Inc., 1987. The family of iso-elastic utility functions takes the basic form $U(V) = \frac{1}{1-\alpha}\left(V^{1-\alpha} - 1\right)$, where V is the terminal value of a $1 investment, and α is the risk aversion parameter.

this framework involves finding the distribution that gives the highest expected utility. We have chosen to use explicit limits on VaR or ES because we have found investors to be more comfortable stating their risk limits explicitly rather than in terms of a risk aversion parameter.

We have formulated the allocation problem under the LHP approximation. This assumes that each asset class will be represented in the portfolio by a large set of relatively small positions in different issuers. This may be appropriate in two situations: for investors who truly intend to maintain a low level of non-systematic risk, or for those who wish to isolate their allocation decisions from their security selection decisions. An alternative approach, which may better suit some more active investors, is to represent the return distribution for each asset class in a way that corresponds to the way they typically manage that asset class. That is, rather than use the default distribution for the asset class as a whole, one could use the (more skewed) default distribution for a portfolio of 20 Baa bonds, if that is anticipated as the typical structure of the Baa portion of the portfolio. Wise and Bhansali[24] take this approach in their analysis of the optimal allocation to corporate bonds, using a utility function approach. They show that the number of bonds assumed to be in the portfolio can strongly influence the optimal allocation, but that this effect decreases as the level of assumed correlation is increased.

6. CONCLUSION

Credit portfolio management, whether the viewpoint is buy-and-hold or total return, is much more complicated than any of the simple abstractions considered in this paper. First and foremost, there is no such thing as a homogeneous pool of issuers. Every issuer has its own unique financial structure and mix of businesses, with exposures to different potential risks – all of which can influence projected default probabilities and recovery rates. Correlations among the default risks of different firms can stem from shared exposures to certain industries, geographical regions, or political factors. Real portfolios cannot diversify among infinitely many issuers, nor will they have exactly the same weights in all securities. Issues related to liquidity or risk-based capital can force even an investor with a long horizon to sell positions in distressed securities.

Despite all these additional considerations, we believe there is value in the simplified models addressed in this paper. In determining their overall allocation to credit sectors, investors must come to grips with the overall level of credit risk they are able to tolerate on a macro level. This "big picture" evaluation can be carried out using the large homogeneous portfolio approximation, leaving many of the details of portfolio construction to a later stage of the process. More complex simulation-based modeling tools such as COMPASS can incorporate more rigorous assumptions about the joint asset distributions of different issuers when implementing a desired exposure with specific positions. Taken together, we believe that these models form a flexible set of tools that can help provide illuminating insights into many more variations of the buy-hold asset allocation problem.

[24] M. B. Wise and V. Bhansali, "Portfolio Allocation to Corporate Bonds with Correlated Defaults", *Journal of Risk*, Fall 2002.

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