Axioma

Flexible is better.™

Axioma Research Paper
No. 004

February, 2008

Constraint Attribution

Robert A. Stubbs, PhD
Dieter Vandenbussche

Understanding the impact of constraints imposed during portfolio construction and being able to quantify that impact to the asset owners is critical.

This research paper demonstrates how to measure the effects that individual constraints have on a portfolio, compared with a portfolio dictated exclusively by a trade-off of forecast risks and returns.

Constraint Attribution

Robert A. Stubbs Dieter Vandenbussche

February 27, 2008

1 Introduction

The use of constraints in quantitative portfolio management has become ubiquitous. Constraints are regularly imposed, both directly and indirectly, by asset owners, regulators, risk managers, trading desks, and, of course, the portfolio managers themselves. Constraints have become such an integral part of the investment management process, that portfolio optimizers have had to adapt their algorithms so as to be able to efficiently solve portfolio construction problems involving hundreds, or even thousands of linear, nonlinear and combinatorial constraints.

Since the imposition of constraints directly affects the portfolio construction process, portfolio managers have become interested in measuring how these constraints cause the optimal portfolio to deviate from a portfolio solely dictated by the trade-off of forecasted risk and return. This deviation has been given various descriptions, including “unrealized alpha”, “opportunity cost”, and “implementation inefficiency”.

An important and commonly used measure of this implementation inefficiency is the Transfer Coefficient (TC), see for example Clarke et al. (2002). However, the Transfer Coefficient only provides an aggregate measure of how all the constraints, considered together, affect the total implementation inefficiency. Clearly, it would be desirable to attribute this implementation inefficiency to constraints individually, rather than to the union of all constraints. With this type of information, we would be able to determine which constraints constitute the biggest obstacles to maximizing the value contained in the forecasted alphas. Quantitative information on specific constraints can provide compelling evidence to justify modifications in the portfolio construction strategy.

For the sake of simplicity, we will assume that our objective is to obtain a portfolio that optimally trades off expected return and risk, using the standard mean-variance utility measure. We will denote an optimal portfolio with respect to this objective and no constraints as the mean-variance (MV) portfolio. The managed portfolio, i.e. the portfolio that is obtained while imposing all constraints, will likely differ from the MV portfolio because of the effect of these various constraints. In this paper, we demonstrate how we can allocate this implementation inefficiency, or opportunity cost, to each individual constraint. There are at least three different methodologies that can be applied to perform this analysis:

- Decompose the backlog portfolio (i.e. the difference between the managed and MV portfolios, see Grinold (2005)) into components corresponding to individual constraints. This decomposition associates a portfolio with each constraint. Each such portfolio can be thought of as the change in holdings of the MV portfolio induced by that constraint.

- Decompose the difference between implied alphas and forecasted alphas into components corresponding to individual constraints. This shows how each constraint modifies the alphas to obtain the implied alphas.

- Decompose the difference between the expected return of the managed portfolio and the MV portfolio into components corresponding to the constraints.
- Decompose the difference between the realized return and risk of a managed strategy and a plain mean-variance strategy in a backtest simulation.

These decompositions are motivated by the work by Grinold (2005). In this paper, we describe a methodology to obtain portfolio decompositions for general portfolio rebalancing problems, allowing for all common types of constraints. Our analysis is based on basic mathematical optimization theory and provides the first consistent techniques for allocating opportunity cost to individual constraints in the portfolio construction process.

In Section 3, we illustrate the different decompositions using a small example. The objective of this section is only to familiarize the reader with the basic properties of each of the decompositions. Section 4 provides a more realistic example, and shows some examples of how the decompositions can provide valuable information to a portfolio manager. For an introduction to the theory behind the decompositions and some of the challenges in their development, see Section 5. Lastly, in Section 6, we describe how to use these decompositions on an ex-post basis, illustrating with a number of use-cases.

2 Previous work

In this section, we describe other research aimed at improving our understanding of how constraints impact the portfolio construction process. One of the main developments in this line of research is the *Transfer Coefficient*, first coined in Clarke et al. (2002), and generalized in Clarke et al. (2006). The transfer coefficient is a single measure that indicates how well the portfolio construction process translated alpha signals into holdings. When the optimal portfolio is obtained through a simple trade-off of expected return and risk, then this measure is equal to 1.0. However, the introduction of any constraints will make this coefficient less than 1.0, indicating that some of the signal is lost because of the constraints. Unfortunately, this measure does not indicate which constraints caused the biggest loss in transfer coefficient. To do this, one could begin to remove constraints and see how this impacts the transfer coefficient, see e.g. Clarke et al. (2004). However, this method is not only tedious when there are many classes of constraints, but the impact of one class of constraints may depend significantly on another, which can make this process ineffective since it depends on the order in which constraints are removed.

Like the work by Grinold (2005) and Scherer and Xu (2007), we propose the use of optimality conditions from the portfolio optimization problem to obtain a view of how individual constraints impact the portfolio construction process. Our main contribution in this arena is the development of a methodology to handle a wide range of constraint classes, include those that are not differentiable, nonlinear, or both.

3 Small Example

In this section, we consider a 4-asset example; the alphas, a benchmark, and a covariance matrix are provided in Table 1. Suppose the reference size is $r = \$100,000$. The objective trades off expected return and active variance. Using a risk aversion coefficient of $\frac{0.5}{r}$, we obtain the MV portfolio given in Table 2.

Now, we introduce some constraints. We require that no asset can be shorted more than 5% (Lower Bound), that holdings in IBM cannot exceed 50% (Upper Bound), and that total net holdings

Table 1: Inputs for small example

	Alpha (%)	Benchmark (%)	Covariance DELL	IBM	MSFT	ORCL
DELL	-5	30	0.02	0.0001	0.002	-0.0001
IBM	7	23	0.0001	0.04	0.0003	0.0009
MSFT	4	12	0.002	0.0003	0.05	0.0001
ORCL	-2	35	-0.0001	0.0009	0.0001	0.02

Table 2: MV and managed portfolio holdings (%): small example

	DELL	IBM	MSFT	ORCL
MV portfolio	-230.39	200.45	101.57	-74.73
Managed Portfolio	-5.00	50.00	60.00	-5.00

must equal the reference size (Budget). The optimal managed portfolio under these constraints is given in Table 2.

To find out exactly how each constraint affected holdings in individual assets, we turn to the holdings decomposition shown in Table 3. Each row in this table corresponds to a portfolio attributed to a constraint in the rebalancing problem. Note that the sum of the attributed portfolios yields the optimal portfolio.

This tells us that the portfolios attributed to the constraints can describe exactly how the constraints caused deviations from the MV portfolio. Obviously, the upper bound on IBM caused a significant reduction in IBM holdings as compared to the MV portfolio. Because the assets are correlated, we also see that the upper bound on IBM impacts holdings in the other assets (the row corresponding to the IBM upper bound has nonzero holdings in all the assets). Lastly, note that since the lower bounds on the holdings for IBM and MSFT were inactive, the portfolios attributed to those constraints are empty.

One important point is that a portfolio associated with a constraint does not describe the change in optimal managed holdings if one were to remove that constraint from the strategy. Instead, these holdings can be thought of as describing the sensitivity of the optimal managed holdings to small changes in the constraint bounds. Consequently, the holdings in a constraint portfolio should be interpreted relative to the holdings in other constraint portfolios.

Table 4 shows the expected returns decomposition. Each row corresponds to the expected return of the portfolio attributed to that term. Note that the sum of the attributed expected return values is the expected return of the optimal portfolio. The table indicates that the lower bound on the DELL holdings causes the biggest loss in expected return.

As with the holdings decomposition, it is important to observe that removing a constraint does not imply a gain in expected return equal to the amount attributed to that constraint. Instead, these numbers give one intuition as to where one can expect the largest gains in expected return by relaxing appropriate constraints. To demonstrate this, for each constraint we also report the optimal expected return if that constraint were removed from the strategy.

Table 5 contains the decomposition of the implied alphas. First observe that the rows corresponding to the alphas and constraints sum to the row of implied alphas. This decomposition

Table 3: Holdings Decomposition (%): small example

		DELL	IBM	MSFT	ORCL
	MV Port	-230.39	200.45	101.57	-74.73
	Budget	-80.54	-39.38	-29.70	-81.78
Lower Bound	DELL	304.66	-0.71	-12.19	1.62
	IBM	0.00	0.00	0.00	0.00
	MSFT	0.00	0.00	0.00	0.00
	ORCL	0.77	-3.26	-0.30	145.08
Upper Bound	IBM	0.50	-107.09	0.61	4.82
	Mngd Port	-5.00	50.00	60.00	-5.00

Table 4: Returns Decomposition: small example

		Expected Return (%)	Expected Return if Removed (%)
	MV Port	31.11	
	Budget	1.72	7.58
Lower Bound	DELL	-15.80	14.75
	IBM	0.00	6.25
	MSFT	0.00	6.25
	ORCL	-3.18	8.74
Upper Bound	IBM	-7.59	7.69
	Mngd Port	6.25	

provides information about how the constraints modify the alphas to obtain the implied alphas. For instance, observe that the upper bound on IBM holdings causes a decrease from the original alpha of IBM. In fact, when we compare ranking of the assets with respect to the original alphas and the implied alphas, we see that MSFT jumped ahead of IBM. A decomposition of the implied alphas can provide information as to why this occurred.

Note that a bound constraint on an asset only impacts the implied alpha of that asset. This stands in contrast to the portfolio decomposition, where the portfolio attributed to a single bound constraint can have holdings in all the assets.

Now that we have illustrated what each decomposition should look like, we turn to a more realistic example to demonstrate how one can use the decompositions.

4 Use-Case

Now consider a long-only rebalancing problem with 500 assets, a \$10 million budget, an objective containing an expected return and active variance term, a budget constraint, a constraint limiting each active position to at most $\pm 2\%$ of the reference size, and constraints on industry and style exposures, limiting them to a 2% deviation from the benchmark. The optimal solution to this rebalancing is a portfolio with a 2.62% tracking error.

Because this strategy contains many constraints, we group the decompositions by constraint

Table 5: Implied Alpha Decomposition (%): small example

		DELL	IBM	MSFT	ORCL
	MV Port	-5.00	7.00	4.00	-2.00
	Budget	-1.67	-1.67	-1.67	-1.67
Lower Bound	DELL	6.07	0.00	0.00	0.00
	IBM	0.00	0.00	0.00	0.00
	MSFT	0.00	0.00	0.00	0.00
	ORCL	0.00	0.00	0.00	2.90
Upper Bound	IBM	0.00	-4.28	0.00	0.00
	Mngd Port	-0.60	1.05	2.33	-0.77

Table 6: Expected Return Decomposition

	Expected Return (%)	Expected Return if removed from strategy (%)
MV Portfolio	596.32	
Long Only	-502.27	17.39
Budget	-18.99	4.85
Asset Bounds	-35.04	4.87
Industry	-18.89	4.81
Style	-16.57	5.49
Optimal Portfolio	4.57	

category instead. In particular, we aggregate the decompositions for the long-only constraints, the active asset position bounds, the industry constraints, and the style constraints.

4.1 Expected Return decomposition

The expected return decomposition for this strategy is shown in Table 6.

From this decomposition, we see that the long only constraints are the most significant cause of deviation from the MV portfolio expected return. Recall that this decomposition does *not* imply that removing the long only constraints will cause a \$ 50 million increase in return. Intuitively, however, these numbers indicate that removing the long only constraints would have the biggest impact on total return. This is borne out in the last column of the table, where each row shows the expected return obtained via the strategy if the given group of constraints is removed from the strategy.

4.2 Implied Alpha Decomposition

In this section, we illustrate the use of the implied alpha decomposition. We selected the 10 assets whose alpha was modified the most by the long-only constraint and relaxed the long only restriction for those assets to allow shorting of at most 0.1% of the budget. Similarly, we selected those assets whose alpha was modified most by the bound constraints on their active positions. We relaxed these constraints to a 3% bound, rather than 2%. After resolving with the new constraints, the resulting

Table 7: Sample Holdings Decomposition (%)

	Asset-1	Asset-2	Asset-3
MV Port	-360.40	453.40	-99.98
Long only	320.92	-343.84	-21.74
Budget	16.21	12.02	-6.92
Asset Bnds	8.70	-114.06	115.44
Industry	-8.56	-11.69	-74.94
Style	23.13	6.24	88.20
Holding	0.00	2.08	0.05

expected return is 4.88%, an increase of more than 30 bps. Note that this is a higher return than if all 2% limits on active positions were dropped entirely. In other words, by using the implied alpha decomposition to judiciously select some assets whose constraints to relax slightly, we obtain better results in terms of return. For the sake of comparison, we also selected 20 random assets and made analogous changes to their constraints. The return only increased to 4.60%, an increase of only 4 bps. This demonstrates the power of the information provided by the implied alpha decomposition.

4.3 Portfolio Decomposition

In this section, we demonstrate how to use the holdings decomposition to gain an understanding of the main drivers behind each asset's position.

In Table 7, we show the holdings decomposition for three different assets. Observe that the MV portfolio significantly shorts Asset-1, hence it is not surprising that the portfolio attributed to the long-only constraints has a significant long position in this asset, since the managed portfolio does not hold it. The MV portfolio holds significantly more of Asset-2 than the managed portfolio. As the decomposition shows, the reduction in holdings is caused mostly by the long-only constraints, which prevent the required leverage, and the asset bound constraints, which prevent an active bet over 2%. In fact, since the benchmark holding for this asset is 0.08%, we see that Asset-2 is holding as much as the asset bound constraint allows.

For Asset-3, the MV portfolio suggests a short position, yet the managed portfolio has a small long position in this asset. As the decomposition indicates, this is again mostly due to the asset bounds constraint. This is explained by the fact that the benchmark holding for Asset-3 is 2.05%, forcing the portfolio to hold 0.05% in order to satisfy the 2% active position limit.

5 Mathematical Details

Suppose the portfolio optimization problem we intend to solve has the form

$$\begin{aligned} &\max \alpha^T x - \frac{1}{2r}(x-b)^T Q(x-b) + \sum_{j\in\mathcal{O}} f_j(x) \\ &g_i(x) \le 0 \quad \forall i \in \mathcal{C}, \end{aligned} \tag{1}$$

where r is the reference size, b is the benchmark, α is a vector of expected returns, and Q is the asset-asset covariance matrix. One can think of the set $\mathcal{O}$ as indexing additional objective terms, such as transaction costs or market impact costs. Similarly, the set $\mathcal{C}$ indexes various constraints, such as budget, transaction cost, or factor exposure constraints. For simplicity, we begin by assuming that all the functions g_i and f_j are differentiable and convex.

If w^* is an optimal portfolio, then first order conditions for the portfolio optimization problem require that

$$\frac{1}{r}Q(w^* - b) = \alpha + \sum_{j\in\mathcal{O}} \nabla f_j(w^*) - \sum_{i\in\mathcal{C}} \lambda_i \nabla g_i(w^*),$$

where λ is vector of dual optimal multipliers (see Bazaraa et al. (1993) for further details on optimality conditions). This equation provides the implied alpha decomposition. The left hand side of this equation is the vector of implied alphas. Each term in the right hand side corresponds to a vector that can be attributed to an objective or constraint.

To obtain the portfolio decomposition, we pre-multiply the equation by rQ^{-1}, to obtain

$$w^* - b = rQ^{-1}\alpha + \sum_{j\in\mathcal{O}} rQ^{-1}\nabla f_j(w^*) - \sum_{i\in\mathcal{C}} \lambda_i rQ^{-1}\nabla g_i(w^*).$$

Now each term on the right hand side of this equation corresponds to a portfolio attributed to some constraint or objective. The term $rQ^{-1}\alpha$ is the active holdings of the MV portfolio. Note that these attributed portfolios sum up to the optimal active holdings.

In particular, we can associate a portfolio w^i with each constraint/objective, where $w^i = -\lambda_i rQ^{-1}\nabla g_i(w^*)\ \forall i \in \mathcal{C}$ and $w^j = rQ^{-1}\nabla f_j(w^*)\ \forall j \in \mathcal{O}$. Denote the MV portfolio as w_{MV}. Using this notation, the portfolio decomposition may be written as

$$w^* = w_{\mathrm{MV}} + \sum_{j\in\mathcal{O}} w^j + \sum_{i\in\mathcal{C}} w^i$$

Lastly, to obtain the returns decomposition, we pre-multiply each term in the last equation by α^T to get

$$\alpha^T w^* = \alpha^T w_{\mathrm{MV}} + \sum_{j\in\mathcal{O}} \alpha^T w^j + \sum_{i\in\mathcal{C}} \alpha^T w^i$$

Though we have illustrated the holdings decomposition by inverting the gradient of a variance objective term, it is easy to see that a holdings decomposition could similarly be obtained by inverting the gradient of a risk objective or constraint.

5.1 Utility decomposition

When the objective function of the portfolio optimization problem is the classic utility function trading off return and variance, Scherer and Xu (2007) showed that one can also decompose the difference in utility between the optimal and MV portfolios. Defining

$$U^{\mathrm{MV}} = \alpha^T w_{\mathrm{MV}} - \frac{1}{2r} w_{\mathrm{MV}}^T Q w_{\mathrm{MV}}$$
$$U^* = \alpha^T w^* - \frac{1}{2r}(w^*)^T Q w^*,$$

they showed that

$$U^{\mathrm{MV}} - U^* = \frac{r}{2}\left(\sum_{i\in\mathcal{C}} \lambda_i \nabla g_i(w^*) Q^{-1} \left(\sum_{j\in\mathcal{C}} \lambda_j \nabla g_j(w^*)\right)\right) = \frac{1}{2}\sum_{i\in\mathcal{C}} \lambda_i \nabla g_i(w^*)^T \left(w_{\mathrm{MV}} - w^*\right).$$

Table 8: Utility Decomposition: small example

		% of difference in utility
	Budget	-8.41
Lower Bound	DELL	66.99
	IBM	0.00
	MSFT	0.00
	ORCL	9.90
Upper Bound	IBM	31.52

Each term in this last sum now describes the contribution of a constraint to the loss in utility caused by adding constraints. We illustrate this decomposition in Table 8 with the example from Section 3. The decomposition tells us that almost 67% of the loss in utility can be attributed to the lower bound on the holdings of DELL. Much like the expected returns decomposition, this does not mean one would gain that much in utility be dropping that constraint. As before, these numbers only indicate the sensitivity of the optimal solution to changes in that constraint.

5.2 Uniqueness of decomposition

In practice, it is possible that a problem has different dual optimal solutions, which implies there are different valid decompositions. Most such cases arise from having equivalent constraints in the problem formulation. For instance, the impact of two equivalent constraints can be shifted between the two without changing the total impact of the constraints. In practice, constraint attribution users will group their constraints into different categories and most often, these equivalent constraints will appear in the same category, negating the effect of having multiple valid decompositions.

5.3 Handling non-differentiable functions

So far, we have assumed that all the objective and constraint functions are differentiable in the holding variables. However, note that even a constraint as simple as an upper bound on the total short position is not differentiable (with respect to the holdings). As long as these functions are convex, one can use the analogue of gradients called *subgradients*. The resulting decomposition will satisfy the same properties as in the differentiable case.

We illustrate handling nondifferentiabilities using a simple example. Consider the following portfolio rebalancing optimization problem:

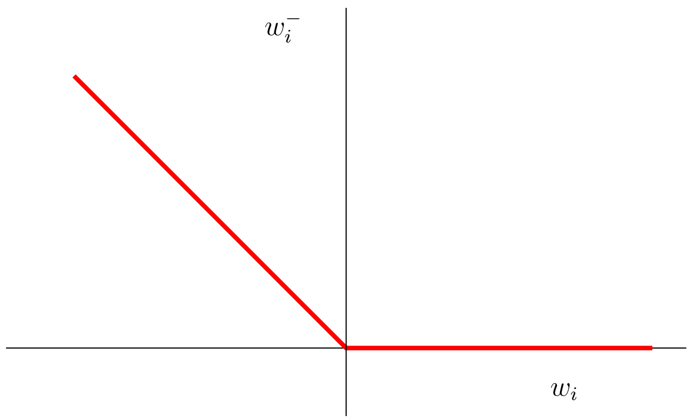
$$\max \alpha^T w - \frac{1}{2r} w^T Q w$$
$$c^T |w - h| \leq b,$$

where h is the vector of initial holdings and $|w - h|$ is applied component-wise.

Terms of the form $c^T|w-h|$ are commonly used to represent transaction costs. Because of the absolute value, this term does not have a partial derivative with respect to w_i if $w_i = h_i$. However, convex optimization theory tells us that first-order optimality conditions are still available by replacing the gradient of $c^T|w-h|$ with an appropriate subgradient. The first order conditions are

$$\alpha - \frac{1}{r} Q w^* - s = 0$$

Figure 1: Plot of w_i^-



where the subgradient, s, is of the form

$$s_i \in \begin{cases} \{-c_i\lambda\} & \text{if } w_i^* < h_i \\ [-c_i\lambda, c_i\lambda] & \text{if } w_i^* = h_i \\ \{c_i\lambda\} & \text{if } w_i^* > h_i, \end{cases}$$

and λ is the optimal dual multiplier for the constraint (see Hiriart-Urruty and Lemaréchal (1993) for details on optimality conditions for nondifferentiable optimization problems).

Note that if $w_i^* = h_i$, then a subgradient must be selected from an interval. Selecting the appropriate value from this interval to satisfy the first order conditions becomes a nontrivial problem when there are multiple constraints and objectives that are not differentiable. Our approach to handle this difficulty is to solve a secondary optimization problem that correctly selects these subgradient values in order to obtain a valid decomposition. This technique is guaranteed to work as long as all the constraints and objective terms are convex, regardless of their differentiability.

5.3.1 Nondifferentiable example

Consider the small example presented in Section 3 and suppose we remove all bound constraints and instead impose a 10% limit on short holdings, written as

$$\sum_i w_i^- \leq 0.1r,$$

where $w_i^- = -\min(0, w_i)$, which is plotted in Figure 1.

This function has a kink at 0, where it is not differentiable. The subdifferential of this function is the interval $[-1, 0]$, and any subgradient must lie in the interval, $[-\lambda, 0]$, where λ is the optimal dual multiplier for the constraint. We now illustrate how this applies to the implied alpha decomposition of this small example, which is provided in Table 9.

The value of the dual variable of the MaxShort constraint is 0.0826. The derivative of the MaxShort constraint with respect to holdings in DELL is -1, since DELL is held short. This explains the value of 8.26% in the implied alpha decomposition for DELL from the MaxShort constraint. On the other hand, both IBM and MSFT are held long, hence the derivative of the MaxShort constraint with respect to these assets is 0, explaining their values for the MaxShort

Table 9: Implied Alpha decomposition (in %) with Max Shorting constraint

	DELL	IBM	MSFT	ORCL
Holdings	-10	97.72	12.29	0
Alpha	-5	7	4	-2
Budget	-4.05	-4.05	-4.05	-4.05
MaxShort	8.26	0	0	5.42
Implied Alpha	-0.79	2.95	-0.05	-0.63

constraint. ORCL is not held, which means the MaxShort constraint is not differentiable with respect to ORCL. The auxiliary optimization problem selects a valid subgradient value to complete the implied alpha decomposition.

5.4 Nonconvex constraints

Portfolio optimization problems often contain constraints that cannot be formulated using convex terms. Common examples include:

- Limiting number of names or trades
- Trade or holding thresholds
- Relative Marginal Contribution to risk limits
- Long-Short ratio constraints
- Lower bounds on long or short holdings

Optimization theory does not support analogous optimality conditions as in the convex case. It is often possible, however, to introduce convex constraints that approximate these nonconvex constraints. These constraints may then be used as surrogates for the original nonconvex constraints to produce an approximate decomposition. Unfortunately, these surrogate constraints often serve to approximate several different nonconvex constraints simultaneously, in which case it becomes to difficult to separate the impact of the different constraints. Regardless, in many cases, we can produce a meaningful decomposition, as long as appropriate constraints are grouped together.

We illustrate with an example covering 10 assets with given expected returns and volatilities. The objective is to maximize expected return minus active variance scaled by a risk tolerance. We have a budget constraint requiring full investment and prohibit shorting. Lastly, a names limit constraint requires that the portfolio holds at most 4 assets. A holding decomposition is shown in Table 10. The decomposition clearly indicates how the names constraint reduces holdings in some assets, and increases holdings in others as it adjusts for the MV portfolio which seeks to hold all assets fairly equally.

6 Ex-Post Constraint Attribution

Thus far, we have considered what happens on an ex-ante basis. However, constraints may have a realized effect different from the predicted. For example, a constraint may reduce expected return

Table 10: Holdings decomposition (in %) with names constraint

Asset	1	2	3	4	5	6	7	8	9	10
MV	9.27	14.99	15.1	10.23	5.13	1.16	9.08	7.98	11.96	15.11
Budget	3.14	2.98	4.45	2.22	2.56	2.26	3.55	2.31	6.1	3.09
Long Only	0	0	0	0	0	1.2	0	0	0	0
Names Limit	-12.41	4.7	7.02	-12.44	-7.69	-4.63	-12.62	-10.29	9.63	4.88
Holdings	0	22.67	26.56	0	0	0	0	0	27.68	23.08

but actually increase realized returns or information ratios. This can occur, for example, if certain biases are present in the expected returns that actually contain no information. In this section, we will look at the effect of each constraint on realized return and realized risk.

In Section 4.3, we looked at a holdings decomposition of a portfolio. In Section 4.1, we considered the effect of each constraint on expected return by taking the inner product of the alphas with each decomposed portfolio component. When we look at returns and risk on an ex-post basis, we follow the same procedure except we take the inner-product of realized returns with the decomposed portfolio components to create a realized return attributable to each component.

In the following subsections, we describe several use cases that illustrate ex-post constraint attribution and how it can be used to not only understand the effect of constraints, but also how to modify strategies to realize further gains. Each of the cases compare quantitative strategies using a backtest where the portfolios are rebalanced monthly from 1995–2007.

6.1 Risk and Return Attribution Methodology

After solving a rebalancing in each period of the backtest, we construct portfolios attributed to different collections of constraints. In particular, let P_{mt} be the portfolio attributed to the m-th collection of constraints and/or objective terms in period t. Let R_{At} be the active return of the managed portfolio and R_{mt} be the return attributed to component m in period t. If r_t is the vector of realized asset returns for period t, then R_{mt} is computed as $r_t^T P_{mt}$. The active return for each time period is equal to the sum of the component returns for that period, i.e.,

$$R_{At} = \sum_m R_{mt}. \tag{2}$$

As with standard performance attribution, we are interested in cumulative contributions of each component. However, the cumulative component returns do not sum to the cumulative active return. We can use standard linking techniques from the performance attribution literature to define the cumulative return of each component m as

$$R_m = \sum_t \beta_t R_{mt}, \tag{3}$$

where the linking coefficient β_t is determined by the particular linking method used. In this paper, we follow the standard approach from Cariño (1999) to define the β's. Linking ensures that the cumulative active return is equal to the sum of the component returns:

$$R_A = \sum_m R_m. \tag{4}$$

Finally, we annualize R_A and scale the component returns R_m appropriately to maintain the equality above, as in Menchero (2006/2007).

Now, let us consider the computation of active risk and its decomposition amongst the component portfolios. The realized active risk over the backtest horizon is computed as

$$\sigma(R_A) = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(R_{At} - \bar{R}_A\right)^2}, \tag{5}$$

where T is the number of periods and $\bar{R}_A$ is the average active return across all periods.

We decompose the ex-post risk by attributing a portion to each constraint or objective term component. We decompose risk according to

$$\sigma(R_A) = \sum_m \sigma(R_m)\rho(R_m, R_A), \tag{6}$$

where $\sigma(R_m)$ is the realized volatility of component R_m and $\rho(R_m, R_A)$ is the realized correlation between component R_m and the active portfolio. This risk attribution methodology was introduced by Menchero and Hu (2006) for classical performance attribution methods that decompose the portfolios based on sectors, styles, etc. We report all risk values on an annualized basis. To convert the standard deviations to annualized numbers, we simply multiply by the square-root of the number of periods per year. For example, if our time-series of returns are monthly, then we would multiply the σ values by $\sqrt{12}$.

6.2 Use Case: Long-Only versus 130/30

Consider a long-only enhanced index fund that has a mandate to track the Russell 1000 within 3%. The quantitatively driven strategy is to maximize expected return subject to being fully invested (Budget), 3% annualized tracking error, limits on active industry bets (Industry Bounds), limits on active asset bets (Asset Bounds), and a limit on turnover in each rebalancing.

The ex-post constraint attribution results of the backtest are provided in Table 11. The first two rows, labeled "Portfolio" and "Benchmark", give the annualized return of managed portfolio and benchmark, respectively. The third row, labeled "Active", gives the annualized active return and standard deviation. Note that the annualized active return is computed as the difference between the annualized return of the managed portfolio and the annualized return of the benchmark. The risk of the active portfolio is computed by first computing the standard deviation of the active returns over all periods and annualizing by multiplying by $\sqrt{12}$.

The remaining rows of the table decompose the annualized active return and annualized active risk amongst portfolios attributed to each class of constraints and objective term as described in Section 6.1. The row labeled "Other" is associated with the portfolio attributed to a set of constraints used in the backtest to ensure that only assets in the benchmark are considered. These constraints are used to force liquidation of any assets that were in the account but are no longer in the benchmark.

The "mean-variance" portfolio (MV) contributed 12.35% to the active return. However, some of the constraints, particularly the long-only constraint, had a negative impact on return. The long-only constraint reduced the active return by 7.28%. Additionally, its contribution to active risk was 1.25%. These two together indicate that the long-only constraint had a very significant negative impact on the information ratio.

Table 11: Long Only - 3% Tracking Error - Russell 1000

	Annualized Return Attribution	Risk Attribution
Portfolio	14.46%	
Benchmark	11.35%	
Active	3.11%	3.22%
MV	12.35%	2.79%
Long Only	-7.28%	1.25%
Industry Bounds	-0.57%	-0.22%
Budget	-1.69%	-0.06%
Max Turnover	-0.16%	-0.01%
Asset Bounds	0.25%	-0.54%
Other	0.21%	0.01%

Table 12: 130/30 - 3% Tracking Error - Russell 1000

	Annualized Return Attribution	Risk Attribution
Portfolio	15.21%	
Benchmark	11.35%	
Active	3.86%	2.72%
MV	7.05%	3.03%
Max Long	-2.53%	-0.12%
Industry Bounds	-0.15%	-0.08%
Budget	0.76%	-0.02%
Max Turnover	-1.54%	0.08%
Asset Bounds	0.17%	-0.14%
Other	0.08%	-0.02%

This negative impact of the long-only constraint suggests relaxing the constraint. Let us now consider a modification of the strategy where we remove the long-only constraint and add a constraint limiting the long value to be at most 130% of the portfolio value to produce a so-called "130/30" strategy.

The backtesting results of this strategy are provided in Table 12. The annualized active return of this strategy was 3.86% and the active risk was 2.72%. Both the return and risk of this strategy compare favorably to that of the long-only strategy to produce an increase in information ratio from 0.97 to 1.42. The max-long and max turnover constraints both have significant negative contributions to return, suggesting possible relaxations of these constraints as well.

6.3 Use Case: Alpha-Eating Constraint

Now, let us consider the same 130/30 strategy as described in Section 6.2 except we add a constraint on the beta of the portfolio saying that it must be within ± 0.01 of the beta of the benchmark. In this test, we used a different methodology to compute the expected returns, ensuring that they have a market timing bias.

The backtest results of this strategy are summarized in Table 13. Note that both the beta, max-

Table 13: 130/30 - Beta Constrained - Russell 1000

		Annualized Return Attribution	Risk Attribution
Portfolio		14.03%	
Benchmark		11.35%	
Active		2.68%	2.74%
	MV	6.23%	3.34%
	Beta	-1.21%	0.02%
	Max Long	-1.52%	-0.17%
	Industry Bounds	-0.02%	-0.01%
	Budget	1.20%	-0.03%
	Max Turnover	-2.37%	-0.13%
	Asset Bounds	0.09%	-0.20%
	Other	0.25%	-0.08%

Table 14: 130/30 - Loosely Beta Constrained - Russell 1000

		Annualized Return Attribution	Risk Attribution
Portfolio		14.64%	
Benchmark		11.35%	
Active		3.28%	2.75%
	MV	6.13%	3.39%
	Beta	-0.07%	0.02%
	Max Long	-1.40%	-0.19%
	Industry Bounds	-0.03%	-0.01%
	Budget	0.61%	-0.02%
	Max Turnover	-2.24%	-0.17%
	Asset Bounds	0.10%	-0.21%
	Other	0.20%	-0.05%

long, and turnover constraints have a negative impact on the active return. While the portfolio manager may not be able to modify the turnover or leverage (max-long) constraint, he may be able to relax the beta constraint as suggested by these attribution results.

Now, let us consider the same set of expected returns and same strategy except we loosen the beta constraint to be within ± 0.05 of the benchmark beta. The results of this attribution are shown in Table 14. By relaxing the beta constraint a little, the annualized active return increased from 2.68% to 3.28% with virtually no change in volatility leading to an increase in information ratio from 0.98 to 1.19. Note that the beta constraint still has a negative contribution to return, but is now very small, suggesting little could be gained by relaxing it further.

6.4 Use Case: Noise Elimination

Thus far, we have only considered cases where we were able to relax a constraint to increase return. Let us now illustrate how constraints can improve returns. Consider the same 130/30 strategy with a different set of expected returns. The results of this backtest are summarized in Table 15. The

Table 15: 130/30 - No Size Constraint - Russell 1000

		Annualized Return Attribution	Risk Attribution
Portfolio		12.17%	
Benchmark		11.35%	
Active		0.82%	3.16%
	MV	3.31%	5.30%
	Max Long	-2.35%	-1.81%
	Industry Bounds	-0.27%	-0.16%
	Budget	1.39%	-0.11%
	Max Turnover	-1.02%	0.07%
	Asset Bounds	-0.11%	-0.08%
	Other	-0.09%	-0.07%

Table 16: 130/30 - Size Constrained - Russell 1000

		Annualized Return Attribution	Risk Attribution
Portfolio		13.80%	
Benchmark		11.35%	
Active		2.45%	3.25%
	MV	3.15%	5.12%
	Size	1.20%	0.15%
	Max Long	-2.77%	-1.62%
	Industry Bounds	-0.27%	-0.15%
	Budget	1.15%	-0.10%
	Max Turnover	0.02%	-0.02%
	Asset Bounds	-0.06%	-0.06%
	Other	0.02%	-0.07%

annualized active return is 0.82% with an information ratio of 0.26.

Now, suppose that an attribution on the expected returns suggests biases on assets with large market caps but that these biases contained no information. In this case, we would want to try to add a constraint on size exposures to eliminate the size bias in the expected returns. We now consider the same strategy with the addition of a constraint on the size exposure taken from the factor risk model used in the risk computations.

From Table 16, we can see that the size constraint had a positive contribution to annualized active return of 1.20% and little contribution to active risk. Adding the size constraint to the strategy increased the information ratio from 0.26 to 0.75.

One could argue that the expected returns should have been modified to eliminate the size bias. Quant managers could potentially do this, but many would consider it easier to just add the constraint. Some quant managers are reluctant to change their alpha construction process if it has a strong track record.

7 Conclusion

Constraints are an integral part of the portfolio construction process. However, as this process becomes more complex, understanding the effect of these constraints is a significant challenge. In this paper, we demonstrate how one can attribute the aggregate effect of imposing constraints to individual constraint and objective terms. In particular, we demonstrate this can be done for any of the classes of constraints that are commonly used in portfolio rebalancing problems. We further illustrate the use of such constraint attributions, both ex-ante and ex-post, and demonstrate how this information can prove valuable to portfolio managers.

The authors would like to thank Sebastian Ceria, Stefan Schmieta, Alper Atamtürk, and Jeff Linderoth for their valuable input into this paper.

References

M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear Programming: Theory and Algorithms*. John Wiley and Sons, 1993.

David R. Cariño. Combining attribution over time. *Journal of Performance Measurement*, 3(4): 5–15, Summer 1999.

R. Clarke, H. de Silva, and S. Thorley. Portfolio constraint and the fundamental law of active management. *Financial Analysts Journal*, 58:48–66, 2002.

R. Clarke, H. de Silva, and S. Sapra. Towards more information-efficient portfolios. *Journal of Portfolio Management*, 31(1):54–63, 2004.

R. Clarke, H. de Silva, and S. Thorley. The fundamental law of active portfolio management. *Journal of Investment Management*, 4(3), 2006.

G. Grinold. Implementation efficiency. *Financial Analysts Journal*, 61:52–64, 2005.

J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex analysis and minimization algorithms. I*. Springer-Verlag, Berlin, 1993.

Jose Menchero. Risk-adjusted performance attribution. *Journal of Performance Measurement*, pages 22–28, Winter 2006/2007.

Jose Menchero and Junmin Hu. Portfolio risk attribution. *Journal of Performance Measurement*, 10(3):22–33, Spring 2006.

B. Scherer and X. Xu. The impact of constraints on value-added. *Journal of Portfolio Management*, 33(4):45–54, 2007.

Axioma

Flexible is better.™

United States and Canada: 212-991-4500
Europe: +44 20 3170 8745
Asia: +852-8203-2790

New York Office
Axioma, Inc.
17 State Street
Suite 800
New York, NY 10004

Phone: **212-991-4500**
Fax: **212-991-4539**

Atlanta Office
Axioma, Inc.
8800 Roswell Road
Building B, Suite 295
Atlanta, GA 30350

Phone: **678-672-5400**
Fax: **678-672-5401**

San Francisco Office
Axioma, Inc.
201 Mission Street
Suite #2230
San Francisco, CA 94105

Phone: **415-614-4170**
Fax: **415-614-4169**

London Office
Axioma, (UK) Ltd.
288 Bishopsgate
London EC2M 4QP

Phone: **+44 20 3170 8745**

Hong Kong Office
Axioma, (HK) Ltd.
Unit C, 17/F
30 Queen's Road Central
Hong Kong

Phone: **+852-8203-2790**
Fax: **+852-8203-2774**

Singapore Office
Axioma, (Asia) Pte Ltd.
30 Raffles Place
#23-00 Chevron House
Singapore 048622

Phone: **+65 6233 6835**
Fax: **+65 6233 6891**

Sales: sales@axiomainc.com
Client Support: support@axiomainc.com
Careers: careers@axiomainc.com