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PORTFOLIO YIELDS AND DURATIONS

This article explores the properties of various measures of portfolio yield and duration. In particular, we examine market weighted averages of security durations, market weighted averages of security yields, dollar duration weighted averages of security yields, and their relation to portfolio internal rate of return. We show that:

- The market weighted average of security yields provides the expected return of a portfolio over the coming period under the assumption of no change in security yields.
- The market weighted average of security durations is the sensitivity of portfolio return to changes in the *dollar-duration* weighted average of security yield changes.
- The dollar duration weighted average of security yields provides a first order approximation of the portfolio internal rate of return.
- This approximation holds with the greatest precision when intra-portfolio yield variations and security convexities are small.

Intra-portfolio yield variations are likely to be large whenever either: 1) the yield curve is steeply sloped, generating large yield variations across maturities, or 2) credit spreads are wide. We provide a second order approximation for the portfolio internal rate of return based on the dollar duration weighted average of security yields, which is appropriate in these situations or for portfolios with extremely convex securities.

Most investors use IRR as a measure of "portfolio yield" if held to maturity. For this to be true, all interim cash flows need to be re-invested at the same IRR. This assumption is also required for the yield-to-maturity of a bond to measure its expected return. The greater the percentage of early cash flows and the greater the slope and volatility of the yield curve, the more troublesome this reinvestment assumption becomes.

The market weighted average of security yields will be closest to the portfolio's internal rate of return in situations in which roll down returns are trivial. Generally, this occurs when the yield curve and term structure of spreads are flat and/or the portfolio's cash flow profile is heavily backloaded. The magnitude of credit spreads, however, does not affect the closeness of the market weighted average of security yields to the portfolio's internal rate of return, provided the term structure of these spreads is flat.

PORTFOLIO YIELD TO MATURITY AND TIME RETURN

To maintain focus on the essential issues, this analysis will consider a zero volatility environment in which all changes in bond yields are perfectly predictable.[1] The current yield curve can have any shape and can change over time, but all such changes are completely deterministic.

Notation

V_{port} and V_i are the current market values of the portfolio and the portfolio's holdings of the ith bond respectively. Let $w_i = V_i / V_{port}$. w_i is bond i's portfolio weight. y_i is the continuously-compounded yield to maturity of bond i at time t. It is the internal rate of return for bond i defined in the following equation:

$$V_i = \sum_j e^{-y_i [t_j - t]} c^i_{t_j} \qquad (1)$$

where $c^i_{t_j}$ is the cash flow that arrives at time t_j from the portfolio's holdings in security i and t_j-t is the number of years until date t_j. Similarly, y_{IRR} is the internal rate of return of the portfolio's cash flows at time t defined from

$$V_{port} = \sum_j e^{-y_{IRR} [t_j - t]} c^{port}_{t_j} \qquad (2)$$

where $c^{port}_{t_j}$ is the portfolio cash flow that arrives at time t_j.

Additionally, define $D_{IRR} = \frac{-1}{V_{port}} \frac{dV_{port}}{dy_{IRR}}$.

D_{IRR} is the sensitivity of the portfolio's return to changes in its internal rate of return.

Expected Returns and Yields

The yield to maturity of an individual security is often used as a gauge of the expected return from holding the security

[1] The appendix treats stochastic bond yields.

to maturity. For a risk-free zero coupon bullet bond, yield to maturity does indeed provide the exact return from holding the security to maturity. For coupon-paying bonds, the issue becomes more complicated. One approach is to view the coupon-paying bond as a portfolio of zero coupon bonds in which each coupon and principal payment is treated as a separate entity. In this case, the original bond's yield to maturity can be viewed as an amalgamation of the expected holding period returns of the zero coupon bonds in this equivalent portfolio. In general, a bond's yield to maturity will be a nonlinear combination of the yields of its component cash flows, which are the spot rates off the corresponding credit curve. However, continuing to view a bond as an equivalent portfolio of zero coupon bonds, we can interpret yield to maturity as being, to a first-order approximation, the dollar duration weighted average of the yields of the bond's component cash flows.

Alternatively, one can view yield to maturity as providing the expected return from holding the bond to maturity subject to the assumption that one will be able to reinvest interim coupons at the bond's current yield to maturity. However, this assumption is typically untenable, particularly when the yield curve exhibits substantial volatility or slope.

The interpretation of internal rate of return becomes especially troublesome for portfolios. Portfolios generally contain bonds that mature at different times. By design, the cash flow stream generated by a bond portfolio (coupon and principal repayments) is often relatively smooth over time. Thus the typical time profile of a portfolio's cash flows will be very different from the time profile of a zero coupon bond's cash flow. These differences exacerbate the problems that the interim cash flows create for the interpretation of portfolio internal rate of return as a measure of the expected return for holding a portfolio until its "maturity."

Additionally, a portfolio may contain bonds with a wide variety of credit qualities. Unless each bond is replaced at maturity with a bond that trades at a similar credit spread, the portfolio internal rate of return may be dramatically affected by the maturity schedule of the bonds in the portfolio. The portfolio's internal rate of return may be affected by any changes in the overall time profile of portfolio's cash flows. These changes could be caused by re-investment decisions or the distribution of cash flows. All of this diminishes the usefulness of the portfolio internal rate of return as a measure of the expected return for holding a portfolio's assets until maturity. Instead, the ability to approximate a portfolio's internal rate of return by the dollar duration weighted average of the yields of its component bonds is particularly relevant for imparting meaning to yield to maturity in a portfolio context.

Expected Return Over Short Time Intervals

Portfolio internal rate of return can be used to calculate expected returns over short horizons. The portfolio's expected return over the next instant in time is given by

$$\text{Portfolio Instantaneous Expected Return per unit time} = y_{IRR} - D_{IRR}\, dy_{IRR} \qquad (3)$$

Recall that D_{IRR} is the sensitivity of the portfolio's return to changes in its internal rate of return and dy_{IRR} is the (expected) change in the portfolio's internal rate of return.[2] The expected portfolio return is the sum of two components: the static portfolio yield (IRR) plus the return coming from the expected capital gain or loss in portfolio value due to the change in the portfolio's IRR yield.

Equation (3), however, is rarely used in practice. Generally, it is simpler to assess a portfolio's expected return over short horizons using a formula based on the market weighted average of the yields of the portfolio's component securities. Here the analysis starts from the following expression for the expected return of an individual bond (bond i) over the next instant in time:

$$\text{Security i Instantaneous Expected Return per unit time} = y_i - D_i\, dy_i \qquad (4)$$

where D_i is bond i's duration, $(-1/V_i)(dV_i/dy_i)$. Since a portfolio's expected return is the market weighted average of the expected returns of its component securities, the portfolio's expected return over the next instant can also be expressed

$$\text{Portfolio Instantaneous Expected Return per unit time} = \sum_i w_i y_i - \sum_i w_i D_i dy_i . \qquad (5)$$

[2] Since we are working in an environment in which all yield changes are known in advance, dy_{IRR} is also equal to the realized yield change. All yields are continuously compounded.

The portfolio's expected return is equal to the sum of the market-weighted averages of:

1) The individual security yields.
2) The expected capital gain or loss return of the individual securities.

At this point, we introduce the following notation:

Let $y_{mw} = \sum_i w_i y_i$

y_{mw} is the market weighted average of the individual security yields.

Let $D_{mw} = \sum_i w_i D_i$.

D_{mw} is the market weighted average of the individual security durations.

Let $dy_{\$dur} = \dfrac{\sum_i w_i D_i dy_i}{D_{mw}}$.

$dy_{\$dur}$ is the dollar duration weighted average of the expected yield changes of the portfolio's component securities.

Equation (5) can be re-expressed as follows: the portfolio's instantaneous expected return at time t is[3]

$$\text{Portfolio Instantaneous Expected Return per unit time} = y_{mw} - D_{mw}\, dy_{\$dur} \tag{6}$$

From Equation (6), the portfolio's expected return can be expressed as the sum of:

- the market weighted average of the individual security yields and
- the (market weighted) portfolio duration times the dollar duration weighted average of the expected individual security yield changes.

If we make the simplifying assumption that the yields of all bonds in the portfolio will remain unchanged over the coming month, then Equation (6) shows that the instantaneous return per unit time is given by the market weighted yield. This is not true for the IRR; even if we assume that all security yields remain unchanged, the portfolio IRR can change as portfolio composition changes due to coupon payments and maturities.

More surprisingly, Equation (6) shows that D_{mw}, the market weighted average of security durations, is the sensitivity of portfolio return to changes in the dollar duration weighted average of individual bond yields.[4]

Note that D_{IRR} is the sensitivity of portfolio return to shifts in the portfolio's internal rate of return. However, D_{mw} is the sensitivity of portfolio return to changes in the dollar duration weighted average of individual bond yields. In general, these two sensitivities will not be identical.

PORTFOLIO INTERNAL RATE OF RETURN AND THE DOLLAR DURATION WEIGHTED AVERAGE OF ASSET YIELDS

We have just seen that the market weighted average of security durations (D_{mw}) is the sensitivity of portfolio return to changes in the dollar duration weighted average of the portfolio's security yields while D_{IRR} is the sensitivity of portfolio return to changes in the portfolio's internal rate of return.

In this section, we show that the dollar duration weighted average of security yields is, to a first order approximation, the same as the portfolio's internal rate of return. Thus, while D_{mw} and D_{IRR} will, in general, not be identical, to a first-order approximation, they are sensitivities to the same variable. In situations in which this first order approximation holds with greatest precision, D_{mw} and D_{IRR} will have very similar values.

[3] If we consider discrete yield changes, rather than infinitesimal yield changes, then the second order approximation of the expected portfolio return is

$$y_{mw} - D_{mw}\,\Delta y_{\$dur} + (1/2)\, C_{mw}\,(\Delta y)^2_{\$dur}$$

where $\Delta y_{\$dur}$ is the expected change in the dollar duration weighted average of the yields of the portfolio's component securities; $(\Delta y)^2_{\$dur}$ is the dollar duration weighted average of the squares of the expected yield changes of the portfolio's component securities; and C_{mw} is the market weighted average of the convexities of the portfolio's component securities.

[4] In this zero volatility environment, Equation (6) provides the portfolio's realized instantaneous return, as well as its expected return, and $dy_{\$dur}$ is both the expected and realized change in the dollar duration weighted average of security yields.

Let $V_i(y)$ be the present value of the cash flows that the portfolio will receive from its investment in security *i* discounted at yield *y*. Recall that y_i is the observed yield to maturity of bond *i*. $V_i(y_i)$ is the current market value of the portfolio's holdings in security *i*.

$$\text{Let } y_{\$dur} = \frac{\sum_i w_i \, D_i y_i}{D_{mw}}.$$

$y_{\$dur}$ is the dollar duration weighted average of the yields of the portfolio's component securities.

Consider a portfolio of *n* securities. By the definition of portfolio yield to maturity, the value y_{IRR} that solves the equation

$$V_{port}(y_{IRR}) = V_1(y_{IRR}) + V_2(y_{IRR}) + ... + V_n(y_{IRR}) \tag{7}$$

is the portfolio's internal rate of return. Note that $V_i(y_{IRR})$ is not equal to the value of the portfolio's holdings of bond *i*. Instead, it is equal to the present value of the cash flows from the portfolio's investment in the ith bond discounted at yield y_{IRR}.

From the standard first-order duration approximation

$$\frac{V_i(y_{IRR}) - V_i(y_i)}{V_i(y_i)} \approx -D_i(y_{IRR} - y_i)$$
$$V_i(y_{IRR}) \approx V_i(y_i) - V_i(y_i)D_i(y_{IRR} - y_i). \tag{8}$$

$V_i(y_i)$ is simply V_i, the current market value of the portfolio's holding's in bond *i*.

From Equations (7) and (8),

$$V_{port}\left(y_{IRR}\right) \approx \sum_i \left[V_i - V_i D_i \left(y_{IRR} - y_i\right)\right]$$
$$V_{port}\left(y_{IRR}\right) \approx \sum_i V_i - \sum_i V_i D_i \left(y_{IRR} - y_i\right) \tag{9}$$

Note that $V_{port}(y_{IRR}) = [V_1 + V_2 + ... + V_n]$; the market value of a portfolio equals the sum of the market values of its component assets. Substituting this into Equation (9) produces

$$\sum_i V_i D_i \left(y_{IRR} - y_i\right) \approx 0. \tag{10}$$

In Equation (10), we assume that the analyst knows all the prices, durations, and yields for individual securities. The only unknown in Equation (10) is the portfolio yield. Solving for y_{IRR},

$$y_{IRR} \approx \frac{V_1 D_1 y_1 + V_2 D_2 y_2 + ... + V_n D_n y_n}{V_1 D_1 + V_2 D_2 + ... + V_n D_n} = y_{\$dur}. \tag{11}$$

From Equation (11), the dollar duration weighted average of bond yields provides the first-order approximation of the portfolio's internal rate of return. To the first order, D_{mw} and D_{IRR} provide return sensitivities to the same entity.

THE ACCURACY OF THE FIRST-ORDER APPROXIMATION OF PORTFOLIO IRR

Equation (11) is based on the first-order duration approximation of bond return. It holds with the greatest precision in situations in which the first-order duration approximation is most accurate. For the current application, these conditions are:

1) Intra-portfolio variations in yield are small (y_{IRR} - y_i is small for all *i*); and
2) Security convexities are close to zero.

First, we consider the effect of intra-portfolio yield variations on the tightness of the match between the dollar duration weighted average of security yields and the portfolio internal rate of return. Figure 1 considers two portfolios. The first portfolio consists of all bullet bonds in the Lehman Gov/Corp Index. The second portfolio has fifty percent of its holdings in the bullet bonds in the Lehman Gov/Corp Index and the remaining fifty percent of its holdings in the bullet bonds in the Lehman High Yield Index. The mixed Gov/Corp, High Yield portfolio can be expected to have significantly more intra-portfolio yield variation than the Gov/Corp only portfolio. The figure presents results for

Figure 1. **Effect of Intra-Portfolio Yield Variation on Dollar Duration Yield and Portfolio IRR**

Date	Portfolio	IRR	Dollar Dur. Yld.	Mkt. Wtd. Yield
8/31/93	Gov/Corp Bullets	5.73	5.70	5.07
	50% Gov/Corp, 50% HY (Bullets)	7.30	7.18	6.74
9/30/98	Gov/Corp Bullets	5.23	5.22	4.97
	50% Gov/Corp, 50% HY (Bullets)	6.69	6.54	6.46

August 31, 1993, and September 30, 1998, dates in which intra-portfolio yield variation were likely to be especially high. The Treasury curve was especially steep in August 1993, generating substantial yield differentials between long and short maturity assets, and the spread sector crash in early Fall 1998 led to wide yield differentials across credit qualities in late September 1998.

The dollar duration weighted average and portfolio internal rate of return match quite closely for the Gov/Corp index at each date. The match is substantially worse for the mixed Gov/Corp, High Yield portfolio. The market weighted average of portfolio yields provides a very poor indicator of portfolio internal rate of return in all cases presented in the figure.

The second-order duration/convexity approximation:

$$V_i(y_{IRR}) \approx V_i(y_i) - V_i(y_i)D_i(y_{IRR} - y_i) + (1/2)V_i(y_i)C_i(y_{IRR} - y_i)^2$$

can be used to show that to a second-order approximation:[5]

$$y_{IRR} \approx y_{\$dur} + \frac{1}{2}\sum_i \frac{w_i C_i}{D_{mw}}(y_{IRR} - y_i)^2 \ . \qquad (12)$$

[5] It can be shown that when one replaces Equation (8) with a second-order duration/convexity approximation, Equation (10) becomes

$$\sum_i V_i D_i (y_{IRR} - y_i) = \frac{1}{2}\sum_i V_i C_i (y_{IRR} - y_i)^2$$

Equation (12) immediately follows.

From Equation (12), the ratio of the market weighted average of security convexities divided by the portfolio's market weighted duration (D_{mw}) provides a measure of the extent to which the dollar duration weighted average of the portfolio's security yields is likely to be appreciably different from the portfolio's internal rate of return.

If all securities in the portfolio have non-negative durations and convexities, then the fraction on the right-hand side of Equation (12) will be non-negative. For securities with non-negative durations and convexities, the dollar duration weighted average of individual security yields will never be greater than the portfolio IRR, provided terms beyond second order are of trivial significance.

Figure 2 separately considers a low convexity subset and a high convexity subset of the two portfolios considered in Figure 1. The low convexity subset consists of maturities of 5 years or less. The high convexity subset consists of maturities of 25 years or greater.

As predicted in Equation (12), the dollar duration weighted yields are always less than or equal to the portfolio internal rates of return in Figures 1 and 2, and the dollar duration yields match the portfolio internal rates of return in Figure 2 more closely for the short maturity portfolios than the long maturity portfolios. The market weighted yields seem to do a better job approximating portfolio internal rates of return for high convexity portfolios. While convexity poses a particular problem for the use of dollar-duration weighted averages as approximations of

Figure 2. **Effect of Portfolio Convexity on Dollar Duration Yield and Portfolio IRR**

Date	Quality	Maturity	Convexity/Duration	IRR	$Dur Yield	MW Yield
8/31/93	Gov/Corp Bullets	Short (≤ 5 yr)	3.5	4.38	4.37	4.21
		Long (≥ 25 yr)	18.9	6.40	6.39	6.39
	50% Gov/Corp, 50% HY (Bullets)	Short (≤ 5 yr)	3.6	6.31	6.21	5.88
		Long (≥ 25 yr)	17.7	7.80	7.61	7.82
9/30/98	Gov/Corp Bullets	Short (≤ 5 yr)	3.6	4.66	4.66	4.65
		Long (≥ 25 yr)	20.3	5.72	5.65	5.70
	50% Gov/Corp, 50% HY (Bullets)	Short (≤ 5 yr)	3.4	6.38	6.29	6.23
		Long (≥ 25 yr)	19.1	7.23	6.97	7.19

portfolio IRR, they do not appear to be an important factor determining the accuracy of market weighted yields as an approximation of portfolio IRR.

SECOND ORDER APPROXIMATION OF PORTFOLIO IRR

While market weighted averages worked well for the high convexity portfolios considered in Figure 2, one cannot rely on them in general to be good approximations of portfolio internal rate of return. For portfolios containing high convexity assets, the more prudent procedure is to solve explicitly for the portfolio's internal rate of return directly from the portfolio's cash flows. This can be easily implemented through the Cash Flow Report feature in Lehman's PC Product.

Failing this, Equation (12) provides a mechanism for generating the second order approximation. Setting Equation (12) to an equality creates a quadratic equation, which can be solved in closed form. The solution, $y_{\mathrm{IRR}}^{\mathrm{so}}$, is the second order approximation to portfolio IRR. Alternatively, one can approximate the solution by replacing y_{IRR} with $y_{\$dur}$ in the right-hand side of Equation (12). Figure 3 uses this short cut to approximate portfolio IRR for the portfolios in Figures 1 and 2 in which dollar duration weighted yields differed noticeably from portfolio IRR. The second order approximations presented in Figure 3 match the portfolio internal rates of return quite well.

PORTFOLIO INTERNAL RATE OF RETURN AND THE MARKET WEIGHTED AVERAGE OF SECURITY YIELDS

We can use Equations (3) and (6) to understand the relationship between the market weighted average of the yields of the portfolio's constituent securities and the portfolio's internal rate of return. Setting the right-hand sides of Equations (3) and (6) equal to each other, we obtain

$$y_{mw} - y_{IRR} = D_{mw}\, dy_{\$dur} - D_{IRR}\, dy_{IRR}. \qquad (13)$$

Recall that $dy_{\$dur}$ and dy_{IRR} refer to the expected changes in portfolio yield per unit time. These changes will tend to be largest when the yield curve or term structure of credit spreads is highly positively or negatively sloped. Equation (13) suggests that the differential between portfolio internal rates of return and market weighted yields will be greater during periods such as August 1993, when the yield curve was steeply sloped, rather than September 1998, when the yield curve was relatively flat. This is confirmed in Figure 4. The September 1998 market weighted yields performed much better than those of August 1993 in matching their respective portfolio internal rate of return, despite the erratic nature of credit spreads in early fall 1998.

The expected change in portfolio yield per unit time ($dy_{\$dur}$ or dy_{IRR}) will invariably be much lower for long maturity assets than for short maturity assets. For instance, the Lehman Treasury spline for 9/30/98 showed no difference in fitted yields between the 20- and 25-years. The difference between the 5-year yield and the 1-month yield from the same 9/30/98 spline was 32 bp.[6] The difference in magnitude of $dy_{\$dur}$ or dy_{IRR} for long maturity portfolios versus short maturity portfolios is much greater than the magnitude of the differences in portfolio duration. Therefore, from Equation (13), one would expect market weighted yields to much more closely approximate portfolio

Figure 3. **Accuracy of Second Order Approximation of Portfolio Internal Rate of Return**

Date	Quality	Maturity	IRR	Dollar Dur. Yield	y^{so}
8/31/93	Mix Gov/Corp HY	All	7.30	7.18	7.31
	Mix Gov/Corp HY	Short	6.31	6.21	6.31
	Mix Gov/Corp HY	Long	7.80	7.61	7.79
9/30/98	Mix Gov/Corp HY	All	6.69	6.54	6.69
	Mix Gov/Corp HY	Short	6.38	6.29	6.39
	Mix Gov/Corp HY	Long	7.23	6.97	7.22

Figure 4. **Accuracy of Market Weighted Yield as an Approximation to Portfolio IRR**

Date	Quality	Maturity	IRR	MW Yield	Diff. (bp)
8/31/93	Gov/Corp	All	5.73	5.07	66
		Short	4.38	4.21	17
	Mix Gov/Corp HY	All	7.30	6.74	56
		Short	6.31	5.88	43
9/30/98	Gov/Corp	All	5.23	4.97	26
		Short	4.66	4.65	1
	Mix Gov/Corp HY	All	6.69	6.46	23
		Short	6.38	6.23	15

internal rates of return for very long maturity portfolios rather than for short (or mixed) maturity portfolios. Figure 2 confirms this. In all cases, the market weighted yield closely approximates portfolio IRR for the long maturity portfolio, but provides a very poor approximation of portfolio IRR for the short maturity portfolios.

THE NUMBERS TODAY

Currently, the fixed income market is experiencing a very flat yield curve and very wide credit spreads. The wide credit spreads imply that differences between portfolio internal rate of return and dollar duration weighted yields should be particularly large, while the flat yield curve implies that market weighted averages should come relatively close to portfolio internal rates of return. Figure 5 presents the numbers as of the end of July 2000.

The market weighted yields outperform dollar duration yields in terms of their ability to approximate portfolio internal rates of return for all portfolios other than those with short maturities. In fact, the market weighted average is able to provide a better IRR approximation than the second order approximation based on Equation (12) in the long maturity, high intra-portfolio yield variation case.

PORTFOLIO DURATION

The portfolio duration

$$D_{IRR} = \frac{-1}{V_{port}} \frac{dV_{port}}{dy_{IRR}}$$

contains a derivative with respect to shifts in the portfolio's internal rate of return, whereas individual security durations

$$\frac{-1}{V_i} \frac{dV_i}{dy_i}$$

contain derivatives with respect to the individual security yields. Since a portfolio IRR is not a linear function of the yields of the individual securities in the portfolio, it follows that D_{IRR} cannot equal D_{mw} in general.

Modified Adjusted Duration in PC Product measures the sensitivity of a bond's return to a parallel shift in the Treasury par curve, holding spreads and volatility parameters constant. Modified Adjusted Duration in PC Product can be expressed as

$$\frac{-1}{V_{port}} \frac{\partial V_{port}}{\partial y_{par}}$$

for a portfolio and as

$$\frac{-1}{V_i} \frac{\partial V_i}{\partial y_{par}}$$

for an individual security where dy_{par} is a unit change in the Treasury par curve.[7] In both cases, the derivative is with respect to the same entity. Therefore, the modified adjusted duration of a portfolio is exactly equal to the market weighted average of the modified adjusted durations of the portfolio's constituent securities.

[6] For 8/31/93, the corresponding numbers for 25-year yield minus the 20-year yield was 13 bp, and the 5-year yield minus the 1-month yield was 139 bp.

[7] Since the dy_{par} is multi-dimensional, $dV_i\ /dy_{par}$ *and* dV_{port}/dy_{par} are gradients. However, linearity is maintained.

Figure 5. **Dollar Duration Yield, Market Weighted Yield and Portfolio IRR,** 7/31/00

Quality	Maturity	Convexity/Duration	IRR	$Dur Yield	MW Yield	y^{so}
Gov/Corp Bullets	All	11.0	6.79	6.75	6.81	6.79
	Short (≤ 5 yr)	3.3	6.84	6.84	6.79	
	Long (≥ 25 yr)	18.9	6.76	6.68	6.74	6.76
50% Gov/Corp,	All	9.4	8.81	8.52	8.68	8.81
50% HY (Bullets)	Short (≤ 5 yr)	3.3	9.11	8.92	8.72	9.12
	Long (≥ 25 yr)	17.7	9.31	8.43	9.23	9.01

The sensitivity of security duration to shifts in security yield is always given by

$$D_i'(y_i) = D_i^2 - C_i.$$

In order to discuss the sensitivity of portfolio duration to changes in portfolio yield, one must identify the specific form of portfolio duration and yield under consideration. The derivative of D_{IRR} with respect to the portfolio's internal rate of return is given by the analogous formula:

$$D_{IRR}^2 - C_{IRR}$$

where C_{IRR} is the convexity calculated directly from the portfolio's cash flows. However, there is no simple formula for the derivative of D_{mw} with respect to shifts in y_{mw} unless one restricts attention to parallel shifts in the yield curve. With parallel shifts in the yield curve, $dy_i = dy_{IRR} = dy_{mw}$. Here one can treat all of these yield differentials interchangeably, and the formula

$$D' = D^2 - C$$

will hold for all forms of portfolio duration.

CONCLUSION

When comparing portfolio and benchmark yields at the start of a given month, the traditional measure of yield employed is the market-value weighted yield. This provides a crude estimate of short-term expected returns under a simple "no change in yields" scenario. Internal rate of return is a measure of the long-term increase in portfolio wealth to be expected in a held-to-maturity context, but is problematic for portfolios with relatively smooth cash flow profiles over time. Portfolio internal rate of return is most often used in applications such as dedication, in which a portfolio is purchased to match a set of liabilities. The highest internal rate of return portfolio to match a given liability stream also has the property of matching the liabilities at the lowest present cost.

The closeness of fit of the dollar duration weighted average of security yields to portfolio internal rate of return depends on security convexities and the cross-sectional variation in yield. This cross-sectional variation in yield is driven by credit spreads and the distribution of constituent securities across the yield curve. The closeness of fit of the market weighted average of security yields to portfolio internal rate of return does not depend on the cross-sectional variation in yields. Instead, it is determined by the rate with which portfolio yields are expected to change as time passes. This effect is largely captured by the slope of the relevant parts of the yield curve and the time-structure of credit spreads. Of course, the fit also depends on the maturity distribution of the various bonds in the portfolio.

In addition, we show that:

- The market weighted average of security durations is the sensitivity of portfolio return to changes in the *dollar duration* weighted average of security yield changes.
- The dollar duration weighted average of security yields provides a first order approximation to portfolio internal rate of return.

We also provide a simple second order approximation of portfolio internal rate of return based on the dollar duration weighted average of security yields.

APPENDIX. STOCHASTIC BOND YIELDS

This section derives the analog to Equation (13) in the presence of stochastic bond yields. Actual interest rates processes are more complicated than those considered in the body of this article. Actual yields evolve stochastically over time. Let μ_i be the expected change in the yield of bond *i* at time *t*. Similarly, let σ_i be the volatility of the yield process for bond *i* at time *t*. The portfolio's expected return over the next instant in time is given by

$$\text{Portfolio Instantaneous Expected Return per unit time} = y_{IRR} - D_{IRR}\mu_{IRR} + \tfrac{1}{2}C_{IRR}\sigma_{IRR}^2 \quad \text{(A1)}$$

where μ_{IRR} is the instantaneous expected change in the portfolio's internal rate of return (y_{IRR}). (This was denoted dy_{IRR} in the section that treated zero volatility interest rate environments.)

σ_{IRR}^2 is the instantaneous volatility of the portfolio's internal rate of return, and C_{IRR} is

$$\frac{1}{V_{port}}\frac{d^2V_{port}}{dy_{IRR}^2}.$$

Equation (A1) replaces Equation (3) under stochastic yields.

Similarly, the expected return on bond *i* over the next instant in time is[8]

$$\text{Security } i \text{ Instantaneous Expected Return per unit time} = y_i - D_i\mu_i + \tfrac{1}{2}C_i\sigma_i^2. \quad \text{(A2)}$$

[8] More formally, let the total change in the yield of security *i*, dy_i, come from the diffusion $dy_i = \mu_i dt + \sigma_i dz_i$ where μ_i and σ_i may be functions of time and current and past yields. The innovation driving the shock to the ith bond, dz_i, may have arbitrary correlation with the innovations driving the yield shocks of other bonds in the portfolio. Note that each bond yield is modeled directly, rather than the instantaneous risk-free rate and spread. However, since μ_i and σ_i are allowed to be arbitrary functions of time and yield history, this form is completely general (except for the exclusion of jump processes). Ito's lemma implies

$$\frac{dV_i}{dt} = \frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial y_i}\frac{\partial y_i}{\partial t} + \frac{1}{2}\frac{\partial^2 V_i}{\partial y_i^2}\sigma_i^2$$

which results in

$$\frac{dV_i}{dt} = V_i y_i - V_i D_i \frac{\partial y_i}{\partial t} + \tfrac{1}{2}V_i C_i \sigma_i^2.$$

Equation (A2) follows.

With stochastic yields, Equation (5) is replaced by the following expression for the portfolio's expected return over the next instant in time:

$$\text{Portfolio Instantaneous Expected Return per unit time} = \sum_i w_i y_i - \sum_i w_i D_i \mu_i + \sum_i w_i \tfrac{1}{2}C_i\sigma_i^2. \quad \text{(A3)}$$

Let $C_{mw} = \sum_i w_i C_i$.

C_{mw} is the market weighted average of the individual security convexities.

$$\text{Let } \sigma_{\$cvx}^2 = \frac{\sum_i w_i C_i \sigma_i^2}{C_{mw}}.$$

$\sigma_{\$cvx}^2$ is the dollar convexity weighted average of the squared yield volatilities of the individual securities.

$$\text{Let } \mu_{\$dur} = \frac{\sum_i w_i D_i \mu_i}{D_{mw}}.$$

$\mu_{\$dur}$ is the dollar duration weighted average of the expected yield changes of the portfolio's component securities.

Equation (A3) can be re-expressed as follows: the portfolio's expected return over the next instant in time is

$$\text{Portfolio Instantaneous Expected Return per unit time} = y_{mw} - D_{mw}\mu_{\$dur} + \tfrac{1}{2}C_{mw}\,\sigma_{\$cvx}^2. \quad \text{(A4)}$$

Note that the market weighted average of security durations continues to provide the sensitivity of portfolio returns to changes in the dollar duration weighted average of the expected changes in the yields of the portfolio's component securities. Here the convexity correction involves multiplying the market weighted average of individual security convexities by the dollar convexity weighted average of the squared volatilities of the individual security yields.

By setting Equations (A1) and (A4) equal to each other, we obtain

$$y_{mw} - y_{IRR} = [D_{mw}\,\mu_{\$dur} - D_{IRR}\,\sigma_{IRR}] + \tfrac{1}{2}[C_{IRR}\,\sigma_{IRR}^2 - C_{mw}\,\sigma_{\$cvx}^2] \quad \text{(A5)}$$

Equation (A5) replaces Equation (13) in the presence of stochastic yields. Incorporating the effects of stochastic yields tends to increase the difference between the portfolio's internal rate of return and the market weighted average yield. Due to diversification effects, unless all bonds in the portfolio have perfectly correlated yields, σ_{IRR}^2 will be less than the dollar duration weighted average value of σ_i^2. All else equal, the second term on the right hand side of Equation (A5) will tend to be positive.