LEHMAN BROTHERS

Fixed Income Research

Sufficient Diversification in Credit Portfolios

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■ We study the impact of rating downgrades on security returns relative to peer groups. Downgrades are clearly identifiable events that represent the main part of security-specific risk. Using 1989-2001 data on the Lehman U.S. Investment-Grade Credit Index, we determine the portfolio diversification required to reduce the expected impact of downgrades to a desired level.

■ Combining Lehman return data with credit transition probabilities published by the rating agencies, we develop a methodology for estimating the distribution of a security's downgrade risk. The analysis proceeds to estimating the absolute risk of a portfolio and, finally, to the portfolio's tracking error relative to a benchmark.

■ The tracking error versus the Credit Index due to downgrades is minimized when tighter position limits are imposed on lower qualities. The optimal ratio of position limits for Aa/Aaa, A and Baa securities is 9:4:1. An optimally structured portfolio of 50 to 100 securities can have a reasonably low tracking error of 42 bp to 29 bp per year, respectively, versus the Credit Index.

■ Recognizing that investors are interested in *total* idiosyncratic risk besides the risk of downgrades only, we study the optimal diversification levels with "natural" volatility taken into account. This is defined as the risk of issuer selection within a peer group with unchanged ratings. When both downgrade risk and "natural" volatility are considered, the optimal ratio of position limits for Aa/Aaa, A, and Baa securities is the significantly less skewed 4:3:1.

■ While increasing diversification reduces risk, it can also reduce expected outperformance. We introduce simple models of the value of credit research to analyze this trade-off. We show that the expected outperformance of the top analyst's pick in each market segment and the cost of credit research determine the optimal portfolio structure. This framework can help investment managers organize their thought process when allocating business resources to credit analysis.

TABLE OF CONTENTS

Publications—L. Pindyck, A. DiTizio, B. Davenport, W. Lee, D. Kramer, J. Threadgill, R. Madison, A. Acevedo, K. Kim, C. Rial, J. Batstone

INTRODUCTION

How many different issuers should an investor have in a credit portfolio?

This question, in one form or another, has long occupied the minds of portfolio managers around the globe. Yet it has taken on a new urgency for credit market veterans as well as for relative newcomers to the debt asset management business. Two main factors motivate this wave of interest in credit risk and diversification: asset shifts to credit markets and a marked increase in spread volatility.

Global demand for credit securities is rising for several reasons. In Europe, where many bond investors traditionally bought only government bonds, the monetary union in 1999 created a much more homogeneous government bond market. This transformation has left less opportunity for portfolio managers to outperform strictly by varying their country/currency mix and has prompted many asset managers to consider extending their investment set to other asset classes, including credit product. A growing number of investors now switch from all-government benchmarks to those including credit products as well, such as the Lehman Brothers Euro-Aggregate and Global Aggregate indices.

The unified market and a greater appetite for credit have led to a large increase in issuance. The European corporate bond market, previously dominated by highly-rated financial issuers, has given a warm reception to issues by lower-rated corporations that had previously been forced to rely exclusively on bank financing. In addition, the single-currency market better supports larger issue sizes. For investors, the greater credit diversity and higher liquidity make the European credit market much more attractive.

In the U.S., where credit product has always been a prominent part of debt portfolios, the changes are less dramatic. Yet, the momentum for credit has been increasing in the U.S. as well. Globally, some investors who have traditionally relied on Treasuries, such as central banks, are now investigating opportunities in other asset classes. U.S. Agency issues, and particularly their benchmark programs, have been the primary beneficiaries, but the search for extra spread has spilled over into the corporate market as well. Some large Asian investors are among those who view corporates as a possible alternative to Treasuries.

Yet, as the demand for credit product increased, the credit markets encountered several difficult years. Spreads widened suddenly in the liquidity crisis of 1998 and then again in 2000. In both 2000 and 2001, a slew of “credit bombs” painfully illustrated the dangers inherent in credit investing.

The credit market volatility of 1997-2001 has increased the caution of investors approaching credit for the first time. As such investors redesign their processes to include credit, they seek a complete understanding of the diversification needed to protect their portfolios from the risks of downgrades and defaults. Even long-time credit managers should revisit this issue in this high spread volatility environment. For managers of structured credit products like CDOs and CLOs, diversification is a critical concern.

In this report, we establish a quantitative framework to address the issues of credit risk and diversification, with an emphasis on the implications for portfolio structure. A model for portfolio credit risk is built using rating transition probabilities from the major rating agencies (Moody's and S&P) and historical return data from the Lehman Brothers U.S. Investment-Grade Credit Index. With this model, we explore how portfolios should be structured to meet various investment goals. Among the problems we address are:

- For a portfolio of N bonds, how many should be purchased from each credit quality class to minimize exposure to downgrade risk?
- How many securities should be held in a credit portfolio to bring the impact of downgrade risk below a given threshold?
- Assuming a simple model for the expected outperformance due to credit research, what number of bonds maximizes the information ratio?
- What percentage of the market should be covered by credit research?

Systematic Risk, Non-systematic Risk, and Downgrade Risk

Managers of credit portfolios face many different risks. Systematic risks include interest rate movements, changes in market volatility, and across-the-board changes in credit spreads, either for the market as a whole or for a particular industry or quality. Non-systematic risk reflects developments that affect specific issuers or securities but not the broader market.[1]

The decision of how much of a portfolio to allocate to the credit markets, or how to allocate the portfolio among different industries and qualities, is tied closely to the study of systematic risk. This risk results from differences between portfolio and benchmark exposures to systematic risk factors such as changes in interest rates and sector spreads. A manager might overweight a sector if its expected outperformance over the market as a whole compensates sufficiently for the added risk. More simply, current spreads (or excess returns) are judged against their historical volatility.

This type of analysis, while important, is not the subject of this study. Our emphasis will be on non-systematic risk, created by differences between portfolio and benchmark exposures to specific issuers and securities. Large allocations to particular issuers make a portfolio vulnerable to credit events at these issuers, while their impact on the highly diversified index is much smaller. Non-systematic risk is often called *diversifiable* risk, *i.e.* reducible by diversification.

The most extreme form of credit event is default. However, the risk of immediate default for an investment-grade bond is extremely small. Developments that increase the market's perception of the probability of downgrade or default are far more typical in the investment-grade market. Examples include disappointing earnings, a large equity buyback, a planned change in financial leverage, a merger announcement, or a rating action.

[1] For a detailed discussion of the Lehman Brothers Risk Model and its applications, see *The Lehman Brothers Multi-Factor Risk Model*, Lehman Brothers, July 1999.

We have chosen to focus on changes in ratings as the credit events of interest in the U.S. investment-grade market. Our index database contains monthly data on all securities in the Lehman Brothers U.S. Investment-Grade Credit Index since 1988, with credit ratings from both Moody's and Standard and Poor's along with prices, durations, and returns. The downgrade data offer a fairly complete view of all serious credit events affecting index securities. Defaults directly from investment grade are rare, but any degradation in an issuer's credit worthiness almost inevitably results first in its market underperformance, and then in its eventual downgrade.

First, we develop an approach for estimating the risk of downgrades in a credit portfolio. The analysis proceeds from the risk of a single security to the absolute risk of a portfolio and finally to tracking error relative to a benchmark. Some simplifying assumptions are made to facilitate analytical treatment. Next, we show how this model can be used to answer the questions raised above concerning optimal portfolio structuring. Simple models of the utility of credit research are introduced to study the trade-off between two effects of increasing diversification: decreased risk and decreased expected outperformance. A set of Appendices provides a complete mathematical presentation of all the models and analytical results.

MODELING DOWNGRADE RISK IN A SINGLE BOND

Historical Approach: Observed Returns of Downgraded Bonds

Rating agencies provide data on the historical frequency of corporate upgrades and downgrades. The data on performance consequences of downgrades is not as readily available. The sequential nature of downgrade announcement complicates measurement of the return impact of downgrades. The rating agencies typically watchlist an issuer in advance of rating changes; the fundamental financial information that can trigger a ratings action is usually available to market participants well before the downgrade. As a result, there is often no apparent negative effect on performance during the month of the actual downgrade. Instead, underperformance may be spread over the course of several months leading up to a downgrade.

To quantify this effect, we have conducted a study of all bonds in the Lehman Brothers U.S. Investment-Grade Credit Index that were downgraded from August 1988 through December 2001.[2] For each downgraded security, we measure performance relative to its peer group over the course of the four quarters preceding the downgrade.

For the purposes of this analysis, we partitioned the U.S. investment-grade credit market by quality, sector, and duration. The credit quality grid is rather coarse with three levels: Aaa-Aa, A, and Baa.[3] There are four sectors: industrial, financial, utilities, and non-U.S issuers. Finally, there are three duration groups: under 4 years, 4-7 years, and over 7 years.

For each security in the index in a given month, we calculate excess return over duration-equivalent U.S. Treasuries. We then average these excess returns over all bonds in each of the 36 cells of the quality-sector-duration partition. For each bond, the cell it belongs to becomes its peer group. We proceed to measure each bond's outperformance relative to its peer group, defined as the difference between the bond's excess return and the peer group's average excess return.

We define a downgrade as a transition from one of our three coarse rating levels to a lower one. A downgrade from one subclass of single-A to another, for example, will not be counted. (This is consistent with the granularity at which the rating agencies publish rating transition matrices.) For each downgrade, we examine the performance of the bond relative to its peer group during the month of the downgrade and over the preceding 11 months. We check that the bond's rating was unchanged over this 11-month period. Downgrades following on the heels of prior downgrades are excluded because of the difficulty of separating the effects of the two events.

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[2] While a return time series for this index, originally called the Corporate Bond Index, extends back to 1973, our archives have detailed analytics beginning only in August 1988.

[3] Our index database lists ratings from Moody's when available. For bonds not rated by Moody's, we use the S&P rating. If neither Moody's nor S&P rate the bond, then we use ratings from Fitch or Duff and Phelps. For simplicity, a single notation denotes the roughly equivalent rating classes from different providers. Thus, Baa is used to denote bonds rated Baa1 through Baa3 by Moody's or BBB+ through BBB- by S&P. The Aaa and Aa ratings have been lumped together, because there are relatively few Aaa credits.

Figure 1 summarizes our findings. Not surprisingly, most of the impact from a downgrade is absorbed in the final few months before the event. The largest underperformance comes in the month of the downgrade and the two months preceding it. As we look further back in time, we find that there is noticeable underperformance 3-5 months before a downgrade, and that the effect can be felt as far back as 8 months before a downgrade. Nine or more months before a downgrade, bonds do not significantly underperform their peer groups.

For example, securities downgraded from Baa experienced an average underperformance of –12.92% to their peer group during the year leading up to the downgrade. This amount gathers unevenly through the year. The average quarterly underperformance was -8.59% in the quarter immediately preceding a downgrade, but only -2.80% and -1.50%, respectively, in the previous two quarters. The t-statistic[4] shows all of these numbers to be statistically significant; no significant underperformance is seen 9-11 months before the downgrade.

[4] The t-statistic is the ratio of the sample mean to the standard error and measures the statistical significance of an observed sample mean. Is it possible that downgraded bonds generally perform the same as their peer groups, but the set of bonds that we sampled just happens to underperform due to random noise? A high t-statistic rules out this possibility. Given that the annual underperformance of bonds downgraded from Baa has an observed sample standard deviation of 22.65% (as shown in Figure 1), we can estimate that the mean over 523 independent observations is subject to a sampling error of 22.65%/ $\sqrt{523}$ =0.99%. The resulting t-statistic of –12.92%/0.99% = -13.0 shows that the mean is quite large compared to the sampling error. A t-statistic whose absolute value is greater than the critical value (the exact critical value depends on the sample size, but is generally close to 2) indicates that there is less than a 5% chance of obtaining such sampling results from a population whose true mean was zero. With a t-statistic of –13.0, this chance is practically zero.

Figure 1. **Average Underperformance due to Downgrades,** August 1988-December 2001

			Underperformance (%)				
			Monthly		**Quarterly and Annual**		
Months Prior to Downgrade	**Initial Quality**	**Number of Observations**	**Mean**	**Std Dev**	**Mean**	**Std Dev**	**T-Stat**
	Aaa-Aa	716	-0.08	1.07	-0.23	1.85	-3.4
0-2	A	974	-0.72	3.28	-2.15	5.67	-11.8
	Baa	523	-2.86	13.34	-8.59	23.10	-8.5
	Aaa-Aa	716	-0.01	0.70	-0.02	1.21	-0.5
3-5	A	974	-0.20	1.40	-0.61	2.43	-7.9
	Baa	523	-0.93	4.40	-2.80	7.61	-8.4
	Aaa-Aa	716	0.00	0.67	-0.01	1.17	-0.2
6-8	A	974	-0.06	0.94	-0.19	1.62	-3.7
	Baa	523	-0.50	2.71	-1.50	4.70	-7.3
	Aaa-Aa	716	-0.01	0.70	-0.03	1.21	-0.7
9-11	A	974	0.04	0.90	0.11	1.56	2.3
	Baa	523	-0.01	1.66	-0.03	2.87	-0.2
	Aaa-Aa	716	-0.02	0.74	-0.30	2.55	-3.1
Full Year	A	974	-0.24	1.90	-2.84	6.58	-13.5
	Baa	523	-1.08	6.54	-12.92	22.65	-13.0

Severe return consequences are usually limited to downgrades from lower-rated credits. The most drastic underperformance is found when bonds are downgraded from Baa to below investment grade. The crossing of the investment grade boundary can create major price dislocations because many portfolios (forced, for example, by the investment policy) must sell into a falling market. For bonds downgraded from single-A, the resulting underperformance in the two to three quarters preceding the event is roughly one fourth of the losses in the Baa sector. The time distribution of these losses roughly mirrors the Baa pattern. For securities rated Aaa and Aa, we did not detect any statistically significant underperformance due to downgrades.

In all cases, the standard deviation of underperformance exceeds the mean underperformance. This means that a downgraded bond could very well do much worse than the average. In fact, several bonds downgraded from Baa lost more than half of their value during the year preceding the downgrade.

Figure 2 highlights the extent to which these results vary over time by breaking down the results by the year in which the downgrade took place. Both the mean peer group underperformance and its standard deviation across all downgraded bonds vary quite a bit from year to year. Interestingly, the peak downgrade losses occur at different times for different credit qualities. This supports the common wisdom that the main source of credit risk in higher qualities is related to specific credit events, while in the lower qualities it is driven mainly by recessions. Accordingly, in the Baa sector, the worst years were 2000 and 2001, while the worst single year for single-A was 1997, the inception year of the "Asian Contagion" and the year of the South Korea downgrade.

As apparent from Figure 2, the years 2000 and 2001 brought extremely high downgrade losses, in terms of both the average loss and its standard deviation,

Figure 2. **Average Underperformance due to Downgrades**
Yearly Results, 1989-2001

	Number of Issues (Issuers)			**Average** (%/month)			**Std Dev** (%/month)		
Year	**Aaa-Aa**	**A**	**Baa**	**Aaa-Aa**	**A**	**Baa**	**Aaa-Aa**	**A**	**Baa**
1989	16 (7)	32 (15)	18 (8)	-0.28	-0.17	-0.35	1.47	0.86	1.71
1990	132 (27)	90 (34)	30 (16)	0.01	-0.23	-1.16	0.42	1.43	6.66
1991	138 (18)	86 (20)	54 (13)	-0.03	-0.08	-0.70	0.54	0.90	2.53
1992	84 (26)	130 (17)	30 (11)	-0.01	-0.10	-0.31	0.89	0.65	1.62
1993	66 (21)	14 (9)	35 (13)	0.03	0.04	-0.25	0.72	0.64	1.39
1994	15 (11)	38 (10)	16 (4)	-0.04	-0.08	-0.31	0.31	1.00	1.84
1995	56 (23)	54 (19)	26 (8)	0.05	-0.08	-0.43	0.95	1.27	1.46
1996	33 (11)	59 (12)	35 (11)	0.02	-0.05	-0.47	0.23	0.47	3.16
1997	6 (7)	107 (31)	10 (5)	-0.05	-0.62	-0.28	0.42	3.28	1.99
1998	50 (16)	70 (29)	46 (16)	-0.08	-0.27	-0.51	0.51	1.31	3.35
1999	42 (15)	64 (30)	54 (14)	0.00	-0.12	-0.71	0.49	0.77	3.58
2000	48 (10)	94 (25)	57 (16)	-0.04	-0.34	-2.65	0.50	2.20	7.80
2001	30 (25)	136 (57)	112 (30)	-0.25	-0.37	-1.94	1.48	2.62	10.08
Totals	**716 (217)**	**974 (308)**	**523 (165)**	**-0.02**	**-0.24**	**-1.08**	**0.74**	**1.90**	**6.54**

especially in the Baa sector. A number of bonds deteriorated rapidly from investment grade toward default, suffering losses as great as 40% in a month.

In addition to the losses observed on downgraded bonds, the modeling of downgrade risk requires one more crucial set of input data - the probabilities of various types of downgrades. The major rating agencies regularly publish annual transition matrices with the probability distribution of a bond's rating at the end of a given year, based on its rating at the start of that year. Our study deals with the total probability of a downgrade, which is obtained as a sum of transition probabilities to all rating categories lower than the initial rating. In Figure 3, we show the total probability of a downgrade from each initial rating group, according to both Standard & Poor (1981-2001) and Moody's (1970-2001). These probabilities are compared to the downgrade frequencies observed for the issuers in the Lehman Brothers U.S. Investment-Grade Credit Index from 1990 through 2001.

As noted above, bonds downgraded during 2000 and 2001 suffered unusually high losses. The severity of the credit events in those two years led Moody's to publish additional 2000- and 2001-only transition matrices. Figure 3 includes results from these matrices as well. But the severity of 2000 and 2001 is not in the downgrade probabilities (both of which are actually lower than the long-term averages), but exclusively in the magnitude of losses, as shown in Figure 2. This conclusion is further illustrated in Figures 4 and 5. While the 2000 and 2001 downgrade/upgrade ratios shown in Figure 4 were lower than those in the recession years of 1990-1991, the list of 25 worst excess returns in the Credit Index between 1989 and 2001 (Figure 5) consists almost entirely of 2000 and 2001 events.

A Simple Model of Downgrade Risk

Using the performance results and downgrade probabilities presented above, we constructed a simple model for the losses bondholders may suffer from downgrades. These losses can be characterized by a two-stage random process. First, there is a probability that a bond will be downgraded; second, there is a fairly wide distribution of losses among downgraded bonds. Our model captures both of these sources of randomness and projects the standard deviation of the resulting loss distribution.

Assume that a bond has a probability p of being downgraded over the coming year. If it does suffer a downgrade, the conditional distribution of peer group underperformance is assumed to have mean μ and standard deviation σ. Parameters p, μ, and σ are all functions of the current credit rating; to reduce clutter, we will

Figure 3. **Total One-Year Downgrade Probabilities**
By Initial Rating, Estimated from Different Sources, in %

Agency	Time Period	Aaa-Aa	A	Baa
Moody's	1970-2001	8.13	5.49	5.70
	2000 Only	6.52	4.95	4.18
	2001 Only	3.28	7.64	5.62
S & P	1981-2001	7.24	5.99	5.69
Lehman Brothers	1990-2001	8.00	7.66	5.15

omit the subscripts as we develop the formulas for bonds of a single quality. As shown in Appendix A, the mean and variance of the loss due to downgrades for a single bond in a particular month are given by:

$$\langle loss \rangle = p\mu \qquad (1)$$
$$\sigma^2_{loss} = p\left(\mu^2 + \sigma^2\right)$$

Figure 4. **Moody's and S&P Downgrade/Upgrade Ratios in the U.S. Investment-Grade Market,** 1989-2001

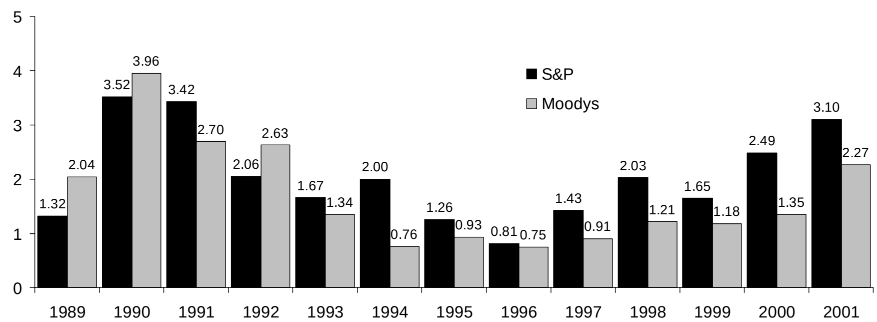

Figure 5. **25 Worst Monthly Performers by Excess Return in the U.S. Investment-Grade Credit Index,** 1989-2001

Issuer	**Excess Return** (bp)	**Month**
Enron Corp.	-7,720	Nov-01
Armstrong World Ind, Inc.	-4,861	Oct-00
Marconi Corporation Plc.	-4,849	Sep-01
SC International	-4,362	Nov-01
Owens Corning	-4,349	Jul-00
Metronet Communications	-4,293	Sep-01
At&T Canada Inc.	-4,071	Sep-01
Xerox Capital Trust I	-3,952	Oct-00
Hilfiger Tommy USA Inc.	-3,530	Apr-00
Conseco Inc.	-3,508	Apr-00
Freeport-McMoran Resource	-3,427	Jan-01
AES Drax Holdings Ltd-Global	-3,391	Dec-01
Comdisco	-3,128	Apr-01
NGC Corp. Capital Trust	-3,127	Dec-01
USG Corporation	-3,069	Oct-00
IBP Inc.	-2,921	Oct-00
Laidlaw	-2,918	Apr-00
Case Corp.	-2,369	Nov-00
Royal Caribbean	-2,323	Sep-01
IKON Office Solutions	-2,204	Oct-00
Kmart	-2,154	Oct-95
Service Corp.	-2,113	Jan-00
Champion Enterprises Inc.	-2,002	Jun-00
USEC	-1,926	Feb-00
Foster Wheeler	-1,902	Dec-00

Figure 6 summarizes the model of single-bond returns. Downgrade probabilities come from the first row of Figure 3, and the mean and standard deviation of loss for downgraded bonds are from the bottom section of Figure 1. The resulting statistics for a bond of a given credit quality are obtained from equation (1). We see, for example, that for a typical Baa-rated security, the possibility of a downgrade over the coming year gives rise to an expected peer group underperformance of -0.74% with a standard deviation of 6.22%.

In Appendix A, a bond's performance relative to its peer group is analyzed in terms of the three possible types of rating movement: downgrade, upgrade, and no change. In this analysis, we assume that we know the probabilities of each of these possible outcomes as well as the conditional mean and standard deviation of relative performance under each. Both the mean and variance of relative performance are thus shown to have three terms, corresponding to these three cases.

We have chosen to focus on the components of mean and variance due to downgrades. On average, of course, the mean outperformance of the peer group (averaged across the entire group) must be zero. The expected underperformance due to downgrades is offset by expected outperformance from the upgraded and unchanged bonds.[5] In terms of variance, the risk of losses due to downgrades is the one of greatest concern to portfolio managers. Variance of outperformance is not nearly as frightening. The isolation of risk due to downgrades can be considered as a form of modeling downside risk as opposed to overall variance.[6] Furthermore, as shown in the next section, when the three contributions to variance are evaluated numerically, we find that the variance due to downgrades is by far the largest component.

[5] This relationship gives us an estimate of the peer group outperformance that can be gained by avoiding downgrades. For example, in Figure 6, the 5.7% probability of being downgraded from Baa generates a peer group underperformance of –0.74%. This is offset by an equal and opposite outperformance coming from the 94.3% probability of no downgrade. The expected outperformance when downgrades are avoided is therefore 0.74% / 94.3% = 0.785%, as shown at the end of Appendix A.

[6] For a benchmarked portfolio, it is not exactly true that there is no downside risk due to upgrades. When index bonds not owned by the portfolio are upgraded, the portfolio underperforms. However, the non-systematic risk due to a particular bond is proportional to the square of the size of the overweight or underweight. Because the overweights for securities in a portfolio are typically much larger than the underweights to index bonds, the risk of downgrades to bonds in the portfolio dominates.

Figure 6. **Parameters of the Model for Downgrade Risk (Annualized)**

Initial Rating	Downgrade Probability	Statistics of Losses Experienced by Downgraded Bonds: Avg	Std Dev	Resulting Statistics for Expected Losses on a Single Bond: Avg	Std Dev
Aaa-Aa	8.13%	-0.30%	2.55%	-0.02%	0.73%
A	5.49	-2.84	6.58	-0.16	1.68
Baa	5.70	-12.92	22.65	-0.74	6.22

While this analysis could be carried out on either a monthly, quarterly or annual basis, we have chosen to work at the annual level. This is the frequency at which the rating agencies typically publish transition matrices. Also, quarterly data might be influenced by seasonal factors, such as the often-observed first quarter outperformance and September/October underperformance of spread sectors.

Alternative Model Based on the Slope of the Quality Spread Curve

In the above analysis, we modeled the return impact of a downgrade based on the observed historical excess returns of downgraded securities. An alternative method for projecting performance consequences of any change in rating could be based on the difference between the average spread levels for the two relevant quality ratings.

Figure 7 shows the average spreads for bonds of different qualities in the Lehman Brothers U.S. Investment-Grade Credit and High-Yield indices as of December 31, 2001. Based on the differences between the average spreads, we can project the returns associated with a particular downgrade. For example, we expect a bond suffering a downgrade from A to Baa over the course of the coming year to see its spread widen by about 75 bp (234-158) over that time. If it has a 5-year spread duration (which is about the average for the Lehman Brothers Credit Index), this will result in a return of –3.77%.

The historical likelihood of these events is reflected by the rating agencies in one-year transition matrices. In Figure 3, we used this data to compute an aggregate number for overall frequency of downgrades; Figure 8 shows the entire transition matrix calculated by Moody's based on data from 1970 through 2001. It shows, for instance, that a bond that started a given year with a single-A rating had a 91.97% chance of remaining in single-A one year later, a 4.84% probability of being downgraded to Baa, and a 0.01% probability of defaulting within one year.

Combining the spread differentials from Figure 7 with the transition probabilities in Figure 8, we can build a rudimentary model of the distribution of returns. Figure 9 shows how this model is used to assess the risk of four hypothetical bonds with 5-year durations. Each column represents a bond of a different quality

Figure 7. **Average Option-Adjusted Spreads by Quality, Lehman Brothers Credit and High Yield Indices**
as of December 31, 2001

Quality	**Number of Issues**	**Average Option-Adjusted Spread** (bp)
Aaa	186	62
Aa	636	92
A	1,585	158
Baa	1,540	234
Ba	480	449
B	493	642
Caa-C	242	2150

whose spread is assumed equal to the average spread for its rating group. Each row in the middle part of Figure 9 gives the return that will be achieved due to a transition to another quality (spread change times duration). For transitions straight to default, we do not assume a total loss of all invested funds. This would ignore the possibility of partial recovery through default proceedings and over-state the expected losses from defaults. We assume a maximum loss of –60% as a result of a downgrade or default, corresponding to an average recovery rate of 40%. To calculate the return statistics, we weight these conditional returns by the transition probabilities from Figure 8 to calculate the mean and standard deviation of returns due to rating transitions. We see that for a Baa bond, the expected loss due to rating transitions is –58 bp with a standard deviation of 426 bp.

Figure 8. **Moody's One-Year Rating Transition Matrix (Adjusted for Withdrawn Ratings),** 1970-2001, in %

Current	**Rating One Year Forward**								**Total**
Rating	**Aaa**	**Aa**	**A**	**Baa**	**Ba**	**B**	**Caa-C**	**Default**	**Downgrades**
Aaa	91.80	7.37	0.81	0.00	0.02	0.00	0.00	0.00	8.20
Aa	1.21	90.73	7.67	0.28	0.08	0.01	0.00	0.02	8.06
A	0.05	2.49	91.97	4.84	0.51	0.12	0.01	0.01	5.49
Baa	0.05	0.26	5.45	88.54	4.72	0.72	0.09	0.16	5.70
Ba	0.02	0.04	0.51	5.57	85.42	6.71	0.45	1.28	8.44
B	0.01	0.02	0.14	0.41	6.69	83.37	2.57	6.79	9.36
Caa-C	0.00	0.00	0.00	0.62	1.59	4.12	68.04	25.63	25.63

Source: Moody's Investors Service.

Figure 9. **Modeled Return Impact of Rating Changes**
Spreads as of December 31, 2001

Initial Characteristics of Hypothetical Bonds				
Quality	**Aaa**	**Aa**	**A**	**Baa**
Duration	5.00	5.00	5.00	5.00
Spread	62	92	158	234
Projected Returns Based on Final Rating Category (%)				
Aaa	0.00	1.52	4.83	8.60
Aa	-1.52	0.00	3.31	7.08
A	-4.83	-3.31	0.00	3.77
Baa	-8.60	-7.08	-3.77	0.00
Ba	-19.37	-17.85	-14.55	-10.78
B	-29.03	-27.51	-24.20	-20.43
C	-60.00	-60.00	-60.00	-60.00
D	-60.00	-60.00	-60.00	-60.00
Model Statistics (bp)				
Average Return	-16	-28	-21	-58
Standard Deviation	64	142	186	426
Expected Return	46	64	137	176
Expected Return/Std Dev	0.72	0.45	0.73	0.41

This result can be viewed as a measure of the extent to which the credit spread compensates investors for the risk of downgrades. The average Baa spread over Treasuries of 234 bp might be interpreted as the expected excess return of Baa credits over Treasuries under a "no change" scenario. However, even absent any systematic change in credit spreads, the effect of ratings transitions carries an expected underperformance of –58 bp. Subtracting this expected loss from the spread of 234 bp, we see that the expected excess return of a Baa bond over Treasuries is only 176 bp. This amount should theoretically compensate investors for taking on the systematic risks of investing in Baa debt: a systematic widening of spreads, or a sudden increase in the rate of downgrades, or defaults. Dividing this expected return by the standard deviation of return due to ratings transitions gives us a measure of expected return per unit of risk. Figure 9 shows that, by this measure, single-A securities have the highest risk-adjusted returns.

How do we compare these results to our observations of peer group underperformance? Recall that in our study of achieved performance of downgraded bonds, we compared their returns to the average return of a peer group consisting of all similar bonds in the index, including the downgraded ones. In this model, we expect the Baa portion of the index to return -58 bp due to ratings transitions. To obtain projected returns relative to this peer group, we shift the whole column of projected returns in Figure 9 by this peer group return. For a Baa bond with an unchanged rating, for example, the return relative to the peer group is 0.58%, while a bond downgraded to Ba will underperform the peer group by –10.20% (= -10.78% - -0.58%). We can then take probability-weighted statistics over all possible downgrade events to obtain the average and standard deviation of performance relative to the peer group among downgraded securities.

For increased accuracy, the analysis illustrated in Figure 9 can be repeated using a finer ratings grid. The difference between the average A spread and the average Baa spread corresponds to a quality difference of a full letter grade, or three "notches". Yet, not every downgrade from A to Baa is accompanied by such a large spread movement. Most downgrades are only a single notch. To address this issue, we replace the transition matrix shown in Figure 8 by a finer-grained transition matrix obtained from S&P, which includes transition probabilities among all credit notches (AA+, AA, AA-, etc.).[7] An array of average spreads for each quality notch as of December 31, 2001 replaces Figure 7.

Figure 10 illustrates the resulting calculation of return statistics relative to the peer group. This computation follows the analysis detailed in Appendix A, in which all ratings transitions are grouped into the three categories of upgrades, downgrades, and no change. Let us focus on the Baa column as an example. For bonds with a Baa rating, there is a 11.8% probability of a downgrade, with a resulting

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[7] Because this matrix includes transitions that are ignored by the broader matrix (such as downgrades from AA to AA-), the probability of an unchanged rating is lower in this matrix. This effect is offset by the fact that the most common transitions are of only a single notch and thus carry smaller spread changes. The matrix used for this analysis was published by S&P in February 2002 and covers data through 2001.

expected return of –832 bp over the coming year if a downgrade occurs. With 78% probability, there will be no change in rating, and no return due to ratings transitions. Upgrades have a probability of 10.2% and a conditional expected return of 225 bp for the year. Computing the weighted expected return from these three cases, we get an overall expected return of –75 bp, which we can consider the expected return of the Baa peer group. Conditional returns for the three cases relative to the peer group can then be calculated as a simple difference. Unchanged bonds will outperform the peer group by 75 bp, downgraded bonds will underperform by –757 bp, and so on. Multiplying these numbers by the probabilities gives the contributions to mean relative performance: for downgrades, we have 11.8% of –757 bp, or –89 bp. The relative performance of the whole peer group must be zero, so the three contributions must cancel each other. The negative performance contributions of downgrades are offset by positive contributions to upgrades and unchanged bonds. Note that the larger of these two positive terms is due to the unchanged bulk of the peer group, which outperforms simply because the peer group return is pulled down by the downgraded bonds. The variance of the relative performance due to downgrades also has contributions from the three possible outcomes. It is clear that here almost all of the variance of performance relative to the peer group comes from the downgraded bonds. This justifies our focus on downgrade risk as opposed to overall risk due to ratings transitions.

Of course, this model is very dependent on the level and slope of the credit curve, which can change dramatically over time. Figure 11 shows the average spreads of

Figure 10. **Spread Differential Model—Calculating Statistics of Return Relative to the Peer Group, due to Ratings Changes**
Fine Transition Matrix, Spreads as of December 31, 2001

Initial Rating	**Aaa-Aa**	**A**	**Baa**
Transition Probabilities			
Upgrade	2.0%	6.8%	10.2%
No Change	86.9%	81.7%	78.0%
Downgrade	11.1%	11.5%	11.8%
Expected Returns due to Rating Transitions			
Upgrade	**53**	**185**	**225**
No Change	0	0	0
Downgrade	**-182**	**-342**	**-832**
Peer Group Return	-19	-27	-75
Contributions to Mean Performance Relative to Peer Group			
Upgrade	1	14	31
No Change	17	22	59
Downgrade	**-18**	**-36**	**-89**
Total	0	0	0
Contributions to Standard Deviations of Relative Performance			
Upgrade	12	67	108
No Change	18	24	66
Downgrade	**135**	**247**	**499**
Total	**136**	**257**	**515**

10-year bullet bonds of different qualities from 1994 through 2001.[8] At the beginning of this period, spreads were quite tight on a historical basis and very stable. Since 1998, we can detect three different types of changes. Spreads have moved higher, they have become more volatile (particularly in 2001), and, perhaps most relevant to our model, the credit curve has steepened considerably. The spread differentials from one quality to the next have increased, giving rise to greater losses for downgraded bonds. Within the investment–grade market, the slope of the curve

[8] Because this model is based on option-adjusted spreads, we do not report results prior to 1994, when our lognormal option pricing model was introduced.

Figure 11. **Historical 10-Year Spreads of U.S. Corporate Bonds, by Quality, versus Off-the-Run Treasury Curve**

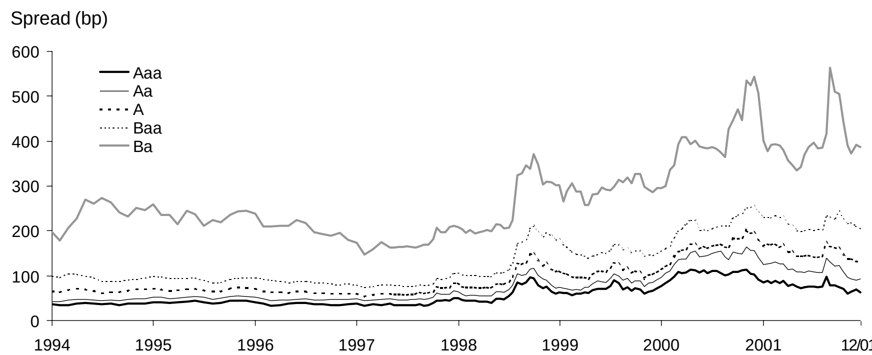


Figure 12. **Spread Differential Model—Results Depend on Spread Levels**

	1994	1995	1996	1997	1998	1999	2000	2001	2002	Avg
Mean Performance of Upgraded Bonds										
Aaa-Aa	6	28	5	21	-1	66	53	75	53	34
A	51	56	49	34	55	90	83	156	185	84
Baa	182	141	170	100	84	166	120	271	225	162
Mean Performance of Downgraded Bonds										
Aaa-Aa	-62	-83	-109	-73	-103	-99	-108	-241	-182	-118
A	-172	-131	-173	-132	-129	-273	-202	-292	-342	-205
Baa	-558	-520	-651	-499	-438	-649	-670	-926	-832	-638
Contribution of Downgrades to Mean Performance										
Aaa-Aa	-6	-8	-11	-8	-11	-10	-11	-24	-18	-12
A	-18	-14	-18	-14	-14	-28	-21	-31	-36	-22
Baa	-60	-56	-70	-53	-47	-69	-71	-100	-89	-68
Std Dev due to Downgrades										
Aaa-Aa	81	86	110	97	98	123	116	159	135	112
A	172	168	191	167	161	208	204	241	247	195
Baa	369	374	421	380	360	442	467	533	499	427
Std Dev Overall										
Aaa-Aa	82	86	111	98	99	124	117	162	136	113
A	174	169	193	168	163	211	206	247	257	199
Baa	382	383	432	386	365	452	475	554	515	438

has been the most volatile between A and Baa; the steep transition from Baa to below investment grade shows even more variability over time.

Figures 9 and 10 illustrate the model using spreads as of the end of 2001. Snapshots of the credit curve taken at different points in time would give very different results. Figure 12 shows the results of the spread differential model with the fine-grained transition matrix using the credit curves at the start of each year from 1994 to 2002. For example, the “2002” column was calculated using spreads as of December 31, 2001. The quantities included in this column are those shown in boldface in Figure 10.

Dependence on time is a major difference between the two models introduced in this section. We have studied historical returns of downgraded bonds over a long time period, and the resulting statistics reflect the net effect of events occurring in very different environments. The model based on spread differentials is based on a snapshot of the credit curve at a given point in time. To compare the two models, we take an average of the model outputs for each of the snapshots shown in Figure 12. These time-averaged results are compared to the results from observed downgrade losses in Figure 13. We see that for single A-rated bonds, the two models agree quite closely. However, the observed losses show greater levels of downgrade risk (both mean and standard deviation) than indicated by the spread differential model for Baa-rated bonds, and lower levels of downgrade risk for Aaa and Aa securities. When bonds are downgraded from investment grade to below investment grade, the market imposes a penalty that is greater than that implied by the spread differentials alone.

We consider the model based on historically observed losses of downgraded bonds to be our primary model. This model requires fewer assumptions and is grounded much more firmly in actual return data. As we proceed to portfolio-level modeling and optimal structuring issues, we will focus on this model.

The model based on spread differentials has one key advantage. By separating the effects of transitions to different quality levels (instead of grouping all downgrades together), it allows us to include the effect of rare events and create a model for the complete distribution of issue-specific returns. We will therefore use this model to calculate confidence intervals for worst case portfolio underperformance.

Figure 13. **Comparing the Two Models of Downgrade Risk**

	From Observed Downgrade Losses 1988-2001	From Credit Spread Differentials (Fine Transition Matrix)	
		Spreads as of 12/31/01	Average 1994-2001
Average Loss			
Aaa-Aa	2	18	12
A	16	36	22
Baa	74	89	68
Standard Deviation			
Aaa-Aa	73	135	112
A	168	247	195
Baa	622	499	427

PORTFOLIO DOWNGRADE RISK: ABSOLUTE AND RELATIVE TO A BENCHMARK

Let us now extend our model from a single bond to a portfolio. We will prove that diversification helps reduce the standard deviation of loss and hence the chance of catastrophic losses.

We start with an equally-weighted portfolio of n bonds of the same credit quality and denote by L_n this portfolio's loss (underperformance of its peer group) due to downgrades. Assume that each bond has the same probability p of a downgrade, the size of the loss has the same distribution for all bonds, and the results are uncorrelated. In this case, the mean loss on the portfolio is the same as for the single bond, but the variance is reduced by a factor of n:

(2)
$$\langle L_n \rangle = p\mu$$
$$\sigma_{L_n}^2 = \frac{1}{n}\sigma_{loss}{}^2$$

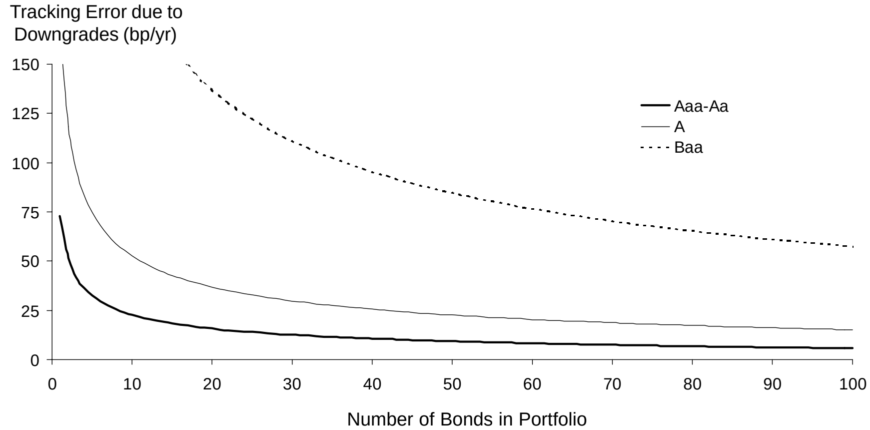
It is important to distinguish between the absolute risk of a portfolio (standard deviation of portfolio return) and the risk relative to a benchmark (standard deviation of performance difference or *tracking error*). Let us assume that the benchmark for this portfolio is a broad-based index of N bonds of the same credit quality. For simplicity, the index is assumed to be equally weighted as well, but consists of a greater number of bonds ($N > n$). To analyze portfolio risk relative to the benchmark, we focus on the difference between the two returns. As the expected performance is the same for both the portfolio and the benchmark, the expected outperformance is zero. The standard deviation of the performance difference, known as tracking error (TE), is given by:

(3)
$$TE^2 = \left(\frac{1}{n} - \frac{1}{N}\right)\sigma_{loss}^2$$

A comparison of the equations for absolute (2) and relative risk (3) shows that when we assume no correlations of downgrade losses among different issuers, there is very little difference between the two risk estimates until the portfolio size starts to approach that of the benchmark. In Figure 14, we plot portfolio risk as a function of the number of bonds in a portfolio, for single-quality portfolios and benchmarks from each rating group. The graph shows the relative risk (tracking error) due to downgrades, but would look much the same for absolute downgrade risk.

Figure 14 clearly demonstrates that due to the higher standard deviation of loss for downgrades from lower qualities, greater diversification is required to achieve a given level of risk. For example, a tracking error of 25 bp/yr can be achieved by a portfolio of 8 securities rated Aaa to Aa, or about 40 single-A bonds, but would require well over 100 bonds in a Baa portfolio.

Figure 14. **Risk due to Downgrades, as a Function of Portfolio Size**
by Credit Quality



Correlations

Of course, downgrades of different issuers are not totally uncorrelated events. During economic downturns, many companies may simultaneously suffer financial hardship that could result in downgrades. An industry-wide slump might lead to downgrades of several companies with similar lines of business. Evaluation of the correlation between any two credits is one of the most challenging problems facing credit risk practitioners.

Practitioners understand that modeling correlations among different industries and issuers is critical, and that by ignoring these correlations an asset manager can severely underestimate risk. We will see that while this is certainly true for measures of absolute risk, it is a far less important issue when evaluating risk relative to a benchmark.

When there are positive correlations among issuers, the effect of diversification is diluted, and the portfolio can not be insulated from risk as well as equation (2) implies. To illustrate this, we introduce a very simple correlation model, in which a single correlation coefficient ρ (rho) represents the correlation between the losses of any two bonds. As shown in Appendix B, the mean and variance of portfolio loss due to downgrades under this model are given by:

(4)
$$\langle L_n \rangle = p\mu$$
$$\sigma_{L_n}^2 = \sigma_{loss}{}^2 \left(\frac{1}{n} + \rho \frac{n-1}{n} \right)$$

Note that in the variance of portfolio downgrade loss shown in equation (4), there is a term $\rho\sigma_{loss}^2$ that does not disappear for a large n. To the extent that two bonds are correlated, diversification offers little reduction of risk. In the extreme case where all bonds are perfectly correlated ($\rho = 1$), the portfolio risk is exactly the same as the risk of a single bond ($\sigma_{L_n} = \sigma_{loss}$), regardless of how many bonds are in the portfolio.

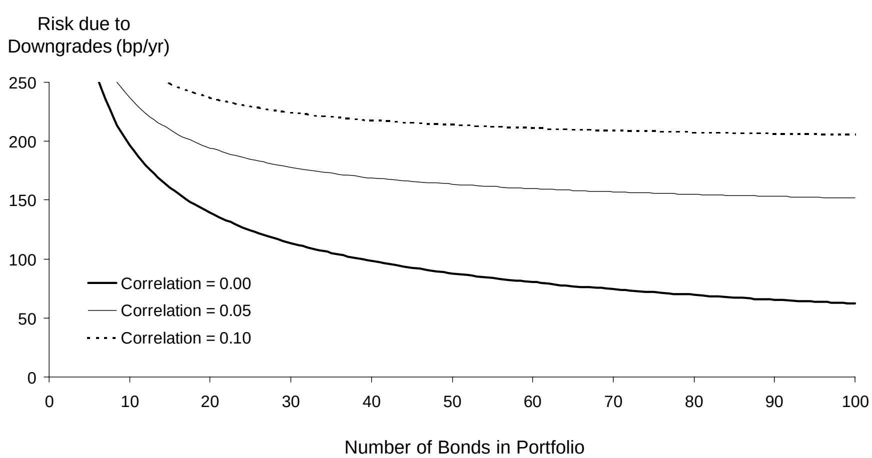
Figure 15 shows how the absolute risk of a Baa portfolio depends on the assumed value of the correlation coefficient ρ. While a portfolio of 100 Baa bonds has an absolute risk of downgrade loss of 62 bp in the uncorrelated case, the assumption of a positive correlation of just 5% more than doubles that risk to 152 bp.

As shown in Figure 15, an increase in the correlations of downgrade losses among different issuers leads to an increased level of absolute risk that cannot be diversified away. However, if risk is measured versus a benchmark, this is no longer true.

Positive correlations among the downgrade risks of different issuers essentially turn these risks into a systematic risk factor that affects the benchmark as well as the portfolio. This risk cannot be eliminated by diversification and increases the likelihood of extreme negative returns for both indices and portfolios. In fact, the increased correlations between bonds imply that the subset of index issuers held in a portfolio will be more likely to track the index closely.[9]

[9] In this study, we focus on the relationship between diversification and non-systematic risk. We implicitly assume that the portfolio matches the systematic risk exposures of the benchmark. For example, a secular widening of corporate spreads will not cause underperformance as long as the portfolio has matched the benchmark's exposure to corporates. For settings in which this is not the case, such as when a portfolio uses credit against an all-government benchmark, there is indeed a systematic risk exposure. In these cases, positive correlation among different credits will certainly increase tracking error.

Figure 15. **Absolute Risk: Standard Deviation of Portfolio Excess Returns due to Downgrades, for Portfolios Containing Different Numbers of Baa Bonds**



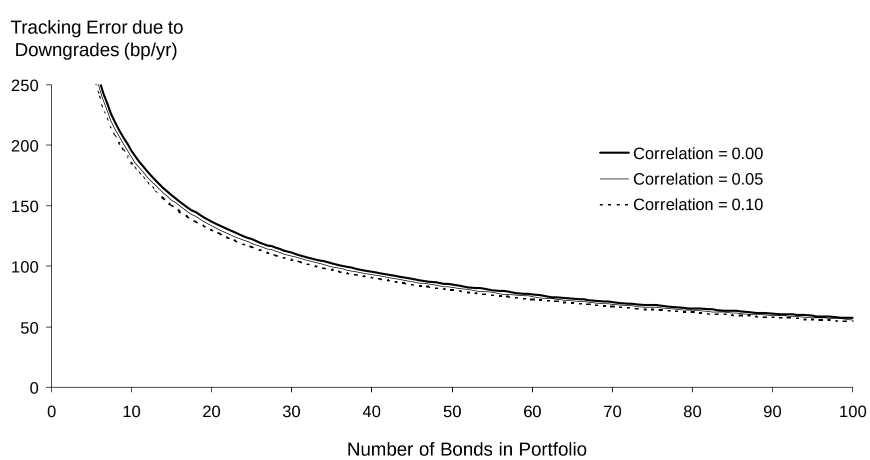
When we revisit the calculation of relative risk with this simple constant correlation assumption, we find that the tracking error is given by:

$$\sigma_{TE}^2 = \sigma_{loss}^2 (1-\rho)\left(\frac{1}{n} - \frac{1}{N}\right), \tag{5}$$

as shown in Appendix B. This function is plotted in Figure 16 for Baa portfolios (against a benchmark of $N = 500$ issuers), using several values of the correlation coefficient ρ. We see that increasing correlations have a much smaller (and opposite) effect on tracking error than on the absolute risk of portfolio loss. Thus, while careful estimates of correlations among the loss risks of different issuers are necessary for accurate measures of absolute risk, such as VAR (absolute magnitude of loss at a given probablility), they are far less critical for projecting tracking errors.[10]

[10] The assumption of a single correlation coefficient that relates any two credits is a gross simplification. It is certainly conceivable that a portfolio could contain a set of bonds that are correlated to each other much more strongly than to the index at large. This could occur, for example, if there is a large concentration of issuers within a single industry. Once again, such a concentration could be viewed as a systematic risk exposure. If the goal is to minimize tracking error, the portfolio should mimic the benchmark's industry exposures to the extent possible. If a particular sector is overweighted to express an investment view, there is always the risk that it will result in underperformance.

Figure 16. **Risk Relative to Benchmark: Correlation Matters Little; Tracking Error due to Downgrades, as a Function of Number of Bonds, Baa Portfolio versus Baa index**



Estimating Confidence Bounds

We characterized the risk due to downgrades using the standard deviation of either absolute or relative return. Another way of looking at risk is in terms of worst case returns or confidence bounds. What is the probability that the portfolio will underperform its benchmark by more than a specified amount? When can an asset manager be 95% certain that it will not?

Very often, such questions are addressed using the standard normal distribution. If underperformance is normally distributed with zero mean, the manager can conclude with 95% confidence that the portfolio will not underperform its benchmark by more than 1.64 times the tracking error. Yet, both of the modeling approaches detailed above for the single bond case make it abundantly clear that excess returns of corporate bonds are far from normally distributed. The large negative tails of the performance distribution could lead to severe underestimates of risk if confidence bounds are based on the normal distribution.

However, if we continue to assume independence between the returns of different bonds, then the law of large numbers allows the distribution of *portfolio* performance to converge towards a normal distribution. Regardless of the underlying distribution for a single bond, an average over n independent, identically distributed variables converges to a normal distribution as n grows towards infinity. It is therefore safe to use confidence bounds based on the normal distribution for very large portfolios, but not for very small ones. To help evaluate how many bonds must be in a portfolio before the normal distribution can be used, we need to model the precise shape of the portfolio return distribution.

To approximate the complete distribution of return relative to the peer group for a single bond, we begin with our estimate of returns due to all possible credit transitions using the spread differential model. To convert this into a continuous distribution, we add an additional source of non-systematic risk that represents the natural spread volatility of bonds whose credit rating remains unchanged.[11] For each possible credit transition, this additional source of volatility is assumed to cause a normally distributed dispersion around the projected return shown in Figure 9. Due to the discrete nature of the credit transitions, however, the overall distribution of returns for a single bond is still very different from normal. We then convolve this distribution upon itself[12] to obtain the distribution for the average return of a portfolio of two identical, independent bonds. By repeating this procedure k times, we can obtain a distribution for a portfolio of 2^k bonds of the same quality. Figure 17(a) shows the distribution obtained for a portfolio of 16 Baa bonds. Because this is a distribution of relative returns, the mean is zero, yet the

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[11] This estimation is based on the observed spread volatilities of bonds that did not experience ratings changes (see Figure 22).

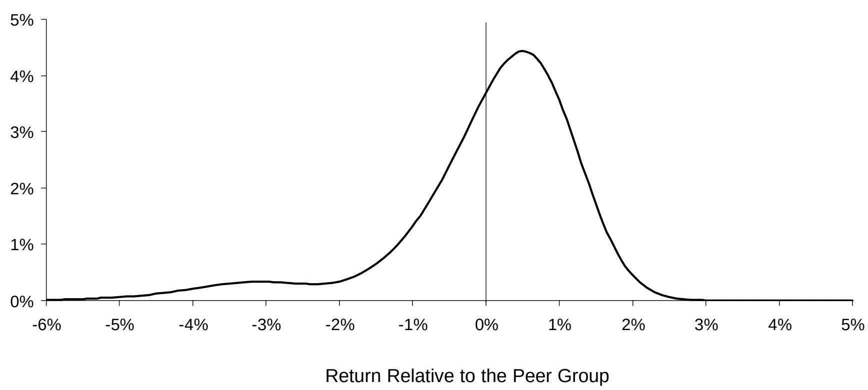
[12] For any two independent random variables X and Y, the distribution of the sum $Z = X + Y$ can be obtained by the convolution of the two distributions: $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$. If $f_X(x)$ is taken to be the distribution of relative return for a 4-bond portfolio, for example, then this relationship can be used to numerically evaluate the distribution for an 8-bond portfolio.

peak occurs at a small positive value corresponding to an unchanged rating. The distribution has a large negative tail, with a non-negligible probability (0.38%) of underperformance by more than 500 bp. The distribution also exhibits a (much smaller) positive tail, corresponding to gains resulting from upgrades. The distribution for a 128-bond portfolio in Figure 17(b) is much more symmetrical and is much closer to the normal distribution.

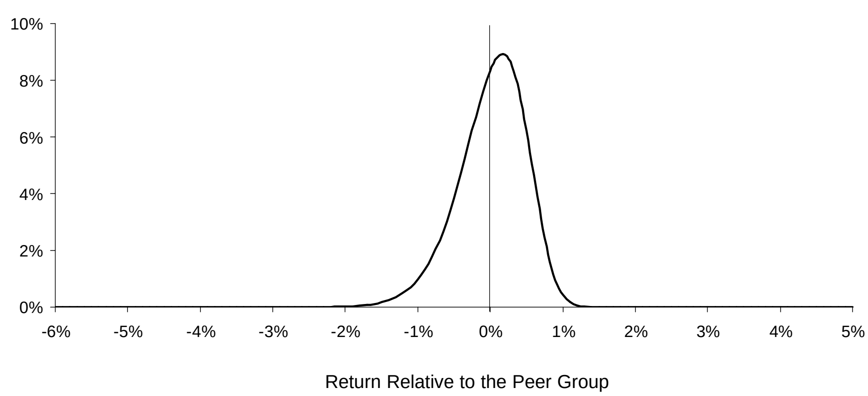
The distribution of portfolio returns can be used to derive confidence bounds on worst case portfolio performance. Figure 18(a) shows that for the 16-bond portfolio, if a high level of confidence is desired, then the use of the normal distribution to determine the confidence bound can severely underestimate the risk of underperformance. For instance, under the normal distribution, the worst case performance with 99% confidence is –3.08%. However, analysis of the full distribution from the model indicates that in order to have 99% confidence of achieving better than the worst case return, the bound must be set at –4.50%. At the 95% level, the confidence bound implied by the normal distribution would be too tight by 75 bp.

Figure 17. **Modeled Distribution of Return Relative to the Peer Group for Portfolios of Baa Securities**

a. **Distribution of Relative Return for a 16-Bond Baa Portfolio**



b. **Distribution of Relative Return for a 128-Bond Baa Portfolio**

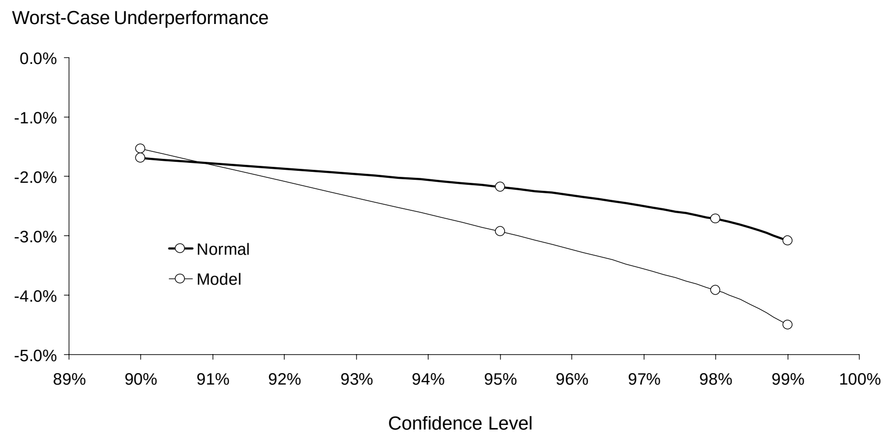


For a portfolio of 128 bonds, as Figure 17(b) demonstrated, the distribution is much closer to normal, and the confidence bound drawn from the normal distribution corresponds much more closely to the one obtained from our model distribution. Combine this with the fact that the tracking error decreases as well, and we find in Figure 18(b) that even at a confidence level of 99%, the confidence bound from our model is only 19 bp worse than the one from the normal distribution.

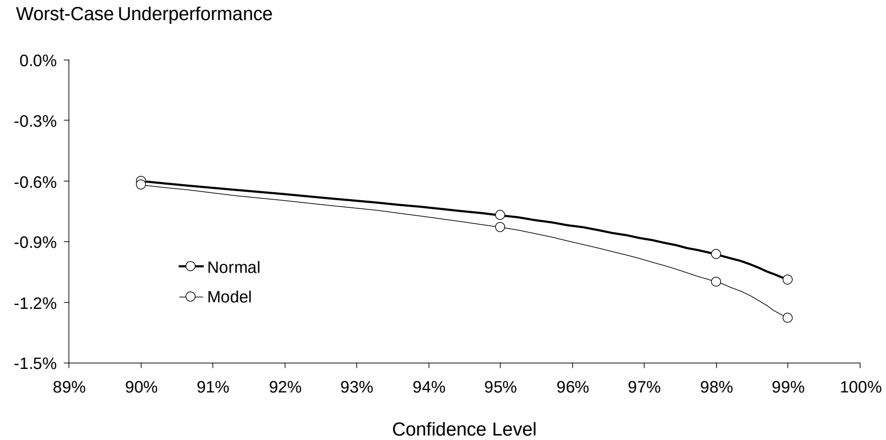
This investigation gives us confidence that for portfolios of moderate size, using the normal distribution to build worst case bounds will produce reasonable results. The level of confidence required will determine how large a portfolio must be to justify the use of this approximation.

Figure 18. **Worst-Case Bounds on Underperformance due to Downgrades for Different Levels of Confidence, from Model Distribution and Estimated from Normal Distribution**

a. **16-Bond Baa Portfolio**



b. **128-Bond Baa Portfolio**



OPTIMAL PORTFOLIO STRUCTURE

Equipped with this model of downgrade risk, we now turn our attention to issues of portfolio structure. We look for the optimal number of bonds in various quality segments of a portfolio and the corresponding diversification constraints that should be imposed on them. We investigate several different formulations of the problem, which correspond to different investment goals. First, for a portfolio containing a given number of bonds, we show how to track an index with the least possible risk due to downgrades. Next, to find the optimal number of bonds, we introduce a model for the value of credit research and maximize the information ratio.

To keep the focus of this study on non-systematic risk, we limit the portfolio optimization problems to a single idealized form. We assume that the benchmark includes all securities in the Lehman Brothers U.S. Credit Index and that the portfolio takes no systematic risks relative to the benchmark. Thus, the portfolio is assumed to match the benchmark exposures to all systematic risk factors, including yield curve, industry, and quality. In addition, we continue to assume that both the portfolio and benchmark are equally weighted within each quality group. The only difference between the portfolio and the benchmark is that the portfolio is formed from only a subset of the index bonds within each quality group.

We illustrate the construction of an idealized $1 billion credit portfolio of 100 bonds in Figure 19. Partitioning the Lehman Brothers U.S. Credit Index by quality, we discover that approximately 26% of the index is concentrated in issues rated Aaa or Aa, 39% is rated A, and about 35% is rated Baa. We assume that the portfolio matches this distribution of market value. The only question is how many bonds will be used to fill the market allotment for each quality, e.g. the $350 million of Baa debt. This example simply assumes the number of bonds of each quality to be proportional to the index weight, so that a uniform position size of about $10 million is maintained across all credit qualities.

Given the number of portfolio securities in each credit quality, we use the models developed in the previous section to separately calculate the tracking error due to

Figure 19. **$1 Billion U.S. Credit Index Proxy (Equal Weights to All Securities)**

Quality	**Percent of Index**	**Number of Bonds**	**Tracking Error due to Downgrades** (bp/yr)	**Position Size** ($ million)	**Position Size** (%)
Aaa-Aa	26.3	26	14	10.0	1.0
A	38.5	39	26	10.0	1.0
Baa	35.2	35	102	10.0	1.0
Total	100.0	100	38		

Worst-Case Bound on Underperformance due to Downgrades (bp, 95% Confidence): **-62** downgrades within each quality. Assuming independence among the tracking errors in the different qualities, the overall tracking error TE can then be calculated from the tracking errors in each quality TE_q according to the index weights w_q:

(6) $$TE^2 = \sum_q w_q^2 TE_q^2$$

We can see that when the portfolio is constructed using equal weights as in Figure 19, the tracking error in the Baa sector is much higher than in the remainder of the portfolio because of the greater risk of loss due to downgrades. Diversification among qualities reduces the portfolio tracking error to just 38 bp/yr. This implies that in the worst case (with 95% confidence), downgrades could cause the portfolio to underperform the index by 62 bp, using the normal distribution as an approximation. However, it is clear that a smaller tracking error could be achieved by a 100-bond portfolio if the Baa portion were spread among a greater number of issuers.

Minimizing Tracking Error

Our first optimization problem can be stated as follows: for a 100-bond portfolio, how many bonds of each quality should be purchased to achieve the lowest tracking error due to downgrades? We formulated this as an integer programming exercise and solved for the number of bonds to be allotted to each quality. The optimal allocation, shown in Figure 20, chooses more than twice as many Baa-rated bonds as the equal-weighted position of Figure 19. The size of each Baa exposure is reduced to $4.8 million, or 0.5% of the portfolio, while the Aa and Aaa exposures are allowed to grow to 4.4% each. (The total allocation to each quality continues to match that of the index.) As a result of this reallocation, we reduced the tracking error due to the risk of downgrades of Baa securities to below investment grade from 102 bp/yr to 69 bp/yr. In the two higher-quality groups, where the downgrade losses are typically smaller, there are slight increases in risk because of increased concentration; the net effect is a lower overall tracking error due to downgrades from 38 bp/yr to 29 bp/yr.

Figure 20. **$1 Billion U.S. Credit Index Proxy (Optimal Allocation of Bonds to Qualities)**

Quality	**Percent of Index**	**Number of Bonds**	**Tracking Error due to Downgrades** (bp/yr)	**Position Size** ($ million)	**Position Size** (%)
Aaa-Aa	26.3	6	30	43.9	4.4
A	38.5	21	36	18.3	1.8
Baa	35.2	73	69	4.8	0.5
Total	100.0	100	29		

Worst-Case Bound on Underperformance due to Downgrades (bp, 95% Confidence): **-48**

We repeated this exercise for portfolios of different numbers of bonds and discovered the results to be quite consistent. As shown in Figure 21, for portfolio sizes ranging from 50 to 500 bonds, the optimal portfolio structure is such that about 72% of the bonds are selected from Baa-rated issuers. In Appendix C, we demonstrate that if we relax the constraint on the number of bonds of each quality being integers, this problem solves in closed form. The optimal solution is such that the ratio of position sizes in different qualities is inversely proportional to the ratio of the respective volatilities of downgrade losses. The volatility of downgrade loss for Baa bonds, as seen in Figure 13, is 622 bp/yr, about 4 times the volatility for single A bonds, and about 9 times that of Aa-Aaa qualities. As a result, the position sizes for single A bonds in the optimal portfolio shown in Figure 20 are almost 4 times the Baa position size of $4.8 M, and those for Aa-Aaa are 9 times as large.[13]

The implication for investment policy is clear. When a position size maximum is imposed as a diversification requirement, insisting on smaller limits for lower quality ratings is more important. In higher qualities, where the risk of catastrophic events is smaller, larger concentrations can be tolerated.

Diversification of "Natural" Spread Volatility

The risk of downgrades, which is the main focus of this paper, is not the only source of non-systematic risk. Even securities that do not experience ratings changes exhibit natural spread volatility. This source of return variance may also motivate portfolio diversification. From the same data set that was used to quantify downgrade risk, we isolated a set of bonds each month whose ratings remained unchanged for at least the next six months. We measured the standard deviation of spread changes across all the bonds within each peer group every month and averaged this quantity over time. The resulting spread volatilities are shown in Figure 22. This volatility tends to increase for lower-rated credits, as was the case for downgrade risk, but in a much less drastic manner. For instance, overall volatility in long spreads was 8 bp/month for Aaa-Aa, 9 bp/month for A, and 13 bp/month for Baa. A similar pattern holds for intermediate spreads. Spreads on short-duration bonds were more volatile and exhibited some variation from this pattern. Short spreads are relatively less stable, probably because small pricing differentials can imply relatively large spread changes for short bonds. We do not consider this to be a very significant effect.

[13] The ratios do not match exactly because Figure 20 is based on an integer number of bonds.

Figure 21. **Structure of Optimal Credit Index Proxies for Different Numbers of Bonds**

Number of Bonds	**50**	**100**	**150**	**200**	**500**
Aaa-Aa	3	6	10	13	34
A	11	21	32	43	114
Baa	36	73	108	144	352
Tracking Error due to Downgrades (bp/yr)	42	29	23	19	10

When we add this variance to the one from downgrade risk and calculate the overall variance, we obtain a set of non-systematic risk volatilities that are much less differentiated by quality than those from downgrade risk alone. This comparison (Figure 23) shows that while downgrade risk dominates in Baa, natural spread volatility may dominate in the higher qualities. The downgrade risk volatilities are obtained from Figure 13, using both the model based on historical returns of downgraded bonds and the model based on spread differentials. The return volatilities for "other non-systematic risk" are obtained from Figure 22 as follows. The short-dated volatilities are disregarded, and the numbers from the two longer cells are averaged to obtain a monthly spread volatility per quality. We then multiply these numbers by a nominal duration of 5 to get a monthly return volatility and by $\sqrt{12}$ to annualize. The resulting annual return volatilities are displayed in the middle column of Figure 23 and then combined with the downgrade risk to obtain the total non-systematic risk.

Figure 22. **Cross-Sectional Spread Change Volatility of Bonds with Unchanged Ratings,** April 1990-December 2001, bp/month

		Industrial	Financial	Utiilty	Non-U.S.	Average
	Aaa-Aa	16	18	15	12	15
Short	A	15	12	18	10	14
	Baa	19	17	19	18	19
	Aaa-Aa	8	8	10	8	8
Intermediate	A	11	9	11	9	10
	Baa	15	12	14	15	14
	Aaa-Aa	8	8	8	7	8
Long	A	9	11	9	7	9
	Baa	14	14	11	12	13

Figure 23. **Downgrade Risk versus Other Non-Systematic Risk**

	Downgrade Risk	Other NonSyst Risk	Total NonSyst Risk
From Observed Performance of Downgraded Bonds			
Aaa-Aa	73	141	159
A	168	165	236
Baa	622	231	664
Position Size Ratio	9:4:1		4:3:1
From Spread Differentials			
Aaa-Aa	112	141	180
A	195	165	256
Baa	427	231	486
Position Size Ratio	4:2:1		3:2:1

We can repeat the portfolio structuring exercise of this section to minimize overall non-systematic risk instead of downgrade risk alone, using either one of the basic models for downgrade risk. The optimal ratio of A and Baa position sizes turns out to be quite stable. For either of the volatility assumptions shown, Baa position limits should be 2 to 3 times smaller than their A counterparts. The relative amount of risk in the Aaa-Aa range depends somewhat more strongly on whether we include non-systematic risk other than downgrade risk and on the downgrade model used. The optimal ratio of these position sizes to those in A-rated bonds ranges from more than 2 to slightly more than 1.

Sufficient Diversification by Sector

Throughout this report, we have focused on the relative degree of diversification required in three broad quality groups. Of course, each of these groups contains a heterogeneous mix of securities from different industries, maturities, and rating tiers. Perhaps our conclusions need to be further refined along one or more of these dimensions? We analyze the performance of downgraded bonds (Figure 1) using several finer-grained partitions.

In Figure 24, each of the three broad quality groups is divided into three duration cells: short (duration 4 years or less), intermediate (4-7 years), and long (over 7 years). We see that, as might be expected, within each quality group both the mean and standard deviation of underperformance tend to increase with duration. Within each duration cell, the pattern is the same as in the overall data set: significant underperformance for bonds downgraded from Baa and A, less so for Aaa-Aa. The ratio of loss volatilities in different qualities, however, is far more pronounced in the short duration cell than in the longer ones. While the overall ratios of volatilities (and hence of the optimal position sizes) were 9:4:1, the ratios by duration cell are 16:7:1 for short durations, 10:3:1 for intermediate, and 6:3:1 for long durations. Less diversification is required in short-duration highly rated securities.

Figure 24. **Additional Studies of Relative Performance of Downgraded Bonds: Duration/Quality Breakdown**
August 1988-December 2001, Annual Results

	Initial Quality	**Observations**	**Mean** (%/mo)	**Std Dev**	**T-Stat**
Short	Aaa-Aa	242	0.01	0.43	0.78
	A	274	-0.13	1.05	-6.97
	Baa	170	-0.94	7.06	-6.04
Intermediate	Aaa-Aa	292	-0.03	0.60	-2.52
	A	393	-0.23	1.88	-8.55
	Baa	219	-1.05	5.81	-9.25
Long	Aaa-Aa	182	-0.06	1.13	-2.49
	A	307	-0.33	2.38	-8.54
	Baa	134	-1.30	6.93	-7.51

Figure 25 shows the dependence of losses due to downgrades on sector. Here the bulk of the downgrade risk in the Baa quality group comes from the industrial and utility sectors. Baa bonds downgraded from these two sectors exhibit the greatest underperformance as well as the greatest volatility of underperformance. The risk in industrials is about double that in the finance and non-U.S. sectors; in utilities it is about double that of industrials. It is interesting to note that if we exclude 2001 data, industrials show the most risk, and the utility sector is about as risky as the finance sector. The explanation for this drastic increase in utility sector risk can be found in Figure 5. Of the 25 worst single-month performers in our sample, four are utility sector issuers that suffered big losses in 2001: Enron, AT&T Canada, AES Drax Holdings and NGC Corp. Clearly, there is room to differentiate by subsector as well. Within the utility category, this short list of issuers covers several very different industries. Within industrials, technology and telecommunications issues are widely perceived to be more risky than energy and consumer products. However, if the data is partitioned too finely, there may not be enough observations in each cell to reach statistically significant results.

The results shown for non-U.S. credits in Figure 25 illustrate the dangers of slicing the data too thin. For example, there are only 27 observations of downgrades from Baa in this sector – not enough to form a statistically significant estimate of downgrade risk. Meanwhile, the results indicate a very large risk of loss for downgrades from single-A. This is due entirely to the Korean crisis of 1997, in which more than two dozen issues were downgraded directly from single-A to below investment grade. Because of the spurious results that can appear when there are too few bonds in a cell, we do not partition the data by duration, sector and quality simultaneously – too many cells would have insufficient data.

In terms of credit quality, how does downgrade risk depend on the finer distinctions among the different tiers of a given letter grade? Figure 26 shows the mean and standard deviation of losses due to downgrades as a function of rating tiers. As in our coarser partition by quality, the general trend is that risk increases with lower

Figure 25. **Additional Studies of Relative Performance of Downgraded Bonds: Sector/Quality Breakdown**
August 1988-December 2001, Annual Results

	Initial Quality	**Observations**	**Mean** (%/mo)	**Std Dev**	**T-Stat**
	Aaa-Aa	174	-0.12	1.02	-5.57
Industrials	A	535	-0.20	1.48	-11.09
	Baa	328	-1.02	5.34	-11.94
	Aaa-Aa	241	0.01	0.71	0.72
Finance	A	227	-0.13	1.02	-6.69
	Baa	65	-0.59	2.98	-5.52
	Aaa-Aa	226	0.02	0.47	1.75
Utility	A	132	-0.21	1.88	-4.43
	Baa	103	-1.75	10.52	-5.86
	Aaa-Aa	75	-0.02	0.33	-1.37
Non-U.S.	A	80	-0.79	4.06	-6.02
	Baa	27	-0.44	2.94	-2.68

Figure 26. **Additional Studies of Relative Performance of Downgraded Bonds: Fine Quality Breakdown**
August 1988-December 2001, Annual Results

Initial Quality	**Observations**	**Mean** (%/mo)	**Std Dev**	**T-Stat**
Aaa-Aa1	321	0.02	0.82	1.28
Aa2	377	-0.02	0.60	-2.69
Aa3	552	-0.03	0.63	-4.36
A1	701	-0.07	1.05	-6.35
A2	756	-0.13	1.28	-9.31
A3	509	-0.10	1.20	-6.66
Baa1	435	-0.36	3.36	-7.79
Baa2	371	-0.19	1.94	-6.53
Baa3	205	-0.56	3.54	-7.88

ratings. There are two anomalies: there seems to be more risk in the A2 tier than in A3, and more in Baa1 than in Baa2. It is not clear what, apart from the limited sample size, might have caused this phenomenon, but we do not believe this to be a significant finding.

Maximizing Information Ratio ("When Is a Portfolio Too Diversified?")

It is apparent from the above discussion that as a portfolio becomes more diversified, the exposure to event risk decreases. At what point should we stop? One approach to finding the "right" number of bonds in a portfolio is to decide how much tracking error due to downgrades can be tolerated. The number of bonds that can deliver a particular tracking error can be looked up in Figure 21.

But what about investors who don't have a preconceived limit on tracking error? Can we find the optimal amount of diversification? One possible conclusion is that a portfolio should own as many issuers as possible. However, we have seen that the benefits of increased diversification decrease as the number of bonds increases. For instance, adding 50 names to the 50-bond portfolio shown in Figure 21 decreases tracking error by 13 bp/yr, while adding yet another 50 bonds gives a further decline of only 6 bp. At the same time, there are several types of costs that are incurred by increasing the number of bonds in a portfolio.

For "fully indexed" portfolios whose goal is to passively replicate index returns, it is indeed optimal to match the index composition as closely as possible. This is the approach used by the largest index funds. In this setting, the limits to diversification are practical ones. If bond positions become so small that the transactions are considered "odd lots," there will be a significant increase in transaction costs. Also, many older and smaller index issues are illiquid. This complicates the effort to maintain index exposures in the face of portfolio inflows and outflows. These considerations prevent all but the largest funds from pursuing the "fully indexed" approach to managing credit portfolios.

A far greater number of portfolios follow an "enhanced indexing" approach, in which the goal is to outperform the index by a modest amount while limiting the tracking error. In this setting, an increase in the number of issuers to be included in the portfolio is likely to entail an increase in the cost of credit research. In addition, a requirement to purchase a greater number of securities will dilute the value of credit research. Once the managers have purchased all highly recommended securities, further diversification is possible only by adding issuers that are considered to be trading rich or those with a neutral (or even negative) outlook. If an asset manager expects the portfolio to outperform the benchmark based on successful security picking, this outperformance will tend to decrease as a greater number of bonds are added to the portfolio.

Qualitatively, then, the "right" amount of diversification will be determined by the trade-off between its two main effects: risk reduction and dilution of outperformance. One way to express this goal quantitatively is to maximize the information ratio, i.e., the ratio of expected outperformance to tracking error. To do this, however, we need to model the value of credit research. Such a model should estimate expected outperformance as a function of the number of bonds in a portfolio. We propose two such models.

A Linear Model for the Value of Credit Research

Our first model for the value of credit research assumes that the expected outperformance of a security is a linear function of analyst preference. Assume that the task given to the credit research team is to rank all the issuers in a particular rating group by preference. Let the variable x represent a given issuer's rank on a smooth scale where 0 represents the most recommended issuer and 100% the least. (The intuition behind this unusual convention is that if one were to buy 5% of the market, one would buy the top 5% by analyst's recommendations.) The function $f(x)$, shown in Figure 27(a), gives expected outperformance as a function of the analyst's ranking. The favorite pick is assumed to outperform by b basis points (In this example b is 78 bp/year; the source of this assumption will be discussed later with Figure 29); the issuer ranked lowest is assumed to underperform by the same amount; and the issuer in the middle earns the index average. Of course, there is plenty of volatility around these values, with magnitude much larger than b. But the assumption is that following the analysts' recommendations should bias the portfolio towards positive outperformance (by finding cheap valuations, predicting tightening, avoiding downgrades, etc.)

Now let us assume that the portfolio is constructed as an equally weighted blend of bonds from the n highest-ranked issuers.[14] The expected outperformance of the resulting portfolio, as a function of the number of bonds, is shown in Figure 27(b).

Intuitively, we see that if very few bonds are used, the outperformance will have an expected value close to b. As more and more bonds are added, the benefits of credit research are diluted. As we approach the middle of the rankings, we are adding bonds with expected outperformance close to zero; if we insist on including

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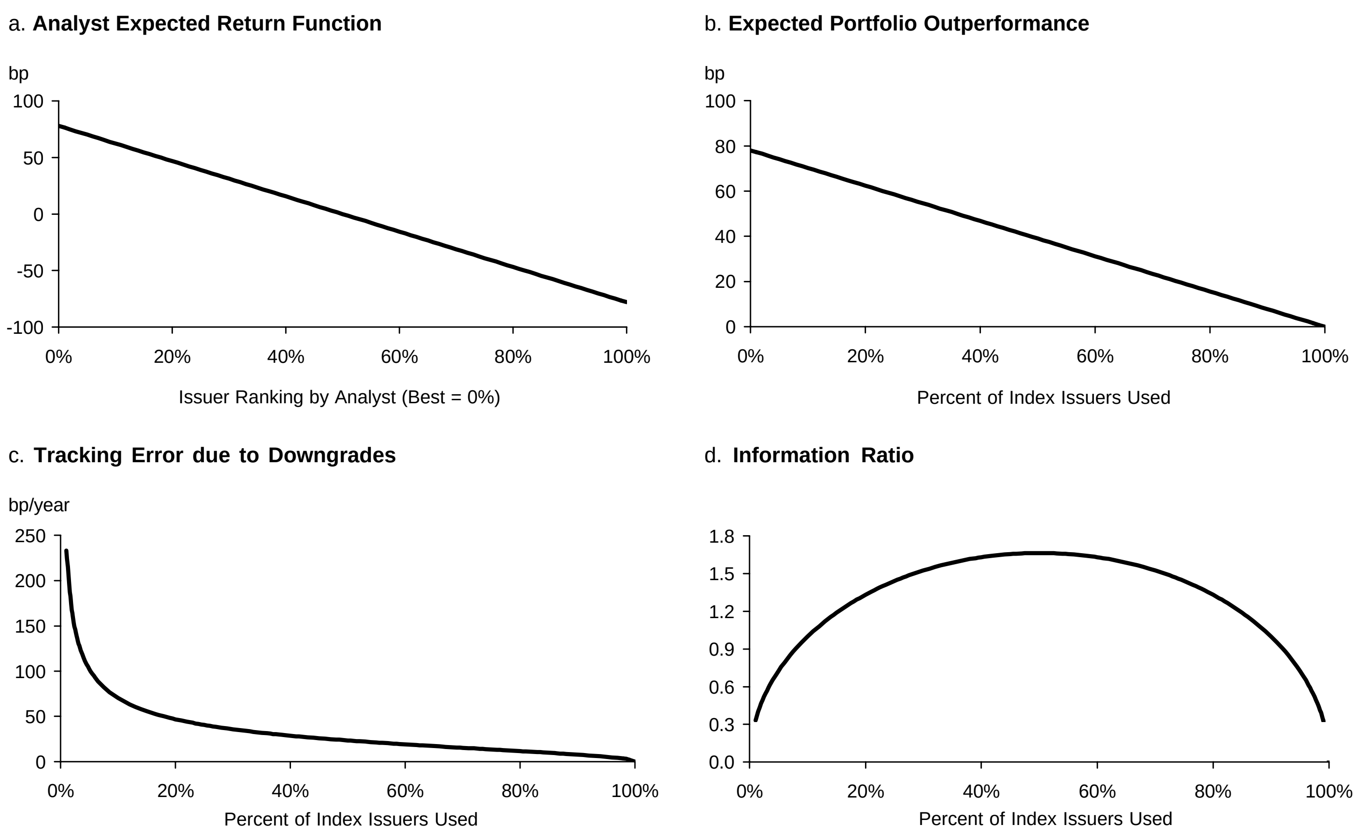
[14] In this section, we deliberately use the words "bonds" and "issuers" interchangeably. We analyze performance in terms of the number of bonds in the portfolio; yet this is compared to the number of issuers in the benchmark. The key determinant of event risk is the exposure to issuers; to maximize issuer diversity for a given number of transactions, a portfolio will hold only one bond from each issuer.

bonds from more than half of the issuers for diversification purposes, we are adding bonds that are expected to underperform. In the limit, when the portfolio includes all the issuers in the index $(n = N)$, the risk is minimized, but the expected outperformance shrinks to zero as well.

In this model, the expected outperformance of the portfolio is a decreasing linear function of the number of issuers selected; the portfolio risk shown in Figure 27(c) also decreases with the number of bonds, but in a non-linear manner. If we look at the expected information ratio (expected outperformance / tracking error), we find that as we increase the number of bonds, the dominant effect at first is the decrease in risk, leading to increasing ratios. As the number of bonds grows, the risk function saturates, and eventually the steadily increasing cost of diversification starts to dominate. Any increase in the number of bonds beyond that point causes the information ratio to decrease. As seen in Figure 27(d), the maximum information ratio occurs at about half of the issuers. For a uniform (single-quality) bond universe, it can be shown that the maximum information ratio is obtained by purchasing bonds from exactly half of the issuers in the index (see the proof in Appendix D). This is true regardless of the value assumed for b, the expected outperformance of the most highly recommended bond.

Figure 27. **Linear Model for the Value of Credit Research, Baa Portfolio versus Baa Index**



When applied to a portfolio and index of mixed quality, we once again find that when evaluating the risk/return trade-off over different qualities simultaneously, it is optimal to diversify more in the lower qualities. For a given set of input parameters, we find the number of bonds of each quality that maximizes the information ratio. Figure 28 shows an example of an optimal portfolio constructed by this method. While about half of the index issuers are purchased overall, this is accomplished by using more than half of the Baa issuers and much less than half of the Aa and Aaa issuers. (The optimal solution would have used even more Baa bonds if the minimum position size of $1 million had not been imposed.) The ratio of position sizes is approximately 7:3:1, not too different from what we observed using tracking error minimization.

These results are very sensitive to the assumed expected outperformance in each quality. The more outperformance is expected in a given quality, the steeper the performance penalty that will result from the dilution of credit research. While this quantity was not a factor in a homogeneous market (where increasing the single parameter just scales all information ratios), introducing different expectations for different qualities can drastically change our results. The more outperformance can be expected from the top picks in a given quality, the more risk in the form of larger position sizes one will be willing to accept there. In Appendix D, we show that the optimal ratio of position sizes in different qualities is proportional to the square root of the ratio of expected outperformance.

How much outperformance can an analyst provide in each credit quality? While this is essentially a subjective question, we investigated several different assumptions, each based on objective quantitative criteria. These assumptions, and the optimal portfolio structure for each one, are shown in Figure 29. The first assumption, that was used in Figure 28, was the outperformance of 49 bp in Aa, 50 bp in A, and

Figure 28. **Optimal Number of Bonds Using the Linear Model,** $1 Billion U.S. Credit Index Proxy

Quality	Expected Outperf of Best Bonds	Number of Index Issuers	Number of Bonds Used	Expected Portfolio Outperformance	Tracking Error due to Downgrades (bp/yr)	Position Size ($ million)	Position Size (%)
Aaa-Aa	49	253	39	41.5	10.8	6.7	0.7
A	50	522	152	35.2	11.5	2.5	0.3
Baa	78	659	352	36.1	22.6	1.0	0.1
Total		1434	543	37.2	9.5		

Figure 29. **Linear Model under Different Outperformance Assumptions,** $2 Billion Portfolio

	Expected Outperf. of Top Issuers			Number of Bonds in Portfolio				Resulting Position Sizes ($ mn)			Expected	Tracking Error due to	Inform.
	Aaa-Aa	A	Baa	Aaa-Aa	A	Baa	Total	Aaa-Aa	A	Baa	Outperf.	Downgrades	Ratio
Imperfect Foresight	0.49%	0.50%	0.78%	92	363	659	1114	5.7	2.1	1.1	0.14%	0.02%	5.71
Equal Outperformance	0.50	0.50	0.50	92	365	659	1116	5.7	2.1	1.1	0.14	0.02	5.78
Proportional to Volatility	0.10	0.23	0.85	72	190	393	655	7.3	4.1	1.8	0.20	0.08	2.39
Avoid Downgrades	0.03	0.16	0.78	122	194	356	672	4.3	4.0	2.0	0.17	0.09	1.93
Equal Position Size	0.01	0.07	1.41	243	355	325	923	2.2	2.2	2.2	0.26	0.09	2.94

78 bp in Baa. These magnitudes are based on the results of our prior research on security selection using "imperfect foresight".[15] In that study, we carried out a historical simulation of security selection in which knowledge of future returns was systematically used to bias selection in favor of better-performing securities at a certain level of "skill." The outperformance levels cited above were achieved using the same level of skill to choose securities within the three quality groups.

In Figure 29, this result is compared to those obtained using several other assumptions, for a $2 billion credit portfolio. The imperfect foresight assumption gives rise to an optimal position size ratio of approximately 5:2:1.[16] If we assume outperformance to be the same in all qualities, the results change very little. Another possible assumption is that the expected outperformance is proportional to volatility, under the theory that greater spread movement implies greater opportunity. This assumption, which increases the performance advantage of Baa, brings the optimal position sizes closer together, for a ratio of about 4:2:1.

A fourth assumption reflects the view that the main role of credit research is to avoid downgrades. As such, we can use our empirical research on the expected peer group underperformance due to downgrades to estimate the expected outperformance from avoiding them. As shown in Appendix A, if downgrades occur with probability p and have mean underperformance μ, the outperformance from avoiding downgrades is given by $-(p/1-p)\mu$.[17] This assumption gives an estimate of expected outperformance that is more than 30 times larger in Baa than in Aa—and yet, the greater risk in Baa still makes it optimal to keep positions smaller there, with a ratio of about 2:2:1.

Just how much more performance in Baa's would we need to make a uniform position size optimal? Using the theoretical solution presented in Appendix D, we find that the ratio of expected outperformance would need to be 1:7:141! When we impose this assumption, our numerical optimization indeed produces a solution with equal position sizes.

Two conclusions may be drawn from Figure 29. First, we see that the optimal ratio of position sizes changes when we assume greater or lower potential for outperformance in various qualities. However, for all reasonable estimates of this performance advantage, it remains optimal to have a stricter diversification constraint (smaller position size) in Baa than in the higher ratings. Second, because of the nature of the linear outperformance function, for all the assumptions

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[15] See "Asset Allocation vs. Security Selection in Credit Markets, Part II: An Imperfect Foresight Study," *The Journal of Portfolio Management*, Vol. 27 No. 1, Fall 2000. The numbers cited here are based on the data summarized in Exhibit 16 of that report.

[16] The results shown on the first line of Figure 29 differ from the solution of Figure 28 because of the difference in portfolio size. For the $2 billion portfolio of Figure 29, the $1 million minimum position size is not a binding constraint.

[17] This quantity differs by a factor of $1-p$ from the underperformance due to downgrades shown in Figure 6, which is equal to $p\mu$. For example, the expected outperformance of 0.785% of avoiding downgrades in Baa is obtained from the data in Figure 6 by dividing 0.74% by the 94.3% probability of no downgrade.

investigated here, the optimal overall number of issuers in the portfolio is still quite large, ranging from approximately half to all of the issuers in the index.

A Piecewise Linear Model

The simple linear model described above allowed us to quantify the trade-off between the costs and benefits of diversification and to demonstrate some qualitative relationships between our assumptions about expected outperformance and the implications for portfolio structure. However, there are some intuitively unsatisfying aspects to this model. The ranking of all issues in the universe on a smooth scale from best to worst does not correspond well to the way most research departments operate. Also, the conclusion that an investor needs to buy half of the issuers in the universe to achieve optimal diversification seems a bit extreme. By making a small modification to the above model, we can remedy these two drawbacks simultaneously.

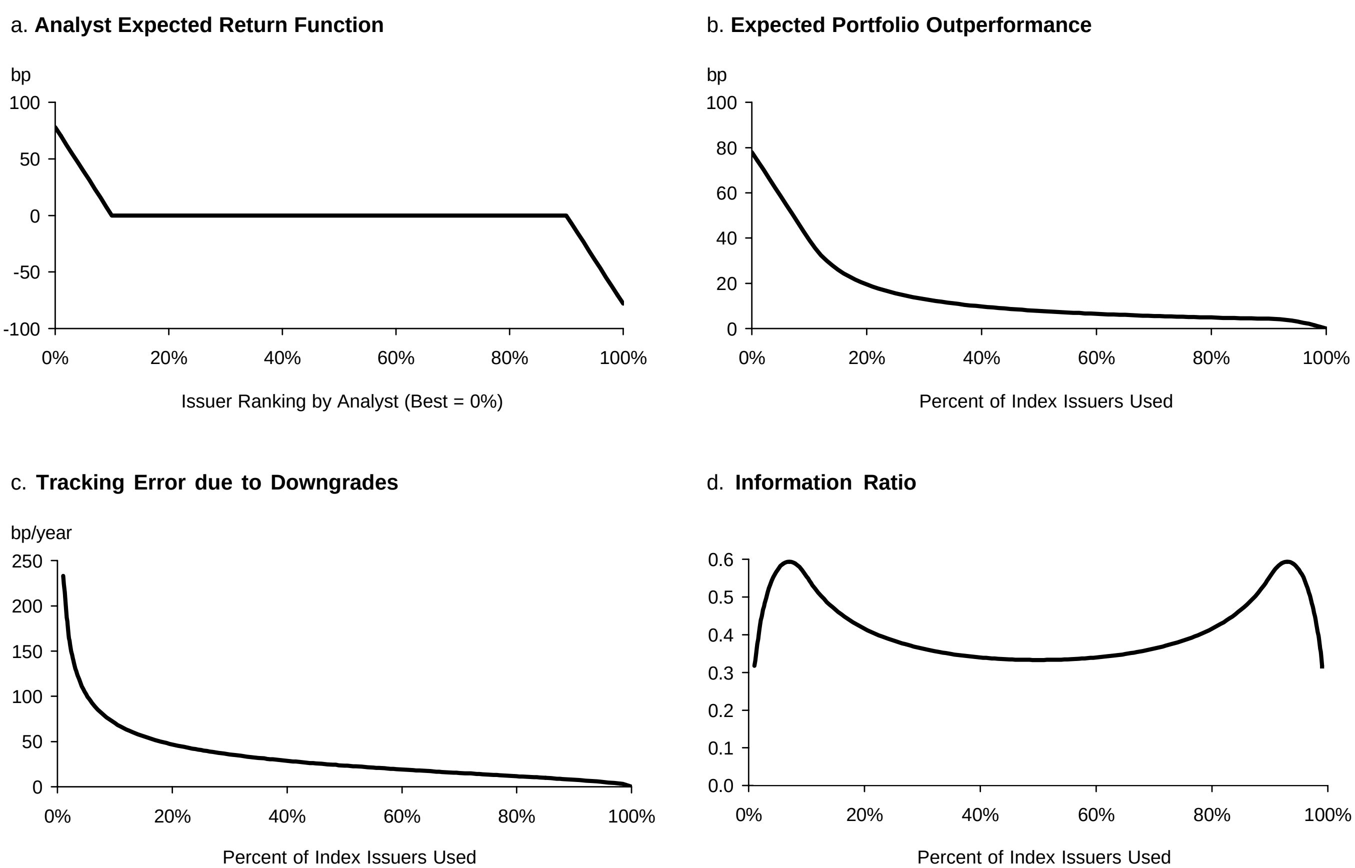
In the piecewise linear model, we continue to assume that when an issuer is researched, it is ranked on a linear scale as before, with expected outperformance b for the favorite selection, and expected underperformance $-b$ for the least recommended issuer. However, we now assume that only a portion of the market is covered by credit research. Half of the covered issuers are expected to outperform and the other half to underperform; for all issuers not covered by research, the expected outperformance is zero. This middle portion of the market now offers us the opportunity to evaluate a pure diversification play. Should a portfolio contain bonds not recommended by research, for the sole purpose of reducing tracking error, at the expense of diluting the outperformance?

Figure 30 illustrates this model for outperformance in the case of a single quality (Baa) and 20% market coverage. The expected outperformance for each bond is shown in (a). We see that zero outperformance is expected from the vast middle portion of the universe, with the top 10% expected to outperform and the bottom 10% expected to underperform. The expected portfolio outperformance for an equally weighted portfolio of the top n issuers is shown in (b). The reduction of expected portfolio outperformance is rather steep as we go from just the top picks to include all recommended securities; after 10% it levels off to a slower descent, and then shoots quickly down towards zero when we start to include "sell" recommendations. The tracking error as a function of the number of bonds is repeated in (c) for comparison; it can be easily seen that it starts out with a steeper descent than the expected portfolio outperformance, but then levels off more sharply. The ratio between the two (the information ratio) is shown in (d). It has a peak at about 7% of the issuers. The optimal information ratio is thus achieved by buying most (but not all) of the recommended bonds, and none of the uncovered issues.

One interesting (and somewhat surprising) aspect of Figure 30 (d) is the symmetrical shape of the information ratio function, with twin peaks at the opposite ends of the spectrum. Both have information ratios of 0.57, but the portfolio using 7% of the issuers in the index achieves an expected outperformance of 51 bp and a tracking error of 88 bp, while the portfolio with 93% of the index has an expected outperformance of 4 bp and a tracking error of 7 bp. Clearly, the two peaks correspond to two very different investment approaches. The first portfolio is pursuing a more active strategy, while the second portfolio is extremely passive. The two uses of credit research illustrated here can be referred to as “picking winners” and “avoiding losers.” We show in Appendix E that as long as the model for expected outperformance is symmetrical, then the information ratio is symmetrical as well, and both approaches can achieve the same range of information ratios.

The case shown in Figure 30, consisting of a homogenous set of issuers from the same credit quality, with a fixed number of bonds covered by research, has a straightforward solution for the optimal number of bonds. The information ratio can easily be expressed as a function of the number of bonds in the portfolio, and we can solve for the maximum. In Appendix E, we derive the formula for the optimal number of bonds in this case, and show that it converges to about one third of the covered issuers (or two thirds of the recommended bonds) when the percentage of the market covered is small, and to about half of the covered issuers (almost all of the recommended bonds) when the percentage covered is large. (This corresponds well to our solution for the linear model, which is the same as the piecewise linear model with 100% market coverage.)

Figure 30. **Piecewise Linear Model—20% Research Coverage of Market, Baa Portfolio versus Baa Index**



Having found that the optimal number of bonds in the portfolio is closely related to the percentage of the market covered by research, the next question is obvious: how much of the market should research cover? In order to make this question meaningful, though, we need to factor in the cost of research to balance against the benefit of increasing the expected portfolio outperformance. As a rough approximation, we assumed an annual research cost of $5,000 per issuer. Unlike all of the other costs and risks assessed thus far in our study, this cost is assumed to be a fixed annual amount per issuer and is not proportional to portfolio size. The performance effect of the cost of research will thus be much greater on small portfolios than on large ones.

With the piecewise linear model, we look at a two-stage optimization problem. We simultaneously solve for the number of index issuers of each quality that should be covered by research as well as for the number of bonds to buy in each quality. In addition to the amount of expected outperformance from the top picks in each quality, input parameters now include the cost of research coverage and the size of the portfolio.

Figure 31 shows the optimal structure for a $1 billion portfolio, using an estimated annual research coverage cost of $5,000, and assuming expected annual outperformance of 49 bp, 50 bp, and 78 bp for the top picks in the Aaa-Aa, A and Baa quality groups, respectively.[18] The cost of research is quite significant for a portfolio of this size and limits the number of issuers that can be covered. The expected annual outperformance of 44 bp is partially offset by the 16 bp spent on research coverage, to achieve a net expected outperformance of 29 bp (after rounding). As we have seen before, a greater number of bonds is chosen from the Baa sector than from the higher qualities, to reduce downgrade risk where it is most significant. The additional implication is that to maintain the expected outperformance, the research budget is shifted toward the Baa sector as well (although not to as great an extent).

Figure 32 shows the results of the model using the same parameters, but for portfolios with different total market values. As portfolio size increases, a greater number of issuers can be covered for approximately the same performance cost in basis points. Thus, for a $2 billion portfolio, we can cover twice as many issuers and purchase more

[18] As described in our discussion of Figure 29, these estimates of outperformance were obtained from our study of security selection using "imperfect foresight."

Figure 31. **Optimal Number of Bonds—Piecewise Linear Model**

Quality	**Expected Outperf. of Best Bonds**	**Number of Index Issuers**	**Percent of Issuers Covered**	**Number of Bonds Recommended**	**Number of Bonds Used**	**Expected Portfolio Outperf**	**Tracking Error due to Downgrades** (bp/yr)
Aaa-Aa	49	253	12	16	4	43	38
A	50	522	15	40	16	40	41
Baa	78	659	30	100	73	49	69
Total		1434		155	93	44	31

Figure 32. **Piecewise Linear Model—Optimal Structure as a Function of Portfolio Size**

Portfolio Size	Percent of Market Covered			Number of Bonds in Portfolio			Expected Outperformance (bp)				
($ billion)	Aaa-Aa	A	Baa	Aaa-Aa	A	Baa	Before Cost	Research Cost	After Cost	Tracking Error (bp/yr)	Inform. Ratio
0.5	6	7	15	2	8	35	44	15	29	45	0.6
1	12	15	30	4	16	73	44	16	29	31	0.9
2	26	32	64	8	36	163	43	16	27	19	1.4
5	100	100	0	54	215	659	21	8	14	4	3.3
10	100	100	0	69	272	659	19	4	15	3	4.4
20	100	100	0	78	309	659	17	2	15	3	5.0

than twice as many bonds as in the $1 billion portfolio, with but a very small decrease in net performance. The increased diversification results in a relatively large reduction in tracking error, and the information ratio increases accordingly.

When portfolio size increases to $5 billion, we find that it is optimal to cover nearly all of the issuers in the index. The net performance effect of 12 bp is less than what was found optimal for the smaller portfolios. With 100% of the market covered, we converge to the case of the linear cost model of the previous section. The number of securities selected is then about half of the issuers in the index, with roughly 60% of the names coming from the Baa quality.

As we move to even larger portfolios, we find an interesting effect that may run counter to the accepted practice. Once a certain level of diversification is reached within Baa's, the model finds it no longer cost-effective to pay for research in this sector. Rather, the maximal information ratio is achieved by switching to a strategy in which the Baa portion of the portfolio is managed totally passively against the index. This may be achieved by buying every issuer in the index or via an index swap. This strategy does not require any credit research expenditure. All efforts at outperformance are concentrated in the higher qualities, where event risk is lower. For money managers that maintain both active and passive funds, this result suggests the strategy of using a passive Baa or high-yield fund as one component of an active credit fund.

CONCLUSION

The models we develop to represent credit risk are quite simple and similar to other transition-matrix-based models that may be found in the literature.[19] In our view, the main contribution of this study lies in the data on observed performance effects of downgrades and in our approach to portfolio structuring. The reported data provides a necessary link between the event probabilities contained in transition matrices and the actual performance implications of these events. Second, while the literature contains many attempts to model credit risk, we have not seen these models applied to higher-level tasks like portfolio structuring, setting investment policy, and establishing research priorities.

One key conclusion of this work is that downgrade risk can be readily understood and managed, especially in a portfolio-vs.-benchmark setting. While the risk of events like downgrades and defaults make the return profile of a single corporate bond far from normally distributed, managers of bond portfolios are aided by the combination of two effects. First, the law of large numbers provides that to the extent that credit events for different issuers are uncorrelated, the losses on a portfolio become closer to normal as the number of securities grows. Second, managing relative to a broad benchmark reduces the role that correlations play in diluting the diversification benefits and increasing risk, because any events that affect large sections of the market impact the benchmark as well. As a result, worst case estimates of portfolio underperformance due to downgrades can be fairly safely constructed based on the normal distribution as long as a sufficient level of diversification has been imposed.

The framework introduced here can serve as a starting point for customization along several different dimensions. For example, Figures 24 through 26 show the results of our study of downgrade losses using finer partitions of the index – by duration, sector, and ratings tier. This approach can be pursued to analyze how much diversification is required within finer industry classifications such as telecommunications or consumer products.

The conclusions of this work depend on a set of assumptions that are quite subjective in nature. We feel more comfortable estimating the volatilities of future downgrade losses based on actual market observations of past downgrade losses. Some investors may prefer to work with spread differentials, which allow risk estimates to react more quickly to changes in the marketplace. Particularly in the area of modeling outperformance due to credit research, one can suggest any number of alternative functional forms that might be more realistic, and each investor may have their own views on how much value they can add by careful security selection within different market segments. The types of portfolio optimization problems to be considered will be different for active and passive investors.

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[19] Two good surveys of the literature on credit models are: “Modelling Credit: Theory and Practice”, by Dominic O’Kane & Lutz Schlögel, Lehman Brothers, February 1, 2001, and “Estimating and Pricing Credit Risk: An Overview” by Duen-Li Kao, *Financial Analysts Journal,* Vol. 56, No. 4 (July/August 2000):pp. 50-66.

Yet from any point of view, the basic message remains the same. Different credit qualities and market segments entail different amounts of non-systematic risk, and these differences should be considered when formulating an investment strategy. The lower the quality, and the greater the amount of risk, the stricter the diversification constraints that should be imposed. From the point of view of downgrade risk alone, we find that the optimal ratio of position sizes in the three quality groups studied (Aaa-Aa : A : Baa) is a rather extreme 9:4:1. In reality, of course, investors are concerned with all sources of non-systematic risk including the potentially significant return volatility of bonds that did not experience a rating change. This total-risk analysis produces a more realistic optimal position ratio of 4:3:1.

When considering the implications for credit research allocation, the conclusions are more dependent on the setting. In active management, the results generally follow the same pattern as for diversification: to support the selection of more securities in the lower qualities, a greater portion of the research budget should be applied. However, our research raises the possibility that in some enhanced indexing applications, it might be appropriate to allocate the bulk of the research budget to active management in the higher qualities and take a purely passive stance in the most risky portion of the market.

APPENDICES

A. Modeling Underperformance due to Downgrades (Single Bond)

In this analysis of a bond's return relative to its peer group, we assume that we know the complete distribution of returns for a typical bond of initial credit quality j, both unconditionally and conditioned on the final credit quality. Let us introduce the following notation:

$$
\begin{aligned}
R_j &= \text{return relative to the peer group for a bond of initial credit quality } j \\
f_j(r) &= \text{probability distribution function for } R_j \\
f_j(r \mid k) &= \text{conditional probability distribution for } R_j \text{ conditioned on final credit quality } k \\
p_{jk} &= \text{transition probability of a bond with initial quality } j \text{ ending in credit quality } k \\
p_{j,DG} &= \text{total probability of downgrade from quality } j = \sum_{k<j} p_{jk} \\
p_{j,UG} &= \text{total probability of upgrade from quality } j = \sum_{k>j} p_{jk} \\
p_{j,NC} &= \text{probability of no change in rating from quality } j = p_{jj} \\
\mu_j &= E[R_j] = \text{mean return relative to peer group for initial quality } j \\
\mu_{jk} &= E[R_j \mid k] = \text{mean relative return for bonds of initial quality } j \text{ and final quality } k \\
\mu_{j,DG} &= E[R_j \mid k<j] = \text{mean relative return for bonds downgraded from initial quality } j \\
\sigma_j^2 &= \text{variance of return relative to peer group for initial quality } j \\
\sigma_{jk}^2 &= \text{variance of relative return for bonds of initial quality } j \text{ and final quality } k \\
\sigma_{j,DG}^2 &= \text{variance of relative return for bonds downgraded from initial quality } j
\end{aligned}
$$

The means and variances of relative returns for upgraded bonds and those with no change in rating are given by $\mu_{j,UG}$, $\mu_{j,NC}$, $\sigma_{j,UG}^2$ and $\sigma_{j,NC}^2$ respectively, and are defined similarly to those for downgraded bonds.

By the fact that returns are defined relative to the peer group, the expected value of this return must be zero. Nevertheless, it is instructive to expand this mean relative return to reflect the conditioning on ratings transitions:

$$\text{(A-1)} \quad \mu_j = 0 = \int_{-\infty}^{\infty} r f_j(r)dr = \int_{-\infty}^{\infty} r \sum_k p_{jk} f_j(r \mid k)dr = \sum_k p_{jk}\mu_{jk}$$

Alternatively, we can look at the effect of all downgrades together. The mean returns conditioned on being downgraded, upgraded or unchanged are given by:

$$\text{(A-2)} \quad \mu_{j,DG} = \sum_{k<j} \frac{p_{jk}}{p_{j,DG}} \mu_{jk} \qquad \mu_{j,UG} = \sum_{k>j} \frac{p_{jk}}{p_{j,UG}} \mu_{jk} \qquad \mu_{j,NC} = \mu_{jj}$$

This allows us to express the mean relative return as follows:

$$\text{(A-3)} \quad \mu_j = 0 = p_{j,DG}\mu_{j,DG} + p_{j,NC}\mu_{j,NC} + p_{j,UG}\mu_{j,UG}$$

Typically, a negative expected return due to downgrades is offset by a positive return due to upgrades. To the extent that expected downgrade losses are larger than the expected gains on upgrades, the unchanged bonds contribute a small positive outperformance of the peer group, to bring the overall expected outperformance to zero. This is illustrated in Figure A1, based on the spread differential model using a fine transition matrix. We see that the mean return due to rating transitions is –75.2 bp, because the losses due to downgrades are much greater than the gains due to upgrades. Once performance is restated relative to this peer group return, we see that in case of either an upgrade or an unchanged rating, a bond will outperform relative to the peer group. The combined contributions to relative performance from these two cases offset the contribution due to downgrades, the -89.4 bp highlighted in Figure A1. This quantity corresponds to the first term of equation (A-3), as well as to the mean loss due to downgrades that appears in Figure 13. Figure A1 also corresponds exactly to the right-most column of Figure 10.

A similar decomposition can be carried out on the variance of relative returns. As the variance of any random variable x is given by $VAR(x) = \sigma_x^2 = E[x^2] - E[x]^2$, we have $E[x^2] = \sigma_x^2 + \mu_x^2$. Keeping in mind also that $\mu_j = 0$ for returns relative to the peer group, we can easily see that

$$\text{(A-4)} \quad \begin{aligned} \sigma_j^2 &= E[R_j^2] - E[R_j]^2 = p_{j,DG} E[R_j^2 \mid DG] + p_{j,NC} E[R_j^2 \mid NC] + p_{j,UG} E[R_j^2 \mid UG] - 0 \\ &= p_{j,DG}\left(\sigma_{j,DG}^2 + \mu_{j,DG}^2\right) + p_{j,NC}\left(\sigma_{j,NC}^2 + \mu_{j,NC}^2\right) + p_{j,UG}\left(\sigma_{j,UG}^2 + \mu_{j,UG}^2\right) \end{aligned}$$

Once again, the first term, the return variance due to downgrades, is typically the largest of the three components. The square root of this first term, the standard deviation of downgrade risk, is the quantity shown in Figure 6, which serves as the basis for the further analysis of portfolio downgrade risk in this paper. In order to include the effect of the entire variance of idiosyncratic returns (return differences from the peer group) instead of just variance due to downgrades, all that is necessary is to use the entire variance σ_j^2 in the remainder of the analysis, rather than just the first term.

The above analysis restates the distribution of returns relative to the peer group by conditioning on rating transitions in two different ways, to reflect the two different

Figure A1. **Mean Relative Returns due to Ratings Transitions for Baa Bonds**
Spread Differential Model, Fine-Grained Transition Matrix,
Spreads as of December 31, 2001

	Probability	**Performance Due to Ratings Transitions** (bp)	**Performance Relative to Peer Group** (bp)	**Contribution to Relative Perf.** (bp)
	p		μ	$p\mu$
Upgrades	10.2%	225.4	300.6	30.8
No Change	78.0%	0.0	75.2	58.6
Downgrades	11.8%	-832.4	-757.2	**-89.4**
Net	100.0%	-75.2	0.0	0.0

modeling approaches used in the text. In both approaches, the rating transition probabilities p_{jk} are modeled using data provided by the rating agencies. However, in the model based on observed returns of downgraded bonds, we directly estimate the quantities $\mu_{j,DG}$ and $\sigma^2_{j,DG}$ by taking sample statistics from our dataset. In the model based on spread differentials, we estimate the quantities μ_{jk} based on current spreads, and combine these across the various ratings to obtain estimates of variance.

According to one interpretation, the value of credit research is primarily in avoiding downgrades. A slight variation of equation (A-3) can be used to find the expected outperformance that can be obtained in this way. Instead of dividing the market into three segments—downgrades, upgrades, and no change—we simply separate downgrades from everything else. Let $\mu_{j,NDG}$ be the mean index outperformance of bonds of quality j that do not suffer downgrades. Then equation (A-3) can be rewritten as

$$\text{(A-5)} \quad \begin{aligned} \mu_j &= 0 = p_{j,DG}\mu_{j,DG} + \left(1 - p_{j,DG}\right)\mu_{j,NDG} \\ \mu_{j,NDG} &= -\frac{p_{j,DG}}{1 - p_{j,DG}}\mu_{j,DG} \end{aligned}$$

B. Portfolio Risk due to Downgrades

Given the mean μ_j and variance σ_j^2 of the return relative to the peer group for a single bond of quality j, we proceed to calculate these quantities for a portfolio. Temporarily omitting the subscript j for simplicity, we start by analyzing a portfolio of securities of a single quality. Let $R(n)$ denote the relative return of a portfolio of n securities of the same quality, with a percentage w_i of portfolio market value in each security i. The relative return of the portfolio is then

(B-1) $$R(n) = \sum_{i=1}^{n} w_i R_i$$

and its mean and variance are given by

(B-2) $$\begin{aligned} E[R(n)] &= \sum_{i=1}^{n} w_i E[R_i] \\ Var(R(n)) &= \sum_{i=1}^{n}\sum_{k=1}^{n} w_i w_k \rho_{ik}\sigma_i\sigma_k = \sum_{i=1}^{n} w_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n} w_i w_k \rho_{ik}\sigma_i\sigma_k \end{aligned}$$

where ρ_{ik} is the correlation between the relative returns of bonds i and k. Let us assume that all of the relative security returns R_i are drawn from the same distribution with known mean μ and variance σ^2, and that the correlation between the returns of any two bonds is given by a single correlation constant ρ. Since the weights w_i sum to one, it is clear that the mean return of the portfolio is just μ. If we make the further assumption that the portfolio is equally weighted (all $w_i = 1/n$), then the expression for variance becomes two sums of like terms:

(B-3) $$Var(R(n)) = \frac{1}{n^2}\left(\sum_{i=1}^{n}\sigma^2 + \sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\rho\sigma^2\right) = \frac{\sigma^2}{n^2}\left(n + n(n-1)\rho\right) = \sigma^2\left(\frac{1}{n} + \rho\frac{n-1}{n}\right)$$

We can see that when the risks associated with all of the bonds in the portfolio are considered independent $(\rho = 0)$, then the variance of non-systematic portfolio returns decreases linearly with the number of bonds in the portfolio. This risk reduction is the desired effect of diversification. However, if we assume some nonzero correlation between the risks (that is, downgrades are likely to occur in clusters), then diversification can reduce risk only to a point. The term related to correlation does not decrease with increasing n, and the variance asymptotically approaches $\rho\sigma^2$ as n grows large. (When all assets are perfectly correlated, no diversification is possible, and portfolio variance is the same as for a single bond.)

For portfolios measured against a benchmark, the critical measure of risk is not the variance of portfolio return, but rather that of the difference between portfolio and benchmark return. Assume that there are N bonds in the benchmark, all of the same single quality as our portfolio, and let W_i denote the weight of bond i in the benchmark. The tracking error $TE(n)$ for our portfolio of n bonds is obtained by replacing the bond's weight in equation (B-2) by the difference in portfolio and benchmark weights. To keep matters simple, we assume that the benchmark is equally weighted as well $(W_i = 1/N)$. We then obtain sums of like terms for pairs of bonds in which zero, one or both of the bonds are in the portfolio, and simplify the result:

$$
\begin{aligned}
TE^2(n) &= \sum_{i=1}^{N}(w_i - W_i)^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N}\sigma_i\sigma_k\rho_{ij}(w_i - W_i)(w_j - W_j) \\
&= \sum_{i\in P}^{n}\left(\frac{1}{n}-\frac{1}{N}\right)^2\sigma^2 + \sum_{i\notin P}^{N-n}\left(0-\frac{1}{N}\right)^2\sigma^2 \\
\text{(B-4)}\quad &+ \rho\sigma^2\sum_{i\in P}^{n}\sum_{\substack{k\in P\\k\neq i}}^{n}\left(\frac{1}{n}-\frac{1}{N}\right)^2 + \rho\sigma^2\sum_{i\in P}^{n}\sum_{k\notin P}^{N-n}\left(\frac{1}{n}-\frac{1}{N}\right)\left(0-\frac{1}{N}\right) + \rho\sigma^2\sum_{i\notin P}^{N-n}\sum_{k\in P}^{n}\left(0-\frac{1}{N}\right)\left(\frac{1}{n}-\frac{1}{N}\right) + \rho\sigma^2\sum_{i\notin P}^{N-n}\sum_{\substack{k\notin P\\k\neq i}}^{N-n}\left(0-\frac{1}{N}\right)^2 \\
&= \sigma^2\left[n\left(\frac{1}{n}-\frac{1}{N}\right)^2 + (N-n)\frac{1}{N^2}\right] + \rho\sigma^2\left[n(n-1)\left(\frac{1}{n}-\frac{1}{N}\right)^2 - 2n(N-n)\frac{1}{N}\left(\frac{1}{n}-\frac{1}{N}\right) + \frac{(N-n)(N-n-1)}{N^2}\right] \\
&= \sigma^2\left(\frac{1}{n}-\frac{1}{N}\right) + \rho\sigma^2\left(\frac{1}{N}-\frac{1}{n}\right) \\
&= \sigma^2(1-\rho)\left(\frac{1}{n}-\frac{1}{N}\right)
\end{aligned}
$$

In the case of tracking error, correlations do not prevent diversification from reducing the risk. On the contrary, positive correlations will decrease tracking error. A high positive correlation may cause return variance to be large for both the portfolio and the benchmark. It will, however, tend to make the securities in the portfolio behave more like the securities not in the portfolio, and therefore track the index more closely. Regardless of the value of ρ, tracking error continues to decrease with increasing n, reaching a minimum of zero when $n = N$, at which point the portfolio matches the benchmark exactly.

We can now relax the assumption that all bonds in the portfolio and the index are from a single quality group. Instead, we divide the index up by quality, and repeat the above analysis to obtain the tracking error within each quality j for an equally-weighted portfolio of n_j securities chosen from the N_j issuers of quality j in the index. We further assume that the portfolio is constructed to exactly match the quality distribution of the index, so that if x_j is the percentage of index market value in quality j, then it is the weight in the portfolio as well. We then find that the overall tracking error is given by

$$\text{(B-5)}\quad TE^2 = \sum_j x_j^2\sigma_j^2(1-\rho)\left(\frac{1}{n_j}-\frac{1}{N_j}\right).$$

Note that this expression for tracking error can be used to obtain estimates of both overall idiosyncratic tracking error (if we use the entire σ_j^2 shown in equation (A-4)) and tracking error due to downgrades (if we limit σ_j^2 to the first term of equation (A-4)). In the body of this paper, the analysis focuses on tracking error due to downgrades.

C. Structuring a Portfolio to Minimize Tracking Error

In Appendix B, we developed the following expression for tracking error:

$$\text{(C-1)}\quad TE^2 = \sum_j x_j^2 \sigma_j^2 (1-\rho)\left(\frac{1}{n_j} - \frac{1}{N_j}\right).$$

For a portfolio of M bonds, how should we choose the number of bonds n_j for each quality to achieve the smallest possible tracking error?

To find the answer to this problem, we first answer a simpler one. Assume we have chosen values for all n_j that satisfy $\sum_j n_j = M$. We now freeze these values for all but two quality groups q and r, and attempt to find the best allocation among these two. Any increase in n_q will result in a corresponding decrease to n_r. We can rewrite (C-1) to isolate these two values as

$$\text{(C-2)}\quad TE^2 = x_q^2 \sigma_q^2 (1-\rho)\left(\frac{1}{n_q} - \frac{1}{N_q}\right) + x_r^2 \sigma_r^2 (1-\rho)\left(\frac{1}{n_r} - \frac{1}{N_r}\right) + \sum_{\substack{j \neq q \\ j \neq r}} x_j^2 \sigma_j^2 (1-\rho)\left(\frac{1}{n_j} - \frac{1}{N_j}\right)$$

and then find the optimal value of n_r (ignoring the fact that this quantity should really be an integer) by setting the derivative to zero:

$$\text{(C-3)}\quad \begin{aligned} \frac{\partial TE^2}{\partial n_q} &= -\frac{x_q^2 \sigma_q^2 (1-\rho)}{n_q^2} - \frac{x_r^2 \sigma_r^2 (1-\rho)}{n_r^2}\frac{\partial n_r}{\partial n_q} + 0 = 0 \\ \frac{x_q^2 \sigma_q^2}{n_q^2} &= \frac{x_r^2 \sigma_r^2}{n_r^2} \end{aligned}$$

Taking square roots (and assuming that all of the quantities shown are positive), we find that the number of bonds selected in a given quality is proportional to both the characteristic return volatility for that quality and its weight in the index:

$$\text{(C-4)}\quad \frac{n_q}{n_r} = \frac{x_q \sigma_q}{x_r \sigma_r}$$

As this relationship should hold for any two quality groups, this relationship can be used to express the number of bonds in every quality group in terms of n_q. As the total number of bonds must remain constant, this allows us to solve for n_q:

$$\text{(C-5)}\quad \begin{aligned} M &= \sum_j n_j = n_q + \sum_{j \neq q} \frac{x_j \sigma_j}{x_q \sigma_q} n_q = n_q \frac{\sum_j x_j \sigma_j}{x_q \sigma_q} \\ n_q &= \frac{x_q \sigma_q}{\sum_j x_j \sigma_j} M \end{aligned}$$

The optimal number of bonds in each quality is a fixed proportion of the overall number of bonds. This is consistent with our numerical results from integer programming. This relationship can be stated even more simply in terms of the position sizes corresponding to the optimal solution. For a portfolio of market value V, the size of each position of quality j is given by

(C-6) $$p_j = \frac{x_j V}{n_j}.$$

Using property (C-4) of the optimal solution, we can now see that to minimize the tracking error, the ratio of position sizes in two quality groups should be

(C-7) $$\frac{p_q}{p_r} = \frac{x_q}{x_r}\frac{n_r}{n_q} = \frac{\sigma_r}{\sigma_q}.$$

That is, the optimal position limit in a given market segment is inversely proportional to the idiosyncratic risk observed there.

D. Structuring a Portfolio to Maximize Information Ratio (Linear Model)

The optimal number of bonds in a portfolio can be expressed as a trade-off between the risk reduction due to diversification and the outperformance that can be expected from concentrating the portfolio in the securities most recommended by credit research. For a population of N bonds, we assume that credit research is applied to rank the bonds from 1 (the best) to N (the worst). We define the outperformance function $f(x)$ as the expected outperformance of a bond with rank x. The first, and simplest, model presented is the linear model shown in Figure 27(a), in which the best bond is expected to outperform by b, the worst bond to underperform by $-b$, and the bond ranked at the halfway point earns the index average:

In our linear model for outperformance, the outperformance z_j in quality j is given by

(D-1) $$f(x) = b\left(1 - \frac{x}{N/2}\right).$$

We now assume that a portfolio manager who decides to build a portfolio of n bonds will select those ranked highest by credit research. Using continuous notation for ease of modeling, we can integrate to find the expected outperformance $z(n)$ of such a portfolio:

(D-2) $$z(n) = \frac{1}{n}\int_0^n f(x)dx = \frac{1}{n}\int_0^n b\left(1 - \frac{x}{N/2}\right)dx = \frac{b}{n}\left(n - \frac{n^2}{N}\right) = b\left(1 - \frac{n}{N}\right).$$

This function, the expected outperformance of a portfolio of n bonds, is illustrated in Figure 27(b). For a very small portfolio, the expected outperformance is close to b, the performance of the top pick. At the other extreme, if we own every bond in the index ($n = N$), outperformance goes to zero.

Of course, all of the parameters may differ by credit quality. Let us introduce subscripts, so that z_j represents the outperformance of a portfolio of n_j securities of credit quality j:

(D-3) $$z_j = b_j\left(1 - \frac{n_j}{N_j}\right).$$

As shown in Appendix B, the tracking error variance v_j for a portfolio of n_j bonds of quality j versus an index of N_j bonds is given by

(D-4) $$v_j = TE_j^2 = \sigma_j^2\left(\frac{1}{n_j} - \frac{1}{N_j}\right)$$

The information ratio for this single-quality portfolio is then given by

(D-5) $IR_j = \frac{z_j}{TE_j} = \frac{z_j}{\sqrt{v_j}}$

To find the number of securities that maximizes the information ratio, we set the derivative equal to zero and solve:

(D-6)
$$\frac{\partial IR_j}{\partial n_j} = \frac{\partial z_j}{\partial n_j}\frac{1}{\sqrt{v_j}} - \frac{1}{2} z_j \frac{1}{v_j\sqrt{v_j}}\frac{\partial v_j}{\partial n_j} = 0$$
$$= \frac{-b_j}{N_j}\frac{1}{\sigma_j\sqrt{\frac{1}{n_j}-\frac{1}{N_j}}} - \frac{1}{2} b_j \frac{\left(1-\frac{n_j}{N_j}\right)}{\sigma_j^2\left(\frac{1}{n_j}-\frac{1}{N_j}\right)\sigma_j\sqrt{\frac{1}{n_j}-\frac{1}{N_j}}}\frac{-\sigma_j^2}{n_j^2}$$
$$= \frac{-b_j}{\sigma_j\sqrt{\frac{1}{n_j}-\frac{1}{N_j}}}\left(\frac{1}{N_j}-\frac{1}{2n_j}\right) = 0$$
$$n_j = \frac{N_j}{2}$$

For a portfolio spanning several qualities, the expression for the optimal number of bonds is more complex, but we can analyze the way in which the optimal allocation depends on the characteristics for each quality. Assuming that the portfolio follows the index weights x_j, the expected outperformance, its variance, and the information ratio are given by

(D-7)
$$z = \sum_j x_j b_j \left(1-\frac{n_j}{N_j}\right)$$
$$v = \sum_j x_j^2 \sigma_j^2 \left(\frac{1}{n_j}-\frac{1}{N_j}\right)$$
$$IR = \frac{z}{\sqrt{v}}$$

Taking the partial derivative with respect to the number of bonds in an arbitrary quality q, we have

(D-8)
$$\frac{\partial IR}{\partial n_q} = \frac{\partial z}{\partial n_q}\frac{1}{\sqrt{v}} - \frac{1}{2} z \frac{1}{v\sqrt{v}}\frac{\partial v}{\partial n_q} = 0$$
$$= -\frac{x_q b_q}{N_q}\frac{1}{\sqrt{v}} - \frac{z}{2v\sqrt{v}}\left(-\frac{x_q^2\sigma_q^2}{n_q^2}\right) = -\frac{x_q}{\sqrt{v}}\left(\frac{b_q}{\cdot N_q} - \frac{z}{2v}\cdot\frac{x_q\sigma_q^2}{n_q^2}\right) = 0$$

Solving for n_q gives

(D-9) $n_q^2 = \frac{z}{2v}\cdot\frac{x_q\sigma_q^2 N_q}{b_q}$

Equation (D-7) does not provide the complete solution for the value of n_q, as both z and v depend on n_q. However, since equation (D-7) applies equally well to every quality, we can use it to find the ratio between the optimal numbers of bonds in any two quality groups q and r, much as we did in equation (C-4):

$$\text{(D-10)}\quad \frac{n_q}{n_r} = \sqrt{\frac{x_q \sigma_q^2 N_q b_r}{x_r \sigma_r^2 N_r b_q}}$$

As in the case of minimizing tracking error, we find that the number of bonds is proportional to both the risk due to downgrades and the index weight. The effect of the linear model for outperformance is to introduce a dependence on the expected outperformance as well, encouraging the manager to own fewer bonds in qualities where the expected outperformance is highest.

Once again, this can be restated as a ratio of position sizes:

$$\text{(D-11)}\quad \frac{p_q}{p_r} = \frac{x_q n_r}{x_r n_q} = \sqrt{\frac{x_q \sigma_r^2 N_r b_q}{x_r \sigma_q^2 N_q b_r}}$$

In this formulation, the position size p_q is once again inversely proportional to the risk σ_q, but now also depends on the expected outperformance b_q and the average index position size ${x_q}/{N_q}$ for quality q.

E. Maximizing Information Ratio (Piecewise Linear Model)

Finding the Optimal Number of Bonds

In the piecewise linear model, we assume that of the N issuers in the market, the analyst covers c of them. He is assumed to form a positive view on half of these issuers, and a negative view on the other half. Within the positive and negative views, there is a linear range of how much the analyst likes the bond, and thus a linear model for expected return. If the entire market is covered $(c = N)$, then this model converges to the linear model above. The outperformance function $f(x)$ for this model is given by

$$\text{(E-1)} \quad f(x) = \begin{cases} b\left(1 - \dfrac{x}{c/2}\right) & x < c/2 \\ 0 & c/2 \le x \le N - c/2 \\ b\left(0 - \dfrac{x - (N - c/2)}{c/2}\right) & x > N - c/2 \end{cases}$$

The expected outperformance $z(n)$ of a portfolio of the analyst's top n picks is obtained by taking the average of $f(x)$ from 0 to n. (Once again, we treat the number of bonds as a continuous variable for ease of modeling.) For $n < c/2$, the outperformance is given by

$$\text{(E-2)} \quad z(n) = \frac{1}{n}\int_0^n f(x)dx = \frac{1}{n}\int_0^n b\left(1 - \frac{x}{c/2}\right)dx = \frac{b}{n}\left(n - \frac{n^2}{c}\right) = b\left(1 - \frac{n}{c}\right).$$

The complete function for outperformance as a function of n is given by

$$\text{(E-3)} \quad z(n) = \begin{cases} b\left(1 - \dfrac{n}{c}\right) & n < c/2 \\ b\left(\dfrac{c}{4n}\right) & c/2 \le n \le N - c/2 \\ b\left(\dfrac{c}{4n}\right) - b\left(\dfrac{(n - (N - c/2))^2}{nc}\right) & n > N - c/2 \end{cases}$$

Using just the first of the three regions, we find that the IR conveniently has a maximum in the first region, so that we can solve for it just based on the first of the above expressions.

Proceeding much as we did in Appendix D, but using this piecewise linear model for outperformance, the outperformance z_j in quality j is given by

$$\text{(E-4)} \quad z_j = b_j\left(1 - \frac{n_j}{c_j}\right),$$

and the tracking error variance v_j is given by

(E-5) $$v_j = TE_j^2 = \sigma_j^2\left(\frac{1}{n_j} - \frac{1}{N_j}\right).$$

The information ratio for this single-quality portfolio is then given by

(E-6) $$IR_j = \frac{z_j}{TE_j} = \frac{z_j}{\sqrt{v_j}}$$

To find the number of securities that maximizes the information ratio, we set the derivative equal to zero and solve:

$$\frac{\partial IR_j}{\partial n_j} = \frac{\partial z_j}{\partial n_j}\frac{1}{\sqrt{v_j}} - \frac{1}{2} z_j \frac{1}{v_j\sqrt{v_j}}\frac{\partial v_j}{\partial n_j} = 0$$

$$= \frac{-b_j}{c_j}\frac{1}{\sigma_j\sqrt{\frac{1}{n_j} - \frac{1}{N_j}}} - \frac{1}{2} b_j \frac{\left(1 - \frac{n_j}{c_j}\right)}{\sigma_j^2\left(\frac{1}{n_j} - \frac{1}{N_j}\right)\sigma_j\sqrt{\frac{1}{n_j} - \frac{1}{N_j}}}\frac{-\sigma_j^2}{n_j^2}$$

(E-7) $$= \frac{-b_j}{\sigma_j\sqrt{\frac{1}{n_j} - \frac{1}{N_j}}}\left(\frac{1}{c_j} - \frac{1}{2n_j}\frac{\left(\frac{1}{n_j} - \frac{1}{c_j}\right)}{\left(\frac{1}{n_j} - \frac{1}{N_j}\right)}\right) = 0$$

$$\frac{1}{c_j} = \frac{1}{2n_j}\frac{\left(\frac{1}{n_j} - \frac{1}{c_j}\right)}{\left(\frac{1}{n_j} - \frac{1}{N_j}\right)} = \frac{1}{2n_j}\frac{\frac{c_j - n_j}{n_j c_j}}{\frac{N_j - n_j}{n_j N_j}} = \frac{1}{2n_j}\frac{c_j - n_j}{N_j - n_j}\frac{N_j}{c_j}$$

We then end up with a quadratic equation for the number of bonds n_j,

(E-8) $$2n_j^2 - 3N_j n_j + N_j c_j = 0$$

for which the solution is given by

(E-9) $$n_j = N_j\left(\frac{3}{4} \pm \frac{1}{4}\sqrt{9 - 8\frac{c_j}{N_j}}\right)$$

Although this gives two solutions, only the one with the minus sign gives a value for which $n_j \le c_j/2$. This is the only range over which $z_j(n)$ is given by (E-4). The other maximum for IR (seen in the graphs) occurs outside of this range. Rather than solving for this maximum by manipulation of the other formulas shown in Equation (E-3), we can obtain it by showing that the information ratio is symmetric.

Proof that IR(n) Is Symmetrical around N/2, i.e. IR(n) = IR(N-n)

Hypothesis: for any model of outperformance for which the function $f(x)$ satisfies

(E-10) $f(x) = -f(N-x)$

(i.e. it is an odd function around $N/2$), the information ratio will be symmetric around $N/2$, satisfying $IR(n) = IR(N-n)$.

Note that both of the models we have considered here (linear and piecewise linear) satisfy this property.

Proof:

Let $F(n)$ denote the cumulative outperformance over the first n bonds:

(E-11) $F(n) = \int_0^n f(x)dx$.

Note that this quantity is very closely related to the average portfolio outperformance $z(n)$ defined in Equation (E-2):

(E-12) $z(n) = \frac{F(n)}{n}$.

Making use of a variable substitution $y = N - x$, as well as the symmetry property of (E-10), we can easily show that cumulative outperformance is zero when all bonds are included:

(E-13)
$$\begin{aligned} F(N) &= \int_0^N f(x)dx = \int_0^{N/2} f(x)dx + \int_{N/2}^N f(x)dx = \int_0^{N/2} f(x)dx + \int_{N/2}^0 f(N-y)(-dy) \\ &= \int_0^{N/2} f(x)dx + \int_{N/2}^0 -f(y)(-dy) = F\left(N/2\right) - F\left(N/2\right) = 0 \end{aligned}$$

Using this result, and following a similar procedure, we can show that the cumulative outperformance function $F(n)$ is symmetric around $N/2$:

(E-14)
$$\begin{aligned} F(N-n) &= \int_0^{N-n} f(x)dx = \int_0^N f(x)dx - \int_{N-n}^N f(x)dx = 0 - \int_n^0 f(N-y)(-dy) \\ &= \int_0^n -f(y)(-dy) = F(n) \end{aligned}$$

Recalling our formula for tracking error from (B-5), we can now express the information ratio for n bonds in terms of the cumulative distribution function:

(E-15)
$$IR(n) = \frac{z(n)}{TE(n)} = \frac{F(n)/n}{\sigma\sqrt{\frac{1}{n} - \frac{1}{N}}} = \frac{F(n)}{\sigma n\sqrt{\frac{N-n}{nN}}} = \frac{F(n)}{\sigma\sqrt{\frac{n(N-n)}{N}}}$$

Making use of the symmetry property of $F(n)$ from Equation (E-14), we can now prove our hypothesis:

(E-16) $$IR(N-n)=\frac{z(N-n)}{TE(N-n)}=\frac{F(N-n)/(N-n)}{\sigma\sqrt{\frac{1}{N-n}-\frac{1}{N}}}=\frac{F(n)}{\sigma(N-n)\sqrt{\frac{N-(N-n)}{(N-n)N}}}=\frac{F(n)}{\sigma\sqrt{\frac{n(N-n)}{N}}}=IR(n)$$

Since $IR(N-n)=IR(n)$, we can conclude that if the information ratio has a maximum at some n, then there should also be a maximum at $N-n$. Thus, the two optimal values for n_j are given by

(E-17) $$\begin{aligned} n_j^{\max 0} &= N_j\left(\frac{3}{4}-\frac{1}{4}\sqrt{9-8\frac{c_j}{N_j}}\right), \qquad n_j^{\max 0}<\frac{c_j}{2} \\ n_j^{\max 1} &= N_j-n_j^{\max 0}=N_j\left(\frac{1}{4}+\frac{1}{4}\sqrt{9-8\frac{c_j}{N_j}}\right), \qquad n_j^{\max 1}>N-\frac{c_j}{2} \end{aligned}$$

As these expressions are not particularly intuitive, it is instructive to examine their asymptotic behavior as the percentage of issuers covered by research becomes very small (close to zero) or very large (close to N).

When the percentage of covered issuers is very small, i.e. $c_j << N_j$, then we can use an approximation for the square root function to obtain

$$n_j^{\max 0}=N_j\left(\frac{3}{4}-\frac{3}{4}\sqrt{1-\frac{8}{9}\frac{c_j}{N_j}}\right)\approx N_j\left(\frac{3}{4}-\frac{3}{4}\left(1-\frac{1}{2}\cdot\frac{8}{9}\frac{c_j}{N_j}\right)\right)=\frac{c_j}{3}$$

and we see that the optimal IR portfolio uses one third of the covered bonds.

When the number of covered issuers is very close to N, i.e. $1-c_j/N_j<<1$, then

$$\begin{aligned} n_j^{\max 0} &= N_j\left(\frac{3}{4}-\frac{1}{2}\sqrt{\frac{9}{4}-2\frac{c_j}{N_j}}\right)=N_j\left(\frac{3}{4}-\frac{1}{2}\sqrt{\frac{1}{4}+2\left(1-\frac{c_j}{N_j}\right)}\right) \\ &= N_j\left(\frac{3}{4}-\frac{1}{4}\sqrt{1+8\left(1-\frac{c_j}{N_j}\right)}\right)=N_j\left(\frac{3}{4}-\frac{1}{4}\left(1+4\left(1-\frac{c_j}{N_j}\right)\right)\right)=c_j-\frac{N_j}{2} \\ &= \frac{c_j}{2}-\frac{N_j-c_j}{2} \end{aligned}$$

Thus when nearly all issuers are covered, the optimal number of bonds to purchase is just slightly less than the number of recommended issuers (which is $c_j/2$).

For a portfolio spanning several qualities, the expression for the optimal number of bonds is more complex. If we assume that the number of bonds chosen in each quality will always be less than the recommended number of issuers $\left(n_j<c_j/2\right)$, then we can proceed much as we did in Appendix D:

$$z = \sum_j x_j b_j \left(1 - \frac{n_j}{c_j}\right)$$

(E-18) $$v = \sum_j x_j^2 \sigma_j^2 \left(\frac{1}{n_j} - \frac{1}{N_j}\right)$$

$$IR = \frac{z}{\sqrt{v}}$$

Taking the partial derivative with respect to the number of bonds in an arbitrary quality q, we have

(E-19) $$\frac{\partial IR}{\partial n_q} = \frac{\partial z}{\partial n_q}\frac{1}{\sqrt{v}} - \frac{1}{2} z \frac{1}{v\sqrt{v}}\frac{\partial v}{\partial n_q} = 0$$
$$= -\frac{x_q b_q}{c_q}\frac{1}{\sqrt{v}} - \frac{z}{2v\sqrt{v}}\left(-\frac{x_q^2\sigma_q^2}{n_q^2}\right) = -\frac{x_q}{\sqrt{v}}\left(\frac{b_q}{\cdot c_q} - \frac{z}{2v}\cdot\frac{x_q\sigma_q^2}{n_q^2}\right) = 0$$

Solving for n_q gives

(E-20) $$n_q^2 = \frac{z}{2v}\cdot\frac{x_q\sigma_q^2 c_q}{b_q}$$

Proceeding as before to find the ratio between the optimal numbers of bonds in any two quality groups q and r, we have

(E-21) $$\frac{n_q}{n_r} = \sqrt{\frac{x_q\sigma_q^2 c_q b_r}{x_r\sigma_r^2 c_r b_q}}\,.$$

Comparing to Equation (D-10), we see that the number of covered issuers has taken the place of the number of issues in the index. However, Equations E-18 through E-21 do not necessarily define the optimal solution for the piecewise linear case, because there is no justification for making the above assumption $\left(n_j < c_j/2\right)$. In fact, our numerical solutions to this problem often violate this condition.

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