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Active Managers Versus Passive Products: How to win the debate...

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The widespread availability of passive products have investors asking: Why pay for active management if I can potentially do just as well with a passive product? Active managers must respond, demonstrating the value of their products and substantiating that value by demonstrating consistent and repeatable investment skill. Performance attribution is a natural starting point for assessing the value of an investment product. One of the issues encountered when using standard attribution methods is that the structure used to decompose returns—a classification scheme for Brinson-based attribution and a linear returns model for factor-based attribution—may not match the investment process of the portfolio manager. In this paper, we contend that investment managers should tailor attribution methods to match their investment processes, highlighting sources of portfolio returns that correspond to their intended bets and showing that these sources are statistically significant overtime. The goals of this paper are two-fold: first, we identify issues commonly encountered with standard, off-the-shelf performance attribution methodologies, and, second, we suggest ways to customize and extend these methodologies to yield attribution results that better match the investment process at hand.

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Contents

1 Introduction

The widespread availability of passive products has investors asking: Why pay for active management if I can potentially do just as well with a passive product? Active managers must respond, demonstrating the value of their products and substantiating that value by demonstrating consistent and repeatable investment skill.

Performance attribution, the process that identifies and quantifies sources of a portfolio's realized return, is a natural starting point for assessing the value of an investment product. Two performance attribution methods commonly used in commercial products are:

- Brinson-based performance attribution, which compares a portfolio's performance to that of a benchmark and decomposes the active return into different allocation and asset selection effects for different groups or classifications of assets.
- Factor-based performance attribution, which decomposes a portfolio's return into systematic (factor) and asset-specific contributions using a linear asset returns model.

One of the issues encountered when using standard, off-the-shelf attribution methods is that the structure used to decompose returns—a classification scheme for Brinson-based attribution and a linear returns model for factor-based attribution—may not match the investment process of the portfolio manager. Consider, for example, a manager who invests in value assets using his or her own proprietary approach for measuring value. It is unlikely that the manager's true exposure to value, as he or she measures it, will be captured in a generic classification scheme or returns model where value is measured differently; as a result, an attribution computed using a generic classification scheme or returns model will not capture the manager's intended "bet" and cannot appropriately attribute returns to that bet.

In this paper, we contend that investment managers should tailor attribution methods to match their investment processes, highlighting sources of portfolio returns that correspond to their intended exposures or bets (demonstrating skill) and showing that these sources are statistically significant (demonstrating consistent and repeatable skill). The goals of this paper are two-fold: first, we identify issues commonly encountered with standard, off-the-shelf performance attribution methodologies, and, second, we suggest ways to customize and extend these methodologies to yield attribution results that better match the investment process at hand.

Custom attribution has been proposed by others in the context of customizing the factors used in the attribution [Blitz (2010); David (2010)]. In this paper, we take a more holistic approach and consider different methods for customizing the structure used to decompose returns in Brinson- and factor-based methodologies, in addition to considering customizations to the attribution methodologies themselves. Moreover, we demonstrate how to analyze sources of return in a hierarchical framework (whether using Brinson- or factor-based methodologies), enabling the practitioner to evaluate, for example, the fundamental sources that drive sector or country return.

We proceed with discussions of Brinson- and factor-based attribution methodologies and consider different ways to extend these methodologies for several sample portfolios. We then compare the Brinson- and factor-based attributions for a particular sample portfolio, evaluating and explaining the differences between the Brinson- and factor-based attribution results.

2 Brinson-Based Attribution

Brinson attribution methodologies (Brinson and Fachler, 1985; Brinson et al., 1986) explain the impact of portfolio management decisions on performance by comparing the portfolio's performance to that of a benchmark and decomposing the excess return into different effects using a classification scheme.

Consider a portfolio with weights $w_a^p, a \in A$, and a benchmark with weights $w_a^b, a \in A$, where A is the set of all assets available for investment. In the classic sector-based attribution, which assumes a top-down investment approach where the manager makes sector weighting decisions followed by stock selection decisions, performance is explained using the following decomposition

$$R^A = \underbrace{\sum_{s \in S}(W_s^p - W_s^b)(r_s^b - R^b)}_{\text{sector allocation effects}} + \underbrace{\sum_{s \in S} W_s^p(r_s^p - r_s^b)}_{\text{selection effects}} \tag{1}$$

where $R^A = \sum_{a \in A}(w_a^p - w_a^b)r_a$ is the portfolio's active return, $W_s^p = \sum_{a \in s} w_a^p$ and $W_s^b = \sum_{a \in s} w_a^b$ are the portfolio and benchmark weight for sector s, $r_s^p = (1/W_s^p)\sum_{a \in s} w_a^p r_a$ and $r_s^b = (1/W_s^b)\sum_{a \in s} w_a^b r_a$ are the portfolio and benchmark return for sector s, and $R^b = \sum_{a \in A} w_a^b r_a$ is the benchmark return.

The first term on the right-hand side of equation (1) represents sector allocation effects, and the second term represents asset selection effects including interaction effects[1]. A positive allocation effect is obtained by either over-weighting an outperforming sector or under-weighting an underperforming sector, and a positive selection effect is obtained by earning a portfolio sector return that is greater than the benchmark sector return.

The Brinson attribution given in equation (1) can be applied to groupings other than sectors, such as countries or value quantiles. The only requirement is that each group contain a distinct (non overlapping) set of assets.

[1]Brinson-based methodologies sometimes decompose active return into three effects (allocation, interaction, and selection), but for the purpose of this paper, since interaction effects tend to be small, we include them in the selection effect.

2.1 Explaining Active Returns: An Example (Value Momentum Portfolio)

Consider a portfolio that makes value, sector momentum, and country momentum bets, herein after referred to as the *Value Momentum Portfolio*. This portfolio was generated as a linear combination of three strategies: the first strategy, accounting for 50% of the final strategy, selects stocks within each sector based on relative value scores while ensuring that there were no sector, country, or other "style" bets; the second strategy, accounting for 25% of the final strategy, weights capitalization-weighted sector portfolios according to a sector momentum signal; and the third strategy, accounting for 25% of the final strategy, weights capitalization-weighted country portfolios according to country-momentum signal that is based on USD total returns of the country portfolios. We analyze this portfolio because it is relatively complex, with (potentially) multiple sources of return to identify and measure, and thus makes for an informative case study.

We begin our analysis by decomposing the return of this portfolio using a sectors model, obtaining the results in Table 1, and a countries model, obtaining the results in Table 2.

Source of Return	Effect	IR	t-stat
Active Return	1.75%	0.95	3.52
Sectors Allocation Effect	0.69%	0.82	3.07
Selection Effect	1.06%	0.62	2.34

Table 1: Value Momentum Portfolio: Brinson attribution with sectors model

Source of Return	Effect	IR	t-stat
Active Return	1.75%	0.95	3.52
Countries Allocation Effect	0.40%	0.40	1.51
Selection Effect	1.35%	0.88	3.28

Table 2: Value Momentum Portfolio: Brinson attribution with countries model

Which attribution tells the right story for this portfolio? The sectors allocation effect (69 bps) is significantly different from zero, while the countries allocation effect (40 bps) is not significantly different from zero; moreover, the contribution from sectors is greater than that of countries; this suggests that sector allocation decisions had a greater impact on returns than country allocation decisions. The selection effect in both models is sizable and significantly different from zero, but may account for other allocation effects not explicitly included in the model (such as value or momentum allocation effects). For these reasons, we conclude that the attribution results in Tables 1 and 2 tell only part of the investment story for the portfolio.

2.2 Extending Brinson-Based Attribution with a Custom Classification Scheme

A Brinson attribution can go beyond the classic sector model using a nested attribution methodology [van de Burgt et al. (2001); Menchero (2004)] that decomposes active return into one or more allocation effects, in addition to selection effects. Unlike standard, off-the-shelf Brinson attribution methodologies, which typically contain one type of allocation effect (e.g., sectors), a nested attribution can account for multiple types of allocation effects, permitting the identification of multiple sources of return.

Suppose the *Value Momentum Portfolio* described previously were constructed using a top-down investment strategy where the manager makes sector weighting decisions, followed by value weighting decisions, followed by security selection decisions. This strategy comprises three levels of decision making, which are summarized in Table 3, and can be modeled using a classification scheme that categorizes assets by their membership to sector-specific value quintiles. Figure 1 presents a simplified version of the classification scheme that corresponds to this top-down, three-step decision making process.

Level	Decision	Choice
1	Sector weighting	Choose weights for 10 sectors
2	Value weighting	Choose value weights within each sector (e.g., choose weights for quintile buckets within each sector bucket)
3	Security selection	Choose securities that will attain the weighting goal set at decision level 2

Table 3: Top-down investment strategy/three-step process

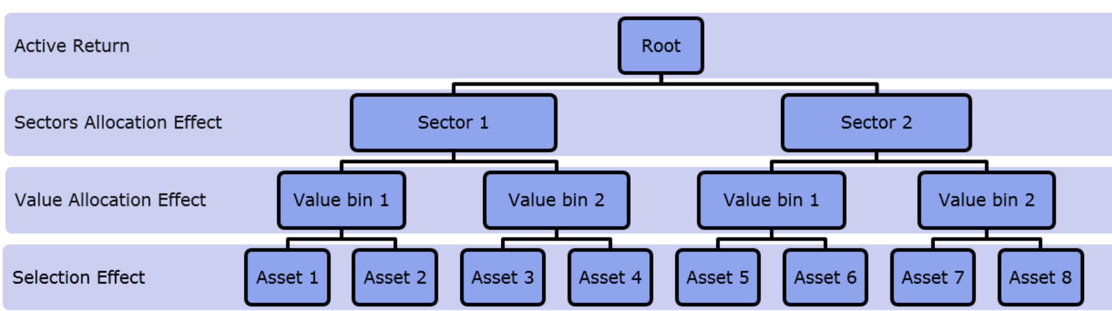


Figure 1: Nested Attribution Classification Scheme

Mathematically, a performance attribution with this classification scheme is computed using the following decomposition

$$R^A = \sum_{s \in S} \left(AE_s^{(1)} + AE_s^{(2)} + SE_s^{(2)} \right) \tag{2}$$

where $AE_s^{(1)} = (W_s^p - W_s^b)(r_s^b - R^b)$ is the sector allocation effect corresponding to group s in the first level of the hierarchy, $AE_s^{(2)} = \sum_{j \in s} W_s^p (W_{s,j}^p / W_s^p - W_{s,j}^b / W_s^b)(r_{s,j}^b - R_s^b)$ is the sub-sector allocation effect corresponding to group s, and $SE_s^{(2)} = \sum_{j \in s} W_{s,j}^p (r_{s,j}^p - r_{s,j}^b)$ is the selection effect. (Note that $W_{s,j}^p = \sum_{a \in j} w_a^p$ and $W_{s,j}^b = \sum_{a \in j} w_a^b$ are the portfolio and benchmark weight for group $j \in s$, and $r_{s,j}^p = (1/W_{s,j}^p) \sum_{a \in j} w_a^p r_a$ and $r_{s,j}^b = (1/W_{s,j}^b) \sum_{a \in j} w_a^b r_a$ are the portfolio and benchmark return for group $j \in s$.) In this decomposition, positive value allocation effects are obtained by over-weighting the outperforming value bins within a sector or underweighting underperforming value bins within a sector, where performance is measured relative to the sector. Value allocation effects provide insight into the groups (bins) that drive performance at the sector level.

Table 4 presents the Brinson attribution of the *Value Momentum Portfolio* using a hierarchical, sectors-value classification scheme (herein after referred to as the sectors-value model), where groups in the first level of the hierarchy correspond to sectors and groups in the second level correspond to value quintiles within sectors. The sectors and value allocation effects are both positive and significantly different from zero, which suggests that the manager made beneficial sector and value weighting decisions, with value decisions accounting for the majority of the active return (132 out of 157 basis points). The stock selection effect, in contrast, is negligible and not statistically different from zero.

Note that the 69 bps sectors allocation effect in Table 4, which was obtained with the sectors-value model, is the same as the sectors allocation effect presented in Table 1, which was obtained with the sectors model. Nested attribution implicitly assumes that investment decisions were made in a top-down fashion; as a result, adding a new level to the classification hierarchy does not impact the effect computed at the parent level.

Source of Return	Effect	IR	t-stat
Active Return	1.57%	0.95	3.52
Sectors Allocation Effect	0.69%	0.82	3.07
Value Allocation Effect	1.32%	0.71	2.66
Selection Effect	-0.26%	-0.15	-0.57

Table 4: Value Momentum Portfolio: Brinson attribution with sectors-value model

Figure 2 presents a breakdown of the attribution effects obtained with the sectors-value model. Here we observe that the total value allocation effect (132 bps) was driven largely by value allocation decisions in financials (38 bps), materials (23 bps), and consumer staples (23 bps). The total sector allocation effect (69 bps) was driven primarily by sector weighting decisions in information technology (24 bps) and materials (20 bps).

Figures 3 and 4 present the average active weight and total effect within each sector-specific value bucket. Here we observe that, within each sector, the manager overweighted assets in the top quintile (assets with the highest value scores) and underweighted assets in all other quintiles. This strategy worked well for assets in the top value quintile, with particularly

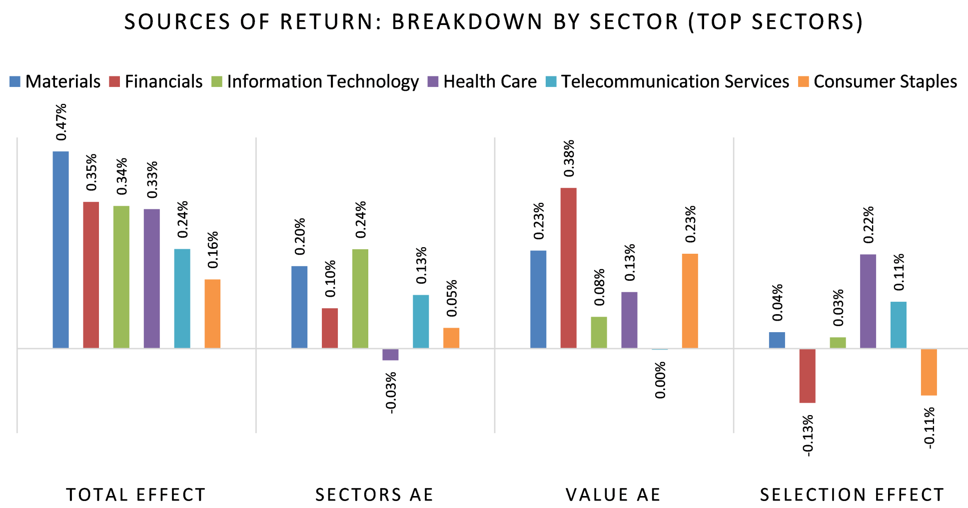


Figure 2: Value Momentum Portfolio: Brinson attribution with sectors-value model (top 6 sectors)

strong performance attributed to the top value quintile in the financial sector. Underweighting the second, third, and fourth value quintile in the financial sector, in contrast, did not work as well, since the effect attributed to these groups is negative.

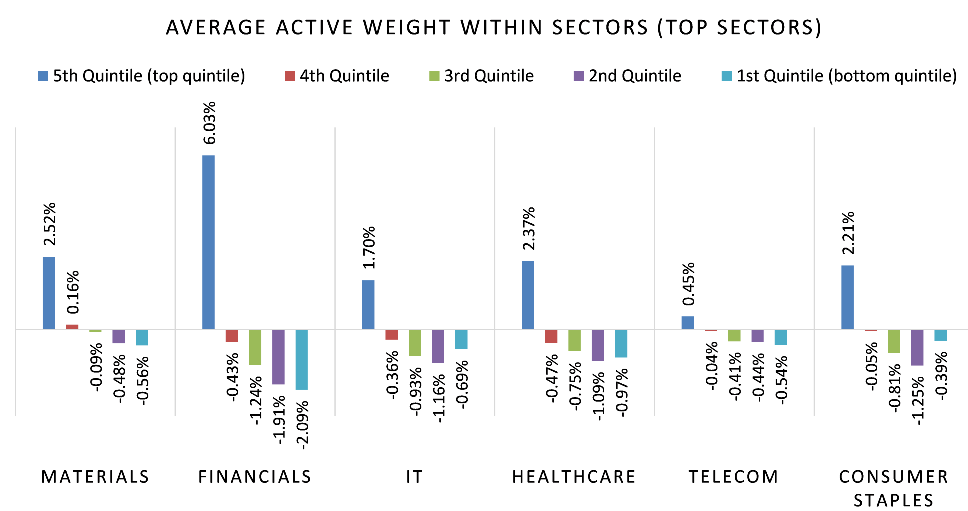


Figure 3: Value Momentum Portfolio: Brinson attribution with sectors-value model (top 6 sectors)

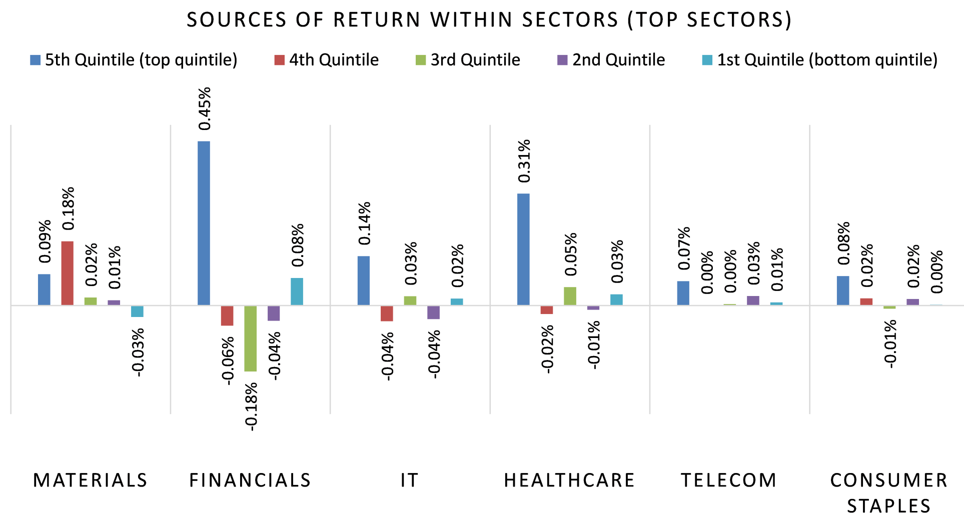


Figure 4: Value Momentum Portfolio: Brinson attribution with sectors-value model (top 6 sectors)

2.3 Incorporating Currency Effects in Multi-Currency Portfolios

Performance must be assessed in a single reporting (or numeraire) currency when the portfolio being analyzed contains assets denominated in more than one currency.

Consider an asset that trades in a local market that is different from the investor's reporting or numeraire market. Letting p^l denote the price of this asset denominated in its local currency l, we can express its price denominated in the numeraire currency n as

$$p^n = p^l x_{nl} \tag{3}$$

where x_{nl} denotes the numeraire exchange rate, expressed in the nth currency per unit of currency l. It follows from equation (3) that the return of the asset denominated in the numeraire currency n can be expressed as

$$\begin{aligned} r^n &= [1 + r^l][1 + e_{nl}] - 1 \\ &= r^l + e_{nl} + r^l e_{nl} \end{aligned}$$

where $e_{nl} = x_{nl}(t)/x_{nl}(t-1) - 1$ is the return of the numeraire exchange rate with respect to the local currency l (Zangari, 2003a).

Thus, when we decompose asset returns denominated in the investor's reporting currency (r^n), we decompose these returns into the following three components:

1. The local return of the asset, denominated in the the local currency (r^l)
2. The return of the numeraire exchange rate relative to the local currency (e_{nl})

3. The product of the local return of the asset and the exchange rate return $(r^l e_{nl})$

The corresponding decomposition of active excess returns at the portfolio level is given by

$$\tilde{R}^A = \tilde{L}^A + \tilde{C}^A + \tilde{Q}^A \tag{4}$$

where $\tilde{R}^A$ denotes the portfolio's active excess return denominated in the numeraire currency, $\tilde{L}^A$ is the active local excess return denominated in local currencies, $\tilde{C}^A$ is the active currency excess return, and $\tilde{Q}^A$ is the product of the local excess return of the portfolio and the currency excess return. Appendix A presents the formulas behind this decomposition. For a more comprehensive review of multi-currency performance attribution, see Menchero and Davis (2009) and Singer and Karnosky (1995).

Active local excess returns, $\tilde{L}^A$, can be further decomposed into allocation and selection effects using a nested classification scheme. For example, decomposing the local returns using a sectors classification scheme would result in the decomposition shown in Figure 5.

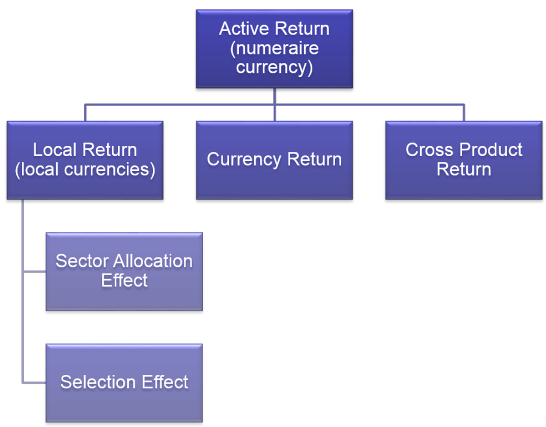


Figure 5: Multi-currency decomposition

What does multi-currency attribution mean for the Value Momentum Portfolio? Figure 6 presents the Brinson attribution of the *Value Momentum Portfolio* using a hierarchical countries-momentum classification scheme, with and without currency effects. In the decomposition without currency effects, the countries allocation effect accounts for 40 bps of the active return, but in the decomposition with currency effects, the countries allocations effect accounts for only 20 bps, with the currency and cross product effects accounting for the remaining 20 bps. This suggests that the return attributed to country bets is in large part explained by currency movements, rather than local market movements.

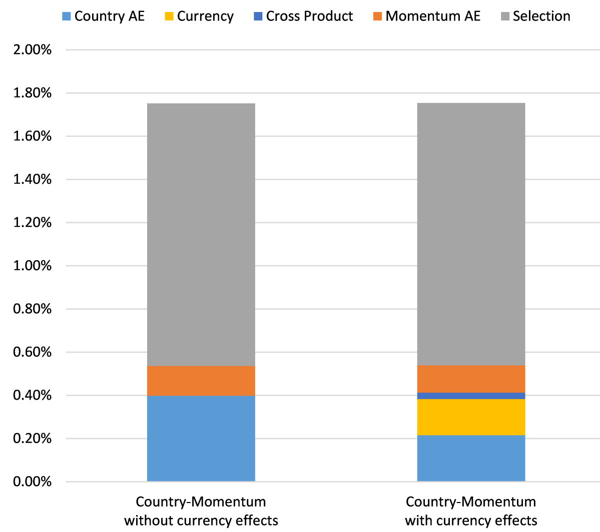


Figure 6: Value Momentum Portfolio: Brinson attribution with the countries-momentum model (with and without currency effects)

2.4 Evaluating Classification Schemes

In the previous sections, we showed that that the sources of return identified in a Brinson attribution depend on the classification scheme employed. When the *Value Momentum Portfolio* was decomposed using a sectors, countries, or country-momentum model, the majority of the active return was attributed to specific effects, and when the portfolio was decomposed with the sectors-value model, the majority was attributed to value allocation effects.

Which attribution is the right attribution? Recall that the *Value Momentum Portfolio* was generated as a linear combination of three strategies: the first strategy, accounting for 50% of the final strategy, selects stocks within each sector based on relative value scores while ensuring that there were no sector, country, or other "style" bets; the second strategy, accounting for 25% of the final strategy, weights capitalization-weighted sector portfolios according to a sector momentum signal; and the third strategy, accounting for 25% of the final strategy, weights capitalization-weighted country portfolios according to country-momentum signal that is based on USD total returns of the country portfolios. In short, the *Value Momentum Portfolio* was created systematically with value, sector momentum, and country momentum weighting decisions, but no asset-specific decisions. Thus, the attribution with the sectors-value classification scheme seems fitting; however, it does not provide insight into the sector momentum and country momentum signals.

Figure 7 presents the Brinson attribution of the *Value Momentum Portfolio* using three different models (sectors-value, sectors-momentum, and country-momentum), where each is presented with and without currency effects. Comparing these results, we see that the sectors and value allocation effects are sizeable (approximately 65 and 135 bps, respectively), and the country, momentum, and currency effects are relatively small (approximately 20 bps each). The selection effect is negligible in the sectors-value attribution, but sizeable in the sector-momentum and country-momentum attributions. Since the selection effect potentially captures "residual" allocation effects, the sizeable selection effect in the sector-momentum and country-momentum attributions likely represents the sector and value bets that are not accounted for in these models.

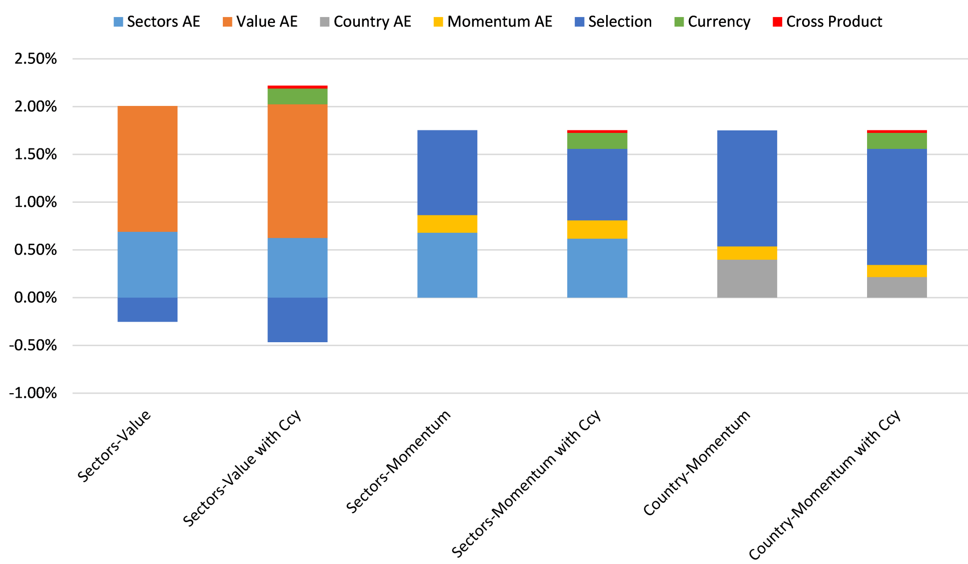


Figure 7: Value Momentum Portfolio: Brinson attribution with the three different models (with and without currency effects)

This leads us to ask the following questions. First, can a Brinson model account for more than two allocation effects? For example, can we look at sectors, countries, value, and momentum effects in the same attribution? Second, how do we measure the significance of Brinson attribution results?

2.4.1 Accounting for More than Two Allocation Effects

The nested attribution methodology discussed previously decomposes active return into one or more allocation effects, in addition to selection effects, using a hierarchical classification scheme. Each level in the hierarchy corresponds to an *effect*, with the effect at the lowest level in the hierarchy corresponding to the well-known selection effect.

Consider a classification scheme with four levels: (1) sectors, (2) value buckets within sectors, (3) momentum buckets within sector-value buckets, and (4) assets within sector-value-momentum buckets. In a nested attribution, momentum allocation and selection effects are computed for each distinct sector-value-momentum group. If there are ten sectors, five value buckets within each sector, and five momentum buckets within each sector-value bucket, we must compute momentum allocation and selection effects of $10 * 5 * 5 = 250$ groups. This begs the question: did the investor actually make individual stock selection decisions within these 250 groups? (We suspect not.)

The nested attribution methodology suffers from the curse of dimensionality: as you add levels to a classification scheme, the number of groups for which effects must be computed grows exponentially. This is problematic for two reasons: first, the number of groups will likely exceed the number of assets, and second, the classification scheme is less likely to mirror the investment process (that is, in the example where assets are grouped by sector, then value bucket within sectors, and then momentum bucket within sector-value buckets, it is assumed that the investor first makes weighting decisions for the 10 sectors, then 50 weighting decisions for the sector-value buckets, and then 250 weighting decisions for the sector-value-momentum buckets—which seems unlikely).

As a result, if a portfolio's returns are driven by more than two or more sources, we recommend using factor-based attribution methods, which are described in the *Factor-Based Attribution* section of this paper.

2.4.2 Measuring the Significance of Effects

The output of a nested attribution comprises time series of "effects" (components of the active return), computed for each asset group in the classification scheme employed. For example, in the classic sectors model (which corresponds to a classification scheme with sector groups only), a sectors allocation effect, $AE^{(1)}_{s,t} = (W^p_{s,t} - W^b_{s,t})(r^b_{s,t} - R^b_t)$, is computed for each sector s and for each period $t = 1, ..., T$. Similarly, a selection effect, $SE^{(1)}_{s,t} = W^p_{s,t}(r^p_{s,t} - r^b_{s,t})$, is computed for each sector s and each period $t = 1, ..., T$. These effects are summarized across sectors, generating the time series $AE^{(1)}_t = \sum_{s \in S} AE^{(1)}_{s,t}$ and $SE^{(1)}_t = \sum_{s \in S} SE^{(1)}_{s,t}$.

Let $e_t, t = 1, ..., T$ denote a generic "effect" series (which, in a classic sectors model, could be a sector-specific effect or an effect summarized across sectors). This series is typically summarized across time by computing the average effect $\bar{e} = (1/T)\sum_t e_t$. To determine if the average effect is statistically different from zero, practitioners often test the null hypothesis that the average effect is zero versus the alternative that it is not zero, forming the following test statistic

$$\text{t-stat} = \frac{\bar{e} - e_0}{s.e.(\bar{e})}$$

where $e_0 = 0$ and $s.e.(\bar{e})$ is the standard error of $\bar{e}$. If certain assumptions about the distribution of e_t hold, under the null hypothesis, this test statistic follows the $t(T-1)$ distribution; thus we can compute the probability of observing a test statistic at least as large as the observed statistic (t-stat). If this probability is greater than the desired significance level of the test, then we conclude that the null hypothesis is true at that significance level. (Alternatively, we can form a confidence interval that consists of a range of values that act as good estimates of the unknown (true) average effect under the null hypothesis, and accept the null hypothesis if the observed test statistic falls within this range.)

T-tests and other tests (parametric and nonparametric) may also be used to determine if the means from two series of effects are different from each other, although a discussion of these tests is beyond the scope of this paper.

Why is it important to measure the significance of effects? Statistical significance gives us confidence in our estimates of average effects and lets us focus only on those estimates that are reliably different from zero (or some other number). More generally, it gives us a bar with which to compare attribution results generated with different classification schemes. If a classification scheme yields an attribution with effects that are statistically significant, the scheme is deemed to be a better fit for the investment process than a classification scheme that yields effects that are not statistically significant.

2.5 Guidelines for Explaining Returns Using Brinson-Based Attribution

Brinson-based attribution methodologies are widely used by portfolio managers, and are thus relatively easy to interpret. To obtain a more detailed view of performance, we recommend using nested attribution with a custom classification scheme that mirrors the investment process. The advantage of using nested attribution is that it permits practitioners to identify and measure more than one systematic source of return; in addition, the hierarchical structure of the classification scheme makes it possible to identify sources that drive performance within a particular category of assets (e.g., in the *Value Momentum Portfolio* presented previously, we were able to identify the sectors that drive value performance).

It is relatively easy to generate custom classification schemes, and for this reason, we believe Brinson-based methodologies readily provide a high-level view of a portfolio's performance. However, because Brinson-based methodologies decompose *active returns*, they are not ideally suited for portfolios that do not track a particular benchmark, such as long-short portfolios.

When explaining total performance (as opposed to active performance), and when explaining a complex investment strategy in which there are more than two or three sources of return that the practitioner would like to identify, we recommend factor-based methodologies, which is the topic of the next section.

3 Factor-Based Attribution

Factor-based attribution (Zangari, 2003b) decomposes a portfolio's return into factor components and an asset-specific, idiosyncratic component using an asset returns model that takes the form

$$r_t = X_{t-1} f_t + \epsilon_t \tag{5}$$

where r_t is an n-dimensional vector of asset returns, f_t is an m-dimensional vector of factor returns, X_{t-1} is an $n \times m$ matrix of factor exposures (each element represents an individual asset's exposure to a particular factor), and ϵ_t is an n-dimensional vector of residual asset returns at time t.[2]

The returns decomposition implied by this model is

$$\underbrace{w_t^T r_t}_{\text{portfolio return}} = \underbrace{w_t^T X_{t-1} f_t}_{\text{factor contribution}} + \underbrace{w_t^T \epsilon_t}_{\text{specific contribution}}$$

where w_t is an n-dimensional vector of portfolios weights at time t.

Clearly, a factor-based decomposition depends on factor exposures (X_{t-1}), factor returns (f_t), and specific returns (ϵ_t), and is therefore sensitive to the ways in which these exposures and returns are constructed.

In the sections that follow, we explore different techniques for formulating and estimating returns models, and evaluate their impact on attribution results. We begin with a brief overview on how returns models are formulated and estimated, identifying sources of flexibility in their construction.

3.1 Formulating and Estimating a Returns Model

3.1.1 A Brief Overview

Consider the returns model given in equation (4). For the purpose of this paper, we assume that factor exposures X_{t-1} are known, and estimate factor returns f_t using cross-sectional regression methods such as ordinary least squares (OLS), weighted least squares (WLS), or other robust methods. Formulating and estimating a returns model in this context involves the following steps:

Step 1: Determining which assets to include in the estimation universe

When estimating factor returns, the practitioner must choose an estimation universe that represents the market he or she wishes to model, while avoiding illiquid and other potentially

[2]The asset returns model in equation (4) should be familiar to practitioners who use factor risk models to measure portfolio risk. A factor risk model is typically created by (1) specifying a linear returns model that relates asset returns to common factor returns and asset-specific returns, (2) estimating the unknown components of the returns model over a pre-specified time period, thereby obtaining time series of factor and specific returns, (3) estimating factor return and specific return covariance matrices from the time series of factor and asset-specific returns, and (4) estimating an asset return covariance matrix using the factor and specific return covariance matrices. This paper is concerned with decomposing a portfolio's return and thus focuses on methods for specifying and estimating a returns model only. For a comprehensive review of factor models, the interested reader is referred to Fabozzi (2012).

"problematic" assets, such as investment trusts and depository receipts. Many practitioners use membership in an appropriate market index as the basis for inclusion in an estimation universe, since benchmark providers typically employ sophisticated selection criteria based on price activity, liquidity, and other business logic that would be difficult to replicate in-house. (Axioma Risk Model Handbook)

Step 2: Determining which factors to include in the model

A returns model will typically include one or more of the following factors:

- A market factor representing the return attributed to the broad market as a whole
- Sector or industry factors representing the return attributed to individual sectors or industries (note that, herein after, we shall refer to such factors as "sector factors," regardless of whether they refer to sectors or industries)
- Style factors representing the return attributed to asset fundamentals (such as value, leverage, and growth) and market-based fundamentals (such as size, liquidity, and momentum)
- Country factors representing the return attributed to individual countries
- Currency factors representing the return attributed to individual currencies

Further, a returns model might include *style factors for each sector or country* in the model. This is typically justified for styles that perform differently in different sectors or countries.

Step 3: Computing factor exposures / standardizing exposures to style factors

If the market factor is modeled as a regression intercept term, then all assets will have unit exposure to that factor, while exposures to sectors, countries, and currencies are typically binary (0 or 1). Exposures to style factors, on the other hand, are continuous; an asset's exposure to liquidity, for example, may be computed as a function of its average trading volume, while its exposure to value may be computed as a function of its common equity and average market capitalization.

Because exposures to styles are expressed in different units, they are typically standardized (by subtracting the mean from each exposure value and then dividing these new values by the standard deviation of the variable), making it easier to interpret and compare exposures, while also ensuring that scaling and/or matrix inversion problems are avoided when estimating the model.

The practitioner has at his or her disposal a variety of options for standardizing exposures. He or she may choose to standardize raw exposures around a capitalization- or equal-weighted mean computed over the entire market. Alternatively, he or she may standardize raw exposures around capitalization- or equal-weighted means computed for each sector.

Let s denote an n-dimensional vector of raw exposures for a particular style, and let x_{t-1}

denote the n-dimensional vector of corresponding standardized exposures. If x_{t-1} is centered around a capitalization-weighted mean computed over the entire market, it will be computed as

$$x_{t-1} = \frac{1}{\sigma_s}\Big(s - (m^T s)u\Big) \tag{6}$$

where σ_s is the standard deviation of the raw exposures s, m is an n-dimensional vector of capitalization weights, and u is an n-dimensional vector of ones.

If x_{t-1} is centered around capitalization-weighted means computed per sector, it will be computed as

$$x_{t-1} = \begin{bmatrix} \frac{1}{\sigma_1}\big(s_1 - (m_1^T s_1)u_1\big) \\ \frac{1}{\sigma_2}\big(s_2 - (m_2^T s_2)u_2\big) \\ \vdots \\ \frac{1}{\sigma_L}\big(s_L - (m_L^T s_L)u_L\big) \end{bmatrix} \tag{7}$$

where s_l denotes an N_l-dimensional vector of raw exposures for sector l, σ_l is the standard deviation of the raw exposures s_l, m_l is an N_l-dimensional vector of capitalization weights for sector l, rescaled to sum to one, and u_l is an N_l-dimensional vector of ones, $l = 1, ..., L$.

Step 4: Estimating the returns model

Recall that our returns model is formulated as

$$r_t = X_{t-1} f_t + \epsilon_t$$

where r_t is an n-dimensional vector of asset returns, f_t is an m-dimensional vector of factor returns, X_{t-1} is an $n \times m$ matrix of factor exposures, and ϵ_t is a vector of residual asset returns at time t.

A natural starting point for estimating the unknown factor returns f_t is to use ordinary least squares (OLS), obtaining the well-known solution

$$\hat{f}_{t,OLS} = (X_{t-1}^T X_{t-1})^{-1} X_{t-1}^T r_t.$$

To ensure that the residuals, ϵ_t, have constant variance across observations[3], it is common practice to scale each asset's residual by the inverse of its residual variance, transforming the OLS problem into a weighted least squares (WLS) problem of the form

$$W_{t-1}^{1/2} r_t = W_{t-1}^{1/2} X_{t-1} f_t + W_{t-1}^{1/2} \epsilon_t, \quad W_{t-1} = diag(1/\sigma_1^2, ..., 1/\sigma_n^2)$$

with the solution

$$\hat{f}_{t,WLS} = (X_{t-1}^T W_{t-1} X_{t-1})^{-1} X_{t-1}^T W_{t-1} r_t,$$

[3]If the residuals have constant variance across observations, they are deemed to be homoskedastic, thereby satisfying one of the conditions of the Gauss-Markov theorem which establishes the superiority of the least squares solution over other linear estimators.

where individual residual variances are estimated using market capitalization or other data. Further, to reduce the effect of outliers in the weighted least squares estimates, the practitioner may choose to employ robust statistical methods to estimate f_t, although a discussion of the impact on using these methods is beyond the scope of this paper.

Step 5: Handling linearly dependent inputs

If a returns model contains a market intercept term, binary (0 or 1) sector exposures, and/or binary (0 or 1) country exposures, then a unique solution to the WLS estimation problem will not exist because the regression-weighted matrix of factor exposures, $W_{t-1}^{1/2}X_{t-1}$, is singular.

One can resolve this issue by imposing constraints on the set of industry and country exposures, forcing the regression-weighted sum of each set of factor returns to be zero (Heston and Rouwenhorst, 1992). For example, let f_{t,S_j} denote the sector returns for sectors $j = 1, ..., P$ and let ω_{t-1,S_j} denote the regression weight, computed from W_{t-1}, in each sector $j = 1, ..., P$. Similarly, let f_{t,C_k} denote the country returns for countries $k = 1, ..., Q$ and let ω_{t-1,C_k} denote the regression weight, computed from the matrix W_{t-1}, in each country $k = 1, ..., Q$. Then, the following constraints would be applied:

$$\sum_{j=1}^{P} \omega_{t-1,S_j} f_{t,S_j} = 0$$

$$\sum_{k=1}^{Q} \omega_{t-1,C_k} f_{t,C_k} = 0$$

Including these constraints in the problem will "force" the broad market return into the market intercept factor, permitting the practitioner to interpret country (sector) factor returns as "excess returns" over the market return of a capitalization-weighted portfolio that is neutral to all sectors and styles (country and styles).

One can also use a *staged regression* to resolve this issue, as discussed in the next step.

Step 6: Employing a staged regression

Another technique that may be employed when formulating and estimating a returns model is a *staged regression* where asset returns are regressed on a subset of factors in the first stage, and the residual obtained from this regression is regressed on the remaining model factors in the second stage. For example, using the returns model in equation (4), let $f_{t,0}$ denote the market factor and let $f_{t,-0}$ denote an $(m-1)$-dimensional vector containing all remaining factors. Similarly, let $X_{t-1,0}$ denote an n-dimensional vector of ones and $X_{t-1,-0}$ denote an $n \times (m-1)$ matrix of factor exposures. Then a staged regression model that estimates the market factor's returns first would comprise the following systems of equations,

$$r_t = X_{t-1,0} f_{t,0} + \mu_t \qquad \text{(stage 1)}$$

and

$$\mu_t = X_{t-1,-0} f_{t,-0} + \epsilon_t. \qquad \text{(stage 2)}$$

If estimated using WLS, these questions would be given by

$$W_{t-1,1}^{1/2} r_t = W_{t-1,1}^{1/2} f_{t,0} X_{t-1,0} + W_{t-1,1}^{1/2} \mu_t \qquad \text{(stage 1, WLS)}$$

and

$$W_{t-1,2}^{1/2} \mu_t = W_{t-1,2}^{1/2} X_{t-1,-0} f_{t,-0} + W_{t-1,2}^{1/2} \epsilon_t \qquad \text{(stage 2, WLS)}$$

where the weighting matrices $W_{t-1,1}$ and $W_{t-1,2}$ need not be the same matrix. [4]

3.1.2 Sources of Flexibility

In summary, formulating and estimating a returns model involves a number of decisions, including the choice of:

1. *Universe of assets from which factor returns are estimated:* When estimating factor returns, practitioners must choose an estimation universe that represents the broad market they wish to model, while avoiding illiquid and other potentially "problematic" assets.

2. *Factors to include in the model:* A returns model may include a market factor, in addition to sets of factors representing sectors, styles (including "alpha" factors), countries, and currencies. If the market factor is included and if it is modeled as a regression intercept term, then all assets will have unit exposure to that factor. Exposures to sectors, countries, and currencies are typically binary (0 or 1), while exposures to style factors are continuous.

3. *Method for standardizing exposures to style factors:* For each style factor, exposures are typically standardized to avoid scaling and/or matrix inversion problems in the regression. When standardizing exposures, the practitioner has a variety of options at his or her disposal (e.g., she or he can choose whether the mean is a capitalization- or equal-weighted mean, computed over the entire market or sectors).

4. *Weighting scheme (W_{t-1}) employed in the WLS regression:* A capitalization weighting scheme will place more emphasis on large-cap assets in the estimation, while an equal weighting scheme will place the same emphasis on all assets.

5. *Constraints imposed in the regression, if there are linearly dependent variables:* If the model includes a market factor in addition to sets of factors representing sectors and countries, then the practitioner may incorporate constraints on the sets of linearly

[4] In this staged regression, if the weighting matrices $W_{t-1,1}$ and $W_{t-1,2}$ are different and if the second stage regression includes industry factors, then one would want to constrain the weighted sum of industry returns to be zero, thereby ensuring that industry returns can be interpreted as excess returns over the market return obtained in the first stage.

dependent variables to resolve issues of perfect multicollinearity. Alternatively, he or she may employ a staged regression.

6. *Ordering/grouping of factors in a staged regression:* If a practitioner employs a staged regression (either to resolve issues of multicollinearity or for reasons related to how he or she interprets the factor returns), then he or she much choose the factors to include at each stage as well the order of the stages.

These choices have a great impact on the factor exposures X_{t-1}, estimated factor returns f_t, and the factor and specific contributions obtained in a factor-based attribution.

We investigate the impact of these choices in the sections that follow.

3.2 Explaining Market Returns

3.2.1 Explaining a Market Capitalization-Weighted Index

Suppose we were to explain a market capitalization-weighted index (a benchmark) using factor-based attribution with a returns model estimated from precisely the same index. What would this attribution tell us? Would the benchmark return be explained by the contribution from the market factor, the sector factors, or both? What about contributions from style factors and the residual component?

To answer these questions, we construct a market capitalization-weighted index for the developed countries (herein after referred to as the *Developed Benchmark*), and estimated four returns models using the *Developed Benchmark* as the estimation universe. We then decompose the returns obtained by the *Developed Benchmark* using these four models.

In the four returns models, we include either sector factors or market, sector, and style factors. If the market factor is included, its return is modeled as the regression intercept term in the first stage of a two-stage regression. We use capitalization weighting in the first stage regression (in all models) and either equal or capitalization weighting in the second stage regression. Style factors are standardized using capitalization-weighted or equal-weighted means, computed over the entire market or per sector. These choices are summarized in Table 6.

	Model 1	Model 2	Model 3	Model 4
Factors				
Market		✓	✓	✓
Sectors	✓	✓	✓	✓
Styles	✓	✓	✓	✓
Two-stage regression (mkt factor estimated in first stage)	No	Yes	Yes	Yes
Weighting scheme	Mkt Cap	Mkt Cap (stage 1) Mkt Cap (stage 2)	Mkt Cap (stage 1) Equal (stage 2)	Mkt Cap (stage 1) Equal (stage 2)
Normalization of style factors	Not applicable	Cap-weighted means	Equal-weighted means	Cap-weighted means per sector

Table 5: Developed Benchmark: Returns Models

Figure 8 presents the factor-based attribution results with these four models. Here we observe that:

- When model 1 is used to decompose returns, the entire benchmark return (437 bps) is attributed to sector factors.
- When model 2 is used to decompose returns, the entire benchmark return (437 bps) is attributed to the market factor.
- When model 3 is used to decompose returns, over 800 bps are attributed to the market and sector factors, with negative returns attributed to the styles (-143 bps) and specific components (-304 bps).
- When model 4 is used to decompose returns, in addition to a market return of 446 bps, substantial returns are attributed to sector factors (294 bps), with a negative specific return (-303 bps) offsetting the sector contribution.

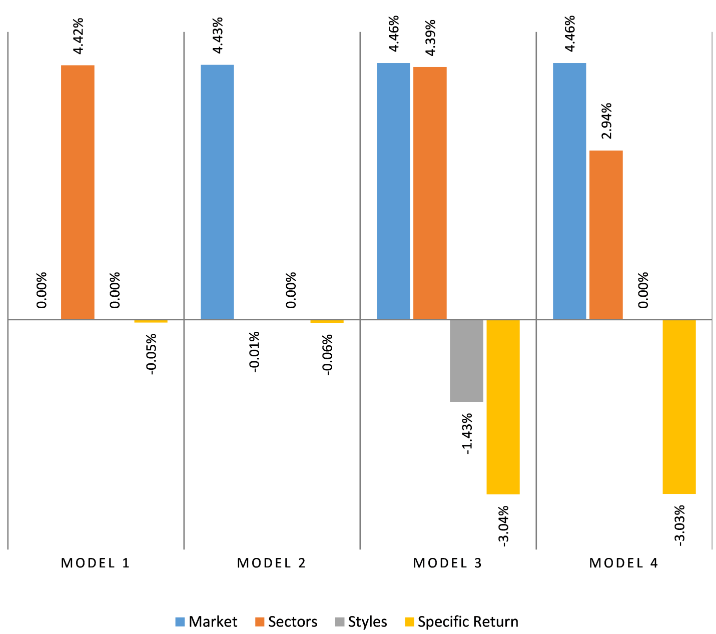


Figure 8: Developed Benchmark: Factor-based attribution of market portfolio with four different models

Why do we observe contributions from styles in model 3, but not in models 1, 2, and 4? Model 1 does not have any style factors. In models 2 and 4, styles are standardized using capitalization-weighted means; as a result the *Developed Benchmark* exposure to these factors is guaranteed to be zero. In model 3, in contrast, styles are standardized using equal-weighted means; the exposure of a capitalization-weighted portfolio to these factors will, therefore, be nonzero. This result is generalized in proposition 1.

Proposition 1. *A capitalization-weighted portfolio will have zero exposure to a style if (1) the style is standardized around capitalization-weighted means and (2) the set of assets over which the style is standardized is the same set of assets as those in the portfolio. Conversely, a non-capitalization-weighted portfolio will have non-zero exposure to a style if the style is standardized around capitalization-weighted means.*

See Appendix C: Propositions for a formal proof.

Why do we observe contributions from sectors in models 3 and 4? In both models 3 and 4, the stage 2 regression uses an equal-weighted weighting matrix; thus the estimated sector returns can be interpreted as "equal-weighted industry excess returns (net of the capitalization-weighted market return obtained in stage 1)". In model 2, in contrast, the estimated sector returns are "capitalization-weighted industry excess returns (net of the capitalization-weighted market return obtained in stage 1)." As a result, the estimated sector returns in models 3 and 4 are nonzero, whereas the estimated sector returns in model 2 are effectively zero. The choice to standardize styles with equal-weighted means (in model 3) and capitalization-weighted means (in model 4) also impacts the estimated sector returns, and explains the difference in sector contributions between models 3 and 4.

Why do we observe negative specific contributions in models 3 and 4? This is simply an artifact of the decomposition. Recall that the specific contribution is computed as

$$w_t^T \epsilon_t = w_t^T r_t - w_t^T X_{t-1} f_t.$$

With inflated factor contributions resulting from the use of equal-weighted regressions in models 3 and 4, the specific contribution must be negative for the decomposition to equal the *Developed Benchmark* return on 437 bps.

3.2.2 Lessons Learned

In the previous section, we decomposed the return of a market index using a returns model that had been estimated from precisely the same index, demonstrating how different aspects of the returns model construction process affect the decomposition. In particular, we showed that the decomposition is highly sensitive to the portfolio's exposure to the style factors used in the model, as well as the estimated factor returns. Estimated factor returns are, in turn, sensitive to choices made in the returns model construction and estimation process.

The lessons we've learned can be generalized around the following returns model construction decisions:

- *Deciding whether or not to include a market intercept:* When we decompose the return of a capitalization-weighted index using a model that does not have a market intercept term (as in Model 1), its return is explained by the sectors; when we decompose the return using a model with a market intercept term (as in Model 2), it is explained by the market. Practitioners who prefer to interpret sector returns as "returns in excess of the market return" should use a market intercept term in a returns model. In addition, including a market intercept and estimating the model using a staged regression where the market factor return is estimated in the first stage gives the practitioner more flexibility (e.g., he or she can use different weighting matrices in the stage 1 and stage 2 cross-sectional regression).

- *Deciding whether or not to standardize style exposures around the capitalization- or equal-weighted mean:* When we decompose the returns of a capitalization-weighted index using a model where style exposures are standardized around capitalization-weighted means (as in Models 2 and 4), the index exposure to each style will be zero and, as a result, the style contribution to returns will be zero. When we decompose the return using a model where style exposures are standardized around non-capitalization-weighted means (as in Model 3), the index exposure to these styles will be nonzero and, as a result, the style contribution will be nonzero.
- *Deciding whether or not to use a capitalization-weighted weighting scheme in cross-sectional regressions:* When we decompose the return of a capitalization-weighted index using a model computed with capitalization-weighted regression weights, its return will be explained entirely by either the market or the sectors (provided style factor exposures are standardized around capitalization-weighted means). When we decompose the return of a capitalization-weighted index using a model computed with non-capitalization-weighted regression weights (as in Model 3), the market and sector contributions will be biased (over-estimated or under-estimated) and the specific contribution will be nonzero.

In short, for a "clean" explanation of the return of an index, the weights in the index need to correspond to the weights used to compute the means around which exposures are standardized, as well as the weights used in the cross-sectional regression. For example, to explain the return of an equal-weighted sector portfolio, one should use equal weights when estimating factor returns.

3.3 Explaining Active Returns

We now turn our attention to active portfolio management and examine active returns, $w_t - b_t$, where b_t is an n-dimensional vector of benchmark weights. The active returns decomposition implied by the returns model given in equation (4) is

$$\underbrace{(w_t - b_t)^T r_t}_{\text{active return}} = \underbrace{(w_t - b_t)^T X_{t-1} f_t}_{\text{factor contribution}} + \underbrace{(w_t - b_t)^T \epsilon_t}_{\text{specific contribution}} .$$

In the sections that follow, we examine the active holdings of the following three portfolios:

1. *Sector Momentum Portfolio*: This portfolio weights capitalization-weighted sector portfolios according to a sector momentum signal.
2. *Value Portfolio*: This portfolio selects value stocks within each sector while remaining neutral to sectors, countries, and other styles.
3. *Country Momentum Portfolio*: This portfolio weights capitalization-weighted country portfolios according to a country momentum signal that is based on USD total returns of the country portfolios.

Once we have examined these three portfolios, we will return to the *Value Momentum Portfolio*, which was discussed in the *Brinson-Based Attribution* section of this paper and is a linear combination of the *Sector Momentum Portfolio* (25%), the *Value Portfolio* (50%), and the *Country Momentum Portfolio* (25%).

To identify practical issues encountered in standard, off-the-shelf factor-based attribution, we evaluate the impact of the following returns model construction decisions:

- When computing style exposures, should styles be centered around capitalization-weighted means computed over the entire universe or capitalization-weighted means computed for each sector?
- Should the returns model include sector-specific style factors (e.g., in a model with m factors and S sectors, should we create $m \times S$ factors)?
- Should the returns model include currencies?

We also consider the following attribution techniques intended to improve attribution results:

- Separating a factor return contribution into intended and residual components
- Reducing the correlation between factor and specific return contributions using adjusted attribution

3.3.1 Explaining a Sector Momentum Portfolio

Consider a *Sector Momentum Portfolio* that weights capitalization-weighted sector portfolios according to a sector momentum signal, taking the form

$$\begin{bmatrix} \eta_1 m_1 \\ \eta_2 m_2 \\ \vdots \\ \eta_3 m_L \end{bmatrix}$$

where m_l denotes an N_l-dimensional vector of capitalization weights for sectors l, rescaled to sum to one, and η_l denotes the weight that corresponds to the sector momentum signal for sector l, $l = 1, ..., L$.

Its active weights relative to the capitalization-weighted index take the form

$$\begin{bmatrix} (\eta_1 - \tau_1)m_1 \\ (\eta_2 - \tau_2)m_2 \\ \vdots \\ (\eta_L - \tau_L)m_L \end{bmatrix}$$

where τ_l denotes the sum of the capitalization weights in sector l,$l = 1, ..., L$.

Should the returns of this portfolio be attributed to sectors, the momentum style, or both?

We begin our analysis of this portfolio by decomposing its return using two returns models (2 and 5), which are identical except that one standardizes styles around capitalization-weighted means and the other standardizes styles around capitalization-weighted means *per sector*. These two returns models are summarized in Table 6. The factor-based attribution results obtained with these models are presented in Figure 9.

	Model 2	Model 5
Factors		
Market	✓	✓
Sectors	✓	✓
Styles	✓	✓
Two-stage regression (mkt factor estimated in first stage)	Yes	Yes
Weighting scheme	Mkt Cap (stage 1) Mkt Cap (stage 2)	Mkt Cap (stage 1) Mkt Cap (stage 2)
Normalization of style factors	Cap-weighted means	Cap-weighted means per sector

Table 6: Sector Momentum Portfolio: Returns Models

Why do we observe contributions from styles in model 2 but not in model 5? In model 5, styles are standardized around capitalization-weighted means per sector; as a result the *Sector Momentum Portfolio* active exposure to these factors is guaranteed to be zero. To see this, note that when raw exposures are standardized around capitalization-weighted means per sector, they take the form

$$\begin{bmatrix} \frac{1}{\sigma_1}\big(s_1 - (m_1^T s_1)u_1\big) \\ \frac{1}{\sigma_2}\big(s_2 - (m_2^T s_2)u_2\big) \\ \vdots \\ \frac{1}{\sigma_L}\big(s_L - (m_L^T s_L)u_L\big) \end{bmatrix}$$

where s_l denotes an N_l-dimensional vector of raw exposures for sector l, σ_l is the standard deviation of the raw exposures s_l, m_l is an N_l-dimensional vector of capitalization weights

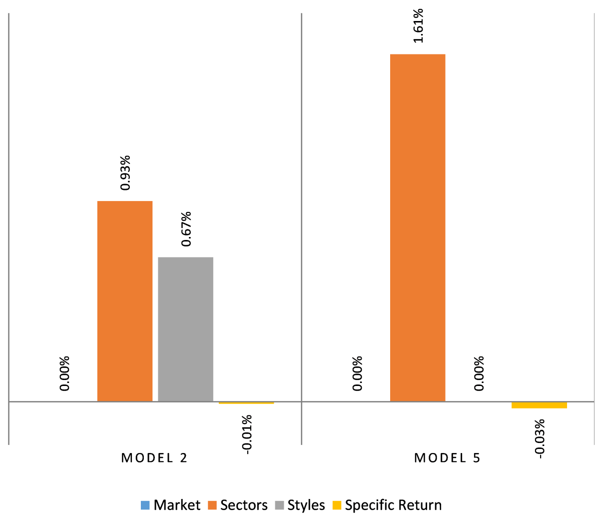


Figure 9: Sector Momentum Portfolio: Factor-based attribution with two different models

for sector l, rescaled to sum to one, and u_l is an N_l-dimensional vector of ones, $l = 1, ..., L$. Thus, the *Sector Momentum Portfolio's* active exposure to a single style factor is

$$\sum_{l=1}^{L} (\eta_l - \tau_l) m_l^T \frac{1}{\sigma_l} \big(s_l - (m_l^T s_l) u_l\big) = \sum_{l=1}^{L} \frac{\eta_l - \tau_l}{\sigma_l} \big(m_l^T s_l - m_l^T s_l\big) = 0.$$

Which individual style factors contribute to the 67 bps style contribution in Model 2? Do the active exposures to these factors represent intended bets? Table 7 decomposes the 67 bps style contribution obtained in Model 2 into individual style factor contributions and presents the average active exposure to these factors. Here we observe that, when exposures are standardized around the capitalization-weighted mean computed over the entire market, the active portfolio is exposed to market sensitivity and momentum, but is (approximately) neutral to size and value; thus, the market-wide sensitivity and momentum factors explain the majority of the style contribution. We can, however, discount these attribution results, since the investor intentionally made sector momentum bets, rather than market-wide momentum bets; the 67 bps style contribution obtained in Model 2 likely accounts for sector-specific style contributions which cannot be captured in Model 2's factor structure.

How do we gain insight into the performance of intended sector momentum bets? To evaluate whether the 161 bps sector contribution obtained in Model 5 comes from our intended sector momentum bets, we can decompose these sector contributions into "intended" and "residual" components using the formula

$$\underbrace{(w_t - b_t)^T X_{t-1,j} f_{t,j}}_{\text{Sector contribution}} = \underbrace{(w_t - \tilde{b}_t)^T X_{t-1,j} f_{t,j}}_{\text{Residual contribution}} + \underbrace{(\tilde{b}_t - b_t)^T X_{t-1,j} f_{t,j}}_{\text{Intended contribution}}. \tag{8}$$

Source of Return	Contribution	Average Exposure
Style Contribution	*0.67%*	
Market Sensitivity	0.53%	-0.0143
Momentum	0.08%	0.0336
Size	0.02%	-0.0018
Value	0.03%	-0.0069

Table 7: Sector Momentum Portfolio: Factor-based attribution (Model 1): Style Contribution

where $X_{t-1,j}$ denotes an n-dimensional vector of exposures to sector j, $f_{t,j}$ denote the return to sector j, and $\tilde{b}$ is an n-dimensional vector of target sector momentum holdings. The intended component corresponds to the active contribution of the target sector momentum holdings, and thus represents the portion of the portfolio's active sector contribution that comes from this intended bet. This decomposition is useful when the target sector momentum holdings, $\tilde{b}$, is part of the construction process, and the manager wants to evaluate the impact of targeting $\tilde{b}$ in his or her investment decision-making process.

Conclusions

We conclude that the active return of the *Sector Momentum Portfolio* is best explained using a returns model that standardizes styles around sector-specific capitalization-weighted means (such as those in Model 5), since this captures the investor's intended exposures. To provide insight into the performance of sector momentum bets, we recommend further decomposing the sector contributions obtained with Model 5 into intended and residual components using the decomposition in equation (5). Using these methods, the return of the *Sector Momentum Portfolio* is explained by sector bets and momentum bets within each sector bet, which is consistent with the investment process.

3.3.2 Explaining a Value Portfolio

Consider a *Value Portfolio* that selects high value stocks within each sector, while remaining neutral to sectors and other styles (size, momentum, and market sensitivity). More specifically, this portfolio, which was generated using an optimizer, is designed to have positive exposure to value, and zero exposure to other styles (size, market sensitivity, and momentum), where exposures are centered around capitalization-weighted means computed per sector, as shown in equation (6).

Would a returns model with a single value factor be sufficient to explain this portfolio's return? Or would sector-specific value factors (i.e., one value factor for each sector) better specify systematic return?

To evaluate these questions, we introduce a new returns model, Model 6, which includes 10 value factors, one for each sector, in addition to a market factor, sector factors, and other style factors (size, momentum, and market sensitivity). This model offers a more granular view of value performance, permitting the identification of the sector or sectors that drive aggregate performance. Figure 10 presents the correlation between estimated sector-specific value factor returns; here we observe that the correlation between factor returns is low, which suggests that the value factor behaves different in each sector, and thus justifies our consideration of Model 6.

	Consumer Discretionary	Consumer Staples	Energy	Financials	Health Care	Industrials	IT	Materials	Telecom	Utilities
Cons. Discretionary	1.000									
Consumer Staples	0.334	1.000								
Energy	0.250	0.244	1.000							
Financials	0.241	0.221	0.085	1.000						
Health Care	0.187	0.182	0.239	0.177	1.000					
Industrials	0.474	0.311	0.236	0.263	0.245	1.000				
IT	0.345	0.341	0.052	0.230	0.121	0.291	1.000			
Materials	0.364	0.123	0.166	0.186	0.160	0.329	0.101	1.000		
Telecom	0.209	0.212	0.104	0.093	0.054	0.164	0.162	0.085	1.000	
Utilities	0.021	0.072	0.080	0.122	0.068	0.206	0.107	0.069	0.126	1.000

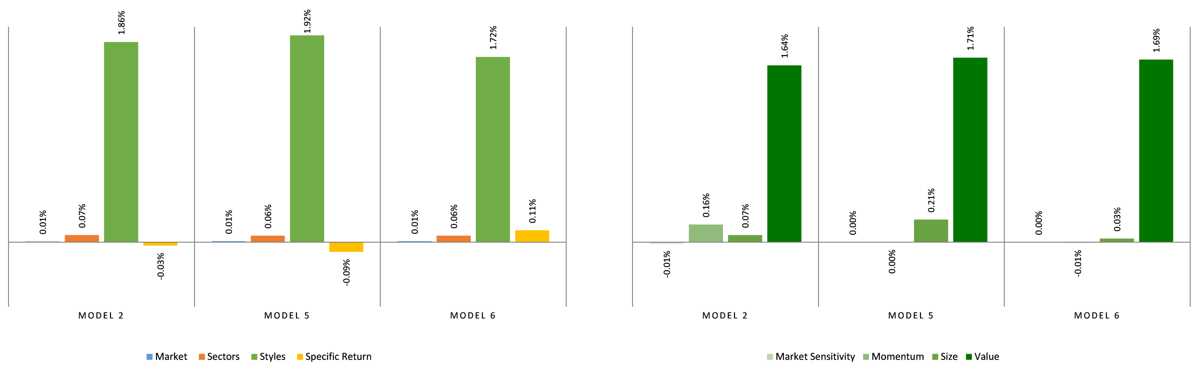
Figure 10: Returns Model 6: Correlation of monthly sector-specific value factor returns

We decompose the active return of the *Value Portfolio* (relative to a capitalization-weighted index) using models 2, 5, and 6. The differences between these models are summarized in Table 8. The adjusted factor-based attribution results obtained with these models are presented in Figure 11.[5] Here we observe that, in each attribution, the active return of the portfolio (190 bps) is almost entirely attributed to styles, with negligible contributions from the market, sector, and specific components, and, further, the value factor accounts for the majority of the style contribution.

	Model 2	Model 5	Model 6
Factors			
Market	✓	✓	✓
Sectors	✓	✓	✓
Styles	✓	✓	✓
Two-stage regression (mkt factor estimated in first stage)	Yes	Yes	Yes
Include sector-specific style factors (second stage)	No	No	Yes
Weighting scheme	Mkt Cap (stage 1) Mkt Cap (stage 2)	Mkt Cap (stage 1) Mkt Cap (stage 2)	Mkt Cap (stage 1) Mkt Cap (stage 2)
Normalization of style factors	Cap-weighted means	Cap-weighted means per sector	Cap-weighted means per sector

Table 8: Value Portfolio: Returns Models

[5] Adjusted factor-based attribution (Stubbs and Jeet, 2013) is a method that reduces the correlation between the factor and specific return contributions that arises when some of the underlying assumptions of the returns model are violated. This method, which is described in Appendix B, moves the portion of the specific contribution that is explained by the factor contribution back into the factor contribution.



(a) Contributions (b) Style Contributions

Figure 11: Value Portfolio: Adjusted factor-based attribution with three different models

What is the impact of including sector-specific value factors (i.e., one value factor for each sector) in Model 6? When performance attribution is run using a model with sector-specific factors, returns are attributed to the individual factors, generating sector-specific style contributions that can be aggregated to evaluate the style as a whole. In the adjusted attribution results obtained with model 6, 169 bps of the 190 bps active return is attributed to value and the remaining 3 bps are attributed to other style factors. Figure 12 presents the breakdown of the 169 bps style contribution by sector. Here we observe that value performance is driven primarily by the financials, heath care, and materials sectors. The ability to "drill down" on style performance by sector, in a manner that is similar to that encountered with Brinson-based attribution, is one of the main benefits of including sector-specific style factors.

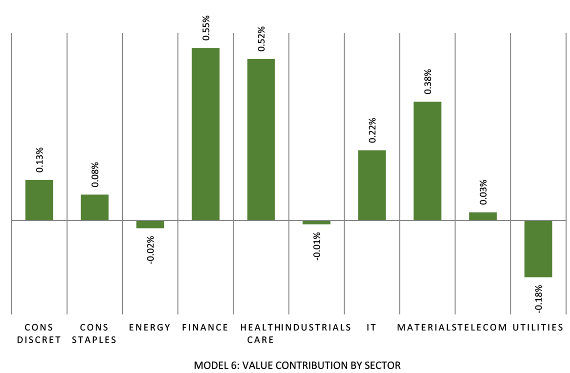


Figure 12: Value Portfolio: Adjusted factor-based attribution: Model 6 Value Contribution by Sector

What is the impact of standardizing styles per sector? The standardization scheme used in Models 5 and 6 is consistent with the standardization scheme used to compute portfolio exposures to factors in the portfolio construction process; as a result, the active exposures obtained with Models 5 and 6 better reflect the portfolio manager's intended bets on sector-specific signals. The attribution results obtained with model 2, in contrast, potentially capture unintended exposures, such as exposures to market-wide style and momentum signals; further, model 2 is not designed to quantify returns in a manner that is consistent with the portfolio's design.

Conclusions

We conclude that the active return of the *Value Portfolio* is best explained by Model 6 for two reasons. First, it standardizes exposures in a manner that is consistent with the portfolio construction process, so the active portfolio exposures in the decomposition match the intended exposures or bets of the investment manager, and, second, by including a value factor for each sector, it provides insight into the sectors that drive value performance.

3.3.3 Explaining a Country Momentum Portfolio

Consider a *Country Momentum Portfolio* that weights capitalization-weighted country portfolios according to a country momentum signal that is based on USD total returns of the country portfolios. Should the return of this portfolio be attributed to countries, the momentum style, or both? How do we determine if the return should be attributed to local markets (countries) or currencies?

To address these questions, we introduce a new returns model, Model 7, which includes factors representing the market, sectors, countries, and currencies, as well as factors representing market-wide style signals (market sensitivity, momentum, and size) and factors representing sector-wide style signals (value). Exposures to market-wide styles are centered around capitalization-weighted means computed over the entire market, and exposures to sector-wide styles are centered around capitalization-weighted means computed for each sector. The features of this model are described in Table 9.

	Model 7
Factors	
Market	✓
Sectors	✓
Styles	✓
Countries	✓
Currencies	✓
Two-stage regression (mkt factor estimated in first stage)	Yes
Include sector-specific style factors (second stage)	Yes (for the value factor only)
Weighting scheme	Mkt Cap (stage 1) Mkt Cap (stage 2)
Standardization of style factors	Cap-weighted means (mkt sensitivity, momentum, and size); cap-weighted means per sector (value)

Table 9: Country Momentum Portfolio: Returns Model

Table 10 presents the attribution results obtained with this model. Here we observe that the return attributed to countries is relatively small (3 bps out of a 153 bps active return) and not significantly different from zero, while the return attributed to currency movements is substantial (63 bps out of a 153 bps active return). This suggests that the manager was betting on currency movements, rather than country momentum.

Source of Return	Return	Risk	IR	t-stat
Portfolio	5.89%	17.05%		
Benchmark	4.36%	16.55%		
Active	1.53%	3.56%	0.43	1.61
Specific Return	0.76%	3.12%	0.24	0.91
Factor Contribution	0.78%	3.10%	0.25	0.94
Style	0.00%	1.30%	0.00	-0.01
Country	0.03%	2.15%	0.01	0.06
Sector	0.11%	0.88%	0.13	0.47
Currency	0.63%	1.93%	0.33	1.23
Market	0.00%	0.02%	0.26	0.99

Table 10: Country Momentum Portfolio: Factor-based attribution with Model 7

Conclusions

In a global portfolio, performance attribution should be conducted with a model that includes both countries and currencies (such as Model 7) to enable the practitioner to distinguish between country and currency effects.

3.3.4 Explaining a Value Momentum Portfolio

We finish our analysis of factor-based attribution for active portfolios by examining the *Value Momentum Portfolio*, which is a linear combination of the *Sector Momentum Portfolio* (25%), the *Value Portfolio* (50%), and the *Country Momentum Portfolio* (25%). (It is the same portfolio that is discussed in the *Brinson-Based Attribution* section of this paper.)

The sources of return we wish to highlight include sector momentum, value, country momentum, and currency. Thus, we need a model that includes style, sector, currency, and country factors. Because the *Value Momentum Portfolio* is constructed using data that has been standardized per sector, we require a returns model with sector-specific style factors where style exposures have also been standardized per sector. Model 7, which is described in Table 9, meets these requirements.

We decompose the active return of the *Value Momentum Portfolio* using Model 7, and then separate country and sector contributions into intended and residual components using the decomposition in equation (8). Table 11 presents the adjusted factor-based attribution results obtained with Model 7, along with the separation of country and sector contributions into intended and residual components. Table 12 decomposes the 129 bps style contribution obtained with Model 7 into individual style contributions and presents the average active exposure to these factors.

Source of Return	Return	Risk	IR	t-stat
Portfolio	6.11%	16.92%		
Benchmark	4.36%	16.55%		
Active	1.75%	1.86%	0.94	3.52
Specific Return	0.05%	1.23%	0.04	0.16
Factor Contribution	1.70%	1.68%	1.01	3.77
Style	1.29%	1.25%	1.03	3.85
Country	-0.07%	0.63%	-0.11	-0.42
Country Momentum (Intended)	0.01%	0.54%	0.02	0.07
Country Residual	-0.08%	0.30%	-0.27	-1.00
Sector	0.31%	0.61%	0.50	1.88
Sector Momentum (Intended)	0.21%	0.42%	0.51	1.90
Sector Residual	0.09%	0.35%	0.27	1.01
Currency	0.17%	0.53%	0.32	1.19
Market	0.00%	0.01%	0.53	1.99

Table 11: Value Momentum Portfolio: Adjusted factor-based attribution with Model 7

Source of Return	Contribution	Avg Exposure	Risk	IR	t-stat
Style	1.29%		1.25%	1.03	3.85
Market Sensitivity	0.16%	-0.0101	0.29%	0.57	2.12
Momentum	0.07%	0.0214	0.17%	0.43	1.60
Size	-0.01%	-0.0405	0.06%	-0.15	-0.57
Value	1.06%	-0.0101	1.25%	0.85	3.17

Table 12: Value Momentum Portfolio: Adjusted factor-based attribution with Model 7: Style Contributions

In Tables 11 and 12, we observe the following:

- The majority of the 175 bps active return obtained by the *Value Momentum Portfolio* is attributed to factors; of the 170 bps factor contribution, the majority is attributed to styles (129 bps), sectors (31 bps), and currencies (17 bps).
- Of the 129 bps style contribution, 106 bps are attributed to value; this suggests that the value bets made by the investor had the largest impact on returns.
- The impact of the sector momentum bets made by weighting capitalization-weighted sector portfolios according to a momentum signal is captured in the "sector momentum (intended)" line item; here we note that the sector momentum overlay had a positive impact on returns (21 bps).
- The impact of the country momentum bets made by weighting capitalization-weighted country portfolios according to a momentum signal is captured in the "country momentum (intended)" line item; here we observe that the contribution from these bets is

negligible; the currency contribution, in contrast, is positive (17 bps), which suggests that the manager was, in fact, betting on currency movements when he constructed the country momentum overlay.

Conclusions

When we use the flexibility available in a returns model appropriately, we can intuitively explain the *Value Momentum Portfolio's* performance. The attribution results in Tables 11 and 12 show that the manager's value positions contribute positively to active return. In addition, it shows that the sector momentum overlay contributes positively to active return. (The country momentum overlay does not contribute to active return, but currency does.) We conclude that the manager has demonstrated investment skill in the areas that contribute positively to active returns (value, sector momentum, and currency), and statistically significant skill (consistent and repeatable skill) in areas that have a high t-statistic, such as value.

3.4 Extending Factor-Based Attribution with a Custom Returns Model

Factor-based attribution methodologies permit practitioners to identify many sources of return simultaneously and is thus ideally suited for explaining complex portfolios and portfolios that have been constructed systematically. Running factor-based attribution methods with commercially available, off-the-shelf returns models, however, may not capture proprietary investment processes where different data are used to construct portfolio weights. As a result, we believe practitioners should tailor the returns model to their investment processes to increase their chances of identifying statistically significant sources of return that correspond to their actual bets.

When constructing a returns model, we recommend that practitioners:

1. Define a market factor that is the benchmark to which performance will be compared
2. Use a regression weighting scheme that is consistent with the weighting scheme used to construct the portfolio (i.e., a capitalization-weighted portfolio is best explained by a returns model that used capitalization weights in the cross-sectional regressions)
3. Standardize style exposures in a manner that is consistent with how the portfolio has been constructed (i.e., if the manager makes bets per sector, then style exposures should be standardized per sector)
4. Include all sources of alpha in the model

Further, when running the attribution, we recommend that practitioners separate contributions into intended and residual components, if a target bet, represented by a vector of weights or holdings, is believed to be part of the contribution.

4 Brinson versus Factor-Based Attribution

In this section, we compare the Brinson and factor-based attributions for the *Value Momentum Portfolio* using the Brinson attribution obtained with the sectors-value classification scheme and the factor-based attribution obtained with model 7. Figure 13 presents the effects/contributions attributed to different aggregate sources, Figure 14 presents the effects/contributions attributed to different sectors, and Figure 15 presents the *value* effects/contributions attributed to different sectors. Here we observe that the distribution of effects/contributions across sources is similar in the Brinson and factor-based attributions. The factor-based attribution accounts for more sources of return than Brinson (note that Figure 13 depicts more data points for the factor-based attribution than it does for the Brinson), but otherwise, the results tell a similar story.

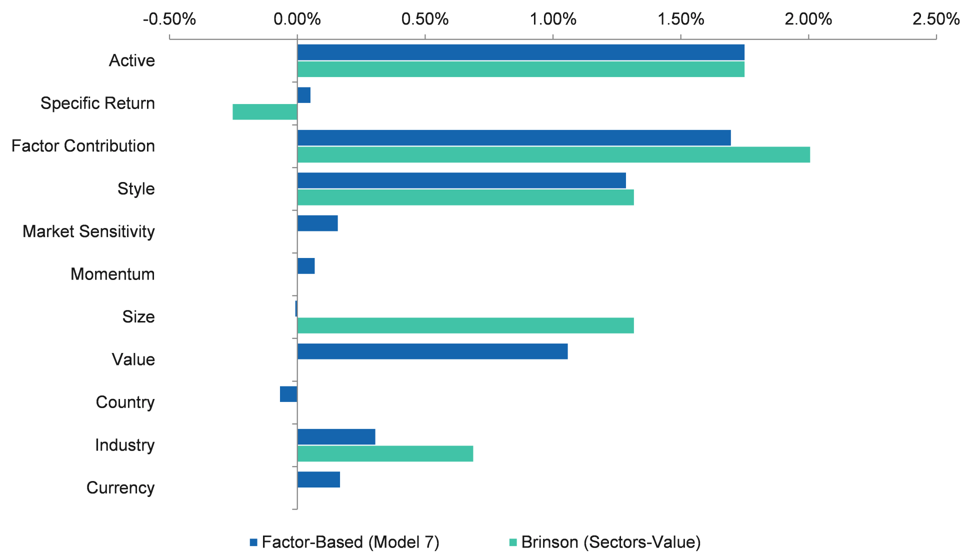


Figure 13: Value Momentum Portfolio: Effects/contributions obtained in a factor-based attribution (using returns model 7) and a Brinson attribution (using the sectors-value classification)

The fact that the Brinson and factor-based attributions presented in Figures 13-15 are similar should not surprise the reader; because we customized the classification scheme and returns model to match the portfolio construction process, we were able to obtain statistically significant results that correspond with the actual bets made by the manager.

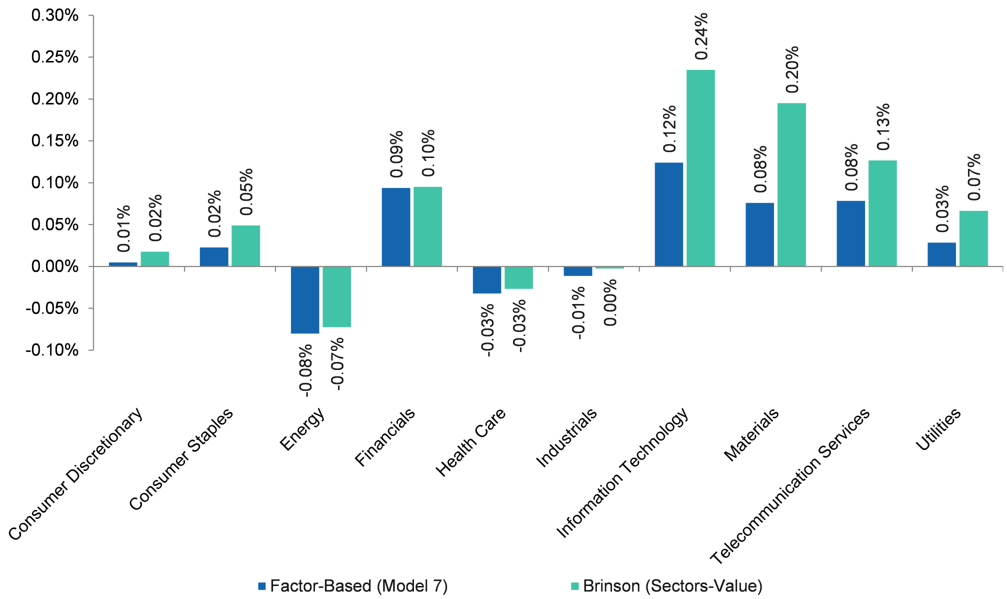


Figure 14: Value Momentum Portfolio: Sector effects/contributions obtained in a factor-based attribution (using returns model 7) and a Brinson attribution (using the sectors-value classification)

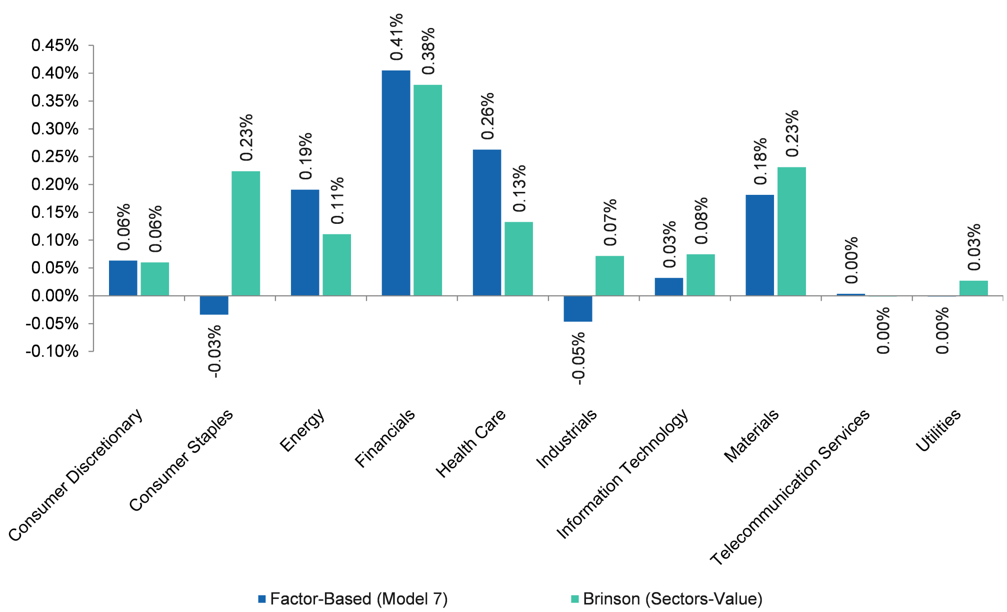


Figure 15: Value Momentum Portfolio: Value effects/contributions obtained in a factor-based attribution (using returns model 7) and a Brinson attribution (using the sectors-value classification)

5 Summary and Conclusions

Performance attribution, when it is done right, should provide insight into the investment process, differentiating it from other products (either active or passive) and demonstrating consistent and repeatable investment skill. By customizing the structure used to decompose returns in either a Brinson- or factor-based attribution, we believe investment managers have a better chance of identifying statistically significant sources of return that coincide with their intended bets.

6 Appendix A: Multiple Currencies in a Brinson Attribution

This appendix presents the formulas used to decompose returns in a multi-currency attribution, using the framework presented in Singer and Karnosky (1995) and Menchero and Davis (2009), where active excess returns at the porfolio level are given by

$$\tilde{R}^A = \tilde{L}^A + \tilde{C}^A + \tilde{Q}^A$$

where $\tilde{R}^A$ denotes the portfolio's active excess return denominated in the numeraire currency, $\tilde{L}^A$ is the active local excess return denominated in local currencies, $\tilde{C}^A$ is the active currency excess return, and $\tilde{Q}^A$ is the cross product.

The portfolio's active excess return is computed as

$$\tilde{R}^A = \sum_{a \in A} (w_a^p - w_a^b)(r_a - \rho_{numeraire})$$

where $\rho_{numeraire}$ is the risk-free rate associated with the numeraire currency (that is, it is the opportunity cost of investing in the numeraire market).

The local excess return is computed as

$$\tilde{L}^A = \sum_{a \in A} (w_a^p - w_a^b)(l_a - \rho_c(a))$$

where l_a is the local return rate of asset a (that is, l_a is denominated in the local currency of asset a), and $\rho_c(a)$ is the risk-free rate associated with the local market of asset a.

Let C denote the set of currencies in the portfolio, and let I_c denote an indicator function for currency $c \in C$ [that is, $I_c(a) = 1$ if asset a is denominated in currency $c, 0$ otherwise]. Then, the active currency excess return is expressed as

$$\tilde{C}^A = \sum_{c \in C} (W_c^p - W_c^b)(r_c - R^{b,c})$$

where $W_c^p = \sum_{a \in A} I_c(a) w_a^p$ and $W_c^b = \sum_{a \in A} I_c(a) w_a^b$ are the portfolio and benchmark weight in currency c, r_c is the excess currency return, and $R^{b,c} = \sum_{c \in C} W_c^b r_c$ is the overall benchmark currency return.

The excess currency return is computed as

$$r_c = \rho_c + e_{numeraire/c} + \rho_c e_{numeraire/c} - \rho_{numeraire}$$

where $e_{numeraire/c}$ is the rate of change of the numeraire exchange rate relative to the local currency.

The cross product is computed as

$$\tilde{Q}^A = \sum_{c \in C} e_{numeraire/c}(G_c^p - G_c^b)$$

where $G_c^x = \sum_{a \in A} I_c(a) w_a^x (l_a - \rho_c(a))$, $x = p, b$, is the active local excess return contribution from all assets.

7 Appendix B: Adjusted Factor-Based Performance Attribution

This appendix describes adjusted factor-based performance attribution (Stubbs and Jeet, 2013), a methodology that reduces the correlation found in the factor and specific return contribution series generated from a standard factor-based performance attribution.

To motivate the need for adjusted factor-based performance attribution, we first discuss the underlying assumptions of an asset returns model and a factor risk model and then evaluate the implications of using a model that violates these assumptions.

The underlying assumptions of an asset returns model and a factor risk model

Recall that an asset returns model is, in most cases, the underlying component of a factor risk model that is generated by

1. Specifying a linear returns model that relates asset returns to common factor returns and a residual component that represents asset-specific returns
2. Estimating the unknown components of the returns model over a pre-specified time period, thereby obtaining time series of factor and residual (specific) returns
3. Estimating factor return and specific return covariance matrices from the time series of factor and residual (specific) returns
4. Estimating an asset return covariance matrix using the factor and specific return covariance matrices

If the returns model is estimated using OLS or one of its variants, such as WLS, a well-specified model assumes that

(A1) The residuals (specific returns) have a mean of zero and are independent of the regressors (the independent variables), thereby ensuring that the estimated parameters are unbiased.

(A2) The residuals (specific returns) are homoskedastic and not correlated between observations, thereby ensuring that the least-squares estimates are superior to other linear estimators. (Note that the assumption of homoskedasticity can be relaxed if WLS is used to estimate the unknown parameters in the model.)

If the returns model is estimated using cross-sectional regressions, assumption A1 implies that the residuals are uncorrelated with the factor exposures. If the returns model is estimated using time-series analysis (where factor returns are typically observed and exposures are estimated), assumption A1 implies that the residuals are uncorrelated with the factor returns. Assumption A2 implies that the residual variance-covariance matrix is a diagonal matrix.

After the unknown parameters of a returns model are estimated, generating a time series of factor and specific returns, the factor return covariance matrix (Σ) and specific covariance matrix (Δ^2) are estimated. (A discussion of covariance matrix estimation methodologies is beyond the scope of this paper; the interested reader should refer to Zangari (2003a) and Santis et al. (2003) for more information on the topic.)

Specifying the asset return covariance matrix in terms the factor and specific return covariance matrices is the last step in generating a factor risk model. This step generally assumes that

(A3) The specific return of asset i is uncorrelated with the specific return of asset j, for all assets $i \neq j$ (note that this assumption is implied by assumption A2 in a cross-sectional regression context)

(A4) The specific return of asset i is uncorrelated with the return of factor k, for all assets i and for all factors k

Under these assumptions, and under the additional assumption that the asset and factor returns have mean zero, the asset return covariance matrix is computed as

$$
\begin{aligned}
var(r) &= var(Xf + \epsilon) \\
&= E[(Xf + \epsilon)(Xf + \epsilon)^T] \\
&= XE[ff^T]X^T + E[\epsilon\epsilon^T] + XE[f\epsilon^T] + E[\epsilon f^T]X^T \\
&= XE[ff^T]X^T + E[\epsilon\epsilon^T] \quad (\text{since } XE[f\epsilon^T] + E[\epsilon f^T]X^T = 0) \\
&= X\Sigma X^T + \Delta^2
\end{aligned}
$$

where Σ is the factor return covariance matrix estimated from the time series of factor returns and Δ^2 is the diagonal matrix of specific variances estimated from the time series of specific returns. Assumption A3 permits the practitioner to specify the specific return covariance matrix Δ^2 as a diagonal matrix. Assumption A4 ensures that $XE[f\epsilon^T] + E[\epsilon f^T]X^T = 0$ and thus permits the practitioner to formulate the asset covariance matrix as $X\Sigma X^T + \Delta^2$.

What are the implications of using a misspecified model?

When assumption A1, A2, A3, or A4 are violated, the factor risk model and underlying returns model are misspecified. Here we focus on the implications of violating assumption A3 (that specific returns are uncorrelated with each other) and assumption A4 (that specific returns are uncorrelated with factor returns).

If assumption A3 is violated (or, equivalently, if assumption A2 in a cross-sectional context is violated), the asset-specific returns are not idiosyncratic (i.e., they have some systematic structure), which suggests that the returns model omits relevant factors. In such cases, the factor returns tend to be underestimated, the asset-specific returns tend to be overestimated, and portfolio risk estimates given by $w^T(X\Sigma X^T + \Delta^2)w$ tend to be underestimated.

If assumption A4 is violated, portfolio risk estimates given by $w^T(X\Sigma X^T + \Delta^2)w$ may be biased. In addition, in a returns decomposition, the factor contribution time series may be correlated with the asset specific contribution time series

$$corr(w_t^T X_{t-1} f_t, w_t^T \epsilon_t) \neq 0$$

Using adjusted attribution to reduce the correlation between factor and specific return contributions

Recognizing that some model misspecification is inevitable in practice, we consider an adjusted attribution to correct for correlated factor and specific return contributions.

Consider an adjusted attribution that uses a factor model with m factors. This adjusted attribution is accomplished by

1. Regressing the time series of specific return contributions on the time series of m factor contributions,
$$w_t^T \epsilon_t = \sum_{i=1}^{m} \beta_i \big(w_t^T X_{t-1,i} f_{t,i}\big) + \mu_t, \quad t = 1, ..., T, \tag{B1}$$
thereby segmenting the specific return time series into two components: a component that is explained by the factor contributions, $\sum_{i=1}^{m} \hat{\beta}_i \big(w_t^T X_{t-1,i} f_{t,i}\big)$, and a residual component, $\hat{\mu}_t = w_t^T \epsilon_t - \sum_{i=1}^{m} \hat{\beta}_i \big(w_t^T X_{t-1,i} f_{t,i}\big)$. Note that $\hat{\mu}_t$ is the new specific return contribution in period t.

2. Substituting the segmentation obtained in (B1) into the returns decomposition,

$$
\begin{aligned}
w_t^T r_t &= \sum_{i=1}^{m} w_t^T X_{t-1,i} f_{t,i} + w_t^T \epsilon_t \\
&= \sum_{i=1}^{m} w_t^T X_{t-1,i} f_{t,i} + \sum_{i=1}^{m} \hat{\beta}_i \big(w_t^T X_{t-1,i} f_{t,i} \big) + \hat{\mu}_t \\
&= \sum_{i=1}^{m} (1+\hat{\beta}_i) \big(w_t^T X_{t-1,i} f_{t,i} \big) + \hat{\mu}_t,
\end{aligned}
$$

thereby moving the portion of the original specific contribution that is explained by the factor contributions back into the factor contributions.

8 Appendix C: Propositions

Proposition 2. *A capitalization-weighted portfolio will have zero exposure to a style if (1) the style is standardized around capitalization-weighted means and (2) the set of assets over which the style is standardized is the same set of assets as those in the portfolio. Conversely, a non-capitalization-weighted portfolio will have non-zero exposure to a style if the style is standardized around capitalization-weighted means.*

Proof. Let s denote an n-dimensional vector of raw exposures for a particular style, and let x_{t-1} denote the n-dimensional vector of corresponding standardized exposures, computed around capitalization-weighted means using the formula

$$
x_{t-1} = \frac{1}{\sigma_s} \Big(s - (m^T s) u \Big)
$$

where σ_s is the standard deviation of the raw exposures s, m is an n-dimensional vector of capitalization weights, and u is an n-dimensional vector of ones. Then, a capitalization-weighted portfolio m will have the following exposure to this style

$$
m^T x_{t-1} = \frac{1}{\sigma_s} \Big(m^T s - (m^T s) m^T u \Big) = \frac{1}{\sigma_s} \Big(m^T s - m^T s \Big) = 0
$$

and a non-capitalization-weighted portfolio $w \neq m$ will have the following exposure

$$
w^T x_{t-1} = \frac{1}{\sigma_s} \Big(w^T s - (m^T s) w^T u \Big) = \frac{1}{\sigma_s} (w - m)^T s \neq 0
$$

□

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