ROBUST OPTIMIZATION

Random matrix theory and risk factor models

Large covariance matrices are hard to estimate, due to the significant number of parameters involved. This issue is magnified when the estimated matrix is then used as an input into other exercises, namely portfolio construction through optimization. Several approaches have been suggested to alleviate this problem, namely the use of the random matrix theory (RMT). In this paper, we start with a covariance coming from a linear factor model and check the effects of applying the RMT to that matrix. Our evidence seems to suggest that imposing a factor structure onto the security covariance matrix significantly increases its robustness, decreasing the value added by the RMT. This highlights the usefulness of factor models to reduce estimation noise. Nevertheless, back-test results on optimized portfolios show that RMT may contribute to a more robust optimal solution. Therefore, this feature is now included in POINT's optimization library.

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Introduction

In finance, the covariance matrix between securities, macro variables or risk factors is widely used by portfolio managers to estimate risk, perform scenario analysis or construct and rebalance portfolios.

While very popular, covariance matrices are hard to estimate in many contexts. For instance, for a universe of 100 securities, the estimation of its sample returns' covariance matrix involves 5,050 parameters. Without extensive historical data on the returns of each individual security, the estimation of these parameters is very noisy. This estimation error blurs our ability to identify the correlation among securities and related statistics, such as the risk of a portfolio of these securities or an appropriate hedge to such a portfolio.

This paper focuses on a particular application for the security covariance matrix: portfolio optimization. In its most standard use – mean variance optimization (Markowitz (1952)) – covariance matrices are applied to find optimal portfolios along the risk/return trade-off. The solution to this problem typically involves taking large positions in securities with a small effect on total portfolio risk, that is, with low volatilities and correlations. Noisy estimation of these statistics leads to "optimal" portfolios that explore this noise, not the true characteristics of each security. The resulting portfolios are abundantly reported as sub-optimal, with ex-post volatilities larger than forecasted at the time of the optimization.

Several methodologies have been proposed to remedy this problem. Imposing structure to the covariance matrix reduces the number of parameters to be estimated out of the same data. The result is a reduction in the estimation error.[1] Ledoit and Wolf (2004) propose a shrinkage estimator, while Michaud (1989) called for an average of re-sampled data (through bootstrapping). Black and Litterman (1992) propose the use of a prior to these statistics, while Jagannathan and Ma (2003) show how introducing certain constraints to the optimization may be equivalent to reducing the sampling error on the covariance matrix. Another suggestion has been to use the random matrix theory (RMT) (see Laloux et al. (1999)) to "clean" spurious differences in the eigenvalues of the covariance matrix.[2]

In this paper, our starting point is a structured covariance matrix coming out of a linear return factor model. This option is intuitive, as factor models are widely used in portfolio management (Lazanas et al. (2011b)). We then compare statistics of optimal portfolios created using this covariance matrix with those coming from the same covariance matrix "smoothed" through the RMT. Previous results (Laloux etc. 1999) have shown that applying the RMT to unstructured and noisy covariance matrices (eg, large equity sample covariance matrices constructed with relatively small samples) significantly improves the robustness of the matrix for purposes such as portfolio construction through optimization. We want to measure to what extent this result holds when the starting point is a covariance matrix with pre-imposed structure (eg, through a linear factor model).

Our results suggest that factor models do significantly reduce the estimation error plaguing sample covariance matrix estimation, decreasing substantially the need for the extra smoothing by the RMT. Nevertheless, evidence here also suggests a more robust solution for optimized portfolios when the RMT is used in a factor model and that for asset classes where these factor models do not apply – or do not lead to a significant reduction on the number of parameters estimated – the RMT may add value.

[1] This reduction is not without a cost: if the structure imposed is not "true," we incur in mis-specification error.
[2] The lasso estimator has also been suggested as a way to reduce the number of non-zero dependencies (and potentially correlations) across variables (Tibshirani (1996)).

This paper is organized as follows. The next two sections describe in more detail the two frameworks used throughout the paper: random matrix theory and linear factor models. For each, we also discuss a simple example to motivate the approach. We then use actual data to present empirical evidence on the effects of the RMT.

Random matrix theory

The RMT suggests that the error while estimating a particular covariance matrix spuriously creates differences in eigenvalues of the covariance matrix. It identifies the range of the error and smoothes that out, so as not to affect materially portfolios constructed using that covariance.

To illustrate this point, let $r_{i,t}$ denote the standardized excess return of security *i* at time *t*. Furthermore, assume that $r_{i,t}$ is pure noise following *i.i.d.* standard normal random variables for all *i = 1, 2,…, N* and *t = 1, 2,…,T*. Written in the matrix form, *R* is the *NxT* matrix denoting a panel data of security returns. $\sum_s = 1/T(RR')$, with dimension *NxN*, is the covariance matrix of the returns. Under the above assumption, $E(\sum_s) = I_N$. However, under finite samples, the empirical realized covariance matrix across these securities will deviate from the identity matrix due to sample estimation errors. To correct this issue, the RMT identifies a range where we cannot statistically distinguish eigenvalues of these matrices from each other. Specifically, it postulates that with finite sample observations, the eigenvalues from a true random matrix lie within an interval determined by *N* and *T*. To be specific, let λ denote the eigenvalues of $\sum_s$. Then that interval ("smoothing region") is given by:

$$\lambda_{\min}^{\max} = \sigma^2 (1 \pm \sqrt{\frac{N}{T}})^2 \tag{1}$$

Here, $\sigma^2 = 1$, denotes the variance of $r_{i,t}$. The density of eigenvalues for this random matrix vanishes beyond the range of $(\lambda_{\min}, \lambda_{\max})$. Intuitively, the length of the interval, $\lambda_{\max} - \lambda_{\min} = 4\sigma^2 \sqrt{N/T}$, is longer if there are more securities in the portfolio and shorter if we observe more history of their returns. This is consistent with the concept of the measurement error: the more securities and less history, the noisier the covariance matrix. The RMT emphasises the fact that only the eigenvalues located outside of $(\lambda_{\min}, \lambda_{\max})$ provide reliable information.

We can apply this result to increase the robustness of any sample covariance matrix. We first need to identify what is the range of eigenvalues of the matrix for which their difference is spurious: $\lambda_{\min}, \lambda_{\max}$. Then, we replace the eigenvalues located within the range with a constant, while preserving the matrix trace, before reconstructing the matrix with these smoothed eigenvalues.

The smoothing range in (1) depends on the volatility of the noisy eigenvalues. In POINT, we have developed an algorithm to estimate this volatility and, therefore, the smoothing area that is portfolio specific.

A simple example

To exemplify this technique, let us look at two examples where *N=10* and *T=50*. Using (1), we find that the bounds for noisy eigenvalues for this sample are $\lambda_{\min} = 0.3056$ and

$\lambda_{\max} = 2.0944$. In the first example, we randomly generate a sample from a multivariate normal with zero correlations. The matrix for this particular sample shows correlations ranging from -0.35 to 0.28 (even though we generate the data with zero correlations). We then calculate the eigenvalues of this matrix and correct them using the boundaries above. Figure 1 shows the profile of the eigenvalues before and after the RMT correction.

Figure 1: Profile of eigenvalues from the correlation matrix with $\rho = 0$

Correlation Matrix	Eigenvalues										
	1	2	3	4	5	6	7	8	9	10	Sum
Original	0.275	0.535	0.623	0.834	0.926	0.992	1.139	1.174	1.647	1.855	10
RMT-corrected	0.275	1.081	1.081	1.081	1.081	1.081	1.081	1.081	1.081	1.081	10

Source: Barclays Research

For this random sample,[3] the RMT identifies all but the lowest eigenvalue as noisy, significantly smoothing out the differences among them. Once the correlation matrix is reconstructed using the corrected eigenvalues, the correlation range decreases significantly: the maximum is now 0.13, while the minimum is -0.20.

In the second example, we draw a similar sample, but now assume a correlation of 0.30 across all variables. The range of correlations in our sample is now between -0.12 and 0.55. The profiles of the eigenvalues before and after the RMT correction are displayed in Figure 2. Compared with the previous case, the RMT is able to identify a significant common source of variation in the variables (that drives the positive correlations) in the highest eigenvalue. It therefore allows that eigenvalue to persist, while smoothing out all others but the smallest one.

Figure 2: Profile of eigenvalues from the correlation matrix with $\rho = 0.3$

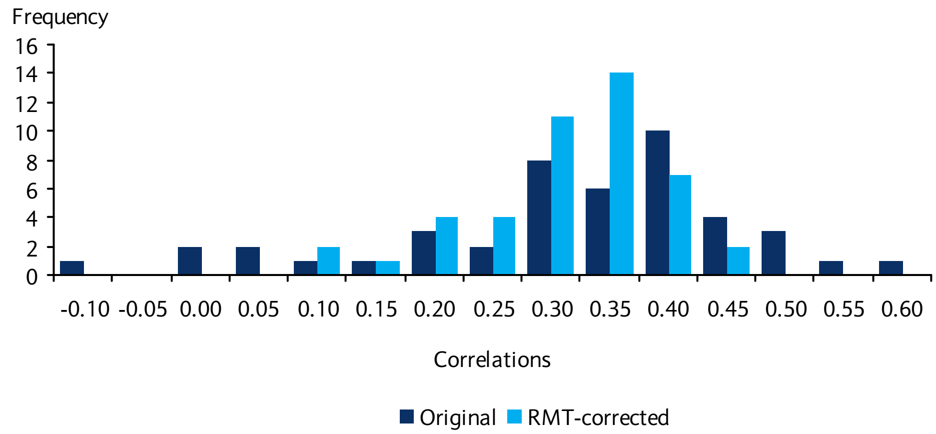
Correlation Matrix	Eigenvalues										
	1	2	3	4	5	6	7	8	9	10	Sum
Original	0.28	0.33	0.40	0.45	0.56	0.69	1.01	1.14	1.41	3.74	10
RMT-corrected	0.28	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	3.74	10

Source: Barclays Research

With the reconstruction of the correlation matrix using the new vector of eigenvalues, the range of correlations decreases: The maximum is 0.42 and the minimum is 0.08. Again, the RMT corrected the random sample correlation towards the true correlations by eliminating noisy differences across eigenvalues. Figure 3 presents the histogram of the 45 correlations in the matrix before and after the RMT correction. As hinted before, the correlations of the treated matrix are closer to their true value (0.3) and much less disperse.

[3] These results are specific to the random sample drawn. Specifically, there are many sample in which all eigenvalues would fall within the RMT interval, leading to equal eigenvalues (after correction) and, therefore, to zero correlations across all random variables.

Figure 3: Correlations on the original and RMT-corrected correlation matrices



Source: Barclays Research

Risk factor model[4]

Factor models are a widely used technique to describe the return process of financial securities and, as we will see, a process to increase the robustness of the estimation (Lazanas et al. (2011b)). Suppose a portfolio manager wants to estimate and analyze the volatility of a large portfolio of securities. A possible method is to compute the volatility of the historical returns of the portfolio and use this measure to forecast future volatility. However, this framework does not take into account the changing nature of the portfolio and does not provide any insight into the relationships between different securities in the portfolio or its major sources of risk. A different method could use the fact that the portfolio's return is a function of security returns and the market weights of the securities in the portfolio. In this setting, the forecasted volatility of the portfolio (σ_P) can be computed as a function of the weights (w) and the covariance matrix ($\sum_s$) of the security returns:

$$\sigma_P^2 = w^T \cdot \sum_s \cdot w$$

This covariance matrix can be decomposed into the individual security volatilities and the correlations between security returns. By analyzing these correlations and volatilities, the manager can gain insight into the portfolio, for example, evaluate the riskiness of different components of the portfolio or understand how it can be diversified. This method requires the estimation of the correlation between each pair of securities. Unfortunately, this means that the number of parameters to be estimated grows quadratically with the number of securities in the portfolio.[5] For most practical portfolios, the relatively large number of issues makes it difficult to estimate the relationship between the security returns in a robust way. Moreover, this framework uses the history of individual issue returns to forecast future volatility. However, the securities' characteristics are generally dynamic; hence, using returns from different time periods may not produce good forecasts.[6] Finally, the analysis does not provide much insight regarding the broad risk factors affecting the portfolio. These drawbacks constitute the motivation for the multi-factor risk models widely used in portfolio management, of which the Global Risk Model (GRM) in POINT is an example.

[4] This section is largely reproduced from Lazanas et al. (2011b). This paper provides also more details about the estimation of the covariance matrix of the systematic risk factors.

[5] As an example, if the portfolio has 10 stocks, we need to estimate 55 parameters; with 100 stocks, we would need to estimate 5,050 parameters.

[6] This is especially the case in crisis periods, when security characteristics can change dramatically over a very short time.

These models allow us to reduce the dimensionality of the estimation problem and increase the robustness of the analysis. They may also increase our intuition regarding the major economic sources of risk in the portfolio.

In the context of factor models, the total return of a security is decomposed into a systematic and an idiosyncratic component. The systematic return is the part of the total return that is due to movements in common risk factors, such as interest rates or sector spreads. The idiosyncratic return, on the other hand, can be described as the residual or security-specific component that cannot be explained by the systematic factors. Under these models, the idiosyncratic return is independent of the systematic return and generally uncorrelated across issuers. Therefore, correlations between securities are driven by their exposures to the systematic risk factors and the correlations between them.

The following equation demonstrates the systematic and idiosyncratic components of security return:

$$r_s = L_s \cdot F + \varepsilon_s \tag{2}$$

The systematic return for security *s* is the product of the loadings of that security (L_s , also called sensitivities) and the returns of the systematic risk factors (F). The idiosyncratic return is given by ε_s [7]. The portfolio volatility can then be written as

$$\sigma_p^2 = L_p^T \cdot \sum_F \cdot L_p + w^T \cdot \Omega \cdot w \tag{3}$$

In equation (3), L_p represents the loadings of the portfolio to the risk factors (determined as the weighted average of individual security loadings) and $\sum_F$ is the covariance matrix of factor returns; w represents the vector of security weights in the portfolio and Ω is the covariance matrix of idiosyncratic security returns. As a result of the generally uncorrelated nature of idiosyncratic returns, the idiosyncratic covariance matrix is diagonal, with all elements outside its diagonal being zero. Therefore, the idiosyncratic risk of the portfolio is diversified away as the number of securities in it increases. This is the diversification benefit attained when combining uncorrelated exposures. We can represent the systematic variance of the portfolio as the first term in (3), $L_p^T \cdot \sum_F \cdot L_p$, and the idiosyncratic variance as the second one, $w^T \cdot \Omega \cdot w$.

For most practical portfolios, the number of risk factors is significantly smaller than the number of securities. Therefore, the number of parameters in $\sum_F$ is much smaller than the number of parameters in $\sum_S$, leading to a generally more robust estimation of the covariance matrix. Moreover, the factors can be designed in such a way that their return distributions are more stable over time than individual security return distributions, resulting in models with better predictability.

We want to understand if the RMT adds significantly more robustness into a framework where a parsimonious linear risk factor model is already employed.

A simple example

As an illustration of the use of the factor model structure for matrix estimation, consider the simulation performed for Figure 2. However, instead of using RMT, suppose that the data generating process (DGP) for r_{it} is defined in the spirit of (3) as:

[7] In the GRM, the risk factor loadings are typically security analytics (eg, durations) and, therefore, known at any point in time. This allows us to capture well the conditional nature of the portfolio, leading to more robust risk factor estimation.

$$r_{it} = F_t + \varepsilon_{it} \tag{4}$$

In this context, we can estimate the factor realizations and its variance as

$$\hat{F}_t = \sum_i r_{it} / N \text{ and } \hat{\sigma}_F^2 = \sqrt{\sum_t \left(F_t - \overline{F}\right)^2 /(T-1)}$$

Moreover, accounting for heteroskedasticity, we can estimate the idiosyncratic volatilities for each security as

$$\hat{\sigma}_{\varepsilon i}^2 = \sqrt{\sum_t \varepsilon_{it}^2 / (T-1)}$$

We can now construct a correlation matrix, using the fact that the correlation between securities *i* and *j* from the DGP (4) is given as:

$$\hat{\rho}_{ij} = \hat{\sigma}_F^2 / (\hat{\sigma}_{ri}\hat{\sigma}_{rj})$$

With the correlation matrix constructed this way, we can look at the structure of its eigenvalues (Figure 4). Comparing with the structure from Figure 2 (which uses the random sample), we can see that the lower eigenvalues are much flatter once the covariance matrix is estimated imposing the structure as in (4). Not surprising, therefore, is the fact that the corrections made by the RMT are much less severe than those in Figure 2. It is important to note that different assumptions regarding the DGP of a particular sample may lead to significantly different conclusions regarding the effect the RMT may have on the final correlation matrix[8]. This example – highly stylized – is presented to provide intuition.

Figure 4: Profile of eigenvalues assuming a one factor model with $\rho = 0.3$

Correlation Matrix	Eigenvalues										
	1	2	3	4	5	6	7	8	9	10	Sum
Original	0.58	0.60	0.61	0.61	0.61	0.63	0.66	0.68	0.71	4.30	10
RMT-corrected	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	4.30	10

Source: Barclays Research

The factor model estimation captures the return volatility and correlation through the factor exposure and correlations. One immediate observation is that the high eigenvalues exceed the predicted $\lambda_{\max}$. The "pure noise" hypothesis in RMT is therefore unsuitable with the inherited correlation implied in the factor model. A more realistic assumption is that the components of the correlation matrix that are orthogonal to the space of the large eigenvalues are dominated by noise. This amounts to renormalization of the return variance $\sigma^2 = 1$, leading to $\sigma^2 = 1 - \sum_{\lambda_i > \lambda_{\max}} \lambda_i / N$. This treatment essentially pushes the smoothing boundaries towards smaller eigenvalues and shrinks the length of the smoothing band, consistent with the preclusion of the systematic returns from the realm of the RMT analysis.[9]

[8] If we assume homoskedasticity, for instance, the implied correlation between any two securities would be the same. This would lead the set of eigenvalues from the factor-based correlation matrix to be the same, except for the higher one. In this case, the application of RMT would be redundant.

[9] This means that in Figure 2 and Figure 4, the boundaries $(\lambda_{\min}, \lambda_{\max})$ should be adjusted for the presence of the large eigenvalue. We keep the boundaries fixed in the examples for illustration of the intuition.

These highly stylized examples provide some intuition regarding the interaction between the RMT and a factor model structure through the prism of their eigenvalues' profile. In the next section, we develop the analysis by looking at real-life data and examples.

Empirical study

Previous research (Laloux etc. 1999) has shown that the RMT significantly reduces the estimation noise of the correlation matrix, leading to better results when used, for instance, in optimization problems. However, in that literature, the original covariance used in the analysis is typically very noisy. For instance, the covariance matrix is constructed directly from individual equity data.

We therefore use two applications to assess the role of RMT in the context of factor models. In the first, we analyze static results when using alternative covariance matrices. In the second, we provide a broader back-testing analysis of optimized portfolios – again using alternative covariance matrices – and look at their characteristics through time. The factor model used throughout this section is the GRM in POINT.

Security vs. factor model covariance matrices

For our first exercise, we use the Dow Jones composite index. In particular, our sample has 55 securities: the constituents of the Dow Jones composite for which we have all data back to February 2007. We construct the security-level (SL) empirical return covariance matrix with an exponential weighted moving average (EWMA) model with a one-year half life (see Gabudean and Schuehle (2011) for more on volatility models). This is one of the estimation procedures used in the GRM calibration, making the comparison between the SL and the GRM factor model (FM) matrix meaningful.[10] We focus on the EWMA covariance in both cases so that the length of the effective observation history is approximately the same.

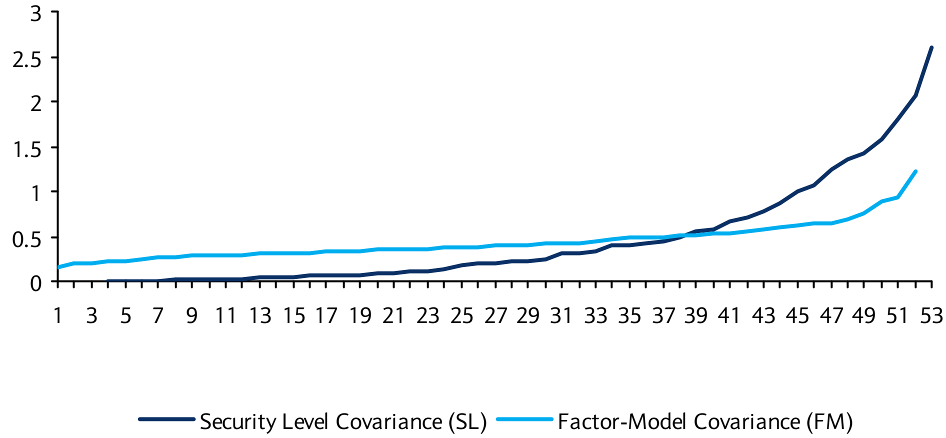
Eigenvalues

We begin our analysis by looking at the profile of the eigenvalues for the two matrices and compare them with their respective RMT smoothing areas. The RMT cleans a substantial number of eigenvalues of the SL covariance. 95% of them in the empirical covariance matrix fall within the smoothing boundary. This is consistent with the findings in the early literature. For instance, Laloux et al. (2000) applies this methodology to the S&P 500 and finds that 94% of the eigenvalues fall in the RMT smoothing region.

We find similar results by looking at the FM covariance matrix: 95% of the eigenvalues are in the smoothing interval for the factor-based covariance. This is not surprising when analyzing broad portfolios. However, there is a significant difference. The range of the smoothing region is significantly larger for the SL matrix (0.003, 2.61) than for the FM one (0.16, 1.21). This suggests that the effect of the RMT smoothing is much smaller for the FM matrix. This is shown in Figure 5, where we see a much flatter profile of eigenvalues for the FM matrix within the smoothing region. With the enhanced robustness coming from the factor model, the RMT shows limited scope for changes in the FM matrix.

[10] The number of systematic factors called by the GRM equity factor risk model is 30, suggesting that the number of parameters to estimate (about 500) is reduced to about one-third of the ones in the SL covariance matrix (about 1,500). See Lazanas et al. (2011a) for more information on the equity model in the GRM.

Figure 5: Dow Jones composites – Profile of eigenvalues in the smoothing range



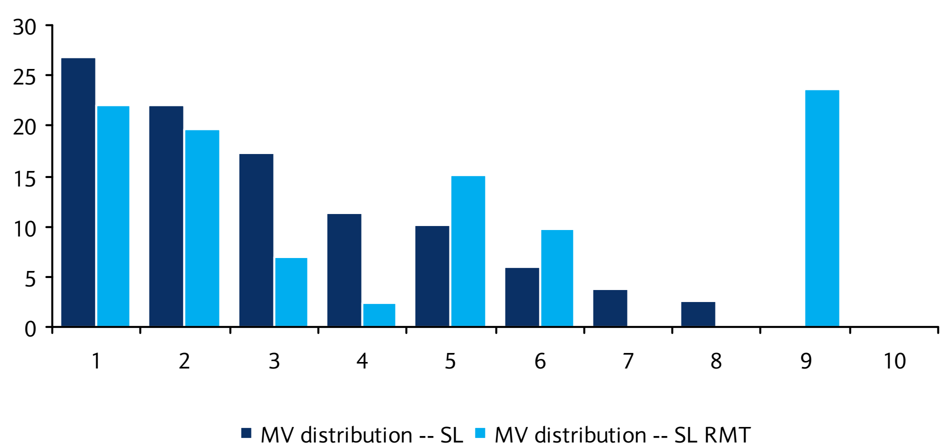
Source: Barclays Research

Optimal Profile

It is known that the potential estimation and specification errors embedded in an empirical covariance matrix are magnified and explored by an optimization that uses it as an input (see Li and Silva (2012) for more on this issue). As the optimized portfolio loads heavily on less risky securities, any noise in the associated small eigenvalues is exaggerated by the security optimization process. The RMT is a good candidate for reducing this bias. Therefore, we want to analyze its effect on the profile of optimal portfolios. To do so, we use the SL and the FM (original and RMT-corrected) to construct the minimum variance portfolio. Then, we compare their characteristics to understand the role of the RMT correction.

First, we analyze the effect of RMT on the solutions to the problem using SL covariance matrix. Figure 6 shows the market value distribution (in descending order) of the securities in the optimal solution based on the SL and the SL-RMT-corrected covariance matrices. Six securities out of 55 are selected in common in the optimal portfolio. However, there are important differences: the largest position constructed with the RMT SL covariance is not selected in the optimal portfolio with the original SL covariance. This suggests the potential for significantly different optimal portfolios when applying RMT to the SL covariance. This point, highlighted in previous literature is not the focus of our paper, but is discussed here for comparison purposes.

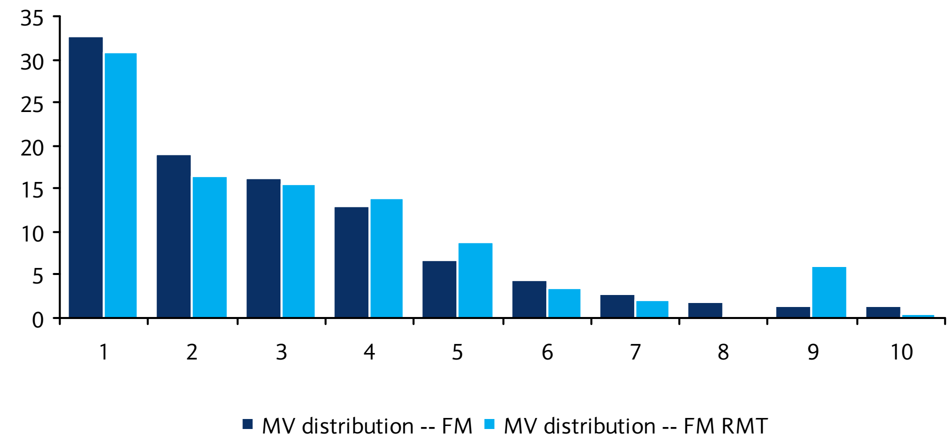
Figure 6: Optimal profile – Security level covariance



Source: Barclays Research

On the contrary, Figure 7 shows that there is little difference between the optimized portfolios from the FM and the FM-RMT-corrected covariance matrices. The profiles are stable, again indicating limited value added by the RMT. One can note, however, a more equally distributed portfolio under the FM-RMT portfolio, especially around major positions.

Figure 7: Optimal profile – Factor model covariance



Source: Barclays Research

Overall, these results suggest that the RMT may significantly affect the SL covariance matrix estimation.[11] However, the effect on the FM covariance matrix is muted. This is consistent with the fact that the POINT factor risk model has already rendered substantial robustness to the covariance estimation process.

Back-testing optimal portfolios – Index replication using the GRM

In this section, we construct a time series of monthly index-replicating portfolios (starting from 2007) using the POINT optimizer. The objective function is to minimize the TEV relative to the benchmark index, using no more than 30 securities. This is an optimization exercise extensively used by asset managers who track particular indices or want to replicate a particular index. For the illustration of RMT's effect on index-replicating optimal portfolios, we use separately the raw FM matrix from the GRM and its RMT-corrected version on the objective function of the optimization problem. We apply the exercise to three different universes: the S&P 500 equity index, the Barclays USD-denominated Emerging Markets index (fixed income) and a multi-asset index that is a volatility-weighted mixture of the Barclays US Treasury index and the S&P 500 index.

We focus on two key metrics to monitor improvements in the optimal solution. The first is the standard deviation of the realized tracking error (the realized TEV). A smaller number demonstrates better index replication. In this case, it suggests a more robust covariance matrix. The other is the standard deviation of the standardized TE (defined as the realized TE divided by the corresponding forecasted TEV), therefore focusing on the accuracy of the risk forecast. In a perfect risk model, this metric should be one.

Our findings show that the optimization solution with RMT achieves a smaller realized TEV, ie, a more robust performance. The improvement on the accuracy of the risk forecast, however, is typically of small magnitude. Moreover, the well-known risk underestimation in

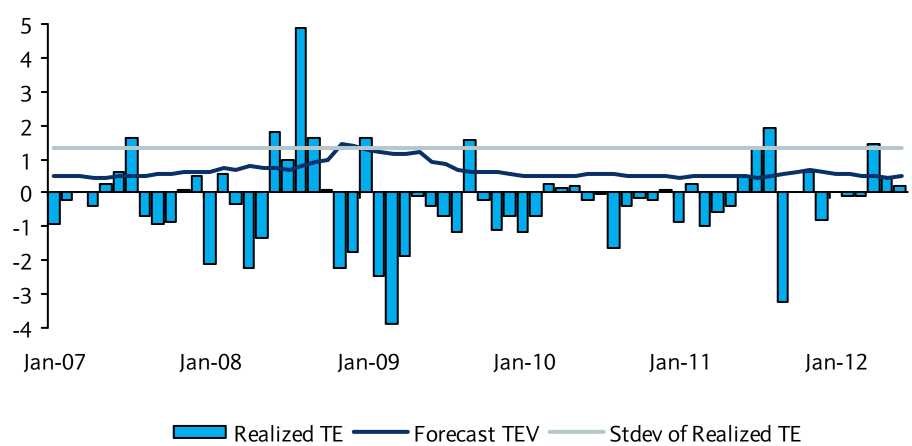
[11] This observation is not a recommendation to use the RMT for portfolio volatility estimation. Instead, the technique can be used to enhance other applications of the covariance matrix, namely optimization.

mean-variance optimizations is not attenuated by the RMT. Correcting this issue requires approaches that will be explored in a forthcoming publication (Li and Silva (2012)).

S&P 500 Equity Index Replication Portfolio

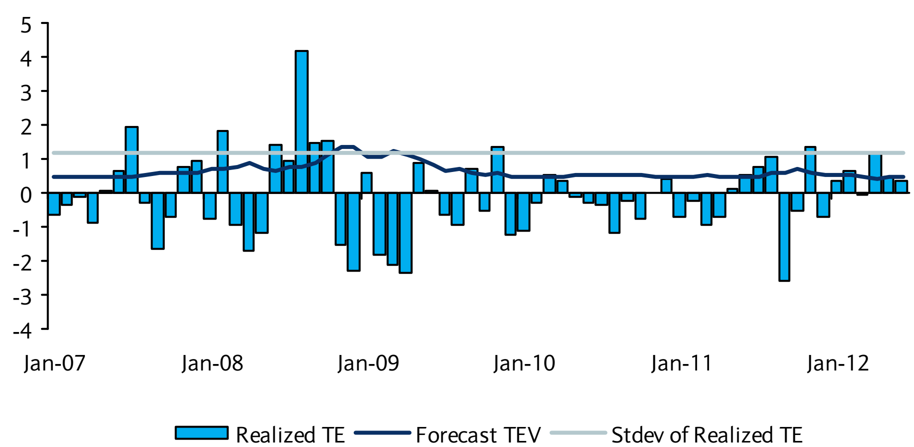
Figure 8-Figure 10 illustrate the back-testing performance of the optimal portfolio that replicates the S&P 500 index. The realized TEV reduces from 1.32% to 1.18% with RMT application, signalling a more robust optimal portfolio. Moreover, other statistics show improvement, as well: the realized TE at the 5 and 95 percentiles shrinks from (-2.2, 1.6) to (-2.0, 1.5) with the RMT. We also observe a small reduction on the standard deviation of the standardized TE, indicating a small improvement of the risk forecast with covariance smoothed with RMT.

Figure 8: Back performance of the optimal portfolio – FM (S&P 500)



Source: Barclays Research

Figure 9: Back performance of the optimal portfolio – FM RMT (S&P 500)



Source: Barclays Research

Figure 10: Back performance of the optimal portfolio (S&P 500)

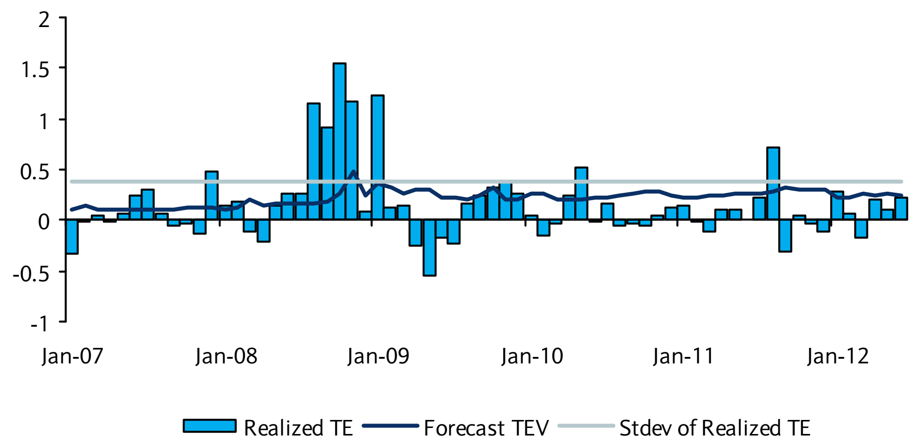
	Optimal Portfolio(%)	Optimal Portfolio with RMT(%)
Forecasted TEV(mean)	0.65	0.63
STD of realized TE	1.32	1.18
STD (real. TE/TEV)	1.88	1.73

Source: Barclays Research

Fixed income emerging market index replication portfolio (EM)

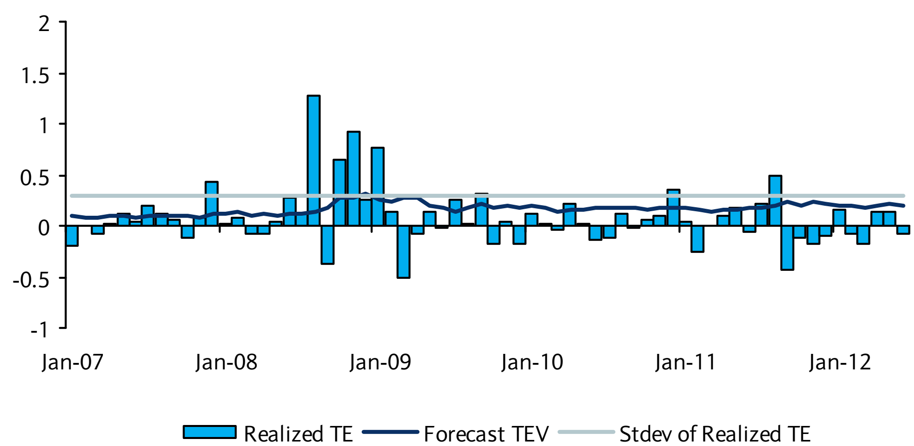
A replication exercise on the Barclays Emerging Market (USD denominated) index shows similar results (Figure 11-Figure 13). The realized TEV reduces from 0.37% to 0.28% with RMT application, indicating again a more robust index replication. The realized TE at the 5 and 95 percentiles shrinks from (-0.3, 1.1) to (-0.2, 0.6) with the RMT. However, the standard deviation of the standardized TE is unchanged. As noted before, the RMT seems to have very limited ability to correct the risk underestimation bias on the optimized portfolios.

Figure 11: Back performance of the optimal portfolio – FM (EM)



Source: Barclays Research

Figure 12: Back performance of the optimal portfolio – FM RMT (EM)



Source: Barclays Research

Figure 13: Back performance of the optimal portfolio (EM)

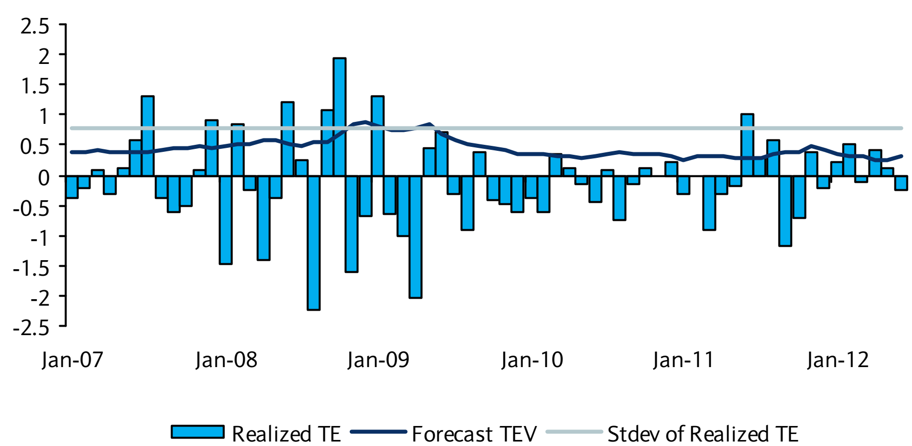
	Optimal Portfolio(%)	Optimal Portfolio with RMT(%)
Forecasted TEV (mean)	0.22	0.16
STD of realized TE	0.37	0.28
STD (real. TE/TEV)	1.67	1.67

Source: Barclays Research

Multi-asset volatility-weighted index replication portfolio

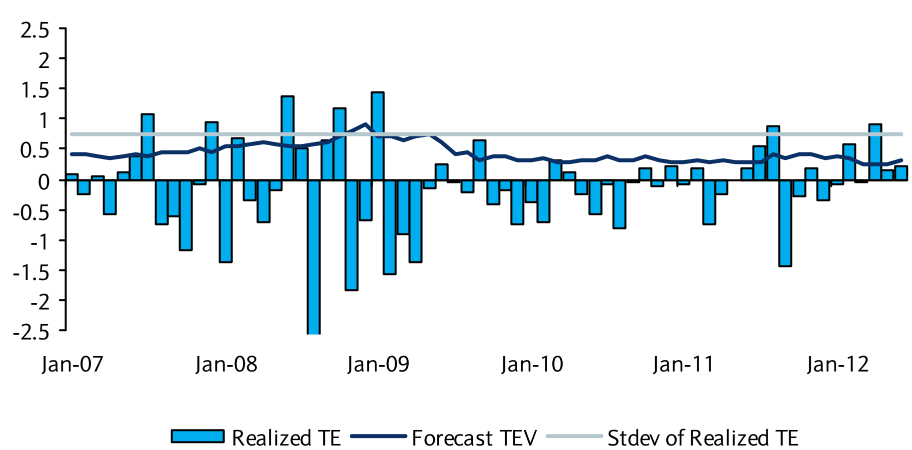
In the last example, we look at the optimal portfolio that replicates a multi-asset index. Figure 14 illustrate the back-testing results. The index is created as a composite of the Barclays US Treasury index and the S&P 500 index, weighted by the index return volatility. Here, the changes are even more muted. The realized TEV of the optimal portfolio reduces from 0.78% to 0.76% with RMT application. Moreover, the standard deviation of the standardized TE actually goes up slightly, showing a marginal deterioration of performance for the RMT. For this mixed portfolio, with two very different asset classes, the RMT seems to struggle to identify and smooth noisy eigenvalues.

Figure 14: Back performance of the optimal portfolio – FM (Multi-asset index)



Source: Barclays Research

Figure 15: Back performance of the optimal portfolio – FM RMT (Multi-asset index)



Source: Barclays Research

Figure 16: Back performance of the optimal portfolio (Multi-asset index)

	Optimal Portfolio(%)	Optimal Portfolio with RMT(%)
Forecasted TEV (mean)	0.44	0.43
STD of realized TE	0.78	0.76
STD (real. TE/TEV)	1.58	1.61

Source: Barclays Research

Conclusion

Several studies defend the use of the random matrix theory to add robustness to the covariance matrix when used in portfolio optimization problems. These focus on conditions with a relatively limited history of security returns. When the history is short for a portfolio with a large number of securities, the arguments goes, sample covariance matrices are poised to be noisy and, therefore, less reliable. The use of these matrices for optimization tends to exacerbate the problem, leading to non-stable optimization solutions, subject to undesirable ex-post tracking performance. These studies found that smoothing the eigenvalues of the covariance matrix significantly improves the property of optimized portfolios.

We apply the random matrix smoothing on covariance matrices estimated using a factor model structure. One primary objective of the factor model is to add robustness by significantly reducing the parameters to be estimated and, therefore, increasing their robustness. Our main finding is that the smoothing suggested by the random matrix theory adds significantly less to a factor-model-based covariance matrix than to a raw empirical covariance of security returns. The robustness added by the structure of the factors leaves small scope for additional improvements through the random matrix theory. Nevertheless, RMT manages to contribute to a more robust optimized solution constructed with factor risk model, as suggested in our back-testing investigation. We show this to be the case for equity, fixed income and marginally for multi-asset portfolios. The value of the random matrix theory can be potentially large in particular cases where the factor structure may not deliver sufficient structure and robustness to the covariance matrix.

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