Fixed Income Attribution Analysis

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# 1 Introduction

The performance of a portfolio is often measured relative to some comparison index, or benchmark. The portfolio management can be either passive or active, i.e. the portfolio manager tries to either replicate or exceed the return of the benchmark. If an active strategy is used, the risk exposure of the portfolio and the benchmark will be different. The purpose of attribution analysis is to determine how much the difference in exposure to certain risk factors has contributed to the difference in return. For fixed income portfolios, the exposure can differ with respect to, for example, a certain issuer, a certain maturity sector or the different types of typical yield curve movements.

In this study, a distinction is made between performance attribution analysis and return attribution analysis. Of the two, the former is more related to the different steps in the management process. A portfolio manager expresses his view on the market by overweighting and underweighting certain market sectors. Within each sector, the manager then picks individual securities, typically deviating from the benchmark’s composition within the sector. Performance attribution analysis tries to answer how much of the excess return can be attributed to each of these activities, asset allocation and security selection. Return attribution analysis is concerned with the return of individual securities. The return is broken down with respect to such factors as accretion, coupon return and yield curve movement. The return due to a specific factor can then be aggregated to market or market sector level for the portfolio and the benchmark.

This report tries to provide a somewhat comprehensive description of how to perform an attribution analysis. After an introduction to bonds in chapter two, the discussion on performance attribution in chapter three to five is essentially general and applies to any type of portfolio. In chapter six, the return attribution model that has been implemented for bonds is described in some detail.

# 2 Some Preliminaries

In this section we consider a bond with coupon payments {*ci*} at times {*ti*} and a principal payment *M* at time of maturity *T*.

## 2.1 Zero-coupon bonds

For zero-coupon bonds the only payment that is made by the issuer is the principal. The rate of interest, the yield, earned by the investor on a zero-coupon bond is called the spot rate. If *Y*(*t*,*T*) is the yield of the bond at time *t* expressed with continuous compounding[[1]](#footnote-1) and *P*(*t*,*T*) is the price of the bond, we have:

. (2.1 )

## 2.2 Coupon bonds

The theoretical price of a coupon bond is the present value of the future cash flows, i.e. coupons and principal. The cash flows are discounted using the corresponding spot rate curve, i.e. for a Treasury bond we use the spot rate curve for zero-coupon Treasuries. The theoretical price of our bond is thus expressed as:

. (2.2)

Normally the coupon payments are received annually or semi-annually.

Bond prices are usually quoted as a percentage of the principal amount. The quoted price is the gross price of the bond minus accrued interest and is referred to as the *clean price*.

 *gross price = clean price + accrued interest*

The accrued interest grows linearly from zero at the time of the last coupon payment to the coupon value at the time of the next coupon payment.



Figure 1.1: Accrued interest

Different bond markets have different day count conventions for calculating the time between two events. The Swedish bond market uses the 30/360 day convention, assuming there are 30 days in a month and 360 days in a year. *Ex-coupon days* are also used in the Swedish bond market. The number of ex-coupon days indicates how many days before the next coupon payment the bond has to be purchased for the purchaser to have right to that coupon. Thus, if the number of ex-coupon days is five, the gross price of the bond will drop five days before each coupon payment. Since the clean price still is supposed to include the part of the next coupon that is left (coupon value minus accrued interest in fig 1.1), the accrued interest will be the negative of that value on an ex-coupon day.

In a purchase, the day money and bonds are exchanged, the settlement date, does not normally coincide with the trade date. Usually, the settlement date is two or three bank days after the trade date.

## 2.3 The implied zero curve

Normally there are no zero-coupons with maturity greater than two years on the market so the spot rate curve *Y*(*t,T*) cannot be observed directly. Instead a curve that is implied by the coupon bond prices is calculated. An implied Treasury zero curve is a curve that, when used to discount the cash flows of Treasury bonds, gives the market prices of the bonds. In practice, a set of potential zero curves *Y*={*Ym* | *m* ∈ i(*m)*} and some numerical method are used to minimize the sum

, (2.3)

where *P*(*i*) is the observed market value of bond i, which may be a coupon or a zero-coupon bond, and *P*(*i*,*m*) is the theoretical price of bond i implied by the curve *Ym*. A model is called residual-free if the error is zero or can be made arbitrarily close to zero. If, on the other hand, the number of parameters in the model is less than the number of bonds used in the derivation of the curve, there will in general be a pricing error. This error can give an indication of which bonds are mispriced.

Zero curves can also be calculated for other sectors of the market, for example the mortgage bonds give an implied mortgage zero curve.

## 2.4 Yield

The yield-to-maturity, or just yield, of a bond is the rate that, when used to discount the future cash flows, gives the market value of the bond. Thus, if *y* is the yield of our bond with market value *P*(*t*), we have:

. (2.4)

The yield-to-maturity is also called redemption yield. For zero-coupon bonds the yield-to-maturity is the same as the spot rate.

## 2.5 Spread

Except for Treasuries, bonds have an associated default risk. Default risk, or credit risk, is the risk that the issuer will not fulfil its payment obligations. A higher default risk means that the investors will require a higher yield, all other things being equal. Thus, the implied mortgage zero curve will be above the implied Treasury zero curve as in figure 2.1. The difference *YM*(*t*, *T*) – *YT*(*t*, *T*) is called the spread and reflects the default risk of the mortgage bonds.

Figure 2.1: The mortgage spread

## 2.6Duration

Duration is a measure of a bond’s price sensitivity to a parallel shift in the spot rate curve. The price of our bond can be expressed as a function of its yield:

****.(2.5)

The Taylor expansion of P(y) is:

. (2.6)

Thus, the return of the bond with respect to a small change Δy is:

 (2.7)

-P′(y)/P(y) is called the duration of the bond and is the first order return sensitivity to a change in yield. For our bond we get the following expression for the duration:

. (2.8)

The duration can be interpreted as the weighted average of the times payments are made, with the weights being the fraction of the bond’s value attributable to the corresponding cash flow.

# 3 Performance Attribution Analysis

## 3.1 Model framework

Performance attribution analysis breaks down the excess return with respect to asset allocation and security selection. A natural and popular method for doing this was suggested by Brinson, Hood and Beebower [1986]. The excess return is expressed as a sum of three terms, corresponding to asset allocation, security selection and an interaction effect.

, (3.1)

*asset allocation security selection interaction effect*

where

*wsP* = the portfolio weight in sector s

*wsB* = the benchmark weight in sector s

*rsP* = the portfolio return in sector s

*rsB* = the benchmark return in sector s.

This procedure can be used repeatedly to break down the difference *rsP*-*rsB* within each sector. For example, if the sectors in the above sum are intervals on the duration axis, the excess return for each interval can be decomposed with respect to Treasuries, corporate bonds and mortgage bonds.

Excess return due to asset allocation is achieved by either overweighting sectors that have a higher return than the total benchmark return (*wsP* > *wsB* and *rsB* > *rB*), or, conversely, underweighting underperforming sectors (*wsP* < *wsB* and *rsB* < *rB*). If we define the active weighting excess return from an individual sector as *es* = (*wsP* – *wsB*)×( *rsB* - *rB*) the asset allocation sum is preferably expressed as:

. (3.2)

### 3.1.1 Example[[2]](#footnote-2)1

Consider a fixed income portfolio, which is managed relative to a benchmark. The portfolio manager has a strategy where he first decides the distribution of the portfolio along the duration axis. He then chooses his sector allocation, i. e. the portfolio weights for government, corporate and mortgage bonds. Finally, individual bonds are picked according to the allocation decisions.

Instead of just using the active return, portfolio return minus benchmark return, to assess the skill of the portfolio manager, we want to do an attribution analysis to see how each allocation decision has contributed to the active return. To analyse the duration strategy the portfolio is split into several sub-portfolios corresponding to different segments of the duration axis. If the manager has used a certain segmentation in his strategy, i. e. he has decided the portfolio weights for the sub-portfolios given by this segmentation, the same segmentation should be used in the analysis. If, on the other hand, the decisions are qualitative rather than quantitative, for example if a barbell strategy is used but without assigning specific weights for different segments, we might choose to split the duration axis into the intervals [0,3], (3,5], (5,10] and (10,∞). Let

, ,  and  *t* = 1, 2, 3, 4,

be the weights and returns of the portfolio and benchmark for the four segments and the measurement period. The excess return due to curve (duration) allocation is

, (3.3)

where *rB* is the total benchmark return and each term is the excess return from the corresponding segment. The rest of the excess return,

, (3.4)

is due to sector allocation and security selection[[3]](#footnote-3)2. The excess return for segment *t* can be broken down in the same way as the total excess return:

, (3.5)

where  () is the part of the portfolio’s (benchmark’s) allocation to duration cell *t* which is in sector *s*, and  () is the portfolio (benchmark) return of the securities in segment *t* and sector *s*. Using (3.5) in (3.4) the sector allocation excess return is

****. (3.6)

The term

 (3.7)

is the excess return due to the decision to allocate  of the portfolio to segment *t* and sector *s*, given that the total allocation to segment *t* is . The residual excess return is due to security selection:

. (3.8)

The term

 (3.9)

is the contribution to the excess return from security selection within segment *t* and sector *s*.

The impact of the management decisions will be best reflected by the analysis if the management process actually is such that the segment allocation is decided first and then the sector allocation within each segment is chosen. If this is not the case, but only the overall sector allocation is set and the allocation within each segment is done more randomly, there is no preferred order to do the attribution analysis. We can still calculate the excess returns due to sector and segment allocation but the values will depend on whether the total excess return *e* first is broken down with respect to segment and then to sector (as above) or vice versa. In the latter case this excess return from allocating  of the portfolio value in sector *s* to segment *t* will be

, (3.10)

where the weights are analogous to those above.

Even if the management process is not as quantitative as the attribution analysis, we can still treat it as if it were. The analysis will show how the excess return would have been realized had the allocation decisions been carried out in the manner the analysis assumes. After getting feedback from an attribution analysis model it is possible that the manager will gradually adapt his management process to the model.

# 4 Calculating the return

## 4.1 The return of a static portfolio

To perform an attribution analysis we need to define the weights and returns for each sector. This is most easily done if there are no cash flows in or out of the portfolio or between different sectors and no income is received from the assets. In that case the return of portfolio *P* for day *t* is simply defined as

, (4.1)

where *MP*(*t*) is the value of portfolio *P* at the beginning if day *t*.

Let *Pi*(*t*) and *Ai*(*t*) be the clean price and accrued interest, respectively, of bond *i* at the beginning of day *t*. Then, the return of bond *i* for day *t,* during which no coupon payments are made, is

. (4.2)

If the market uses ex-coupon days, the following modification has to be made to the formula:

, (4.3)

where *CPi*(*t*) is equal to the next coupon if *t* less than the coupon date, and the settlement date (= *t* + 3 bank days for the Swedish bond market) is larger than or equal to the first ex-coupon date[[4]](#footnote-4)1; otherwise *CPi*(*t*) = 0.

### Example

The Swedish government bond SGB1030 has maturity 15, June 2001 and a coupon of 13.

The number of ex-coupon days on the Swedish bond market is five (including non-banking days), which means that the settlement day has to be June 9 or earlier for the buyer to receive the June 15 coupon. The settlement day is three bank days after the trade date so after June 6 the coupon will not be included in the price of the bond. On June 6, 2000 the gross price of the bond was 121,03 SEK. After that date the coupon is no longer reflected by the gross price of the bond, which on June 13, 2000 had sunk to 108,13. When calculating the return of the bond, the coupon has to be added to the end value as the period end is before the coupon date. The return of SGB 1030 over the period June 6, 2000 – June 13, 2000 was (108,13+13-121,03)/121,03 = 8,3 bp.

Let *Ni*(*t*) be the position of portfolio *P* in bond *i* at the beginning of day *t*. Then the return and weight of portfolio *P* in market sector *s* are

 (4.4)

and

, (4.5)

where *Bs*is the set of securities that fall in sector s and we have used the fact that *Ni*(*t*+1) = *Ni*(*t*) since no securities are bought or sold during day *t*..

## 4.2 Incorporating cash-flows

When calculating the return of a portfolio or a sub-portfolio, it makes life easier if the portfolio or sub-portfolio has not been subject to cash flows during the measurement period.

Suppose, however, that some time in the period [*t, t + n*] the portfolio manager made a tactical bet, putting a bigger part of his portfolio in the fixed income sector at the expense of equities. This means that there has been a cash flow from the equity portfolio to the fixed income portfolio. That cash flow has to be taken into account when calculating the return for the two sub-portfolios. The total portfolio, however, has only had cash flows between sectors, and the standard formula for the portfolio return can be used:

. (4.6)

Had money been invested in or taken out of the portfolio that formula would not be satisfactory. If cash flows occurred at the beginning or the end of the period we can make a simple adjustment to the formula,

, (4.7)

where  and  are the cash flows at the start and end of the period. They are positive for cash flows into the sector. This way we get a return based on just the assets that were in the portfolio for the whole period. The above formula can be generalized to handle cash flows at any time during the period. We assume that the value of the assets in the portfolio grows linearly from day *t* today *t* + *n*. If a cash flow *C* occurs on day *t* + *k*, at the end of the period this cash flow will have grown to

. (4.8)

With = 0, (4.6) takes the form

, (4.9)

and by solving for *rP* we get

. (4.10)

This formula is often referred to as the *modified Dietz method*. In general, if there are cash flows {*Ci*} occurring on days {*t*+*ki*} the return is

 , (4.11)

where .

The method does not take into account the changes in the rate of return over the period. By calculating the return on a daily basis and compounding the results we get a better measure of the whole period return.

## 4.3 Calculating the daily return

Calculating the return when there are cash flows involved is somewhat arbitrary. The standard[[5]](#footnote-5)1 is to place the cash flows *C*(*t*) in the denominator, corresponding to cash flows at the beginning of the day. The return of day *t*, is

, (4.12)

where *C*(*t*) is the net cash flow into the portfolio during day *t*. The return from day 1 to day

*n* + 1 is

. (4.13)

But, as we will see, we cannot rely on a single formula to cover all possible situations.

### 4.3.1 A portfolio consisting of one security

#### We have not yet considered trading profit/loss (P/L) or positions that have negative market value. To illustrate these issues[[6]](#footnote-6)2 we consider a portfolio consisting of just one type of security, for example a portfolio of Ericsson B shares. First, we introduce some notation:

*N(t)* number of shares in the portfolioat the start of day *t*

*P(t)* valuation price at time *t*

#### *NB(t)* number of shares bought between times *t* and *t* + 1

*NS(t)* number of shares sold between times *t* and *t* + 1

*PB(t)* average price of the *NB*(*t*) shares bought

*PS(t)* average price of the *NS*(*t*) shares sold

Now, the trading P/L *π(t)* is defined as

. (4.14)

When calculating the return we assume that shares are bought and sold at the same price, thus separating the return due to trading P/L. This means that the cash flow is defined as follows:

#### . (4.15)

The return for the period [*t, t* +1]is

. (4.16)

If the numerator and denominator in (4.15) have different signs[[7]](#footnote-7)3 the return will not make sense. If that can be resolved by putting the cash flow *C*(*t*) in the numerator, the return is set to

. (4.17)

(4.16) is also used for the return if *N*(*t*+1) = 0. There are cases, though, where none of the above formulas will work.

The returns discussed this far do not include trading P/L. The actual cash flow into the portfolio equals the cash flow *C*(*t*) minus the trading P/L *π*(*t*)[[8]](#footnote-8)4:

*total cash flow into portfolio = *. (4.18)

A return including trading P/L could be defined as

, (4.19)

and the return *r*P/L due to trading P/L would then be

. (4.20)

If the number of shares at the time *t*+1 is negative, i.e. *N*(*t*+1) < 0, the formula (4.16) is altered according to

#### , (4.21)

where the assumption is made that the denominator also is less than zero. Or, if (4.17) is used,

the same change is made if *N*(*t*) < 0:

. (4.22)

### 4.3.2 A general portfolio

The discussion in the previous section can be generalised to cover any portfolio. The portfolio return is

, (4.23)

where the sum is over all securities in the portfolio and the assumption is made that the numerator and denominator have the same sign. For most real world portfolios this should be true as the daily inflows or outflows normally are much smaller than the portfolio value. However, for a small sector of the portfolio it might not be true. The return of a portfolio can also be written as a weighted sum of the sub-portfolio returns,

. (4.24)

Assuming that the sector returns can be defined according to (4.16) and (4.21) the weights are

 (4.25)

where

 (4.26)

with summation over all the securities belonging to sector *s*.

# 5 Multi-period attribution analysis

If an attribution analysis is made over a long period it is possible that the sector weights will change considerably from the beginning weights. In that case, one might ask whether the attribution analysis gives a true picture of how the management decisions have contributed to the excess return. Consider the following example:

The market is divided into two sectors, equity and fixed income. The following notation is used for the weights and returns of the portfolio:

 the return in the equity (fixed income) sector for the whole period.

 the return in the equity (fixed income) sector for the first half of the period.

 the return in the equity (fixed income) sector for the second half of the period.

 the weight in the equity (fixed income) sector at the start of the period.

 the weight in the equity (fixed income) sector at the middle of the period.

Similarly, for the benchmark the whole period return in the equity sector is denoted .

The asset allocation manager has chosen a passive strategy, i. e. at the start of the measurement period we have:

 , . (5.1)

The returns for the portfolio and benchmark over the measurement period were:

 ,  ,  , . (5.2)

The excess return for the period is *wePre* and an attribution analysis over the whole period shows that this is solely due to security selection within the equity sector. However, suppose that the return for the portfolio and the benchmark for the first half of the period were:

 ,  ,  , . (5.3)

This means that the returns for the second half of the period must have been:

 ,  ,  , . (5.4)

The portfolio weights at the middle of the period were,

 , . (5.5)

As for the whole period, the excess return of  for the first half is attributed to the equity manager’s security selection ability. The difference in equity return between the portfolio and the benchmark leads to a drift from passive weighting in the first period. By not rebalancing the portfolio we get an excess return of  in the second period. If  and  are large this excess return might be considerable. It is, however, not accounted for by the whole period attribution analysis.

## 5.1 Linking attribution results

The above example shows that there is an incentive to perform the attribution analysis on short periods. To get an analysis for a longer period we need a way of combining the short period results. There is no one or obvious way of doing this. Consider *N* subsequent periods and assume that the excess return *et* for each period *t* is broken down with respect to some set of factors that explain all of the excess return. If the excess return for period *t* due to factor *i* is *et,i* we have

.  (5.6)

The problem of combining attribution results arises because we are dealing with discrete returns that compound over time according to

 , . (5.7)

Thus, the excess return *e* is not equal to the sum of the single-period excess returns,

. (5.8)

Therefore, if we define the excess return *ei* attributable to factor *i* simply as the sum of the single-period excess returns *et,i* from factor *i,* we *will not* generally have

. (5.9)

A method that achieves the above equality, i.e. the excess return over all periods is fully explained by the factors used in the single period attribution, was presented by Cariño [1999].

His idea is to use the continuous compounding returns which are additive over time. For each periodthe factor *kt* is defined as

 (5.10)

Since

= and (5.11)

, (5.12)

we have

. (5.13)

The difference we are interested in is  and therefore we define

 (5.14)

so that

 (5.15)

Now, the excess return due to factor *i* is defined as

 (5.16)

This way the total excess return *e = rP - rB* is completely explained by the *N* factors. It should be pointed out that this is by no means the “correct” method for linking attribution results over several periods. When dealing with discretely compounding returns, it is difficult, if even possible, to isolate the effect of a single factor in a multi-period analysis.

# 6 Return Attribution Analysis

Return attribution analysis is an *ex post* analysis that tries to express the portfolio return as a sum, where each term is associated with a specific risk factor that the portfolio is exposed to.

. (6.1)

Thus, the analysis has a connection to factor models that are used to relate the risk and return of a security to a set of common factors that affect the return of similar securities. In the Arbitrage Pricing Theory (APT), developed by Stephen Ross in 1975, the return of asset *i* is given by

, (6.2)

where tilde (~) denotes a random variable and

 = expected return of asset i

*=* the *jth* common factor with mean zero

*βij* = the sensitivity of the return of asset *i* to the common factor 

 = a mean-zero factor, specific for asset *i*, that is uncorrelated with the common factors

and with the specific factors of the other assets.

For fixed income portfolios, the main sources of risk are the movements of the yield curve. It is therefore natural to define the common factors as different types of yield curve movements.

The value of these factors for a given period are obtained by decomposing the yield curve movement for the period with respect to the movements corresponding to the factors. Normally, as in Litterman and Scheinkman (1991), three factors are used to explain the return due to yield curve movements[[9]](#footnote-9)1. Litterman and Scheinkman use a statistical method called principal component analysis (PCA) to derive three uncorrelated factors, or yield curve movements. This way the factors are chosen optimally, in the sense that no other set of three uncorrelated factors explains as much of the variance of historical return of the bonds used in the derivation of the factors. However, PCA does not always give simple movements that are intuitive for the fixed income managers. If the manager learns that 50% of his return is due to factor one, that information is not worth much to him unless he understands how the factor one movement corresponds to his portfolio strategy, i. e. his distribution of the portfolio along the yield curve. The curve movements that are part of the fixed income jargon are called *level*, *slope* and *curvature* or sometimes *shift*, *twist* and *butterfly.* They are not precisely defined but qualitatively correspond to a parallel shift, a change in steepness and a change in curvature of the yield curve as illustrated in figure 6.1.



Figure 6.1: The different types of yield curve movement

## 6.1 Price return

The most natural division of the return of a bond is into price return and coupon return. The price return is due to changes in the clean price and the coupon return comes from coupon payments and changes in accrued interest.

Clean price changes are a result of the passage of time and yield curve movements. A decomposition of the price return is made by subsequently changing the yield curve used in the pricing of the bond, where each change corresponds to a factor that affects the clean price of the bond. This decomposition will depend not only on the definition of the factors but also on the yield curve that is used in the pricing of the bonds. We can either use a curve that is given by the yield of the bonds, and some interpolation method for the yields between the maturity grid points, or a zero curve implied by the bonds. In the latter case the factor returns will also depend on the method that is used in the derivation of the zero curve.

### 6.1.1 Accretion and rolldown return

The time component of the price return is defined as the return that would be obtained if the curve looked the same at the beginning and end of the period. This means that, for the period [*t*1, *t*2], the curve  is translated by the amount *t2*-*t*1:

. (6.3)

If *Pi*,*t*(*Y*(⋅))is the clean price of bond *i* at time *t*, given the yield curve *Y*(⋅), the time component of the price return for bond *i* is:

*time component of price return* = , (6.4)

where *GPi,t* is the gross price of bond *i* at time *t*. The curve will be evaluated at the times of all the cash flows of the bond if *Y*(⋅)is a zero curve or only at the time of maturity of the bond if *Y*(⋅)is the yield of coupon bonds. The time component can be further divided into *accretion* and *rolldown* returns. The accretion return is defined as the return that would be obtained if the yield of the bond at time *t*2 was the same as at time *t1*. The residual is the rolldown return. For a positively sloping curve, we have

, (6.5)

which means that the yield corresponding to the clean price  will be less than the bond’s yield at time *t*1, providing a positive rolldown return.

### 6.1.2 Curve return

We denote the return due to curve movements as *curve return*. To be meaningful, the decomposition of the curve return should be such that the portfolio manager can understand the qualitative impact of his strategy on the different return components. Therefore, simple types of yield curve movements, as in fig 6.1, are used. Different models have different definitions of the shift, twist and butterfly factors. For a given set of factors the decomposition of the total curve movement into the three factor movements can also differ.

Apart from being intuitive, the factors should be able to explain most of the curve movement. This means that the residual term *ε*(*T*) in the decomposition,

, (6.6)

should be small in relation to the shift, twist and butterfly components *s*(*T*), *t*(*T*) and *b*(*T*). The character of the yield curve dynamics is such that most of the curve change is usually explained by the parallel shift component[[10]](#footnote-10)2.

When the curve decomposition is made the shift return is defined as

*shift return* = , (6.7)

where

. (6.8)

The other return components are defined in the same manner, for example

*twist return =* , (6.9)

where

. (6.10)

We call the curve return from the residual movement *ε*(*T*) *shape return*. The shape return is:

*shape return* = ,(6.11)

where

.

### 6.1.3 Spread return

For bonds with credit risk a part of the price return can also be attributed to the change in spread, see fig 2.1. The spread change can be defined as the residual after the movement of the default free curve has been subtracted from the total movement of the credit curve. The spread return is then given by

*spread return* = , (6.12)

where

, (6.13)

where  is the Treasury zero curve at time *t*. The shape return, i. e. the curve return due to the residual movement *ε*(*T*), is given by

*shape return* = . (6.14)

## 6.2 Coupon return

Coupon return is the return due to coupon payments and changes in accrued interest. The coupon return for the period [*t*1, *t*2] is:

*coupon return* = , (6.15)

where  is the sum of the coupons recieved from holding bond *i* during the time interval (*t1*, *t2*] and  is the accrued interest of bond *i* at time *t*.If the market uses ex-coupon days the same adjustment as in section 4.1 is made:

*coupon return* = , (6.16)

where *CPi,t*  is equal to the next coupon if *t* is less than the coupon date, and the settlement date is larger than or equal to the first ex-coupon date[[11]](#footnote-11)3; otherwise *CPi*,*t*= 0.

Note that the coupon return always is positive and is only subject to default risk, it does not depend on the yield curve movements.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Total bond return | | | | | | | | |
| Price Return | | | | | | | Coupon Return | |
| Time Return | | Curve Return | | | | | Coupon Payments | Accrued Coupon change |
| Accretion | Rolldown | Shift | Twist | Butterfly | Spread | Shape |

Table 6.2 Decomposition of bond return (the size of the buckets does not correspond to their relative contribution to the total return)

## 6.3 The implemented model

### 6.3.1 Deriving the implied zero curve

The model that has been implemented use implied Treasury zero curves. A residual free model, meaning that the curves give the correct price for all the input bonds (see section 2.3), is used to obtain the zero curves for the start and end of the measurement period. Let

 *i* = 1, 2,..., *N*

be the maturities of the *N* input bonds with *Ti* < *Tj*if *i* < *j*. In general, *N* parameters are needed to define the curve if all input bonds are to be correctly priced. The parameters are taken to be the value of the zero curve *Y* at the times of maturity of the *N* bonds. We denote these values

** *i* = 1, 2,..., *N*. (6.17)

An interpolation method is used to obtain the curve value at other points. The implementation supports linear, log-linear and Hermitian interpolation. According to Valtonen (1998), Hermitian interpolation gives the best results of the three methods and was therefore used for most tests.

Denote the parameter vector {*z1*,…,*zN*} by **z** and let *Pi,t*(**z)** be the price of bond *i* at time *t* implied by the yield-curve given by **z**. We want to find the parameter values that make the error terms

 *i* = 1, 2,..., *N*

equal to zero or as close to zero as is desired. This is done using the multidimensional Newton-Raphson method.

### 6.3.2 Defining the curve change components

One way of defining the shift, twist and butterfly movements, is to use a statistical method called principal component analysis (PCA). To get an idea of how PCA works, consider a vector **x** of *n* stochastic variables that are not uncorrelated. In Appendix A it is shown that this vector can be expressed as

, (6.18)

where the stochastic variables {*yi, i* = 1, 2, …, *n*} are uncorrelated and the constant vectors {**a**i, *i* = 1, 2, …, *n*} make up an orthonormal basis in R*n*. This means that the sum oft he variances of the correlated variables is

. (6.19)

If the variables are ordered so that Var(*yi*) ≥ Var(*yj*) for *i* ≥ *j* and most of the variance of the *xi*:s comes from the *k* first variables we can approximate **x** by these variables, the principal components, and their corresponding vectors, the principal axes:

**ε** , (6.20)

where the residual vector **ε** = **a***k*+1*yk*+1+…+ **a***nyn*is orthogonal to **x - ε**. By only considering the principal components, the number of variables has been reduced from *n* to *k* while the vectors, or principal axis, {**a***i*,  *i* = 1, 2, …, *k*}corresponding to these *k* variables span a subspace of R*n* that contain most of the variance of **x**.

For the purpose of decomposing the yield curve movement, the correlated variables are defined as the movements of the implied Treasury zero curve at prescribed points on the time axis. Let

 (6.21)

be the movement of the zero curve at time *T* for the period [*t*1, *t*2]. Choose a set of points on the time axis, {*Ti* *i* = 1, 2, …, *n*} and consider the vector

. (6.22)

Now, the uncorrelated variables {*yi*,  *i* = 1, 2, …, *k*} and the orthogonal vectors {**a***i*,  *i* = 1, 2, …, *k*} are defined as described in Appendix A and the vector Δ**Y** is expressed as

. (6.23)

The dynamics of the yield curve is such that it is sufficient to consider the first three terms when decomposing the yield curve movement. Loosely speaking, most of the variance of Δ**Y**

|  |  |
| --- | --- |
| Figure 6.3: The principal axes of the daily zero curve movement. Based on a data sample of quotes from Jan 1, 2000 to September 18, 2000. | Figure 6.4 Principal axes based on quotes from June 1, 1998 to December 1, 1998. |

is along the first principal axis and we define this axis as the shift movement. The second and third principal axes are defined as the twist and butterfly movements, respectively.

As the distribution of the zero curve movement Δ*Y*(*Ti*)is not known, a data sample of implied zero curves is used to calculate the principal axes and components. Figure 6.3 shows the shift, twist and butterfly vectors for one day zero curve movements at the points 1, 2 , 3, …, 10 years. Notice the similarity to figure 6.1. The vectors are based on zero curves from January 2, 2000 to February 25, 2000 implied by the quotes on Swedish government bonds. If another data sample is used, the principal axes might not look as nice as in Figure 6.3. Figure 6.4 shows the axes for the period June 1, 1998 to December 1, 1998. Since it is important that the vectors have simple characteristics, the implemented model does not use PCA. Instead, three orthonormal vectors are chosen ad hoc. The zero curve movement at *n* equidistant points, {*Ti* = *T*1 + (*i* - 1)(*Tn* – *T*1)/(*n* – 1), *i* = 1, 2, ..., *n*} is studied. Figure 6.5 shows these vectors for *n* = 10 with the first point at 1 year and the last point at 10 years. Generally, the shift, twist and butterfly vectors are defined as[[12]](#footnote-12)3

, (6.24)

**τ**  and (6.25)

, (6.26)

Figure 6.5: The shift, twist and butterfly vectors (not normalized) for 10 equidistant points between one year and ten years.

The idea is to express the vector Δ**Y** consisting of the yield curve movements at the *n* points, {*Ti*, *i* = 1, 2, …, *n*}, as a linear combination of **s**, **τ** and **b** and a residual vector **ε**:

**τ***yτ* **ε**. (6.27)

It is natural to let **s***ys* + **τ***yt* + **b***yb* be the projection of Δ**Y** onto the subspace of Rn that is spanned by **s**, **τ** and **b**. Since these vectors are orthonormal, this means that the variables *ys*,*yt* and *yb* should be defined as follows:

, **τ** and . (6.28)

The trade-off for not using PCA is that these variables in general will not be uncorrelated. Correlation measurements have been carried out and the results are presented in the next section.

When the vector has been decomposed according to (6.27) and (6.28), the shift twist and butterfly components are used to define the corresponding components of the whole curve movement. First, the shift function *s* is constant:

 . (6.29)

The twist function is a straight line that crosses the time axis at (*Tn* + *T*1)/2. The slope of the line is

. (6.30)

Thus, the twist function is

 . (6.31)

Finally, the butterfly function is a piecewise linear function with a peak at (*T*1 + *Tn*)/2:

, (6.32)

where

. (6.33)

Figure 6.6 shows the shift, twist and butterfly functions for *ys*= *yτ* = *yb* = 1.

Figure 6.6: The shift, twist and butterfly curves based on the vectors in figure 6.5 with

*ys*= *yτ* = *yb* = 1.

These functions represent movements of the zero curve and are used to define the curves resulting from these movements. The shift curve is obtained by adding the constant shift function to the original curve at the maturity grid points.

 *i* = 1, 2, …, *N*. (6.34)

Interpolation is used to get the curve value at other points. This means that the shift will not be exactly parallel, unless we use linear interpolation, but in most cases the difference



is close to s for *t* < *TN*. The shift curve is then translated to get the curve used for pricing bonds at time *t2*.

 . (6.35)

The twist and butterfly curves are obtained in the same manner. For example, the twist curve is obtained by translating the curve , which has grid point values

 *i* = 1, 2, …, *N*: (6.36)

 . (6.37)

Using the shift, twist and butterfly curves in the pricing of a bond, the return due to the corresponding curve movements can be calculated. For example, the butterfly return of bond *i* is:

*butterfly return* = . (6.38)

When dealing with credit curves the total curve movement is decomposed into Treasury curve movement and spread change. The definitions of the shift, twist and butterfly components are therefore as above, i. e. based on the Treasury curves. The curve used in the calculation of the shift return of the credit bonds is obtained in the same manner as the Treasury shift curve. First, the shift value is added to credit curveat the maturity grid points,

 *i* = 1, 2, …,*m*, (6.39)

where the *Ti*:s are the maturities of the *m* input bonds used in the derivation of the credit curve. The resulting curve is then translated:

 . (6.40)

To obtain the curve used in the calculation of the spread and shape return (see section 6.1.3), the Treasury curve movement is added to the credit curve at the grid points:

. *i* = 1, 2, …, *m* (6.41)

(6.12) and (6.14) are then used to calculate the two return components.

## 6.4 Results

The model described in the previous section has been tested on data from the Swedish bond market for the period January 1, 1995, to October 11, 2000. Handelsbanken’s bond index for the Swedish government bonds was used in the derivation of the Treasury zero curve. A problem with the derivation method is that it will not work if two or more of the input bonds have the same maturity. If that was the case, only one of those bonds was used to obtain the curve, which therefore will not give the correct price for the other bonds with that maturity. This normally resulted in large rolldown and shape returns when applying the model to those bonds.

Figure 6.7 shows the quotes and data for the Swedish government bonds for September 5, 1998, and October 5, 1998. The implied zero curves based on this data are shown in Figure 6.8. The bond SGB1041 is somewhat isolated having a maturity more than five years later than the next longest bond. This can make it difficult to decide where to position the last point, *Tn*, of the *n* points that are used when defining the shift, twist and butterfly movements.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| BondName | Maturity | Coupon | Quote  98-09-05 | Quote  98-10-05 |
| SGB1036 | 2000-05-05 | 10,25 | 4,375 | 4,295 |
| SGB1030 | 2001-06-15 | 13 | 4,475 | 4,335 |
| SGB1039 | 2002-04-12 | 5,5 | 4,545 | 4,34 |
| SGB1033 | 2003-05-05 | 10,25 | 4,635 | 4,37 |
| SGB1042 | 2004-01-15 | 5 | 4,705 | 4,39 |
| SGB1035 | 2005-02-09 | 6 | 4,785 | 4,43 |
| SGB1038 | 2006-10-25 | 6,5 | 4,875 | 4,495 |
| SGB1037 | 2007-08-15 | 8 | 4,875 | 4,49 |
| SGB1040 | 2008-05-05 | 6,5 | 4,925 | 4,555 |
| SGB1043 | 2009-01-28 | 5 | 4,965 | 4,605 |
| SGB1034 | 2009-04-20 | 9 | 4,935 | 4,57 |
| SGB1041 | 2014-05-05 | 6,75 | 5,16 | 4,84 |

Table 6.7: Data and quotes for the Swedish Figure 6.8: Implied zero curves based on government bonds in the HMSA index. the data in figure 6.7

In most of the testing, the first point, *T1*, was set at two years. As a rule, the last point should not be later than the latest maturity of the bonds used in the derivation of the zero curve. Otherwise, the value Δ*Y*(*Tn*) will be obtained through extrapolation, something that should be avoided.

Figure 6.9 shows the results of a return attribution analysis based on the data in figure 6.7 with *T*1 at two years, *Tn* at ten years and a total number of points of ten. The figures for the HMSA index are based on the nominal outstanding amount (as given by this index) of each bond. Notice the relatively large residual return, or shape return, of the bond SGB1041. This is a result of only considering the movement of the curve up to the ten-year point when defining the shift, twist and butterfly components. The twist and butterfly returns for SGB1041 are also relatively large due to this. If, instead, *T*10is set at the time of maturity of SGB1041, the analysis gives the results shown in figure 6.10. The twist, butterfly and residual returns for SGB1041 have decreased at the expense of a larger residual return for most of the other bonds (in terms of absolute value). Only three bonds have a positive twist return – a result of the twist function being positive before the point (*T*10 – *T*1)/2, which now is further away on the time axis. The differences between the two analysises means that it is important to consider which bonds one are interested in when doing the analysis.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| BondName | Accretion | Rolldown | Shift | Twist | Butterfly | Shape | Total Price Return | Coupon | Total Return |
| SGB1036 | -0,37% | 0,01% | 0,45% | -0,24% | -0,11% | 0,00% | -0,26% | 0,71% | 0,45% |
| SGB1030 | -0,47% | 0,00% | 0,73% | -0,29% | -0,10% | -0,02% | -0,15% | 0,81% | 0,66% |
| SGB1039 | -0,06% | 0,01% | 0,99% | -0,30% | -0,04% | -0,02% | 0,58% | 0,41% | 0,98% |
| SGB1033 | -0,28% | 0,02% | 1,17% | -0,24% | 0,07% | -0,05% | 0,70% | 0,63% | 1,33% |
| SGB1042 | -0,01% | 0,01% | 1,42% | -0,16% | 0,20% | -0,07% | 1,39% | 0,37% | 1,76% |
| SGB1035 | -0,06% | 0,01% | 1,63% | -0,01% | 0,28% | -0,10% | 1,76% | 0,42% | 2,19% |
| SGB1038 | -0,06% | 0,01% | 1,93% | 0,32% | 0,06% | -0,01% | 2,24% | 0,43% | 2,68% |
| SGB1037 | -0,14% | 0,00% | 2,11% | 0,49% | -0,08% | 0,04% | 2,42% | 0,51% | 2,93% |
| SGB1040 | -0,07% | 0,03% | 2,28% | 0,73% | -0,24% | -0,13% | 2,59% | 0,44% | 3,03% |
| SGB1043 | 0,00% | 0,00% | 2,48% | 1,02% | -0,45% | -0,23% | 2,82% | 0,38% | 3,19% |
| SGB1034 | -0,14% | -0,02% | 2,30% | 0,88% | -0,40% | -0,10% | 2,51% | 0,51% | 3,03% |
| SGB1041 | -0,05% | 0,07% | 3,18% | 2,78% | -1,96% | -0,86% | 3,16% | 0,44% | 3,60% |
|  |  |  |  |  |  |  |  |  |  |
| HMSA | -0,20% | 0,01% | 1,45% | 0,16% | -0,11% | -0,08% | 1,23% | 0,56% | 1,79% |

Table 6.9 A return attribution analysis for the Swedish government bonds for the period

September 5, 1998, to October 5, 1998.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| BondName | Accretion | Rolldown | Shift | Twist | Butterfly | Shape | Total Price Return | Coupon | Total Return |
| SGB1036 | -0,37% | 0,01% | 0,49% | -0,16% | -0,17% | -0,07% | -0,26% | 0,71% | 0,45% |
| SGB1030 | -0,47% | 0,00% | 0,80% | -0,22% | -0,19% | -0,07% | -0,15% | 0,81% | 0,66% |
| SGB1039 | -0,06% | 0,01% | 1,07% | -0,25% | -0,17% | -0,02% | 0,58% | 0,41% | 0,98% |
| SGB1033 | -0,28% | 0,02% | 1,28% | -0,25% | -0,09% | 0,02% | 0,70% | 0,63% | 1,33% |
| SGB1042 | -0,01% | 0,01% | 1,54% | -0,25% | 0,01% | 0,10% | 1,39% | 0,37% | 1,76% |
| SGB1035 | -0,06% | 0,01% | 1,77% | -0,22% | 0,16% | 0,09% | 1,76% | 0,42% | 2,19% |
| SGB1038 | -0,06% | 0,01% | 2,09% | -0,13% | 0,50% | -0,16% | 2,24% | 0,43% | 2,68% |
| SGB1037 | -0,14% | 0,00% | 2,29% | -0,08% | 0,63% | -0,28% | 2,42% | 0,51% | 2,93% |
| SGB1040 | -0,07% | 0,03% | 2,48% | -0,01% | 0,60% | -0,44% | 2,59% | 0,44% | 3,03% |
| SGB1043 | 0,00% | 0,00% | 2,70% | 0,08% | 0,53% | -0,49% | 2,82% | 0,38% | 3,19% |
| SGB1034 | -0,14% | -0,02% | 2,50% | 0,05% | 0,43% | -0,30% | 2,51% | 0,51% | 3,03% |
| SGB1041 | -0,05% | 0,07% | 3,46% | 0,68% | -0,56% | -0,44% | 3,16% | 0,44% | 3,60% |
|  |  |  |  |  |  |  |  |  |  |
| HMSA | -0,20% | 0,01% | 1,58% | -0,13% | 0,08% | -0,10% | 1,23% | 0,56% | 1,79% |

Table 6.10: The same analysis as in figure 6.9 but now with *T*10 equal to the maturity of

SGB1041.

### 6.4.2 Measure of ‘goodness-of-fit’ and correlation

In the implemented model, the vector Δ**Y** consisting of the zero curve movements at the *n* points {*Ti*, *i* = 1, 2, …, *n*} is decomposed into the shift, twist and butterfly vectors illustrated in figure 6.5:

**τ***yτ* **ε**. (6.27)

If principal component analysis is used, the orthonormal vectors {**a***i*, *i* = 1, 2, …, *n*} in

 (6.42)

are chosen so that the variables {*yi*, *i* = 1, 2, …, *n*} are uncorrelated. Formula (6.19),

 (6.19)

can be used to get an idea of how well the vector Δ**Y** is approximated by a subset of the orthonormal vectors. Let the variables be ordered so that Var(*yi*) ≥Var(*yj*) for *i* ≥ j and denote by *tk* the part of the total variance that comes from the first *k* variables:

 (6.43)

The higher the value of *t*k the better the vector Δ**Y** is approximated by the first *k* terms in (6.19).

Like the principal axes, the vectors **s**, **τ** and **b** are orthonormal but were chosen for their simple appearance and we do not know how well they approximate the vector Δ**Y**. Nor do we know the correlation between the corresponding variables *ys*, *yτ* and *yb*. In a similar manner as at is shown in Appendix A that (6.19) holds, it can be shown that

 (6.44)

Thus, we can introduce the measures of ‘goodness-of-fit’, *ts*, *tτ* and *tb*, which are analogous to *t*1, *t*2 and *t*3 for the principal components.

,  and  (6.45)

Table 6.11 shows these measures for our components and the principal components of the daily zero curve movement at 10 equidistant points between two years and ten years. The values are based on quotes for Swedish government bonds from June 1, 1998 to September 18, 2000.

|  |  |  |
| --- | --- | --- |
|  | Goodness-of-fit | |
|  | Ad hoc | PC |
| Shift | *ts* =0,916 | *t*1 = 0,922 |
| Twist | *tτ* =0,985 | *t*2 = 0,987 |
| Butterfly | *tb* =0,992 | *t*3 = 0,994 |
| Table 6.11: Measures of ‘goodness-of- fit’ for ad hoc components and principal components of the daily zero curve movement. Based on data from June 1, 1998 to September 18, 2000 | | |

|  |  |  |  |
| --- | --- | --- | --- |
|  | s | τ | b |
| s | 1 | 0,24 | -0,37 |
| τ | 0,24 | 1 | -0,17 |
| b | -0,37 | -0,17 | 1 |
| Table 6.12: The correlation matrix for the ad hoc components. Based on the same data as the values in table 6.11. | | | |

The theory for principal components states that the measures of the ad hoc components cannot be as high as the corresponding measures for the principal components, see Jolliffe [1986] pages 9-10. However, the values do not differ much, suggesting that the ad hoc decomposition in (6.27) is satisfactory. The fact that the difference is not greater is not surprising if we look at the principal axes for the data sample (figure 6.13). The axes are very similar to the pre-defined vectors illustrated in figure 6.5.

|  |  |
| --- | --- |
| Figure 6.13 The principal axes for the daily zero curve movement at ten equidistant points between two years and ten years. Based on data from June 1, 1998 to September 18, 2000. | Figure 6.14 The principal axes for the monthly zero curve movement at ten equidistant points between two years and ten years. Based on data from June 1, 1998 to September 18, 2000. |

Table 6.15 shows these measures for our components and the principal components of the monthly zero curve movement at 10 equidistant points between two years and ten years. The values are based on data from June 1, 1998 to October 10, 2000.

|  |  |  |
| --- | --- | --- |
|  | Goodness-of-fit | |
|  | Ad hoc | PC |
| Shift | *ts* =0,933 | *t*1 = 0,935 |
| Twist | *tτ* =0,991 | *t*2 = 0,993 |
| Butterfly | *tb* =0,996 | *t*3 = 0,998 |
| Table 6.15: Measures of ‘goodness-of-fit’ for ad hoc components and principal components of the monthly zero curve movement. Based on data from June 1, 1998 to October 10, 2000 | | |

|  |  |  |  |
| --- | --- | --- | --- |
|  | s | τ | b |
| s | 1 | 0,00 | -0,49 |
| τ | 0,00 | 1 | -0,11 |
| b | -0,49 | -0,11 | 1 |
| Table 6.16: The correlation matrix for the ad hoc components. Based on the  same data as the values in table 6.14. | | | |

As shown if figure 6.15, the principal axes of the monthly zero curve movement are also similar to the pre-defined vectors.

# 7 Summary

In this report is described how to perform an attribution analysis for a bond portfolio. The popular framework for attributing the excess return over a benchmark – performance attribution – is presented in chapter three. It should be quite straightforward to use once the weights and returns of the portfolio in the different sectors have been defined. However, this is not as straightforward if the portfolio has been subject to cash flows. Formulas for determining time-weighted rates-of-return and corresponding weights of the sectors are presented in chapter four. They require keeping track of the cash flows in or out of the sub-portfolios used in the attribution analysis. The performance attribution part of the report ends with a description of how to link attribution results from several periods.

A return attribution model is suggested for attributing a bond’s return to different types of yield curve movements. So called shift, twist and butterfly movements are defined based on three orthonormal vectors. The model has been implemented and tested on data from the Swedish government bond market. ‘Goodness-of-fit’ comparisons with principal components show that the shift, twist and butterfly vectors capture most of the variance of the yield curve movement.

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# Appendix A

Principal component analysis

Let

 (A.1)

be a vector of *n* random variables, which are linearly independent but not uncorrelated. Suppose we would like to express this vector as a linear combination of *n* random variables (with mean zero) that are uncorrelated:

, (A.2)

where the constant vectors

 *i* = 1, 2, …, *n* (A.3)

are orthonormal. To see that this is possible, define the matrix **A** as the solution to the equation

, (A.4)

where  is the covariance matrix of  and the *λi*:s are the eigenvalues of the covariance matrix with *λi* ≥ *λj* for *i* < *j*. That such a solution exists is guaranteed by the spectral theorem since the covariance matrix is symmetric. The theorem also states that **A** is an orthonormal matrix, i. e.

. (A.5)

Now, define the vector  as

. (A.6)

By expressing the covariance matrix as

, (A.7)

(A.4) gives the following equality:



. (A.8)

Thus, the stochastic variables that make up the vector  are uncorrelated and

. (A.9)

By inverting (A.6) we see that we have achieved what we set out to do:

,

where **A** is orthonormal and the components of  are uncorrelated. Moreover, the sum of the correlated variables variance can be written as



. (A.10)

Now, assume that for some *k < n* it holds that (remember that *λi* ≥ *λj* for *i* < *j*)

.

This means that most of the variance of the correlated variables comes from , , …,  and . These variables may therefore be called the principal components of .

1. In practice, discrete compounding is used. For a comprehensive discussion on different market conventions, see Brown [1998]. [↑](#footnote-ref-1)
2. 1 This example follows closely Dynkin, Hyman and Vankudre [1998]. [↑](#footnote-ref-2)
3. 2 The common convention of including the interaction effect in the security selection term is used. [↑](#footnote-ref-3)
4. 1 If the settlement date is larger than or equal to the first ex-coupon date, the quoted price is exclusive the corresponding coupon payment. However, if the day we value our portfolio is before the coupon date that coupon payment should still be reflected by our portfolio value, hence the adjustment to the formula. [↑](#footnote-ref-4)
5. 1 GIPS, AIMR [↑](#footnote-ref-5)
6. 2 The example is taken from Valtonen, [2000] [↑](#footnote-ref-6)
7. 3 This will happen if, for example, 0<*NS*(*t*)<*N*(*t*)and *PS*(*t*)>*P*(*t*)*N*(*t*)/*NS*(*t*) [↑](#footnote-ref-7)
8. 4 Assuming transaction costs are zero. [↑](#footnote-ref-8)
9. 1 For an introduction to term structure factor models see Kuberek [1998] [↑](#footnote-ref-9)
10. 2See section 6.4.2, Measure of ‘goodness-of-fit’ and correlation.. [↑](#footnote-ref-10)
11. 3 See footnote 1 in section 4.1 [↑](#footnote-ref-11)
12. 3 The definitions are only for *n* an even number and *n* ≥ 4. [↑](#footnote-ref-12)