赵云天-11811715-HW4

1. 推导Rodrigues' formula （罗德里格斯公式）：

$$e^{\hat{a}\theta} = \boldsymbol{E} + \hat{\boldsymbol{a}}\sin\theta + \hat{\boldsymbol{a}}^2(1-\cos\theta).$$

We first prove a lemma:

for $\forall x \in \mathbb{R}^3$, if $\|x\|_2 = 1$, we have

$$([x]^\wedge)^3 = -[x]^\wedge$$
$$([x]^\wedge)^4 = -([x]^\wedge)^2$$

The proof is straight forward:

let $x = (x_1, x_2, x_3)^T$, $x^T x = 1$

then $[x]^\wedge = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$

thus
$$([x]^\wedge)^3 = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}[x]^\wedge$$
$$= \begin{pmatrix} -x_3^2-x_2^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & -x_3^2-x_1^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & -x_1^2-x_2^2 \end{pmatrix}[x]^\wedge$$
$$= \begin{pmatrix} -x_3^2-x_2^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & -x_3^2-x_1^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & -x_1^2-x_2^2 \end{pmatrix}\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & x_3^3+x_2^2x_3+x_1^2x_3 & -x_3^2x_2-x_2^3-x_1^2x_3 \\ -x_3^3-x_1^2x_3-x_2^2x_3 & 0 & x_1x_2^2+x_1x_3^2+x_1^3 \\ x_2x_3^2+x_1^2x_2+x_2^3 & -x_1x_3^2-x_1^3-x_1x_2^2 & 0 \end{pmatrix}$$
$$= (x_1^2+x_2^2+x_3^2)\begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$$
$$= \|x\|_2(-[x]^\wedge)$$
$$= -[x]^\wedge$$

with the proof, another part $([x]^\wedge)^4 = -([x]^\wedge)^2$ is trivil

we then do some math:

Maclaurin Series is the Taylor expansion at zero.

which $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}, \quad 1-\cos x = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{(2m-1)!},$$

And by definition $e^{A\theta} = \sum_{k=0}^{\infty} \frac{(A\theta)^k}{k!}$,

for $A = [x]^\wedge$, and $\|x\|_2 = 1$,

$$A^k = \begin{cases} A, & k = 4n+1 \\ A^2, & k = 4n+2 \\ -A, & k = 4n+3 \\ -A^2, & k = 4n+4 \end{cases} \quad \text{for } n = 0, 1, 2, \cdots$$

then $e^{A\theta} = I + A^2 + A\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\theta^{2n-1}}{(2n-1)!} + A^2\sum_{m=1}^{\infty} \frac{(-1)^m\theta^{2m-1}}{(2m-1)!}$

$$= I + A\sin x + A^2(1-\cos x)$$

By this we prove that $e^{\hat{a}\theta} = E + \hat{a}\sin\theta + (\hat{a})^2(1-\cos\theta)$

2. 令坐标系{b}最初与世界坐标系{s}重合，现要求坐标系{b}绕单位转轴 a_1 = (0 0.866 0.5) 旋转θ_1 =30度 (0.524 rad)，求旋转后得到的新坐标系的姿态矩阵*R*. (要求计算具体数值)

This can be calculated by Rodrigues' formula

$$R = e^{\hat{a}\theta} = E + \hat{a}\sin\theta + (\hat{a})^2(1-\cos\theta)$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix} + (1-\frac{\sqrt{3}}{2})\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & -\frac{3}{4} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{1}{4} & \frac{3}{4}+\frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{4}-\frac{3}{8} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4}-\frac{3}{8} & \frac{1}{4}+\frac{3\sqrt{3}}{8} \end{pmatrix}$$

3. 如果上述结果R再绕旧坐标系中的转轴a_2 旋转角度θ_2，求第二次旋转后得到的姿态矩阵*R'*. (无需计算数值)

则 $R' = R(a_2, \theta_2)R$

$$= [I + \hat{a}_2\sin\theta_2 + (\hat{a}_2)^2(1-\cos\theta_2)]R$$

4. 如果第二次旋转是绕新坐标系中的转轴a_2旋转角度 θ_2，求第二次旋转后得到的姿态矩阵*R'*. (无需计算数值)

$$R' = RR(a_2, \theta_2)$$
$$= R[I + \hat{a}_2\sin\theta_2 + (\hat{a}_2)^2(1-\cos\theta_2)]$$