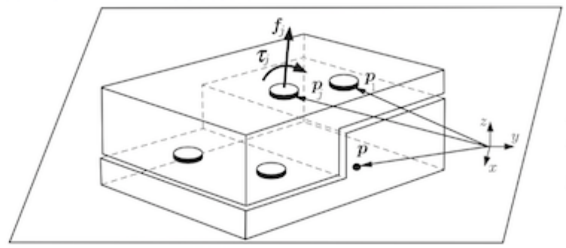
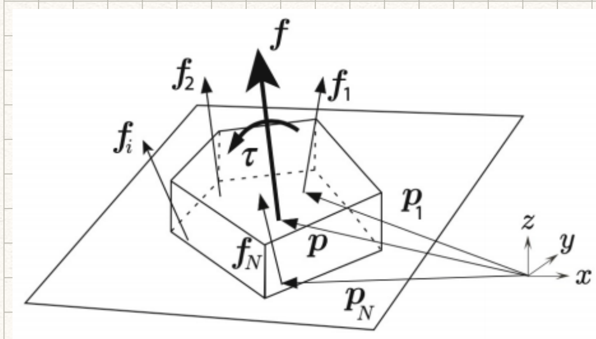
1. 推导下述一般情况的ZMP公式



$\boldsymbol{f}_j = [f_{jx}\ f_{jy}\ f_{jz}]^T$
$\boldsymbol{\tau}_j = [\tau_{jx}\ \tau_{jy}\ \tau_{jz}]^T$
$\boldsymbol{p}_j = [p_{jx}\ p_{jy}\ p_{jz}]^T$

some words
作业我想用来顺顺便复习上课内容 会推很多其他公式 但是可能会对助教批改造成影响 因此和作业相关的部分会用这个标记出来 感谢，见谅！

The derivation is strated from the general discrite (finite many) case below:



The spatial forces f_i are equavient to a single wrench (f and τ) at point p. If p is ZMP, the τ_x, τ_y should be 0 (where $\vec{\tau} \equiv [\tau_x\ \tau_y\ \tau_z]^T$)

thus
$$\begin{cases} f = \sum_{i=1}^{N} f_i & \cdots ① \\ \tau = \sum_{i=1}^{N} (p_i - p) \times f_i & \cdots ② \end{cases}$$

for $a \equiv [a_x\ a_y\ a_z]^T$, $b \equiv [b_x\ b_y\ b_z]^T$

$$a \times b = [a]^\wedge b = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

then ① gives
$$\begin{cases} f_x = \sum_{i=1}^{N} f_{ix} \\ f_y = \sum_{i=1}^{N} f_{iy} \\ f_z = \sum_{i=1}^{N} f_{iz} \end{cases}$$

$$\tau = \sum_{i=1}^{N} p_i \times f_i - p \times f_i$$

② gives
$$\begin{cases} \tau_x = \sum_{i=1}^{N} p_{iy} f_{iz} - p_{iz} f_{iy} - (p_y f_{iz} - p_z f_{iy}) = 0 & \cdots ③ \\ \tau_y = \sum_{i=1}^{N} p_{iz} f_{ix} - p_{ix} f_{iz} - (p_z f_{ix} - p_x f_{iz}) = 0 & \cdots ④ \\ \tau_z = \sum_{i=1}^{N} p_{ix} f_{iy} - p_{iy} f_{ix} - (p_x f_{iy} - p_y f_{ix}) \end{cases}$$

③ gives $\sum_{i=1}^{N} p_{iy} f_{iz} - p_{iz} f_{iy} = p_y \sum_{i=1}^{N} f_{iz} - p_z \sum_{i=1}^{N} f_{iy}$

④ gives $\sum_{i=1}^{N} p_{iz} f_{ix} - p_{ix} f_{iz} = p_z \sum_{i=1}^{N} f_{ix} - p_x \sum_{i=1}^{N} f_{iz}$

we have to assume $p_z = 0$ (that ZMP is on the ground to slove about equations).

Then $$p_x = \frac{\sum_{i=1}^{N} p_{ix} f_{iz} - p_{iz} f_{ix}}{\sum_{i=1}^{N} f_{iz}} \quad \cdots ⑤$$

$$p_y = \frac{\sum_{i=1}^{N} p_{iy} f_{iz} - p_{iz} f_{iy}}{\sum_{i=1}^{N} f_{iz}} \quad \cdots ⑥$$

This is a little different from the formula in lecture page 9:

$$\boldsymbol{p} = \frac{\sum_{i=1}^{N} \boldsymbol{p}_i f_{iz}}{\sum_{i=1}^{N} f_{iz}}$$

That's due to $p_{iz} \equiv 0$. If $p_{iz} \equiv 0$ holds in ⑤ and ⑥

$$p_x = \frac{\sum p_{ix} f_{iz}}{\sum f_{iz}}, \quad p_y = \frac{\sum p_{iy} f_{iz}}{\sum f_{iz}}, \quad p_z = 0 = \frac{\sum p_{iz} f_{iz}}{\sum f_{iz}}$$

$$p = \frac{\sum p_i f_{iz}}{\sum f_{iz}}$$ which is the discrete case as in lecture page 8.

$$p_x = \frac{\int_S \xi \rho(\xi, \eta) dS}{\int_S \rho(\xi, \eta) dS}$$
$$p_y = \frac{\int_S \eta \rho(\xi, \eta) dS}{\int_S \rho(\xi, \eta) dS}$$
$$\xrightarrow[\text{still on the ground}]{\text{discrectized}}$$
$$\boldsymbol{p} = \frac{\sum_{i=1}^{N} \boldsymbol{p}_i f_{iz}}{\sum_{i=1}^{N} f_{iz}}$$

(the similarity in form).

But both cannot use in general case.

we will use ⑤ and ⑥, as we still assume ZMP on the ground.

recap from ⑤ and ⑥,

$$p_x = \frac{\sum_{i=1}^{N} p_{ix} f_{iz} - p_{iz} f_{ix}}{\sum_{i=1}^{N} f_{iz}} \quad \cdots ⑤$$

$$p_y = \frac{\sum_{i=1}^{N} p_{iy} f_{iz} - p_{iz} f_{iy}}{\sum_{i=1}^{N} f_{iz}} \quad \cdots ⑥$$

OK, actually ⑤ and ⑥ are also invalid. Analyse is similar.

But the expression is very complicated.

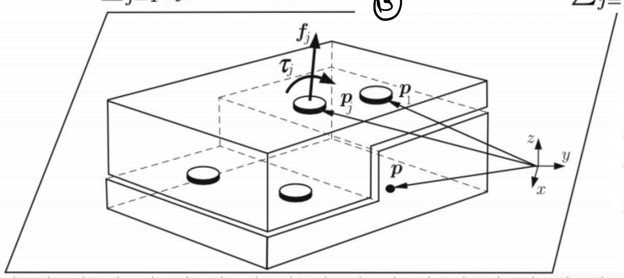
There are two rigid bodies contacting each other where one of them also contacts the ground. The forces and moments applied by one rigid body to the other are measured at multiple points.

The moment about the point $\boldsymbol{p} = [p_x\ p_y\ p_z]^T$ is $\boldsymbol{\tau}(\boldsymbol{p}) = \sum_{j=1}^{N} (\boldsymbol{p}_j - \boldsymbol{p}) \times \boldsymbol{f}_j + \boldsymbol{\tau}_j$.

The position of the ZMP can be obtained by

$$p_x = \frac{\sum_{j=1}^{N} \{-\tau_{jy} - (p_{jz} - p_z) f_{jx} + p_{jx} f_{jz}\}}{\sum_{j=1}^{N} f_{jz}} \quad Ⓐ \qquad p_y = \frac{\sum_{j=1}^{N} \{\tau_{jx} - (p_{jz} - p_z) f_{jy} + p_{jy} f_{jz}\}}{\sum_{j=1}^{N} f_{jz}} \quad Ⓑ$$

The assumption is as shown left. we will prove Ⓐ and Ⓑ.



$$\boldsymbol{f}_j = [f_{jx}\ f_{jy}\ f_{jz}]^T$$
$$\boldsymbol{\tau}_j = [\tau_{jx}\ \tau_{jy}\ \tau_{jz}]^T$$
$$\boldsymbol{p}_j = [p_{jx}\ p_{jy}\ p_{jz}]^T$$

ZMP still have to be on the ground, or at least p_z is known.
But we need ZMP for understand the GRF, which definately on the ground.

$$f = \sum_{i=1}^{N} f_i \underset{\text{vector}}{\overset{\text{scaler}}{\rightleftharpoons}} \begin{cases} f_x = \sum_{i=1}^{N} f_{ix} \\ f_y = \sum_{i=1}^{N} f_{iy} \\ f_z = \sum_{i=1}^{N} f_{iz} \end{cases}$$

$$\tau = \sum_{i=1}^{N} (p_i - p) \times f_i + \tau_i = \sum_{i=1}^{N} p_i \times f_i - p \times f_i + \tau_i$$

vector form $\upharpoonleft\downharpoonright$ scaler form

$$\tau_x = \sum_{i=1}^{N} p_{iy} f_{iz} - p_{iz} f_{iy} - (p_y f_{iz} - p_z f_{iy}) + \tau_{ix} = 0 \cdots ⑦$$
$$\tau_y = \sum_{i=1}^{N} p_{iz} f_{ix} - p_{ix} f_{iz} - (p_z f_{ix} - p_x f_{iz}) + \tau_{iy} = 0 \cdots ⑧$$
$$\tau_z = \sum_{i=1}^{N} p_{ix} f_{iy} - p_{iy} f_{ix} - (p_x f_{iy} - p_y f_{ix}) + \tau_{iz} = 0$$

we even don't need to make $p_z = 0$ will we have

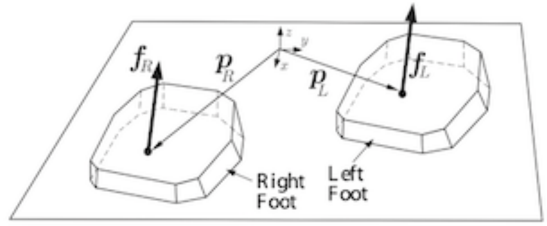
$$\sum_{i=1}^{N} p_{iy} f_{iz} - p_{iz} f_{iy} + p_z f_{iy} + \tau_{ix} = p_y \sum_{i=1}^{N} f_{iz}$$
$$\sum_{i=1}^{N} p_{iz} f_{ix} - p_{ix} f_{iz} + p_z f_{ix} + \tau_{iy} = -p_x \sum_{i=1}^{N} f_{iz}$$

simplify do some conversion will we have the result:

A and B :

$$p_x = \frac{\sum_{j=1}^{N} \{-\tau_{jy} - (p_{jz} - p_z) f_{jx} + p_{jx} f_{jz}\}}{\sum_{j=1}^{N} f_{jz}} \quad p_y = \frac{\sum_{j=1}^{N} \{\tau_{jx} - (p_{jz} - p_z) f_{jy} + p_{jy} f_{jz}\}}{\sum_{j=1}^{N} f_{jz}}$$

2. 推导双腿支撑期的ZMP公式



$$\boldsymbol{p}_R = [p_{Rx}\ p_{Ry}\ p_{Rz}]^T$$
$$\boldsymbol{f}_R = [f_{Rx}\ f_{Ry}\ f_{Rz}]^T$$
$$\boldsymbol{f}_L = [f_{Lx}\ f_{Ly}\ f_{Lz}]^T$$

We will use the following formula to show formula of ZMP with both feet.

$$p_x = \frac{\sum_{j=1}^{N} \{-\tau_{jy} - (p_{jz} - p_z) f_{jx} + p_{jx} f_{jz}\}}{\sum_{j=1}^{N} f_{jz}} \quad p_y = \frac{\sum_{j=1}^{N} \{\tau_{jx} - (p_{jz} - p_z) f_{jy} + p_{jy} f_{jz}\}}{\sum_{j=1}^{N} f_{jz}}$$

As $\tau_{Ly} = \tau_{Lx} = \tau_{Ry} = \tau_{Rx} = 0$

$p_{Lz} = p_{Rz} = p_z$

then $p_x = \dfrac{p_{Lx} f_{Lz} + p_{Rx} f_{Rz}}{f_{Lz} + f_{Rz}}$ $\quad p_y = \dfrac{p_{Ry} f_{Rz} + p_{Ly} f_{Lz}}{f_{Rz} + f_{Lz}}$ $\square$

3. 推导 ? 处的公式

Let us assume that a rigid body floats in the space and rotates without affected by external forces. The rotational velocity of a rigid body can be expressed by the angular velocity vector $\boldsymbol{\omega}$. Let us also assume that the origin of the reference coordinate system coincides with its center of mass. The velocity at a point in the rigid body can be expressed by

$$\boldsymbol{v}_i = \boldsymbol{v}(\boldsymbol{p}_i) = \boldsymbol{\omega} \times \boldsymbol{p}_i.$$

$$\mathcal{L}_i = \boldsymbol{p}_i \times \mathcal{P}_i.$$

$$\mathcal{L} = \sum_i \boldsymbol{p}_i \times (m_i \boldsymbol{\omega} \times \boldsymbol{p}_i) = \sum_i m_i \boldsymbol{p}_i \times (-\boldsymbol{p}_i \times \boldsymbol{\omega}) \overset{?}{=} (\sum_i m_i \widehat{\boldsymbol{p}}_i \widehat{\boldsymbol{p}}_i^T)\, \boldsymbol{\omega}.$$

$\sum_i m_i p_i \times (-p_i \times \omega)$, where $p_i \equiv [p_{ix}\ p_{iy}\ p_{iz}]^T$

$\omega \equiv [\omega_x\ \omega_y\ \omega_z]^T$

thus:

$$\sum_i m_i p_i \times (-p_i \times w) = \left(\sum_i m_i \underbrace{\begin{bmatrix} 0 & -p_{iz} & p_{iy} \\ p_{iz} & 0 & -p_{ix} \\ -p_{iy} & p_{ix} & 0 \end{bmatrix}}_{\hat{p}_i} \underbrace{\begin{bmatrix} 0 & p_{iz} & -p_{iy} \\ -p_{iz} & 0 & p_{ix} \\ p_{iy} & -p_{ix} & 0 \end{bmatrix}}_{\hat{p}_i^T} \right) \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

$$= \left(\sum_i m_i \hat{p}_i \hat{p}_i^T \right) w \quad \square$$