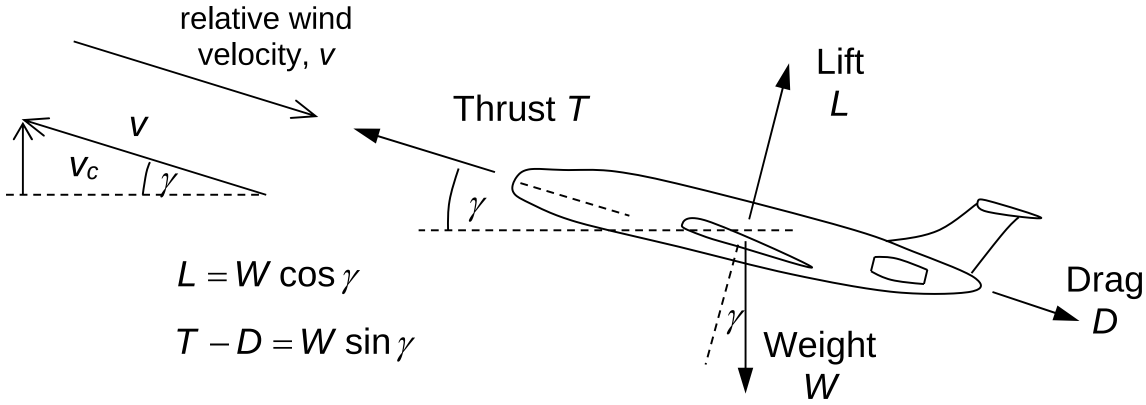
# C2 Aircraft Flight and Propulsion – Lecture 3

Prof L. He 4 lectures

## Climbing flight

Consider an aircraft climbing at an angle $\gamma$ with the thrust acting in the direction of velocity. The rate of climb $v_c$ is the vertical component of velocity.




$$L = W\cos\gamma$$

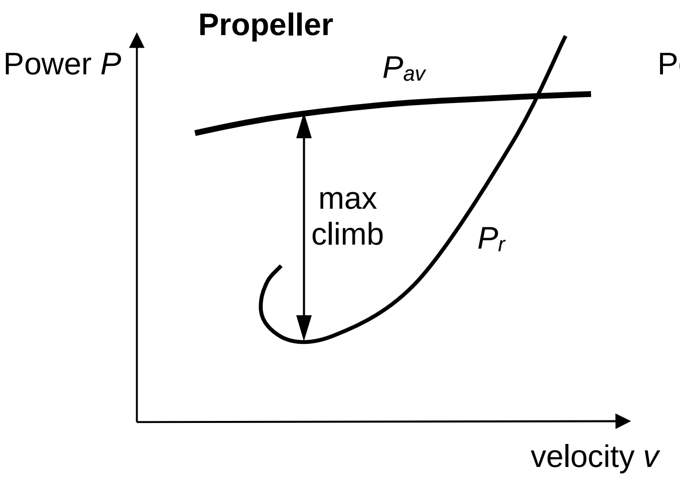
$$T - D = W\sin\gamma$$

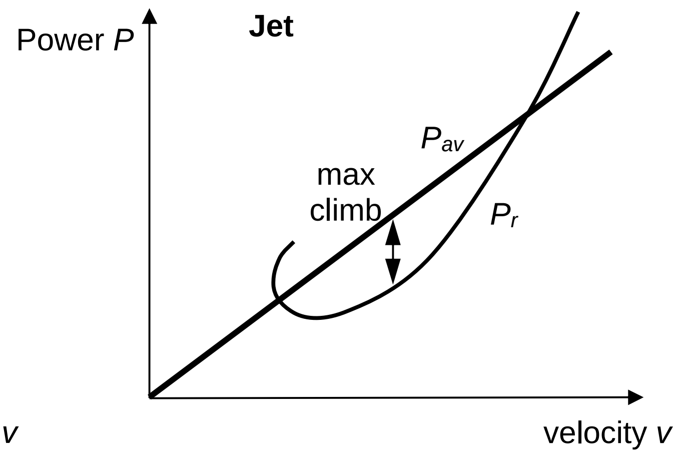
$(T-D)$ is the excess thrust required to climb.

$(T-D)v = W\,v\sin\gamma = W\,v_c$ is the excess power required to climb.

Rate of climb: $$v_c = \frac{(T-D)v}{W} = \frac{\text{excess power}}{\text{weight}}. \qquad (3\text{-}1)$$

The maximum rate of climb is when excess power $(P_{av} - P_r)$ is a maximum.



For a propeller-driven aircraft, the approximation that $P_{av}$ = constant gives maximum rate of climb $v_{c\,max}$ at the velocity for minimum power, which occurs at $\left(\frac{C_L^{3/2}}{C_D}\right)_{\max}$ (lecture 2). For a jet, the climb rate can get small if the aircraft is flying too slowly, and the flight velocity for maximum rate of climb is higher than the velocity for minimum power.

As altitude increases, $P_{av}$ and $T_{av}$ decrease and maximum rate of climb falls. The rate of climb is zero at maximum altitude.

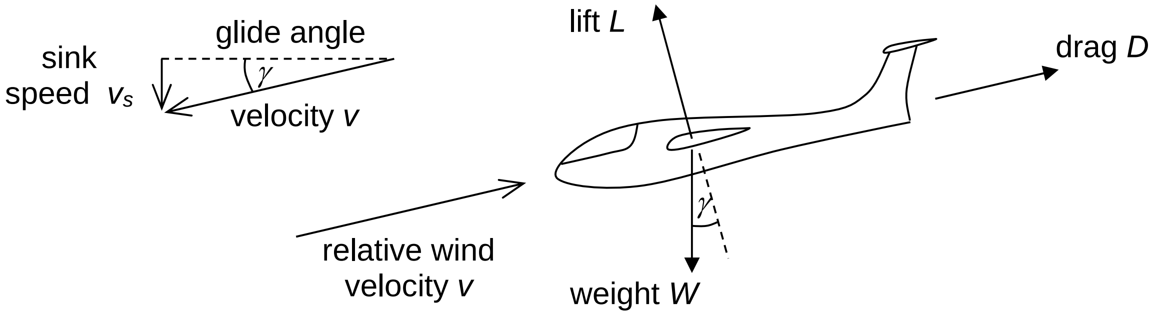
Angle of climb, $\gamma = \sin^{-1}\left(\frac{v_c}{v}\right)$

$$\sin\gamma = \frac{(T-D)}{W} = \frac{\text{excess thrust}}{\text{weight}} \qquad \text{(3-2)}$$

The angle of climb is a maximum when excess thrust ($T_{av} - T_r$) is a maximum.

For a jet aircraft, where $T_{av}$ is approximately constant with velocity at a given altitude, maximum angle of climb $\gamma_{max}$ is at minimum drag, i.e. at $v_{md}$ or $\left(\frac{L}{D}\right)_{\max}$.

## Gliding flight

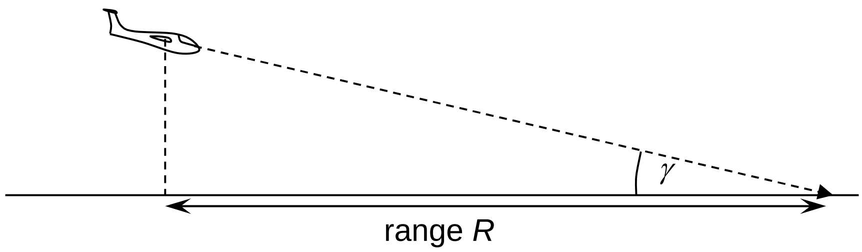



With gliding flight there is no engine so $L$, $W$, and $D$ are in equilibrium:

$$D = W \sin\gamma \qquad \text{and} \qquad L = W\cos\gamma$$

$$\frac{\sin\gamma}{\cos\gamma} = \frac{D}{L} \quad \rightarrow \quad \tan\gamma = \frac{1}{L/D} \tag{3-3}$$

Glide angle $\gamma$ is a function only of $L/D$; to minimise glide angle $\rightarrow$ maximise $L/D$ (i.e. maximise $C_L/C_D$). This is the same condition for minimum thrust in level flight, which is reasonable since $W\sin\gamma$ is the 'forward thrust' for the glider. Flying at minimum glide angle will maximise <u>range</u>, the horizontal distance covered by the aircraft.




To determine the required glide velocity: $L = \frac{1}{2}\rho v^2 S C_L = W\cos\gamma$

$$v = \sqrt{\frac{2}{\rho C_L}\frac{W}{S}\cos\gamma}$$

Flying at constant glide angle corresponds to flying at constant $L/D$, and hence constant incidence, and constant $C_L$. Glide velocity therefore decreases with altitude due to increasing density; equivalent airspeed (and indicated airspeed) remains constant.

$C_L$ for maximum $L/D$ is given by $(C_L)_{max\ L/D}$ = $\sqrt{\dfrac{\pi A C_{D0}}{k}}$ (from lecture 2), so the velocity for minimum glide angle is given by:

$$v_{\min\gamma} = \left(\frac{2}{\rho}\frac{W}{S}\cos\gamma\right)^{\frac{1}{2}}\left(\frac{k}{\pi A C_{D0}}\right)^{\frac{1}{4}}$$

For sail planes, usually $\dfrac{L}{D}$ > 15, so $\gamma < \tan^{-1}\left(\dfrac{1}{L/D}\right)$= 3.8°, and cos$\gamma$ > 0.998 ≈ 1

so to a close approximation, $$v_{\min\gamma} \approx \left(\frac{2}{\rho}\frac{W}{S}\right)^{\frac{1}{2}}\left(\frac{k}{\pi A C_{D0}}\right)^{\frac{1}{4}} \qquad (3\text{-}4)$$

The <u>sink rate</u>, $v_s$ is given by: $$v_s = v\sin\gamma = \frac{v}{(L/D)}\cos\gamma$$

$$v = \sqrt{\frac{2L}{\rho C_L S}} \qquad \rightarrow \qquad v_s = \sqrt{\frac{2W\cos\gamma}{\rho C_L S}}\frac{1}{(C_L/C_D)}\cos\gamma$$

$$v_s = \left(\frac{2W}{\rho S}\right)^{\frac{1}{2}}\frac{1}{\left(C_L^{3/2}/C_D\right)}\cos^{3/2}\gamma$$

For minimum sink rate, maximise $\left(\dfrac{C_L^{3/2}}{C_D}\right) \rightarrow \left(\dfrac{C_L^{3/2}}{C_D}\right)_{\max} = \dfrac{1}{4}\left(\dfrac{3\pi A}{k}\right)^{\frac{3}{4}}\dfrac{1}{(C_{D0})^{\frac{1}{4}}}$.

This is the same condition for minimum power in level flight, (lecture 2), which is reasonable since during glide, potential energy $Wz$ is used at a rate $Wv_s$.

Minimum sink rate $\rightarrow$ $$v_{s\min} = 4\left(\frac{2W}{\rho S}\right)^{\frac{1}{2}}\left(\frac{k}{3\pi A}\right)^{\frac{3}{4}}C_{D0}^{1/4}\cos^{3/2}\gamma \qquad (3\text{-}5)$$

and using the approximation that $\cos\gamma \approx 1$: $$v_{s\,\min} \approx 4\left(\frac{2W}{\rho S}\right)^{\frac{1}{2}}\left(\frac{k}{3\pi A}\right)^{\frac{3}{4}} C_{D0}^{1/4}$$

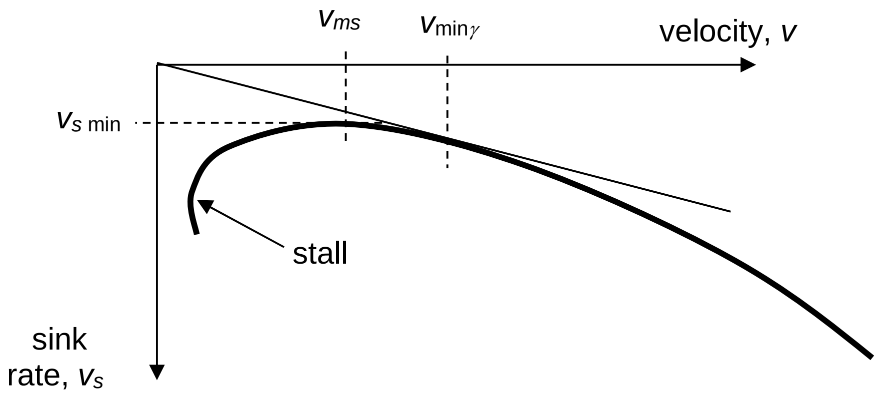
At $\left(\frac{C_L^{3/2}}{C_D}\right)_{\max}$, the drag coefficient $C_D\left(= C_{D0} + \frac{kC_L^2}{\pi A}\right) = 4C_{D0}$ (lecture 2), hence

$$C_L = \left(\frac{3\pi A C_{D0}}{k}\right)^{\frac{1}{2}}.$$

The glide velocity for minimum sink rate $v_{ms}$ is therefore given by:

$$v = \sqrt{\frac{2L}{\rho C_L S}} \qquad \rightarrow \qquad v_{ms} = \left(\frac{2}{\rho}\frac{W}{S}\cos\gamma\right)^{\frac{1}{2}}\left(\frac{k}{3\pi A C_{D0}}\right)^{\frac{1}{4}}$$

$$v_{ms} = \left(\frac{1}{3}\right)^{\frac{1}{4}} v_{\min\gamma} = 0.76 v_{\min\gamma}$$




The velocity for minimum sink is lower than the velocity for best glide. For this reason, glider pilots fly at $v_{ms}$ in thermals (regions of hot air rising) in order to gain altitude, then accelerate to $v_{\min\gamma}$ for best glide to maximise range when flying to the next thermal.

**<u>Specific fuel consumption (sfc)</u>**

For a piston-driven propeller engine, the rate of fuel consumption is a function of the power output. The specific fuel consumption is defined as

$$\text{sfc}, c = -\frac{\dot{W}_f}{P_{av}} = \frac{\text{Weight of fuel consumed / second}}{\text{Brake power}} \qquad (\text{m}^{-1}) \qquad (3\text{-}6)$$

For a jet engine, the fuel consumption is approximately proportional to thrust, and the specific fuel consumption is defined as

$$\text{sfc}, c = -\frac{\dot{W}_f}{T_{av}} = \frac{\text{Weight of fuel consumed / second}}{\text{Thrust}} \qquad (\text{s}^{-1}) \qquad (3\text{-}7)$$

**<u>Specific air range (SAR)</u>**

Range $R$ is defined as the horizontal distance covered over the ground during flight. The specific air range of an aircraft is defined as the rate of increase of range with respect to the weight of fuel consumed:

$$\text{SAR} = -\frac{dR}{dW_f} \qquad (3\text{-}8)$$

This can be written: $\text{SAR} = -\frac{dR/dt}{dW_f/dt} = -\frac{v}{\dot{W}_f} = \frac{\text{Airspeed}}{\text{Rate of fuel consumption}}$

For a propeller-driven aircraft, $\text{SAR} = -\frac{v}{\dot{W}_f} = \frac{v}{cP_{av}} = \frac{\eta_p v}{cP_r}$

In steady level flight, $P_r = T_r$ v, and $T_r = \frac{W}{(C_L/C_D)}$

$$\boxed{\underline{\text{propeller}}: \quad \text{SAR} = \frac{\eta_p}{c}\left(\frac{C_L}{C_D}\right)\frac{1}{W}} \qquad (3\text{-}9)$$

i.e. specific air range is a maximum at max $C_L/C_D$, i.e. at minimum drag speed.

For a <u>jet</u> engine: $$\text{SAR} = -\frac{v}{\dot{W}_f} = \frac{v}{cT_{av}}$$

In steady level flight, $T = D$, and $$\text{SAR} = \frac{v}{cD} = \frac{v}{cD} \times \frac{L}{L} = \frac{v}{c}\left(\frac{C_L}{C_D}\right)\frac{1}{W}$$

$$v = \sqrt{\frac{2W}{\rho C_L S}} \rightarrow$$

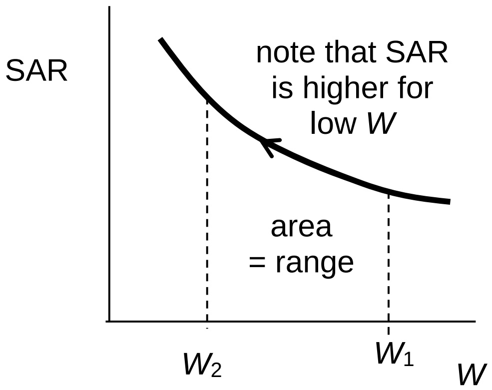
$$\boxed{\text{jet:} \qquad \text{SAR} = \frac{1}{c}\left(\frac{C_L^{\frac{1}{2}}}{C_D}\right)\left(\frac{2}{\rho S}\right)^{\frac{1}{2}} \frac{1}{\sqrt{W}}} \qquad (3\text{-}10)$$

**<u>Range</u>**

The weight of an aircraft $W$ at any instant is given by its empty fuel weight $W_0$, plus the weight of fuel $W_f$ carried

$$W = W_0 + W_f$$

As the aircraft flies, it consumes fuel at a rate $-\frac{dW_f}{dt}$, hence the aircraft's weight decreases at the same rate, $-\frac{dW}{dt} = -\frac{dW_f}{dt}$.




The specific air range can be written

$$\text{SAR} = -\frac{dR}{dW}$$

The range is given by:

$$R = \int_1^2 dR = \int_1^2 \frac{\partial R}{\partial W}\, dW$$

For an aircraft taking off with weight $W_1$ and landing with weight $W_2$:

$$R = \int_{W1}^{W2} -SAR\, dW = \int_{W2}^{W1} SAR\, dW$$

The Breguet range equation - for a propeller-driven aircraft

$$R = \int_{W2}^{W1} SAR\, dW = \int_{W2}^{W1} \frac{\eta_p}{c}\left(\frac{C_L}{C_D}\right)\frac{1}{W}\, dW$$

Assuming that the specific fuel consumption and propeller efficiency are constant during flight, and that the aircraft is flown at a fixed incidence (*L*/*D* remains constant):

$$\boxed{\text{Breguet range equation:} \qquad R = \frac{\eta_p}{c}\left(\frac{C_L}{C_D}\right)\ln\left(\frac{W_1}{W_2}\right) \qquad \underline{\text{propeller}}} \qquad (3\text{-}11)$$

Since $W = L = \frac{1}{2}\rho v^2 S C_L$ , the aircraft flies slower as *W* reduces.

This shows that *R* for a propeller-driven aircraft is independent of density, and hence of altitude if $\eta_p$, *c* and $C_L/C_D$ are not affected.

To maximise range:

(i) use an efficient propeller $\rightarrow$ maximise $\eta_p$;

(ii) use an efficient engine $\rightarrow$ minimise *c*;

(iii) fly at $(C_L/C_D)_{max}$, i.e. at minimum drag speed $v_{md}$.

Range equation for a jet aircraft

$$R = \int_{W2}^{W1} SAR\, dW = \int_{W2}^{W1} \frac{1}{c}\left(\frac{C_L^{\frac{1}{2}}}{C_D}\right)\left(\frac{2}{\rho S}\right)^{\frac{1}{2}} \frac{1}{\sqrt{W}}\, dW$$

Flying at constant incidence (constant $C_L^{\frac{1}{2}}/C_D$), constant altitude ($\rho$ is constant), and assuming that *c* is constant,

$$\boxed{R = 2\left(\frac{2}{\rho S}\right)^{\frac{1}{2}} \frac{1}{c}\left(\frac{C_L^{\frac{1}{2}}}{C_D}\right)\left(\sqrt{W_1} - \sqrt{W_2}\right) \qquad \underline{\text{jet}}} \qquad (3\text{-}12)$$

To maximise range for a jet at constant altitude:

(i) use an efficient engine → minimise $c$;

(ii) fly at $(C_L^{1/2}/C_D)_{max}$ → from the drag polar $\left(C_D = C_{D0} + \dfrac{kC_L^2}{\pi A}\right)$,

$$\frac{C_D}{C_L^{1/2}} = \frac{C_{D0}}{C_L^{1/2}} + \frac{kC_L^{3/2}}{\pi A}$$

$$\frac{d}{dC_L}\left(\frac{C_D}{C_L^{1/2}}\right) = -\frac{1}{2}\frac{C_{D0}}{C_L^{3/2}} + \frac{3}{2}\frac{kC_L^{1/2}}{\pi A} = 0 \quad \rightarrow \quad C_L = \left(\frac{\pi A C_{D0}}{3k}\right)^{\frac{1}{2}}$$

$$v = \sqrt{\frac{2W}{\rho C_L S}} \quad \rightarrow \quad v = \left(\frac{2W}{\rho S}\right)^{\frac{1}{2}}\left(\frac{3k}{\pi A C_{D0}}\right)^{\frac{1}{4}} = (3)^{\frac{1}{4}} v_{md} = 1.32 v_{md}$$

(iii) fly at high altitude since $R \propto \dfrac{1}{\sqrt{\rho}}$ → this is the main reason for the high cruising altitudes of jet aircraft.

<u>Maximise altitude to maximise range for constant altitude cruise</u>

For a constant altitude cruise, the altitude and hence range $R$ are limited by the thrust available and the weight at the start of cruise $W_1$:

The thrust available is given by

$$T = (\dot{m}_a + \dot{m}_f)v_e - \dot{m}_a v_\infty + (p_e - p_\infty)A_e .$$

The $(p_e - p_\infty)A_e$ term is generally much smaller than the rate of change of momentum terms, and $\dot{m}_f << \dot{m}_a$ hence $T \approx \dot{m}_a(v_e - v_\infty)$ or $T \approx \rho_\infty A_\infty v_\infty (v_e - v_\infty)$. To a reasonable approximation, the thrust available at altitude is then given by $T_{av} = \dfrac{\rho}{\rho_0} T_0$, where $T_0$ and $\rho_0$ are the thrust available and density at sea level.

Maximum altitude occurs when $T_{av} = T_r = D$

$$T_{av} = \frac{\rho}{\rho_0} T_0 = D \times \frac{L}{L} = \frac{C_D}{C_L} W_1$$

$$\boxed{\rho = \rho_0 \frac{C_D}{C_L} \frac{W_1}{T_0}} \qquad (3\text{-}13)$$

More powerful engines enable flight at higher altitudes; also lower weight enables higher altitudes.

<u>Maximum range at constant altitude</u>

$$R = 2\left(\frac{2T_0}{\rho_0 S}\right)^{\frac{1}{2}} \frac{1}{c}\left(\frac{C_L}{C_D}\right)^{\frac{1}{2}} \left(\frac{C_L^{\frac{1}{2}}}{C_D}\right) \frac{1}{\sqrt{W_1}}\left(\sqrt{W_1} - \sqrt{W_2}\right)$$

$$R = 2\left(\frac{2T_0}{\rho_0 S}\right)^{\frac{1}{2}} \frac{1}{c}\left(\frac{C_L}{C_D^{3/2}}\right) \frac{1}{\sqrt{W_1}}\left(\sqrt{W_1} - \sqrt{W_2}\right) \qquad (3\text{-}14)$$

To maximise range for constant altitude flight, maximise $(C_L^{1/2}/C_D)_{max}$

$$\rightarrow \quad C_L = \left(\frac{\pi A C_{D0}}{3k}\right)^{\frac{1}{2}} \quad \text{and } C_D = \frac{4}{3} C_{D0}$$

$$R_{\max,\rho=\text{const}} = \frac{3}{2^{3/2} c C_{D0}} \left(\frac{\pi A T_0}{k \rho_0 S}\right)^{\frac{1}{2}} \left(1 - \frac{\sqrt{W_2}}{\sqrt{W_1}}\right) \qquad (3\text{-}15)$$

<u>Cruise climb</u>

From the condition for maximum altitude for a given weight $W$: $\rho = \rho_0 \dfrac{C_D}{C_L}\dfrac{W}{T_0}$,

as the weight of the aircraft decreases during flight due to the consumption of fuel, the aircraft is able to fly at higher altitudes (lower $\rho$). Substituting this expression for $\rho$ into the expression of SAR for a jet-powered aircraft,

$$\mathrm{SAR_{cc}} = \frac{1}{c}\left(\frac{C_L}{C_D^{3/2}}\right)\left(\frac{2T_0}{\rho_0 S}\right)^{\frac{1}{2}}\frac{1}{W}$$

Maximum range now occurs at $\left(\dfrac{C_L}{C_D^{3/2}}\right)_{\max}$ $\rightarrow$ this is at $\left(\dfrac{C_D}{C_L^{2/3}}\right)_{\min}$

From the drag polar $\rightarrow$ $\dfrac{C_D}{C_L^{2/3}} = \dfrac{C_{D0}}{C_L^{2/3}} + \dfrac{kC_L^{4/3}}{\pi A}$

Differentiate: $\dfrac{d}{dC_L}\left(\dfrac{C_D}{C_L^{2/3}}\right) = -\dfrac{2}{3}\dfrac{C_{D0}}{C_L^{5/2}} + \dfrac{4}{3}\dfrac{kC_L^{1/3}}{\pi A} = 0$

$$\rightarrow \qquad C_L = \left(\frac{\pi A C_{D0}}{2k}\right)^{\frac{1}{2}} \quad \text{and } C_D = \frac{3}{2}C_{D0}$$

$$v = \sqrt{\frac{2W}{\rho C_L S}} \qquad \rightarrow \qquad v = \left(\frac{2W}{\rho S}\right)^{\frac{1}{2}}\left(\frac{2k}{\pi A C_{D0}}\right)^{\frac{1}{4}} = (2)^{1/4} v_{md} = 1.19 v_{md}$$

So for cruise-climb, fly slower and at a higher $C_L$ compared to constant altitude cruise.

$$\mathrm{SAR_{cc}} = \left(\frac{2}{3}\right)^{\frac{3}{2}}\left(\frac{\pi A T_0}{k\rho_0 S}\right)^{\frac{1}{2}}\frac{1}{cC_{D0}}\frac{1}{W} \qquad (3\text{-}16)$$

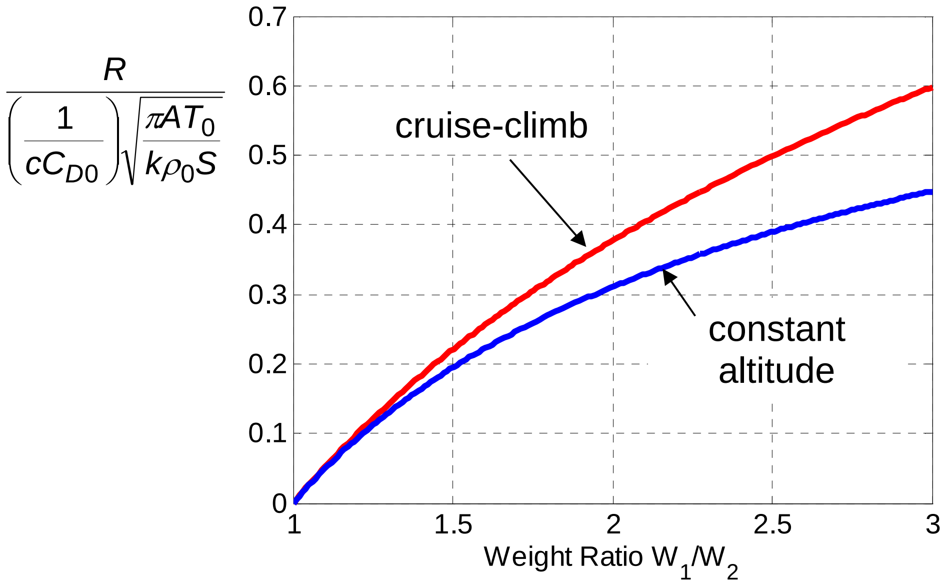
## Cruise-climb range

Range: $$R = \int_{W2}^{W1} SAR\, dW = \int_{W2}^{W1} \left(\frac{2}{3}\right)^{\frac{3}{2}} \left(\frac{\pi A T_0}{k \rho_0 S}\right)^{\frac{1}{2}} \frac{1}{c C_{D0}} \frac{1}{W} dW$$

$$\boxed{R = \left(\frac{2}{3}\right)^{\frac{3}{2}} \left(\frac{\pi A T_0}{k \rho_0 S}\right)^{\frac{1}{2}} \frac{1}{c C_{D0}} \ln\left(\frac{W_1}{W_2}\right)} \qquad (3\text{-}17)$$

c.f. maximum constant altitude range:

$$R_{\max,\rho=\text{const}} = \frac{3}{2^{3/2} c C_{D0}} \left(\frac{\pi A T_0}{k \rho_0 S}\right)^{\frac{1}{2}} \left(1 - \frac{\sqrt{W_2}}{\sqrt{W_1}}\right)$$




Cruise-climb gives a useful increase of range when $W_1/W_2$ is large

i.e. with a heavy fuel load.

## Endurance

Endurance is defined as the length of time that an aircraft can remain airborne on one tank of fuel. This will be important for air surveillance missions, e.g. maritime patrol, the monitoring a hurricane, or if an aircraft had to wait to land at a destination.

From specific air range:

$$\text{SAR} = -\frac{dR}{dW} = -\frac{dR}{dt}\frac{dt}{dW} = -v\frac{dt}{dW}$$

$$dt = -\frac{\text{SAR}}{v}dW$$

Endurance: $$\boxed{T_E = \int_{W2}^{W1} \frac{\text{SAR}}{v} dW} \qquad (3\text{-}18)$$

$$v = \sqrt{\frac{2W}{\rho C_L S}} \qquad \rightarrow \qquad T_E = \int_{W2}^{W1} \sqrt{\frac{\rho S}{2}} C_L^{1/2} \frac{\text{SAR}}{\sqrt{W}} dW$$

Endurance – propeller

$$T_E = \int_{W2}^{W1} \sqrt{\frac{\rho S}{2}} C_L^{1/2} \frac{\text{SAR}}{\sqrt{W}} dW$$

$$\text{SAR} = \frac{\eta_p}{c}\left(\frac{C_L}{C_D}\right)\frac{1}{W}$$

$$T_E = \int_{W2}^{W1} \frac{\eta_p}{c}\sqrt{\frac{\rho S}{2}}\left(\frac{C_L^{3/2}}{C_D}\right)\frac{1}{W^{3/2}} dW$$

Assuming all but $W$ is constant in flight:

$$\boxed{T_E = \frac{\eta_p}{c}\sqrt{2\rho S}\left(\frac{C_L^{3/2}}{C_D}\right)\left(\frac{1}{\sqrt{W_2}} - \frac{1}{\sqrt{W_1}}\right) \text{ \underline{propeller}}} \qquad (3\text{-}19)$$

To maximise $T_E$ (for a propeller-driven aircraft):

(i) use efficient propeller $\rightarrow$ maximise $\eta_p$;

(ii) use an efficient engine $\rightarrow$ minimise $c$;

(iii) fly at $(C_L^{3/2}/C_D)_{\max}$, i.e. at minimum power speed $v_{mp}$;

(iv) $T_E \propto \rho^{1/2}$ $\rightarrow$ fly low.

<u>Endurance – jet</u>

$$T_E = \int_{W2}^{W1} \frac{\text{SAR}}{v} dW$$

From the specific air range:

$$\text{SAR} = -\frac{v}{\dot{W}_f} = \frac{v}{cT_{av}} = \frac{v}{cD}$$

$$\rightarrow \qquad \frac{\text{SAR}}{v} = \frac{1}{cD} \times \frac{L}{L} = \frac{1}{c}\frac{C_L}{C_D}\frac{1}{W}$$

$$T_E = \int_{W2}^{W1} \frac{1}{c}\left(\frac{C_L}{C_D}\right)\frac{1}{W} dW$$

assuming all but $W$ constant in flight:

$$\boxed{T_E = \frac{1}{c}\left(\frac{C_L}{C_D}\right)\ln\left(\frac{W_1}{W_2}\right) \quad \underline{\text{jet}}} \qquad \text{(3-20)}$$

Note that for a jet, endurance is independent of density $\rightarrow$ fly at any altitude.

To maximise $T_E$ (for a jet):

(i) use an efficient engine $\rightarrow$ minimise $c$;

(ii) fly at $(C_L/C_D)_{\max}$, i.e. at minimum drag speed $v_{md}$;

Note that the Range and Endurance equations are approximations that make a number of assumptions, however they are an extremely useful guide.