B15 Limits of Controller Performance

Tutorial Questions

1. **State Space Model**.

   (a) For the proper, but not strictly proper, system

   $$G(s) = \frac{2s^2 + 4s + 1}{s^2 + 5s + 2}$$

   express the transfer function as the sum of a fixed gain and a strictly proper transfer function.

   (b) Write down the state space model for the system.

   (c) Use the `tf` comand to create a transfer function model in MATLAB. Convert this to a state space model using `ss` and compare it with the model you have derived. The MATLAB model will not be exactly the same as yours, but convince yourself that the two models are equivalent.

2. **BIBO Stability**. For the system

   $$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 9y(t) = u(t)$$

   (a) Write down the transfer function and work out the impulse response.

   (b) Use the impulse response to show that this system is not bounded-input, bounded-output (BIBO) stable.

   (c) Obtain an expression for $y(t)$ when $u(t) = \sin 3t$ and confirm that the output is not bounded when this (bounded) input is applied to the system.

   (d) Use the frequency response of the system to show that the $\mathcal{H}_\infty$ norm of the system is not finite.

   (e) Explain the physical reason why this system is not BIBO stable.

3. **Nyquist Plots**.

   (a) Draw (i.e. do *not* use MATLAB) the Nyquist plot for

   $$G(s) = \frac{K}{s^2(s+1)}$$

(b) Explain why this system is unstable for all values of K when unity feedback is applied.

(c) In the lectures, when there are poles on the imaginary axis, we indented the D-contour into the right half plane. Would we still be able to use the Nyquist criterion if we indented the D-contour into the left half plane?

4. **Internal Stability**

(a) Write down the state space models for the system

$$G(s) = \frac{s-2}{s+3}$$

and the controller

$$C(s) = \frac{1}{s-2}$$

(b) Work out the state space model when these two models are connected in series.

(c) Work out the eigenvectors of the $\mathbf{A}$ matrix and use them to transform the state space model of the combined system so that $\tilde{\mathbf{A}}$ matrix of the transformed signal is diagonalised.

(d) Show that the combined system has an unobservable, unstable mode

5. **MATLAB Commands**. Use MATLAB to repeat the same steps as in the previous question for the system

$$G(s) = \frac{1}{s-2}$$

and the controller

$$C(s) = \frac{s-2}{s+3}$$

Comment on the difference between this system and the system in the previous question.

6. **Loop Shaping** A control system is to be designed for

$$G(s) = \frac{1}{s(s^2+5s+2)}$$

(a) Use MATLAB to determine the magnitude of the complementary sensitivity for the closed loop

$$|T(\mathrm{j}\omega)| = \frac{|G(\mathrm{j}\omega)|}{|1 + G(\mathrm{j}\omega)|}$$

at $\omega = 1$ rad.s^{-1}. Check that the closed loop system is stable.

(b) Design a controller of the form

$$C(s) = K\frac{s+a}{s+b}$$

that will reduce $|T(\mathrm{j}\omega)|$ for the compensated system by a factor of (about) 4 at $\omega = 1$ rad.s^{-1}, while ensuring that the closed loop system remains stable.

(c) How does including the controller affect the performance of the closed loop?

7. **Bode Integral Formula** Suppose that

$$G(s) = \frac{1}{s^2 - s + 4}$$

We want to design a controller $C(s)$ so that the feedback loop is internally stable and

- $|S(\mathrm{j}\omega)| \leq \varepsilon$ for $0 \leq \omega < 0.1$ rad.s^{-1}
- $|S(\mathrm{j}\omega)| \leq 2$ for $0.1 \leq \omega < 5$ rad.s^{-1}
- $|S(\mathrm{j}\omega)| = 1$ for $5 \leq \omega < \infty$ rad.s^{-1}

Find a (positive) lower bound on the achievable ε.

8. **Robust Stability**. A system is described by the transfer function

$$\tilde{G}(s) = \frac{1}{s^2 + as + 1}$$

where a is an uncertain parameter in the range $0.4 \leq a \leq 0.8$.

(a) By writing

$$a = 0.6 + 0.2\Delta \quad -1 \leq \Delta \leq 1$$

show that $\tilde{G}(s)$ can be expressed in terms of an inverse multiplicative uncertainty

$$\tilde{G}(s) = \frac{G(s)}{1 + \Delta W(s)G(s)}$$

where

$$G(s) = \frac{1}{s^2 + 0.6s + 1} \qquad W(s) = 0.2s$$

(b) Use the small gain theorem to show that when the system is connected in feedback with a controller $C(s)$, the closed loop response of the perturbed system will be stable provided that $||W(s)G(s)S(s)||_\infty < 1$.

(c) If a proportional controller is used, with $C(s) = k$, determine the range of k for closed loop to be stable for the perturbed system. (You may find it easier to use the MATLAB `bode` function to determine the ∞-norm)

9. **Multiplicative Uncertainty**. Suppose that a nominal model of a system is described by the transfer function $G_0(s)$, but the actual system includes an unmodelled time delay τ. The exact value of τ is not known, but it lies in the range $0 \le \tau \le 0.1$. The unmodelled time delay can be expressed as a multiplicative uncertainty, so that the transfer function of the actual system satisfies

$$G(s) = [1 + W(s)\Delta(s)]\, G_0(s)$$

where $\|\Delta\|_\infty \le 1$

(a) Show that

$$\left|e^{-j\tau\omega} - 1\right| \le |W(j\omega)| \qquad \text{for all } \omega \text{ and for } 0 \le \tau \le 0.1$$

(b) Use MATLAB to plot $\left|e^{-j\tau\omega} - 1\right|$ over the range $10^{-1} \le \omega \le 5 \times 10^2$ for $\tau = 0.1$ (i.e. the largest allowable time delay)

(c) Show that

$$W(s) = \frac{0.21s}{0.1s + 1}$$

is a suitable weighting function for this uncertainty.