C201 Viscous Flow and Turbulence

Lecture 2

Part 2: parallel shear flows

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1 Applications of the RANS equations

We have derived the equations ruling the conservation of mass, momentum and energy for the Reynolds average of a turbulent flow. The equations we have derived are general.

We have learned that the TKE is sustained at the expense of the kinetic energy of the mean flow through the work done by the Reynolds stress in presence of velocity gradients. In isotropic homogeneous turbulence TKE must therefore decay in the streamwise direction because the mean flow is uniform.

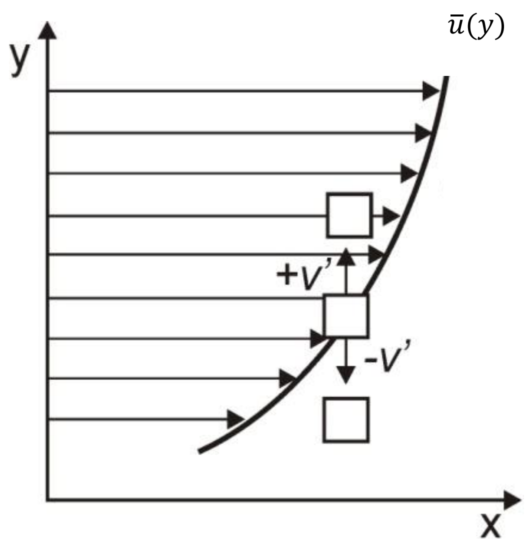
We now turn our attention to a class of flows collectively known as parallel shear flows.

Parallel shear flows are only marginally more complex than isotropic homogeneous turbulence, but are of considerable practical interest because they appear in the flow fields of vehicles as well as most engineering devices.

2 Momentum transfer in turbulent flows – the Reynolds stress

We have mentioned that in the RANS equation

$$U_j \cdot \frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\overline{u_i' u_j'} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j}$$



$v' > 0 \rightarrow u' < 0$

$v' < 0 \rightarrow u' > 0$

Figure 1

the term $\overline{u_i' u_j'}$ is the Reynolds stress and it represents the average effect of turbulence on the transfer of momentum between adjacent parts of the flow.

To see how the Reynolds stress arises we can make refer to the diagram in Figure 1. We are considering a flow with a finite velocity gradient $\partial\bar{u}/\partial y$. We consider the effect of an eddy displacing a parcel of fluid from the position y to the position $y + dy$. Because the parcel of fluid is being displaced in the positive y-direction, the eddy must have induced a positive fluctuation v' in the y-velocity component: $v' > 0$. Furthermore, since the velocity gradient is positive, once the parcel of fluid arrives at its destination will be slower than its surrounding and it will appear as a negative streamwise velocity fluctuation u': $u' < 0$. For this case we must have therefore $u'v' < 0$. Similar reasoning applied to a negative y-displacement still leads to the conclusion that $u'v' < 0$.

We see that the presence of turbulent eddies against a non-uniform background flow induces a correlation between the velocity components u' and v'. For the case shown (positive velocity gradient) the correlation is negative and it results in a negative Reynolds stress.

It is common – but by no means universal! - for the Reynolds stress to be oriented in the direction opposite the velocity gradient.

3 Parallel shear flows

Parallel shear flows have a two-dimensional statistically stationary mean flow with a primary flow direction. We will take this flow direction to be the x-direction. The gradients of all flow quantities in the streamwise direction are small compared to the gradients in the direction orthogonal to it, which we will take to be the y-direction. A corollary of this property is that streamline curvature is weak and no pressure gradient exists in the y-direction. Unlinke homogeneous isotropic turbulence, parallel shear flows can sustain turbulence because of they have velocity gradients and have non-vanishing Reynolds stresses. Furthermore, the mean velocity gradient defines the integral length scale as well as the turbulent velocity scale.

Examples of parallel shear flows are

- Wakes
- Jets
- Boundary layers

We now wish to write a specialised form of the RANS equation for parallel shear flows.

We can make use of Prandtl's approximation because the longitudinal extent of the flow much larger than transversal length scale and 1) omit the second derivatives in the streamwise direction 2) omit the momentum equation in the y-direction. Since we are operating in two space dimensions we also dispense with index notation and we write:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y}\overline{u'v'} - \frac{1}{\rho}\frac{\partial P}{\partial x}$$

For wakes and jets the RANS equation is further simplified because these flows take place in an infinite body of fluid at rest at a large distance from the area where gradients are present and therefore the pressure is essentially uniform:

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\nu\frac{\partial^2 U}{\partial y^2}-\frac{\partial}{\partial y}\,\overline{u'v'}$$

4 Turbulent jets

A jet is a stream of fluid injected in otherwise stagnant surroundings, which are assumed to be of infinite extent and not to constrain the downstream evolution of the jet.

As the jet penetrates its surroundings it gradually mixing with the ambient fluid, so that the velocity profile becomes smoother and smoother and spreads further and further away from the jet centreline.

In a turbulent jet the interface between the ambient fluid and the injected fluid is wrinkled as the jet breaks up into eddies. The area of intense mixing is called the mixing zone. The eddies of the mixing zone are responsible for the Reynolds stress distribution of the jet.

The mixing zone does not include the whole volume of the jet: there is a relatively short portion of the jet, near the injection location where essentially no mixing takes place. This portion of the jet is called the potential core (See Figure 2)

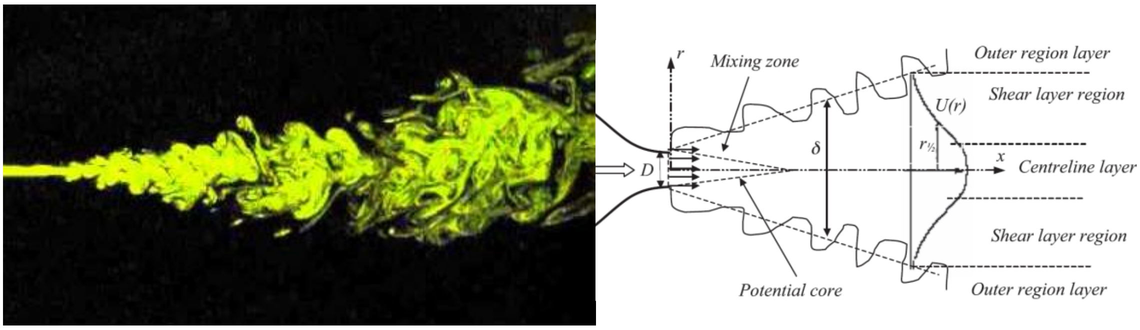


Figure 2

To fix ideas we examine the state of the flow at a distance x downstream of the origin of the jet. We denote with R the jet radius and we define R as the distance from the centerline where the mean velocity decays to half the centerline value U_s:

$$R(x) \rightarrow U(x,R) = 1/2U_s(x)$$

We conduct our analysis under the assumption that the jet spreads slowly and therefore

$$R \ll x$$

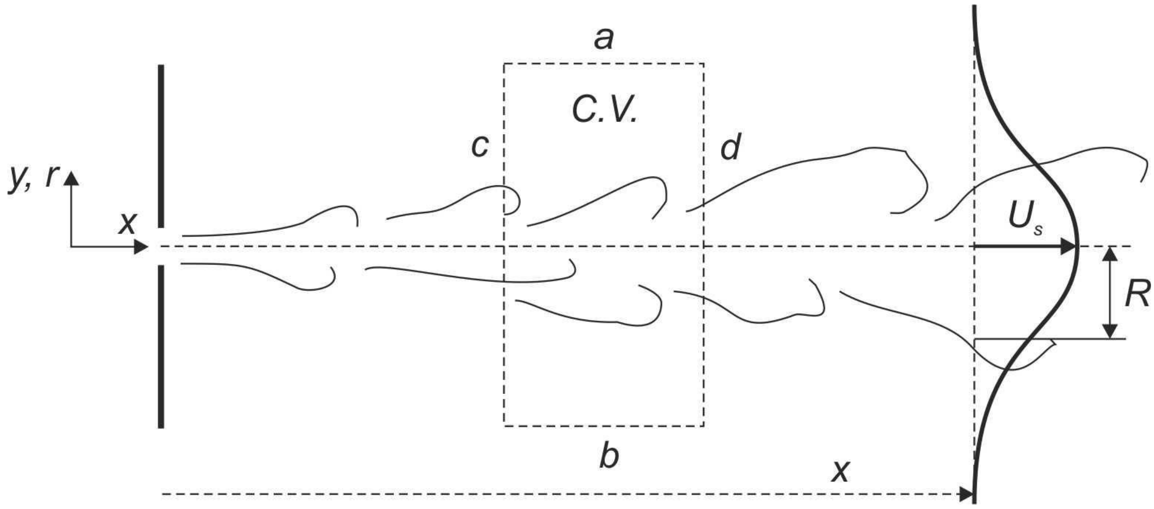


Figure 3

In order to analyse the jet we use the control volume $abcd$ in Figure 3.

If the sides a and b are pushed at a large distance from the centreline, no momentum flux crosses them. Furthermore, as already mentioned, the pressure is uniform because the jet is injected in a quiescent environment.

From these considerations we conclude that the momentum flux crossing the sides c and d of the control volume must be the same. We have reached the important conclusion that the momentum flux of a jet is invariant. More precisely, given any plane $S \rightarrow x = constant$

$$\int_S \rho U^2 dS = M = const$$

If the shape of the velocity profile does not change and M is a constant then the quantities

$$\int_S \rho U dS = f(x)$$

$$\frac{1}{2}\int_S \rho U^2 U dS = g(x)$$

cannot be constant, but must be functions of x. In particular $f(x)$ is the mass flux and $g(x)$ is the kinetic energy flux. In a jet the mass flux increases because the jet entrains fluid from its surroundings, whereas the kinetic energy decreases because it is transferred to turbulence and ultimately dissipated by viscosity.

We can now expand these considerations to obtain scaling laws relating the distance travelled by the jet, its width and its centreline velocity.

We start from the continuity equation:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0$$

If U scales with U_s and the derivatives with respect to x and y scale with x and R, respectively, then:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0\rightarrow\frac{U_s}{x}\sim\frac{V}{R}$$

Additionally, momentum conservation leads to

$$2R\rho U_s^2=M$$

At origin the initial width and velocity are r and U_0 respectively:

$$RU_s^2=rU_0^2$$

$$U_s^2=\frac{r}{R}U_0^2$$

We can now invoke the RANS equations to examine how R scales with x.

We study first the laminar case, where $\overline{u'v'}=0$. Using the same approximations for the derivatives, i.e.

$$\frac{\partial}{\partial x}\approx\frac{1}{x}$$

and using the scaling law for V derived from continuity

$$\frac{V}{R}\approx\frac{U}{x}$$

we find

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\nu\frac{\partial^2 U}{\partial y^2}\rightarrow\frac{U_s^2}{x}\sim\nu\frac{U_s}{R^2}$$

We can now use the condition that the momentum flux is constant obtained above, i.e.

$$RU_s^2=rU_0^2$$

to solve for R and U_s separately in terms of r, U_0 and x:

$$R\sim\nu^{2/3}r^{-1/3}U_0^{-2/3}x^{2/3}$$

$$U_s\sim\nu^{-1/3}r^{2/3}U_0^{4/3}x^{-1/3}$$

Having practised with laminar jets we now move on to the turbulent case:

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y}-\overline{u'v'}\right)$$

At high Reynolds number the viscous contribution to the momentum flux is negligible and only the Reynolds stress is responsible for spreading the momentum of the jet away from the centreline:

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{\partial}{\partial y}\overline{u'v'}$$

On dimensional grounds we use the approximation

$$\overline{u'v'}\sim U_s^2$$

This approximation is justified physically on the grounds that the centreline velocity is the only velocity scale available to us. Returning to the momentum equation and using the same approximations for the derivatives, we find:

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{\partial}{\partial y}\overline{u'v'}$$

$$\frac{U_s^2}{x}\sim\frac{U_s^2}{R}$$

We have reached the important conclusion that in a turbulent jet, R is proportional to x:

$$R\sim x$$

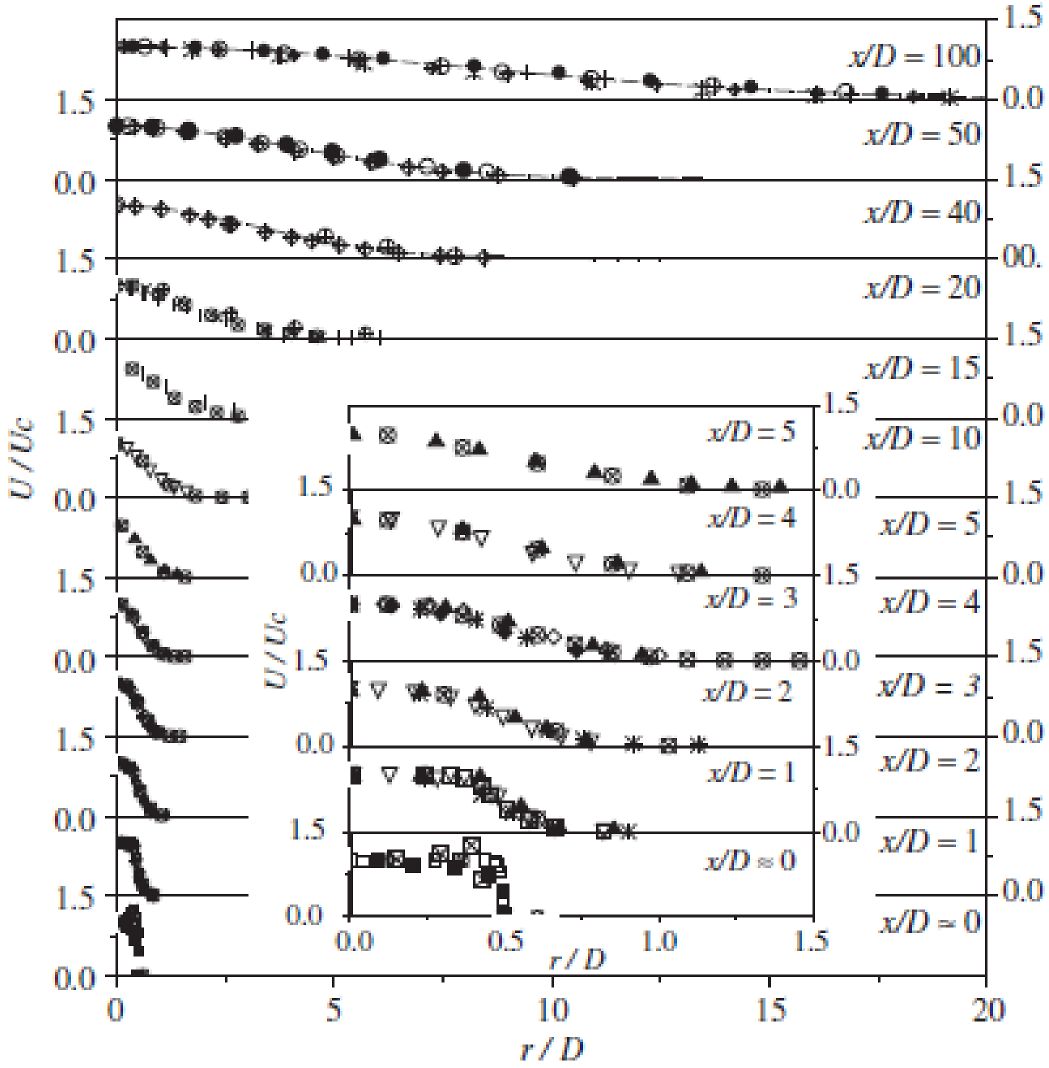
So a turbulent jet spreads considerably faster than a laminar jet, where $R \approx x^{2/3}$. We also notice that we have operated in the limit of large Reynolds number. The viscosity therefore disappears from our estimate. Experimentally, for planar jets

$$R\sim 0.08x$$

We can now turn to the momentum equation to find

$$RU_s^2=rU_0^2\rightarrow U_s\sim U_0r^{1/2}x^{-1/2}$$

The faster spreading rate of a turbulent jet corresponds to a faster decay of the centerline velocity compared to the laminar case, where $U_s\sim x^{-1/3}$.



5 Wakes

A wake is a region behind an obstacle in an otherwise uniform flow. The fluid in the wake is slowed down by the drag of the object creating the wake, and has therefore a smaller velocity than its surroundings.

The analysis of wakes and the construction of scaling laws proceeds in a similar way to the analysis of jets, except there are now two velocities to contend with: the velocity of the uniform far field, U_0, and the defect velocity in the wake U_s.

Because there are no external forces acting on the fluid apart from the drag of the body creating the wake, the momentum flux deficit must be constant along the streamwise direction:

$$\rho \int_S U(U_0 - U)dS = M$$

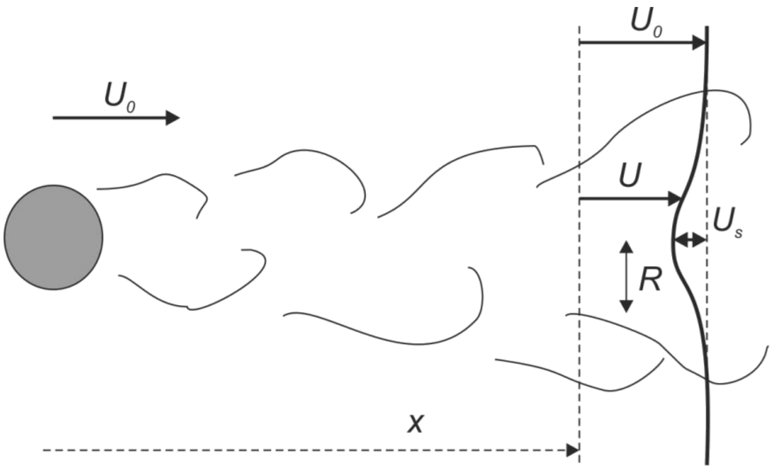


Figure 4

The momentum defect M is equal to the drag, D, acting on the body.

$$D = c_d \frac{1}{2} \rho U_0^2 \pi r^2$$

where r is radius of the body. For Reynolds numbers between 1000 and 300000 c_d is almost constant and is not far from unity

$$c_d \approx 1$$

Based on this observation we can write

$$\int_S U(U_0 - U) dS = \frac{1}{2} U_0^2 \pi r^2$$

For a planar wake the momentum conservation equation is:

$$U_0 U_s R \sim r U_0^2 \rightarrow U_s \sim \frac{r U_0}{R}$$

5.1 Scaling laws

In order to obtain scaling laws we return to the RANS equations and we examine first the case of a laminar wake. We adopt the scaling

$$\frac{U_s}{x} \sim \frac{V}{R}$$

implied by continuity. The momentum equation

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \nu \frac{\partial^2 U}{\partial y^2}$$

implies the scaling law

$$U_0 \frac{U_s}{x} \sim \nu \frac{U_s}{R^2}$$

Or, equivalently:

$$\frac{U_0}{x} \sim \frac{\nu}{R^2}$$

Comparing this scaling with the momentum integral

$$U_s \sim \frac{r U_0}{R}$$

we obtain:

$$R \sim \nu^{1/2} U_0^{-1/2} x^{1/2}$$

$$U_s \sim \nu^{-1/2} r U_0^{3/2} x^{-1/2}$$

For the turbulent case we resort to the limit of very large Reynolds number and use the estimate

$$\overline{u'v'} \sim U_s^2$$

We select U_s as the velocity scale for turbulent fluctuations because turbulence is sustained by a velocity gradient proportional to U_s/R. Furthermore U_0 is unsuitable as a turbulent velocity scale because it would make our estimate not invariant under Galileian transformations.

Based on this estimate, the RANS equation becomes:

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{\partial}{\partial y} \overline{u'v'}$$

whence

$$U_0 \frac{U_s}{x} \sim \frac{U_s^2}{R}$$

Or, equivalently:

$$\frac{U_0}{x} \sim \frac{U_s}{R}$$

Comparing this scaling with the scaling inferred from the constancy of the momentum deficit:

$$U_s \sim \frac{r U_0}{R}$$

We obtain separate scaling laws for U_s and R in terms of r, U_0 and x.

$$R \sim x^{1/2} r^{1/2}$$

$$U_s \sim x^{-1/2} r^{1/2} U_0$$

Differently from the case of a jet, we find that a turbulent and a laminar wake spread at the same rate ($x^{1/2}$) but they scale in different ways with respect to the free stream velocity and size of the object creating the wake.

6 Turbulent jets and wakes: checklist

Jets and wakes are examples of parallel shear flows.

Parallel shear flows have almost rectilinear streamlines in the mean flow and no pressure gradient

For jets, we use distance downstream of orifice $x,$ initial radius $r,$ local radius R and velocity U_s to describe the flow. The momentum flux integral $M = \int_S U^2 ds$ is constant. The jet entrains fluid from its surrounding environment. The scaling laws for turbulent jets are:

$$R \sim x$$

$$U_s \sim U_0 r^{1/2} x^{-1/2}$$

For wakes we use the distance downstream of the obstacle $x,$ initial radius $r,$ local radius R and velocity defect $U_s,$ free stream velocity U_0. The momentum flux integral $M = \int_S U(U_0 - U) ds$ is constant. The scaling laws are:

$$R \sim x^{1/2} r^{1/2}$$

$$U_s \sim x^{-1/2} r^{1/2} U_0$$