A10586W1

TRINITY TERM 2022

SECOND PUBLIC EXAMINATION

Honour School of Engineering Science

CONTROL SYSTEMS (Paper B15)

07 June 2022 9:30 a.m. – 11:00 a.m.

Answers to not more than **THREE** *questions should be submitted.*

Answer each question in a separate booklet.

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Note that:

- *The approximate allocation of marks is given in the margin.*
- *Permitted calculator series: Casio fx-83, Casio fx-85, Sharp EL-531.*
- *A copy of Engineering Tables & Data (HLT) is provided.*

1. Consider the following linear, time-invariant system,

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha^2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \alpha + 1 \end{bmatrix}, \quad \text{and } \alpha \leq 0.$$

(a) Compute the eigenvalues and eigenvectors of A as a function of parameter α. For which values of α is matrix A diagonalisable?

[2 marks]

(b) (i) Show that for $\alpha < 0$ the state transition matrix is given by

$$\mathrm{e}^{At} = \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} (\alpha^2 - \alpha)\mathrm{e}^{\alpha t} & \mathrm{e}^{\alpha^2 t} - \mathrm{e}^{\alpha t} \\ 0 & (\alpha^2 - \alpha)\mathrm{e}^{\alpha^2 t} \end{bmatrix}.$$

(ii) Compute the state transition matrix for the case where $\alpha = 0$.

[3 marks]

(c) Let $u(t) = 0$ for all t. For which values of α is the system stable? Justify your answer.

[3 marks]

(d) Set $\alpha = 0$ and explain why the following claim is true: There exists an input $u(t)$ that can drive the state of the system from $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Compute one such input that achieves the desired state transfer.

[5 marks]

(e) Now setting $\alpha = -1$. Does there exists an input $u(t)$ that can drive the state of the system from $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} \mathrm{e}^{-1} \\ 0 \end{bmatrix}$? Justify your answer.

Hint: Compute the state solution $x(t)$ for $\alpha = -1$.

[3 marks]

2. Consider the following linear, time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t) \; and$$
$$y(t) = Cx(t)\,,$$

where

$$A = \begin{bmatrix} -10 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(a) From the state space representation of the system, determine which states are controllable and which are observable.

[2 marks]

(b) Compute a gain matrix K of a state feedback controller $u(t) = Kx(t)$ such that the eigenvalues of the closed-loop system are -5 and -10.

[3 marks]

(c) Consider an infinite horizon linear quadratic regulator (LQR) problem

$$\begin{aligned} &\text{minimize } \int_0^\infty \left(x(t)^{\mathrm{T}} Q x(t) + u(t)^{\mathrm{T}} u(t) \right) \mathrm{d}t\,, \\ &\text{subject to } \dot{x}(t) = Ax(t) + Bu(t) \text{ for all } t, \\ &\text{and } x(0) = x_0 : \text{ given.} \end{aligned}$$

Here A and B are the given state space matrices, and $Q \geq 0$ is a matrix of appropriate dimensions. Determine some Q so that $K = \begin{bmatrix} 0 & -2 \end{bmatrix}$ is the gain matrix of the optimal LQR controller.

Hint: Use the given K and the expression of the optimal LQR controller to determine the structure of a matrix $P > 0$ that constitutes the solution of the associated algebraic Riccati equation.

[6 marks]

(d) Consider another state space representation with the same A and C matrices from part (c), but with $B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Compute the LQR controller for the problem of part (c) if $Q = \begin{bmatrix} 24 & 0 \\ 0 & 2 \end{bmatrix}$.

[3 marks]

(e) Consider the controllers of parts (c) and (d). Identify which of them is impossible to implement in practice for the corresponding system. Justify your answer.

[2 marks]

3. The angular velocity of a shaft attached to a motor is described by the state space model

$$\dot{x}(t) = -Ax(t) + Bu(t) + w(t)\,,$$

where $x(t)$ is the angular velocity, $u(t)$ is the torque generated by the motor and $w(t)$ is the process noise, which is zero mean white noise with variance $\mathrm{E}[w(t)w(t+\tau)^{\mathrm{T}}] = Q\,\delta(\tau)$.

The angular velocity is measured using a tachometer, where the measurement equation is

$$y(t) = Cx(t) + v(t)\,,$$

where $y(t)$ is the measured angular velocity and $v(t)$ is the measurement noise, which is zero mean with variance $\mathrm{E}[v(t)v(t+\tau)^{\mathrm{T}}] = R\,\delta(\tau)$.

(a) When $A = 1$, $B = 1$, $C = 1$, $Q = 1$ and $R = 1$, solve the continuous-time algebraic Riccati equation,

$$0 = A\,\overline{P} + \overline{P}\,A^{\mathrm{T}} + Q - \overline{P}\,C^{\mathrm{T}}R^{-1}C\,\overline{P}\,,$$

and derive the steady state Kalman filter for the continuous-time system.

[4 marks]

(b) The continuous-time system is sampled with sample period T. Assuming that a zero-order hold is used so that the input $u(t)$ remains constant between samples, determine A_{d} and B_{d} for the discrete-time state space model:

$$\begin{aligned} x_{k+1} &= A_{\mathrm{d}}x_k + B_{\mathrm{d}}u_k + w_k\,, \\ y_k &= Cx_k + v_k\,, \end{aligned}$$

where x_k, u_k, y_k, w_k and v_k are the state, input, measurement, process noise and measurement noise at the kth sample, respectively.

[5 marks]

(c) Solve the discrete-time algebraic Riccati equation of the sampled system for $\overline{P}_{\mathrm{d}}$,

$$\overline{P}_{\mathrm{d}} = A_{\mathrm{d}}\,\overline{P}_{\mathrm{d}}\,A_{\mathrm{d}}^{\mathrm{T}} + Q_{\mathrm{d}} - A_{\mathrm{d}}\,\overline{P}_{\mathrm{d}}\,C^{\mathrm{T}}\left(C\,\overline{P}_{\mathrm{d}}\,C^{\mathrm{T}} + R_{\mathrm{d}}\right)^{-1} C\,\overline{P}_{\mathrm{d}}\,A_{\mathrm{d}}^{\mathrm{T}}\,.$$

when $Q_{\mathrm{d}} = (1 - \mathrm{e}^{-T})^2$ and $R_{\mathrm{d}} = 1$. Derive the steady state Kalman filter for the combined state and measurement update version of the discrete-time filter.

[7 marks]

4. (a) A system with transfer function

$$G(s) = \frac{10}{s-1},$$

is connected in feedback with a controller

$$C_1(s) = \frac{1}{2}.$$

The system is subject to *additive* uncertainty $\Delta_1(s)$, where $\|\Delta_1(s)\|_\infty \leq 1$, as shown in Figure 1.

(i) Use the small gain theorem to show that the closed loop system will be stable provided that

$$\|C_1(s)S(s)\|_\infty < 1,$$

where $S(s)$ is the sensitivity function.

(ii) Show that the closed loop system is robustly stable.

[6 marks]

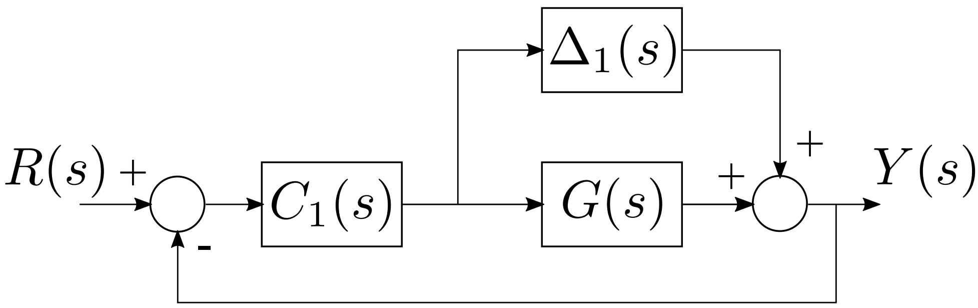

Figure 1

(b) The system is now modified, as shown in Figure 2, so that $G(s)$ is subject to *multiplicative* uncertainty $\Delta_2(s)$, where $\|\Delta_2(s)\|_\infty \leq 1$.

(i) Use the small gain theorem to show that the closed loop system will be stable provided that

$$\|T(s)\|_\infty < 1,$$

where $T(s)$ is the complementary sensitivity function.

(ii) Is the closed loop system is robustly stable? Justify your answer.

[6 marks]

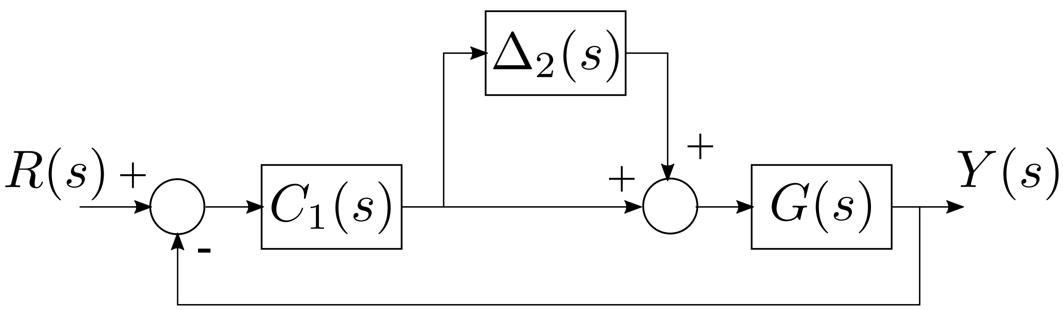

Figure 2

(c) In attempt to improve the closed loop performance, the controller is replaced by

$$C_2(s) = \frac{s-1}{s+1} \,.$$

(i) Show that the closed loop transfer function from $R(s)$ to $Y(s)$ for the nominal (unperturbed) system is stable.

(ii) Comment on the suitability of this controller.

[4 marks]