C2 Aerothermal Eng - Wing Theory and Compressible Flow

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Lecture 2 Aerofoil Characteristics

- Fourier series solution for the circulation distribution / associated aerofoil characteristics for a given camber line shape and incidence.
- Control flaps are analysed
- Lift enhancement and stall prevention.

Calculating circulation for a given aerofoil shape

- In lecture one we saw how the upwash equation gives us the aerofoil shape if we know the circulation distribution, it would be more useful to be able to calculate the circulation distribution for a defined aerofoil shape.
  - To solve this inverse problem we use a Fourier Series solution.
- General Fourier Series loading function: lets's let the loading function:

$$\frac{\gamma(\theta)}{U} = A_0 \cot\frac{\theta}{2} + \sum_{n=1}^{\infty} A_n \sin(n\theta).$$

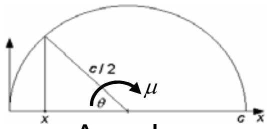
(1) We do not want cosine terms because we want zero circulation at the trailing edge (Kutta condition).

(2) The inclusion of the cot term allows the superposition of flat plate loading as the incidence is varied (Lecture 1).

Tools from lecture 1:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\pi U}\int_0^c \frac{\gamma(\eta)}{(\eta - x)}\mathrm{d}\eta = \frac{1}{2\pi U}\int_0^\pi \gamma(\mu)\frac{\sin\mu}{(\cos\theta - \cos\mu)}\mathrm{d}\mu$$

Upwash Eqn.



Angular coordinate system

$$\textit{from HLT}: \int_0^\pi \frac{\cos(n.\mu)}{(\cos\mu - \cos\theta)}\mathrm{d}\mu = \frac{\pi\sin(n\theta)}{\sin\theta}$$

Glauert Integral

Calculating circulation for a given aerofoil shape (2)

Subtitution: 1st subtitute the general loading into the upwash equation:

$$\frac{dy}{dx}=\frac{1}{2\pi}\int_0^{\pi}\left[A_0\cot\frac{\mu}{2}+\sum_{n=1}^{\infty}A_n\sin(n\mu)\right]\frac{\sin\mu}{(\cos\theta-\cos\mu)}d\mu$$

Rewrite as 2 integral groups :

$$\frac{dy}{dx}=\frac{1}{2\pi}\left[A_0\int_0^{\pi}\frac{\cot\left(\frac{\mu}{2}\right)\sin\mu}{(\cos\theta-\cos\mu)}d\mu+\sum_{n=1}^{\infty}A_n\int_0^{\pi}\frac{\sin(n\mu)\sin\mu}{(\cos\theta-\cos\mu)}d\mu\right]$$

Recall from Lecture 1 $\quad \cot(\mu/2)\ \sin\mu = 1+\cos\mu$

While from HLT p. 7 $\quad 2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Putting $A = n\mu$, $B = \mu$: $\quad \sin(n\mu)\sin\mu = \frac{1}{2}\{\cos((n-1)\mu) - \cos((n+1)\mu)\}$

Also from HLT p.7 $\quad \sin A - \sin B = 2\sin(A+B)/2.\cos(A-B)/2$

so that $\sin((n-1)\mu) - \sin((n+1)\mu) = -2\sin\mu\cos(n\mu)$

Calculating circulation for a given aerofoil shape (3)

Hence we can rewrite the equation as:

$$\frac{dy}{dx}=\frac{1}{2\pi}\left[A_0\int_0^{\pi}\frac{1+\cos\mu}{(\cos\theta-\cos\mu)}d\mu+\frac{1}{2}\sum_{n=1}^{\infty}A_n\left(\int_0^{\pi}\frac{\cos((n-1)\mu)}{(\cos\theta-\cos\mu)}d\mu-\int_0^{\pi}\frac{\cos((n-1)\mu)}{(\cos\theta-\cos\mu)}d\mu\right)\right]$$

Finally we apply the Glauert Integral:

$$\frac{dy}{dx}=-\frac{1}{2\pi}\left[A_0\left(0-\frac{\pi\sin\theta}{\sin\theta}\right)+\frac{1}{2}\sum_{n=1}^{\infty}A_n\left(\frac{\pi\sin(n-1)\theta}{\sin\theta}-\frac{\pi\sin(n+1)\theta}{\sin\theta}\right)\right]$$

So, substituting $\sin\{(n-1)\theta) - \sin(n+1)\theta\} = -2\sin\theta\cos(n\theta)$ we obtain

$$\boxed{\frac{dy}{dx}=-\frac{A_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}A_n\cos n\theta.}$$

This is the second standard key result (worth memorising), it follows from the upwash equation.
The coefficient A_0 is the *only* one which can vary without altering the shape of the aerofoil. If A_0 alone varies, then *the aerofoil tilts without changing shape*.

Calculating circulation for a given aerofoil shape (4)

Fourier Integral Equations for Coefficients A_n

The camber line of an aerofoil will be specified either as an analytical function or as a table of co-ordinates. In either case we can find values of the slope dy/dx as a function of θ. Conventional Fourier analysis is then used to isolate each coefficient A_n in turn.

1st × each side by $\cos m\theta$ and integrate w.r.t. θ

$$\int_0^\pi \frac{dy}{dx}\cos m\theta \, d\theta = -\frac{A_0}{2}\int_0^\pi \cos m\theta \, d\theta + \frac{1}{2}\sum_{n=1}^{\infty} A_n \int_0^\pi \cos n\theta \cos m\theta \, d\theta.$$

Only terms where m=n are non-zero integrals

$$\int_0^\pi \frac{dy}{dx}\cos m\theta \, d\theta = \frac{1}{2}A_n \cos^2 m\theta \, d\theta = \frac{1}{2}A_n \frac{\pi}{2}. \Rightarrow \boxed{A_n = \frac{4}{\pi}\int_0^\pi \frac{dy}{dx}\cos m\theta \, d\theta}$$

When m=0 [$\cos 0\theta = 1$]:

$$\int_0^\pi \frac{dy}{dx}\, d\theta = -\frac{1}{2}A_0 \, d\theta = -\frac{1}{2}A_0 \pi. \Rightarrow \boxed{A_0 = -\frac{2}{\pi}\int_0^\pi \frac{dy}{dx}\, d\theta}$$

Remember we can use these coefficients to determine the circulation:

$$\frac{\gamma(\mu)}{U} = A_0 \cot\frac{\mu}{2} + \sum_{n=1}^{\infty} A_n \sin(n\mu).$$

Using circulation to calculate Lift

The circulation is $\gamma(\mu) = UA_0 \cot\frac{\theta}{2} + U\sum_{n=1}^{\infty} A_n \sin(n\theta), \quad x = \frac{c}{2}(1-\cos\theta).$

$$\text{Lift } l = \rho U \int_0^C \gamma(x).dx = \frac{\rho U^2 c}{2}\left[A_0 \int_0^\pi \cot\left(\frac{\theta}{2}\right).\sin\theta \, d\theta + \sum_{n=1}^{\infty} A_n \int_0^\pi \sin(n\theta).\sin\theta \, d\theta\right]$$

Only n=1 will be non-zero

$$\text{Lift } l = \frac{\rho U^2 c}{2}\left[A_0 \int_0^\pi (1+\cos\theta)d\theta + \int_0^\pi A_1 \sin^2\theta \, d\theta\right]$$

The Lift per unit span is therefore only made up of 2 terms:

$$\text{Lift } l = \frac{\pi\rho U^2 c}{2}\left[A_0 + \frac{A_1}{2}\right] \qquad or \qquad C_l = \frac{2l}{\rho U^2 c} = \pi\left[A_0 + \frac{A_1}{2}\right].$$

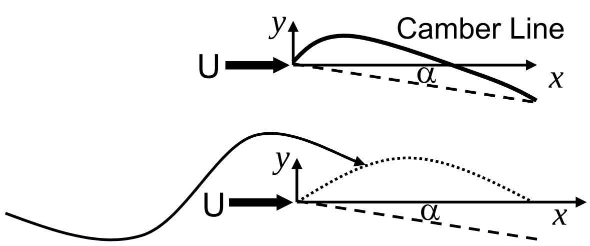
Significance:

For any Camber line

C_l depends on: A_0 **(=2α),** the incidence effect,

and A_1 the <u>fundamental component of the camber line</u>

(Note No lift when $\alpha = -A_1/4$)



Pitching Moment about Leading Edge

From lecture 1, if we define x at the leading edge = 0 then $m_0 = -\rho U \int_0^C \gamma(x)(x-0)\mathrm{d}x.$

Once again the circulation is $$\gamma(\mu) = UA_0 \cot\frac{\theta}{2} + U\sum_{n=1}^{\infty} A_n \sin(n\theta), \quad x = \frac{c}{2}(1-\cos\theta).$$

Hence: $$m_0 = -\rho U \int_0^{\pi} \left(UA_0 \cot\frac{\theta}{2} + U\sum_{n=1}^{\infty} A_n \sin(n\theta) \right) \frac{c}{2}(1-\cos\theta)\frac{c}{2}\sin\theta \mathrm{d}\theta.$$

$$\Rightarrow m_0 = -\frac{\rho U^2 c^2}{4}\left[\int_0^{\pi} A_0 \cot\frac{\theta}{2}\sin\theta(1-\cos\theta)\mathrm{d}\theta + \sum_{n=1}^{\infty} A_n \int_0^{\pi} \sin(n\theta)(1-\cos\theta)\sin\theta \mathrm{d}\theta \right]$$

Substitute cot (½ θ) (1 – cosθ) sinθ = sin²θ and

sin(nθ) (1 – cosθ) sinθ = sin (nθ) sinθ - ½ sin(nθ) sin(2θ)

$$m_0 = -\frac{\rho U^2 c^2}{4}\left[\int_0^{\pi} A_0 \sin^2\theta \,\mathrm{d}\theta + \sum_{n=1}^{\infty} A_n \int_0^{\pi} \sin(n\theta)(\sin\theta)\mathrm{d}\theta - \frac{1}{2}\int_0^{\pi} \sin(n\theta)\sin(2\theta)\mathrm{d}\theta \right]$$

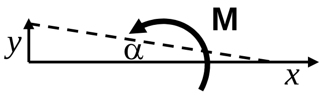
$$m_0 = -\frac{\rho U^2 c^2}{4}\left[A_0\frac{\pi}{2} + A_1\frac{\pi}{2} - A_2\frac{\pi}{4} \right] = -\frac{\pi\rho U^2 c^2}{8}\left[A_0 + A_1 - \frac{A_2}{2} \right]$$

Pitching Moment about Leading Edge

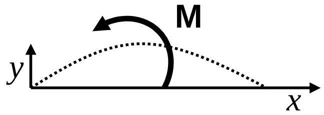
We can define a pitching moment coefficient about the leading edge C_{mo}:

$$m_0 = -\frac{\pi\rho U^2 c^2}{8}\left[A_0 + A_1 - \frac{A_2}{2} \right] \Rightarrow C_{m_0} = \frac{m_0}{\left(\frac{\rho U^2 c^2}{2}\right)} = -\frac{\pi}{4}\left[A_0 + A_1 - \frac{A_2}{2} \right].$$

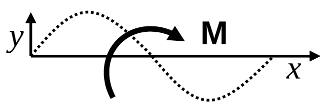
C_m depends on:

A_0 

A_0=2a

plus A_1 

Fundamental

plus A_2 

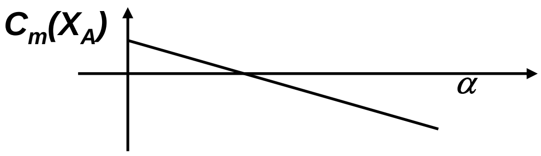
First Harmonic

The Aerodynamic Centre

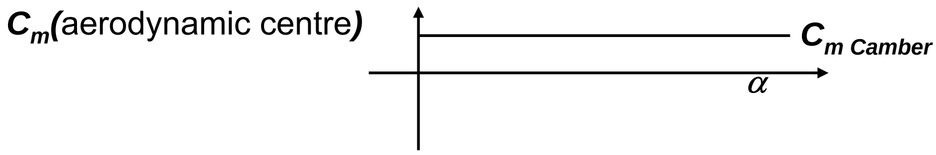
About any point $x=x_A$, the pitching moment coefficient $\boldsymbol{C_m}$ could be written as

$$C_m = C_{m\ Flat\text{-}Plate} + C_{m\ Camber}.$$

This varies with α, but only $C_{m\ Flat\text{-}Plate}$ varies with α, not $C_{m\ Camber}$.



Hence if we put $C_m = C_m(x/4)$ where $C_{m\ Flat\text{-}Plate} = 0$, C_m is constant with α. This defines the aerodynamic centre.



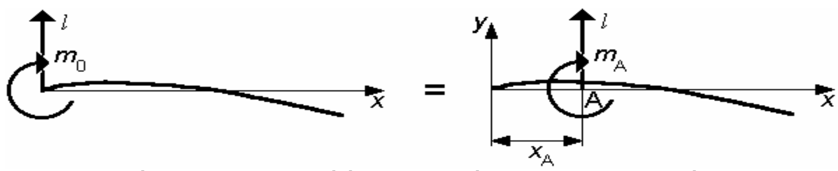
Aerodynamic centre: the point on the chord line where C_m is independent of the angle of incidence α.

This point is very useful in calculations of wing stability etc. Real aerofoils have Aerodynamics centre very close to $\boldsymbol{x=c/4}$.

The Aerodynamic Centre

More rigorously, we can define the aerodynamic centre as the point A on the chord line where C_m is invariant with changing lift.

$$\frac{\mathrm{d}C_{mA}}{\mathrm{d}C_l} = 0$$



The equivalence illustrated in the diagram may be expressed by equating moments about A:

$m_A = m_0 + l\, x_A$, or dividing by ½ $\rho\, U^2 c^2$: $C_{mA} = C_{m0} + C_l x_A/c$.

Differentiate w.r.t. C_l

$$\frac{\mathrm{d}C_{mA}}{\mathrm{d}C_L} = \frac{\mathrm{d}C_{m0}}{\mathrm{d}C_L} + \frac{x_A}{c}, \text{if A} = \text{aerodynamic centre}, \frac{\mathrm{d}C_{mA}}{\mathrm{d}C_L} = 0, \text{hence}, \frac{x_A}{c} = -\frac{\mathrm{d}C_{m0}}{\mathrm{d}C_l}.$$

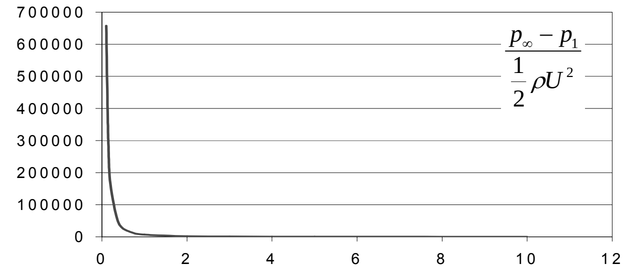
We know that $C_{m_0} = -\frac{\pi}{4}\left[A_0 + A_1 - \frac{A_2}{2}\right]$ and $C_l = \pi\left[A_0 + \frac{A_1}{2}\right]$. It is also noted for a rigid aerofoil shape, only A_0 may vary with incidence and lift. All other coefficients are constants.

So $$\frac{x_A}{c} = -\frac{\mathrm{d}C_{m0}/\mathrm{d}A_0}{\mathrm{d}C_l/\mathrm{d}A_0} = \frac{\pi/4}{\pi} = \frac{1}{4}.$$

The result is as before, the aerodynamic centre lies at the quarter chord point.

The effect of A_0 on pressure distribution and the risk of stalling

The **cot (½ θ)** term in the general circulation distribution is infinite at the leading edge. This is the term which adds incidence to any aerofoil or which, on its own, represents a flat plate at incidence $A_0 / 2$. We now consider what it does to the surface pressure distribution.



In Lecture 1, Bernoulli's equation was applied to determine the pressure difference between the upper and lower surfaces. We now apply the same equation to find the upper surface pressure relative to the undisturbed pressure datum

$$p_1 + \tfrac{1}{2}\rho\,(U + \Delta u)^2 = p_\infty + \tfrac{1}{2}\rho\, U^2$$

so that $$(p_\infty - p_1)/(\tfrac{1}{2}\rho U^2) = 2\Delta U/U + (\Delta U/U)^2$$

In Lecture 1 the supervelocity Δu was shown to be $\Delta u = \gamma(x)/2$

while for this discussion $\gamma(x)/U = A_0 \cot(\tfrac{1}{2}\theta)$.

Combining these, we show that the local pressure is strongly related to the circulation function.

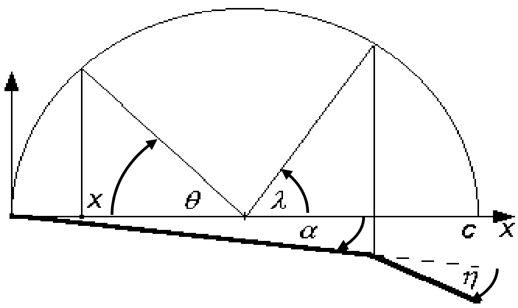
$$\frac{p_\infty - p_1}{\frac{1}{2}\rho U^2} = A_0 \cot\left(\frac{\theta}{2}\right) + \frac{1}{2}\left[A_0 \cot\left(\frac{\theta}{2}\right)\right]^2$$

The diagram shows there is a steep adverse pressure gradient and a serious risk that the upper surface boundary layer will separate, causing the aerofoil to stall. Hence **we must try to keep A_0 as low as possible.**

Aerodynamic Controls: The trailing edge flap

Movable flaps are used for aerodynamic controls and also to increase wing camber on take-off and landing.

A simple example we will model is a flap at the end of a flat plate. This illustrates the analysis techniques involved, and also provides an exercise in the Fourier analysis.



Loading Function: $$\frac{\gamma(\theta)}{U} = A_0 \cot\frac{\theta}{2} + \sum_{n=1}^{\infty} A_n \sin(n\theta).$$

Slope: $$\frac{dy}{dx} = -\frac{A_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} A_n \cos n\theta.$$

Fourier Coefficients:

Geometry:

Incidence = α

Flap deflection = η

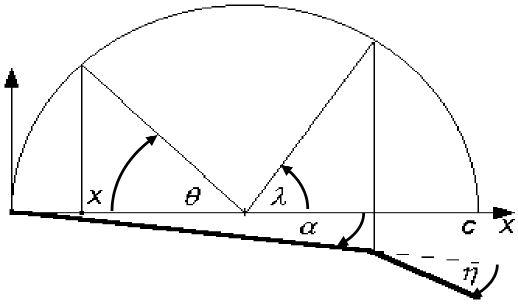
$x = \tfrac{1}{2} c\,(1 - \cos\theta)$

At the hinge $\theta = \pi - \lambda$

$$A_0 = -\frac{2}{\pi}\int_0^{\pi} \frac{dy}{dx}\, d\theta = -\frac{2}{\pi}\left[(-\alpha)(\pi - \lambda) + (-\alpha - \eta)(\pi - (\pi - \lambda))\right] \Rightarrow A_0 = \frac{2}{\pi}\left[\alpha + \lambda/_{\pi}\,\eta\right]$$

Aerodynamic Controls: The trailing edge flap

Fourier Coefficients not A_0:



$$A_n = \frac{4}{\pi}\int_0^{\pi} \frac{\mathrm{d}y}{\mathrm{d}x}\cos m\theta \,\mathrm{d}\theta$$

$$A_n = \frac{4}{\pi}\int_0^{\pi-\lambda} (-\alpha)\cos m\theta \,\mathrm{d}\theta + \int_{\pi-\lambda}^{\pi} (-\alpha-\eta)\cos m\theta \,\mathrm{d}\theta$$

$$A_n = -\frac{4}{\pi}\left[\underbrace{\int_0^{\pi} \alpha \cos m\theta \,\mathrm{d}\theta}_{\to 0} + \int_{\pi-\lambda}^{\pi} \eta \cos m\theta \,\mathrm{d}\theta\right]$$

$$A_n = -\frac{4\eta}{\pi} \times \frac{\sin m\pi - \sin m(\pi-\lambda)}{m} = -\frac{4\eta}{m\pi}\left[\underbrace{\sin m\pi - \sin m\pi \cos m\lambda}_{0} + \cos m\pi \sin m\lambda\right]$$

$$A_n = -\frac{4\eta}{m\pi}(-1)^m \sin m\lambda$$

$$\boxed{A_0 = \frac{2}{\pi}\left[\alpha + \lambda/_{\pi}\,\eta\right] \qquad A_1 = +\frac{4\eta}{\pi}\sin\lambda \qquad A_2 = -\frac{2\eta}{\pi}\sin 2\lambda \qquad A_3 = \frac{4\eta}{3\pi}\sin 3\lambda}$$

Aerodynamic Controls: The trailing edge flap

Lift Coefficient:

$$C_l = \pi\left(A_0 + \frac{A_1}{2}\right) = 2\pi\left(\alpha + \frac{\lambda}{\pi}\eta\right) + 2\eta\sin\lambda \quad or \qquad C_l = 2\pi\alpha + 2\eta(\lambda + \sin\lambda)$$

Flat Plate

Flap: linear relationship for given size

$$\alpha = \frac{C_l - 2\eta(\lambda + \sin\lambda)}{2\pi}$$

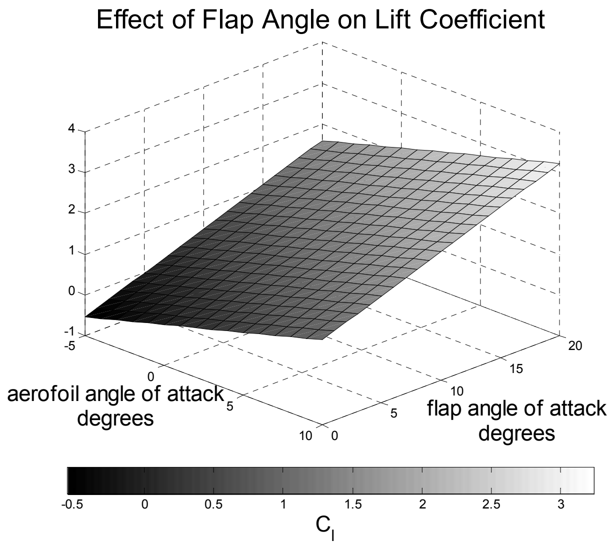
At same C_l flap deployment reduces α

Notes on results:

1/ The change in C_L is proportional to η, the flap angle

2/ The theory is only valid for small η, no separations etc.

3/ As this is potential flow we can superimpose the result for any camber line shape.



Aerodynamic Controls: The trailing edge flap

Pitching moment coefficient about Aerodynamic Centre:

$\boldsymbol{C_{mA}} = \boldsymbol{C_{m0}} + \boldsymbol{C_l} \times \frac{1}{4}$ (standard relationship see 2-8 to 2-10)

$\boldsymbol{C_{mA}} = -\pi/4[A_0+A_1-A_2/2] + \pi/4[A_0+A_1/2].$ (from 2-8and 2-6)

$\boldsymbol{C_{mA}} = -\pi/8[A_1-A_2].$

Substituting in coefficients for the flap : $C_{mA} = \frac{\pi}{8}(A_1 - A_2) = -\frac{\eta}{4}[2\sin\lambda + \sin 2\lambda]$

NB: The change in C_m is also proportional to η, the flap angle

λ is fixed for a given aerofoil geometry