B15
Question 1

1.

Consider the linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where A and B are matrices of appropriate dimension.

(a) For each of the following cases compute the zero input transition from an arbitrary initial state $x(0) = x_0$.

(i) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$;

(ii) $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$;

(iii) $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution:

(i) Matrix A is Nilpotent and $A^k = 0$ for $k \geq 2$. The state transition matrix is thus given by

$$\begin{aligned} \Phi(t) = e^{At} &= I + At + \frac{A^2t^2}{2!} + \dots \\ &= I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

The zero input transition is then given by $x(t) = \Phi(t)x_0$.

(ii) Matrix A is Nilpotent and $A^k = 0$ for $k \geq 3$, while $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. The state transition matrix is thus given by

$$\begin{aligned} \Phi(t) = e^{At} &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots \\ &= I + At + \frac{A^2t^2}{2!} = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{t^2}{2} & t & 1 \end{bmatrix}. \end{aligned}$$

The zero input transition is then given by $x(t) = \Phi(t)x_0$.

(iii) Matrix A is such that

$$A^k = \begin{cases} I & \text{if } k\text{: even;} \\ A & \text{if } k\text{: odd,} \end{cases}$$

where I is a 2×2 identity matrix. The state transition matrix is thus given by

$$
\begin{aligned}
\Phi(t) = e^{At} &= I + At + \frac{A^2t^2}{2!} + \dots \\
&= I(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots) + A(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots) \\
&= I\cosh t + A\sinh t \\
&= \begin{bmatrix} \cosh t - \sinh t & \sinh t \\ 0 & \cosh t + \sinh t \end{bmatrix} \\
&= \begin{bmatrix} e^{-t} & \frac{e^t - e^{-t}}{2} \\ 0 & e^t \end{bmatrix},
\end{aligned}
$$

where the second last equality is due to the given series expansion in the hint. The zero input transition is then given by $x(t) = \Phi(t)x_0$.
Alternatively, as A is diagonalizable (since eigenvalues are distinct), one could compute the state transition matrix by means of $\Phi(t) = e^{At} = We^{\Lambda t}W^{-1}$, where Λ is a diagonal matrix with the eigenvalues on the diagonal and W has the eigenvectors as its columns. However, using the hint prevents from computing the eigenvectors.

[5 marks]

(b) For the system in part (a.i):

(i) Show that the system is controllable by computing the controllability Gramian.

(ii) Determine the minimum energy controller to transfer the state from $x(0) = x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x(1) = x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Solution:

(i) The controllability gramian is given by

$$
\begin{aligned}
W_c(t) &= \int_0^t e^{A\tau}BB^\top e^{A^\top\tau}d\tau \\
&= \int_0^t \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix} d\tau \\
&= \int_0^t \begin{bmatrix} \tau^2 & \tau \\ \tau & 1 \end{bmatrix} d\tau = \begin{bmatrix} \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{bmatrix}.
\end{aligned}
$$

Notice that $\det(W_c(t)) = \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12} \neq 0$ for all $t > 0$. Hence, $W_c(t)$ is invertible and as a result the system in part (a.i) is controllable.

(ii) The minimum energy controller to transfer the system's state from $x(0) = x_0$ to x_1 is given for any $\tau \in [0, 1]$ by

$$\begin{aligned} u(\tau) &= B^\top e^{A^\top(1-\tau)} W_c(1)^{-1} x_1 \\ &= 12 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1-\tau & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= 10 - 18\tau. \end{aligned}$$

[4 marks]

(c) For the system in part (a.ii):

(i) Show that the system is controllable by computing the controllability matrix.

(ii) Design a state feedback controller so that all eigenvalues of the closed loop system are placed at -1.

Solution:

(i) The controllability gramian is given by

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $\det(P) \neq 0$ (P is a square matrix), hence the system in part (a.ii) is controllable.

(ii) Placing the eigenvalues of the closed loop system at -1 results in the target characteristic polynomial $\lambda^3 + 3\lambda^2 + 3\lambda + 1$. Consider a feedback gain matrix $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ (row vector since we have a scalar input), that results in a closed loop matrix

$$A + BK = \begin{bmatrix} k_1 & k_2 & k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of the closed loop system is in turn given by

$$\det(\lambda I - (A + BK)) = \lambda^3 - k_1\lambda^2 - k_2\lambda - k_3.$$

Equating the coefficients of the target and the characteristic polynomial, we obtain that $k_1 = -3$, $k_2 = -3$ and $k_3 = -1$.

[4 marks]

(d) For the system in part (a.iii):

(i) Is the system stable, asymptotically stable, or unstable? Justify your answer.

(ii) If the output equation is

$$y(t) = Cx(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

show that the associated transfer function is given by

$$G(s) = \frac{1}{s+1}.$$

(iii) This transfer function has only one pole at -1. What does this imply about stability, and how does this compare with your answer in part (d.i)?

[3 marks]

Solution:

(i) Matrix A is triangular, hence the eigenvalues are its diagonal entries, i.e., -1 and 1. Both of them are real but one is positive, hence the (autonomous) system is unstable.

(ii) To show this, notice that the system's transfer function is given by

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \frac{1}{s(s+1)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{s+1}. \end{aligned}$$

(iii) The system has one negative real pole, hence the transfer function is stable. However, it can be computed that the system is uncontrollable (the controllability matrix is given by $P = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$), and there is a pole-zero cancellation of the term corresponding to the uncontrollable part. The system is hence not internally stable.

Hint: For the case of part (a.iii) you can use the fact that

$$\begin{aligned} \sinh t &= \frac{e^t - e^{-t}}{2} = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots \\ \cosh t &= \frac{e^t + e^{-t}}{2} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots . \end{aligned}$$

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Question 2

2.

Consider the following infinite horizon optimal control problem with $\mu > 0$:

$$\begin{aligned} &\text{minimize} \int_0^\infty \big(z(t)^2 + \mu u(t)^2\big)dt \\ &\text{subject to } \ddot{z}(t) = u(t), \text{ for all } t, \\ &\qquad\qquad z(0), \dot{z}(0) : \text{ given.} \end{aligned}$$

Let $y(t) = z(t) \in \mathbb{R}$ denote the output and $u(t) \in \mathbb{R}$ the input of the underlying system, respectively.

(a) (i) Write $\ddot{z}(t) = u(t)$, $y(t) = z(t)$ in state space form by determining matrices A, B, C and D such that

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where $x(t)$ is the state vector.

(ii) Determine matrices Q and R as a function of μ so that the cost criterion can be written in the form

$$\int_0^\infty \big(x(t)^\top Q x(t) + u(t)^\top R u(t)\big)dt.$$

[2 marks]

Solution:

(i) Setting the state vector as $x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}$, we have that the state space matrices are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ \text{and}\ \ D = 0.$$

(ii) By inspection of the cost criterion and the state vector defined above, we have that

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ \text{and}\ \ R = \mu.$$

(b) State and solve the algebraic Riccati equation associated with this infinite horizon linear quadratic regulation (LQR) problem.

[6 marks]

Solution:
Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ (notice that the off-diagonal terms are the same as P is symmetric) be the solution of the algebraic Riccati equation. We then have that

$$\begin{aligned}
& PA + A^\top P + Q - PBR^{-1}B^\top P = 0 \\
\Rightarrow & \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
& \qquad - \frac{1}{\mu} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = 0 \\
\Rightarrow & \begin{bmatrix} 1 - \frac{1}{\mu} p_2^2 & p_1 - \frac{1}{\mu} p_2 p_3 \\ \star & 2p_2 - \frac{1}{\mu} p_3^2 \end{bmatrix} = 0,
\end{aligned}$$

where the term denoted by $\star$ is equal to the second element of the first row due to symmetry. Equating each of the three distinct entries with zero, we obtain three equations with three unknowns, namely, p_1, p_2 and p_3. We thus have that

$$\begin{aligned}
1 - \frac{1}{\mu} p_2^2 = 0 \ &\Rightarrow \ p_2 = \mu^{\frac{1}{2}} \\
2p_2 - \frac{1}{\mu} p_3^2 = 0 \ &\Rightarrow \ p_3 = \sqrt{2}\mu^{\frac{3}{4}} \ \text{ or } \ p_3 = -\sqrt{2}\mu^{\frac{3}{4}} \\
p_1 - \frac{1}{\mu} p_2 p_3 = 0 \ &\Rightarrow \ p_1 = \sqrt{2}\mu^{\frac{1}{4}} \ \text{ or } \ p_1 = -\sqrt{2}\mu^{\frac{1}{4}},
\end{aligned}$$

where from the first equation we can only have a positive solution for p_2, as by the second one $p_2 = \frac{1}{2} p_3^2 \geq 0$. We then have two solutions:

$$P = \begin{bmatrix} \sqrt{2}\mu^{\frac{1}{4}} & \mu^{\frac{1}{2}} \\ \mu^{\frac{1}{2}} & \sqrt{2}\mu^{\frac{3}{4}} \end{bmatrix} \ \text{ or } \ P = \begin{bmatrix} -\sqrt{2}\mu^{\frac{1}{4}} & \mu^{\frac{1}{2}} \\ \mu^{\frac{1}{2}} & -\sqrt{2}\mu^{\frac{3}{4}} \end{bmatrix}.$$

Notice that only the first solution is positive semidefinite.

(c) Does the algebraic Riccati equation admit a unique positive semidefinite solution? If yes, justify whether this is anticipated.

[3 marks]

Solution: Notice that the penalty matrix on the state Q can be written as $Q = C^\top \bar{Q} C$, where $\bar{Q} = 1$, hence, it is positive definite. Moreover, pairs (A, B) and (A, C) are controllable and observable, respectively. To see this notice that

$$\begin{aligned}
\begin{bmatrix} B & AB \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ \Rightarrow \ \text{full rank}, \\
\begin{bmatrix} C \\ CA \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ \Rightarrow \ \text{full rank}.
\end{aligned}$$

Therefore, the algebraic Riccati equation is expected to admit a unique positive semidefinite solution (indeed that was the case in part (c)).

(d) Compute the optimal LQR controller and the poles of the closed loop system. Comment on the effect of $\mu \to \infty$ to the stability of the closed loop system.

[3 marks]

Solution:
Denote by $\bar{P}$ the unique positive semidefinite solution of the algebraic Riccati equation. The optimal LQR controller is then given by

$$\begin{aligned} u^\star(t) &= -R^{-1}B^\top \bar{P}x(t) \\ &= -\frac{1}{\mu}\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}\mu^{\frac{1}{4}} & \mu^{\frac{1}{2}} \\ \mu^{\frac{1}{2}} & \sqrt{2}\mu^{\frac{3}{4}} \end{bmatrix} x(t) \\ &= -\begin{bmatrix} \mu^{-\frac{1}{2}} & \sqrt{2}\mu^{-\frac{1}{4}} \end{bmatrix} x(t). \end{aligned}$$

The closed loop system is thus given by

$$\dot{x}(t) = Ax(t) + Bu^\star(t) = \begin{bmatrix} 0 & 1 \\ -\mu^{-\frac{1}{2}} & -\sqrt{2}\mu^{-\frac{1}{4}} \end{bmatrix} x(t).$$

The poles of the closed loop system correspond to the eigenvalues of the closed loop system matrix, which are given by the complex conjugate pair

$$\lambda_{1,2} = -\frac{1}{\sqrt{2}}\mu^{-\frac{1}{4}} \pm \frac{1}{\sqrt{2}}\mu^{-\frac{1}{4}}j.$$

Notice that the eigenvalues/poles have both negative real part, hence the system is asymptotically stable. As $\mu \to \infty$ then the poles tend to zero, hence the system becomes stable and not asymptotically stable. This is anticipated since as $\mu \to \infty$ the control effort becomes "expensive" in the cost criterion. As a result $u^\star(t)$ tends to the zero signal, and the closed loop system tends to the open loop one whose eigenvalues are both equal to zero.

(e) Consider the LQR optimal control problem with $Q \succ 0$ and dynamics

$$\dot{x}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} x(t) + \begin{bmatrix} -4 \\ 8 \end{bmatrix} u(t),\ x(0): \text{ given.}$$

Do you anticipate the optimal LQR cost to be finite in this case? Justify your answer.

[2 marks]

Solution:
This is an infinite horizon optimal control problem and the state is penalized in the cost as $Q \succ 0$. Hence the cost remains finite if the system is controllable. In this case the controllability matrix is given by $P = \begin{bmatrix} -4 & -12 \\ 8 & 24 \end{bmatrix}$. Its columns/rows are linearly dependent, hence the system is not controllable. As a result the cost is not expected to be finite.