B15 Kalman Filtering
Tutorial Questions

1. **Semi-autonomous braking system.** As part of the design of a semi-autonomous braking system you are asked to estimate the distance to the car in the front. Your experimental car is equipped with two sensing systems - an optical and an ultrasound distance measuring device. The sensors provide unbiased distance measurements $y_1(t)$ and $y_2(t)$ of the real distance $x(t)$, but are affected by zero-mean, Gaussian white noise processes $v_1(t)$ and $v_2(t)$, so that the measurement equation is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(t) + \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

where

$$E\left[\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \begin{bmatrix} v_1(t+\tau) & v_2(t+\tau) \end{bmatrix}\right] = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \delta(\tau)$$

An estimate $\hat{x}(t)$ of the actual distance is found by taking a linear combination of the two measurements, so that

$$\hat{x}(t) = \alpha y_1(t) + (1-\alpha)\, y_2(t)$$

Define the estimation error as $e(t) = x(t) - \hat{x}(t)$ and find the value of α that minimises $E\left[e(t)^\top e(t)\right]$, the variance of the estimation error. Note that in this question, there are no state dynamics

2. Now modify the model in Question 1 by including the dynamics of the system. Assume that the distance between the cars, $x(t)$ is constant, so that

$$\dot{x}(t) = 0$$

Note that $A = 0$ and $Q = 0$ for this model. Determine the (scalar) Riccati equation for this system and integrate to derive an expression for the covariance of the estimation error, $P(t)$. Use this to compute the (time varying) Kalman filter estimate. Comment on the result as $t \to \infty$.

3. **Voyager position estimation.** The Voyager 1 probe has now left our Solar system and is about 137AU (Astronomical Units) from Earth (http://voyager.jpl.nasa.gov

It is travelling at roughly constant speed but is subject to random perturbations due to external effects such as the gravitational pulls of passing asteroids. These effects can be modelled as a white noise, $w(t)$ with $E\left[w(t)w(t+\tau)^\top\right] = Q\,\delta(\tau)$. NASA's Deep Space Network of radio antennas can be used to estimate the velocity of the probe by utilising the Doppler effect, but the measurement is subject to error that can also be modelled as white noise $v(t)$ with $E\left[v(t)v(t+\tau)^\top\right] = R\,\delta(\tau)$. We will express the actual velocity of Voyager 1 as a deviation from a nominal velocity, which is calculated from knowledge of celestial dynamics and gravity. Choose the state $x(t)$ to represent the deviation of the velocity due to the effects of the random acceleration disturbances $w(t)$. Form the system dynamics by assuming that the Deep Space Network measurement is simply the deviation $x(t)$ corrupted by (additive) noise $v(t)$, and hence derive the time varying Kalman filter covariance matrix $P(t)$ and time varying Kalman gain $L(t)$. Comment on the behaviour of $P(t)$ and $L(t)$ as $t \to \infty$.

4. **DC Motor.** A D.C. motor driving an inertial load has a transfer function model

$$\frac{\Theta(s)}{U(s)} = \frac{1}{s\,(s+1)}$$

where $\Theta(s)$ is the Laplace transform of $\theta(t)$, the angle of the motor shaft and $U(s)$ is the Laplace transform of the input $u(t)$, which is the current applied to the motor.

A noisy measurement of the angular rotary is available

$$y(t) = \theta(t) + v(t)$$

where $E[v(t)\,v(t+\tau)] = 1\,\delta(\tau)$. In addition there is also an unknown torque disturbance $w(t)$ that causes perturbations to the angular acceleration perturbations where $= E[w(t)\,w(t+\tau)] = 1\,\delta(\tau)$. Derive the steady-state Kalman filter gain by solving the Riccati equation.

Note. It may be easier to find the steady state Kalman gain by using MATLAB to calculate the solution to the algebraic Riccati equation. The MATLAB function `K = lqr(A,B,Q,R)` finds the solution to the algebraic Riccati equation for the Linear Quadratic Regulator, but by exploiting the duality of the

LQR and the Kalman filter, the same function can be used to find the steady state solution to the Riccati equation for the Kalman filter.

5. **LQG Design.** The landing-approach configuration of the F16 aircraft has unstable longitudinal dynamics given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} -0.0507 & -3.861 & 0-32.17 & \\ -0.00117 & -0.5164 & 1 & 0 \\ -0.000129 & 1.4168 & -0.4932 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -0.0717 \\ -0.1645 \\ 0 \end{bmatrix} \delta_e(t)$$

where the state vector is

$$x(t) = [\delta_u(t) \;\; \alpha(t) \;\; q(t) \;\; \theta(t)]^\top$$

and the input is $\delta_e(t)$, which is the input to the elevator. The measurement is $\theta(t)$, which is the pitch angle that is obtained from the aircraft's attitude and heading reference system(AHRS), so that

$$y(t) = [0 \;\; 0 \;\; 0 \;\; 1]\, x(t) + v(t)$$

with $E\left[v(t)v(t+\tau)\right] = 1\,\delta(\tau)$.

(a) Using the `K = lqr(A,B,Q,R)` function in MATLAB, or by other means, design a full-state feedback controller that stabilises the dynamics, using the following (infinite horizon) cost function

$$J = \int_0^\infty \left(10^{-3}\delta_u^2(t) + 10^{-3}\alpha^2(t) + q^2(t) + 10^{-3}\theta^2(t)\right) + \delta_e^2(t)\ \mathrm{d}t$$

Note that the MATLAB function `lqr` gives $-K$ (not K).

(b) Since the full state is not available, it is necessary to design a Kalman filter to estimate the states. The aircraft is subject to wind gusts that affect only $\delta_u(t)$, which can be modelled as zero mean Gaussian white noise with covariance $E[w(t)w(t+\tau)^\top] = 1\,\delta(\tau)$. Find the corresponding Kalman filter gain vector.

(c) Compute the corresponding LQG controller transfer function from the measurement to the elevator input $\delta_e(t)$.

6. **Discrete-time Model**. The dynamics of the position of a mass attached to a damper are described by

$$m\ddot{p}(t) = -c\dot{p}(t) + u(t)$$

where $p(t)$ is the position, m is the mass, c is the damping coefficient and $u(t)$ is the force applied to the mass.

   (a) Using the state $x(t) = [p(t) \quad \dot{p}(t)]^\top$, write down a state space model in the form

   $$\dot{x}(t) = Ax(t) + Bu(t)$$

   for the case where $m = 1$ and $c = 1$ and show that A is singular.

   (b) When the input is generated by a zero-order hold, the discrete-time model of the state with sample time T becomes

   $$x_{k+1} = A_\mathrm{d} x_k + B_\mathrm{d} u_k$$

   where

   $$A_\mathrm{d} = \mathrm{e}^{AT} \qquad\qquad B_\mathrm{d} = A^{-1}\left(\mathrm{e}^{AT} - I\right)B$$

   For the matrix A, find the matrix of eigenvalues, Λ, and the matrix of eigenvectors, V, such that $A = V\,\Lambda\,V^{-1}$, and determine A_d for the case where $T = 1$.

   (c) Because A is singular, B_d cannot be calculated directly. By showing that

   $$B_\mathrm{d} = V \begin{bmatrix} \frac{\mathrm{e}^{\lambda_1 T}-1}{\lambda_1} & 0 \\ 0 & \frac{\mathrm{e}^{\lambda_2 T}-1}{\lambda_2} \end{bmatrix} V^{-1} B$$

   and considering the limit as one of the eigenvalues approaches zero, calculate B_d.

7. **Discrete Random Walk** The state space model of a discrete random walk is given by

$$x_{k+1} = x_k + w_k$$
$$y_k = x_k + v_k$$

where $x_k \in \mathbb{R}$ is the location, with $E[w_k w_{k+\ell}] = Q\,\delta_\ell$ and $E[v_k v_{k+\ell}] = R\,\delta_\ell$

(a) Write down the Kalman filter to create the state estimate $\hat{x}_k$ for this system in the form of a prediction and measurement update step

(b) For the case where $Q = 1$ and $R = 1$, use MATLAB to simulate the system and the Kalman filter to show that $\hat{x}_k$ tracks x_k. Use the `randn` function in MATLAB to generate Gaussian white noise

8. **Extended Kalman filter.** At $t = kT$, the state of a simplified model of the movement of a robot vehicle can be described in terms of its location (X_k, Y_k) and its heading ϕ_k, so that $x_k = [X_k \;\; Y_k \;\; \phi_k]^\top$. The input to the robot is its velocity, which is described by its speed V_k and tangential angle ψ_k, so that $u(t) = [V_k \;\; \psi_k]^\top$.

   The motion of the vehicle can be described by the nonlinear, discrete-time model
$$\begin{bmatrix} X_{k+1} \\ Y_{k+1} \\ \phi_{k+1} \end{bmatrix} = \begin{bmatrix} X_k + T\,V_k \cos\left(\phi_k + \psi_k\right) \\ Y_k + T\,V_k \sin\left(\phi_k + \psi_k\right) \\ \phi_k + \frac{T\,V_k}{D} \sin\psi_k \end{bmatrix} + w_k$$
   where T is the time between samples, D is the (fixed) distance between the wheels of the robot and $w_k \in \mathbb{R}^3$ is the process noise.

   The measurement is provided by the range (distance) r_k and bearing (angle) θ_k to a fixed beacon at $(X_\mathrm{B}, Y_\mathrm{B})$, so that
$$\begin{bmatrix} r_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} \sqrt{(X_\mathrm{B} - X_k)^2 + (Y_\mathrm{B} - Y_k)^2} \\ \tan^{-1}\left(\frac{Y_\mathrm{B} - Y_k}{X_\mathrm{B} - X_k}\right) - \phi_k \end{bmatrix} + v_k$$
   where $v_k \in \mathbb{R}^2$ is the measurement noise

   By expressing the model in the form
$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + w_k \\ y_k &= h(x_k, u_k) + v_k \end{aligned}$$
   derive an Extended Kalman Filter for this system

Answers and Hints

1. $\alpha = \frac{R_2}{R_1+R_2}$

2. $P(t) = \frac{P(0)}{1+P(0)t\left(\frac{1}{R_1+R_2}\right)}$

3. $P(t)$ satisfies the Riccati equation $\dot{P}(t) = Q - \frac{P(t)^2}{R}$. Integrate this to find $P(t)$

4. $L = (0.7321 \;\; 0.2679)^\top$

5. $K = (-0.0297 \;\; 7.3550 \;\; 8.0564 \;\; 3.5987)$, $L = (-64.7058 \;\; 0.9176 \;\; 1.0689 \;\; 1.4621)^\top$

6.
$$A_\mathrm{d} = \begin{bmatrix} 1 & 0.6321 \\ 0 & 0.3679 \end{bmatrix} \qquad B_\mathrm{d} = \begin{bmatrix} 0.3679 \\ 0.6321 \end{bmatrix}$$