A10586W1

TRINITY TERM 2021

SECOND PUBLIC EXAMINATION

Honour School of Engineering Science

CONTROL SYSTEMS (Paper B15)

Monday 07 June 2021 Opening Time: 09:30am UK Time

Mode of Completion: Handwritten

Answers to not more than **THREE** *questions should be submitted.*

Start each question on a new page. You must upload your answer files into the submission box for the relevant question. Questions with sub-sections (e.g. Q1a & Q1b) should all be included in the answer file for the corresponding overall question number and uploaded into the submission box for the relevant question.

You have 1 hour and 30 minutes writing time to complete the paper and up to 30 minutes technical time to upload your answer files.

Note that:

- *The approximate allocation of marks is given in the margin.*
- *You are permitted to use the following material(s):*
- *Engineering Tables & Data (HLT) - Candidate to provide.*

1. Consider the linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \ ,$$

where A and B are matrices of appropriate dimension.

(a) For each of the following cases compute the zero input transition from an arbitrary initial state $x(0) = x_0$.

(i) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$;

(ii) $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$;

(iii) $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

[5 marks]

(b) For the system in part (a.i):

(i) Show that the system is controllable by computing the controllability Gramian.

(ii) Determine the minimum energy controller to transfer the state from $x(0) = x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x(1) = x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

[4 marks]

(c) For the system in part (a.ii):

(i) Show that the system is controllable by computing the controllability matrix.

(ii) Design a state feedback controller so that all eigenvalues of the closed loop system are placed at -1.

[4 marks]

(d) For the system in part (a.iii):

(i) Is the system stable, asymptotically stable, or unstable? Justify your answer.

(ii) If the output equation is

$$y(t) = Cx(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \ ,$$

show that the associated transfer function is given by

$$G(s) = \frac{1}{s+1} \ .$$

(iii) This transfer function has only one pole at -1. What does this imply about stability, and how does this compare with your answer in part (d.i)?

Hint: For the case of part (a.iii) you can use the fact that

$$\sinh t = \frac{e^{t} - e^{-t}}{2} = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots$$
$$\cosh t = \frac{e^{t} + e^{-t}}{2} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots .$$

[3 marks]

2. Consider the following infinite horizon optimal control problem with $\mu > 0$:

$$\begin{aligned} &\text{minimize} \int_0^\infty (z(t)^2 + \mu u(t)^2)dt \\ &\text{subject to } \ddot{z}(t) = u(t), \text{ for all } t, \\ &\qquad z(0), \dot{z}(0): \text{ given.} \end{aligned}$$

Let $y(t) = z(t) \in \mathbb{R}$ denote the output and $u(t) \in \mathbb{R}$ the input of the underlying system, respectively.

(a) (i) Write $\ddot{z}(t) = u(t)$, $y(t) = z(t)$ in state space form by determining matrices A, B, C and D such that

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \ , \\ y(t) &= Cx(t) + Du(t) \ , \end{aligned}$$

where $x(t)$ is the state vector.

(ii) Determine matrices Q and R as a function of μ so that the cost criterion can be written in the form

$$\int_0^\infty (x(t)^\top Q x(t) + u(t)^\top R u(t))dt \ .$$

[2 marks]

(b) State and solve the algebraic Riccati equation associated with this infinite horizon linear quadratic regulation (LQR) problem.

[6 marks]

(c) Does the algebraic Riccati equation admit a unique positive semidefinite solution? If yes, justify whether this is anticipated.

[3 marks]

(d) Compute the optimal LQR controller and the poles of the closed loop system. Comment on the stability of the closed-loop system as $\mu \to \infty$.

[3 marks]

(e) Consider an LQR optimal control problem with $Q \succ 0$ and dynamics

$$\dot{x}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} x(t) + \begin{bmatrix} -4 \\ 8 \end{bmatrix} u(t), \ x(0): \ \text{given}$$

Do you anticipate the optimal LQR cost to be finite in this case? Justify your answer.

[2 marks]

3. You are a designer in the *IntraCity EVTOL* company and have been asked to design an integrated flight control and propulsion system for their latest quadcopter design. Your colleague in flight physics had already worked out that the pitch axis dynamics are approximately given by

$$I_{yy}\dot{q}(t) = lu(t) + I_c d(t) \ ,$$

where $q(t)$ is the pitch rate, $u(t)$ is the thrust differential and $d(t)$ models the gyroscopic torque disturbance due to the rotational coupling of the roll and yaw axes. The terms I_{yy} is the pitch axis inertia, l is the distance between two rotors on the pitch axis and I_c is the inertia coupling. The disturbance takes the form

$$d(t) = \int w(t)\mathrm{d}t$$

where $w(t)$ is a zero mean random process with variance Q.

(a) (i) Taking the pitch rate as the output from the system and defining the state as $x(t) = [q(t)\ d(t)]^{\mathrm{T}}$, derive a state-space model of the system.

(ii) Show that the system is not completely controllable but that it is observable.

[4 marks]

(b) A sensor provides a measurement of the pitch rate. The sensor is subject to zero-mean, white noise whose variance is R. The on-board Inertial Measurement Unit (IMU) uses a Kalman filter to provide an estimate of the pitch rate, $\hat{q}(t)$. For $Q = 1$, $R = 1$, $l = 4$, $I_{yy} = 2000$ and, $I_c = 2000$, derive an expression for L (do not use MATLAB), the steady state Kalman filter gain matrix.

[8 marks]

(c) It is necessary to track the pitch rate command q_c with zero steady state error. Sketch a control scheme that achieves this without using an additional integrator.

[4 marks]

4. (a) State the small gain theorem.

[2 marks]

(b) A system is described by the transfer function

$$\tilde{P}(s) = \frac{s + a}{s^2 + 2s + 1}$$

where $0.8 \le a \le 1.2$ is an unknown parameter.
Find $W(s)$ so that this system can be expressed as $\tilde{P}(s) = P(s) + \Delta W(s)$, where $-1 \le \Delta \le 1$ and

$$P(s) = \frac{s + 1}{s^2 + 2s + 1} \quad .$$

[2 marks]

(c) The system is controlled using feedback as in Figure 1, where $C(s)$ is the transfer function of the controller. Show that the closed loop system can be rearranged to the form in Figure 2, where $S(s)$ is the sensitivity transfer function.

[6 marks]

(d) Suppose that $C(s) = K$, where $K > 0$ is a proportional gain. Using your solution from part (b), show that the closed-loop system is stable for all possible values of a.

[6 marks]

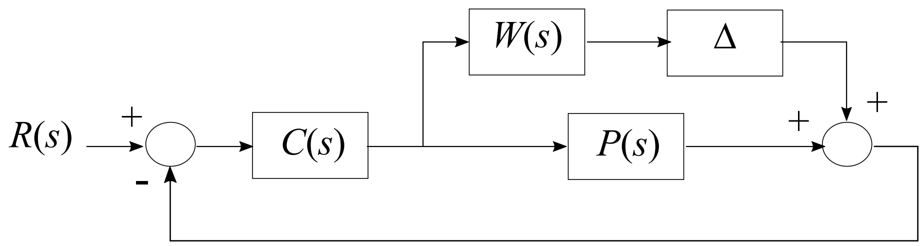

Figure 1

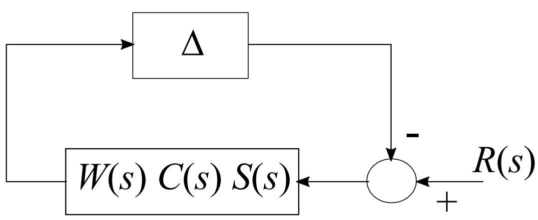

Figure 2