1. Consider the following linear, time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha^2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \alpha + 1 \end{bmatrix}, \text{ and } \alpha \leq 0.$$

(a) Compute the eigenvalues and eigenvectors of A as a function of parameter α. For which values of α is matrix A diagonalizable?

[2 marks]

(b) (i) Show that for $\alpha < 0$ the state transition matrix is given by

$$e^{At} = \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} (\alpha^2 - \alpha)e^{\alpha t} & e^{\alpha^2 t} - e^{\alpha t} \\ 0 & (\alpha^2 - \alpha)e^{\alpha^2 t} \end{bmatrix}.$$

(ii) Compute the state transition matrix for the case where $\alpha = 0$.

[3 marks]

(c) Let $u(t) = 0$ for all t. For which values of α is the system stable? Justify your answer.

[3 marks]

(d) Let $\alpha = 0$ and $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Explain why the following claim is true: There exists an input $u(t)$ that can drive the state of the system from $x(0)$ to $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Compute one such input that achieves the desired state transfer.

[5 marks]

(e) Let $\alpha = -1$ and $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Does there exists an input $u(t)$ that can drive the state of the system from $x(0)$ to $x(1) = \begin{bmatrix} e^{-1} \\ 0 \end{bmatrix}$? Justify your answer.
Hint: Compute the state solution $x(t)$ for $\alpha = -1$.

[3 marks]

Solution:

(a) Since A is a triangular matrix, the eigenvalues correspond to the elements on the diagonal. Hence, the eigenvalues are $\lambda_1 = \alpha$ and $\lambda_2 = \alpha^2$. If $\alpha < 0$, then the eigenvalues are distinct, hence A is diagonalizable. The eigenvectors in this case, computed by means of $Aw_i = \lambda_i w_i$, for $i = 1, 2$, are given by

$$w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } w_2 = \begin{bmatrix} 1 \\ \alpha^2 - \alpha \end{bmatrix}.$$

If $\alpha = 0$ (same conclusion also holds if $\alpha = 1$ had we allowed for positive values of α), then we have a repeated eigenvalue since $\lambda_1 = \lambda_2$, thus the algebraic eigenvalue multiplicity is 2. However, we only have one eigenvector, namely, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This implies that the geometric eigenvalue multiplicity is 1, hence strictly smaller than the algebraic one, implying that A in this case is non-diagonalizable.

[2 marks]

(b) (i) Case $\alpha < 0$: By part (a), for negative values of α matrix A is diagonalizable, hence we can decompose it as

$$A = W\Lambda W^{-1}, \quad \text{with } \Lambda = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix} \text{ and } W = \begin{bmatrix} 1 & 1 \\ 0 & \alpha^2 - \alpha \end{bmatrix},$$

where Λ is a diagonal matrix whose diagonal entries are the eigenvalues, and W is a matrix whose columns are the eigenvectors. Note that W is invertible, as its column (the eigenvectors) are linearly independent as an effect of A being diagonalizable.

Therefore, the state transition matrix, given by the matrix exponential, is

$$\begin{aligned} e^{At} = We^{\Lambda t}W^{-1} &= \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} 1 & 1 \\ 0 & \alpha^2 - \alpha \end{bmatrix} \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\alpha^2 t} \end{bmatrix} \begin{bmatrix} \alpha^2 - \alpha & -1 \\ 0 & 1 \end{bmatrix} \\ &= \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} (\alpha^2 - \alpha)e^{\alpha t} & e^{\alpha^2 t} - e^{\alpha t} \\ 0 & (\alpha^2 - \alpha)e^{\alpha^2 t} \end{bmatrix}. \end{aligned}$$

(ii) Case $\alpha = 0$: By part (a), matrix A is non-diagonalizable, however, it is Nilpotent and is given by $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This implies that $A^k = 0$ for all $k \geq 2$. The state transition matrix in this case is given by the truncated series expansion of the matrix exponential, i.e.,

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

[3 marks]

(c) We distinguish two cases. For any $\alpha < 0$ (in fact for any $\alpha \neq 0$ is positive values of α were allowed), $\lambda_2 = \alpha^2 > 0$. As such, since one of the eigenvalues is real and positive, the system is unstable.

For the case where $\alpha = 0$, and since $u(t) = 0$, the state solution is given by

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0).$$

There exist initial conditions such that $\lim_{t\to\infty} \|x(t)\| = \infty$. To see this, notice that for $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have that $x(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$. Hence, the system is unstable also when $\alpha = 0$.

Overall, the autonomous system is unstable for any $\alpha \leq 0$ (in fact for any $\alpha \in \mathbb{R}$).

[3 marks]

(d) For $\alpha = 0$ we have that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore, the system is controllable since the controllability matrix

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

is full rank. Hence, indeed there exists a control input trajectory $u(t)$ that can drive the state from $x(0)$ to $x(1)$. Once such controller is the minimum energy one. To compute it, consider first the controllability gramian

$$\begin{aligned} W_c(t) &= \int_0^t e^{A\tau} B B^\top e^{A^\top \tau} d\tau \\ &= \int_0^t \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix} \\ &= \int_0^t \begin{bmatrix} \tau^2 & \tau \\ \tau & 1 \end{bmatrix} = \begin{bmatrix} \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{bmatrix}, \end{aligned}$$

where we used the fact that $e^{A\tau} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$ from part (b), case (ii). The minimum energy controller, for any $\tau \in [0, t]$, is given then by

$$\begin{aligned} u(\tau) &= B^\top e^{A^\top (t-\tau)} W_c(t)^{-1} x(1) \\ &= \frac{12}{t^4} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t-\tau & 1 \end{bmatrix} \begin{bmatrix} t & -\frac{t^2}{2} \\ -\frac{t^2}{2} & \frac{t^3}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 12t(\frac{t}{2} - \tau). \end{aligned}$$

Since we want to steer the system to $x(1)$ at $t = 1$, the minimum energy controller (and hence one controller) to achieve this is $u(\tau) = 6 - 12\tau$. The time variable is arbitrary, hence we can equivalently write it as $u(t) = 6 - 12t$.

[5 marks]

(e) For $\alpha = -1$, we have $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. As such, the control input does not influence the state evolution of the system. The latter coincides then with the zero input transition, i.e.,

$$x(t) = e^{At} x(0) = \frac{1}{2} \begin{bmatrix} 2e^{-t} & e^t - e^{-t} \\ 0 & 2e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}.$$

where for e^{At} we used the expression of part (b), case (i), with $\alpha = -1$. From the state solution at $t = 1$, we obtain $x(1) = \begin{bmatrix} e^{-1} \\ 0 \end{bmatrix}$. Since this result is independent of the choice of $u(t)$ (recall that the input in this case does not influence the state), the desired state transfer is possible for any control input.

[3 marks]

2. Consider the following linear, time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t),$$

where

$$A = \begin{bmatrix} -10 & 0 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; \text{and} \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(a) From the state space representation of the system determine which states are controllable and which are observable.

[2 marks]

(b) Compute a gain matrix K of a state feedback controller $u(t) = Kx(t)$ such that the eigenvalues of the closed-loop system are -5 and -10.

[3 marks]

(c) Consider an infinite horizon linear quadratic regulator (LQR) problem

$$\begin{aligned} \text{minimize} \quad & \int_0^\infty (x(t)^\top Q x(t) + u(t)^\top u(t))dt \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu(t), \quad \text{for all } t, \\ & x(0) = x_0 : \text{ given}, \end{aligned}$$

where A and B are the given state space matrices, and $Q \geq 0$ is a matrix of appropriate dimension. Determine a matrix Q so that $K = \begin{bmatrix} 0 & -2 \end{bmatrix}$ is the gain matrix of the optimal LQR controller.

Hint: Use the given K and the expression of the optimal LQR controller to determine the structure of a matrix $P > 0$ that constitutes the solution of the associated algebraic Riccati equation.

[6 marks]

(d) Consider another state space representation with the same A and C matrices, but with $B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Compute the LQR controller for the problem of part (c) if $Q = \begin{bmatrix} 24 & 0 \\ 0 & 2 \end{bmatrix}$.

[3 marks]

(e) Consider the controllers of parts (c) and (d). Comment on which of them is impossible to implement in practice for the corresponding system. Justify your answer.

[2 marks]

Solution:

(a) By computing the controllability and observability matrices it can be seen that the system is neither controllable nor observable. By the state space representation we can see that the control input influences only the second state, while the two states evolve independently as A is diagonal. As such, $x_2(t)$ is controllable, while $x_1(t)$ is uncontrollable. Moreover, in the output of the system we can observe directly $x_1(t)$, which is thus observable, and not $x_2(t)$. Since states evolve independently, $x_2(t)$ is an unobservable state. Overall,

- $x_1(t)$: uncontrollable + observable;
- $x_2(t)$: controllable + unobservable;

The same conclusion can be reached by noticing that the given system is directly related to the Kalman decomposition form. In fact treating $x_1(t)$, $x_2(t)$ as the states denoted by $x_3(t)$ and $x_2(t)$, respectively, in Theorem 3 of the B15 Lecture Notes, we directly reach the aforementioned conclusion.

Moreover, notice that the eigenvalues corresponding to the uncontrollable and the unobservable states (-10 and -1, respectively) are both negative, hence the system is stabilizable and detectable.

[2 marks]

(b) Let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, and consider a state feedback controller $u(t) = Kx(t)$. The closed loop system matrix becomes

$$A + BK = \begin{bmatrix} -10 & 0 \\ 2k_1 & -1 + 2k_2 \end{bmatrix}.$$

We design k_1 and k_2 by means of the pole placement procedure. To this end, the target eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = -10$. The eigenvalues of $A + BK$ can be directly computed as -10 and $-1 + 2k_2$, since the closed loop system matrix is triangular. It thus suffices to equate these eigenvalues with the target ones without computing the characteristic polynomials. This leads to $k_2 = -2$, while the choice of k_1 is arbitrary.

The latter could be anticipated, as the system is uncontrollable but stabilizable, and the eigenvalue corresponding to the uncontrollable part (-10) is included in the target eigenvalues. This can be also verified by inspection of the state space representation: the first state evolves autonomously and decays towards the origin at a rate -10, while the second state in closed loop would evolve according to $\dot{x}_2(t) = -5x_2(t) + 2k_1x_1(t)$. The dominant term is the first one (as $x_1(t)$ decays at rate -10), leading to the fact that second state will decay towards the origin at a rate -5.

[3 marks]

(c) The LQR controller is $u(t) = -R^{-1}B^\top Px(t) = -B^\top Px(t)$, where the second equality is since the input penalty matrix R here is one, and P is a positive semidefinite solution to the algebraic Riccati equation. Therefore, letting $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$, since it is symmetric, the LQR gain is

$$-B^\top P = -\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -2p_2 & -2p_3 \end{bmatrix}.$$

Equating this with the given gain $K = \begin{bmatrix} 0 & -2 \end{bmatrix}$, we obtain that $p_2 = 0$ and $p_3 = 1$. As such, P is in the form

$$P = \begin{bmatrix} p_1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By the algebraic Riccati equation we have

$$PA + A^\top P + Q - PBR^{-1}B^\top P = 0 \;\Rightarrow\; Q = PBR^{-1}B^\top P - PA - A^\top P$$
$$\Rightarrow\; Q = \begin{bmatrix} 20p_1 & 0 \\ 0 & 6 \end{bmatrix}.$$

Variable p_1 is free, hence multiple choices for Q exist; for $p_1 = 1$ we get one such matrix, namely, $Q = \begin{bmatrix} 20 & 0 \\ 0 & 6 \end{bmatrix}$.

[6 marks]

(d) Let again $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. For the matrices given in this part of the question the algebraic Riccati equation becomes

$$
\begin{aligned}
& PA + A^\top P + Q - PBR^{-1}B^\top P = 0 \\
\Rightarrow\ & \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} -10 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -10 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 24 & 0 \\ 0 & 2 \end{bmatrix} \\
& \qquad - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = 0.
\end{aligned}
$$

Due to symmetry, we obtain the following three equations with three unknowns:

$$
\begin{aligned}
-20p_1 + 24 - 4p_1^2 = 0 &\ \Rightarrow\ p_1 = 1 \text{ or } p_1 = -6, \\
-11p_2 - 4p_1p_2 = 0 &\ \Rightarrow\ p_2 = 0, \\
-2p_3 + 2 - 4p_2^2 = 0 &\ \Rightarrow\ p_3 = 1.
\end{aligned}
$$

Therefore, we obtain two different solutions for P, of which $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ that corresponds to $p_1 = 1$ is the positive semidefinite one. The LQR controller is thus $u(t) = -R^{-1}B^\top Px(t) = -2x_1(t)$.

[3 marks]

(e) The controller of part (c) is $u(t) = Kx(t) = -2x_2(t)$. However, $x_2(t)$ is unobservable (see part (a)), hence we cannot access that state to realize the controller. This is not the case for the system and the controller of part (d).

[2 marks]