B15 Linear Dynamic Systems and Optimal Control Example Paper 2: Solutions

Kostas Margellos

kostas.margellos@eng.ox.ac.uk

Michaelmas Term 2021

Questions

1. Consider an LTI system whose state evolves according to

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_1(t) - 2x_2(t) + u(t).\end{aligned}$$

   (a) Verify that it is controllable.

   (b) Design a state feedback controller that places the eigenvalues of the closed loop system at -2 and -4.

Solution:

(a) Setting $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ we can write the system dynamics as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ &= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).\end{aligned}$$

The system is of order 2, hence, the controllability matrix is then given by

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow \operatorname{rank}(P) = 2,$$

hence P is full rank. As a result the system is controllable.

(b) Notice that not only is the given system controllable, but it is also in controllable canonical form. We will exploit it to compute the state feedback control gains without following the four step pole placement procedure.

To this end, let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ be the state feedback gain matrix. Since A is in controllable canonical form, the coefficients in its last row are the (negation) of the coefficients of the characteristic polynomial of A, namely,

$$\text{characteristic polynomial of } A\text{:} \quad \lambda^2 + a_1\lambda + a_2,$$

where $a_1 = 2$ and $a_2 = -1$.

The target set of eigenvalues for the closed loop system is $\{-2, -4\}$. Therefore, the target characteristic polynomial of the closed loop system is given by

$$\begin{aligned}\text{target characteristic polynomial:} \quad & \lambda^2 + d_1\lambda + d_2 \\ & =(\lambda + 2)(\lambda + 4) = \lambda^2 + 6\lambda + 8.\end{aligned}$$

Hence, $d_1 = 6$ and $d_2 = 8$. We have shown in p. 69 of the notes that the gains k_1 and k_2 (as the system is already in controllable canonical form no coordinate transformation is needed) that would result in the target characteristic polynomial for the closed loop system matrix $A + BK$ are given by

$$k_1 = a_2 - d_2 = -9 \text{ and } k_2 = a_1 - d_1 = -4.$$

It can be verified that using $K = \begin{bmatrix} -9 & -4 \end{bmatrix}$, the eigenvalues of the closed loop matrix $A + BK$ coincide with the target ones. If the system happens to be in controllable canonical form we can always compute the state feedback controller gains by means of this procedure.

2. Consider the following LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$= \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} u(t),$$
$$y(t) = Cx(t) = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} x(t).$$

(a) Verify that the system is controllable.

(b) Determine K such that the state feedback $u(t) = Kx(t)$ results in a closed loop system with three eigenvalues at -2.

Solution:

(a) The system is of order 3. Hence, the controllability matrix is then given by

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \Rightarrow \text{rank}(P) = 3,$$

hence P is full rank. As a result the system is controllable.

(b) The system is of order 3 with a single input, so we seek a control gain matrix of the form $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$. The closed loop matrix is thus given by

$$A + BK = \begin{bmatrix} 3k_1 & 3k_2 & 3k_3 + 2 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

To determine k_1, k_2 and k_3 so that the closed loop system has three eigenvalues at -2, we apply the pole placement procedure. We thus have:

Step 1. The target set of eigenvalues is $\{\lambda_1, \lambda_2, \lambda_3\} = \{-2, -2, -2\}$.

Step 2. The target characteristic polynomial is given by

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = (\lambda+2)^3 = \lambda^3+6\lambda^2+12\lambda+8.$$

Step 3. The characteristic polynomial of $A+BK$ is given by

$$\begin{aligned}&\det(\lambda I-(A+BK))\\ &\qquad = \lambda^3+(-3k_1-1)\lambda^2+(3k_1-3k_2)\lambda+(3k_2-6k_3-4).\end{aligned}$$

Step 4. Equating the coefficients of the polynomials of Step 2 and Step 3 we obtain

$$\begin{aligned}-3k_1-1=6 &\;\Rightarrow\; k_1=-\frac{7}{3}\\ 3k_1-3k_2=12 &\;\Rightarrow\; k_2=-\frac{19}{3}\\ 3k_2-6k_3-4=8 &\;\Rightarrow\; k_3=-\frac{31}{6}.\end{aligned}$$

The resulting system of equations admitted a unique solution, as the given system had a single input and was shown to be controllable.

3. Consider the LTI system of Question 2.

   (a) Verify that the system is observable.

   (b) If $\hat{x}(t)$ denotes the state estimated by means of a linear state observer with gain matrix L, determine the dynamics of the estimation error $e(t)=x(t)-\hat{x}(t)$.

   (c) Determine L such that the dynamics of the estimation error have three eigenvalues at -3.

Solution:

(a) The system is of order 3. Hence, the observability matrix is then given by

$$Q=\begin{bmatrix}C\\CA\\CA^2\end{bmatrix}=\begin{bmatrix}0&0&2\\0&4&2\\4&4&2\end{bmatrix}\quad\Rightarrow\;\operatorname{rank}(Q)=3,$$

hence Q is full rank. As a result the system is observable.

(b) The dynamics of a linear state observer are given by

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t))\\ \hat{y}(t) &= C\hat{x}(t) + Du(t),\end{aligned}$$

where L is the observer gain matrix. The dynamics of the estimation error $e(t) = x(t) - \hat{x}(t)$ are thus given by

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t)\\ &= Ax(t) + \cancel{Bu(t)} - A\hat{x}(t) - \cancel{Bu(t)} - L(y(t) - \hat{y}(t))\\ &= A(x(t) - \hat{x}(t)) - L(Cx(t) + \cancel{Du(t)} - C\hat{x}(t) - \cancel{Du(t)})\\ &= (A - LC)(x(t) - \hat{x}(t))\\ &= (A - LC)e(t).\end{aligned}$$

(c) The system is of order 3 and has a single output, so we seek an observer gain matrix of the form $L = \begin{bmatrix}\ell_1\\ \ell_2\\ \ell_3\end{bmatrix}$. The estimation error system involves then matrix

$$A - LC = \begin{bmatrix}0 & 0 & 2-2\ell_1\\ 1 & 0 & -2\ell_2\\ 0 & 2 & 1-2\ell_3\end{bmatrix}.$$

To determine ℓ_1, ℓ_2 and ℓ_3 so that the estimation error matrix $A-LC$ has three eigenvalues at -3, we apply the pole placement procedure with $A - LC$ in place of $A + BK$. We thus have:

Step 1. The target set of eigenvalues is $\{\lambda_1, \lambda_2, \lambda_3\} = \{-3, -3, -3\}$.

Step 2. The target characteristic polynomial is given by

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = (\lambda + 3)^3 = \lambda^3 + 9\lambda^2 + 27\lambda + 27.$$

Step 3. The characteristic polynomial of $A - LC$ is given by

$$\det(\lambda I - (A - LC)) = \lambda^3 + (2\ell_3 - 1)\lambda^2 + 4\ell_2\lambda + (4\ell_1 - 4).$$

Step 4. Equating the coefficients of the polynomials of Step 2 and Step 3 we obtain

$$\begin{aligned} 2\ell_3 - 1 = 9 \;&\Rightarrow\; \ell_3 = 5 \\ 4\ell_2 = 27 \;&\Rightarrow\; \ell_2 = \frac{27}{4} \\ 4\ell_1 - 4 = 27 \;&\Rightarrow\; \ell_1 = \frac{31}{4}. \end{aligned}$$

The resulting system of equations admitted a unique solution, as the given system had a single output and was shown to be observable.

4. Consider the transfer function

$$G(s) = \frac{\omega_0^2}{s^2 + \omega_0 s + \omega_0^2}.$$

(a) Determine the poles of $G(s)$ and specify their damping ratio.

(b) Determine a realization (A, B, C, D) of $G(s)$.

(c) Compute the gains of a state feedback controller as a function of ω_0 so that the closed loop system has a complex conjugate pair of eigenvalues with damping ratio $\frac{1}{\sqrt{2}}$.

(d) Wha was the purpose of this controller?

Hint: Recall that the general description of a complex conjugate pole (eigenvalue) pair is given by $-\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$, where ζ denotes the damping ratio.

Solution:

(a) The poles of $G(s)$ are the roots of its denominator, i.e.,

$$s_{1,2} = -\frac{1}{2}\omega_0 \pm j\frac{\sqrt{3}}{2}\omega_0.$$

Comparing with the general description for a complex conjugate pole pair, we deduce that the damping ratio is $\zeta = \frac{1}{2}$.

(b) Denote by $U(s)$ and $Y(s)$ the Laplace transform of an input signal $u(t)$ and an output signal $y(t)$, respectively, such that $G(s) = \frac{Y(s)}{U(s)}$. We then have that

$$\begin{aligned} & (s^2 + \omega_0 s + \omega_0^2)Y(s) = \omega_0^2 U(s) \\ \Rightarrow \ & s^2 Y(s) + \omega_0 s Y(s) + \omega_0^2 Y(s) = \omega_0^2 U(s) \\ \Rightarrow \ & \ddot{y}(t) + \omega_0 \dot{y}(t) + \omega_0^2 y(t) = \omega_0^2 u(t), \end{aligned}$$

where the last step follows by taking the inverse Laplace transform, assuming zero initial conditions. Setting $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$ we obtain the following set of equations:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\omega_0^2 x_1(t) - \omega_0 x_2(t) + \omega_0^2 u(t) \\ y(t) &= x_1(t). \end{aligned}$$

Setting $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, we obtain the following state space description of this system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\omega_0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix} u(t), \\ y(t) &= Cx(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \end{aligned}$$

Matrices (A, B, C, D) constitute a realization of $G(s)$.

(c) The system is of order 2 with a single input, so we seek a control gain matrix of the form $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. The closed loop matrix is thus given by

$$A + BK = \begin{bmatrix} 0 & 1 \\ (k_1 - 1)\omega_0^2 & \omega_0(k_2\omega_0 - 1) \end{bmatrix}.$$

To determine k_1 and k_2 we apply the pole placement procedure. We thus have:

Step 1. For a damping ratio $\zeta = \frac{1}{\sqrt{2}}$ the target eigenvalues are given by

$$\lambda_{1,2} = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2} = -\frac{1}{\sqrt{2}}\omega_0 \pm j\frac{1}{\sqrt{2}}\omega_0.$$

Step 2. The target characteristic polynomial is given by

$$(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 + \sqrt{2}\omega_0\lambda + \omega_0^2.$$

Step 3. The characteristic polynomial of $A+BK$ is given by

$$\begin{aligned}\det(\lambda I - (A+BK)) \\ &= \lambda^2 + \omega_0(1-k_2\omega_0)\lambda + (1-k_1)\omega_0^2.\end{aligned}$$

Step 4. Equating the coefficients of the polynomials of Step 2 and Step 3 we obtain

$$\begin{aligned}\omega_0(1-k_2\omega_0) = \sqrt{2}\omega_0 \;&\Rightarrow\; k_2 = \frac{1-\sqrt{2}}{\omega_0}\\ (1-k_1)\omega_0^2 = \omega_0^2 \;&\Rightarrow\; k_1 = 0.\end{aligned}$$

The resulting system of equations admitted a unique solution, as the given system had a single input and it can be verified that it is controllable.

(d) The purpose of this state feedback controller was to increase the damping from $\frac{1}{2}$ to $\frac{1}{\sqrt{2}}$.

5. Consider the transfer function $G(s)$ of Question 4, and the realization computed in part (b). Compute the gains of a linear state observer as a function of ω_0 so that the estimation error dynamics are 10 times faster (with the same damping ratio) than the dynamics of the closed loop system computed in Question 4.

Solution: Notice that the characteristic polynomial associated to the closed loop system of Question 4 has roots equal to $-\frac{1}{\sqrt{2}}\omega_0 \pm j\frac{1}{\sqrt{2}}\omega_0$.

Since the system is of order 2 with a single output, let $L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$ be the observer gain matrix.

To achieve state estimation error dynamics that are 10 times faster than the dynamics of that system, we would like the eigenvalues of

$$A - LC = \begin{bmatrix} -\ell_1 & 1 \\ -\ell_2 - \omega_0^2 & -\omega_0 \end{bmatrix},$$

to be $-\frac{10}{\sqrt{2}}\omega_0 \pm j\frac{10}{\sqrt{2}}\omega_0$. That way the real part of these eigenvalues is 10 times higher in magnitude compared to the one of the closed loop system in Question 4. This in turn implies that designing gains ℓ_1 and ℓ_2 so that we achieve these target eigenvalues will lead to the desired behaviour of the state estimation error. To achieve this we follow the pole placement procedure.

Step 1. The target eigenvalues are given by $\lambda_{1,2} = -\frac{10}{\sqrt{2}}\omega_0 \pm j\frac{10}{\sqrt{2}}\omega_0$.

Step 2. The target characteristic polynomial is given by

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + 10\sqrt{2}\omega_0\lambda + 100\omega_0^2.$$

Step 3. The characteristic polynomial of $A - LC$ is given by

$$\det(\lambda I - (A - LC)) = \lambda^2 + (\ell_1 + \omega_0)\lambda + (\ell_1\omega_0 + \ell_2 + \omega_0^2).$$

Step 4. Equating the coefficients of the polynomials of Step 2 and Step 3 we obtain

$$\begin{aligned} \ell_1 + \omega_0 = 10\sqrt{2}\omega_0 \;&\Rightarrow\; \ell_1 = (-1 + 10\sqrt{2})\omega_0 \\ \ell_1\omega_0 + \ell_2 + \omega_0^2 = 100\omega_0^2 \;&\Rightarrow\; \ell_2 = (-10\sqrt{2} + 100)\omega_0^2. \end{aligned}$$

The resulting system of equations admitted a unique solution, as the given system had a single output and it can be verified that it is observable.

6. Let T be a given time horizon length, and consider the following finite horizon optimal control problem:

$$\begin{aligned} \text{minimize} \ & \int_0^T u(t)^2 dt + x(T)^2 \\ \text{subject to} \ & \dot{x}(t) = x(t) + u(t), \text{ for all } t \in [0, T], \\ & x(0) = 1. \end{aligned}$$

(a) Determine matrices A and B corresponding to the state space description of the system's dynamics. Determine matrices Q, R and Q_T so that the cost criterion can be written in the form

$$\int_0^T \big(x(t)^\top Q x(t) + u(t)^\top R u(t)\big) dt + x(T)^\top Q_T x(T).$$

(b) State and solve the Riccati differential equation associated with this finite horizon linear quadratic regulation (LQR) problem.

(c) Compute the optimal LQR controller and the associated optimal cost.

Solution:

(a) From the system dynamics that appear as constraints in the optimal control problem, it follows that

$$A = 1 \ \text{ and } \ B = 1,$$

while from the cost criterion we have that

$$R = 1, \ Q = 0 \ \text{ and } \ Q_T = 1.$$

Notice that all quantities are scalars as we have an one-dimensional system.

(b) For the quantities determined in part (a), the Riccati differential equation takes the form

$$\begin{aligned} -\dot{P}(t) &= P(t)A + A^\top P(t) + Q - P(t)BR^{-1}B^\top P(t) \\ &= 2P(t) - P(t)^2 = P(t)(2 - P(t)), \end{aligned}$$

with $P(T) = Q_T = 1$. Notice that in this case $P(t)$ is a scalar. We can solve the resulting equation using separation of variables, i.e.,

$$
\begin{aligned}
&\frac{1}{P(2-P)} dP = -dt \;\Rightarrow\; \int \frac{1}{P(2-P)} dP = -\int dt \\
&\text{[using partial fraction expansion]} \\
\Leftrightarrow\; &\int \Big(\frac{1}{2P} + \frac{1}{2}\frac{1}{2-P} \Big) dP = -t + \text{constant} \\
\Rightarrow\; &\frac{1}{2}\ln(|P|) - \frac{1}{2}\ln(|2-P|) = -t + \text{constant} \\
&\text{[using } P(T) = 1 \Rightarrow \text{constant} = T\text{]} \\
\Leftrightarrow\; &\ln\Big|\frac{P}{2-P}\Big| = 2(T-t) \;\Rightarrow\; P(t) = \frac{2e^{2(T-t)}}{e^{2(T-t)}+1}.
\end{aligned}
$$

(c) The optimal LQR controller is then given by

$$
u^\star(t) = -R^{-1}B^\top P(t)x(t) = -\frac{2e^{2(T-t)}}{e^{2(T-t)}+1}x(t),
$$

while the associated optimal cost is

$$
J(u^\star) = x(0)^\top P(0)x(0) = \frac{2e^{2T}}{e^{2T}+1}.
$$

Notice that $\lim_{T\to\infty} P(t) = 2$, which coincides with one of the two solutions of the algebraic Riccati equation of Example 20 in the notes, that has the same dynamics and the same running cost. However, this is not the one that corresponds to the solution of the infinite horizon problem in Example 20, as here $Q_T = 1 \neq 0$.

7. Consider the following infinite horizon optimal control problem:

$$
\begin{aligned}
&\text{minimize } \frac{1}{2}\int_0^\infty \Big(x_1(t)^2 + \frac{1}{8}u(t)^2\Big)dt \\
&\text{subject to } \dot{x}_1(t) = x_2(t),\; \dot{x}_2(t) = -x_1(t) + u(t),\; \text{for all } t, \\
&\qquad x_1(0), x_2(0):\; \text{given.}
\end{aligned}
$$

Let $y(t) = x_1(t)$ denote the output of the underlying LTI system.

(a) Determine matrices A, B and C corresponding to the state space description of the system's dynamics. Determine matrices Q and R so that the cost criterion can be written in the form

$$\int_0^\infty \big(x(t)^\top Q x(t) + u(t)^\top R u(t)\big) dt.$$

(b) State and solve the algebraic Riccati equation associated with this infinite horizon linear quadratic regulation (LQR) problem.

(c) Does the algebraic Riccati equation admit a unique positive semidefinite solution? If yes, justify whether this is anticipated.

(d) Compute the optimal LQR controller.

Solution:

(a) From the system dynamics that appear as constraints in the optimal control problem, it follows that

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

while from the cost criterion we have that

$$R = \frac{1}{16} \text{ and } Q = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ (notice that the off-diagonal terms are the same as P is symmetric) be the solution of the algebraic Riccati equation. We then have that

$$\begin{aligned}
& PA + A^\top P + Q - PBR^{-1}B^\top P = 0 \\
\Rightarrow & \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
& \qquad - 16 \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = 0 \\
\Rightarrow & \begin{bmatrix} -2p_2 + \frac{1}{2} - 16p_2^2 & p_1 - p_3 - 16p_2p_3 \\ \star & 2p_2 - 16p_3^2 \end{bmatrix} = 0,
\end{aligned}$$

where the term denoted by $\star$ is equal to the second element of the first row due to symmetry. Equating each of the three distinct entries with zero, we obtain three equations with three unknowns, namely, p_1, p_2 and p_3. We thus have that

$$\begin{aligned} -2p_2 + \frac{1}{2} - 16p_2^2 = 0 \;&\Rightarrow\; p_2 = \frac{1}{8} \\ 2p_2 - 16p_3^2 = 0 \;&\Rightarrow\; p_3 = -\frac{1}{8} \text{ or } p_3 = \frac{1}{8} \\ p_1 - p_3 - 16p_2p_3 = 0 \;&\Rightarrow\; p_1 = -\frac{3}{8} \text{ or } p_3 = \frac{3}{8}, \end{aligned}$$

where from the first equation we can only have a positive solution for p_2, as by the second one $p_2 = 8p_3^2 \geq 0$. We then have two solutions:

$$P = \begin{bmatrix} -\frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} \end{bmatrix} \text{ or } P = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix}.$$

Notice that only the second solution is positive semidefinite, i.e., the unique positive semidefinite solution of the algebraic Riccati equation.

(c) Notice that the penalty matrix on the state Q can be written as $Q = C^\top \bar{Q} C$, where $\bar{Q} = 0.5$, hence, it is positive definite. Moreover, pairs (A, B) and (A, C) are controllable and observable, respectively. To see this notice that

$$\begin{aligned} \begin{bmatrix} B & AB \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;\Rightarrow\; \text{full rank}, \\ \begin{bmatrix} C \\ CA \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\Rightarrow\; \text{full rank}. \end{aligned}$$

Therefore, the algebraic Riccati equation is expected to admit a unique positive semidefinite solution (indeed that was the case in part b).

(d) Denote by $\bar{P}$ the unique positive semidefinite solution of the algebraic Riccati equation. The optimal LQR controller is then given by

$$\begin{aligned} u^\star(t) &= -R^{-1} B^\top \bar{P} x(t) \\ &= -16 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -2x_1(t) - 2x_2(t). \end{aligned}$$

8. Consider the following LQR problem with $\mu > 0$:

$$\text{minimize} \int_0^{\infty} \left(\mu^2 x(t)^2 + u(t)^2\right) dt$$
$$\text{subject to } \dot{x}(t) = u(t), \text{ for all } t,$$
$$x(0) = x_0.$$

Let $y(t) = x(t)$ denote the output of the underlying LTI system.

(a) Compute the optimal LQR controller.

(b) Comment on the effect of the choice of μ on the behaviour of the closed loop system state $x(t)$.

Hint: For part (a) adapt the solution of the Riccati equation computed in Examples 18 & 19 in the notes to account for the presence of parameter μ in the cost function.

Solution:

(a) The given infinite horizon optimal control problem is the same with Example 19 in the notes, with the difference that $Q = \mu^2$. This in turn induces some changes in the solution $P(t)$ of the Riccati differential equation computed in Example 18 in the notes. In particular, $P(t)$ will not be the solution of $-\dot{P}(t) = \mu^2 - P(t)^2$ with $P(T) = Q_T = 0$. Following exactly the same steps with the derivation in Example 18, we obtain that

$$\frac{1}{\mu^2 - P^2} dP = -dt \;\Rightarrow\; \int \frac{1}{\mu^2 - P^2} dP = -\int dt$$

[using partial fraction expansion]

$$\Leftrightarrow \int \left(\frac{1}{2\mu}\frac{1}{\mu - P} + \frac{1}{2\mu}\frac{1}{\mu + P}\right) dP = -t + \text{constant}$$

$$\Rightarrow -\frac{1}{2\mu}\ln(|\mu - P|) + \frac{1}{2\mu}\ln(|\mu + P|) = -t + \text{constant}$$

[using $P(T) = 0 \Rightarrow$ constant $= T$]

$$\Leftrightarrow \ln\left|\frac{\mu + P}{\mu - P}\right| = 2\mu(T - t) \;\Rightarrow\; P(t) = \mu\frac{e^{2\mu(T-t)} - 1}{e^{2\mu(T-t)} + 1}.$$

Therefore, we have that $\bar{P} = \lim_{T\to\infty} P(t) = \mu$ (as opposed to $\bar{P} = 1$ in Example 19). It can be verified that this is a solution to the algebraic Riccati equation.

The optimal LQR controller is then given by

$$u^\star(t) = -R^{-1}B^\top \bar{P}x(t) = -\mu x(t).$$

(b) Note that as μ tends to zero, then the optimal controller tends to zero as well. This is anticipated, since in that case the cost tends to $\int_0^\infty u(t)^2 dt$, hence, the optimal solution is $u^\star(t) = 0$ for all t, resulting in $x(t) = x_0$ for all t. In other words, the optimal action is "not to move", as in this limiting case only the input (and not the state) is penalized in the cost function. On the other hand, if μ tends to infinity, then the control effort is "cheaper" in terms of cost with respect to how the state is penalized. As such the optimal controller (and also the cost) tends to infinity. Overall, μ allows trading between penalizing the state versus the control input.

9. OPTIONAL: Consider an (open loop) LTI system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

with n states, a single input and a single output. Assume that a state feedback controller $u(t) = Kx(t) + r(t)$ is designed with $K \in \mathbb{R}^{1\times n}$. The closed loop system is then given by

$$\begin{aligned}\dot{x}(t) &= (A + BK)x(t) + Br(t),\\ y(t) &= (C + DK)x(t) + Dr(t).\end{aligned}$$

(a) Show that if the open loop system is controllable, then the closed loop system is controllable as well.

(b) Use your answer in Question 1 to construct a counterexample of an open loop system that is observable, while the closed loop is not.

Note: The condition of part (a) is in fact an "if and only if" one.

Solution:

(a) If the open loop system is controllable, and since it has a single input, then there exists an invertible coordinate transformation that renders the system in controllable canonical form (see p. 68 in the notes). As such, in the new coordinates, the closed loop system matrices would be $\hat{A} + \hat{B}\hat{K}$ and $\hat{B}$, where $\hat{K} = \begin{bmatrix} \hat{k}_1 & \dots & \hat{k}_n \end{bmatrix}$ is a row vector including the new control gains, and

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \hat{k}_1 - a_n & \hat{k}_2 - a_{n-1} & \hat{k}_3 - a_{n-2} & \dots & \hat{k}_n - a_1 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

with $a_1, \dots, a_n$ being the coefficents of the characteristic polynomial of A. Notice that the closed loop system is also in controllable canonical form. The controllability matrix associated with the closed loop system matrices $(\hat{A} + \hat{B}\hat{K}, \hat{B})$ would then be

$$\hat{P} = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \star & \dots & \star \\ 1 & \star & \star & \dots & \star \end{bmatrix} \in \mathbb{R}^{n \times n},$$

where all entries indicated by $\star$ depend on $a_1, \dots, a_n$ and $\hat{k}_1, \dots, \hat{k}_n$. Due to the triangular structure of $\hat{P}$ we have that $\mathrm{rank}(\hat{P}) = n$, hence $\hat{P}$ is full rank. The rank of the controllability matrix remains unaffected by the coordinate transformation[1], hence the controllabil-

[1] To see this, let T denote the coordinate transformation. Assisting p. 68-69 in the notes, we then have that $(\hat{A} + \hat{B}\hat{K})^k \hat{B} = T(A + BK)^k B$ for all $0 \leq k \leq n-1$. We thus have that $\hat{P} = TP$, where P is the controllability matrix in the original coordinates. As T is invertible, if $\hat{P}$ is full rank, then we must have that P is full rank as well.

ity matrix of the original system would be full rank as well. As a result the closed loop system would be controllable.

(b) Consider the system of Question 1, with the computed gain matrix $K = \begin{bmatrix} -9 & -4 \end{bmatrix}$, and recall that

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \text{ and } A + BK = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}.$$

Amend to it the output equation

$$\begin{aligned} y(t) = &\ Cx(t) + Du(t) = 2x_1(t) + x_2(t) \\ \Rightarrow &\ C = \begin{bmatrix} 2 & 1 \end{bmatrix},\ D = 0. \end{aligned}$$

We then have that the observability matrix of the open loop system is given by

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \operatorname{rank}(Q) = 2.$$

Therefore, the open loop system is observable. Consider now the observability matrix of the closed loop system:

$$\begin{bmatrix} C + DK \\ (C + DK)(A + BK) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -8 & -4 \end{bmatrix}.$$

Note that the rank of this matrix is 1 as the second row is a linear combination (multiple by a factor of -4) of the first one. As a result the closed loop system is not observable.