B15 Linear Dynamic Systems and Optimal Control
Example Paper 1: Solutions

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Questions

1. Consider the amplifier of Figure 1, where $v_i(t)$ is the input voltage and $v_o(t)$ the output one. Denote by $v_{C_1}(t)$ and $v_{C_2}(t)$ the voltage across the capacitor C_1 and C_2, respectively, and assume that the amplifier is ideal.

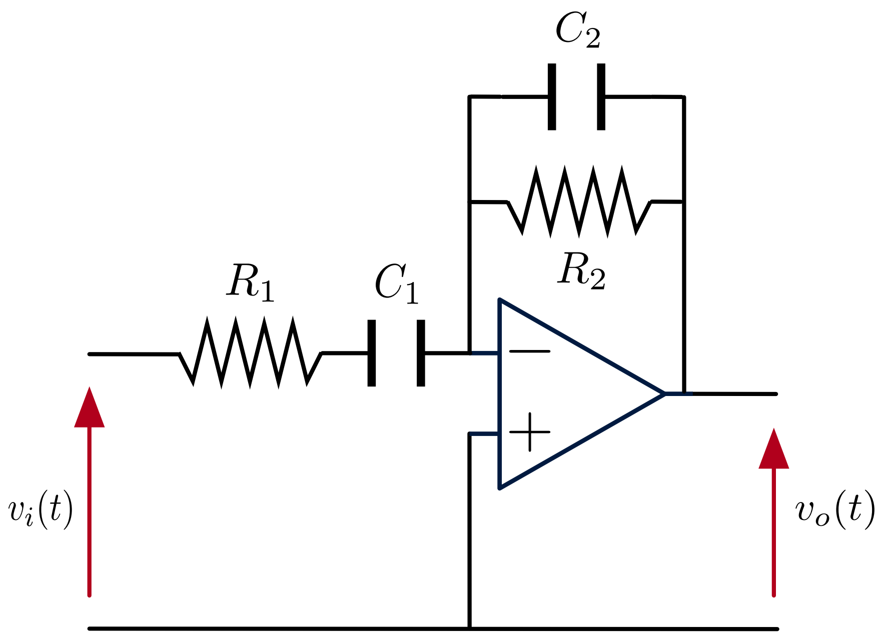

Figure 1: Amplifier circuit.

   (a) Derive the ordinary differential equations (ODEs) that capture the evolution of $v_{C_1}(t)$ and $v_{C_2}(t)$.

   (b) Provide a state space description of the amplifier circuit. What is the order of the resulting system?

   (c) Is the resulting system autonomous? Is it linear? Justify your answers.

Solution:

(a) Denote by $i_{C_1}(t)$ the current that flows through R_1 and C_1, by $i_{R_2}(t)$ the current that flows through R_2, and by $i_{C_2}(t)$ the current that flows through C_2. All currents flow towards the right direction. Moreover, denote by v_- and v_+ the voltage at the negative and positive terminal of the amplifier, respectively. As a consequence of the fact that the amplifier is ideal we have that (i) $v_- = v_+$; (ii) there is no current across the amplifier's input terminals.

By consequence (i), and since $v_+ = 0$ we have that $v_- = v_+ = 0$. Therefore, $v_- - v_o(t) = -v_o(t)$ is the voltage across the terminals of R_2. We thus have that

$$i_{C_1}(t) = C_1 \frac{dv_{C_1}(t)}{dt},\ i_{C_2}(t) = C_2 \frac{dv_{C_2}(t)}{dt} \ \text{ and } i_{R_2}(t) = -\frac{v_o(t)}{R_2}.$$

By Kirchhoff's voltage law (KVL) we have that

$$\begin{aligned} & v_i(t) = R_1 i_{C_1}(t) + v_{C_1}(t) \\ & \quad \Rightarrow\ v_i(t) = R_1 C_1 \frac{dv_{C_1}(t)}{dt} + v_{C_1}(t). \qquad \texttt{[KVL]} \end{aligned}$$

By consequence (ii), Kirchhoff's current law (KCL) at v_- results in

$$\begin{aligned} & i_{C_1}(t) = i_{R_2}(t) + i_{C_2}(t) \\ & \quad \Rightarrow\ C_1 \frac{dv_{C_1}(t)}{dt} = -\frac{v_o(t)}{R_2} + C_2 \frac{dv_{C_2}(t)}{dt} \\ & \quad \Rightarrow\ \frac{v_i(t)}{R_1} - \frac{v_{C_1}(t)}{R_1} = \frac{v_{C_2}(t)}{R_2} + C_2 \frac{dv_{C_2}(t)}{dt}, \qquad \texttt{[KCL]} \end{aligned}$$

where the last implication follows from the second one by substituting for $C_1 \frac{dv_{C_1}(t)}{dt}$ from the `KVL` equation, and using the fact that $v_- - v_o(t) = -v_o(t)$ is also the voltage across the terminals of C_2, hence $v_0(t) = -v_{C_2}(t)$.

The two equations labeled as `KVL` and `KCL` constitute a pair of coupled ODEs that capture the evolution of $v_{C_1}(t)$ and $v_{C_2}(t)$ (depending also on the external input $v_i(t)$).

(b) Let $u(t) = v_i(t)$ be the input, and $y(t) = v_o(t) = -v_{C_2}(t)$ be the output of the system. Denote the system state by

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{bmatrix}.$$

Under these variable assignments, we can solve the `KVL` and `KCL` equations with respect to $\frac{dv_{C_1}(t)}{dt}$ and $\frac{dv_{C_2}(t)}{dt}$, respectively, and stacking them in matrix form. We thus obtain the following state space representation:

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ -\frac{1}{R_1C_2} & -\frac{1}{R_2C_2} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_1C_2} \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 0 & -1 \end{bmatrix} x(t).$$

This is a 2nd order system, as we have two states.

(c) The system is not autonomous, as its evolution depends on the external input $u(t) = v_i(t)$. It is linear as the right-hand side in the description of $\dot{x}(t)$ (vector field) and $y(t)$ is linear both with respect to $x(t)$ and $u(t)$.

2. Consider the following dynamical system

$$\ddot{z}(t) = 1 - \frac{1}{(z(t) + z^\star)^2} u(t),$$

where $z^\star > 0$ is a fixed parameter.

(a) Find the constant input $u(t) = u^\star$ for all $t \geq 0$ that renders 0 an equilibrium of the system, i.e., starting at $z(t) = 0$ the system does not move.

(b) Use $x(t) = \begin{bmatrix} z(t) & \dot{z}(t) \end{bmatrix}^\top$ and $y(t) = z(t)$ as the state vector and system output, respectively. Write the given dynamical system in state space form. Is the resulting system linear?

(c) Linearize the system around $x^\star = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ and the value for $u^\star$ computed in part (a).

Solution:

(a) Since we would like $z(t) = 0$ to be an equilibrium of the system, at the equilibrium we would have $\ddot{z}(t) = \dot{z}(t) = 0$. We thus have

$$0 = 1 - \frac{1}{(z^\star)^2} u^\star \;\Rightarrow\; u^\star = (z^\star)^2.$$

(b) Let $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^\top = \begin{bmatrix} z(t) & \dot{z}(t) \end{bmatrix}^\top$ and $y(t) = z(t)$. We thus have that $\dot{x}_1(t) = x_2(t)$ and $\dot{x}_2(t) = \ddot{z}(t) = 1 - \frac{1}{(x_1(t)+z^\star)^2} u(t)$. Therefore, the given system can be written in state space form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ 1 - \frac{1}{(x_1(t)+z^\star)^2} u(t) \end{bmatrix},$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Due to the fact that $\dot{x}_2(t)$ involves a product between the state $x_1(t)$ and the input $u(t)$, we infer that that the system is nonlinear. Moreover, nonlinearity is also introduced by the fact that the inverse of $x_1(t)$ as well as its square appear in the expression for $\dot{x}_2(t)$.

(c) Let $\begin{bmatrix} f_1(x(t), u(t)) \\ f_2(x(t), u(t)) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}$. Consider the perturbed input and states, which under the choice of the operating point become

$$\begin{aligned} x_{1,p}(t) &= x_1(t) - x_1^\star \;\Rightarrow\; x_{1,p}(t) = x_1(t), \\ x_{2,p}(t) &= x_2(t) - x_2^\star \;\Rightarrow\; x_{2,p}(t) = x_2(t), \\ u_p(t) &= u(t) - u^\star \;\Rightarrow\; u_p(t) = u(t) - (z^\star)^2. \end{aligned}$$

The linearized system is then given by

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^\star, u^\star) & \frac{\partial f_1}{\partial x_2}(x^\star, u^\star) \\ \frac{\partial f_2}{\partial x_1}(x^\star, u^\star) & \frac{\partial f_2}{\partial x_2}(x^\star, u^\star) \end{bmatrix} x(t) + \begin{bmatrix} \frac{\partial f_1}{\partial u}(x^\star, u^\star) \\ \frac{\partial f_2}{\partial u}(x^\star, u^\star) \end{bmatrix} u_p(t) \\ &= \begin{bmatrix} 0 & 1 \\ \frac{2}{z^\star} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -\frac{1}{(z^\star)^2} \end{bmatrix} u_p(t), \end{aligned}$$

where

$$\frac{\partial f_2}{\partial x_1}(x^\star, u^\star) = \frac{2u(t)}{(x_1(t)+z^\star)^3}\Big|_{x_1(t)=x_1^\star=0, u(t)=u^\star} = \frac{2u^\star}{(z^\star)^3} = \frac{2}{z^\star},$$
$$\frac{\partial f_2}{\partial u}(x^\star, u^\star) = -\frac{1}{(x_1(t)+z^\star)^2}\Big|_{x_1(t)=x_1^\star=0} = -\frac{1}{(z^\star)^2}.$$

3. Recall that the state transition matrix for linear time invariant (LTI) systems with starting time zero is given by

$$\Phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^kt^k}{k!} + \ldots$$

Show that it satisfies the following properties:

(a) $\Phi(0) = I$.

(b) $\frac{d}{dt}\Phi(t) = A\Phi(t)$.

(c) For any $t_1, t_2 \in \mathbb{R}$, $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$.

(d) $\Phi(t)\Phi(-t) = \Phi(-t)\Phi(t) = I$.
What is the role of $\Phi(-t)$ in this case?

Solution:

(a) This follows by substituting in the Taylor series expansion $t = 0$.

(b) Differentiate both sides of the Taylor series expansion with respect to t. This leads to

$$\begin{aligned}\frac{d}{dt}\Phi(t) &= A + A^2t + \ldots + \frac{A^kt^{k-1}}{(k-1)!} + \ldots \\ &= A\Big(I + At + \ldots + \frac{A^{k-1}t^{k-1}}{(k-1)!} + \ldots\Big) = A\Phi(t),\end{aligned}$$

where the second equality follows by pulling out A as a common factor, and the last one since the quantity in the parentheses is the Taylor series expansion of the matrix exponential, i.e., $\Phi(t)$.

(c) Let t_1 and t_2 be arbitrary, and consider the zero input transition of an LTI system with $\dot{x}(t) = Ax(t)$ and initial state x_0. Consider the following two candidate state solutions for $t \geq t_2$:

$$x_A(t) = \Phi(t)x_0 \text{ and } x_B(t) = \Phi(t - t_2)\Phi(t_2)x_0.$$

Notice that first state solution is the one that we would obtain if we start from x_0 at time $t = 0$. The interpretation of the second solution is that we start from $\Phi(t_2)x_0$ at $t = t_2$ (i.e., from $x_A(t_2)$), and the state transition matrix is given by $\Phi(t - t_2)$, as the initial time is shifted to $t = t_2$. In other words, the second solution is as if we "restart" the system at $t = t_2$.

Both these solutions satisfy the system's ODE for $t \geq t_2$. To see this, notice that both initial condition at $t = t_2$ are the same, i.e.,

$$x_A(t_2) = \Phi(t_2)x_0 \text{ and } x_B(t_2) = \Phi(0)\Phi(t_2)x_0 = \Phi(t_2)x_0,$$

where we used the fact that $\Phi(0) = I$ from part (a). Moreover, we have that

$$\begin{aligned}
\dot{x}_A(t) &= \dot{\Phi}(t)x_0 = A\Phi(t)x_0 = Ax_A(t), \\
\dot{x}_B(t) &= \dot{\Phi}(t - t_2)\Phi(t_2)x_0 = A\Phi(t - t_2)\Phi(t_2)x_0 = Ax_B(t),
\end{aligned}$$

where in both cases we used property (b) for the derivative of the state transition matrix. Therefore, both candidate solutions satisfy the system's ODE. By existence and uniqueness of solutions for LTI systems, they have to be the same, leading to $\Phi(t) = \Phi(t-t_2)\Phi(t_2)$. Setting $t = t_1 + t_2$, we obtain that

$$\Phi(t_1 + t_2) = \Phi(t_1 + t_2 - t_2)\Phi(t_2) = \Phi(t_1)\Phi(t_2).$$

Pictorially, this is illustrated in Figure 2. The interpretation of this fact is that due to existence and uniqueness of solutions to LTI systems, transferring the state directly from x_0 at $t = 0$ to some state $\Phi(t_1+t_2)x_0$ at $t = t_1+t_2$, is equivalent to first transferring the state to some intermediate state $\Phi(t_2)x_0$ at $t = t_2$, and then starting from that state perform a transfer with duration t_1 units of time, leading to the state $\Phi(t_1)\Phi(t_2)x_0$.

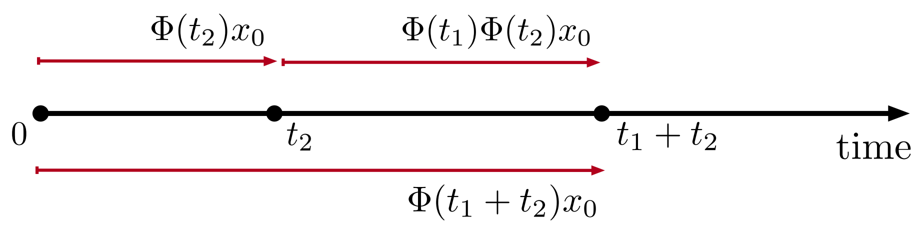

Figure 2: Pictorial illustration of the fact that $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$.

(d) Consider the property of part (c), which holds for arbitrary t_1, t_2. Setting $t_1 = t$ and $t_2 = -t$, we obtain

$$\Phi(0) = \Phi(t)\Phi(-t) \;\Rightarrow\; I = \Phi(t)\Phi(-t),$$

where the implication is since $\Phi(0) = I$ by part (a). Repeating the same argument with the roles of t_1 and t_2 reversed, shows that $\Phi(-t)\Phi(t) = I$. We thus have that matrix $\Phi(-t)$ plays the role of the inverse of the state transition matrix $\Phi(t)$.

4. For each case below comment on whether matrix A is diagonalizable, and determine the matrix exponential e^{At}.

(a) $A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$, where $\omega \neq 0$.

(b) $A = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}$, where $\sigma \neq 0$.

(c) $A = \begin{bmatrix} \lambda_1 & \lambda_2 - \lambda_1 \\ 0 & \lambda_2 \end{bmatrix}$, where $\lambda_1,\ \lambda_2 \neq 0$.

(d) $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution:

(a) For $\omega \neq 0$, matrix A is guaranteed to be nonzero. Its eigenvalues are given by $\det(\lambda I - A) = 0$, which leads to

$$\det\left(\begin{bmatrix} \lambda & \omega \\ -\omega & \lambda \end{bmatrix}\right) = 0 \;\Rightarrow\; \lambda^2 = -\omega^2.$$

We then have that matrix A has a complex conjugate eigenvalue pair, namely, $\lambda_1 = j\omega$ and $\lambda_2 = -j\omega$. Recall that having distinct eigenvalues is a sufficient condition for having linearly independent eigenvectors (see discussion below Definition 6 in the notes' appendix). As a result, A is diagonalizable.

In this case, rather than computing the eigenvectors of A and diagonalizing it, it is more efficient to compute e^{At} directly by means of its Taylor series expansion. To see this, notice that for $k = 0, 1, 2, \ldots,$

$$A^{2k} = \begin{bmatrix} (-1)^k\omega^{2k} & 0 \\ 0 & (-1)^k\omega^{2k} \end{bmatrix},$$
$$A^{2k+1} = \begin{bmatrix} 0 & -(-1)^k\omega^{2k+1} \\ (-1)^k\omega^{2k+1} & 0 \end{bmatrix}.$$

This can be shown either by computing a few even and odd powers of A and recognizing the pattern or, more formally, by means of induction[1]. We then have that

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k\omega^{2k}t^{2k}}{(2k)!} & -\sum_{k=0}^{\infty} \frac{(-1)^k\omega^{2k+1}t^{2k+1}}{(2k+1)!} \\ \sum_{k=0}^{\infty} \frac{(-1)^k\omega^{2k+1}t^{2k+1}}{(2k+1)!} & \sum_{k=0}^{\infty} \frac{(-1)^k\omega^{2k}t^{2k}}{(2k)!} \end{bmatrix}$$
$$= \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix},$$

[1] We will show via induction the expression for A^{2k}; the one for A^{2k+1} is analogous.
`Base case (`$k = 0$`).` Substituting $k = 0$ in the given expression for A^{2k}, we obtain that $A^0 = I$, which is trivially satisfied.
`Induction hypothesis.` Assume that the given expression for A^{2k} holds for an arbitrary k.
`Show the claim for the (`$k+1$`)-th case.` Notice that $A^2 = \begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix}$. We have that

$$A^{2(k+1)} = A^{2k}A = \begin{bmatrix} (-1)^k\omega^{2k} & 0 \\ 0 & (-1)^k\omega^{2k} \end{bmatrix} \begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix}$$
$$= \begin{bmatrix} (-1)^{k+1}\omega^{2(k+1)} & 0 \\ 0 & (-1)^{k+1}\omega^{2(k+1)} \end{bmatrix}.$$

where in the last equality we used the fact that $\sum_{k=0}^{\infty} \frac{(-1)^k \omega^{2k} t^{2k}}{(2k)!}$ and $\sum_{k=0}^{\infty} \frac{(-1)^k \omega^{2k+1} t^{2k+1}}{(2k+1)!}$ are the Taylor series expansion of $\cos \omega t$ and $\sin \omega t$, respectively.

(b) For $\sigma \neq 0$, matrix A is guaranteed to have nonzero diagonal elements. Its eigenvalues are given by $\det(\lambda I - A) = 0$, which leads to

$$\det\left(\begin{bmatrix} \lambda - \sigma & \omega \\ -\omega & \lambda - \sigma \end{bmatrix}\right) = 0 \;\Rightarrow\; \lambda^2 - 2\sigma\lambda + (\sigma^2 + \omega^2) = 0$$
$$\Rightarrow \lambda_1 = \sigma + j\omega \text{ and } \lambda_2 = \sigma - j\omega.$$

The eigenvalues are distinct, hence matrix A is diagonalizable.

In this case, rather than computing the eigenvectors of A and diagonalizing it, it is more efficient to compute e^{At} by means of the following observation: Let $A_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$, and notice that A_1 and A_2 commute, i.e., $A_1 A_2 = A_2 A_1$. We then have that

$$\begin{aligned} e^{At} = e^{A_1 t} e^{A_2 t} &= \begin{bmatrix} e^{\sigma t} & 0 \\ 0 & e^{\sigma t} \end{bmatrix} \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} \\ &= \begin{bmatrix} e^{\sigma t} \cos \omega t & -e^{\sigma t} \sin \omega t \\ e^{\sigma t} \sin \omega t & e^{\sigma t} \cos \omega t \end{bmatrix}, \end{aligned}$$

where for $e^{A_1 t}$ we used the expression for the matrix exponential of diagonal matrices, and for $e^{A_2 t}$ we used the result of part (a).

(c) Matrix A is triangular, hence its eigenvalues are its diagonal entries. Therefore, the eigenvalues of A are λ_1 and λ_2. Since $\lambda_1 \neq \lambda_2$, the eigenvalues are distinct, hence A is diagonalizable. By direct calculation we can compute the following eigenvectors corresponding to these eigenvalues:

$$\text{eigenvector corresponding to } \lambda_1\text{: } \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\text{eigenvector corresponding to } \lambda_2\text{: } \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since A is diagonalizable, setting

$$W = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

we can write $A = W\Lambda W^{-1}$. We then have that

$$\begin{aligned} e^{At} = We^{\Lambda t}W^{-1} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} - e^{\lambda_1 t} \\ 0 & e^{\lambda_2 t} \end{bmatrix}. \end{aligned}$$

(d) Matrix A is diagonal, hence its eigenvalues are its diagonal entries. Therefore, $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Therefore, eigenvalues are repeated with algebraic multiplicity equal to 3. To infer whether matrix A is diagonalizable, we need to compute its eigenvectors. These can be computed as

$$Aw = \lambda w \;\; \Leftrightarrow \;\; \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \;\; \Rightarrow \; w_2 = 0, \; w_3 = 0.$$

We then have only one eigenvector, namely,

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

leading to a geometric multiplicity of 1. Since the geometric multiplicity is strictly smaller than the algebraic one, we conclude that A is non-diagonalizable.

However, notice that A is a Nilpotent matrix. In particular,

$$A^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } A^k = 0, \text{ for all } k \geq 3.$$

Directly from the Taylor series expansion, we then have that

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = I + At + \frac{1}{2}A^2t^2 = \begin{bmatrix} 1 & t & t^2 + 3t \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix}.$$

5. The so called Wien oscillator is the circuit shown in Figure 3 with $k > 1$. Denote by $v_{C_1}(t)$ and $v_{C_2}(t)$ the voltage across the capacitor C_1 and C_2, respectively, and assume that the amplifier is ideal.

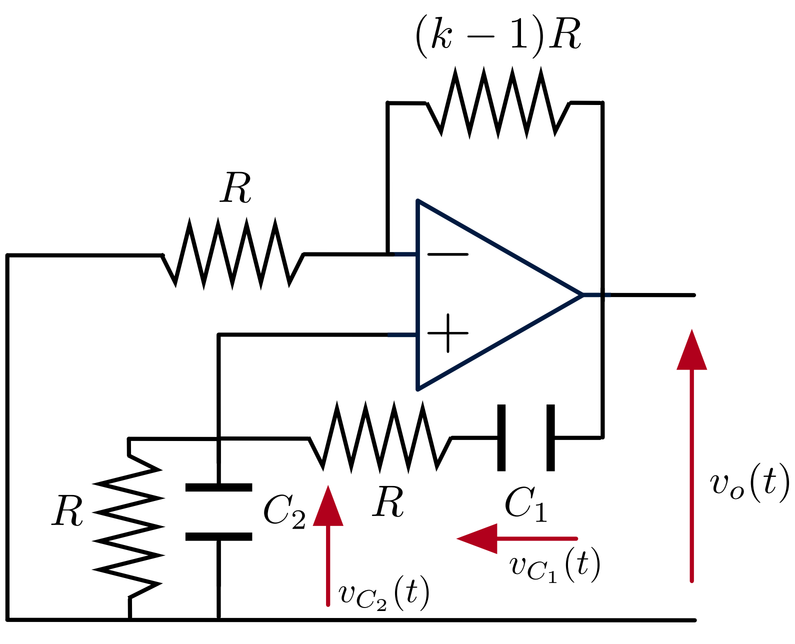

Figure 3: Wien oscillator.

(a) Let $x(t) = \begin{bmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{bmatrix}$ denote the state vector, and $y(t) = v_o(t)$ the output of the Wien oscillator circuit (notice that there is no input).

Show that its state space representation is given by

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1-k}{RC_1} \\ \frac{1}{RC_2} & \frac{k-2}{RC_2} \end{bmatrix} x(t),$$
$$y(t) = \begin{bmatrix} 0 & k \end{bmatrix} x(t).$$

(b) If $C_1 = C_2 = C$, determine the range of values for the parameter k for which the resulting system is stable, asymptotically stable, or unstable.

Solution:

(a) Denote by v_- and v_+ the voltage at the negative and positive terminal of the amplifier, respectively. As a consequence of the fact that the amplifier is ideal we have that (i) $v_- = v_+$; (ii) there is no current across the amplifier's input terminals.

Notice that $v_- = v_+ = v_{C_2}(t)$, and recall that the current across C_1 and C_2 is given by $C_1 \frac{dv_{C_1}(t)}{dt}$ and $C_2 \frac{dv_{C_2}(t)}{dt}$, respectively. By the Kirchhoff's current law (KCL), we have that

$$\texttt{KCL @}v_-\texttt{:} \quad \frac{v_{C_2}(t)}{R} + \frac{v_{C_2}(t) - v_o(t)}{(k-1)R} = 0 \;\Rightarrow\; v_o(t) = k v_{C_2}(t)$$
$$\texttt{KCL @}v_+\texttt{:} \quad \frac{v_{C_2}(t)}{R} + C_2 \frac{dv_{C_2}(t)}{dt} + C_1 \frac{dv_{C_1}(t)}{dt} = 0.$$

Moreover, by Ohm's law we have that

$$\texttt{Ohm's law:} \quad v_{C_2}(t) - v_{C_1}(t) - v_o(t) = RC_1 \frac{dv_{C_1}(t)}{dt}.$$

Substituting `KCL @`v_- in `Ohm's law`, we obtain that

$$\frac{dv_{C_1}(t)}{dt} = -\frac{1}{RC_1} v_{C_1}(t) + \frac{1-k}{RC_1} v_{C_2}(t),$$

which is the first ODE in the given state space description.

Substituting the latter equation in `KCL @`v_+, we obtain

$$\frac{dv_{C_2}(t)}{dt} = \frac{1}{RC_2} v_{C_1}(t) + \frac{k-2}{RC_2} v_{C_2}(t),$$

which is the second ODE in the given state space description. Finally, note that the output equation is given directly by `KCL` $@v_-$.

(b) Setting $C_1 = C_2 = C$, to comment on the stability of the system we compute its eigenvalues. To this end,

$$\det\left(\begin{bmatrix} \lambda + \frac{1}{RC} & \frac{k-1}{RC} \\ -\frac{1}{RC} & \lambda - \frac{k-2}{RC} \end{bmatrix}\right) = 0$$

$$\Rightarrow \lambda^2 + \frac{3-k}{RC}\lambda + \frac{1}{(RC)^2} = 0.$$

Its roots are then given by

$$\lambda_{1,2} = \frac{1}{2RC}\Big(k - 3 \pm \sqrt{(k-1)(k-5)}\Big).$$

We can then distinguish the following cases according to the values of k (recall that $k > 1$):

- $1 < k < 3$: Eigenvalues are distinct and form a complex conjugate pair with negative real part.
- $k = 3$: Eigenvalues are distinct and imaginary, i.e., $\lambda_{1,2} = \pm\frac{1}{RC}j$.
- $3 < k < 5$: Eigenvalues are distinct and form a complex conjugate pair with positive real part.
- $k = 5$: Eigenvalues are repeated and are both positive, i.e., $\lambda_{1,2} = \frac{1}{RC}$.
- $k > 5$: Eigenvalues are are distinct, real and positive. To see this notice that $\sqrt{(k-1)(k-5)} = \sqrt{(k-3)^2 - 4} < k - 3$.

Notice that for all $k > 1$ with $k \neq 5$ the associated system matrix is diagonalizable as the eigenvalues are distinct. For $k = 5$ we have two repeated eigenvalues, and it can be computed that we only have one eigenvector, namely, $[-2\ \ 1]^\top$, hence the associated matrix is non-diagonalizable. We thus have that the system is

– Stable if and only if all eigenvalues have non-positive real part, i.e., for $1 < k \leq 3$.

– Asymptotically stable if and only if all eigenvalues have negative real part, i.e., for $1 < k < 3$.

– Unstable if at least one eigenvalue has positive real part, i.e., for $k > 3$. This becomes an "if and only if" condition if in addition $k \neq 5$, i.e., for all cases where the system's matrix is diagonalizable.

6. Consider an LTI system whose state $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ evolves according to

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

where $u(t)$ is an external input.

(a) Does there exist a control input u (as a function of time) such that we can drive the system state from $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $x(2\pi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

(b) Consider the following piecewise constant input:

$$u(t) = \begin{cases} u_1 & \text{if } 0 \leq t \leq \frac{2\pi}{3}; \\ u_2 & \text{if } \frac{2\pi}{3} \leq t \leq \frac{4\pi}{3}; \\ u_3 & \text{if } \frac{4\pi}{3} \leq t \leq 2\pi. \end{cases}$$

Do there exist u_1, u_2 and u_3 such that we can drive the system state from $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $x(2\pi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

Hint: Note that the system's "A" matrix is in the form of the one in Question 3(a).

Solution:

(a) By the given system let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We are able to drive the system from any initial state at time $t = 0$ to any terminal state at $t = 2\pi$ (hence also for the given states), if it is controllable. This is the case if and only if the controllability matrix (this is a 2nd order system)

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is full rank. Indeed, $\mathrm{rank}(P) = 2$ as its rows are linearly independent. Here, P is a square matrix, so we could equivalently verify that $\det(P) = -1 \neq 0$.

(b) Setting $\omega = -1$ in Question 3(a), the state transition matrix of the system is given by

$$\Phi(t) = e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \Rightarrow \Phi(2\pi - \tau) = \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix},$$

while $\Phi(2\pi)$ is a 2×2 identity matrix. The state solution of the system at $t = 2\pi$ is thus given by

$$\begin{aligned}
x(2\pi) = {} & \Phi(2\pi)x_0 + \int_0^{2\pi} \Phi(2\pi - \tau)Bu(\tau)d\tau \\
\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = {} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^{2\pi} \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau)d\tau \\
\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = {} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^{2\pi} \begin{bmatrix} -\sin\tau \\ \cos\tau \end{bmatrix} u(\tau)d\tau \\
\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = {} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^{\frac{2\pi}{3}} \begin{bmatrix} -\sin\tau \\ \cos\tau \end{bmatrix} u_1 d\tau \\
& + \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \begin{bmatrix} -\sin\tau \\ \cos\tau \end{bmatrix} u_2 d\tau + \int_{\frac{4\pi}{3}}^{2\pi} \begin{bmatrix} -\sin\tau \\ \cos\tau \end{bmatrix} u_3 d\tau.
\end{aligned}$$

Computing the integral we obtain that

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} \cos\tau \\ \sin\tau \end{bmatrix}\Bigg|_0^{\frac{2\pi}{3}} + u_2 \begin{bmatrix} \cos\tau \\ \sin\tau \end{bmatrix}\Bigg|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} + u_3 \begin{bmatrix} \cos\tau \\ \sin\tau \end{bmatrix}\Bigg|_{\frac{4\pi}{3}}^{2\pi} \\
\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} -\frac{3}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix} + u_3 \begin{bmatrix} \frac{3}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}.
\end{aligned}$$

The last equality yields the following system of equations for u_1, u_2 and u_3:

$$\begin{bmatrix} 3 & 0 & -3 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

This system has fewer equations than unknowns, however, it is consistent thus admitting an infinite number of solutions. One of these is $u_1 = \frac{2}{3}$, $u_2 = \frac{1}{3}$ and $u_3 = 0$. Therefore, we can indeed achieve the desired state transfer by means of the suggested piecewise constant input signal. It turns out that this result is more general, and for an n-th order system, there exist at least $n-1$ switching instances such that the resulting piecewise constant input signal renders the system controllable.

7. (a) Consider a system whose state evolves according to

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= u(t), \end{aligned}$$

where $u(t)$ is a control input. Let $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Do there exist u_1 and u_2 such that $u(t) = u_1 t + u_2$ can drive the system state from $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

(b) Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. If (A, B) is controllable, would (A^2, B) be controllable as well? Justify your answer.

Solution:

(a) To decide whether the state transfer is possible (using some input) we need first to verify whether the given system is controllable. To this end, notice that it can be equivalently represented by

$$\dot{x}(t) = Ax(t) + Bu(t), \text{ where } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The controllability matrix is thus given by

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This is full rank (same P as in Question 5(a)), hence the system is controllable. As a result, the state transfer between the given $x(0)$ and $x(1)$ is possible; we will now show whether this can be achieved via the candidate input $a_1 t + a_2$.

To this end, notice that A is a Nilpotent matrix, hence the state transition matrix of the system is given by

$$\Phi(t) = e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \Rightarrow \Phi(1-\tau) = \begin{bmatrix} 1 & 1-\tau \\ 0 & 1 \end{bmatrix}.$$

The state solution of the system at $t = 1$ is thus given by

$$\begin{aligned}
x(1) = {} & \Phi(1)x_0 + \int_0^1 \Phi(1-\tau)Bu(\tau)d\tau \\
\Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} = {} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^1 \begin{bmatrix} 1 & 1-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau)d\tau \\
\Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} = {} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^1 \begin{bmatrix} 1-\tau \\ 1 \end{bmatrix} (u_1\tau + u_2)d\tau \\
\Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} = {} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^1 \begin{bmatrix} -u_1\tau^2 + (u_1 - u_2)\tau + u_2 \\ u_1\tau + u_2 \end{bmatrix} d\tau.
\end{aligned}$$

Computing the integral we obtain that

$$\begin{aligned}
\begin{bmatrix} 0 \\ 1 \end{bmatrix} = {} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} (-u_1\frac{\tau^3}{3} + (u_1 - u_2)\frac{\tau^2}{2} + u_2\tau)\big|_0^1 \\ (u_1\frac{\tau^2}{2} + u_2\tau)\big|_0^1 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} = {} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{6}u_1 + \frac{1}{2}u_2 \\ \frac{1}{2}u_1 + u_2 \end{bmatrix}.
\end{aligned}$$

The last equality yields the following system of equations for u_1 and u_2:

$$\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}.$$

Tis system of equations admits a unique solution, namely, $u_1 = 18$ and $u_2 = -8$. Therefore, it is indeed possible to achieve the desired state transfer by means of the suggested input.

(b) Even if (A, B) is controllable, (A^2, B) is not necessarily controllable. To show this, a counterexample is sufficient. To this end consider matrices A and B from part (a). Since A is a 2×2 Nilpotent matrix, $A^2 = 0$. We thus have that the controllability matrix corresponding to (A^2, B) would be

$$\begin{bmatrix} B & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The latter is not full rank, hence (A^2, B) is not controllable.

8. Consider the following two systems:

$$\text{system } S\text{:} \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$, and

$$\text{system } \hat{S}\text{:} \quad \begin{aligned} \dot{\hat{x}}(t) &= A^\top \hat{x}(t) + C^\top \hat{u}(t), \\ \hat{y}(t) &= B^\top \hat{x}(t) + D^\top \hat{u}(t), \end{aligned}$$

where $\hat{x}(t) \in \mathbb{R}^n$, $\hat{u}(t) \in \mathbb{R}^p$ and $\hat{y}(t) \in \mathbb{R}^m$.

(a) Show that S is controllable if and only if $\hat{S}$ is observable.

(b) Show that S is observable if and only if $\hat{S}$ is controllable.

Solution:

(a) We will prove this by showing that the observability matrix of $\hat{S}$ (denote it by $\hat{Q}$) has the same rank with the controllability matrix of S (denote it by P). To this end, since for any integer k, $(A^\top)^k = (A^k)^\top$

(this can be shown by means of induction), we have that

$$\hat{Q} = \begin{bmatrix} B^\top \\ B^\top A^\top \\ \vdots \\ B^\top (A^\top)^{n-1} \end{bmatrix} = \begin{bmatrix} B^\top \\ B^\top A^\top \\ \vdots \\ B^\top (A^{n-1})^\top \end{bmatrix} \in \mathbb{R}^{nm\times n}.$$

However, we have that

$$\mathrm{rank}(\hat{Q}) = \mathrm{rank}(\hat{Q}^\top) = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = \mathrm{rank}(P).$$

Hence, P is full rank if and only if $\hat{Q}$ is full rank, thus showing that S is controllable if and only if $\hat{S}$ is observable.

(b) We will prove this by showing that the controllability matrix of $\hat{S}$ (denote it by $\hat{P}$) has the same rank with the observability matrix of S (denote it by Q). To this end, since for any integer k, $(A^\top)^k = (A^k)^\top$ (this can be shown by means of induction), we have that

$$\begin{aligned}\hat{P} &= \begin{bmatrix} C^\top & A^\top C^\top & \dots & (A^\top)^{n-1}C^\top \end{bmatrix} \\ &= \begin{bmatrix} C^\top & A^\top C^\top & \dots & (A^{n-1})^\top C^\top \end{bmatrix} \in \mathbb{R}^{n\times np}.\end{aligned}$$

However, we have that

$$\mathrm{rank}(\hat{P}) = \mathrm{rank}(\hat{P}^\top) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \mathrm{rank}(Q).$$

Hence, Q is full rank if and only if $\hat{P}$ is full rank, thus showing that S is observable if and only if $\hat{S}$ is controllable.

9. Consider the transfer function

$$G(s) = \frac{1}{(s+1)(s+2)}.$$

(a) Is (A, B, C, D) with

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \end{bmatrix},\ D = 0,$$

a realization of $G(s)$? Is the system with the matrices (A, B, C, D) above controllable and observable?

(b) Is (A, B, C, D) with

$$A = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}, \; D = 0,$$

also a realization of $G(s)$? Is the system with the matrices (A, B, C, D) above controllable and observable?

(c) Comment on the effect that lack of controllability or observability may have on the realization of a transfer function.

Solution:

(a) The transfer function corresponding to the particular tuple of matrices is given by

$$\begin{aligned} C(sI - A)^{-1}B + D &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \\ &= \frac{1}{s^2 + 3s + 2} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(s+1)(s+2)}. \end{aligned}$$

This coincides with $G(s)$, hence the given tuple of matrices constitutes a realization of $G(s)$.

The associated controllability matrix is given by

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \;\Rightarrow\; \mathrm{rank}(P) = 2,$$

while the observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;\Rightarrow\; \mathrm{rank}(Q) = 2.$$

Notice that both of them are full rank.

(b) The transfer function corresponding to the particular tuple of matrices is given by

$$C(sI-A)^{-1}B+D=\begin{bmatrix}0 & 1 & 3\end{bmatrix}\begin{bmatrix}s+6 & 11 & 6\\ -1 & s & 0\\ 0 & -1 & s\end{bmatrix}^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix}+0$$

$$=\frac{1}{s^3+6s^2+11s+6}\begin{bmatrix}0 & 1 & 3\end{bmatrix}\begin{bmatrix}s^2 & \star & \star\\ s & \star & \star\\ 1 & \star & \star\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$=\frac{1}{(s+1)(s+2)(s+3)}(s+3)=\frac{1}{(s+1)(s+2)}.$$

Note that only the first element of B is nonzero, hence, we only need to compute the first column of $(sI-A)^{-1}$.

Due to the pole-zero cancellation, the resulting transfer function coincides with $G(s)$, hence the given tuple of matrices constitutes another realization of $G(s)$.

The associated controllability matrix is given by

$$P=\begin{bmatrix}B & AB & A^2B\end{bmatrix}=\begin{bmatrix}1 & -6 & 25\\ 0 & 1 & -6\\ 0 & 0 & 1\end{bmatrix}\quad\Rightarrow\ \text{rank}(P)=3,$$

while the observability matrix is given by

$$Q=\begin{bmatrix}C\\ CA\\ CA^2\end{bmatrix}=\begin{bmatrix}0 & 1 & 3\\ 1 & 3 & 0\\ -3 & -11 & -6\end{bmatrix}\quad\Rightarrow\ \text{rank}(Q)<3,$$

since $\det(Q)=0$. Therefore, the observability matrix is not full rank, and as a result the system is not observable.

(c) Notice that the lack of observability in part (b) – a similar argument would hold if the system was uncontrollable instead – resulted in a pole zero cancellation. Hence, even though the matrix tuples of parts (a) and (b) are both valid realizations of $G(s)$, the order of the system in part (a) turns out to be lower than that of part (b). In general, lack of controllability or observability is reflected by pole-zero cancelations and leads to a non-minimal realization of the system's transfer function.

10. OPTIONAL: Consider the controllability Gramian

$$W_c(t) = \int_0^t e^{A\tau} B B^\top e^{A^\top \tau} d\tau \in \mathbb{R}^{n\times n}.$$

Show that if it is invertible for a particular $\bar{t}$, i.e., $W_c(\bar{t}) \succ 0$, then it is invertible for any t, i.e., $W_c(t) \succ 0$, for all $t \in \mathbb{R}$.

Note: The fact that $W_c(t) \succ 0$, for all $t \geq \bar{t}$ is easier to show compared to the case where $t < \bar{t}$.

Solution: Assume that $W_c(\bar{t})$ is invertible for some $\bar{t}$. By Fact 10 in the notes this is equivalent to the system being controllable over $[0, \bar{t}]$, while by Fact 11 this is in turn equivalent to the controllability matrix P being full rank. We show that $W_c(t)$ will then be invertible for all t as well, and we will do this separately for $t > \bar{t}$ and $t \leq \bar{t}$.

Case $t \geq \bar{t}$: Since $W_c(\bar{t})$ is invertible, we have that $W_c(\bar{t}) \succ 0$. However, we also have that

$$x^\top \Big(W_c(t) - W_c(\bar{t}) \Big) x = \int_{\bar{t}}^t x^\top e^{A\tau} B B^\top e^{A^\top \tau} x d\tau \geq 0,$$

while the inequality holds as the integrand is non-negative since we have that $x^\top e^{A\tau} B B^\top e^{A^\top \tau} x = \|B^\top e^{A^\top \tau} x\|^2$, and $\bar{t} < t$. Hence we have that

$$x^\top \Big(W_c(t) - W_c(\bar{t}) \Big) x \geq 0, \text{ for all } x \neq 0 \;\Rightarrow\; W_c(t) \succeq W_c(\bar{t}) \succ 0,$$

where the last inequality is strict since $W_c(\bar{t}) \succ 0$. This in turn implies that $W_c(t) \succ 0$, i.e., $W_c(t)$ is invertible for all $t \geq \bar{t}$.

Case $t < \bar{t}$: For the sake of contradiction assume that exists $t^\star < \bar{t}$ such that $W_c(t^\star)$ is not positive definite. This implies that $W_c(t^\star)$ would not be invertible, or in other words, there would exist $\bar{x} \neq 0$ such that

$W_c(t^\star)\bar{x} = 0$. By the definition of the controllability gramian, we then have that (see proof of Fact 11)

$$W_c(t^\star)\bar{x} = 0 \;\Leftrightarrow\; B^\top e^{A^\top \tau}\bar{x} = 0, \text{ for all } \tau \in [0, t^\star].$$

Performing a Taylor series expansion of the last expression around $\tau = 0$ as in the proof of Fact 11 in the notes, this would be equivalent to $P\bar{x} = 0$, where P is the controllability matrix. However, $\bar{x}$ is nonzero; this in turn implies that P is not full rank, establishing a contradiction as we have already shown that P is full rank (as a result of $W_c(\bar{t})$ being invertible). We thus have that $W_c(t)$ will have to be invertible for all $t < \bar{t}$.