Lecture 4 Aerofoils in high speed flight.

- Definitions of subsonic, transonic and supersonic flow.
- Shock-expansion method from B19 lecture course for surface pressures, lift and drag.
- Small perturbation theory and the Prandtl-Glauert Rule for supersonic flow.
- Use of Prandtl-Glauert rule to estimate Critical mach number.

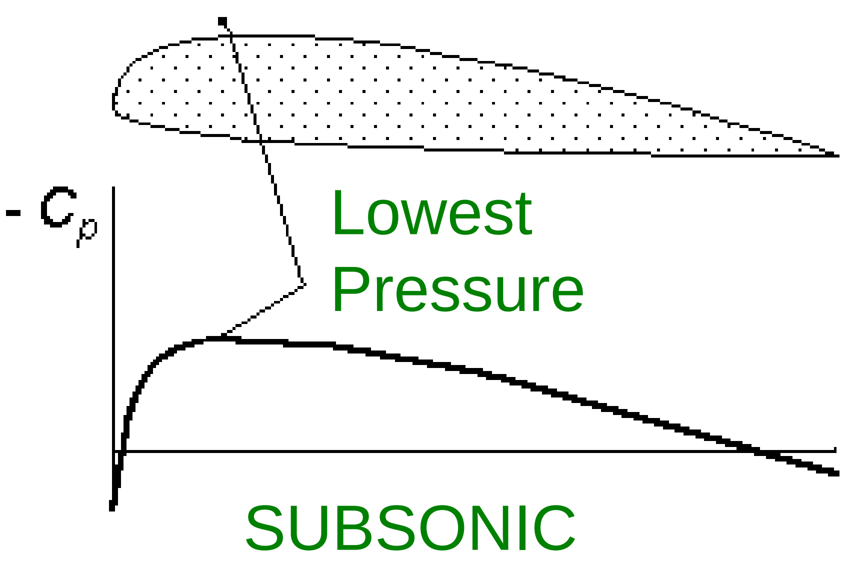
Subsonic, Transonic and Supersonic Flow

Wings at high mach numbers

Lectures 1-3 restricted to incompressible flows. (say $M < 0.3$).

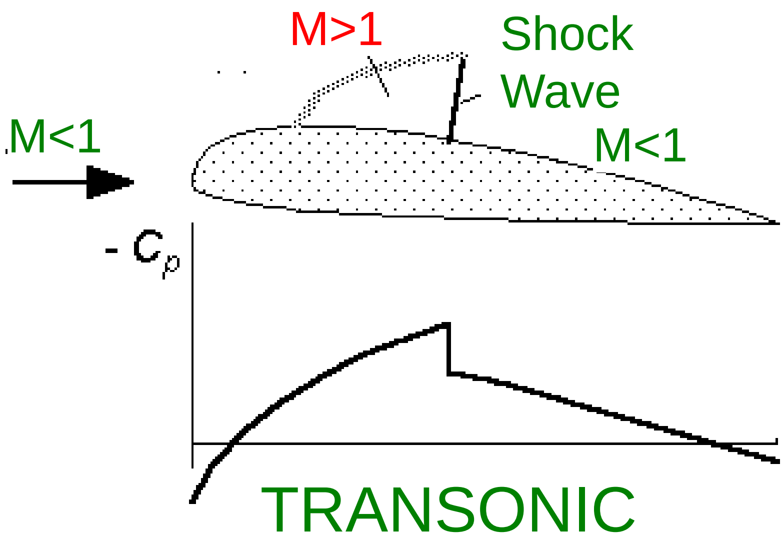
This lecture is to to explain some aspects of wing behaviour in high subsonic-, transonic- and supersonic flows.

Definitions of subsonic, transonic and supersonic flow



The speed of air over a wing, particularly on the upper surface, is faster than the free stream speed and the pressures are lower. If the highest speed <u>at any point</u> remains below the local speed of sound, then the flow is subsonic. The **Critical Mach number** is the (subsonic) free stream Mach number when sonic flow first occurs at any point on the aerofoil.

Transonic Flow



At Mach numbers above critical, the free stream is still subsonic, but regions of local supersonic flow appear near the surfaces where the local air speed is highest. Where the locally supersonic flows revert to subsonic, shock waves occur. These becomes progressively stronger, and move towards the tailing edge as the free stream approaches sonic speed.

The sharp pressure rise across these shock waves may cause the boundary layer to separate, causing unexpected changes in lift and pitching moment. The separation may also be unsteady, causing "buffeting".

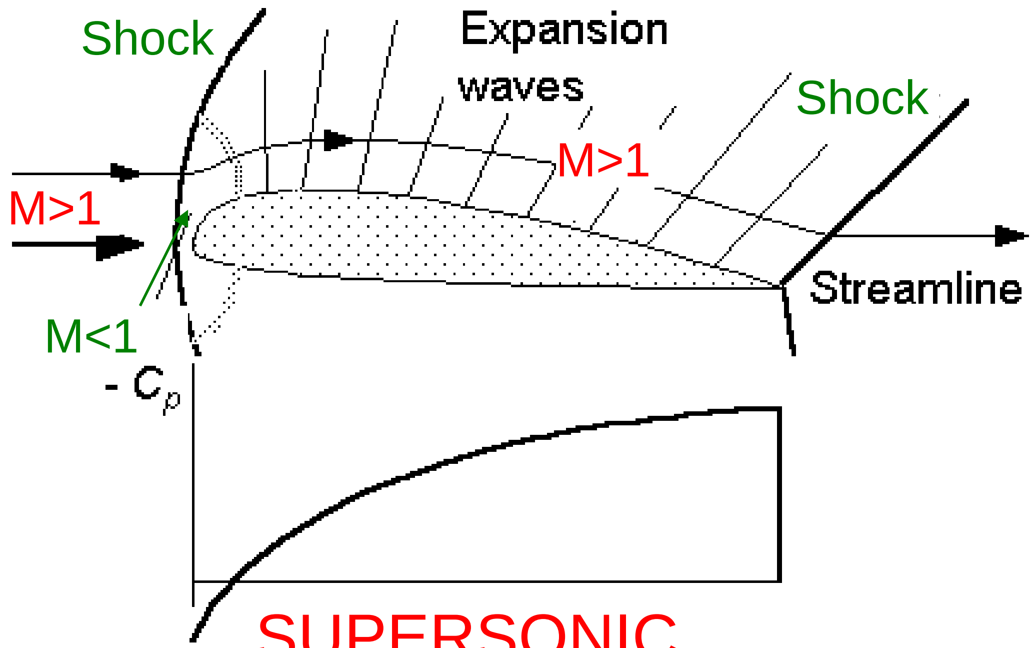
Supersonic Flow: Free stream Mach number >= 1

Flow is:

• turned and slowed by shock waves

turned and accelerated by expansion waves.

•Pressures and centre of pressure are totally different from subsonic flow.

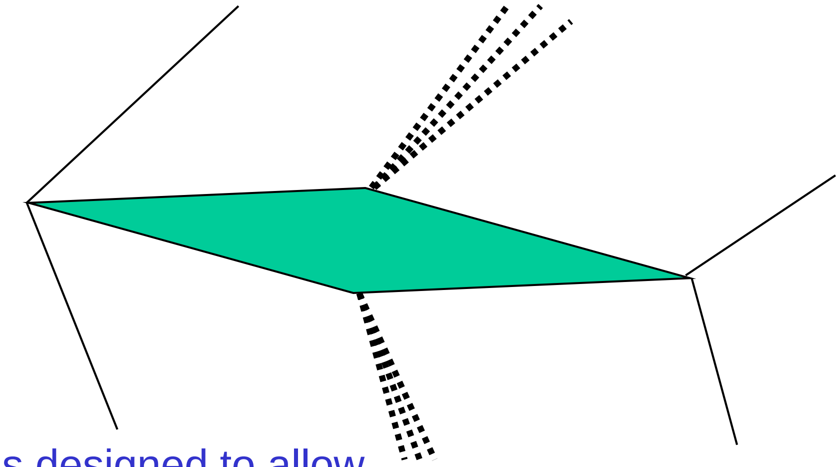


Any blunt leading edge demands large flow deflections:

•the shock wave will detach

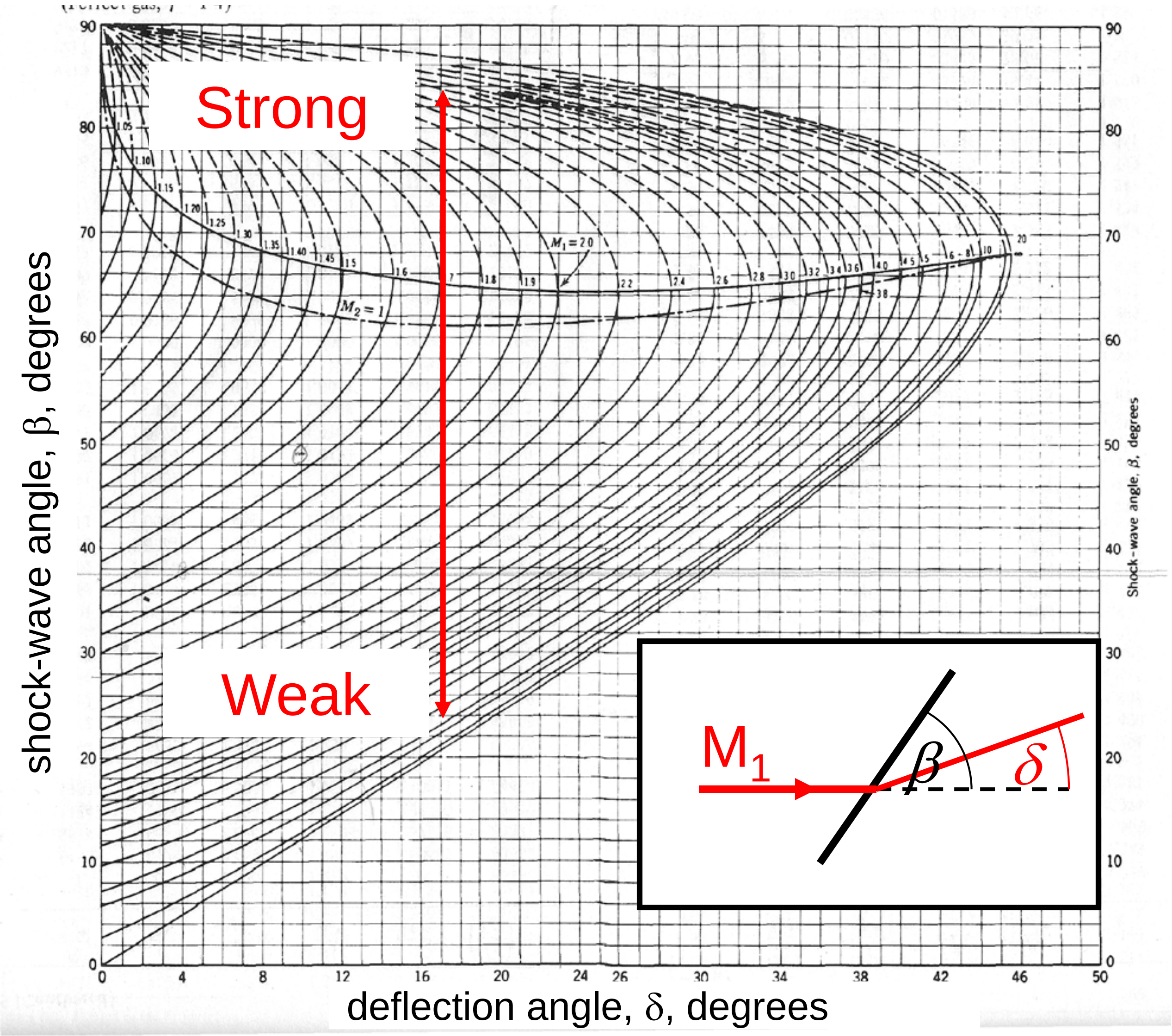
•produce a smaller deflection

•subsonic flow can turn further to pass round the aerofoil



Supersonic wing is designed to allow attached shock at the leading edge

Oblique Shock Tables: Shock-wave vs. flow deflection angle



The shock chart on H.L.T p.122. shows no solution when the deflection angle is too large.

The shock is no longer straight and oblique, but detached from the surface and curved.

The weak shock normally occurs, but the strong shock will occur if the down stream pressure is independently increased.

Analysis of SUPERSONIC FLOW: Shockwaves

This material was covered in the B19 Gas Dynamics Course

For oblique shock waves:

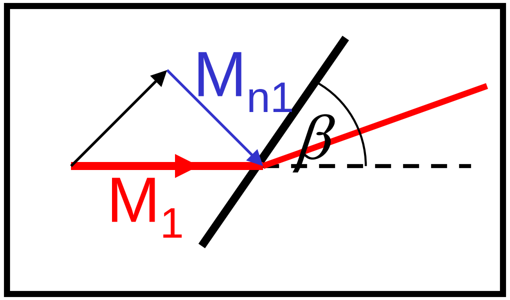
$$p_2 / p_1 = \frac{2\gamma M_{n_1} - (\gamma - 1)}{(\gamma + 1)},$$

and

$$\frac{V_{n_2}}{V_{n_1}} = \rho_1 / \rho_2 = \frac{2 - (\gamma - 1) M_{n_1}^2}{(\gamma + 1) M_{n_1}^2},$$

where

$$M_{n_1} = M_1 \sin\beta.$$



Remember:

The flow components || to the shock are unchanged

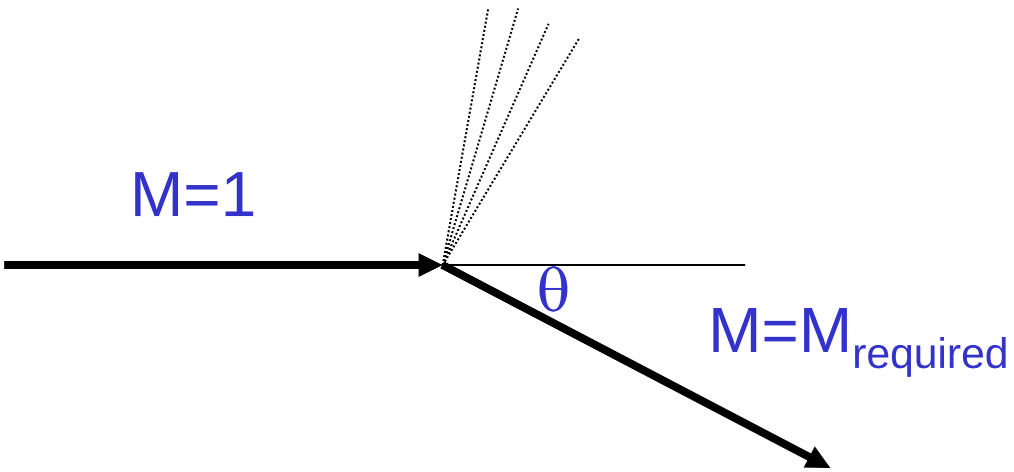
The flow components perpendicular to the shock behave like a normal shock (apply momentum & continuity)

Analysis of SUPERSONIC FLOW: Expansion Waves

This material was covered in the B19 Gas Dynamics Course

For isentropic flow:

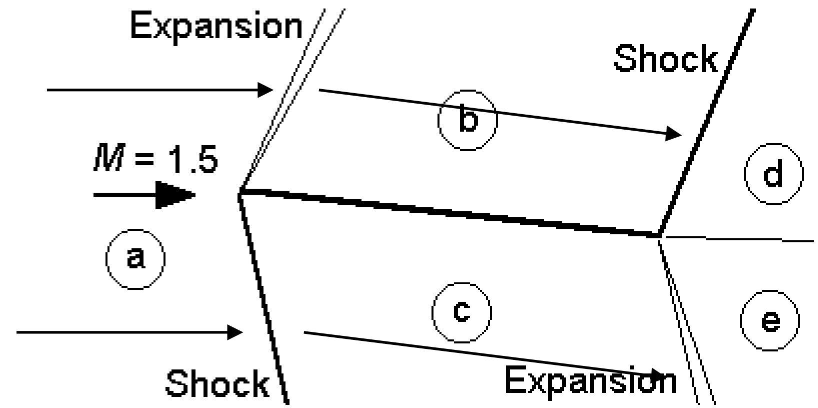
$$\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho_0}{\rho}\right)^{\gamma-1} = \frac{T_0}{T} = 1 + \frac{1}{2}(\gamma-1)M^2$$



$$\theta = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left(\frac{(\gamma-1)(M^2-1)}{\gamma+1}\right) - \tan^{-1}\sqrt{(M^2-1)}$$

Theta is the Prandtl – Meyer Angle

The shock expansion method



Example - Flat Plate Aerofoil

Find the pressures and the lift coefficient on a flat plate at 5° incidence in air at M =1.5

Region a – b

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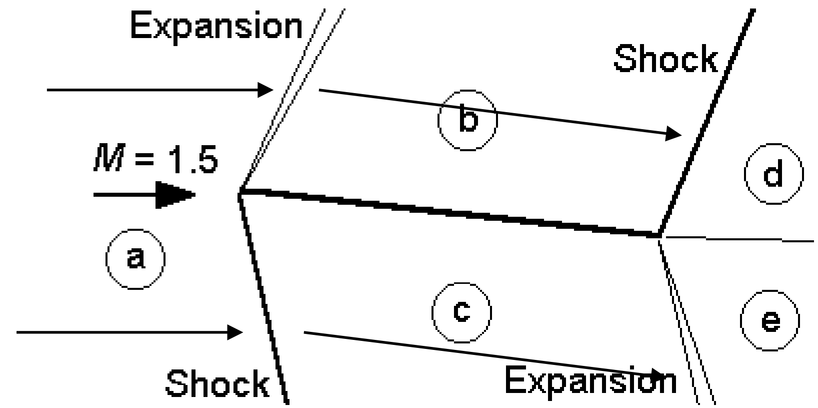
	M	$\frac{p}{p_0}$	$\frac{\rho}{\rho_0}$	$\frac{T}{T_0}$	$\frac{a}{a_0}$	$\frac{A^*}{A}$	$\frac{\frac{\rho}{2}V^2}{p_0}$	θ
a	1.50	.2724	.3950	.6897	.8305	.8502	.4290	11.91
	1.51	.2685	.3909	.6868	.8287	.8453	.4285	12.20
	1.52	.2646	.3869	.6840	.8270	.8404	.4279	12.49
	1.53	.2608	.3829	.6811	.8253	.8354	.4273	12.79
	1.54	.2570	.3789	.6783	.8236	.8304	.4266	13.09
	1.55	.2533	.3750	.6754	.8219	.8254	.4259	13.38
	1.56	.2496	.3710	.6726	.8201	.8203	.4252	13.68
	1.57	.2459	.3672	.6698	.8184	.8152	.4243	13.97
	1.58	.2423	.3633	.6670	.8167	.8101	.4235	14.27
	1.59	.2388	.3595	.6642	.8150	.8050	.4226	14.56
	1.60	.2353	.3557	.6614	.8133	.7998	.4216	14.86
	1.61	.2318	.3520	.6586	.8115	.7947	.4206	15.16
	1.62	.2284	.3483	.6558	.8098	.7895	.4196	15.45
	1.63	.2250	.3446	.6530	.8081	.7843	.4185	15.75
	1.64	.2217	.3409	.6502	.8064	.7791	.4174	16.04
	1.65	.2184	.3373	.6475	.8046	.7739	.4162	16.34
	1.66	.2151	.3337	.6447	.8029	.7686	.4150	16.63
b	1.67	.2119	.3302	.6419	.8012	.7634	.4138	16.93
	1.68	.2088	.3266	.6392	.7995	.7581	.4125	17.22
	1.69	.2057	.3232	.6364	.7978	.7529	.4112	17.52

+5°

The shock expansion method



Example - Flat Plate Aerofoil

Find the pressures and the lift coefficient on a flat plate at 5° incidence in air at M =1.5

Region a: Undisturbed free stream: use table on H.L.T. p.99 with M = 1.5

$$\frac{p_a}{p_0} = 0.2724 \qquad \theta_a = 11.91^\circ$$

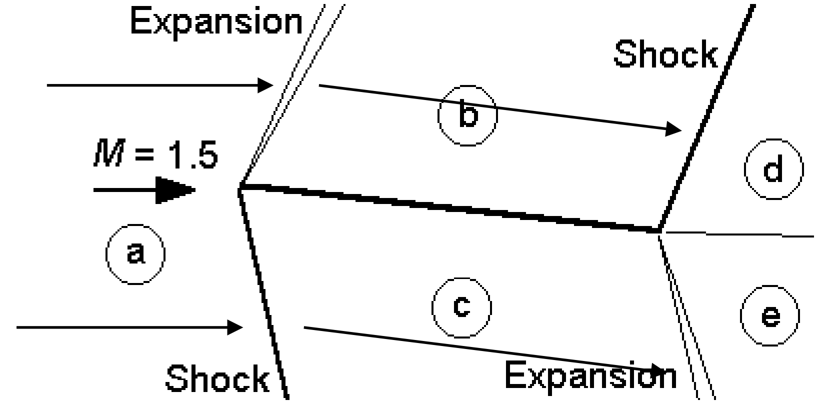
θ = Prantl-Meyer Angle (=0 at M=1)

Region b: Isentropic expansion from region a with 5° deflection

$$\theta_b = 16.91^\circ \quad (11.91^\circ + 5^\circ) \qquad \frac{p_b}{p_0} = 0.2121 \qquad M_b = 1.67$$

The shock expansion method

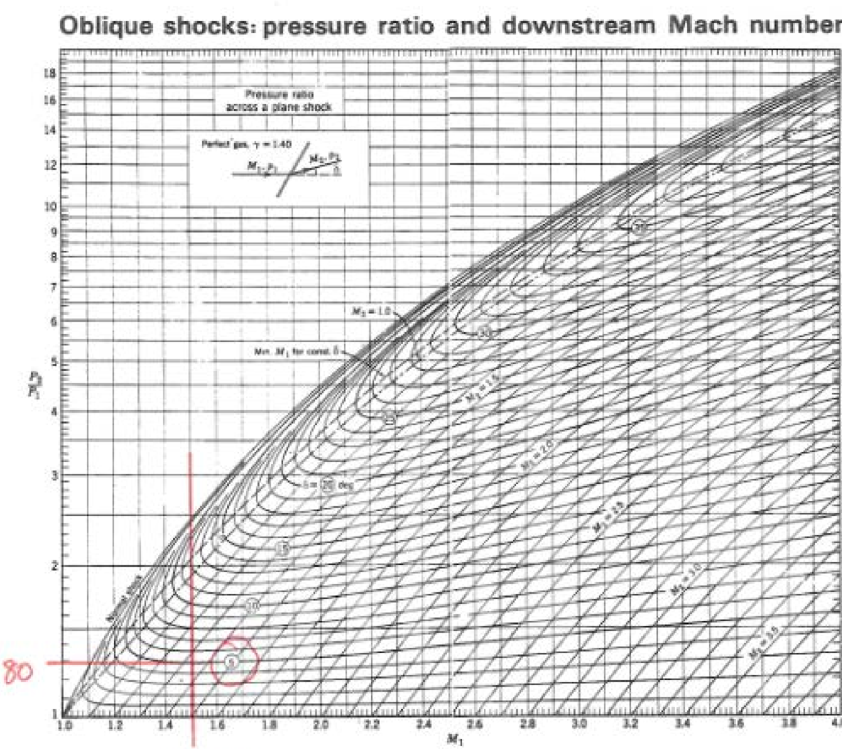
Find the pressures and the lift coefficient on a flat plate at 5° incidence in air at M =1.5



Example - Flat Plate Aerofoil

Region c: Oblique shock from region a with 5° flow deflection.

Enter chart on H.L.T p. 123 with M = 1.5 and δ = 5°



$$\frac{p_c}{p_a} = 1.2780$$

$P_2/P_1 = 1\cdot2780$

$M_1 = 1\cdot5$

Lift and Drag Coefficient

The pressure difference p_c - p_b acting over the chord length c, causes a force f per unit span, normal to the flat plate. Dividing by the dynamic pressure, gives the usual coefficient form.

$$C_f = \frac{(p_c - p_b)c}{\frac{1}{2}\rho_a v_a^2 c}$$

Recalling that $v_a^2 = M_a^2 \frac{\gamma p_a}{\rho_a}$ we write: $C_f = \frac{\frac{p_c}{p_a} - \frac{p_b}{p_a}}{\frac{1}{2}\gamma M_a^2} = \frac{\frac{p_c}{p_a} - \frac{p_b}{p_0}\frac{p_0}{p_a}}{\frac{1}{2}\gamma M_a^2}$

Substitute in:

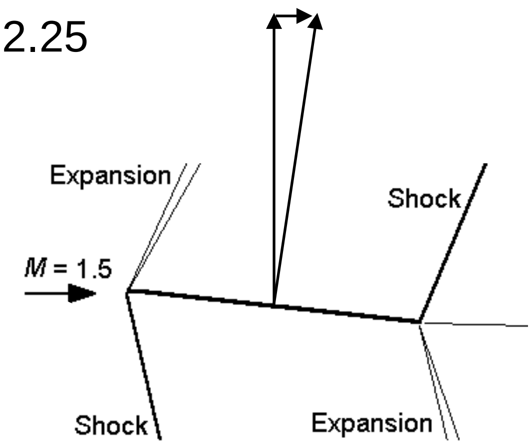
$\frac{p_a}{p_0} = 0.2724$ $\frac{p_b}{p_0} = 0.2121$ $\frac{p_c}{p_a} = 1.2780$

$$C_f = \frac{1.2780 - \frac{0.2121}{0.2724}}{\frac{1}{2} \times 1.4 \times 2.25} = 0.317$$

The lift $\boldsymbol{l}$ and wave drag $\boldsymbol{d_w}$ per unit span are then defined by

$l = f\cos\alpha$ and $d_w = f\sin\alpha$ respectively.

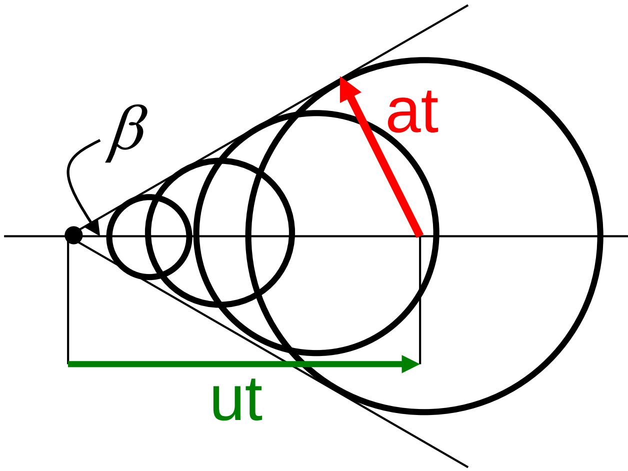
Taking components of $\boldsymbol{C_f}$: $C_l = 0.316$ $C_{d_w} = 0.028$



Derivation of Small Perturbation Pressure Equation

This approximation is based upon the fact that, for small flow deflections, shock waves became weak, and may be approximated by isentropic compression waves. It allows a simpler calculation. *'If the pressure waves are weak and flow deflections are small, the differential form of the equations will do.'* Therefore, we start by describing a weak expansion wave.

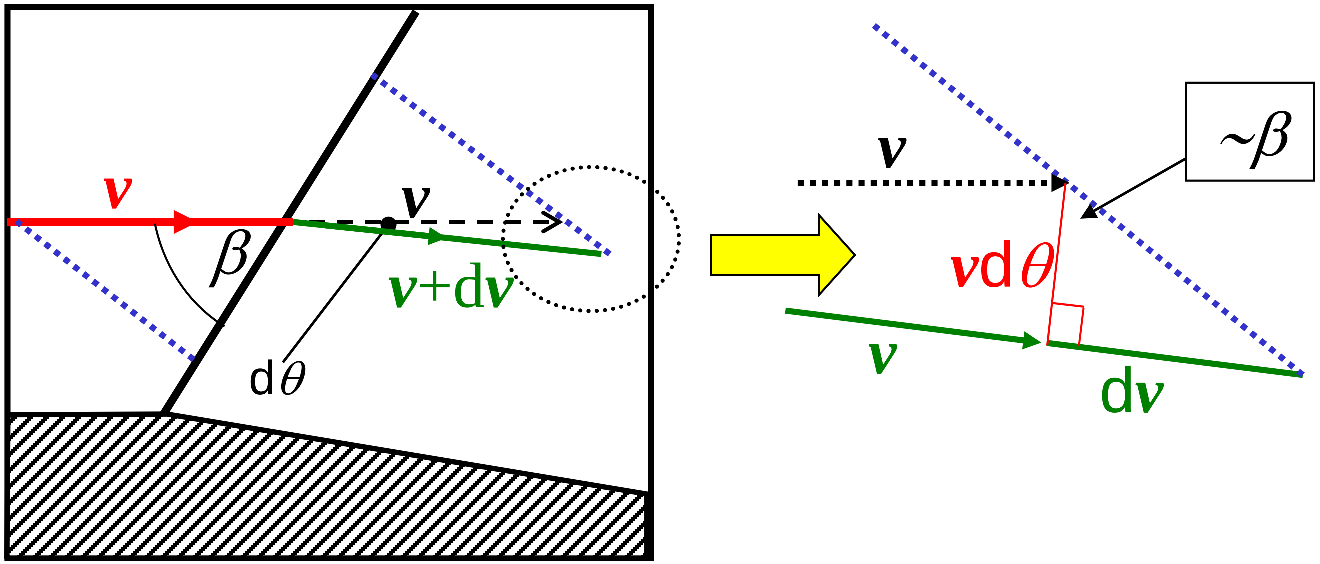
If the disturbance in the flow field is ***infinitessimal***, then the disturbance propagates through the flow as a Mach wave, propagating at speed a in all directions from the disturbed flow, and moving at speed u in the flow direction. Hence, from the diagram the Mach Wave angle is characterised as



$$\sin\beta = \frac{at}{ut} = \frac{a}{u} = \frac{1}{M} \qquad (a = sonic\ velocity)$$

Derivation of Small Perturbation Pressure Equation

The argument behind this derivation is, *'If the pressure waves are weak and flow deflections are small, the differential form of the equations will do.'* Therefore, we start by describing a weak expansion wave.



Now consider deflection through a small angle $d\theta$. From the diagram:

$$\sin\beta = \frac{1}{M}$$

$$\frac{dv}{v} = \tan\beta \; d\theta = \frac{\sin\beta}{\sqrt{1-\sin^2\beta}} d\theta = \frac{\frac{1}{M}}{\sqrt{1-\left(\frac{1}{M}\right)^2}}$$

$$\frac{\mathrm{d}v}{v} = \frac{\mathrm{d}\theta}{\sqrt{M^2-1}} \qquad \text{...... eqn (1)}$$

Derivation of Small Perturbation Pressure Equation

The appearance of velocity is awkward since we have been dealing with Mach numbers, for isentropic inviscid flow:

The energy of the flow can be described by $Q - W_s = \Delta\left[C_p T + \frac{v^2}{2} + gz\right]$

As the system is adiabatic and there is no shaft work E= $C_p T + \frac{v^2}{2} = Constant$

As $C_p\left(\frac{\gamma - 1}{\gamma}\right) = R$, E= $\frac{\gamma R T}{\gamma - 1} + \frac{v^2}{2} = Constant.$

If we differentiate this $\frac{\gamma R dT}{\gamma - 1} + vdv = 0.$

and $\times \frac{\gamma - 1}{\gamma R T}$ we obtain $\frac{dT}{T} + \frac{\gamma - 1}{\gamma R T} M^2 \gamma R T \frac{dv}{v} = 0.$ $\frac{dT}{T} + (\gamma - 1) M^2 \frac{dv}{v} = 0 \ldots\ldots$ eqn (2)

Derivation of Small Perturbation Pressure Equation

$$\frac{dv}{v} = \frac{d\theta}{\sqrt{M^2 - 1}} \quad \ldots\ldots \text{ eqn (1)}$$

$$\frac{dT}{T} + (\gamma - 1)M^2 \frac{dv}{v} = 0. \quad \ldots\ldots \text{ eqn (2)}$$

Furthermore for an isentropic perfect gas

$$\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = \frac{T_0}{T} \quad \Rightarrow \frac{\gamma - 1}{\gamma}\frac{dp}{p} = \frac{dT}{T} \quad \ldots\ldots \text{ eqn (3)}$$

Eliminate ***v*** and ***T*** from (1), (2) & (3) to give

$$\frac{\gamma - 1}{\gamma}\frac{dp}{p} = (1 - \gamma)M^2 \frac{dv}{v}$$

$$\frac{\gamma - 1}{\gamma}\frac{dp}{p} = (1 - \gamma)M^2 \frac{d\theta}{\sqrt{M^2 - 1}}$$

$$\frac{dp}{p} = -\gamma M^2 \frac{d\theta}{\sqrt{M^2 - 1}} \quad \ldots\ldots \text{ eqn (4)}$$

Derivation of Small Perturbation Pressure Equation

By definition: pressure coefficient $$C_p = \frac{\mathrm{d}p}{\frac{1}{2}\rho v^2} = \frac{\mathrm{d}p}{\frac{1}{2}\left(\frac{p}{RT}\right)M^2\gamma RT}$$

$$= \frac{\mathrm{d}p}{\frac{1}{2}\gamma p M^2}$$

Substitute for d***p*** using eqn 4: $$\frac{\mathrm{d}p}{p} = -\gamma M^2 \frac{\mathrm{d}\theta}{\sqrt{M^2-1}} \quad \text{...... eqn (4)}$$

$$C_p = \frac{\mathrm{d}p}{\frac{1}{2}\gamma p M^2} = \frac{-\gamma M^2 \frac{\mathrm{d}\theta}{\sqrt{M^2-1}} p}{\frac{1}{2}\gamma p M^2} = \frac{-2\mathrm{d}\theta}{\sqrt{M^2-1}} \quad \text{...... eqn (5)}$$

NB dp is change from a→b or a→c, but not b→c in previous example

dθ is the angle of the local surface <u>relative to the free stream</u>. The sign is positive for an expansive deflection. Thus for the flat plate in the previous example, d**θ** is 0.08727rad, (5°) on the top surface, giving a negative pressure by eqn. 5, and conversely positive on the lower surface.

Small Perturbation Pressure Equation

$d\theta$ is the angle of the local surface <u>relative to the free stream</u>. The sign is positive for an expansive deflection. Thus for the flat plate in the previous, $d\theta$ is 0.08727rad, (5°) on the top surface, giving a negative pressure by eqn. 5, and conversely positive on the lower surface.

$$C_p = \frac{-2d\theta}{\sqrt{M^2 - 1}}$$

For the flat plate

$$C_f = C_{p,l} - C_{p,u}$$

$$C_f = \frac{4\left|d\theta\right|}{\sqrt{M^2 - 1}}$$

$$C_f = \frac{4 \times 0.087266}{\sqrt{1.5^2 - 1}} = 0.3122$$

This compares well with the value obtained from the shock – expansion method = 0.317

Approximation for Small Flow Deflections

For M_1 = 1.5, the table below compares shock pressure ratios with the isentropic flow tables (H.L.T. p.80), or the equations derived above

Deflection	Shock (M_1 = 1.5)	Isentropic compression (M_1 = 1.5)		
$\theta°$	p_2/p_1	p_2/p_0	p_1/p_0	p_2/p_1
4.0	1.2165	0.331612	0.27240	1.2173
6.0	1.3433	0.36883	0.274611	1.3431

High speed subsonic flow

Prandtl-Glauert Similarity Rule For Subsonic Flow

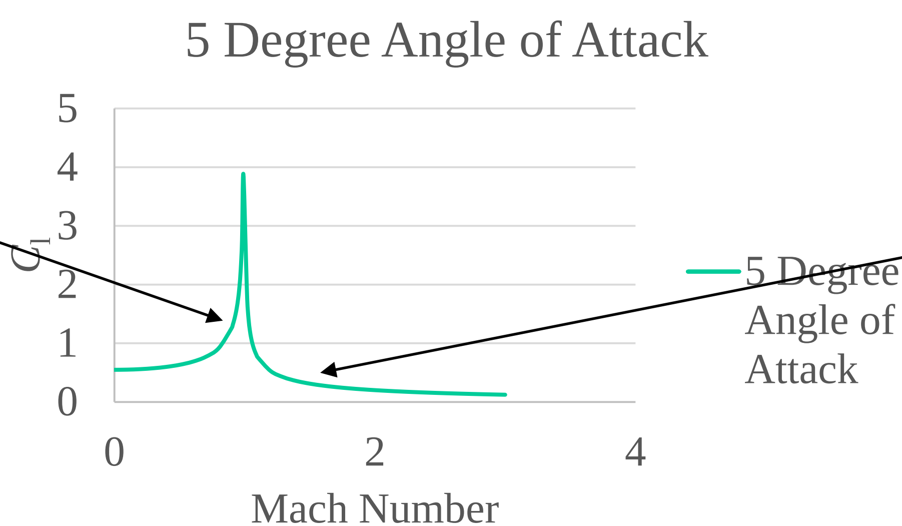
For supersonic flow, small perturbation theory describes the variation of pressure, and hence of lift and drag. The specific effect of Mach number is the 'Supersonic Similarity Rule'

$$C \propto \frac{1}{\sqrt{M^2 - 1}}$$

Although we shall not discuss it, there exists an equivalent small perturbation theory for subsonic flow. This leads to the Prandtl-Glauert Similarity Rule:

$$C = \frac{C_i}{\sqrt{1 - M^2}}$$

$$C = \frac{C_i}{\sqrt{1 - M^2}}$$



$$C_f = \frac{4|d\theta|}{\sqrt{M^2 - 1}}$$

High speed subsonic flow

We have no exact calculation to validate this equation, but as in the supersonic case, the small perturbation approximation is limited to thin aerofoils, specifically when

$$\frac{\left(t/c\right)}{\sqrt{1-M^2}} << 1$$

i.e. the nearer to sonic conditions, the thinner the aerofoil needs to be

The Prantl-Glauert rule relates pressure or force coefficients *C*, at high subsonic free stream Mach numbers *M*, to the corresponding coefficient C_i for incompressible flow.

High speed subsonic flow

It is extremely simple to apply the Prandtl Glauert rule

$$C = \frac{C_i}{\sqrt{1 - M^2}}$$

Example: Effect of Mach number on Lift curve slope

The lift curve slope of aerofoils in incompressible flow, is 2π per radian. Find the corresponding lift curve slope at a free stream Mach number of 0.7.

$C_{l,M=0} = 2\pi\alpha$ For a flat plate, the only component contributing to lift vs slope

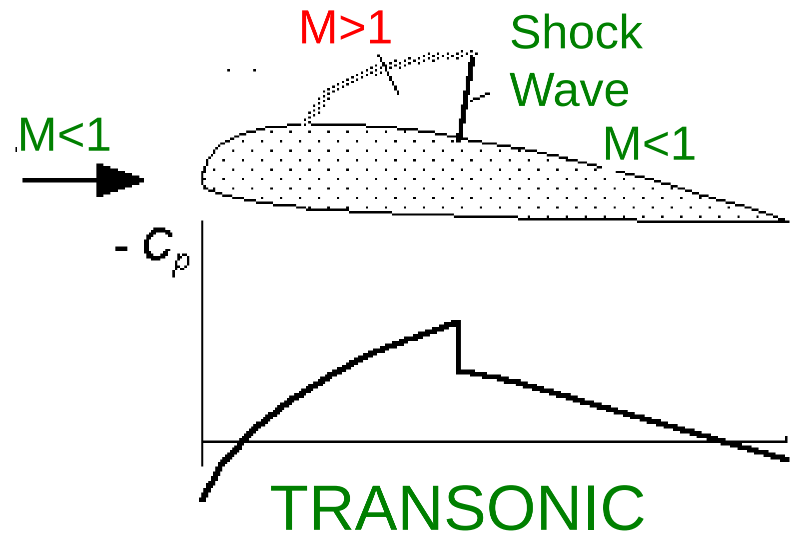
$$\frac{\mathrm{d}C_l}{\mathrm{d}\alpha} = \frac{\mathrm{d}\dfrac{C_{l,M=0}}{\sqrt{1-M^2}}}{\mathrm{d}\alpha} = \frac{\mathrm{d}\dfrac{2\pi\alpha}{\sqrt{1-M^2}}}{\mathrm{d}\alpha} \frac{2\pi}{\sqrt{1-0.7^2}} = 8.798 / \text{radian}$$

cf. 6.28 / radian

Critical Mach Number

Over the upper surface of an aerofoil the incompressible pressure coefficient is usually negative with a minimum $C_{p,i}$ at a particular location. As the free stream Mach number M_∞ increases, the local Mach number M at the peak suction point, will reach unity locally. This marks the beginning of transonic flow. The Critical Mach Number M_c, is defined as the value of M_∞ when the local surface Mach number M, first reaches 1.0.

The aim of the following analysis, is to predict the critical mach number M_C, in terms of the lowest pressure coefficient $C_{p,i}$ in incompressible flow.



Equations:

Subsonic similarity: $$C_p = \frac{C_{p,i}}{\sqrt{1-M^2}}. \qquad \text{...... eqn (b1)}$$

By definition $$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty v_\infty^2}. \qquad \text{...... eqn (b2)}$$

Also by definition $$v_\infty^2 = M_\infty^2 \gamma R T_\infty = M_\infty^2 \frac{\gamma p_\infty}{\rho_\infty}. \qquad \text{...... eqn (b3)}$$

Critical Mach Number

Starting with a constant energy equation for regions with no shocks: $C_p T + \frac{v^2}{2} = C_p T_0$

As before: $C_p\left(\frac{\gamma - 1}{\gamma}\right) = R$ hence $\frac{\gamma R T}{\gamma - 1} + \frac{v^2}{2} = \frac{\gamma R T_0}{\gamma - 1}$

Substitute $v^2 = M^2 \gamma R T$ yields $\frac{T_0}{T} = \left(1 + \frac{1}{2}(\gamma - 1)M^2\right)$

For isentropic conditions $\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$

Thus the local isentropic energy equation is: $\frac{p_0}{p} = \left(1 + \frac{1}{2}(\gamma - 1)M^2\right)^{\frac{\gamma}{\gamma-1}}$, …… eqn (b4)

and for the mainstream where $M = M_\infty$ $\frac{p_0}{p_\infty} = \left(1 + \frac{1}{2}(\gamma - 1)M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}$…… eqn (b5)

Critical Mach Number

Now to find the critical Mach number, set the local Mach number at a point over the blade to 1 and the mainstream flow set to $M_\infty = M_c$. Also for air put $\gamma = 1.4$. Substitute

$$v_\infty^2 = M_\infty^2 \frac{\gamma p_\infty}{\rho_\infty} \qquad \text{into } C_p = \frac{p - p_\infty}{\tfrac{1}{2}\rho_\infty v_\infty^2}. \quad \text{to eliminate } v$$

$$C_p = \left(\frac{p}{p_\infty}\right) - 1 \Big/ 0.7 M_c^2 \qquad \text{...... eqn (b6)}$$

taking the ratio of

$$\frac{\dfrac{p_0}{p_\infty} = \left(1 + \dfrac{1}{2}(\gamma - 1)M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}}{\dfrac{p_0}{p} = \left(1 + \dfrac{1}{2}(\gamma - 1)M^2\right)^{\frac{\gamma}{\gamma-1}}}$$

with M=1, M_∞=M_c

$$\frac{p}{p_\infty} = \left(\frac{1 + 0.2 M_c^2}{1.2}\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{...... eqn (b7)}$$

Critical Mach Number (2)

Finally we combine eqns 1, 6 and 7 and equate the pressure coefficient to the equivalent for non-compressible flow using subsonic similarity arguments.

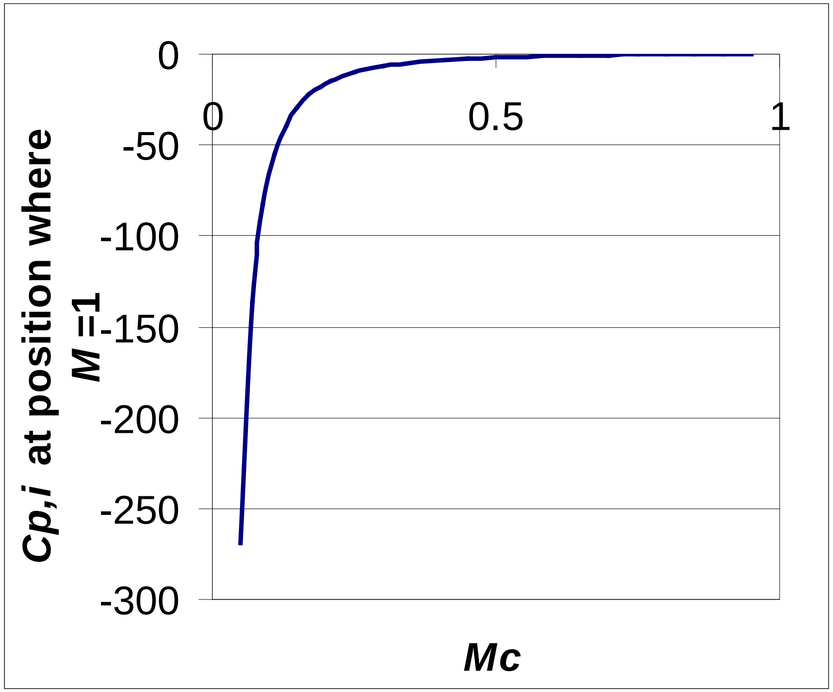
$$C_p = \frac{C_{p,i}}{\sqrt{1-M^2}} \Rightarrow C_{p,i} = C_p\sqrt{1-M_c^2}$$

$$C_{p,i} = \frac{\left(\dfrac{p}{p_\infty}\right)-1}{0.7M_c^2}\sqrt{1-M_c^2}$$

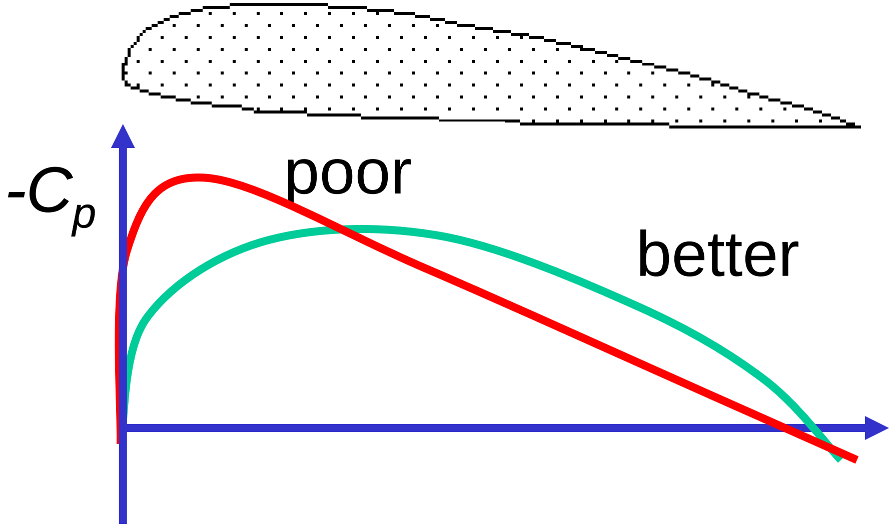
$$C_{p,i} = \frac{\left(\dfrac{1+0.2M_c^2}{1.2}\right)^{3.5}-1}{0.7M_c^2}\sqrt{1-M_c^2}$$

How can we raise the critical Mach number?

$$C_{p,i} = \frac{\left(\frac{1+0.2M_c^2}{1.2}\right)^{3.5} - 1}{0.7M_c^2}\sqrt{1-M_c^2}$$



The equation shows that the critical Mach number will be increased if the minimum incompressible pressure coefficient can be raised. To achieve a specified lift coefficient, we require a particular area inside the pressure distribution. Therefore, design the aerofoil to give a flat-topped pressure distribution without a prominent suction peak.

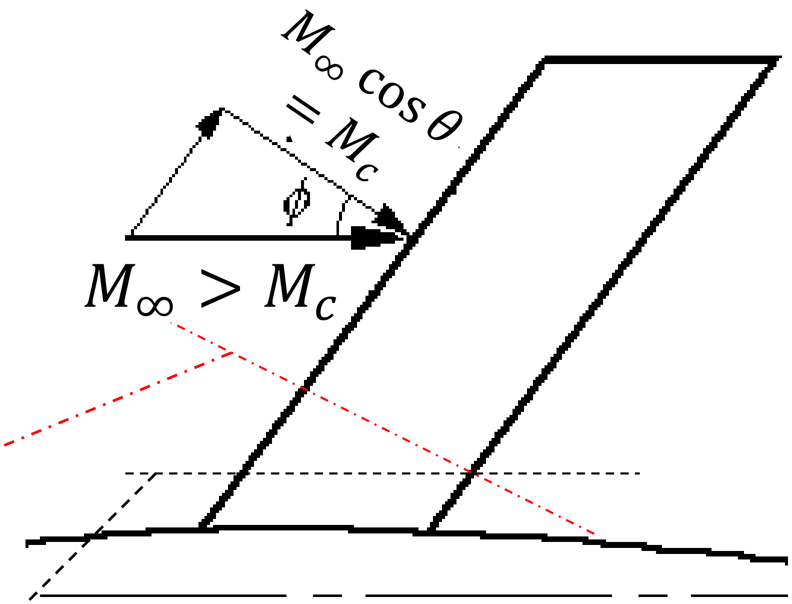


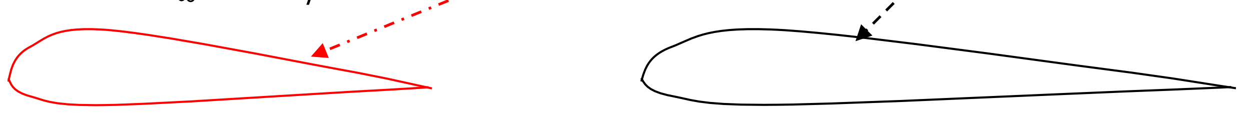
How can we raise the critical Mach number?

Alternatively we can try to reduce the approach velocity normal to the blade. We know that only the normal component of velocity approaching a surface which deflects the flow is involved in characterising the 1-D flow over the surface. So if we sweep back the blade surface, then the component of the normal flow is reduced.

Sweepback

If a two dimensional aerofoil is swept back at a sweepback angle ϕ, then the onset flow has a component $v_\infty \cos\phi$ perpendicular to the leading edge. The parallel component plays no part in the generation of pressures etc. Therefore we calculate the flow purely in terms of $M_\infty \cos\phi$.





Absolutely the last example!

Example

An aerofoil is designed with a minimum incompressible pressure coefficient of -0.3 at the design incidence. Find the sweep-back angle necessary to allow this aerofoil to operate with sub-critical flow when M = 0.95

$$C_{p,i} = \frac{\left(\dfrac{1+0.2M_c^2}{1.2}\right)^{3.5} - 1}{0.7M_c^2}\sqrt{1-M_c^2}$$

gives M_c in terms of $C_{p,i}$ for an unswept wing.

M_c	$C_{p,i}$
0.8	-0.26078
0.79	-0.28438
0.783659	-0.3
0.78	-0.30916

The sweep angle ϕ is then found from

$M \cos\phi = M_c$

$0.95 \cos\phi = 0.7836$, so that $\phi = 34.4°$