C201 Viscous Flow and Turbulence
Lecture 2
Part 1: Kolmogorov’s 1941 theory

Luca di Mare
St John’s College

1 Homogeneous isotropic turbulence

Homogeneous isotropic turbulence has properties independent of position and spatial direction. Two-point correlations only depend on separation distance but not on the direction of the separation vector

A single scalar function describes the energy content of the flow[1].

Turbulence in wind tunnels is a good approximation of isotropic homogeneous turbulence and in fact it was the study of flow in wind tunnels that spurred advantages in the understanding of turbulent flows and, in particular, homogeneous isotropic turbulence.

Kolmogorov’s theory, which is one of the very few closed-form results in the theory of turbulence, refers to homogeneous isotropic turbulence.

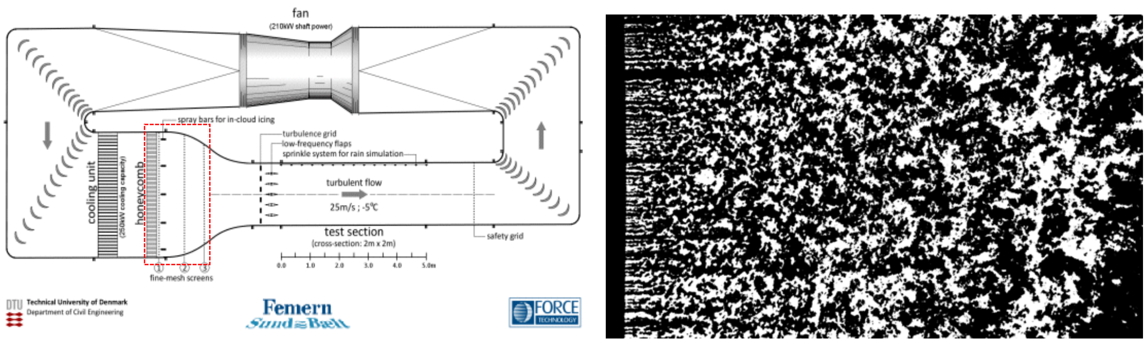

Figure 1: wind tunnel turbulence.

1.1 TKE budget in homogeneous isotropic turbulence

The mean flow in isotropic homogeneous turbulence is uniform. Also the statistical properties of the flow are uniform, and this considerably simplifies the kinetic energy budgets.

The kinetic energy budget of the mean flow simply tells us that the kinetic energy of the mean flow is uniform:

[1] We have created such a velocity field in the examples for the sections on correlations.

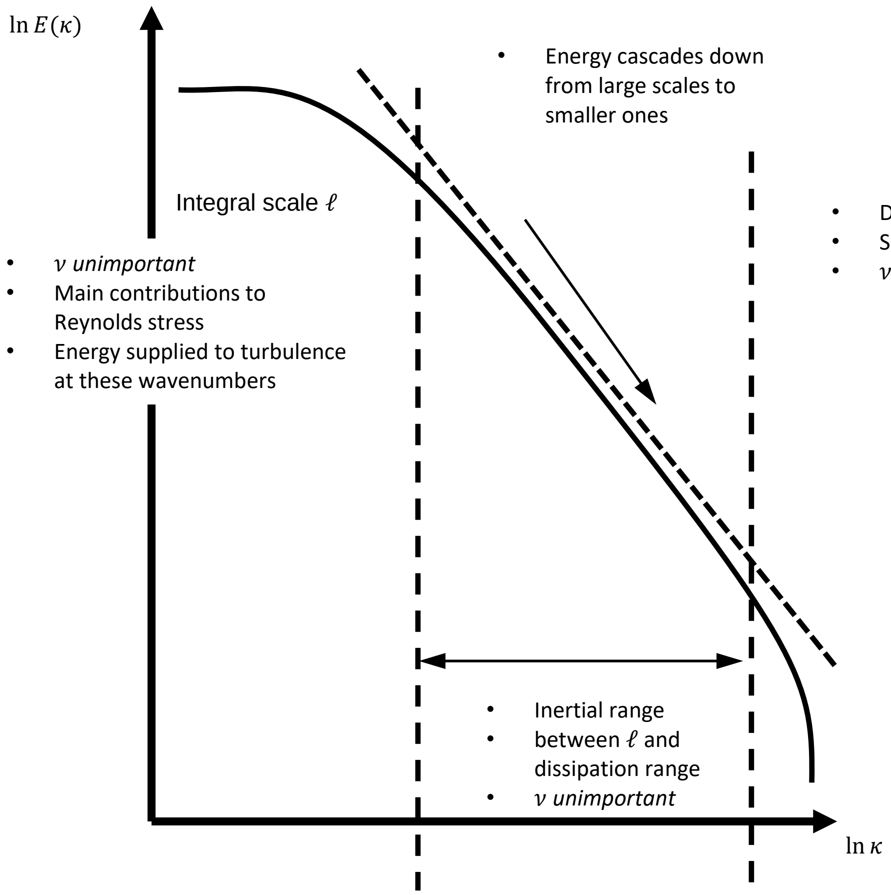

Figure 2: energy spectrum in homogeneous isotrotopic turbulence

$$U_j \frac{\partial K}{\partial x_j} = -\frac{1}{\rho} U_i \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 K}{\partial x_j \partial x_j} - \nu \frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j} - \frac{\partial}{\partial x_j} U_i \overline{u_i' u_j'} + \overline{u_i' u_j'} \frac{\partial U_i}{\partial x_j}$$

The TKE budget tells us that TKE must decay because production is inactive but dissipation is still taking place.

$$U_j \frac{\partial k}{\partial x_j} = -\frac{1}{\rho} \overline{u_i' \frac{\partial p'}{\partial x_i}} + \nu \frac{\partial^2 k}{\partial x_j \partial x_j} - \nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} - \frac{\partial}{\partial x_j} \overline{u_j' u_i' u_i'} - \overline{u_i u_j'} \frac{\partial U_i}{\partial x_j}$$

The decay rate is dictated by the dissipation rate rate

$$\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

1.2 The energy spectrum of turbulent flows

The typical shape of the energy spectrum in homogeneous isotropic turbulence is shown in Figure 2. The spectrum can be split into three main zones.

1.2.1 Energy carrying structures

The zone at low wave-numbers contains eddies of size comparable to the integral scale. These flow structures are those that give the largest contribution to the TKE and are called “energy carrying”. Energy is initially supplied to the turbulent flow at these low wave-numbers. Viscosity plays no role in the evolution of eddies at these wavenumbers.

1.2.2 The dissipation range

The range of high wave-numbers is called the dissipation range. In this range we find flow structures of sufficiently small size that their motion is affected by viscosity. It is here that the energy drained by the mean flow is finally dissipated into heat.
The spectrum in the dissipation range decays rapidly.

1.2.3 The inertial range

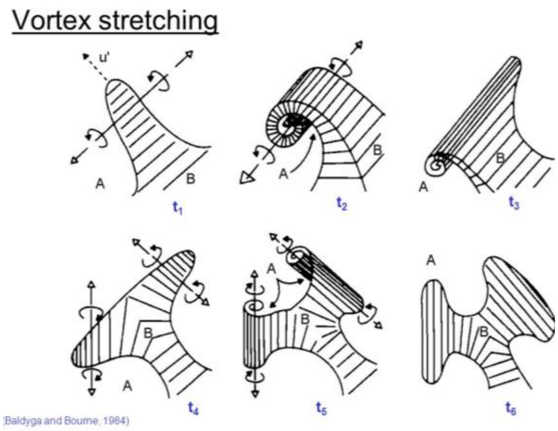

Figure 3: vortex stretching

The range of wave-numbers between the integral scale and the dissipation range is called the inertial subrange. In the inertial subrange we find structures of decreasing size as we move to the right of the spectrum. Viscosity plays no role in their evolution. The main process taking place in the inertial subrange is the transfer of energy from large scales to small scales. This process is known as energy cascade. The energy cascade takes place via an essentially inviscid mechanism of vortex stretching. Vortex stretching happens when flow structures are repeatedly folded and stretched by the turbulent flow field (see Figure 3). The effect of repeated stretching and folding is modify the shape of the original flow structure into a convoluted pattern of sheets and filaments, which, because of their smaller size compared to the original structure, appear as high wave-number structures on the energy spectrum. Kolmogorov's theory describes the properties of the energy spectrum in the inertial range.

2 Kolmogorov's 1941 (K41) theory

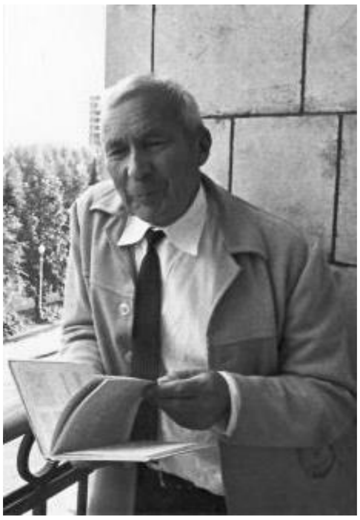

Figure 4: Andrey Kolmogorov

Andrey Kolmogorov (25 April 1903 – 20 October 1987) was a soviet mathematician. He had a fairly difficult start in life: he was brought up by his sister and worked as apprentice in the railways for a while before being able to join university. Once he did, he did not immediately study mathematics, but started studying history, and even wrote a dissertation on the structure of land ownership in Russia in the 17th century. Once he finally started studying mathematics he wrote several important papers when still an undergraduate. His interest in turbulence stemmed from his interest in statistical mechanics.
The 1941 theory is named thus because it appeared in a paper published by Kolmogorov in that year. The theory is

based on very simple assumptions about the turbulent flow field:

- The energy cascade is isotropic
- the small scales of motion are determined uniquely by the kinematic viscosity and the dissipation rate
- Most of the energy spectrum is unaffected by viscosity ("universal")

So the K41 theory is based entirely on few physical insights and dimensional analysis.

2.1 Small scales

The small scale behaviour of turbulence is described by Kolmogorov's microscale η. η determined by ν and ε. If we inspect the dimensions of the quantities at play we see

$$\eta = [\ell]$$
$$\nu = [\ell^2\, t^{-1}]$$
$$\varepsilon = \nu\, \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial u_j}} = [\,\ell^2\, t^{-3}]$$

If we want to combine the dissipation rate and the viscosity into a quantity with the physical dimensions of length the appropriate exponents are:

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

η is the length scale of the eddies where viscous dissipation takes place.
We can also define a Kolmogorov's time scale τ_η. Again, on dimensional grounds

$$\tau_\eta = \varepsilon^{-\frac{1}{3}}\, k^{-\frac{2}{3}} = \varepsilon^{-\frac{1}{3}}\, \eta^{\frac{2}{3}} = \varepsilon^{-\frac{1}{3}} \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}\frac{2}{3}} = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$$

τ_η is the life-span of eddies with size η. Combining η and τ_η we can obtain a velocity scale, Kolmogorov's velocity scale u_η:

$$u_\eta = \frac{\eta}{\tau_\eta} = \frac{\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}}{\left(\frac{\nu}{\varepsilon}\right)^{1/2}} = (\nu\varepsilon)^{1/4}$$

u_η represents the velocity RMS fluctuation for eddies of size η.

2.2 The slope of the spectrum in the inertial subrange

In the inertial range $E(\kappa)$ determined by ε and κ but not by ν. Inspecting the physical dimensions of the quantities involved:

$$E(\kappa) = [\ell^3\, t^{-2}\,]$$
$$\varepsilon = \nu\, \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial u_j}} = [\,\ell^2\, t^{-3}]$$
$$\kappa = [\ell^{-1}]$$

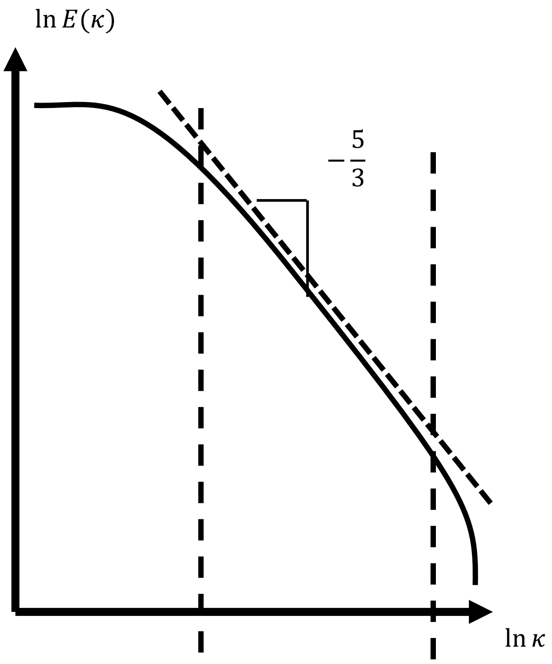

Figure 5: slope of the energy spectrum in the inertial range.

We try to combine ε and κ to form a quantity with the dimensions of the power density function.

$$E(\kappa) = C\varepsilon^{\alpha}\kappa^{\beta}$$

The exponents α and β must obey:

$$2\alpha - \beta = 3$$
$$-3\alpha = -2$$

Hence:

$$\alpha = 2/3$$
$$\beta = \frac{4}{3} - 3 = -\frac{5}{3}$$

Finally

$$E(\kappa) = C\,\varepsilon^{2/3}\kappa^{-5/3}$$

We have determined that the energy spectrum has slope $-5/3$ in the inertial subrange, as shown in Figure 5

2.3 Eddy turnover time in the inertial subrange

From the spectrum in the inertial range

$$E(\kappa) = C\,\varepsilon^{2/3}\kappa^{-5/3}$$

we can form a characteristic time

$$\tau_e = \frac{kE(k)}{\varepsilon} \sim \varepsilon^{-1/3}\,k^{-2/3}$$

This is the time eddies of size wavenumber κ survive before being destroyed by vortex stretching.

2.4 Kolmogorov's scale compared to the integral scale

In a turbulent flow with integral scale ℓ and velocity RMS fluctuation u', the dissipation rate must scale like

$$\varepsilon = \frac{u'^3}{\ell}$$

Since the Kolmogorov scale is known

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

We can eliminate ε to find Kolmogorov's scale in terms of the integral scale and the turbulence Reynolds number:

$$\eta = \frac{\ell}{Re_t^{\frac{3}{4}}}$$

$Re_t = u'\ell/\nu$ is the Reynolds number of the turbulent fluctuations

3 Energy cascade for different type of turbulent flows.

The K41 theory was derived under the hypothesis of homogeneity and isotropy of the inertial range.
It is remarkable that data obtained in widely different flows obeys the findings of the theory to a high degree of accuracy, even for flow that are definitely not homogeneous nor isotropic.
The fact that all turbulent flows comply with K41 indicates that at wave-numbers immediately above those corresponding to the integral scale turbulence is isotropic.
The images below show data from a selection of flows, a research compressor and even a tidal channel. All these flows obey Kolmogorov's theory.

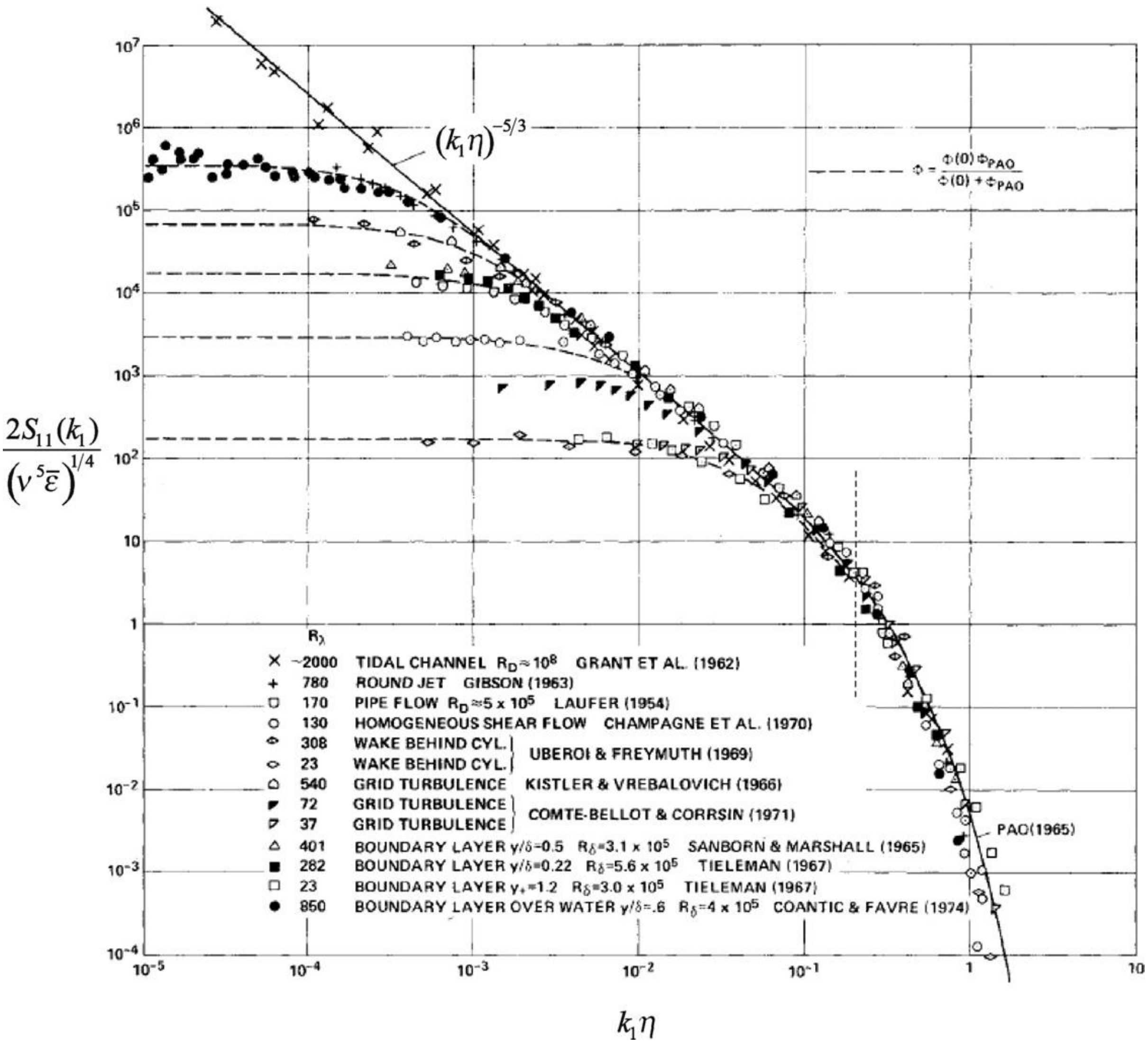

Figure 6: turbulence spectra from a broad variety of flows. All data obey Kolmogorov's scaling laws.

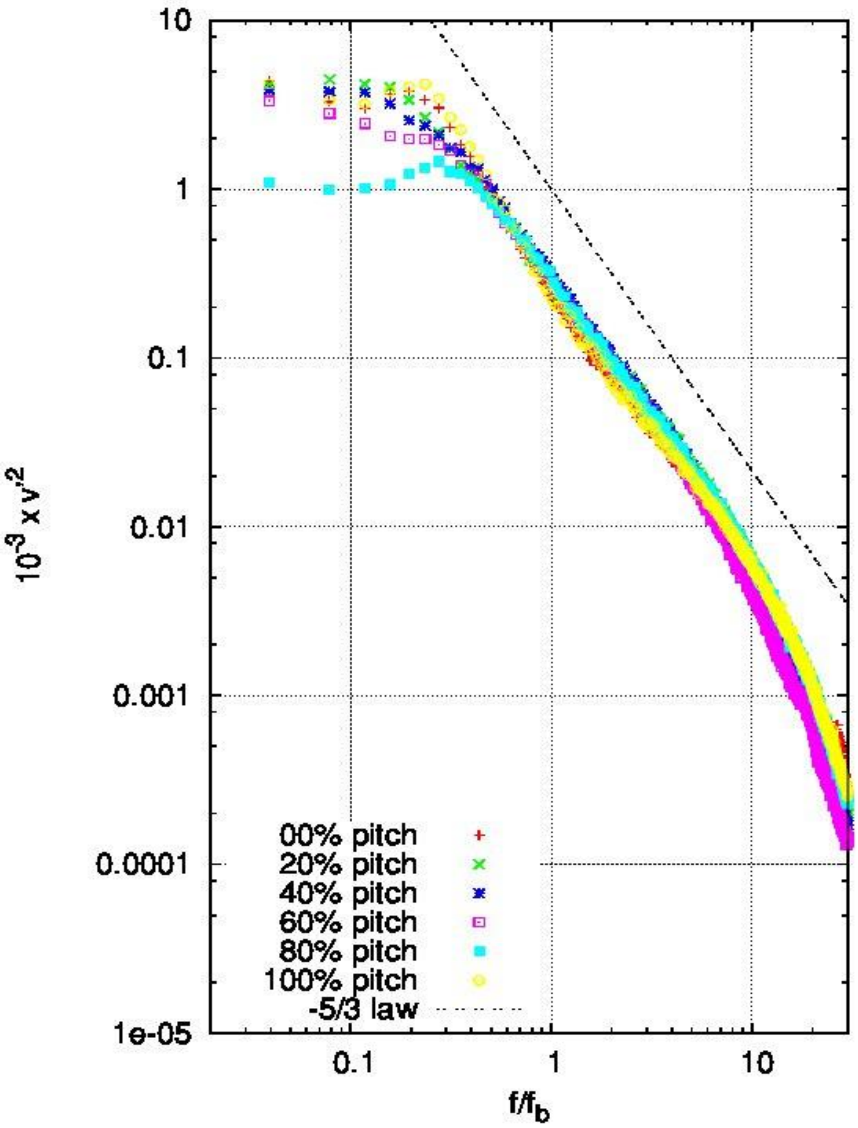

Figure 7: turbulence spectra at several locations in a research compressor.

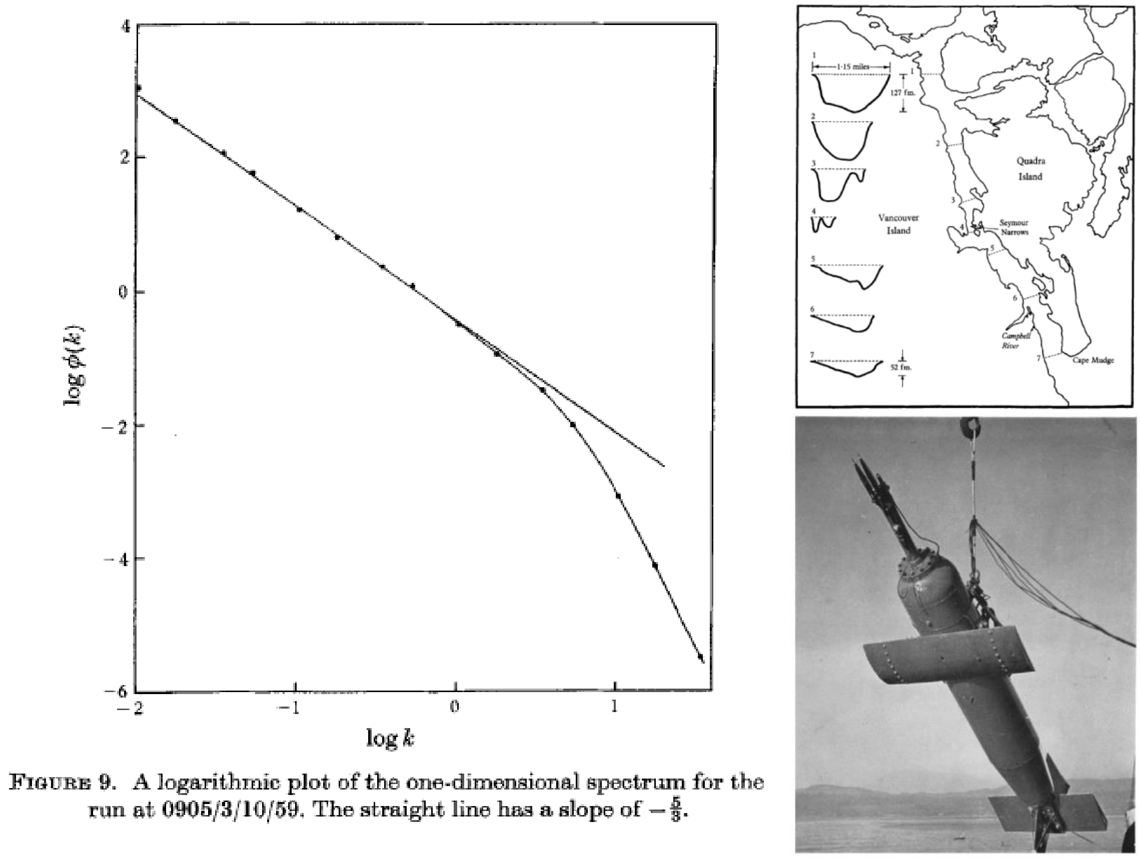

Figure 8: turbulence data from a tidal channel. The picture at the bottom right shows the probe use to take data. It is a small submarine towed behind a large ship.

4 Homogeneous isotropic turbulence and K41: checklist

Homogeneous isotropic turbulence has properties independent of position and spatial direction. TKE in homogeneous isotropic turbulence decays at a rate $\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$. No production is available to sustain turbulence because the mean flow is uniform. The smallest (Kolmogorov's) scales of turbulence depend only on ε and ν: $\eta = (\nu^3/\varepsilon)^{1/4}$. The energy spectrum has a range of wave-numbers where viscosity does not matter. In this range (inertial subrange) the energy spectrum scales with $\varepsilon^{2/3}\kappa^{-5/3}$. In the inertial subrange energy is transferred from larger eddies to smaller eddies by vortex stretching