C201 Viscous Flow and Turbulence

Lecture 2

Part 3: turbulent boundary layers

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1 Boundary layers

In the previous lecture we have examined the downstream evolution of wakes and jets. We found that these flows can be described by a single length scale and we have obtained scaling laws for the centreline velocity – or velocity defect – and the width of the jet/wake in terms of distance from the origin, initial size and strength. We obtained these laws from considerations of conservation of momentum flux – or momentum deficit flux – continuity and, crucially, on order-of-magnitude estimates for the Reynolds stress. We obtained those estimates under the assumption of large Reynolds number, so that the viscosity does not appear in the scaling laws for turbulent jets and wakes.

We now turn to the study of boundary layers. The main difference between a boundary layer on one hand and wakes and jets on the other is the presence of the wall. Because of the wall, there is a very thin layer of fluid where the velocity gradient is large compared to the other regions of the boundary layer. Furthermore the flow is dominated by viscosity in this region. The effect of this near-wall layer is that the Reynolds number does not disappear from the scaling laws for boundary layer even in the limit of large Reynolds numbers.

1.1 Wall units

Before we start studying the properties of boundary layers we introduce some nomenclature. We assume that the wall is impermeable and that it obeys the no-slip condition. We also assume that the boundary layer develops on a flat plate in the absence of external forces or pressure gradients. Then

$$\tau_w = \mu \frac{dU}{dy}$$

Is the wall shear stress. From τ_w we can obtain a velocity u_τ

$$\tau_w = \rho u_\tau^2$$

u_τ is related to the far field velocity U_∞ by the usual definition of the local friction coefficient

$$c_f = \frac{\tau_w}{\frac{1}{2\rho U_\infty^2}} \rightarrow u_\tau = \sqrt{\frac{c_f}{2}} U_\infty$$

Because the friction coefficients varies slowly with the Reynolds number – in fact it varies more slowly the higher the Reynolds number - u_τ is a weak function of U_∞. We will give a very precise meaning to the adjective "weak" a little later on.

The velocity u_τ can be used to form a Reynolds number:

$$Re_\tau = \frac{\delta u_\tau}{\nu}$$

Similarly u_τ can be used to form non-dimensional values based on the distance from the wall or the velocity:

$$y^+ = \frac{y u_\tau}{\nu}$$

$$u^+ = U/u_\tau$$

Non-dimensional quantities based on u_τ are called wall units and are customarily denoted by the $+$ superscript.

1.2 Magnitude of the wall units

Before we proceed further it is useful to obtain some numerical value for the size of the wall units. We use as reference Schlichting's correlation

$$C_f = \left[2log_{10}\left(R_{e,x}\right) - 0.65\right]^{-2.3}, Re_x < 10^9$$

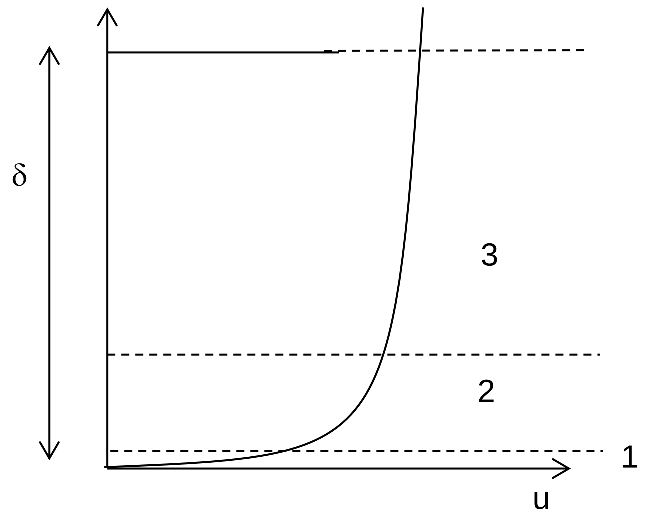
for the friction coefficient in a turbulent boundary layer and estimate the wall units at a distance $x = 1m$ from the leading edge for different values of the free-stream velocity. We assume the fluid to be air: $\rho = 1.2 kg/m^3$, $\mu = 1.82\ 10^{-5} kg/ms$

$U_\infty[m/s]$	10	50	100
Re_x	$6.6 \cdot 10^5$	$3.36 \cdot 10^6$	$6.66 \cdot 10^6$
$y \to y^+ = 1$	$3.6 \cdot 10^{-5}$	$7.7 \cdot 10^{-6}$	$4.1 \cdot 10^{-6}$

we see that the size of a wall unit in a turbulent boundary layer at high-Reynolds number is exceedingly small compared to the size of the boundary layer.

2 The structure of turbulent boundary layers

We can now introduce the general structure of turbulent boundary layers. A turbulent boundary layer is composed of three layers. With reference to Figure 1, the first layer (1) is known as the laminar sublayer. The outmost part of the boundary layer is known as the outer layer (3). Between the outer layer and the laminar sublayer there is a transition region known as the logarithmic layer (2).



The structure of turbulent boundary layer is more easily illustrated by showing the velocity profile in semi-logarithmic scale, as shown in Figure 2. The purpose of the semi-logarithmic scale is to expand the area corresponding to small values of y^+, i.e. the viscous sublayer and contract the outer layer. We notice immediately from Figure 2 that the when plotted in semi-logarithmic coordinates, the velocity profile in the logarithmic layer is a straight line. We now set off to explain this interesting finding.

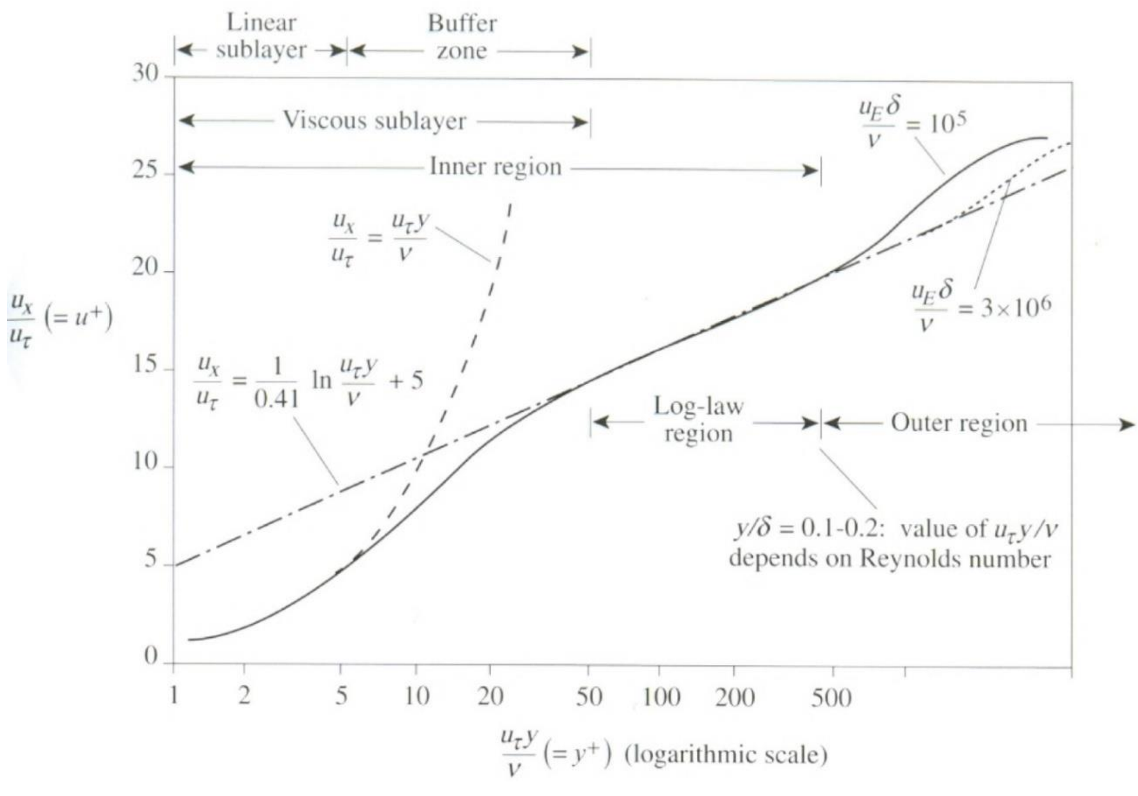


Figure 2: the structure of turbulent boundary layers

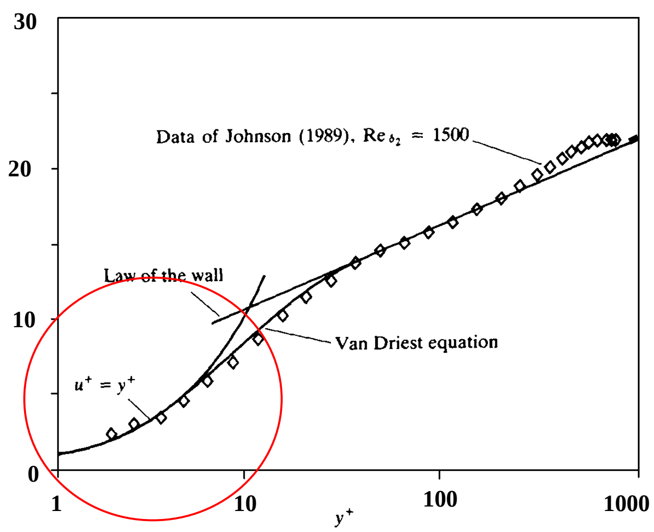


Figure 3

2.1 Laminar sublayer

In the laminar sublayer the flow is influenced by the no-slip boundary condition at the wall

$$U = V = u' = v' = 0$$

In the laminar sublayer turbulence does not enhance the transport of momentum because the small scale of the sublayer prevents the formation of turbulent structures. The proximity of the wall also prevents convective transport from playing a significant role. In the laminar sublayer the momentum equation therefore reduces to

$$\mu \frac{d^2U}{dy^2} = 0$$

$$\mu \frac{dU}{dy} = \tau_W$$

And, in nondimensional terms

$$U^+ = y^+$$

The thickness of the viscous sublayer is in the order of 5 to 10 wall units. Based on the numerical example above we see that the viscous sublayer represents a very thin layer of fluid in the immediate proximity of the wall.

2.2 The outer layer

The momentum budget of the outer layer is dominated by the balance between the Reynolds stress and the convective transport of momentum

$$U\frac{\partial U}{\partial x} = -\frac{\partial\, \overline{u'v'}}{\partial y} - \frac{1}{\rho}\frac{\partial P}{\partial x}$$

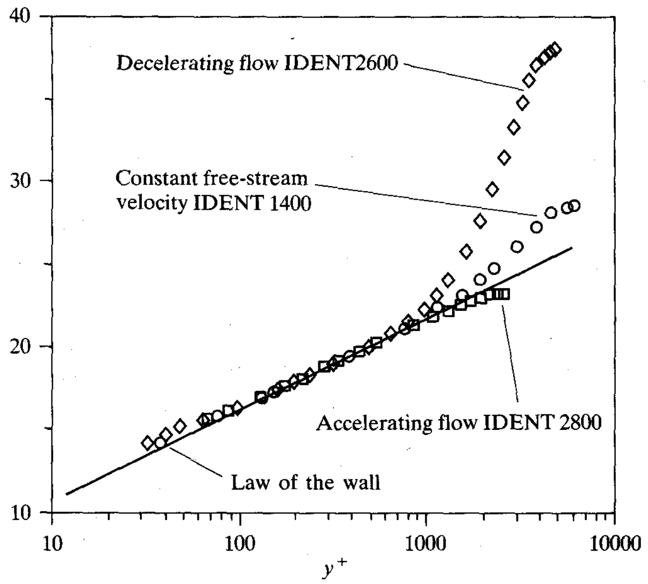
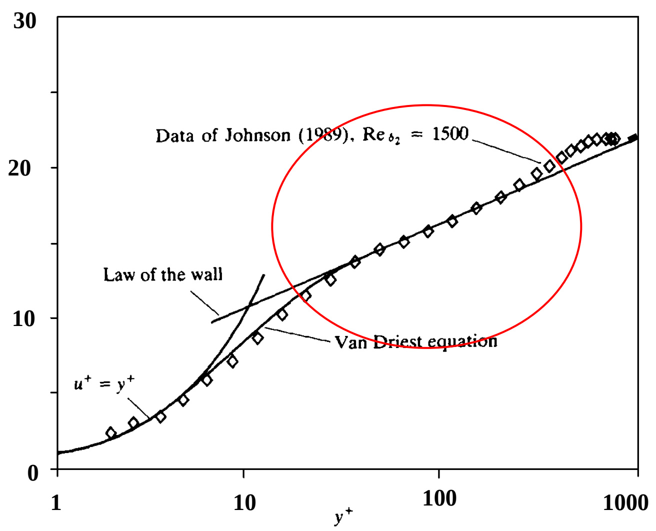


Figure 4

The integral properties of the boundary layer (thicknesses, shape factor) are largely determined by the shape of the velocity profile in the outer layer, because this is (geometrically) the largest portion of the boundary layer itself. The momentum equation above suggests that the velocity profile in this part of the boundary layer is not easily described in universal terms and it bears the effect of the past history of the boundary layer, its Reynolds stress distribution and the pressure gradient. This observation is observed experimentally, as we can see from the data in Figure 1.

Cole (1956) observed that the velocity defect in the outer region behaves in a manner analogous to a turbulent wake and prosed a parameterization for the velocity profile known as the Cole (1956) “wake” velocity profile

2.3 The log-law region



In the layer between the viscous sublayer and the outer layer we expect the velocity profile to be determined by the Reynolds stress. Experimental evidence indicates that the Reynolds stress must scale like u_τ^2 and is independent of U_∞. The momentum balance implies

$$\frac{dU}{dy} = f_3(y,\, u_\tau)$$

We notice that the differential relation representing the momentum balance contains three quantities but only two dimensions, so it leads us to form a single non-dimensional group:

$$\frac{y}{U_\tau}\frac{dU}{dy} = A$$

If A is constant, then

$$\frac{U}{u_\tau} = A \ln y + B$$

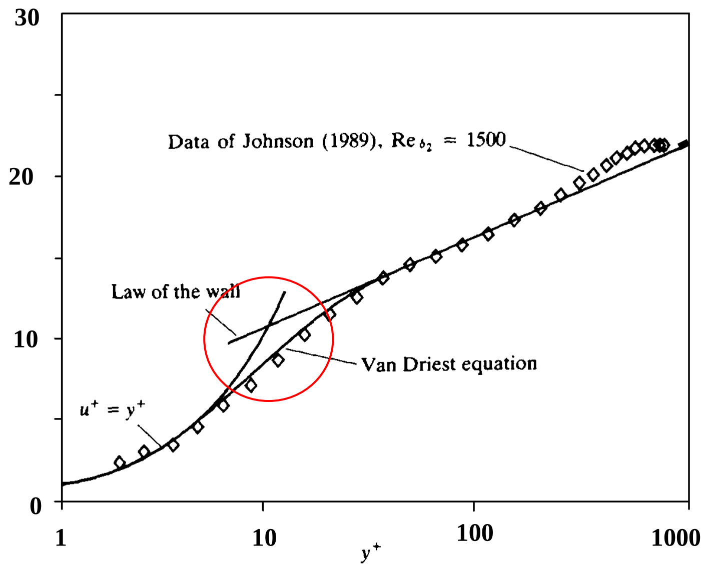
This equation is precisely the straight profile in semilogarithmic coordinates we have first seen in Figure 2

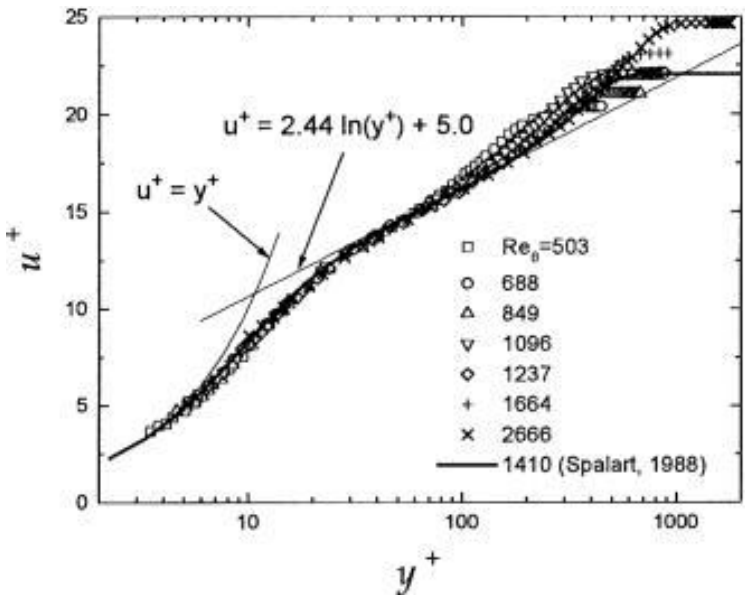
2.4 Buffer region ($5 < y^+ < 30$)

The buffer region is confined between the laminar sublayer and log-law region. In this region the Reynolds and viscous stress comparable. Neither viscous law of the wall nor the log law apply

$$u^+ \neq y^+ \qquad u^+ \neq \frac{1}{k} lny^+ + \beta$$

but the velocity profile is still universal: turbulent boundary layer velocity profiles in the inner region collapse when scaled using law of the wall in the buffer region.





3 Turbulent boundary layers: checklist

The structure of turbulent boundary layers is made of three layers:

Viscous sublayer

Logarithmic layer

Outer layer

Boundary layers are conveniently described using wall units

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}},\ u^+ = \frac{U}{u_\tau},\ y^+ = \frac{y u_\tau}{\nu}$$

When using these units, data in the inner region of all turbulent boundary layer collapse and become independent of quantities such as free-stream velocity, pressure gradient, thickness etc.

Law of the wall describes the universal behaviour of boundary layers in the inner region and it applies in the viscous layer, the logarithmic layer and the buffer layer.

In the laminar sublayer

$$u^+ = y^+$$

In the logarithmic layer

$$u^+ = \frac{1}{k}\ln y^+ + \beta$$

$$k = 0.41,\ \beta \approx 5$$