C2 Aerothermal Eng - Wing Theory and Compressible Flow

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Summary of course

There are four lectures and one examples class. Prerequisites are elementary potential flow theory from A4 Fluid Dynamics and certain equations from B19 Gas dynamics.

C2 – Wing Theory and Compressible Flow

Summary of course

Lecture 1 Potential flow theory for thin aerofoils

- Array of vortex elements can induce a curved streamline.
- Relationship between circulation and surface slope for a thin aerofoil.
- The Glauert integral.
- Special case of a flat plate.
- Definition of pressure distribution and associated lift and pitching moment coefficients

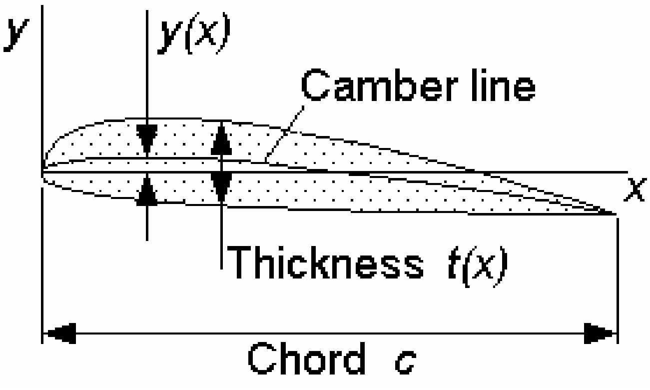
Books and Tools

Theory of wing sections : including a summary of airfoil data, Abbott and Von Doenhoff,1959,New York : Dover.

http://virtualskies.arc.nasa.gov/aeronautics/

https://en.wikipedia.org/wiki/Lifting-line_theory.

Potential Flow Theory for Thin Aerofoils



Aerofoil shape and Basic Definitions

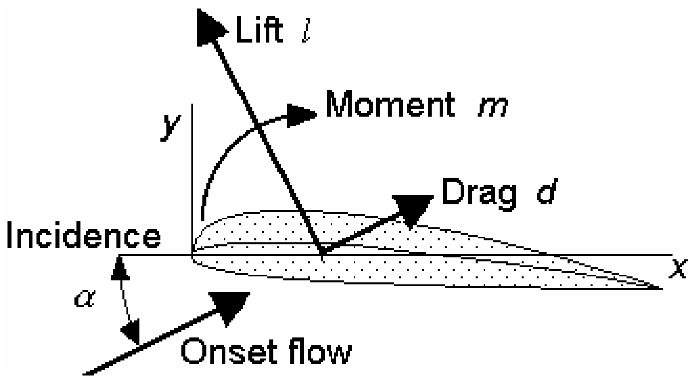
An **aerofoil** is two-dimensional. It represents the cross-section of a wing whose span is infinite. Aerofoils are usually drawn in co-ordinates aligned with a nominal undisturbed flow direction and the **chord line** from the leading edge to the trailing edge may be inclined at a small angle to the x axis. The basic dimension is the **chord** length *c*.

An aerofoil has a distribution of **thickness** $t(x)$ and of **camber** $y(x)$. The **camber line** is mid-way between the upper and lower surfaces.

Potential Flow Theory for Thin Aerofoils

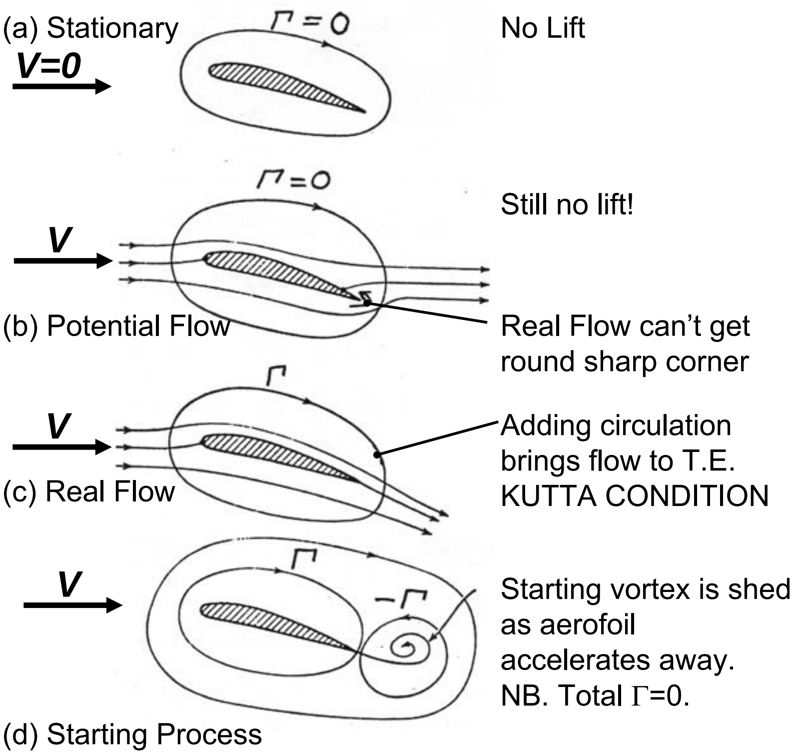
Once the shape is defined, an aerofoil may be operated at any **angle of incidence** α by defining an undisturbed (**onset**) flow vector inclined at α to the x axis.

In this airstream, the aerofoil experiences an aerodynamic force f per unit span. The magnitude and direction of f vary with flow speed, air density and the angle of incidence. By convention, this aerodynamic force is defined by a **lift** component l per unit span *normal* to the onset stream direction and a **drag** component d per unit span *parallel* to the onset flow direction.
In addition, depending upon the point of application of f, there may also be a **pitching moment** m per unit span.
The sign convention for m is **nose-up-positive.**



Why does an Aerofoil Produce Lift?

This can be described by circulation / vorticity. First let us consider a stationary aerofoil:

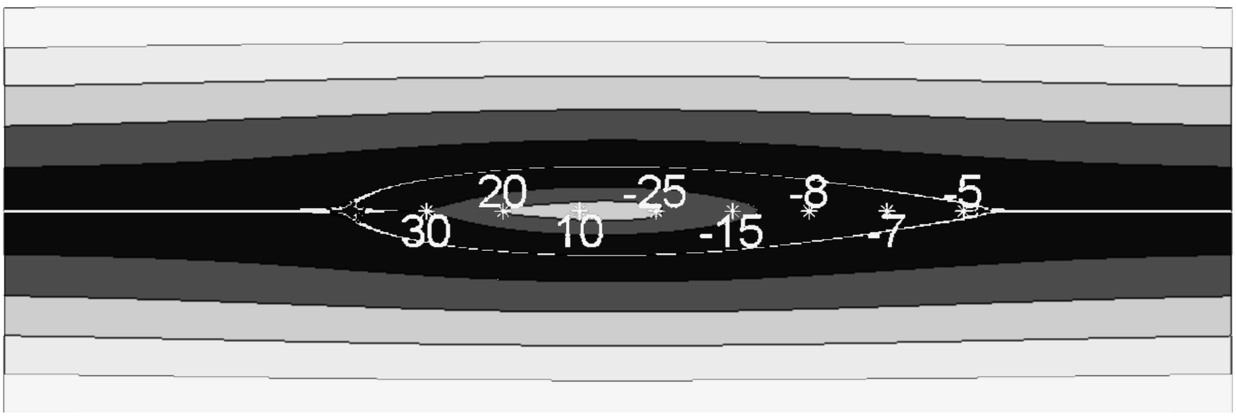


2 Potential Flow Tools: 2nd Year Revision

2.1 Distributed sources and sinks

In the second year fluid dynamics course, it was shown that a source and a sink of equal strength fixed in a uniform stream generate a flow pattern in which one streamline forms a closed oval shape.

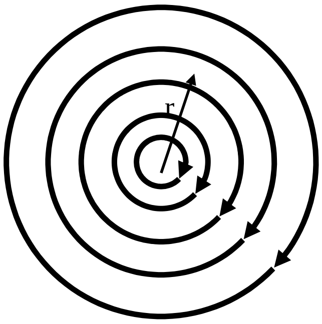
Extending this idea intuitively: any number of sources and sinks arranged along a line, will form a closed streamline *if the total source - sink outflow is zero*. It is not difficult to imagine that by choosing a distribution of source and sink elements, the shape of the closed streamline could be made slender like an aerofoil.



We have no time to apply this and is for example only.

2 Potential Flow Tools: 2nd Year Revision

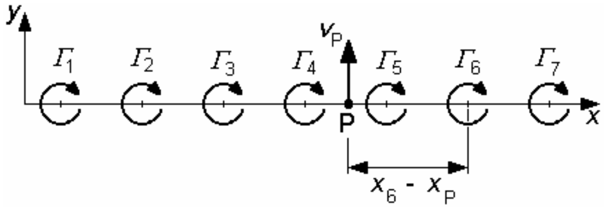
2.2 (a) Irrotational Vortex of strength Γ:



$$v(r) = \frac{\Gamma}{2\pi r}$$

Vorticity = 0
except at r=0

(b) Extend this to a row of vortices VORTEX ROW



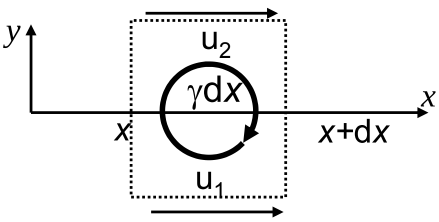
At any point P the velocity vectors from all vortices must be summed up. Along the x-axis u_p=0.

$$v_p = \sum_{n=1}^{N} \frac{\Gamma_n}{2\pi(x_n - x_p)}$$

The sign convention is a personal choice here, we are taking clockwise positive.

2 Potential Flow Tools: 2nd Year Revision

2.2 (c) Now extend this to a continuous UNIFORM VORTEX SHEET:



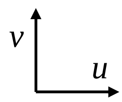
The vorticity is γ / unit length, V=0

Going around the dotted perimeter:

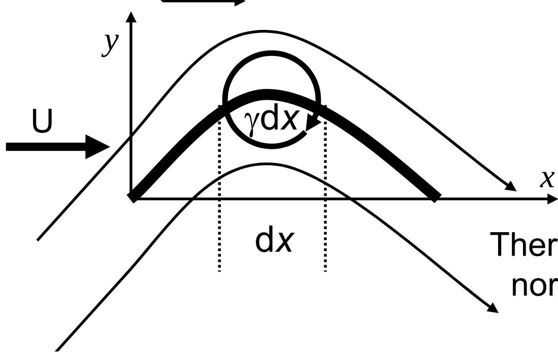
Circulation: $\gamma dx = u_2 dx - u_1 dx \implies \gamma = u_2 - u_1$.

By symmetry $u_2 = -u_1 = ½\gamma$.

2.2 (d) Finally consider the general case of a CURVED NON-UNIFORM VORTEX SHEET along a streamline.



Since the Vortex sheet is along a streamline: $\dfrac{dy}{dx} = \dfrac{v}{(U \pm u)} \approx \dfrac{v}{U}$



U = freestream velocity

u=increment of x-velocity

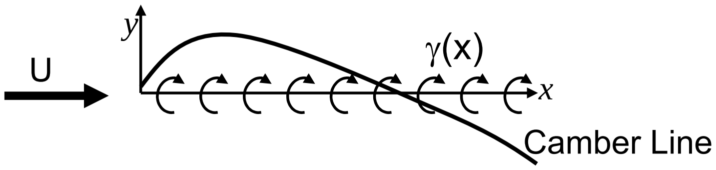
V= y velocity

There is no velocity component normal to the sheet

3 Thin Aerofoil Theory

3.1 Assumptions used in thin aerofoil theory

1. Ignore the thickness of the aerofoil and replace it by a VORTEX SHEET
2. Camber is small so LAY THE VORTEX SHEET ALONG THE X-AXIS
3. Make the incident flow PARALLEL TO THE X-AXIS (Later we can rotate the coordinate system

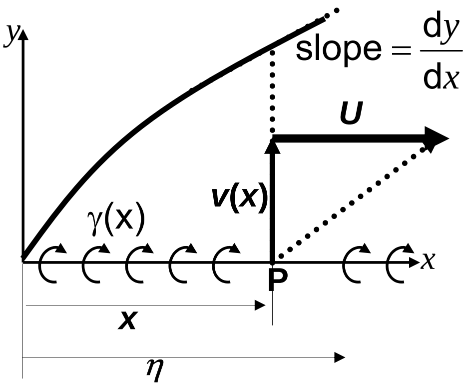


4. |u| << U and |v| << U i.e. the perturbations in the flow velocity are small.

Given the assumptions above we now have the tools to show how a distribution of vorticity along the vortex sheet affects:

- Camber line shape *y(x)*
- Surface velocities, u,v
- Surface pressures *P*
- lift per unit span, *l*
- pitching moment per unit span, *m*

The Upwash Equation
– Vertical induced velocity v and camber line shape y(x)



At **P** the vertical induced velocity **v(x)** is due only to the distributed vortex elements $\gamma(\eta).\mathrm{d}\eta$ which are 'smeared' along the x-axis over the extent of the chord of the blade 0→c. $\frac{dy}{dx}$

$$v(x) = \int_0^c \frac{\gamma(\eta)}{2\pi(\eta - x)}\mathrm{d}\eta$$

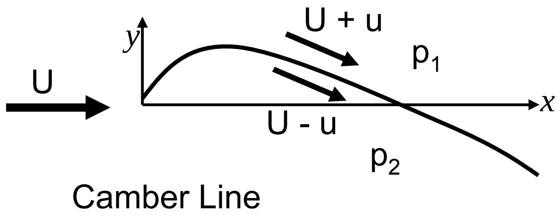
Note η is a dummy variable we use for integration.

From the previous definition of the slope along a streamline we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v(x)}{U} = \frac{1}{2\pi U}\int_0^c \frac{\gamma(\eta)}{(\eta - x)}\mathrm{d}\eta,$$

which can be integrate w.r.t. *x* to obtain *y(x)***.**

Surface Pressures and Velocities, Lift and Moment



Bernoulli can be applied everywhere except across the vortex sheet. $u(x) = ½\,\gamma(x)$.

So apply Bernoulli above and below the line:

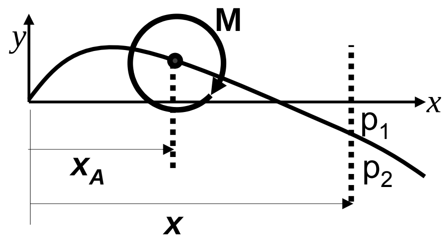
$$p_1 + ½\,\rho(U+u(x))^2 = p_2 + ½\,\rho(U-u(x))^2$$

So: $p_2 - p_1 = 2\,\rho\,U\,u(x) = \rho\,U\,\gamma(x)$.

Lift per unit Span

Lift $l = \int_0^C (p_2 - p_1).dx = \rho U \int_0^C \gamma(x).dx \Rightarrow l = \rho U \Gamma$: here Γ is the total circulation.

Pitching Moment per unit Span



$$m = -\int_0^C (p_2 - p_1)(x - x_A)\,\mathrm{d}x$$

$$m = -\rho U \int_0^C \gamma(x)(x - x_A)\,\mathrm{d}x$$

Uniform Loading Distribution (Physically Unrealistic but easy)

Vorticity: $\gamma(x) = \gamma$ = *constant*.

Circulation: $\Gamma = \gamma \times c$

Lift : $L = \rho\,U\,\Gamma = \rho\,U\,\gamma\,c \Rightarrow \quad C_l = \dfrac{l}{1/2\rho U^2 c} = \dfrac{2\gamma}{U}$

So the loading must be chosen for the desired lift coefficient C_l.

We have already seen that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2\pi U}\displaystyle\int_0^c \dfrac{\gamma(\eta)}{(\eta - x)}\mathrm{d}\eta = \dfrac{1}{2\pi U}\displaystyle\int_0^c \dfrac{C_l U}{2(\eta - x)}\mathrm{d}\eta = \dfrac{C_l}{4\pi}\displaystyle\int_0^c \dfrac{\mathrm{d}\eta}{(\eta - x)}$.

Problem at $\eta = x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta\to 0}\frac{C_l}{4\pi}\left[\int_0^{x-\delta}\frac{\mathrm{d}\eta}{(\eta - x)} + \int_{x+\delta}^{c}\frac{\mathrm{d}\eta}{(\eta - x)}\right]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta\to 0}\frac{C_l}{4\pi}[\ln(\delta) - \ln(x) + \ln(c - x) - \ln(\delta)]$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{C_l}{4\pi}[\ln(c - x) - \ln(x)]}$$

Integrate dy/dx to get y: (remember ∫ln x = xlnx –x + const).

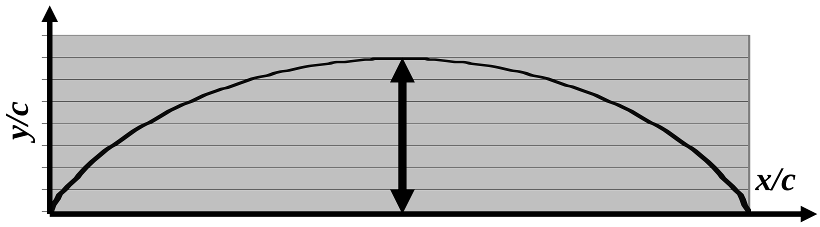
$$y = \frac{C_l}{4\pi}\{-(c-x)\ln(c-x) + (c-x) - x\ln x + x\} + A$$

or

$$y = \frac{C_l}{4\pi}\{-(c-x)\ln(c-x) - x\ln x\} + K(const)$$

Boundary Condition: y=0 at x=0

$$K = \frac{C_l c \ln c}{4\pi} \quad \text{so} \quad \frac{y}{c} = -\frac{C_l}{4\pi}\left[\left(1-\frac{x}{c}\right)\ln\left(1-\frac{x}{c}\right) + \frac{x}{c}\ln\frac{x}{c}\right]$$



An unusual aerofoil shape!

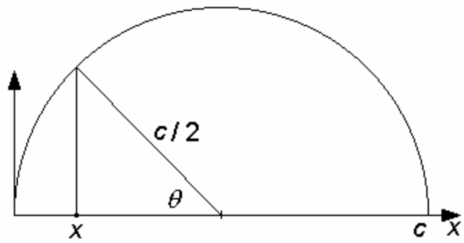
Depth of camber line $= \frac{C_l c \ln 2}{4\pi}$ is proportional to the design lift coefficient C_l.

It would not give the same $\gamma(x)$ = constant distribution at any other C_l.

Thin Aerofoils: A new coordinate system

Consider a new coordinate system which is more convenient for computing chord lines etc. from a given vorticity distribution:

$$x = \frac{c}{2}(1-\cos\theta).$$



Now consider a 2nd arbitrary loading distribution

$$\gamma(x) = 2U\alpha\cot\left(\frac{\theta}{2}\right)$$

To find the camber line shape use the previous definition for the slope of the line with a substitution for x, and the dummy variable η

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\pi U}\int_0^c \frac{\gamma(\eta)}{(\eta - x)}\mathrm{d}\eta = \frac{1}{2\pi U}\int_0^\pi \frac{2U\alpha\cot\left(\frac{\mu}{2}\right)\frac{c}{2}\sin\mu}{\frac{c}{2}(1-\cos\mu) - \frac{c}{2}(1-\cos\theta)}\mathrm{d}\mu = \frac{\alpha}{\pi}\int_0^\pi \frac{\cot\left(\frac{\mu}{2}\right)\sin\mu}{\frac{c}{2}(\cos\theta - \cos\mu)}\mathrm{d}\mu$$

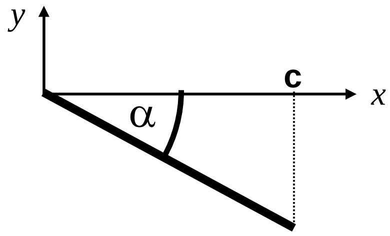
$$\boxed{\eta = \frac{c}{2}(1-\cos\mu).} \quad \boxed{\mathrm{d}\eta = \frac{c}{2}\sin\mu.\mathrm{d}\mu.}$$

N.B. $$\cot\left(\frac{\mu}{2}\right)\sin\mu = \frac{\cos\left(\frac{\mu}{2}\right)}{\sin\left(\frac{\mu}{2}\right)} 2\sin\frac{\mu}{2}\cos\frac{\mu}{2} = 2\cos^2\frac{\mu}{2} = 1 + \cos\mu$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\alpha}{\pi}\int_0^{\pi}\frac{\cos(0\mu)}{(\cos\mu - \cos\theta)}\mathrm{d}\mu + \int_0^{\pi}\frac{\cos(1.\mu)}{(\cos\mu-\cos\theta)}\mathrm{d}\mu$$

$$\textit{from HLT}: \int_0^{\pi}\frac{\cos(n.\mu)}{(\cos\mu-\cos\theta)}\mathrm{d}\mu = \frac{\pi\sin(n\theta)}{\sin\theta}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\alpha}{\pi}\left[\frac{\pi\times 0}{\sin\theta}+\frac{\pi\sin\theta}{\sin\theta}\right] = -\alpha \quad i.e.\ \frac{\mathrm{d}y}{\mathrm{d}x} = -\alpha$$



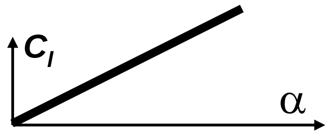
This is a flat Plate at angle α to the flow!

Flat Plate Lift Coefficient

Lift per unit span $l = \rho U \int_0^c \gamma(x).\mathrm{d}x = \rho U \int_0^{\pi} 2U\alpha\cot\left(\frac{\theta}{2}\right).\frac{c}{2}\sin\theta\mathrm{d}\theta$

$$l = \rho U\int_0^{\pi} 2U\alpha\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.\frac{c}{2}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathrm{d}\theta = 2\rho U^2 c\alpha\int_0^{\pi}\cos^2\frac{\theta}{2}\mathrm{d}\theta = \rho U^2 c\alpha\pi$$

The flat plate lift coefficient: $C_l = \dfrac{l}{\frac{1}{2}\rho U^2 c} = 2\pi\alpha$



Flat Plate Pitching Moment per unit span about the leading edge

$$m_0 = -\rho U\int_0^c \gamma(x).x.\mathrm{d}x$$

$$= -\rho U\int_0^{\pi} 2U\alpha\cot\left(\frac{\theta}{2}\right).\frac{c}{2}(1-\cos\theta)\frac{c}{2}\sin\theta\mathrm{d}\theta, \quad \left[NB:\ \cot\left(\frac{\theta}{2}\right).\sin\theta = 1+\cos\theta\right]$$

$$= -\frac{\rho U^2 c^2\alpha}{2}\int_0^{\pi}(1+\cos\theta)(1-\cos\theta)\mathrm{d}\theta \Rightarrow m_0 = -\frac{\pi\rho U^2 c^2\alpha}{4}$$

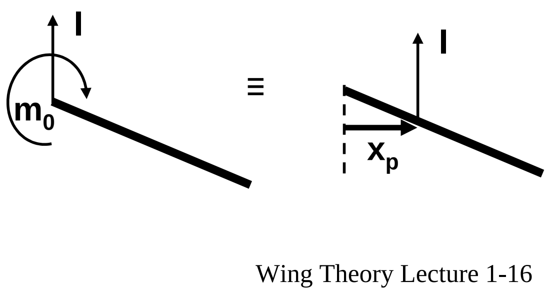
The flat plate leading edge moment coefficient: $C_{m_0} = \dfrac{m}{\frac{1}{2}\rho U^2 c^2} = -\dfrac{\pi\alpha}{2}$

Q: Is there a point where *m*=0?

$$\frac{\pi\rho U^2 c^2\alpha}{4} = \pi\rho U^2 c\alpha\ x_p$$

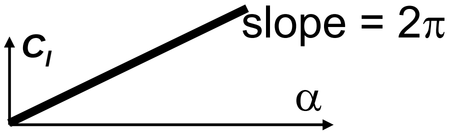
$$\boxed{x_p = \frac{c}{4}}$$

Hence we say that the centre of pressure is at ¼ chord from the leading edge.



Flat Plate Loading – some final notes

•A change in C_l only effects the incidence. The camber line does not alter.

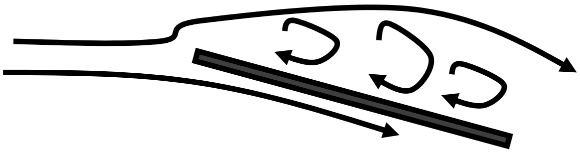


•Centre of Pressure at $x_p = c/4$ is independent of a. $m=0$ about x_p

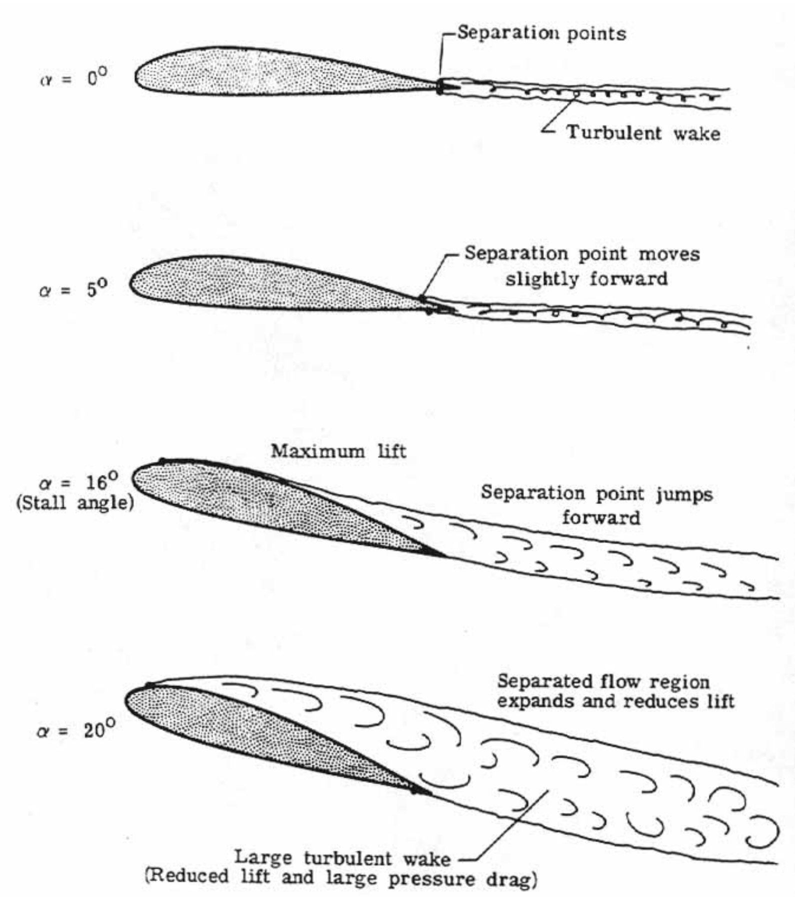
•Flat plate loading function could be very useful – remember we are developing potential flow, so add it to other loading functions to model a change of incidence.

$$\gamma(x) = 2U\alpha \cot\left(\frac{\theta}{2}\right)$$

•Flat plate loading gives very high velocities and low pressures near the leading edge. The upper surface pressure gradients are very adverse. This implies boundary layers will separate and cause early stall at low angles of attack α.



Why is separation a problem?



A simple way of thinking about the pressure drop achieved on the suction surface, is to consider the curvature of the flow as is moves around the leading edge. Small radius of curvature = large pressure gradient, and hence a low pressure at the upper surface of the blade.

from: http://virtualskies.arc.nasa.gov/aeronautics/3.html