C2 – Wing Theory and Compressible Flow

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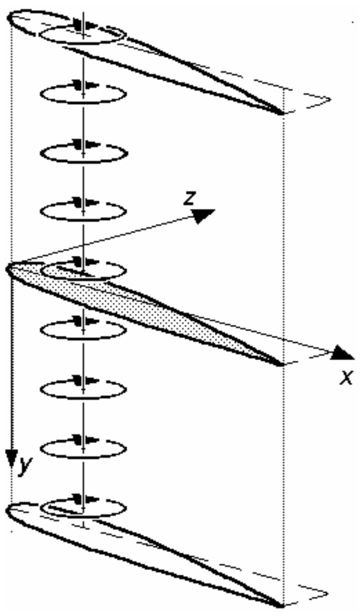
Michaelmas 2021

Summary of course

Lecture 3 Wings of finite span

- 3-D vortex lines with circulation conservation.
- Representation of finite wings by nested ‘horseshoe’ vortex lines.
- The lifting line model and downwash equation for high aspect ratio wings.
- Circular co-ordinates / Fourier series for spanwise circulation distribution.
- Dependence of lift and drag coefficients
- Ideal elliptic loading
- Aeroplane drag equation is established.

Introduction to lifting line theory



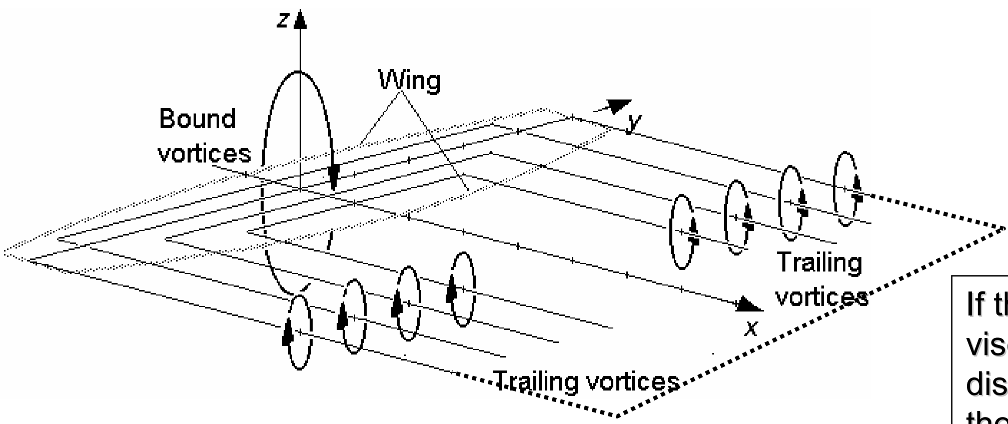
In the previous lectures, a two dimensional aerofoil has been modelled by a distribution of circulation over the chord from leading edge to trailing edge.

The first step in the discussion of whole wings is to recognise that an aerofoil is a wing of infinite span. In the same way a two dimensional vortex is not just a “dot with circulation” on a diagram, but a straight line extending to infinity in both directions normal to the paper. The vortex line shown in the sketch is one of an array distributed along the chord.

Theorem: ***The strength of a vortex line does not vary along its length. Consequently a vortex line cannot end but must form a closed loop with constant circulation.***

Finite Wings: Accommodating vortex lines

Finite wings may be represented by an array of vortex lines distributed along the chord and extending across the span, **but the lines cannot end at the wing tips**.



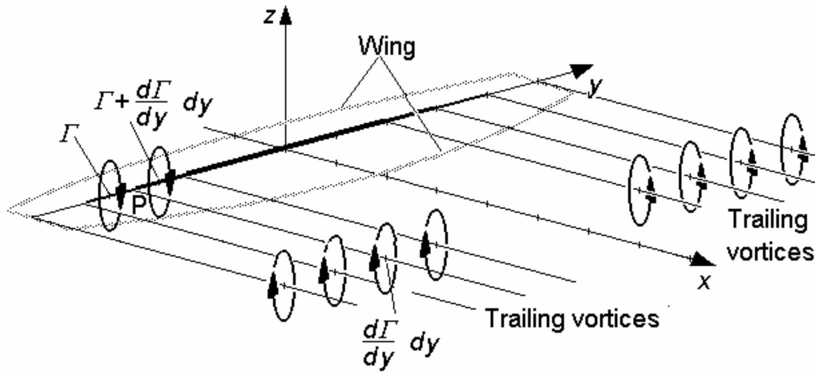
If there was no viscous dissipation then these trailing vortices would extend to the starting vortex to form a closed loop

Wing-Tip vortices are an obvious feature of wings of Finite Span

Development of lifting line theory

An Approximation for Wings of High Aspect Ratio
For a wing of plan area *S*, and span *b*, the aspect ratio *A* is defined by: $A = \dfrac{b^2}{S}$

A wing of high aspect ratio has large span and small chord.



Simplification achieved by assuming:

- Chord is small *cf* span
- Camber line shape not important

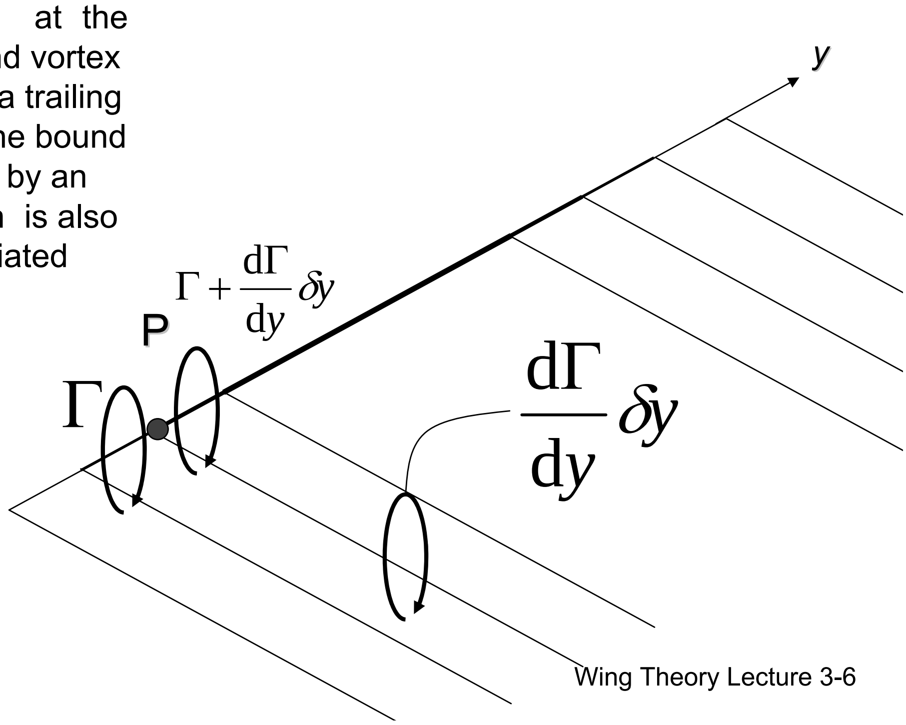
$$\boxed{i.e.\ \Gamma = \int_0^c \gamma(x).dx}$$

We model the blade by **ignoring the chordwise distribution of bound vortices and pretending that they all lie on a single spanwise lifting line.**

The diagram has a lifting line of variable thickness to illustrate how the circulation on each "horseshoe vortex" element contributes to a bound circulation which increases towards the centre.

The vortex continuity relation

This illustration of variable vortex strength incorporates a statement of the fundamental continuity property of vortex lines. On the diagram, the point P lies at the junction between the bound vortex line (**the lifting line**) and a trailing vortex line. At that point the bound vortex strength increases by an amount $(d\Gamma/dy)\ \delta y$ which is also the strength of the associated trailing vortex element.



The Downwash Equation

To determine the total induced velocity potential flow integrates to sum the induced velocity contributions element-by-element along the length of a vortex line. However, the result can be deduced very easily by physical reasoning.

Recall from lecture 1 that a two-dimensional vortex Γ causes a circular velocity w(θ) at radius *r*

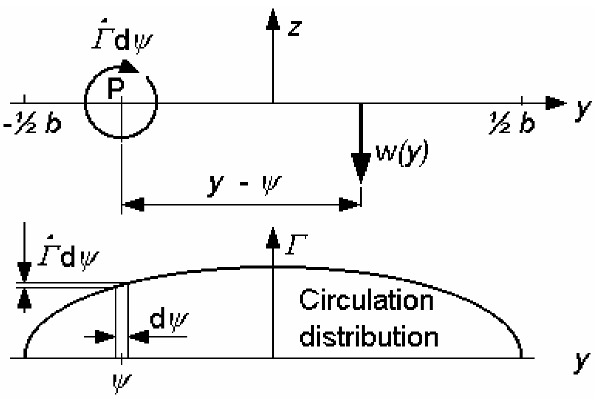
where $$w(\theta) = \frac{\Gamma}{2\pi r}$$

Having considered three dimensional flows we are now aware that a two dimensional vortex is a line-vortex extending to infinity above and below the paper. The difference here is that the trailing vortices also extend to infinity ***but in only one direction.***

Therefore the induced velocity <u>in the y-z plane</u> due to a trailing vortex is **HALF** that of a doubly infinite, two-dimensional vortex.

$$w(\theta) = \frac{\Gamma}{4\pi r}$$

The Downwash Equation (2)



Applying this result we write the vertically <u>downward</u> induced velocity contribution d*w* at an arbitrary position *y* on the lifting line, due to the vortex element trailing from P (Position ψ on the *y* axis).

Using the abbreviation $\frac{\mathrm{d}\Gamma}{\mathrm{d}\psi} = \dot{\Gamma}$

ψ is a dummy *y* variable.

It follows that $$\mathrm{d}w = \frac{\dot{\Gamma}\,\mathrm{d}\psi}{4\pi(y-\psi)}$$

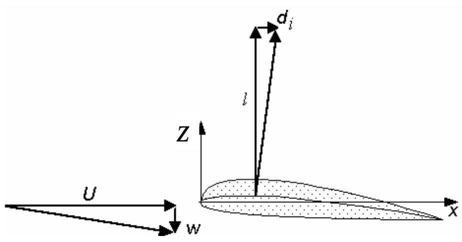
Finally we integrate from *y* = -½ *b* to *y* = ½ *b* to include contributions from all trailing vortex elements

$$w(y) = \frac{1}{4\pi}\int_{-\frac{1}{2}b}^{\frac{1}{2}b} \frac{\dot{\Gamma}\,\mathrm{d}\psi}{y-\psi}$$

N.B. This is only the induced downwash, not the whole velocity field

This downwash equation should be understood and memorised

Induced Drag



The importance of the downwash is because of the resulting change in direction of the onset flow: although initially horizontal, it is deflected downwards as it reaches the wing. However, at each spanwise location, the local aerofoil lift l (y) per unit span is still perpendicular to the local onset stream. Thus it is not perpendicular to the undisturbed flow, but is inclined backwards at the local downwash angle $w(y) / U$.

The convenient way to treat this inclined lift is to divide it into components, retaining the definition of lift perpendicular to the undisturbed flow and introducing an **induced drag** d_i per unit span.

$$d_i = l(y)\frac{w(y)}{U}$$

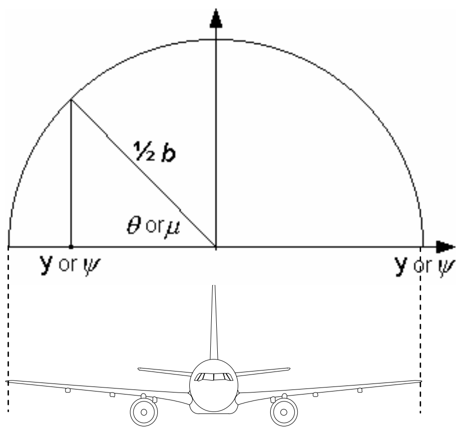
The total induced drag is found by integrating across the whole span

$$D_i = \int_{-\frac{b}{2}}^{\frac{b}{2}} l(y)\frac{w(y)}{U}\mathrm{d}y, \text{ recall that } l(y) = \rho U\Gamma(y), \text{ we can substitute to obtain:}$$

$$D_i = \rho U \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y)\frac{w(y)}{U}\mathrm{d}y.$$

To find the lift and drag induced for a given wing we need an expression for Γ(y)

Analysis using Fourier series for circulation distribution (along the lifting line)



Just like when we calculated the slope of an aerofoil we make a coordinate substitution to carry out this analysis:

y = $-\frac{1}{2} b \cos\theta$

dy = $\frac{1}{2} b \sin\theta \; \mathrm{d}\theta$

ψ = $-\frac{1}{2} b \cos\mu$

dψ = $\frac{1}{2} b \sin\mu \; \mathrm{d}\mu$

Suggested Fourier series

$$\Gamma(\psi) = 2bU\sum_{n=1}^{\infty} B_n \sin(n\mu), \qquad (n = 1,3,5,\ldots).$$

Odd coefficients only are used because an asymmetrical circulation distribution would cause the aircraft to roll.

Differentiate with respect to linear distance

$$\dot{\Gamma}(\psi) = \frac{\mathrm{d}\Gamma}{\mathrm{d}\psi} = \frac{\mathrm{d}\Gamma}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\psi} = \frac{2bU\sum_{n=1}^{\infty} nB_n\cos(n\mu)}{\frac{1}{2}b\sin\mu} = 4U\sum_{n=1}^{\infty}\frac{nB_n\cos(n\mu)}{\sin\mu}.$$

Lift coefficient (revert to lifting surface concept for a moment)

The pressure difference p_2 - p_1 is determined by the local vortex intensity $\gamma(x,y)$.

$$p_2 - p_1 = \rho \, U \, \gamma(x,y)$$

The lift L is the integral of this pressure difference over the whole wing area.

$$L = \int\int (p_2 - p_1)\mathrm{d}x\,\mathrm{d}y = \rho U \int\int \gamma(x,y)\mathrm{d}x\,\mathrm{d}y = \rho U \int \Gamma(y)\mathrm{d}y$$

Using the definition of lift coefficient C_L for a wing of area S, we write

$$C_L = \frac{L}{\tfrac{1}{2}\rho U^2 S} = \frac{2}{US}\int \Gamma(y)\mathrm{d}y$$

Substitute the standard Fourier series for $\Gamma(y)$, noting that $\mathrm{d}y = \tfrac{1}{2} b \sin\theta \; \mathrm{d}\theta$

$$C_L = \frac{2}{US}\int \Gamma(y)\mathrm{d}y = \frac{2}{US}\int_0^{\pi} 2bU\sum_{n=1}^{\infty} B_n \sin(n\theta) \times \tfrac{1}{2} b \sin\theta \,\mathrm{d}\theta$$

When integrated, all terms give zero, except for n = 1 when the integral is ½ π.

$$C_L = \frac{2b^2}{S}\int_0^{\pi}\sum_{n=1}^{\infty} B_n \sin(n\theta)\sin\theta\,\mathrm{d}\theta = \frac{2b^2}{S}\times B_1 \times \frac{\pi}{2} = \pi\left(\frac{b^2}{S}\right)B_1 = \pi A B_1$$

A is the aspect ratio

There is no contribution to lift from any Fourier term other than the first

Downwash

Earlier in the lecture we defined the downwash equation:

$$w(y) = \frac{1}{4\pi}\int_{-\frac{1}{2}b}^{\frac{1}{2}b} \frac{\dot{\Gamma}\,\mathrm{d}\psi}{y-\psi}$$

y = - ½ b cos θ, dy = ½ b sin θ dθ

ψ = - ½ b cos μ, dψ = ½ b sin μ dμ

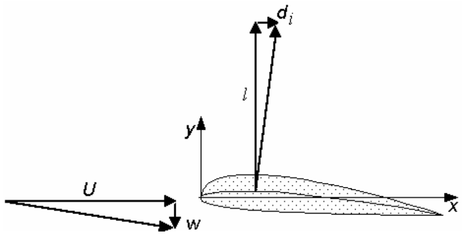
We can now substitute the general expression for $\dot{\Gamma}(\psi) = 4U\sum_{n=1}^{\infty}\frac{nB_n\cos(n\mu)}{\sin\mu}$.

$$\frac{w(y)}{U} = \frac{1}{4\pi U}\int_0^{\pi}\frac{4U\sum_{n=1}^{\infty}\frac{nB_n\cos(n\mu)}{\sin\mu}\frac{\mathrm{b}}{2}\sin\mu\,\mathrm{d}\mu}{\left(-\frac{\mathrm{b}}{2}\cos\theta\right)-\left(-\frac{\mathrm{b}}{2}\cos\mu\right)} = \frac{1}{\pi}\sum_{n=1}^{\infty} nB_n\int_0^{\pi}\frac{\cos(n\mu)\,\mathrm{d}\mu}{\cos\mu - \cos\theta}$$

As before: *from HLT*: $\displaystyle\int_0^{\pi}\frac{\cos(n.\mu)}{(\cos\mu - \cos\theta)}\mathrm{d}\mu = \frac{\pi\sin(n\theta)}{\sin\theta}$ Glauert Integral

$$\boxed{\frac{w(y)}{U} = \sum_{n=1}^{\infty} n B_n \frac{\sin(n\theta)}{\sin\theta}}$$

Induced Drag



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$$D_i = \int_{-\frac{b}{2}}^{\frac{b}{2}} l(y)\frac{w(y)}{U}\mathrm{d}y, \text{recall that } l(y) = \rho U \Gamma(y), \text{ we can substitute to obtain:}$$

$$D_i = \rho U \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y)\frac{w(y)}{U}\mathrm{d}y. \text{ We know that } \Gamma(y) = 2bU\sum_{m=1}^{\infty} B_n \sin(m\theta); \frac{w(y)}{U} = \sum_{n=1}^{\infty} n\,B_n \frac{\sin(n\theta)}{\sin\theta}.$$

Note m and n can vary independently

Induced Drag (2)

Thus: $D_i = \rho U \int_0^{\pi} 2bU \sum_{m=1}^{\infty} B_n \sin(m\theta) \sum_{n=1}^{\infty} n\,B_n \frac{\sin(n\theta)}{\sin\theta}\frac{b}{2}\sin\theta \mathrm{d}\theta$

This integral can be reduced as only the term where m=n produces a non-zero integral:

$$D_i = \rho U^2 b^2 \int_0^{\pi} \sum_{n=1}^{\infty} nB_n^2 \sin(n\theta)\sin(n\theta)\,\mathrm{d}\theta = \rho U^2 b^2 \sum_{n=1}^{\infty} nB_n^2\ \frac{\pi}{2}$$

We can then write the Induced Drag Coefficient C_{Di} as:

$$C_{D_i} = \frac{D_i}{\frac{1}{2}\rho U^2 S} = \pi A \sum_{m=1}^{\infty} nB_n^2$$

Unfortunately we see that the induced drag coefficient depends on all the cofficients of the Fourier series we have defined to characterise the Vortex distribution along the Lifting line.

Minimising the Induced Drag

We now have 2 equations: 1 decribing the lift to the wing, and the other the induced drag. For an aircraft of a fixed weight, W, the required lift is fixed. So the lift coefficient:

$$C_L = \frac{W}{\frac{1}{2}\rho U^2 S} = \pi A B_1 \Rightarrow B_1 = \frac{W}{\frac{1}{2}\rho U^2 S \pi A}, \text{and importantly is fixed.}$$

To minimise the induced drag coefficient we want to minimise:

$$C_{D_i} = \pi A \sum_{n=1}^{\infty} n B_n^2 = \pi A B_1^2 + \pi A \sum_{n=3}^{\infty} n B_n^2 \quad \textit{also} \quad \frac{C_L^2}{\pi A} + \pi A \sum_{n=3}^{\infty} n B_n^2$$

Therefore C_{Di} takes its smallest value when all the terms $B_3 \ldots B_\infty$ (apart from B_1) = 0. This gives us:

$$\Gamma(y) = 2bUB_1 \sin\theta \;\text{ where }\; y = -b\cos\theta. \quad \text{This is known as elliptical loading}$$

It not only creates the minimum induced drag, but creates constant downwash across the span

$$\frac{w(y)}{U} = \sum_{n=1}^{\infty} n B_n \frac{\sin(n\theta)}{\sin\theta} = B_1$$

Application to the design of wings – The Aeroplane Drag Equation

Induced drag is a side-effect of lift. Total drag also includes **skin friction drag** and **form drag** due to boundary layer separation on non-streamlined shapes. These are not predicted by potential flow theory.

Therefore for aircraft performance estimation we add an arbitrary term C_{D0} to represent these and thus represent the total drag by

$$C_D = C_{D0} + K\frac{C_L^2}{\pi A}$$

C_{D0} = **Profile drag coefficient** (Skin friction + form drag)

K = **Induced drag factor** (K = 1 for elliptic loading or K >1 for any other loading)

Designing a wing

For any spanwise location the local section lift coefficient (reduced by downwash) is formed by contributions from several terms:

- Aerofoil lift at 0^o incidence
- Lift from wing incidence
- Lift from local wing twist incidence
- Reduction of lift by induced downwash

We can more formally describe this by the equation

$$C_l(y) = C_{l,0}(y) + 2\pi\alpha_0 + 2\pi\lambda(y) - 2\pi\sum_{n=1}^{\infty}\frac{nB_n\sin(n\theta)}{\sin\theta}.$$

NB: this is lift per unit span

The local lift coefficient may also be written in terms of the local circulation, (lect. 2.)

$$C_l(y) = \frac{\rho U\Gamma(y)}{\frac{1}{2}\rho U^2 c(y)} = \frac{4b}{c(y)}\sum_{n=1}^{\infty}B_n\sin(n\theta).$$

Equating these two expressions for $C_l(y)$

$$\frac{4b}{c(y)}\sum_{n=1}^{\infty}B_n\sin(n\theta) = C_{l,0}(y) + 2\pi\alpha_0 + 2\pi\lambda(y) - 2\pi\sum_{n=1}^{\infty}\frac{nB_n\sin(n\theta)}{\sin\theta}.$$

This is the wing design equation. We do not so lve it, we look at it and take decisions about the type of section we want [$C_{l\,0}(y)$] the taper and twist variation [$c(y)$ & $l(y)$] and the type of loading that will be acceptable [Choice of Fourier coefficients B_n]

Using the wing design equation

Wing Design Eqn: $$\frac{4b}{c(y)}\sum_{n=1}^{\infty}B_n\sin(n\theta) = C_{l,0}(y) + 2\pi\alpha + 2\pi\lambda - 2\pi\sum_{n=1}^{\infty}\frac{nB_n\sin(n\theta)}{\sin\theta}.$$

1st let's consider the case of elliptical loading where only B_1 is non-zero.
The wing design equation becomes:

$$\frac{4b}{c(y)}B_1\sin\theta = C_{l,0}(y) + 2\pi(\alpha_0 + \lambda(y) - B_1).$$

Some possible choices

a) The wing may be tapered but with sections of similar shape ($C_{l,0}(y)$=constant). If there is no twist either ($\lambda(y) = 0$) , then the entire RHS. of the equation will be independent of y and θ. Consider what choice of c(y) would make the LHS independent of θ also.

b) If the wing has no taper and all sections are identical, consider what twist function $l(y)$ will satisfy the equation. The answer will depend upon the choice of B_1 so the design will be right for one lift coefficient (Flight speed) only.

Calculating Fourier Coefficients for a Specified Wing Shape

To solve for a large number of unknown coefficients using the wing design equation requires the application of that equation a large number of times at a large number of different points (Values of θ) along the span. Clearly it is necessary to limit the Fourier series to a finite number of coefficients B_n.

1st rewrite the equation to obtain a single term in B_n.

$$\sum_{n=1}^{\infty} B_n \left\{ \frac{2\pi n \sin(n\theta)}{\sin\theta} + \frac{4b}{c(y)} \sin(n\theta) \right\} = C_{l,0}(y) + 2\pi(\alpha + \lambda(y))$$

At position on the wing matched θ, the coefficients of B_n (term in { }) can be calculated, let's refer to them by $C(n,\theta)$. Similarly the RHS of the equation can be evaluated at a single value of y (or θ); let's call the terms on the RHS $D(\theta)$. Thus:

$$\sum_{n=1}^{\infty} B_n C(n,\theta) = D(\theta)$$

We must form this equation at as many positions along the wing as the number of coefficients of B_i which we want to compute.

Example

Remember these increment in odd numbered steps

The use of this equation is best illustrated with an example:
Limit the number of coefficients to 4. B_1, B_3, B_5, B_7 and write 4 equations by repeating the last equation above for 4 spanwise positions $\theta_1 = 0$, $\theta_2 = \pi/8$, $\theta_3 = \pi/4$, $\theta_4 = 3\pi/8$.

For $\theta = \theta_1$ $\quad B_1 C_{11} + B_3 C_{13} + B_5 C_{15} + B_7 C_{17} = \quad D_1$
For $\theta = \theta_2$ $\quad B_1 C_{21} + B_3 C_{23} + B_5 C_{25} + B_7 C_{27} = \quad D_2$
For $\theta = \theta_3$ $\quad B_1 C_{31} + B_3 C_{33} + B_5 C_{35} + B_7 C_{37} = \quad D_3$
For $\theta = \theta_4$ $\quad B_1 C_{41} + B_{23} C_{43} + B_5 C_{45} + B_7 C_{47} = \quad D_4$

In Matrix form this is

$$\begin{bmatrix} C_{11} & C_{13} & C_{15} & C_{17} \\ C_{21} & C_{23} & C_{25} & C_{27} \\ C_{31} & C_{33} & C_{35} & C_{37} \\ C_{41} & C_{43} & C_{45} & C_{47} \end{bmatrix} \begin{Bmatrix} B_1 \\ B_3 \\ B_5 \\ B_7 \end{Bmatrix} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}$$

where $\quad C_{nm} \quad = \quad \dfrac{2\pi n \sin(n\theta_m)}{\sin\theta_m} + \dfrac{4b}{c(\theta_m)} \sin(n\theta_m)$

and $\quad D_m \quad = \quad C_{l0}(\theta_m) + 2\pi\{\alpha_0 + \lambda(\theta_m)\}$

Lift curve slope reduction for finite wings (Elliptic loading).

2-D Aerofoil lift curve

In lecture 2 we saw that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{A_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} A_n \cos n\theta \text{ and } C_l = \pi\left[A_0 + \frac{A_1}{2}\right].$$

It was also noted that for a rigid aerofoil, all terms expect for A_0 are fixed for a rigid wing, which merely tilts to vary the incidence α. Thus for an infinitesimal increase of incidence $\mathrm{d}\alpha$ the slope everywhere changes by $-\mathrm{d}\alpha$ which yields

$$\mathrm{d}\alpha = ½\,\mathrm{d}A_0$$

also: $\mathrm{d}C_l = 2\pi\,\mathrm{d}\alpha = \pi\ \mathrm{d}A_0$

Thus $\dfrac{\mathrm{d}C_L}{\mathrm{d}\alpha} = 2\pi.$

The lift curve of a rigid 2-D aerofoil is, therefore, is a straight line of slope 2π which may be described in general by

$$C_l = C_{l\,0} + 2\,\pi\,\alpha$$

Lift curve for finite wings

When this aerofoil is part of a finite wing, the wing has a nominal incidence α_0 and the local section may be twisted relative to this by a further angle $\lambda(y)$. However, the incidence relative to the local flow is less than $\alpha_0 + \lambda(y)$ because of the downwash angle $w(y)/U$.

For elliptic loading this downwash is constant $\dfrac{w(y)}{U} = \displaystyle\sum_{n=1}^{\infty} nB_n \frac{\sin(n\theta)}{\sin\theta} = B_1$

and the effective local incidence $\alpha(y)$ is $\alpha(y) = \alpha_0 + \lambda(y) - \dfrac{C_L}{\pi A}$

Now from our definition $C_l = C_{l\,0} + 2\,\pi\,\alpha$ (y): $C_l(y) = C_{l,0}(y) + 2\pi\left(\alpha_0 + \lambda(y) - \dfrac{C_L}{\pi A}\right)$

The total lift L is found by integrating the local lift across the span so that in dimensionless form

$$C_L = \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[C_{l,0}(y) + 2\pi\left(\alpha_0 + \lambda(y) - \frac{C_L}{\pi A}\right)\right]\frac{c(y)}{S}\mathrm{d}y$$

$$= \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[C_{l,0}(y) + 2\pi\lambda(y)\right]\frac{c(y)}{S}\mathrm{d}y + \left(2\pi\alpha_0 - \frac{2C_L}{A}\right)\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{c(y)}{S}\mathrm{d}y$$

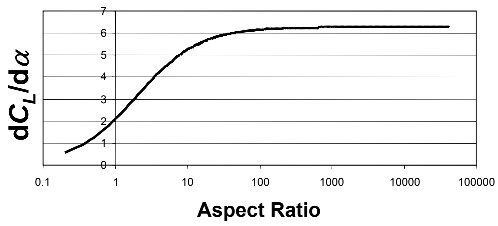
Lift curve for finite wings (2)

Integrating the chord across the span gives the wing area S so the second integral is unity. Collecting coefficients of C_L on the LHS:

$$C_L\left(1+\frac{2}{A}\right)=2\pi\alpha_0+\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[C_{l,0}(y)+2\pi\lambda(y)\right]\frac{c(y)}{S}\mathrm{d}y$$

Consider the final term for a rigid wing $C_{l,0}$ is a constant, and $\lambda(y)$ is constant. If we want to find the lift curve i.e. the rate of change of lift coefficient with nominal incidence, then when we differentiate the final term w.r.t. α_0, it will disappear. So:

$$\frac{\mathrm{d}C_L}{\mathrm{d}\alpha_0}=2\pi\left(\frac{A}{A+2}\right)$$



As expected this becomes equal to the two-dimensional aerofoil result when $A = \infty$