C201 Viscous Flow and Turbulence

Lecture 4

Part 2: Transition

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1 Back to Moody's chart

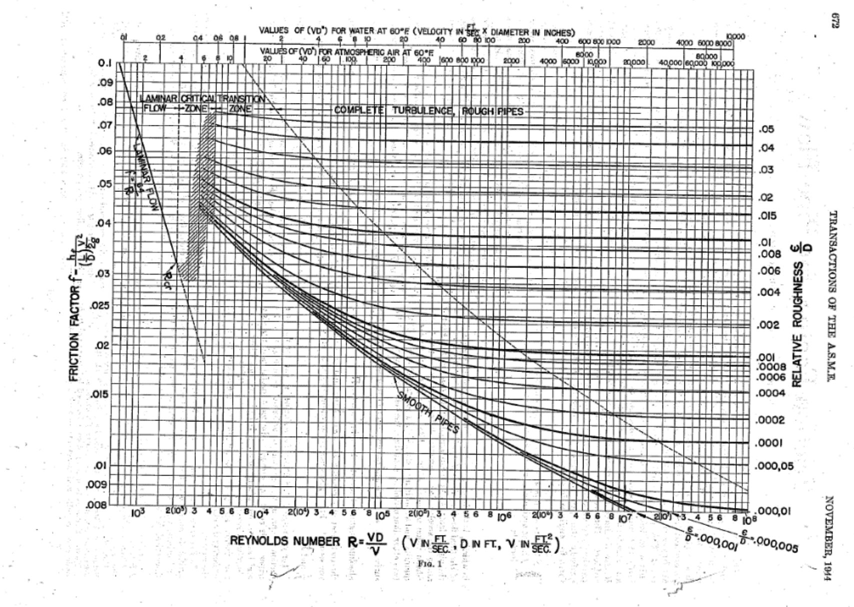

Figure 1: Moody, L. F. (1944), Friction factors for pipe flow, Transactions of the ASME, 66 (8): 671–684

Let's take a look back at Moody's chart. In lecture 1 we justified the need to study turbulent flows on the basis of the differences in friction, heat transfer and mass transfer characteristics between turbulent flows and laminar flows, exemplified by drastically different scaling laws for the friction coefficient in the two flow regimes shown by Moody's chart.

The question now arises about the shaded area between the two flow regimes: how and why does a laminar flow become turbulent?

The phenomena leading a laminar flow to turn into a turbulent one are generally called transition. Transition is largely determined by the Reynolds number, but other factors are also at play: wall curvature, condition of the free stream, surface roughness etc. all contribute to transition. It is because of the sensitivity of transition to the details of the flow that transitional flows are not indicated in Moody's chart by a single line but by a shaded area.

1.1 Transition in practice

Transition is in fact a phenomenon of considerable practical importance. In modern gas turbines, like the Trent XWB in advances in aerodynamic design and

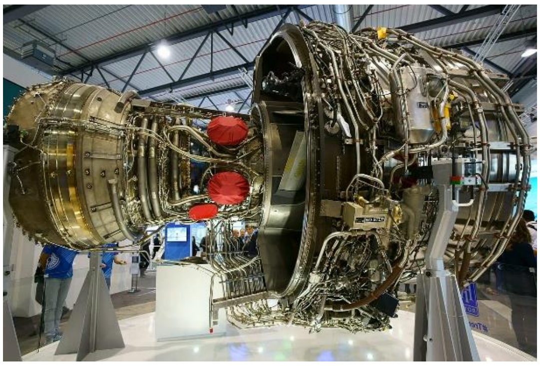

Figure 2: top left: Rolls-Royce Trent XWB. The front stages of the IP compressor exhibit transitional flow. Top right: the Boeing 787: the flow over the wings is transitional at cruise. Bottom: the Rolls-Royce BR725 (fan set only) with transition patches (small dark rectangle on each blade) to force transition and prevent vibrations.

manufacturing mean that the flow near the leading edges of the front stages in compressors is so well controlled that it says laminar for some distance downstream of the leading edge, before undergoing transition to turbulence. This behaviour makes the performance characteristics of the compression system dependent on the Reynolds number (unlike what you saw in your A4 module) and mean that performance improves at altitude, where Reynolds numbers are lower and laminar patches are larger.

Similar advances in wing design and manufacture mean that the wings of modern large long-haul aircraft, like the Boeing 787 or the Airbus A350, present extensive areas of laminar flow and are transitional in flight at cruise conditions. As with power plants, a reduction in the portion of the boundary layer that is turbulent leads to a reduction in drag. Transitional flow is encountered on all large wind turbines and the design of aerofoils for these machines must take transition in account if a competitive product is to be arrived at.

Transition in practical devices is not always beneficial. In the Rolls-Royce BR710 engine, transitional flow on the fan blades during operation at high altitude and high Mach numbers created unsteadiness at frequencies close to the natural frequencies of the blades and ultimately led to fatigue cracks. On the BR725 this problem is avoided by applying a transition strip (small rough patch) to each blade to force transition well upstream of the passage shock. Your lecturer is one of the people who designed the transition patches.

2 Reynolds' experiment

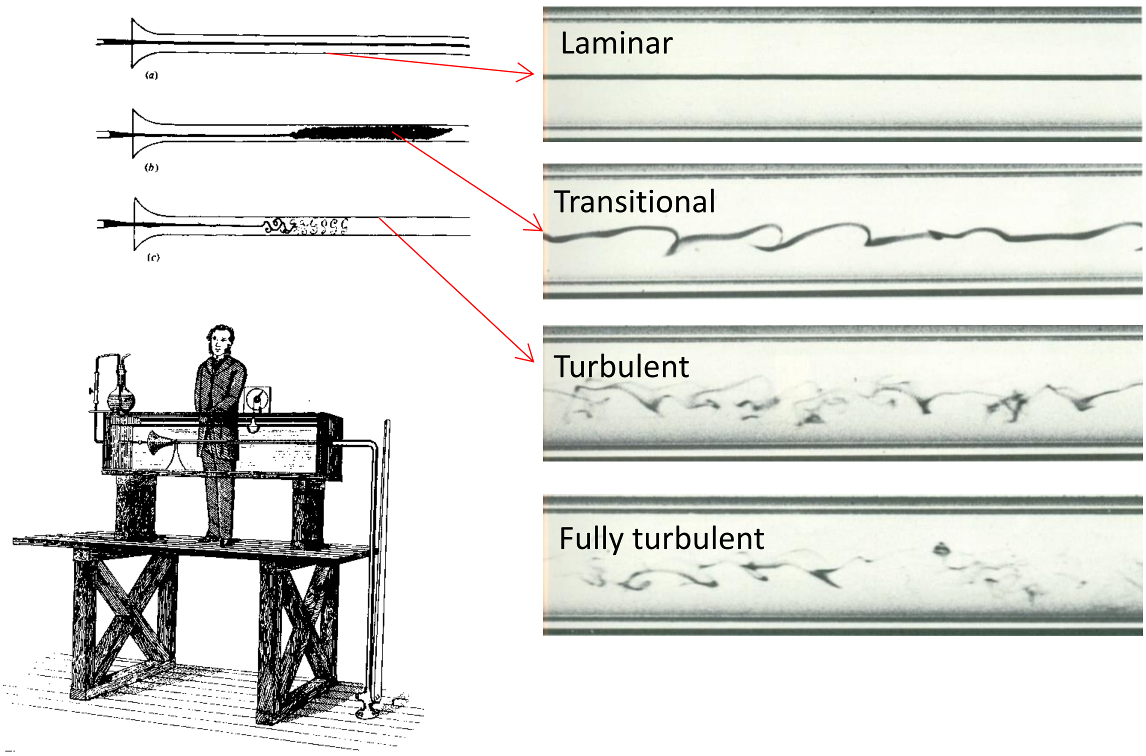

Figure 3: sketches of Reynolds experiment and its outcomes

The first rigorous study on transition is Reynolds' experiment (see Figure 3). Reynolds experiment consisted in injecting a stream of ink in a transparent pipe where water was flowing. At low velocities, the stream of ink flows steadily along a straight trajectory and does not mix appreciably with the surrounding water. As the velocity of the water flow is increased the ink steam starts meandering about the pipe centreline. At even higher speeds, the ink steam breaks up in thin filaments and rapidly mixes with the water. The controlling parameter for the behaviour of the ink is really the Reynolds number. Reynolds found that transition to turbulence would take place in his experiment at a Reynolds number close to 2300.

Reynolds was painfully aware of the critical Reynolds number on small perturbations entering the water pipe from its intake – a wooden bellmouth – and remarked on the care needed to minimise disruption to the flow upon entrance to the glass pipe. Reynolds experiment was repeated in the '60 by Eckman using Reynolds original equipment – still preserved at the University of Manchester. Eckman managed to preserve laminar flow up to a Reynolds number of 44000, but only after carefully smoothing the bellmouth to remove any flow non-uniformity.

2.1 Flow instabilities

What Reynolds observed is that when the Reynolds number is large enough, small perturbations induced in the flow from its surroundings start growing, to the point that they replace the original flow pattern with a new flow pattern. When this happens there are multiple flow patterns that are compatible with the Navier-Stokes equations at a given Reynolds number and boundary conditions. Some of these patterns are not observed because they are unstable, in the sense that if the flow starts from one of these patterns and is disturbed, the disturbance we introduce will grow until it is of sufficient amplitude to alter the flow pattern an transform it into one of the other possible solutions. This phenomenon is very common in nature.

We encounter flow instabilities in the Earth atmosphere – they determine the appearance of wave-shaped clouds – in rotating enclosures, like the internal cavities of engines or even the atmospheres of giant gas planets or stars. A succession of flow instabilities determines the behavior of wakes behind objects placed in a uniform steady flow.

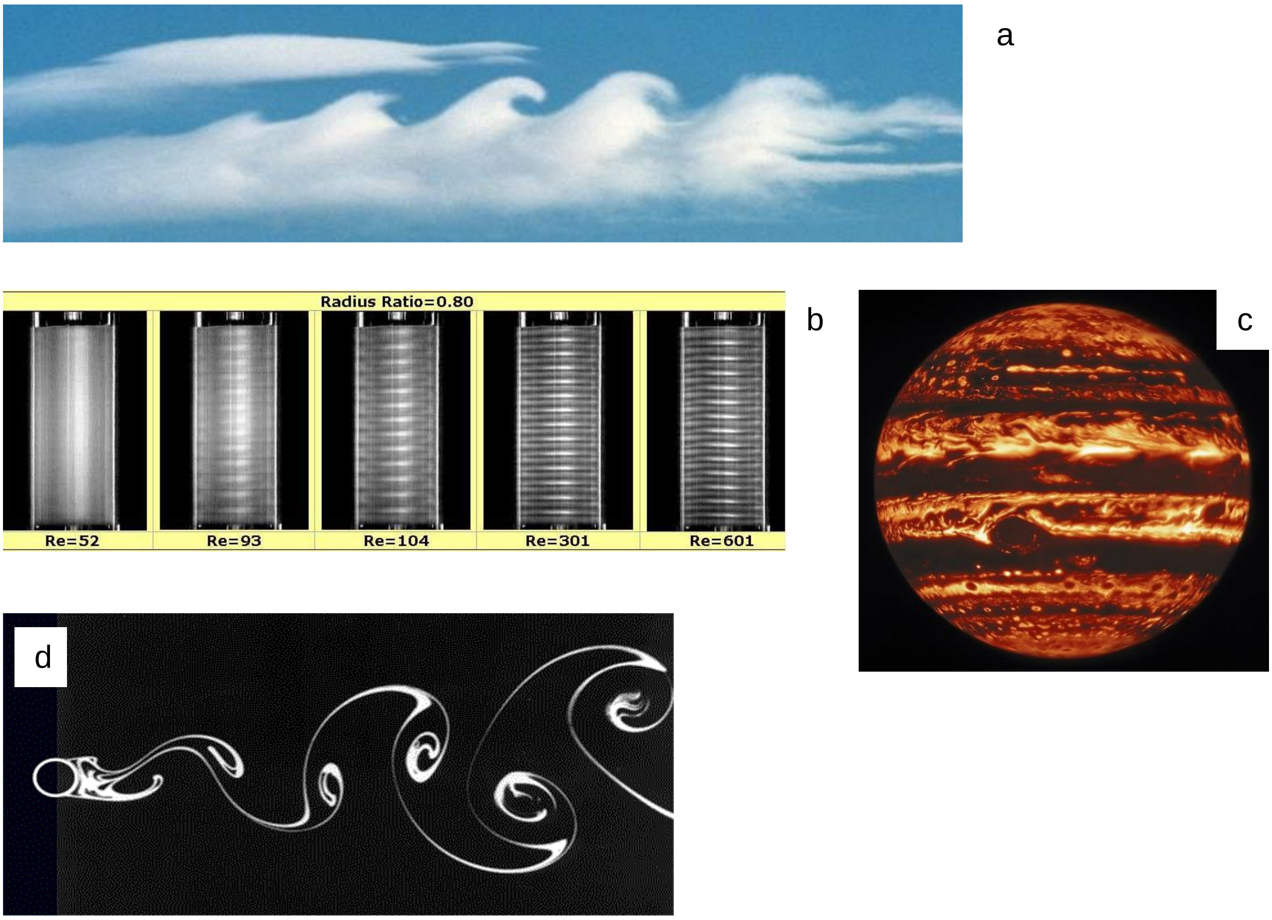

Figure 4: a) flow instabilities determine the shape of clouds. b) flow instabilities in rotating systems control the dynamics of flow in rotating cavities in engines and in the atmosphere of giant gaseous plantes like Jupiter or stars. c) wakes of blunt bodies break up into discrete vortices and ultimately turbulence because of flow instabilities.

3 Linear stability theory

In this section we develop formal tools to study the evolution of small perturbations. In particular we are interested in finding out if a small perturbation grows – in which case the flow is (linearly) unstable – or decays – which case the flow is linearly stable. We expect the stability properties of the flow to be determined by parameters such as the Reynolds number, the geometry, pressure gradients, the addition or removal of fluid at the walls etc.

3.1 The linearised Navier-Stokes equations

Let's start from the Navier-Stokes equations

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \nu \frac{\partial u_i}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial p}{\partial x_i}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

and assume that the flow variables are composed of a steady component (denoted by the uppercase letters) and a perturbation (denoted by a prime):

$$u_i = U_i + \epsilon u_i'$$

$$p = P + \epsilon p'$$

ϵ is here a small parameter and we will use it to decide which terms in the equations can be discarded. We further assume that the steady components represent a solution of the steady Navier-Stokes:

$$U_j \frac{\partial U_i}{\partial x_j} = \nu \frac{\partial U_i}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial P}{\partial x_i}$$

$$\frac{\partial U_i}{\partial x_i} = 0$$

Substituting the perturbed flow variables in the Navier-Stokes and removing terms corresponding to the steady equations we find

$$\epsilon \frac{\partial u_i{}'}{\partial t} + \epsilon\, U_j \frac{\partial u_i'}{\partial x_j} + \epsilon u_j{}' \frac{\partial U_i}{\partial x_j} + \epsilon^2 u_j{}' \frac{\partial u_i'}{\partial x_j} = \epsilon\nu \frac{\partial u_i'}{\partial x_j \partial x_j} - \epsilon \frac{1}{\rho}\frac{\partial p'}{\partial x_i}$$

$$\epsilon \frac{\partial u_i'}{\partial x_i} = 0$$

We notice that the momentum equation contains only one term scaling with ϵ^2. In the limit of small ϵ this term is negligible and therefore, within first order in ϵ we find

$$\frac{\partial u_i{}'}{\partial t} + \ U_j \frac{\partial u_i'}{\partial x_j} + u_j{}' \frac{\partial U_i}{\partial x_j} = \nu \frac{\partial u_i'}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial p'}{\partial x_i}$$

$$\frac{\partial u_i'}{\partial x_i} = 0$$

This set of equations is the set of the linearised Navier-Stokes equations. We find that they are linear equations in the perturbation variables u_i', p' and are influenced by the flow pattern through convection velocity of the base flow and its gradient.

If we are interested in the growth of the perturbations we can assume

$$u_i' = \hat{u}_i e^{st}$$

$$p' = \hat{p}\, e^{st}$$

which yields

$$s\hat{u}_i + U_j \frac{\partial\, \hat{u}_i}{\partial x_j} + \hat{u}_j \frac{\partial U_i}{\partial x_j} = \nu \frac{\partial\, \hat{u}_i}{\partial x_j \partial x_j} - \frac{1}{\rho} \frac{\partial\, \hat{p}}{\partial x_i}$$

$$\frac{\partial\, \hat{u}_i}{\partial x_i} = 0$$

We have written a homogeneous liner system in the variables $\hat{u}_i, \hat{p}$. The situation we have created is not dissimilar from the situation you encountered in your A1 and P1 studies when studying homogeneous linear systems of the type

$$A(s)x = 0$$

Non-trivial solutions only exist for certain values of the parameter s which make $A(s)$ singular. s is an eigenvalue and x is in this case an eigenvector – or a family of eigenvectors. The nature of the flow perturbations we will find to be stable of unstable is very similar: they grow or decay in time without changing shape, very much like eigenvectors are magnified by multiplied by the matrix they belong to, but do not change direction.

3.2 The stability of an inviscid shear layer

Let's first tackle an inviscid problem. Let' set $\nu \to 0$ and let's consider a flow taking place between two impermeable plates at the locations $y = \pm b$. The mean flow takes

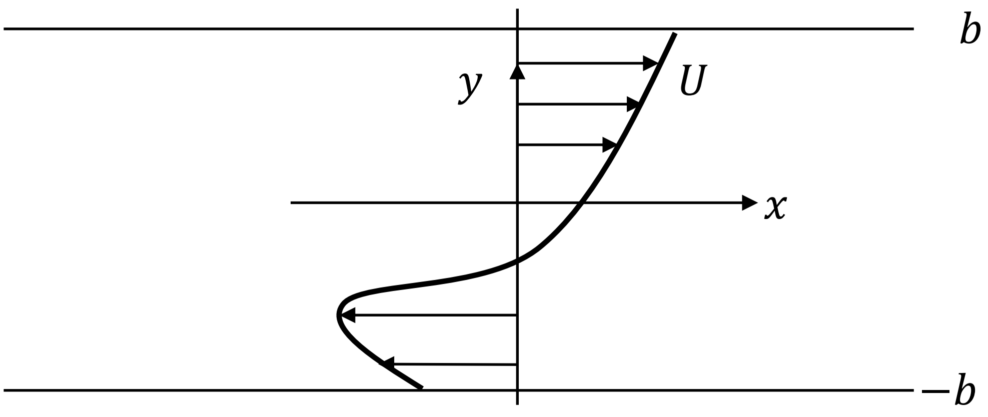

Figure 5: flow configuration for inviscid shear layer between parallel plates.

place along the x direction only and is a function of the y variable only. The flow domain is infinite in the x and z directions (see Figure 5). The velocity profile can be written as

$$U_i = \begin{bmatrix} U(y) \\ 0 \\ 0 \end{bmatrix}$$

Because the flow domain is infinite in the x and z directions we can write

$$\hat{u}_i = \tilde{u}_i \mathrm{e}^{i(kx+mz)}$$

$$s = -ick$$

where k and m are two wavenumbers. We can also switch back to the x, y, z, u, v, w notation as the equations are very compact anyway:

$$ik(U-c)\tilde{u} + \tilde{v}\frac{dU}{dy} = -ik\tilde{p}$$

$$ik(U-c)\tilde{v} = -\frac{d\tilde{p}}{dy}$$

$$ik(U-c)\tilde{w} = -im\tilde{p}$$

$$ik\tilde{u} + im\tilde{w} + \frac{d\tilde{v}}{dy} = 0$$

3.3 Squire's transformation

The set of equations for the inviscid shear layer represent a three-dimensional disturbance. We can reduce the problem to a 2D problem by introducing the following transformation, due to Squire:

$$\kappa\tilde{\tilde{u}} = k\tilde{u} + m\tilde{w}$$

$$\kappa = \sqrt{k^2 + m^2}$$

$$\tilde{\tilde{p}}/\tilde{\kappa} = \tilde{p}/k$$

Squire's transformation replaces the perturbation velocities $\tilde{u}$ and $\tilde{w}$ with a combined perturbation $\tilde{\tilde{u}}$ formed by the weighted average of the two. Each velocity is weighted by its wavenumber. Similarly the two wavenumbers k and m are replaced by a single wavenumber, which represents the magnitude of the wave-number vector and the pressure is rescaled.

We can now combine the equations for $\tilde{u}$ and $\tilde{w}$ to find:

$$ik(U-c)(k\tilde{u} + m\tilde{w}) + k\tilde{v}\frac{dU}{dy} = -i(k^2 + m^2)\tilde{p}$$

Simplifying and substituting Squire's transformation yields

$$i(U-c)\kappa\tilde{\tilde{u}} + \tilde{v}\frac{dU}{dy} = -i\kappa\tilde{\tilde{p}}$$

$$i\kappa(U-c)\tilde{v} = -\frac{d\tilde{\tilde{p}}}{dy}$$

$$i\kappa\tilde{\tilde{u}} + \frac{d\tilde{v}}{dy} = 0$$

This set of equations is formally identical to those we would have obtained starting from the original 3D problem setting $\tilde{w} = 0$ and $m = 0$, therefore represent a 2D stability problem. This is Squire theorem: to study a 3D stability problem it is sufficient to study a 2D problem.

3.4 Rayleigh's equation

Now use continuity to eliminate $\tilde{\tilde{u}}$

$$-(U-c)\frac{d\tilde{v}}{dy} + \tilde{v}\frac{dU}{dy} = -i\kappa\tilde{\tilde{p}}$$

$$i\kappa(U-c)\tilde{v} = -\frac{d\tilde{\tilde{p}}}{dy}$$

and finally eliminate $\tilde{\tilde{p}}$ by differentiating the first equation and combining with the second

$$-(U-c)\frac{d^2\tilde{v}}{dy^2} - \frac{dU}{dy}\frac{d\tilde{v}}{dy} + \frac{d\tilde{v}}{dy}\frac{dU}{dy} + \tilde{v}\frac{d^2U}{dy^2} = -i\kappa\frac{d\tilde{\tilde{p}}}{dy}$$

$$i\kappa(U-c)\tilde{v} = -\frac{d\tilde{\tilde{p}}}{dy}$$

The resulting equation

$$(U-c)\left(\frac{d^2}{dy^2} - \kappa^2\right)\tilde{v} + \frac{d^2U}{dy^2}\tilde{v} = 0$$

is Rayleigh's stability equation (1880). This equation describes the evolution of small vertical perturbations in an inviscid, two-dimensional shear layer. Most of the theory of stability and large parts of transition theory were established by the start of WWII The publications of some papers was delayed by WWI.

3.5 The role of inflection points

We would like to know if a perturbation $\tilde{v}$ can grow or must decay by using Rayleigh's equation. First rewrite Rayleigh's equation as

$$\tilde{v}'' - \kappa^2\tilde{v} - \frac{U''}{U-c}\tilde{v} = 0$$

Now multiply by $\tilde{v}^*$ and integrate between $\pm b$

$$\int_{-b}^{b} (|\tilde{v}'|^2 + \kappa^2 |\tilde{v}|^2)\, dy + \int_{-b}^{b} \frac{U''}{U - c}\, |\tilde{v}|^2\, dy = 0$$

The quantity $\tilde{v}\,\tilde{v}^* = |\tilde{v}|^2$ is the energy of the perturbation. We used a similar technique when we introduced turbulence energy spectra. Looking at the equation above we realise that the first integral must be a positive real number. The second integral is instead a complex number in general because the eigenvalue c is in general complex. If we let $c = \alpha + i\beta$ then the imaginary part of the second integral and therefore of the left hand side of the equation is

$$-\beta \int_{-b}^{b} \frac{U''}{|U - c|^2}\, |\tilde{v}|^2\, dy$$

This integral can only be 0 if U'' has both positive and negative values in the interval $]-b, b[$. This means that we cannot have non-trivial solutions of Rayleigh’s equation without inflection points in the base velocity profiles. The presence of inflection points in the velocity profiles is a necessary – but not sufficient – condition for the existence of flow instabilities in shear layers

3.6 Neutrally stable mode for sinusoidal velocity profile

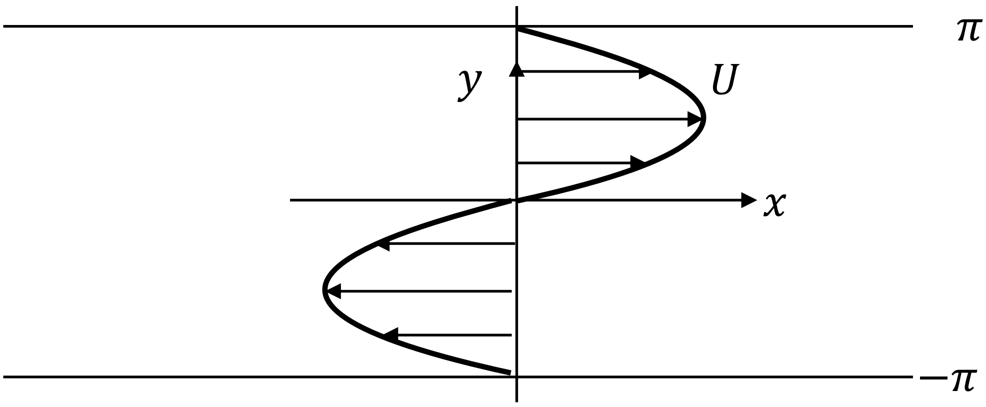

Figure 6: sinusoidal velocity profile in inviscid shear layer.

Take $b = \pi$ and $U = \sin y$. If we set $c = 0$ we are looking at a mode that does neither grow nor decay in time. Such a mode is called a neutrally stable mode and represents the boundary between stable and unstable disturbances. The Rayleigh’s equation for the neutrally stable mode is

$$\sin y \left(\frac{d^2}{dy^2} - \kappa^2 \right) \tilde{v} + \sin y\, \tilde{v} = 0$$

Or equivalently:

$$\frac{d^2 \tilde{v}}{dy^2} + (1 - \kappa^2)\tilde{v} = 0$$

The boundary conditions are:

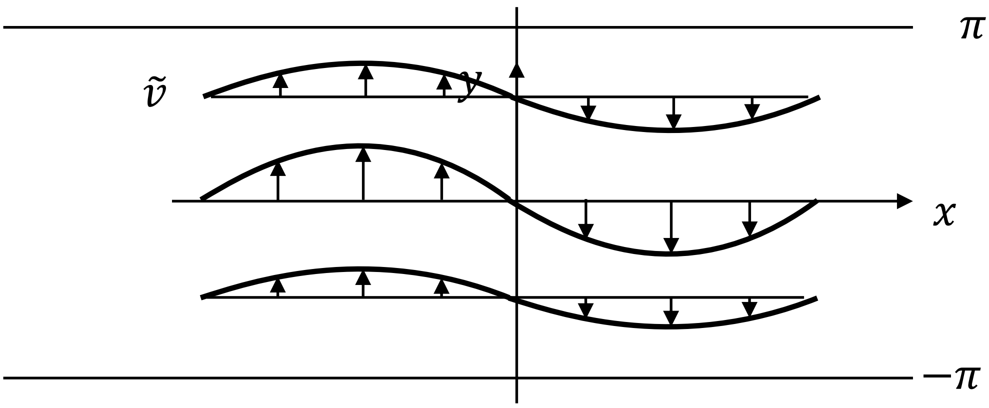

Figure 7: neutrally stable mode for the sinusoidal velocity profile

$$\tilde{v}(\pm\pi) = 0$$

To find the corresponding $\tilde{v}$ perturbation we can take:

$$\tilde{v} = \cos ny$$

The boundary conditions are satisfied for $n = 1/2$ and Rayleigh's equation demands

$$-n^2 + 1 - \kappa^2 = 0$$

whence

$$\kappa = \kappa_s = \frac{1}{2}\sqrt{3}$$

It is possible to show that for wavenumbers approaching κ_s from below, the eigenvalue c has imaginary part

$$Im(c) = \sqrt{3}(\kappa_s - \kappa)$$

and corresponds to unstable modes. The streamline pattern of the unstable disturbance for a shear layer (but with slightly different boundary conditions from

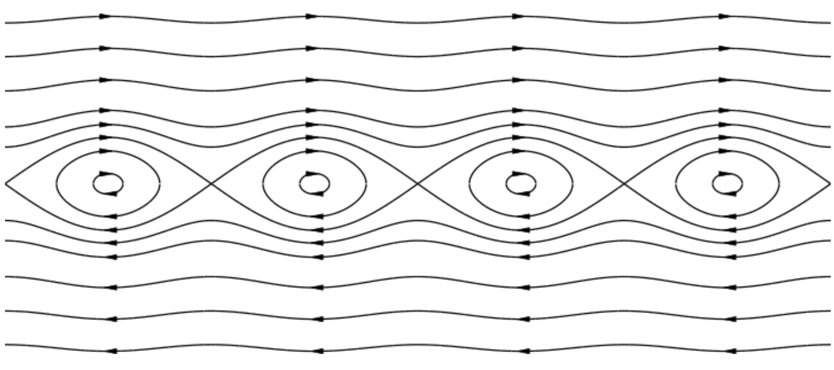

Figure 8: Rayleigh's cat's eye flow pattern

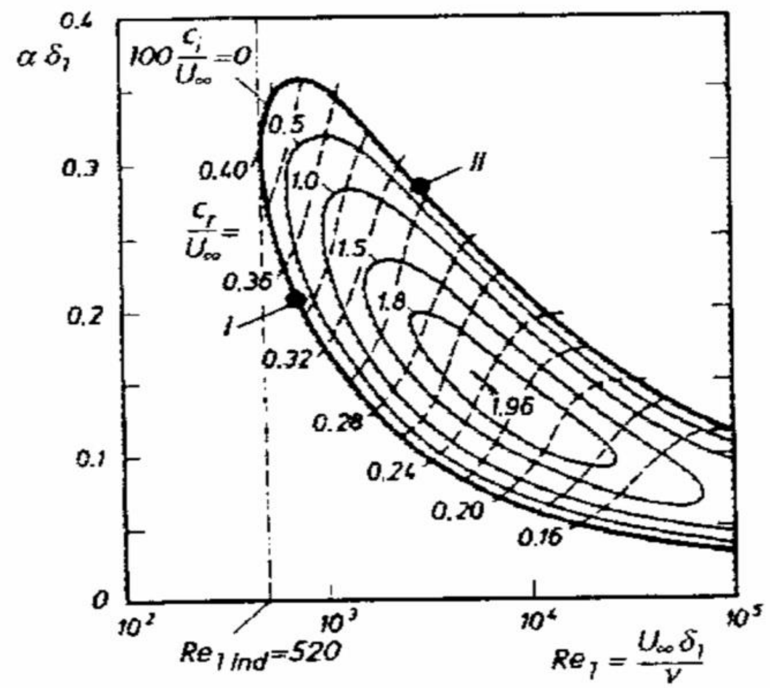

Figure 9: stability diagram for the Orr-Sommerfeld equation

those we used in our derivation) is presented in Figure 8. This pattern is responsible for the appearance of the clouds in Figure 4 a).

3.7 The Orr-Sommerfeld equation and boundary layer stability

The stability of boundary layers and viscous free shear flows is governed by an equation similar to Rayleigh's equation. The derivation is similar, but the viscous terms are not discarded. Furthermore, because boundary layers and free shear flows spread laterally, we cannot use the vertical velocity perturbation $\tilde{v}$ as main variable, but we must use a perturbation stream-function ψ. The Orr-Sommerfeld equation is

$$\frac{i\kappa}{Re}\left(\frac{d^2}{dy^2}-\kappa^2\right)^2\psi=(U-c)\left(\frac{d^2}{dy^2}-\kappa^2\right)\psi-U''\psi$$

The term on the left-hand side of the equation comes from the Laplacian of the velocity perturbations and contains the Reynolds number. The right hand side of the equation is identical to Rayleigh's equation.

The unstable modes of the Orr-Sommerfeld equations are waves and are known as Tollmein-Schlichting waves. A typical stability diagram for the Orr-Sommerfeld equation is shown in Figure 9. The axis on the stability diagram are the Reynolds number (horizontal) and the reduced wave-number (vertical). On the left of the Reynolds number 520 no instability is possible. The boundary of the oval region on the right of $Re=520$ is the neutral stability boundary, i.e. the locus of combinations of Reynolds number and wave-number for which a neutrally stable mode is found. The points within the oval region have at least one unstable mode. The numbers on the contours indicate the value of the imaginary part of the corresponding

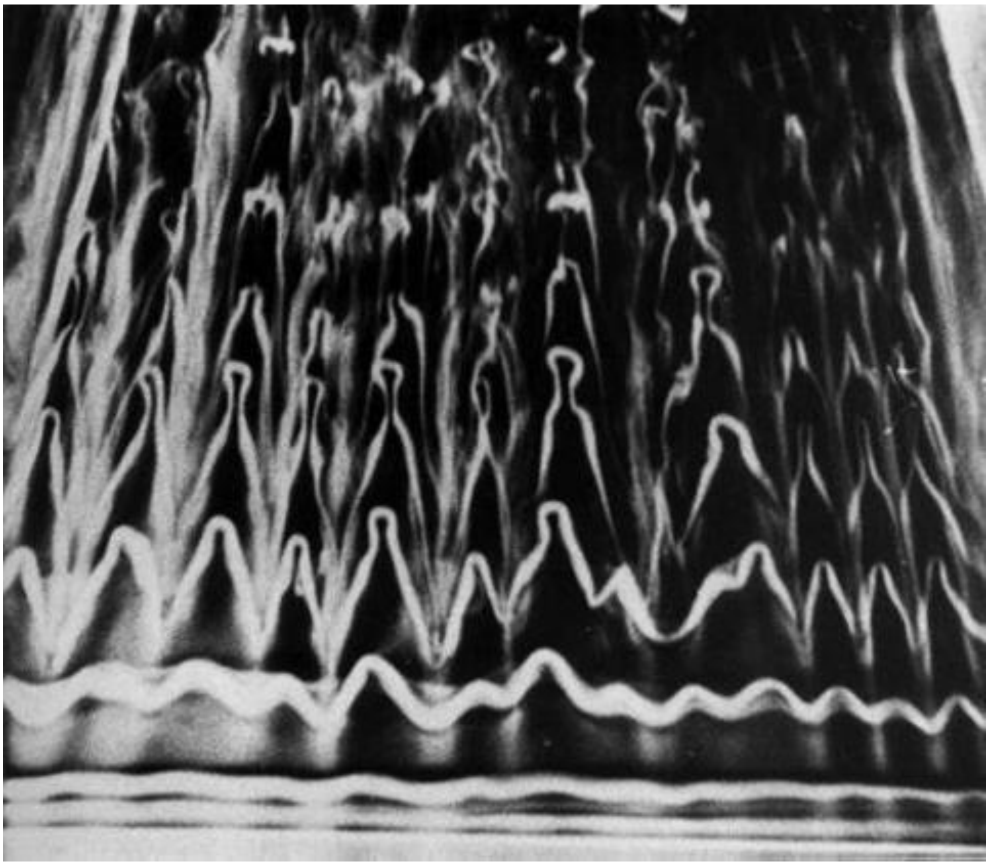

Figure 10: growth of TS waves into λ vortices. Smoke visualization of transition in a flat plate boundary layer. The flow is from the bottom of the image to the top.

eigenvalue. The absence of unstable modes at low Reynolds numbers indicates that for thin boundary layers the effect of viscosity is sufficient to damp all flow perturbations. As the boundary layer becomes thicker, viscosity can no longer act on long wave-length disturbances, which are allowed to grow.

3.8 Peculiar results of linear stability theory

Linear stability theory has some peculiar results: pipe flow is unconditionally stable to small disturbances. Similarly inviscid flow near a wall is unconditionally stable if the velocity profile has no inflection point.

These peculiar results indicate that in some instances transition to turbulence is not due to infinitesimally small perturbations growing exponentially, but to perturbations of finite size. This was certainly the case in Reynolds' experiment, where perturbations of finite size were being introduced by small – but crucially finite – imperfections in the equipment.

3.9 Tollmein-Schlichting waves, λ vortices and large structures in turbulent boundary layers

Tollmien-Shclichting waves start as small two-dimensional disturbances in boundary layers. Their initial shape is highlighted by the white lines near the bottom of the smoke visualization in Figure 10. As the waves travel downstream they grow in amplitude and evolve into roughly a triangular pattern (second row of lines from the bottom). As the waves become of comparable extent to the thickness of the boundary layer, their tips are caught by the free stream and they are stretched into structures resembling the Greek letter λ, hence the name λ vortices. As the vortices are stretched further and they are caught into each other's flow field, they degenerate into turbulence.

3.10 The role of λ-vortices in turbulent boundary layers

Large vortices are ultimately responsible for sustaining turbulence and drawing energy from the mean flow. They are also responsible for creating the Reynolds stress that maintains the momentum balance of the boundary layer. Their action takes place through two main types of events: ejections and sweeps. In an ejection event, slow fluid is raised from the viscous sublayer far above the wall. In sweep events fast fluid is dragged down towards the wall. These events are those we described when we justified the formation of the Reynolds stress in a non-uniform flow (see Figure 11).

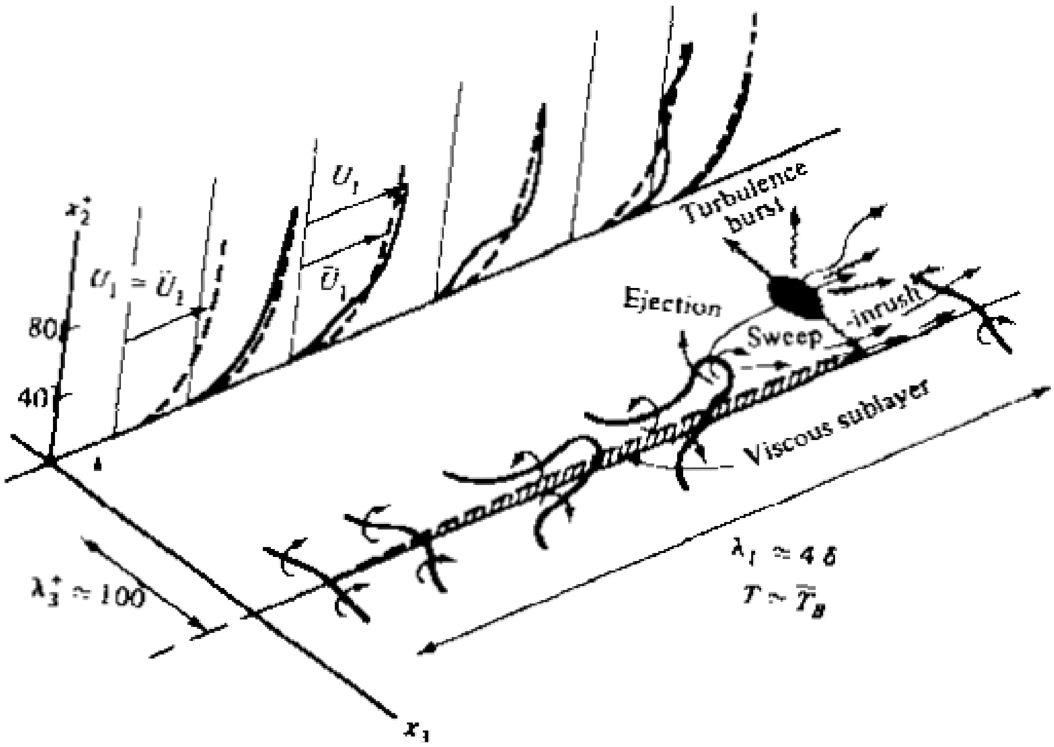

Figure 11: sweep and ejection events from Hinze, 1975

4 Other paths to turbulence

The growth of TS waves is not the only path to transition available to a boundary layer.

4.1 Curved walls

Boundary layers flowing past curved walls are subject to a flow instability due to the pressure gradient associated with the streamline curvature. This flow instability removes the span-wise uniformity of the flow and breakes the flow into an array of adjacent vortices, roughly adhering to the wall and with cores oriented in the streamwise direction, and organized in counter-rotating pairs. These vortices are called Götler vortices. Götler vortices can coil around one-another and promote transition to turbulence even when TS waves cannot grow.

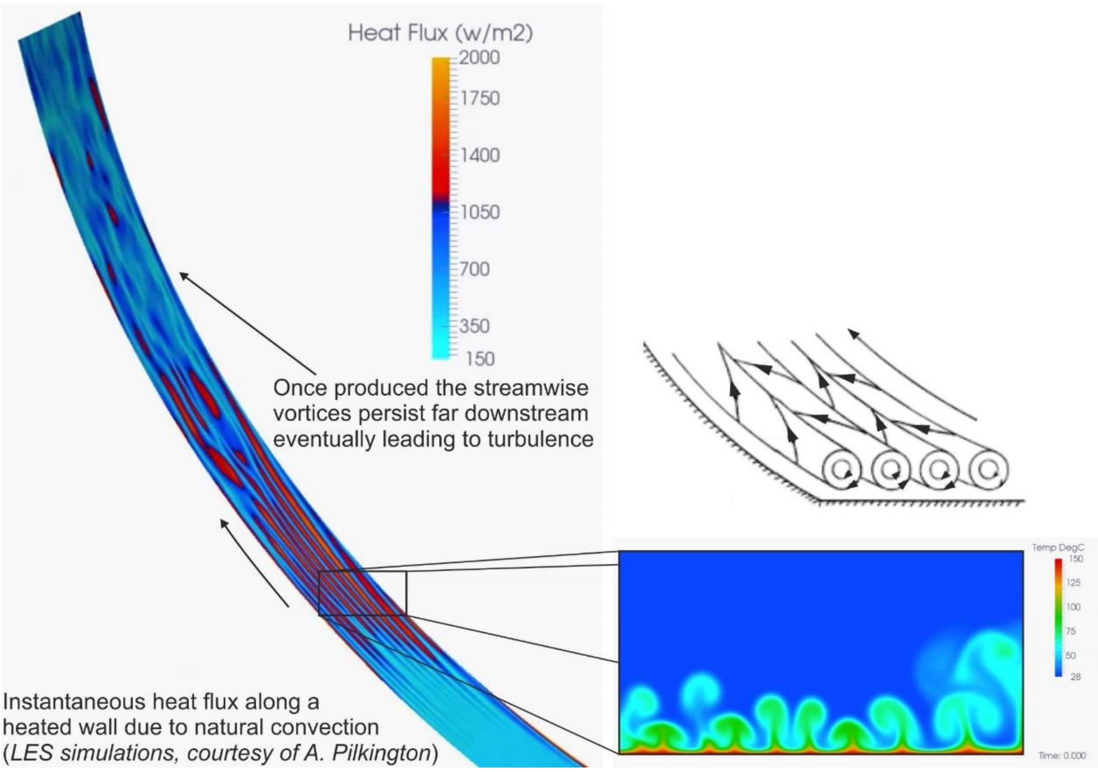

Figure 12: Götler vortices on a curved surface.

4.2 Bypass transition

Bypass transition is encountered at low-Reynolds numbers when the free-stream flow carries high turbulence intensity levels. Such a situation is encountered on turbine blades, where the Reynolds number is kept low by relatively low air densities but high turbulence intensity levels in the free-stream are due to the wakes of upstream blade-rows. Bypass transition appears in the form of turbulent spots which coalesce to form a fully developed turbulent flow.

4.3 Separation induced transition

Separation induced transition takes place where a laminar boundary layer is subjected to an adverse pressure gradient and separates when it is still laminar. The separated section of the boundary layer undergoes transition via the mechanism we have studied for the stability of a shear layer and is facilitated by the presence of an inflection point in the mean velocity profile. The weaker stability condition for the separated flow means It can undergo transition when the attached part of the boundary layer is still stable. As a result the flow is turbulent when reattachment takes place. Separation induced transition is often observed on highly-loaded turbine blades, where a laminar separation bubble exists on the pressure side because of the high camber of the blades. Compressor blades also experience separation induced transition when operating at high incidence.

5 Checklist

Reynolds' experiment is the first documented evidence of studies into transition. Reynolds' experiment showed the existence of a Reynolds number and demonstrated the sensitivity of transition to turbulence to external perturbations

Transition to turbulence happens because small disturbances grow into flow structures. The linear stability theory studies the evolution of infinitesimally small disturbances. This theory relies on the linearized Navier-Stokes equations and produces leads to the solution of eigenproblems where the eigenvalue is a growth rate and the eigenvector is a flow pattern. The flow pattern grows exponentially at the rate dictated by the eigenvalue.

The linearized Navier-Stokes equations for an inviscid flow lead to Rayleigh's equation. Rayleigh's equation shows that an inflection point is needed in the mean velocity profile of shear layer for instabilities to occur. When instabilities occur they have the shape of Rayleigh's cat's eye pattern.

The linearized Navier-Stokes equations for a boundary layer or a viscous shear layer lead to the Orr-Sommerfeld equation. The solutions of the Orr-Sommerfeld equations are the Tollmein-Schlichting waves. The TS waves start as small two-dimensional disturbances and grow into three-dimensional vortices shaped like the Greek letter λ, hence the name λ-vortices. λ vortices are ultimately responsible for sustaining turbulence and forming the Reynolds stress.

Other routes to transition are available to laminar boundary layers: Gotler vortices (on curved surfaces), bypass transition (in a turbulent far-field) or separation induced (in presence of adverse pressure gradients).