



The CENTRE for EDUCATION  
in MATHEMATICS and COMPUTING  
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## ***2021 Fryer Contest***

**April 2021**  
(in North America and South America)

**April 2021**  
(outside of North America and South America)

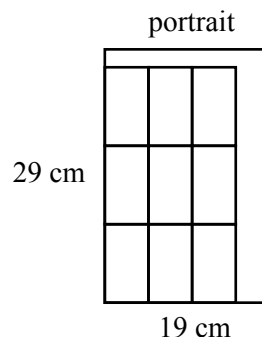
*Solutions*

1. (a) Each business card has dimensions  $5\text{ cm} \times 9\text{ cm}$  and thus has area  $5\text{ cm} \times 9\text{ cm} = 45\text{ cm}^2$ .
- (b) Each single page with dimensions  $20\text{ cm} \times 27\text{ cm}$  has area  $20\text{ cm} \times 27\text{ cm} = 540\text{ cm}^2$ . Each business card has area  $45\text{ cm}^2$ .

The entire page is used with no waste and so the number of business cards that can be printed without overlap is  $\frac{540\text{ cm}^2}{45\text{ cm}^2} = 12$ .

- (c) We begin by first considering the portrait page layout.

The width of each card is  $5\text{ cm}$  and the width of the page is  $19\text{ cm}$ . Since 3 adjacent cards give a combined width of  $15\text{ cm}$  (less than  $19\text{ cm}$ ) and 4 adjacent cards give a combined width of  $20\text{ cm}$  (greater than  $19\text{ cm}$ ), then the greatest number of adjacent cards that can be printed horizontally across the page in this layout is 3.

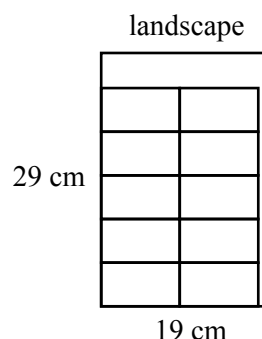


The height of each card is  $9\text{ cm}$  and the height of the page is  $29\text{ cm}$ . Since 3 adjacent cards give a combined height of  $27\text{ cm}$  (less than  $29\text{ cm}$ ) and 4 adjacent cards give a combined height of  $36\text{ cm}$  (greater than  $29\text{ cm}$ ), then the greatest number of adjacent cards that can be printed vertically on the page in this layout is 3.

Thus the portrait page layout allows a maximum of  $3 \times 3 = 9$  business cards to be printed on a single page, as shown in the diagram above.

Next, we consider the landscape page layout.

The width of each card is  $9\text{ cm}$  and the width of the page is  $19\text{ cm}$ . Since 2 adjacent cards give a combined width of  $18\text{ cm}$  (less than  $19\text{ cm}$ ) and 3 adjacent cards give a combined width of  $27\text{ cm}$  (greater than  $19\text{ cm}$ ), then the greatest number of adjacent cards that can be printed horizontally across the page in this layout is 2.



The height of each card is  $5\text{ cm}$  and the height of the page is  $29\text{ cm}$ . Since 5 adjacent cards give a combined height of  $25\text{ cm}$  (less than  $29\text{ cm}$ ) and 6 adjacent cards give a combined height of  $30\text{ cm}$  (greater than  $29\text{ cm}$ ), then the greatest number of adjacent cards that can be printed vertically on the page in this layout is 5.

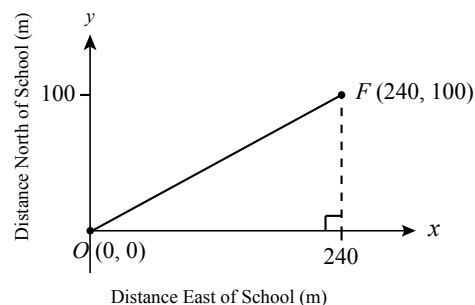
Thus the landscape page layout allows a maximum of  $2 \times 5 = 10$  business cards to be printed on a single page, as shown.

Therefore, the landscape page layout allows the greatest number of business cards to be printed on a single page.

2. Throughout the solution to this problem, all coordinates represent lengths in metres.

- (a) Let the point  $A$  be positioned at  $(240, 0)$ . Then  $OA$  represents a horizontal distance of 240 m,  $AF$  represents a vertical distance of 100 m, and  $\triangle OAF$  is right-angled at  $A$ , as shown. By the Pythagorean Theorem,  $FO^2 = OA^2 + AF^2$  or  $FO^2 = (240 \text{ m})^2 + (100 \text{ m})^2 = 67\,600 \text{ m}^2$  and so  $FO = \sqrt{67\,600 \text{ m}^2} = 260 \text{ m}$ .

Therefore, the distance along the straight path from the school to Franklin's home is 260 m.

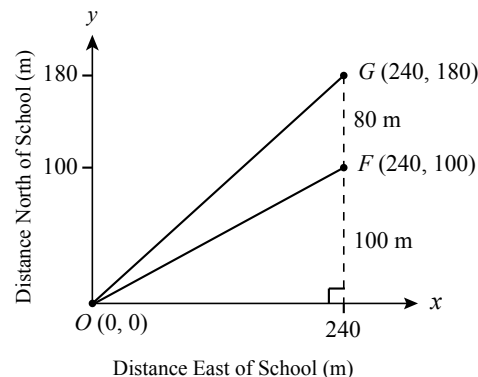


- (b) On Monday, Franklin walks from school straight home (a distance of 260 m) at a constant speed of 80 m/min. Since time equals distance divided by speed, it takes Franklin  $\frac{260 \text{ m}}{80 \text{ m/min}} = \frac{13}{4}$  (or 3.25) minutes to walk home on Monday.

- (c) Points  $F$  and  $G$  have the same  $x$ -coordinate, and so the distance between Franklin's home and Giizhig's home equals the difference in their  $y$ -coordinates, or 80 m, as shown.

Franklin and Giizhig meet halfway between their two homes, which is a distance of 40 m from each home.

From part (b), it takes Franklin  $\frac{13}{4}$  (or 3.25) minutes to walk straight home from school when walking at a speed of 80 m/min. It takes Franklin an additional  $\frac{40 \text{ m}}{80 \text{ m/min}} = \frac{1}{2}$  (or 0.5) minutes to walk from his home to the halfway point between their two homes.



Thus, it takes Franklin  $\frac{13}{4} + \frac{1}{2} = \frac{15}{4}$  (or 3.75) minutes to walk from the school to his home and then to the halfway point between their two homes.

Since Giizhig leaves the school at the same time as Franklin, and they meet halfway between the two homes, then it also takes Giizhig  $\frac{15}{4}$  minutes to walk from the school to her home and then to the halfway point between their homes.

As in part (a),  $\triangle OAG$  is right-angled at  $A$ , and so by the Pythagorean Theorem,  $GO^2 = OA^2 + AG^2$  or  $GO^2 = (240 \text{ m})^2 + (180 \text{ m})^2 = 90\,000 \text{ m}^2$  and so  $GO = \sqrt{90\,000 \text{ m}^2} = 300 \text{ m}$ .

Therefore, the distance along the straight path from the school to Giizhig's home to the halfway point between the two homes is  $300 \text{ m} + 40 \text{ m} = 340 \text{ m}$ .

Since Giizhig's speed,  $g$  m/min, equals distance divided by time, then

$$g = 340 \div \frac{15}{4} = 340 \times \frac{4}{15} = \frac{272}{3} \text{ or } 90\frac{2}{3}.$$

3. (a) When  $R_3$  is performed on the list 5, 2, 3, 1, 4, 6, the order of the first three numbers in the list is reversed, and so the new list is 3, 2, 5, 1, 4, 6.
- (b) The two Reverse Operations change the order of the first four numbers in the list, 1, 2, 3, 4, while the last two numbers, 5 and 6, do not change. Thus, it is reasonable to consider

that the first Reverse Operation may be  $R_4$ .

After performing  $R_4$  on the original list, the list becomes 4, 3, 2, 1, 5, 6.

Performing  $R_2$  on this list gives the final list 3, 4, 2, 1, 5, 6.

Thus, the two Reverse Operations are  $R_4$  followed by  $R_2$ .

(It is worth noting that there are no other ways to do this using exactly two Reverse Operations.)

- (c) (i) Each Reverse Operation, with the exception of  $R_6$ , does not change the number in the last position of the list. Further,  $R_6$  reverses the order of the entire list and so the first number in the list becomes the last number in the list.

That is, the number 3 moves to the last position only when  $R_6$  is performed on a list in which 3 is in the first position.

Is it possible to perform a Reverse Operation on the original list that moves 3 to the first position of the list?

If  $R_3$  is performed on the original list, 1, 2, 3, 4, 5, 6, the list becomes 3, 2, 1, 4, 5, 6.

If  $R_6$  is then performed on this list, the list becomes 6, 5, 4, 1, 2, 3, as required.

Thus, performing  $R_3$  followed by  $R_6$  results in 3 ending up in the last position of the list and so 2 Reverse Operations achieve the desired result.

- (ii) Since in (i) we found a way to achieve the desired result using 2 Reverse Operations, we need only to explain why performing 1 Reverse Operation is not possible.

As described in (i), the number 3 moves to the last position only when  $R_6$  is performed on a list in which 3 is in the first position. Since 3 is not in the first position of the original list, then performing 1 Reverse Operation can never achieve the desired result.

Together, (i) and (ii) show that the minimum number of Reverse Operations that need to be performed on the list 1, 2, 3, 4, 5, 6 so that 3 ends up in the last position is 2.

- (d) Performing the Reverse Operations  $R_5$ ,  $R_2$ ,  $R_6$ , in order, changes the original list from 1, 2, 3, 4, 5, 6 to 5, 4, 3, 2, 1, 6, and then to 4, 5, 3, 2, 1, 6, and finally to 6, 1, 2, 3, 5, 4.

Thus, 3 Reverse Operations can achieve a list of the desired form.

Can the desired result be achieved by performing fewer than 3 Reverse Operations?

As similarly demonstrated in (c), performing the Reverse Operations  $R_4$  and  $R_6$ , in order on the original list, will move the number 4 to the last position in the list.

This is the minimum number of Reverse Operations needed to move 4 to the last position in the list and further, these are the only 2 Reverse Operations that move 4 to the last position in the list. (Can you see why?)

However, performing  $R_4$  and  $R_6$ , in order, changes the original list from 1, 2, 3, 4, 5, 6 to 4, 3, 2, 1, 5, 6, and then to 6, 5, 1, 2, 3, 4. The second last number in this list is not 5, and so it is not possible to achieve the desired result in 2 Reverse Operations.

Since it is clearly not possible to achieve the desired result in 1 Reverse Operation, then the minimum number of Reverse Operations is 3.

The 3 Reverse Operations  $R_6$ ,  $R_3$ ,  $R_6$  also achieves a list of the desired form. Performing  $R_6$ ,  $R_3$ ,  $R_6$ , in order, changes the original list from 1, 2, 3, 4, 5, 6 to 6, 5, 4, 3, 2, 1 to 4, 5, 6, 3, 2, 1, and finally to 1, 2, 3, 6, 5, 4.

Therefore, starting from r1c9, there is thus 1 path to each of r6c4 (5  $\nwarrow$  steps), r11c9 (5  $\uparrow$  steps), and r6c14 (5  $\nearrow$  steps).

(ii) 4 of one step and 1 of another

There are 3 choices for the type of step to be taken 4 times (we call this choice  $x$ ) and, for each of these choices, there are 2 choices for the type of step to be taken 1 time (we call this choice  $y$ ).

There are thus  $3 \times 2 = 6$  choices for the endpoint of the path (each choice of  $x$  and  $y$  gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of  $x$  and  $y$ , there are 5 ways to arrange  $xxxxy$  since there are 5 possible positions for the  $y$ , after which the  $x$ 's fill the remaining spots.

Therefore, starting from r1c9, there are thus 5 paths to each of r7c5 ( $x = \nwarrow, y = \uparrow$ ), r6c6 ( $x = \nwarrow, y = \nearrow$ ), r7c13 ( $x = \nearrow, y = \uparrow$ ), r6c12 ( $x = \nearrow, y = \nwarrow$ ), r10c8 ( $x = \uparrow, y = \nwarrow$ ), r10c10 ( $x = \uparrow, y = \nearrow$ ).

(iii) 3 of one step and 2 of another

There are 3 choices for the type of step to be taken 3 times (we call this choice  $x$ ) and, for each of these choices, there are 2 choices for the type of step to be taken 2 times (we call this choice  $y$ ).

There are thus  $3 \times 2 = 6$  choices for the endpoint of the path (each choice of  $x$  and  $y$  gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of  $x$  and  $y$ , there are 10 ways to arrange  $xxxyy$  since there are 10 ways to choose the positions for the two  $y$ 's:

1st and 2nd, 1st and 3rd, 1st and 4th, 1st and 5th,  
2nd and 3rd, 2nd and 4th, 2nd and 5th,  
3rd and 4th, 3rd and 5th,  
4th and 5th

after which the  $x$ 's fill the remaining spots.

Therefore, starting from r1c9, there are thus 10 paths to each of r8c6 ( $x = \nwarrow, y = \uparrow$ ), r6c8 ( $x = \nwarrow, y = \nearrow$ ), r8c12 ( $x = \nearrow, y = \uparrow$ ), r6c10 ( $x = \nearrow, y = \nwarrow$ ), r9c7 ( $x = \uparrow, y = \nwarrow$ ), r9c11 ( $x = \uparrow, y = \nearrow$ ).

(iv) 3 of one step, 1 of a second type, and 1 of the third type

There are 3 choices for the type of step to be taken 3 times (we call this choice  $x$ ). After  $x$  is chosen, the types of steps to be taken 1 time each are fixed (we call these  $y$  and  $z$  in some order).

There are thus 3 choices for the endpoint of the path (each choice of  $x$  gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of  $x$ ,  $y$  and  $z$ , there are 20 ways to arrange  $xxxyz$  since there are 5 ways to choose the position for the  $y$ , after which there are 4 ways to choose the position for the  $z$ , after which the  $x$ 's fill the remaining spots.

Therefore, starting from r1c9, there are thus 20 paths to each of r7c7 ( $x = \nwarrow, y = \uparrow, z = \nearrow$ ), r7c11 ( $x = \nearrow, y = \nearrow, z = \uparrow$ ), r9c9 ( $x = \uparrow, y = \nwarrow, z = \nearrow$ ).

(v) 2 of one step, 2 of a second type, and 1 of the third type

There are 3 choices for the type of step to be taken 1 time (we call this choice  $z$ ). After  $z$  is chosen, the types of steps to be taken 2 times each are fixed (we call these  $x$  and  $y$ ).

There are thus 3 choices for the endpoint of the path (each choice of  $z$  gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of  $x$ ,  $y$  and  $z$ , there are 30 ways to arrange  $xyyz$  since there are 10 ways to choose the positions for the  $x$ 's (as there were 10 ways to choose 2 of 5 positions for the  $y$ 's above), after which there are 3 ways to choose the position for the  $z$  (there are three positions left), after which the  $y$ 's fill the remaining spots.

Therefore, starting from r1c9, there are thus 30 paths to each of r8c8 ( $x = \nwarrow$ ,  $y = \uparrow$ ,  $z = \nearrow$ ), r7c9 ( $x = \nearrow$ ,  $y = \nwarrow$ ,  $z = \uparrow$ ), r8c10 ( $x = \uparrow$ ,  $y = \nearrow$ ,  $z = \nwarrow$ ).

Therefore, in summary, there are 6 distinct hexagons (r7c7, r7c11, r9c9, r8c8, r7c9, r8c10) which are the endpoints of at least 20 paths of 5 steps; in other words,  $n = 6$ .



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## ***2014 Fryer Contest***

**Wednesday, April 16, 2014**  
(in North America and South America)

**Thursday, April 17, 2014**  
(outside of North America and South America)

*Solutions*





1. (a) There are 99 positive integers from 1 to 99.  
 The first 9 of these (the integers from 1 to 9) are each single-digit integers and thus contribute  $1 \times 9 = 9$  digits to the given integer.  
 The final 90 positive integers (the integers from 10 to 99) are each 2-digit integers and thus contribute  $2 \times 90 = 180$  digits to the given integer.  
 Therefore, the given integer has  $9 + 180 = 189$  digits in total.
  - (b) From part (a), the integer formed by writing the positive integers from 1 to 99 in order next to each other has 189 digits.  
 In addition to these first 99 positive integers, the given integer has the positive integers from 100 to 199 written in order next to one another.  
 Since there are  $199 - 100 + 1 = 100$  of these and each is a 3-digit integer, then they contribute  $3 \times 100 = 300$  more digits.  
 Therefore, the given integer has  $189 + 300 = 489$  digits in total.
  - (c) From part (b), the positive integers from 1 to 199 form a 489-digit integer.  
 To obtain 1155 digits, another  $1155 - 489 = 666$  digits are required.  
 Assuming that each of the integers from 200 to  $n$  is a 3-digit integer (we will confirm this), then another  $\frac{666}{3} = 222$  integers are needed to obtain 1155 digits in total.  
 (Since 222 integers beyond the integer 199 are needed to obtain 1155 digits in total, then each of these integers is indeed a 3-digit integer.)  
 Therefore,  $n$  is 222 integers past 199, or  $n = 199 + 222$  and so  $n = 421$ .  
 If the positive integers from 1 to 421 are written in order next to each other, the resulting integer has 1155 digits.
  - (d) From part (c), the positive integers from 1 to 421 form a 1155 digit integer.  
 To obtain 1358 digits, another  $1358 - 1155 = 203$  digits are required.  
 Writing the 68 integers that follow 421 will add another  $3 \times 68 = 204$  digits to our total (since each of the 68 integers that follow 421 is a 3-digit integer).  
 The 68<sup>th</sup> integer after 421 is  $421 + 68 = 489$ .  
 However, we only require another 203 digits, not 204. Since the 204<sup>th</sup> digit is the final digit of 489, which is 9, then the 203<sup>rd</sup> digit is 8.  
 Therefore, if the positive integers from 1 to 1000 are written in order next to each other, the 1358<sup>th</sup> digit of the resulting integer is 8.
2. (a) The sum of the 3 angles in any triangle is  $180^\circ$ .  
 Therefore,  $\angle BAC = 180^\circ - \angle ABC - \angle ACB = 180^\circ - 60^\circ - 50^\circ = 70^\circ$ .
  - (b) Since  $BD$  bisects  $\angle ABC$ , then  $\angle DBC = \frac{60^\circ}{2} = 30^\circ$ .  
 Since  $CD$  bisects  $\angle ACB$ , then  $\angle DCB = \frac{50^\circ}{2} = 25^\circ$ .  
 Since the sum of the 3 angles in any triangle is  $180^\circ$ ,  

$$\angle BDC = 180^\circ - \angle DBC - \angle DCB = 180^\circ - 30^\circ - 25^\circ = 125^\circ.$$
  - (c) In  $\triangle SQR$ , let the measure of  $\angle SQR$  be  $x^\circ$ . Since  $\triangle SQR$  is isosceles with  $QS = RS$ , then  $\angle SRQ = \angle SQR = x^\circ$ .  
 The sum of the 3 angles in  $\triangle SQR$  is  $180^\circ$ , so  $\angle SQR + \angle SRQ + \angle QSR = 180^\circ$ , or  $x^\circ + x^\circ + 140^\circ = 180^\circ$  or  $2x = 40$ , and so  $x = 20$ .  
 Since  $QS$  is the angle bisector of  $\angle PQR$ , then  $\angle PQS = \angle SQR = x^\circ$ .  
 Similarly, since  $RS$  is the angle bisector of  $\angle PRQ$ , then  $\angle PRS = \angle SRQ = x^\circ$ .  
 In  $\triangle PQR$ ,  $\angle PQR = \angle PQS + \angle SQR = (2x)^\circ$  and  $\angle PRQ = \angle PRS + \angle SRQ = (2x)^\circ$ .  
 The sum of the 3 angles in  $\triangle PQR$  is  $180^\circ$ , so  $\angle QPR = 180^\circ - \angle PQR - \angle PRQ$ , or  $\angle QPR = 180^\circ - (2x)^\circ - (2x)^\circ = 180^\circ - (4x)^\circ = 180^\circ - (4 \times 20)^\circ$  or  $\angle QPR = 100^\circ$ .

- (d) We begin by assuming that it is possible that  $\angle QSR = 80^\circ$ .

Proceeding as we did in part (c), if we let the measure of  $\angle SQR$  be  $y^\circ$ , then  $\angle SRQ = \angle SQR = y^\circ$ .

In  $\triangle SQR$ ,  $\angle SQR + \angle SRQ + \angle QSR = 180^\circ$ , or  $y^\circ + y^\circ + 80^\circ = 180^\circ$  or  $2y = 100$ , and so  $y = 50$ .

In  $\triangle PQR$ ,  $\angle PQR = \angle PQS + \angle SQR = (2y)^\circ$  and  $\angle PRQ = \angle PRS + \angle SRQ = (2y)^\circ$  ( $QS$  and  $RS$  are angle bisectors).

Thus,  $\angle QPR = 180^\circ - \angle PQR - \angle PRQ$ , or  $\angle QPR = 180^\circ - (2y)^\circ - (2y)^\circ = 180^\circ - (4y)^\circ$ , and so  $\angle QPR = 180^\circ - (4 \times 50)^\circ$  or  $\angle QPR = -20^\circ$ .

Since all angles must have positive measure, then the only assumption that we made, (that  $\angle QSR = 80^\circ$ ), must be false.

Therefore, it is not possible that  $\angle QSR = 80^\circ$ .

3. (a) Since the base  $BC$  of  $\triangle ABC$  is horizontal, its length is given by the positive difference between the  $x$ -coordinates of points  $B$  and  $C$ , or  $10 - 0 = 10$ .

The height of  $\triangle ABC$  is given by the length of the vertical line segment from point  $A$  to the base  $BC$ , which is the  $y$ -coordinate of point  $A$ , or 9 (since  $BC$  lies along the  $x$ -axis). Thus, the area of  $\triangle ABC$  before the first move in the game is made is  $\frac{1}{2} \times 10 \times 9 = 45$ .

- (b) At this point in the game, the area of  $\triangle ABC$  is  $\frac{1}{2} \times 10 \times 7$  or 35 (since the length of the base is still 10 and the height is 7).

The person who makes the area of  $\triangle ABC$  equal to 25 wins the game.

The base of  $\triangle ABC$ ,  $BC$ , remains fixed at length 10 throughout the game since only point  $A$  can move.

The area of  $\triangle ABC$  is 25 when the height of the triangle is 5, since  $\frac{1}{2} \times 10 \times 5 = 25$ .

The height of  $\triangle ABC$  is 5 when the  $y$ -coordinate of point  $A$  is 5.

That is, the only way for a player to win the game is to make the move that changes the  $y$ -coordinate of point  $A$  to 5.

It is now Dexter's turn to move. Dexter can move  $A$  down one unit to the point  $(2, 6)$  or he may move  $A$  left one unit to the point  $(1, 7)$ .

If Dexter moves  $A$  down one unit to the point  $(2, 6)$ , then on the next turn Ella can move  $A$  down one more unit to the point  $(2, 5)$  and win the game (since the  $y$ -coordinate would be 5).

Alternatively, if Dexter moves  $A$  one unit left to the point  $(1, 7)$ , then on the next turn Ella can move  $A$  one more unit left to the point  $(0, 7)$  (Ella would not choose to move  $A$  down one unit to  $(1, 6)$  since then Dexter could win on his next move).

On the next turn, Dexter is forced to move  $A$  down one unit to  $(0, 6)$  since moving  $A$  left would make the  $x$ -coordinate negative which is not permitted.

On the next move, Ella can move  $A$  down one more unit to  $(0, 5)$  and win the game.

We have shown that for all possible moves that Dexter can make, Ella can always win the game from the point  $(2, 7)$ .

- (c) (i) Beginning with point  $A$  at  $(6, 9)$ , the player who moves second, Geoff, has the winning strategy.

This winning strategy will be first described and then justified in parts (ii) and (iii) below.

- (ii) There are many different ways to describe Geoff's winning strategy.

One way to describe it is to say that on Geoff's turn, he will perform the same move (that is, move  $A$  either left or down) that his opponent Faisal just performed in the previous move.

- (iii) Why does Geoff win the game every time if he always matches the move that Faisal just performed?

Begin by assuming that Geoff can always match Faisal's move (an assumption that we will prove later on).

At the start of the game, the  $y$ -coordinate of point  $A$  is 9.

If Faisal moves down to 8, then Geoff will match and move down to 7.

If Faisal moves down to 6, then Geoff will match and move down to 5.

From part (b), we know that the player who makes the  $y$ -coordinate of point  $A$  equal to 5 wins the game.

Thus Geoff wins the game, assuming that Geoff can always move down after Faisal and is never forced to move down before Faisal.

Next, we must prove our assumption that Geoff can always match Faisal's move.

First, consider moves down.

A move down followed by a matching move down is always possible since the game will end when the  $y$ -coordinate of  $A$  is 5 (and as was already shown, it is Geoff who makes this final move).

That is, the  $y$ -coordinate of  $A$  will never be made negative and Geoff can always match any down move that Faisal makes.

Finally, we consider moves left.

At the start of the game, the  $x$ -coordinate of point  $A$  is 6, an even integer.

If at any time in the game Faisal moves  $A$  left one unit, then the  $x$ -coordinate becomes 5, an odd integer.

On his next turn, Geoff matches Faisal's move (left), and the  $x$ -coordinate becomes 4, an even integer again.

That is, each time Faisal moves  $A$  to the left, the  $x$ -coordinate becomes an odd integer, and since Geoff matches Faisal's move he returns the  $x$ -coordinate to an even integer.

Since the smallest  $x$ -coordinate that is permitted is an even integer, 0 (coordinates cannot be made negative), it is Geoff who will (if required) make the last possible move to the left.

On his next turn, Faisal is then forced to move  $A$  downward.

Therefore we have shown that Geoff will always be able to match Faisal's move, and that as a result, he will be the player who moves point  $A$  so that the  $y$ -coordinate becomes 5, thus winning the game.

4. (a) The eight subsets of the set  $\{1, 2, 3\}$  are:  $\{\}$ ,  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{1, 2\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$ , and  $\{1, 2, 3\}$ . Therefore the eight subset sums are: 0, 1, 2, 3, 3, 4, 5, and 6.

- (b) When constructing subsets, any element of the original set may either be included in the subset or not included in the subset.

That is, when constructing a subset from the set  $\{1, 2, 3, 4, 5\}$ , there are two choices for each of the 5 elements of the set; include it in the subset or do not.

Since there are two choices for each of the 5 elements, there are a total of  $2^5$  or 32 different subsets that can be created.

Consider the number of these 32 subsets which contain the number 1.

That is, when counting the number of these subsets, there is only one choice for the number one (include it in the subset), while there are still two choices for each of the numbers 2, 3, 4, and 5.

Thus, there are  $2^4$  or 16 subsets that contain the number 1 (and so there are  $32 - 16 = 16$  subsets which do not contain the number 1).

This is true in general for each of the elements of the original set; the number 2 will be included in 16 of the 32 subsets, the number 3 will be included in 16 subsets, and likewise for the numbers 4 and 5.

Since the element 1 is contained in exactly 16 subsets, then these 16 occurrences of 1 contribute exactly  $1 \times 16 = 16$  to the sum of the subsets that they appear in.

Similarly, since the element 2 is contained in exactly 16 subsets, then these 16 occurrences of 2 contribute exactly  $2 \times 16 = 32$  to the sum of the subsets that they appear in.

Since each of the 5 elements is contained in exactly 16 subsets, then the total contribution of all 5 elements in all possible subsets is

$$(1 \times 16) + (2 \times 16) + (3 \times 16) + (4 \times 16) + (5 \times 16) = 16(1 + 2 + 3 + 4 + 5) = 16(15) = 240.$$

Thus, the sum of all of the subset sums of  $\{1, 2, 3, 4, 5\}$  is 240.

- (c) Each subset of  $\{1, 3, 4, 5, 7, 8, 12, 16\}$  consists of zero or more of the odd integers  $\{1, 3, 5, 7\}$  and zero or more of the multiples of four  $\{4, 8, 12, 16\}$ .

If a subset is formed using only numbers that are divisible by 4, then the subset sum must also be divisible by 4 (since the sum of multiples of 4 is a multiple of 4).

So then every subset of  $\{4, 8, 12, 16\}$  must have a subset sum that is divisible by 4.

As in part (b), each of the 4 elements will appear in  $2^3 = 8$  of the  $2^4 = 16$  subsets. The sum of all subset sums using only elements from the set  $\{4, 8, 12, 16\}$  is then

$$(4 \times 8) + (8 \times 8) + (12 \times 8) + (16 \times 8) = 8(4 + 8 + 12 + 16) = 320.$$

Any subset includes 0, 1, 2, 3 or 4 of the elements of  $\{4, 8, 12, 16\}$ .

The sum of these elements is divisible by 4.

For the sum of all elements in the subset to be divisible by 4, the sum of the remaining (odd) elements must be divisible by 4.

Some combinations of the remaining (odd) elements  $\{1, 3, 5, 7\}$  produce sums which are divisible by 4.

Specifically, these are  $\{1, 3\}$ ,  $\{1, 7\}$ ,  $\{3, 5\}$ ,  $\{5, 7\}$ ,  $\{1, 3, 5, 7\}$ , and the empty set  $\{\}$ .

(No other subsets of  $\{1, 3, 5, 7\}$  have a sum divisible by 4 since each of the individual elements is odd, thus the sum of any 3 of them is odd, and the only pairs remaining are  $\{1, 5\}$  and  $\{3, 7\}$ , whose sums are 6 and 10).

Since each of these subsets has a sum which is divisible by 4, then combining them (that is, using all of their elements) with any of the 16 subsets of  $\{4, 8, 12, 16\}$  (including the empty set) will produce a new subset whose sum is also divisible by 4.

What remains is to determine the sum of all of these subset sums.

First consider combining both of the elements  $\{1, 3\}$  with each of the 16 subsets of  $\{4, 8, 12, 16\}$ . As we previously determined, the sum of all of the subsets of  $\{4, 8, 12, 16\}$  is 320.

Combining both of the elements  $\{1, 3\}$  with each of these subsets adds  $1 + 3 = 4$  to each of the subset sums.

Since there are 16 subsets, then the sum of all subset sums of this form is  $(4 \times 16) + 320$  or 384.

Similarly, combining both of the elements  $\{1, 7\}$  with each of the subsets of  $\{4, 8, 12, 16\}$  adds  $1 + 7 = 8$  to each of the subset sums.

Since there are 16 subsets, then the sum of all subset sums of this form is  $(8 \times 16) + 320$  or 448.

The sum of all subset sums which combine both  $\{3, 5\}$  with all subsets of  $\{4, 8, 12, 16\}$  is  $(8 \times 16) + 320$  or 448.

The sum of all subset sums which combine both  $\{5, 7\}$  with all subsets of  $\{4, 8, 12, 16\}$  is

$(12 \times 16) + 320$  or 512.

The sum of all subset sums which combine all of  $\{1, 3, 5, 7\}$  with all subsets of  $\{4, 8, 12, 16\}$  is  $(16 \times 16) + 320$  or 576.

Finally, combining the empty set with each of the subsets of  $\{4, 8, 12, 16\}$  adds nothing to each of the subset sums and so the sum of all subset sums is 320  $((0 \times 16) + 320 = 320)$ .

Since there are no other combinations of elements which produce subset sums that are divisible by 4, the sum of all required subset sums is  $384+448+448+512+576+320 = 2688$ .



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## ***2013 Fryer Contest***

**Thursday, April 18, 2013**  
(in North America and South America)

**Friday, April 19, 2013**  
(outside of North America and South America)

*Solutions*

1. (a) Ann's average score for these two games was  $\frac{103+117}{2} = \frac{220}{2}$  or 110.

(b) *Solution 1*

Since the average score of Bill's three games was 115, then the sum of his three scores was  $115 \times 3 = 345$ .

The sum of Bill's scores in his first two games was  $108 + 125 = 233$ .

Therefore, Bill's score in his third game was  $345 - 233 = 112$ .

*Solution 2*

Let  $x$  represent Bill's score in the third game.

Since the average of the three games was 115, then  $\frac{108 + 125 + x}{3} = 115$ .

Solving for  $x$ , we get  $108 + 125 + x = 115 \times 3$  or  $233 + x = 345$ , so  $x = 345 - 233 = 112$ .

Therefore, Bill's score in his third game was 112.

(c) *Solution 1*

Since the average score of Cathy's first three games was 113, then the sum of her first three scores was  $113 \times 3 = 339$ .

If the average score of Cathy's first five games was 120, then the sum of her five scores was  $120 \times 5 = 600$ .

Therefore, the sum of Cathy's scores in games four and five was  $600 - 339 = 261$ .

Since Cathy scored the same in her fourth and fifth games, then each of these scores would equal  $\frac{261}{2} = 130.5$ .

However, in bowling, each score is a whole number so this is not possible.

Therefore, it is not possible for Cathy to have an average score of 120 in these five games.

*Solution 2*

Since the average score of Cathy's first three games was 113, then the sum of her first three scores was  $113 \times 3 = 339$ .

Let  $y$  represent Cathy's score in game four and her score in game five (since they were the same score).

Since the average of the five games was 120, then  $\frac{339 + y + y}{5} = 120$ .

Solving for  $y$ , we get  $339 + 2y = 120 \times 5$  or  $339 + 2y = 600$ , so  $2y = 261$  and  $y = 130.5$ .

However, in bowling, each score is a whole number so this is not possible.

Therefore, it is not possible for Cathy to have an average score of 120 in these five games.

2. (a) The outside of the field consists of two straight sides each of length 100 m joined by two semi-circular arcs each of diameter 60 m.

Thus, the perimeter of the field is equal to the total length of the two straight sides, or 200 m, added to the circumference of a circle with diameter 60 m, or  $\pi(60)$  m.

Thus, the perimeter of the field is  $(200 + 60\pi)$  m.

(b) Amy may run along the perimeter of the field from  $C$  to  $D$  in two possible ways.

No matter which way Amy chooses, she will travel along one of the two straight sides, and one of the two semi-circular arcs.

That is, Amy will travel exactly one half of the total perimeter of the field, or  $(100 + 30\pi)$  m, in travelling from  $C$  to  $D$ .

Billais runs in a straight line distance from  $C$  to  $D$ .

Since each straight 100 m side is perpendicular to each 60 m diameter, then  $CD$  is the hypotenuse of a right triangle with the other two sides having length 100 m and 60 m.

By the Pythagorean Theorem,  $CD^2 = 100^2 + 60^2$  or  $CD = \sqrt{13600}$  m (since  $CD > 0$ ).  
Thus, Amy travels  $100 + 30\pi - \sqrt{13600} \approx 78$  m farther than Billais.

- (c) The outside of the track consists of two straight sides each of length 100 m joined by two semi-circular arcs each of diameter  $60 + x + x$  or  $(60 + 2x)$  m.

Thus, the perimeter of the outside of the track is equal to the total length of the two straight sides, or 200 m, added to the circumference of a circle with diameter  $(60 + 2x)$  m, or  $\pi(60 + 2x)$  m.

Thus, the perimeter of the outside of the track is  $(200 + (60 + 2x)\pi)$  m.

Since the perimeter of the outside of the track is 450 m, then  $200 + (60 + 2x)\pi = 450$ .

Solving this equation for  $x$ , we get  $(60 + 2x)\pi = 250$  or  $60 + 2x = \frac{250}{\pi}$  or  $2x = \frac{250}{\pi} - 60$  or  $x = \frac{1}{2}(\frac{250}{\pi} - 60)$ , and so  $x = \frac{125}{\pi} - 30 \approx 9.789$ .

Rounded to the nearest whole number, the value of  $x$  is 10.

3. (a) The four-digit number  $51A3$  is divisible by 3 when the sum of its digits is divisible by 3.

The sum of its digits is  $5 + 1 + A + 3 = 9 + A$ .

The values for the digit  $A$  such that  $9 + A$  is divisible by 3 are 0, 3, 6, or 9.

- (b) The four-digit number  $742B$  is divisible by 2 when it is even.

Thus,  $742B$  is divisible by 2 if  $B$  is 0, 2, 4, 6, or 8.

The four-digit number  $742B$  must also be divisible by 3.

The number  $742B$  is divisible by 3 when  $7 + 4 + 2 + B$  or  $13 + B$  is divisible by 3.

Thus,  $742B$  is divisible by 3 if  $B$  is 2, 5 or 8.

The only values of the digit  $B$  such that the four-digit number  $742B$  is divisible by 6 (divisible by both 2 and 3), are  $B = 2$  or  $B = 8$ .

- (c) An integer is divisible by 15 when it is divisible by both 5 and 3 (since the product of 5 and 3 is 15 and since 5 and 3 have no common factor).

The integer  $1234PQPQ$  is divisible by 5 when its units digit,  $Q$ , is 0 or 5.

The integer  $1234PQPQ$  is divisible by 3 when the sum of its digits,  $1 + 2 + 3 + 4 + P + Q + P + Q$  or  $10 + 2P + 2Q$ , is divisible by 3.

We proceed by checking the two cases,  $Q = 0$  and  $Q = 5$ .

When  $Q = 0$ , the sum of the digits  $10 + 2P + 2Q$  is  $10 + 2P$ .

Therefore, when  $Q = 0$  the values of  $P$  for which  $10 + 2P$  is divisible by 3 are 1, 4 and 7.

When  $Q = 5$ , the sum of the digits  $10 + 2P + 2Q$  is  $10 + 2P + 10$  or  $20 + 2P$ .

Therefore when  $Q = 5$ , the values of  $P$  for which  $20 + 2P$  is divisible by 3 are 2, 5 and 8.

The values of digits  $P$  and  $Q$  (written as  $(P, Q)$ ) for which  $1234PQPQ$  is divisible by 15 are  $(1, 0)$ ,  $(4, 0)$ ,  $(7, 0)$ ,  $(2, 5)$ ,  $(5, 5)$ , or  $(8, 5)$ .

- (d) An integer is divisible by 12 when it is divisible by both 4 and 3 (since the product of 4 and 3 is 12 and since 4 and 3 have no common factor).

In the product  $2CC \times 3D5$ ,  $3D5$  is odd and thus cannot be divisible by 4.

Therefore,  $2CC$  must be divisible by 4 and so must also be divisible by 2.

That is,  $C$  must be an even number. We proceed by checking  $C = 0, 2, 4, 6, 8$ .

If  $C = 0$ , 200 is divisible by 4 and so  $C = 0$  is a possibility.

If  $C = 2$ , 222 is not divisible by 4 and so  $C = 2$  is not a possibility.

If  $C = 4$ , 244 is divisible by 4 and so  $C = 4$  is a possibility.

If  $C = 6$ , 266 is not divisible by 4 and so  $C = 6$  is not a possibility.

Finally, if  $C = 8$ , 288 is divisible by 4 and so  $C = 8$  is a possibility.

We now proceed by using these three cases,  $C = 0$ ,  $C = 4$  and  $C = 8$ , to determine possible values for  $D$ .

When  $C = 0$ , the product  $2CC \times 3D5$  becomes  $200 \times 3D5$ .



The number 200 is not divisible by 3 since the sum of its digits, 2, is not divisible by 3. Therefore,  $3D5$  must be divisible by 3 and so  $8 + D$  is divisible by 3.

The possible values for  $D$  such that  $8 + D$  is divisible by 3 are 1, 4 and 7.

In this case, there are 3 possible pairs of digits  $C$  and  $D$  for which the product  $2CC \times 3D5$  is divisible by 12.

When  $C = 4$ , the product  $2CC \times 3D5$  becomes  $244 \times 3D5$ .

The number 244 is not divisible by 3 since the sum of its digits, 10, is not divisible by 3. Therefore,  $3D5$  must be divisible by 3 and as was shown in the first case, there are 3 possible pairs of digits  $C$  and  $D$  for which the product  $2CC \times 3D5$  is divisible by 12.

When  $C = 8$ , the product  $2CC \times 3D5$  becomes  $288 \times 3D5$ .

The number 288 is divisible by 3 since the sum of its digits, 18, is divisible by 3.

Therefore, the product  $288 \times 3D5$  is divisible by 3 independent of the value of  $D$ .

That is,  $D$  can be any digit from 0 to 9.

In this case, there are 10 possible pairs of digits  $C$  and  $D$  for which the product  $2CC \times 3D5$  is divisible by 12.

The total number of pairs of digits  $C$  and  $D$  such that  $2CC \times 3D5$  is divisible by 12 is  $3 + 3 + 10 = 16$ .

4. (a) Since the dot starts at  $(0, 0)$  and ends at  $(1, 0)$ , it moves a total of 1 unit right and 0 units up or down.

Since the dot moves 0 units up or down, then the number of  $\uparrow$  moves equals the number of  $\downarrow$  moves.

Since the total number of moves is at most 4, then the moves include 0  $\uparrow$  and 0  $\downarrow$  (0 moves in total), or 1  $\uparrow$  and 1  $\downarrow$  (2 moves in total), or 2  $\uparrow$  and 2  $\downarrow$  (4 moves in total).

Since the dot moves 1 unit right, then the number of  $\rightarrow$  moves is 1 more than the number of  $\leftarrow$  moves.

Since the total number of moves is at most 4, then there are 1  $\rightarrow$  and 0  $\leftarrow$  (1 move in total), or 2  $\rightarrow$  and 1  $\leftarrow$  (3 moves in total).

If there are 0 up/down moves, there can be 1 or 3 left/right moves.

If there are 2 up/down moves, there can be 1 left/right move.

If there are 4 up/down moves, there can be 0 left/right moves, which isn't possible.

With 0 up/down and 1 left/right moves, the sequence of moves is  $\rightarrow$ .

With 0 up/down and 3 left/right moves, the sequence of moves is  $\leftarrow\rightarrow\rightarrow$  or any arrangement of these moves.

With 2 up/down and 1 left/right moves, the sequence of moves is  $\uparrow\downarrow\rightarrow$  or any arrangement of these moves.

There are 3 possible arrangements of the moves  $\leftarrow\rightarrow\rightarrow$ . (This is because the  $\leftarrow$  can be the first, second or third moves, and the two remaining moves are both  $\rightarrow$ .)

There are 6 possible arrangements of the moves  $\uparrow\downarrow\rightarrow$ . (This is because there are three moves that can go first, and then two remaining moves that can go second, and then one move that can go third, and so  $3 \times 2 \times 1 = 6$  combinations of moves.)

Therefore, there are  $1 + 3 + 6 = 10$  possible ways for the dot to end at  $(1, 0)$ .

- (b) If the dot makes exactly 4 moves, these moves can consist of

- any combination of 4 right/left moves, or
- any combination of 3 right/left moves and any 1 up/down move, or
- any combination of 2 right/left moves and any combination of 2 up/down moves, or

- any 1 right/left move and any combination of 3 up/down moves, or
- any combination of 4 up/down moves.

Given a specific set of moves, any arrangement of these moves will move the dot to the same point, so we can think, for example, of all of the right/left moves occurring first followed by the up/down moves.

Four right/left moves can be 4 right (ending at  $x = 4$ ), 3 right and 1 left (ending at  $x = 3 - 1 = 2$ ), 2 right and 2 left (ending at  $x = 0$ ), 1 right and 3 left (ending at  $x = 1 - 3 = -2$ ), or 4 left (ending at  $x = -4$ ).

Three right/left moves can be 3 right, 2 right and 1 left, 1 right and 2 left, or 3 left, ending at  $x = 3, 1, -1, -3$ , respectively.

Two right/left moves can be 2 right, 1 right and 1 left, or 2 left, ending at  $x = 2, 0, -2$ , respectively.

One right/left move can be 1 right or 1 left, ending at  $x = 1, -1$ , respectively.

No moves right or left will end at  $x = 0$ .

Similarly, four up/down moves can end at  $y = 4, 2, 0, -2, -4$ , three up/down moves can end at  $y = 3, 1, -1, -3$ , two up/down moves can end at  $y = 2, 0, -2$ , one up/down move can end at  $y = 1, -1$ , and no moves up or down will end at  $y = 0$ . (To see this, we can repeat the previous argument, replacing “right/left” with “up/down” and changing  $x$ -coordinates to  $y$ -coordinates.)

Given the total number of moves in each of the horizontal and vertical direction, any combination of the resulting  $x$ - and  $y$ -coordinates is possible, since the horizontal and vertical moves do not affect each other.

Therefore, if the dot makes exactly 4 moves, we have

- any combination of 4 right/left moves and no moves up or down ( $x = 4, 2, 0, -2, -4$ , and  $y = 0$ ), ending at the points  $(4, 0), (2, 0), (0, 0), (-2, 0), (-4, 0)$ , or
- any combination of 3 right/left moves and any 1 up/down move ( $x = 3, 1, -1, -3$ , and  $y = 1, -1$ ), ending at the points  $(3, 1), (3, -1), (1, 1), (1, -1), (-1, 1), (-1, -1), (-3, 1), (-3, -1)$ , or
- any combination of 2 right/left moves and any combination of 2 up/down moves ( $x = 2, 0, -2$ , and  $y = 2, 0, -2$ ), ending at the points  $(2, 2), (2, 0), (2, -2), (0, 2), (0, 0), (0, -2), (-2, 2), (-2, 0), (-2, -2)$ , or
- any 1 right/left move and any combination of 3 up/down moves ( $x = 1, -1$ , and  $y = 3, 1, -1, -3$ ), ending at the points  $(1, 3), (1, 1), (1, -1), (1, -3), (-1, 3), (-1, 1), (-1, -1), (-1, -3)$ , or
- any combination of 4 up/down moves and no moves right or left ( $x = 0$ , and  $y = 4, 2, 0, -2, -4$ ), ending at the points  $(0, 4), (0, 2), (0, 0), (0, -2), (0, -4)$ .

After careful observation of the list of ending points above, we see that a number of points can be reached in more than one way.

In counting the total number of distinct points we must take care to count the same point more than once.

In the first bullet above, there are 5 distinct points.

In the second bullet, there are 8 new points that have not yet been counted.

In the third bullet there are 9 points, however, we have already counted the 3 points  $(2, 0), (0, 0)$ , and  $(-2, 0)$ .

In the fourth bullet there are 8 points, however, we have already counted the 4 points  $(1, 1), (1, -1), (-1, 1)$ , and  $(-1, -1)$ .

In the final bullet there are 5 points, however, we have already counted the 3 points

$(0, 2)$ ,  $(0, 0)$ , and  $(0, -2)$ .

Therefore, the total number of possible points is  $5 + 8 + (9 - 3) + (8 - 4) + (5 - 3) = 25$ .

- (c) The dot can get to  $(-7, 12)$  in 19 moves: 7 left and 12 up.

We need at least 19 moves for the dot to get to  $(-7, 12)$ , since we need at least 7 moves to the left and at least 12 moves up (with possibly more moves that cancel each other out).

The dot can also get to  $(-7, 12)$  in 21 moves: 7 left, 12 up, 1 left, 1 right.

The dot can also get to  $(-7, 12)$  in 23 moves: 7 left, 12 up, 2 left, 2 right.

In a similar way, the dot can get to  $(-7, 12)$  in  $19 + 2m$  moves for any non-negative integer  $m$ : 7 left, 12 up,  $m$  left,  $m$  right.

As  $m$  ranges from 0 to 40, the expression  $19 + 2m$  takes every odd integer value from 19 to 99.

This is 41 integer values.

We have shown that if  $k < 19$ , then the dot cannot get to  $(-7, 12)$  in  $k$  moves, and if  $k \geq 19$  and  $k$  is odd, then the dot can get to  $(-7, 12)$  in  $k$  moves.

Finally, we show that if  $k \geq 19$  and  $k$  is even, then the dot cannot get to  $(-7, 12)$  in  $k$  moves. This will complete our solution, and give an answer of 41, as above.

Suppose that  $k \geq 19$  is even and that the dot can get to  $(-7, 12)$  in  $k$  moves.

These  $k$  moves include  $h$  right/left moves and  $v$  up/down moves, with  $k = h + v$ .

Since  $k$  is even, then  $h$  and  $v$  are both even or both odd. (This is because Even + Even is Even, Odd + Odd is Even, and Odd + Even is Odd.)

Suppose that  $h$  is even. The  $h$  right/left moves include  $r$  right moves and  $l$  left moves, which gives  $h = r + l$  and  $r - l = -7$ , since the dot ends with  $x$ -coordinate  $-7$ .

Since  $h$  is even and  $h = r + l$ , then  $r$  and  $l$  are both even or both odd.

But if  $r$  and  $l$  are both even or both odd, then  $r - l$  must be even.

This disagrees with the fact that  $r - l = -7$ .

Therefore,  $h$  cannot be even, and so  $h$  is odd.

Since  $h$  is odd, then  $v$  is odd.

But the  $v$  up/down moves include  $u$  up moves and  $d$  down moves, which gives  $v = u + d$  and  $u - d = 12$ .

Since  $v = u + d$  is odd, then  $u$  and  $d$  are odd and even or even and odd, which means that  $u - d$  is odd, and so cannot be equal to 12.

Therefore, none of the possibilities work.

This means that we cannot get to  $(-7, 12)$  in an even number of moves.

Thus, there are 41 positive integers  $k$  with  $k \leq 100$  for which the dot can reach  $(-7, 12)$  in  $k$  moves.

- (d) Consider any point  $P$  at which the dot can end in 47 moves. The dot can also end at this point in 49 moves by adding, for example, a cancelling pair such as  $\rightarrow\leftarrow$  onto the end of the sequence of 47 moves ending at  $P$ .

Therefore, each of the 2304 points at which the dot can end in 47 moves is a point at which the dot can end in 49 moves.

Any other point  $Q$  at which the dot can end in 49 moves cannot be arrived at in 47 moves. Consider any sequence of moves that takes the dot to  $Q$ . It cannot include both a right and a left move, or both an up and a down move, otherwise these moves could be “cancelled” and removed, giving a sequence of 47 moves ending at  $Q$ .

Thus, a sequence of moves ending at  $Q$  includes  $h$  horizontal moves and  $v$  vertical moves (with  $h + v = 49$ ), where the horizontal moves are either all right or all left moves, and the vertical moves are either all up or all down moves.

If  $h = 0$ , then  $v = 49$ , so the dot can end at  $(0, 49)$  (49 moves up) or at  $(0, -49)$  (49 moves

down).

Similarly, if  $v = 0$ , the dot can end at  $(49, 0)$  or  $(-49, 0)$ .

If  $h$  and  $v$  are both not zero, then there are both horizontal and vertical moves.

We consider the case where there are  $h$  right moves and  $v$  up moves, with  $h + v = 49$ .

Since  $h \geq 1$  and  $v \geq 1$ , then  $h$  can take any value from 1 to 48 giving values of  $v$  from 48 to 1, and points  $(1, 48), (2, 47), \dots, (47, 2), (48, 1)$ .

This is exactly 48 distinct points.

Similarly, in each of the cases of right moves and down moves, left moves and up moves, and left moves and down moves, we obtain 48 distinct points.

Therefore, the number of points  $Q$  at which the dot can end in 49, but not 47 moves, is  $2 + 2 + 4 \times 48 = 196$ .

This gives a total of  $2304 + 196 = 2500$  points at which the dot can end in exactly 49 moves.



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## ***2012 Fryer Contest***

**Thursday, April 12, 2012**  
(in North America and South America)

**Friday, April 13, 2012**  
(outside of North America and South America)

*Solutions*

1. (a) Candidate A received  $\frac{1008}{5600} \times 100\% = 0.18 \times 100\%$  or 18% of all votes.

(b) *Solution 1*

Since  $\frac{3}{5} \times 100\% = 0.60 \times 100\%$ , Candidate B received 60% of all votes.

Since Candidates C and D tied, they equally shared the remaining  $100\% - 60\% = 40\%$  of the votes.

Therefore, Candidate C received  $\frac{1}{2}$  of 40% of the votes, or 20% of all votes.

*Solution 2*

Since Candidate B received  $\frac{3}{5}$  of all votes, then Candidates C and D shared the remaining  $1 - \frac{3}{5} = \frac{2}{5}$  of all votes.

Candidates C and D tied, thus they shared equally the remaining  $\frac{2}{5}$  of the votes.

Therefore, Candidate C received  $\frac{1}{2}$  of  $\frac{2}{5}$  or  $\frac{1}{2} \times \frac{2}{5} = \frac{2}{10} = \frac{1}{5}$  of the votes.

Since  $\frac{1}{5} \times 100\% = 0.20 \times 100\%$ , Candidate C received 20% of all votes.

(c) *Solution 1*

At 10:00 p.m., 90% of 6000 votes or  $\frac{90}{100} \times 6000 = 5400$  votes had been counted.

Of those 5400 votes that had been counted, Candidate E received 53%.

Therefore at 10 p.m.,  $\frac{53}{100} \times 5400 = 2862$  votes had been counted for Candidate E.

Since there were only 2 candidates, the remaining  $5400 - 2862$  or 2538 votes must have been counted for Candidate F.

Thus, there were  $2862 - 2538$  or 324 more votes counted for Candidate E than for Candidate F.

*Solution 2*

At 10:00 p.m., 90% of 6000 votes or  $\frac{90}{100} \times 6000 = 5400$  votes had been counted.

Of those 5400 votes that had been counted, Candidate E received 53%.

Since there are only 2 candidates, then Candidate F must have received the remaining  $100\% - 53\%$  or 47%.

Thus, Candidate E received  $53\% - 47\%$  or 6% more votes than Candidate F.

Since there were a total of 5400 votes that had been counted at 10:00p.m., then Candidate E received 6% of 5400 or 324 more votes than Candidate F.

- (d) Candidate H received 40% of the votes and Candidate J received 35% of the votes.

Thus, the only other candidate, G, received the remaining  $100\% - 40\% - 35\% = 25\%$  of the votes.

Since Candidate G received 2000 votes representing 25% of all votes cast, then the total number of votes cast was  $2000 \times 4 = 8000$  (since  $25\% \times 4 = 100\%$ ).

Thus, Candidate H received 40% of 8000 votes or  $\frac{40}{100} \times 8000 = 3200$  votes.

2. (a) Factoring gives  $112 = 2 \times 56 = 2 \times 2 \times 28 = 2 \times 2 \times 2 \times 14 = 2 \times 2 \times 2 \times 2 \times 7$ .

So, the prime factorization of 112 is  $2 \times 2 \times 2 \times 2 \times 7$  or  $2^4 \times 7$ .

- (b) For every perfect square, each of its prime factors occurs an even number of times (see the *Note* at the end of part (d) for a brief explanation of this).

From part (a), the prime factorization of 112 is  $2^4 \times 7$ .

We are asked for the smallest value of the positive integer  $u$  so that  $112 \times u$  or  $2^4 \times 7 \times u$  is a perfect square.

The prime factor 2 already occurs an even number of times, (four times), in the factorization of 112.

Thus, no additional factors of 2 are needed to make  $2^4 \times 7 \times u$  a perfect square.

However, the prime factor 7 occurs only once.

Since all prime factors must occur an even number of times, then at least one additional

factor of 7 is needed for  $2^4 \times 7 \times u$  to be a perfect square.

Therefore, the smallest positive integer  $u$  that makes the product  $112 \times u$  a perfect square, is 7:

$$112 \times u = 2^4 \times 7 \times u = 2^4 \times 7 \times 7 = 2^4 \times 7^2 = (2^2 \times 7) \times (2^2 \times 7).$$

- (c) Since  $5632 = 512 \times 11$  and  $512 = 2^9$ , then the prime factorization of 5632 is  $2^9 \times 11$ .

Again, every perfect square has each of its prime factors occurring an even number of times. We are asked for the smallest value of the positive integer  $v$  so that  $5632 \times v$  or  $2^9 \times 11 \times v$  is a perfect square.

The prime factor 2 occurs an odd number of times, (nine times), in the factorization of 5632.

Thus, at least one additional factor of 2 is needed to make  $2^9 \times 11 \times v$  a perfect square.

The prime factor 11 occurs only once.

Thus, at least one additional factor of 11 is needed for  $2^9 \times 11 \times v$  to be a perfect square.

Therefore, the smallest positive integer  $v$  that makes the product  $5632 \times v$  a perfect square, is  $2 \times 11$  or 22:

$$5632 \times v = 2^9 \times 11 \times v = 2^9 \times 11 \times 2 \times 11 = 2^{10} \times 11^2 = (2^5 \times 11) \times (2^5 \times 11).$$

- (d) For every perfect cube, the number of times that each of its prime factors occurs is a multiple of 3 (see the *Note* below for a brief explanation of this).

From part (a), the prime factorization of 112 is  $2^4 \times 7$ .

We are asked for the smallest value of the positive integer  $w$  so that  $112 \times w$  or  $2^4 \times 7 \times w$  is a perfect cube.

The prime factor 2 occurs four times in the factorization of 112.

Thus, the smallest number of additional factors of 2 needed to make  $2^4 \times 7 \times w$  a perfect cube is two (since 6 is the smallest multiple of 3 that is greater than 4).

The prime factor 7 occurs only once.

Thus, the smallest number of additional factors of 7 needed to make  $2^4 \times 7 \times w$  a perfect cube is two (since 3 is the smallest multiple of 3 that is greater than 1).

Therefore, the smallest positive integer  $w$  that makes the product  $112 \times w$  a perfect cube, is  $2^2 \times 7^2$  or 196:

$$112 \times w = 2^4 \times 7 \times w = 2^4 \times 7 \times 2^2 \times 7^2 = 2^6 \times 7^3 = (2^2 \times 7) \times (2^2 \times 7) \times (2^2 \times 7).$$

*Note:* Every positive integer greater than 1 can be written as a unique product of prime numbers (this is known as the Fundamental Theorem of Arithmetic!).

Every perfect square,  $P$ , is the product of a positive integer,  $n$ , with itself.

That is,  $P = n \times n$ .

By the Fundamental Theorem of Arithmetic,  $n$  can be written as a product of prime numbers.

Since  $P = n \times n$ , the prime factors of  $P$  are matching pairs of prime factors of  $n$ .

Thus, the prime factors of every perfect square occur an even number of times.

This argument similarly extends to every perfect cube,  $C$ .

Since  $C = n \times n \times n$  for some positive integer  $n$ , then the prime factors of  $C$  occur in sets of three matching prime factors of  $n$ .

Thus for every perfect cube, the number of times that each of the prime factors occurs is a multiple of 3.

3. (a) The first row contains the integers 1 through 6.  
 Each successive row contains the next six integers, in order, that follow the largest integer in the previous row.  
 Thus, the largest integer in any row is six times the row number.  
 Therefore, the largest integer in row 30 is  $6 \times 30 = 180$ .
- (b) By a similar argument to part (a), it follows that the largest integer in row 2012 is  $6 \times 2012 = 12\,072$ .  
 We find the other numbers in the row by counting backwards.  
 Thus, the six integers in row 2012 are 12 072, 12 071, 12 070, 12 069, 12 068, 12 067.  
 The sum of the six integers in row 2012 is

$$12\,072 + 12\,071 + 12\,070 + 12\,069 + 12\,068 + 12\,067 = 72\,417.$$

- (c) Again from part (a), the largest integer in any row is six times the row number.  
 Thus to find the approximate row in which 5000 appears, we divide 5000 by 6.  
 Since  $\frac{5000}{6} = 833\frac{1}{3}$ , and  $6 \times 833 = 4998$ , then the largest integer in row 833 is 4998.  
 Therefore, row 834 contains the next six consecutive integers from 4999 to 5004.  
 (We can check this by recognizing that  $6 \times 834 = 5004$ .)  
 Thus, the integer 5000 appears in row 834.  
 Next, we recognize that all even numbered rows list the largest integer in the row beginning in column A through to the smallest integer in column F.  
 Since row 834 is an even numbered row, then the integers are listed in the order 5004, 5003, 5002, 5001, 5000, 4999, with 5004 beginning in column A.  
 Therefore, the integer 5000 appears in row 834, column E.
- (d) The largest integer in row  $r$  is  $6 \times r$  or  $6r$ .  
 Since each row contains six consecutive integers, counting backwards the remaining five integers in the row are,  $6r - 1, 6r - 2, 6r - 3, 6r - 4, 6r - 5$ .  
 Thus, the sum of the six integers in row  $r$  is

$$6r + (6r - 1) + (6r - 2) + (6r - 3) + (6r - 4) + (6r - 5) = 36r - 15.$$

Since we require the sum of the six integers in the row to be greater than 10 000, then  $36r - 15 > 10\,000$  or  $36r > 10\,015$  or  $r > \frac{10\,015}{36}$ , and so  $r > 278\frac{7}{36}$ .  
 But the row number  $r$  must be a whole number, so  $r \geq 279$ .  
 Since we also require the sum of the six integers in the row to be less than 20 000, then  $36r - 15 < 20\,000$  or  $36r < 20\,015$  or  $r < \frac{20\,015}{36}$ , and so  $r < 555\frac{35}{36}$ .  
 But the row number  $r$  must be a whole number, so  $r \leq 555$ .  
 Therefore, the rows in which the six integers have a sum greater than 10 000 and less than 20 000 are 279, 280, 281, ..., 555.  
 This gives  $555 - 279 + 1$  or 277 rows that satisfy the requirements.



4. (a) The point  $A$  lies on the sphere, vertically above the centre of the sphere,  $O$ .

Similarly, point  $B$  lies on the sphere, vertically below the centre of the sphere.

The top and bottom faces of the cylinder touch the sphere at points  $A$  and  $B$  respectively, as shown.

The segment  $AB$  passes through the centre of the sphere.

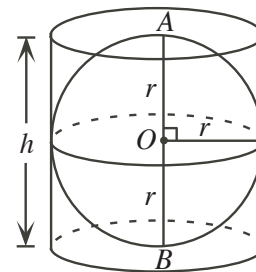
Since  $OA$  is a radius of the sphere, it has length  $r$ .

Similarly,  $OB$  has length  $r$ .

Thus, the length of segment  $AB$  is  $2r$ .

However, segment  $AB$  also represents the perpendicular distance between the top and bottom faces of the cylinder and thus has length equal to the height of the cylinder,  $h$ .

Therefore, the equation relating the height of the cylinder to the radius of the sphere is  $h = 2r$ .



- (b) The volume of the sphere is given by the formula  $\frac{4}{3}\pi r^3$ .

Since the volume of the sphere is  $288\pi$ , then  $\frac{4}{3}\pi r^3 = 288\pi$  or  $4\pi r^3 = 3 \times 288\pi$ , and so  $r^3 = \frac{3 \times 288\pi}{4\pi} = 216$ .

The volume of the cylinder is given by the formula  $\pi r^2 h$ .

From part (a),  $h = 2r$ .

Thus, the volume of the cylinder is  $\pi r^2 h = \pi r^2(2r) = 2\pi r^3$ .

Since  $r^3 = 216$ , then the volume of the cylinder is  $2\pi(216) = 432\pi$ .

- (c) The shape of the space that Darla is able to travel within is determined by the set of points that are exactly 1 km from the nearest point on the cube.

The surface of the cube is comprised of three types of points.

These are points on a face of the cube, points on an edge of the cube, and points at a vertex of the cube.

To determine the volume of space that Darla is able to occupy, we will consider each of these three types of surface points as separate cases.

#### *Case 1 - Points on a face of the cube.*

The question we must answer is, "What set of points is exactly 1 km away from the nearest point that is on a face of the cube?"

Consider that if Darla begins from any point on a face of the cube, the maximum distance that she can travel is 1 km.

To travel 1 km away from that point, but not be nearer to any other point on the cube, Darla must travel in a direction perpendicular to the face of the cube.

If Darla travels 1 km in a direction perpendicular to the face of a cube, beginning from each of the points on a face of the cube, then the shape of this space that she can occupy is another cube of side length 1 km.

This new cube extends directly outward from the original cube.

Since this can be repeated from each of the 6 faces of the original cube, then in this case Darla can occupy a volume of space equal to  $6 \times 1 \times 1 \times 1$  or  $6 \text{ km}^3$ , as shown in Figure 1.

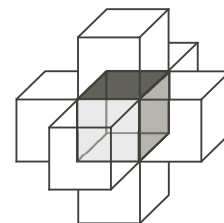


Figure 1

*Case 2 - Points on an edge of the cube.*

The question we must answer is, “What set of points is exactly 1 km away from the nearest point that is on an edge of the cube?” Consider point  $A$ , the midpoint of the edge on which it lies. Let points  $B$  and  $C$  be the midpoints of their respective edges also, as shown in Figure 2.

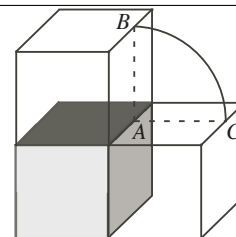


Figure 2

Darla can travel to both points  $B$  and  $C$  since point  $A$  is on the original cube, 1 km away from each of these points.

However, Darla can also travel from  $A$  to any point on the arc  $BC$ . This arc is one quarter of the circumference of the circle with centre  $A$ , radius 1 km, and passing through points  $B$  and  $C$  (since  $\angle BAC = 90^\circ$ ).

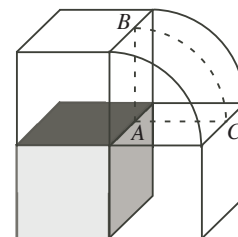


Figure 3

Darla can repeat this movement, beginning from any point on this edge. Thus, this shape of space that can be occupied is one quarter of a cylinder of radius 1 km and height 1 km, and has a volume of  $\frac{1}{4}\pi(1)^2(1) = \frac{1}{4}\pi \text{ km}^3$  (see Figure 3).

Since this can be repeated from any point on each of the 12 edges of the original cube, then in this case Darla can occupy a volume of space equal to  $12 \times \frac{1}{4}\pi$  or  $3\pi \text{ km}^3$ , as shown in Figure 4.

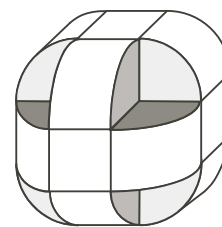


Figure 4

*Case 3 - Points at a vertex of the cube.*

The question we must answer is, “What set of points is exactly 1 km away from the nearest point that is at a vertex of the cube?” Consider point  $P$ , a vertex of the original cube. Let points  $Q$ ,  $R$  and  $S$  be vertices of external cubes, as shown in Figure 5.

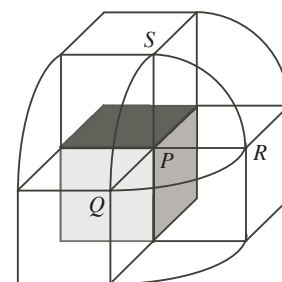


Figure 5

Darla can travel to points  $Q$ ,  $R$  and  $S$  since point  $P$  is on the original cube, 1 km away from each of these points.

From Case 2, we also know that Darla can travel anywhere along the arcs  $QR$ ,  $RS$  and  $SQ$ .

However, Darla can also travel up to 1 km outward from  $P$  to any point on the 3-dimensional surface contained within these 3 arcs.

Since  $\angle SPQ = \angle SPR = \angle QPR = 90^\circ$ , this surface is one eighth of the surface of the sphere with centre  $P$  and radius 1 km (see Figure 5).

Thus, the volume of the space that can be occupied is  $\frac{1}{8} \times \frac{4}{3}\pi(1)^3 = \frac{1}{6}\pi \text{ km}^3$ .

Since this can be repeated from each of the 8 vertices of the original cube, then in this case Darla can occupy a volume of space equal to  $8 \times \frac{1}{6}\pi$  or  $\frac{4}{3}\pi \text{ km}^3$ , as shown in Figure 6.

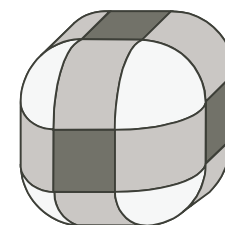


Figure 6

The total volume of space that Darla can occupy is the sum of the volume of space given by the 3 cases above.

That is, Darla can occupy a volume of space equal to  $6 + 3\pi + \frac{4}{3}\pi$  or  $(6 + \frac{13\pi}{3}) \text{ km}^3$ .



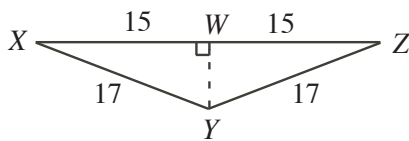
*The CENTRE for EDUCATION in  
MATHEMATICS and COMPUTING*

## *2011 Fryer Contest*

Wednesday, April 13, 2011

*Solutions*

1. (a) The 5<sup>th</sup> term of the sequence is 14 and the common difference is 3.  
Therefore, the 6<sup>th</sup> term of the sequence is  $14 + 3$  or 17, and the 7<sup>th</sup> term of the sequence is  $17 + 3$  or 20.
  - (b) Each term after the 1<sup>st</sup> is 3 more than the term to its left.  
To get from the 1<sup>st</sup> term to the 31<sup>st</sup> term, we move 30 times to the right.  
Therefore, the 31<sup>st</sup> term is  $30 \times 3$  more than the 1<sup>st</sup> term, or  $2 + 30(3) = 92$ .
  - (c) Using our work from part (b), we want to determine how many times 3 must be added to the first term 2 so that the result is 110.  
That is, how many 3s give  $110 - 2$  or 108.  
Since  $108 \div 3 = 36$ , then 36 3s must be added to 2 to give 110.  
Therefore if the last term is 110, then the number of terms in the sequence is  $36 + 1$  or 37.
  - (d) If 1321 appears in the sequence, then some integer number of 3s added to the first term 2 is equal to 1321.  
That is, we want to determine how many 3s give  $1321 - 2$  or 1319.  
Since 3 does not divide 1319 ( $\frac{1319}{3} = 439\frac{2}{3}$ ), there is no integer number of 3s that when added to the first term 2 is equal to 1321.  
Thus, 1321 does not appear in the sequence.
2. (a) (i) Since  $AB = AC$ , then  $\triangle ABC$  is isosceles.  
Therefore, altitude  $AD$  bisects the base  $BC$  so that  $BD = DC = \frac{14}{2} = 7$ .  
Since  $\angle ADB = 90^\circ$ ,  $\triangle ADB$  is right angled.  
By the Pythagorean Theorem,  $25^2 = AD^2 + 7^2$  or  $AD^2 = 25^2 - 7^2$  or  $AD^2 = 625 - 49 = 576$ , and so  $AD = \sqrt{576} = 24$ , since  $AD > 0$ .
  - (ii) The area of  $\triangle ABC$  is  $\frac{1}{2} \times BC \times AD$  or  $\frac{1}{2} \times 14 \times 24 = 168$ .
  - (b) (i) Through the process described,  $\triangle ADB$  is rotated  $90^\circ$  counter-clockwise about  $D$  to become  $\triangle PDQ$ .  
Similarly,  $\triangle ADC$  is rotated  $90^\circ$  clockwise about  $D$  to become  $\triangle RDQ$ .  
Through both rotations, the lengths of the sides of the original triangles remain unchanged.  
Thus,  $PD = AD = 24$  and  $RD = AD = 24$ .  
Since  $P$ ,  $D$  and  $R$  lie in a straight line, then base  $PR = PD + RD = 24 + 24 = 48$ .
  - (ii) *Solution 1*  
When  $\triangle ADC$  is rotated  $90^\circ$  clockwise about  $D$ , side  $DC$  becomes altitude  $DQ$  in  $\triangle PQR$ .  
Therefore,  $DQ = DC = 7$ .  
Thus, the area of  $\triangle PQR$  is  $\frac{1}{2} \times PR \times DQ$  or  $\frac{1}{2} \times 48 \times 7 = 168$ .  
*Solution 2*  
As determined in part (a), the area of  $\triangle ABC = 168$ .  
The area of  $\triangle ABC$  is equal to the area of  $\triangle ADB$  added to the area of  $\triangle ADC$ .  
Thus, the sum of the areas of these two smaller triangles is also 168.  
Through the rotations described, triangles  $ADB$  and  $ADC$  remain unchanged.  
Since  $\triangle PDQ$  is congruent to  $\triangle ADB$ , their areas are equal.  
Since  $\triangle RDQ$  is congruent to  $\triangle ADC$ , their areas are equal.  
Therefore, the area of  $\triangle PQR$  is equal to the sum of the areas of triangles  $PDQ$  and  $RDQ$ , which equals the sum of the areas of triangles  $ADB$  and  $ADC$  or 168.
  - (c) Since  $XY = YZ$ , then  $\triangle XYZ$  is isosceles.  
Draw altitude  $YW$  from  $Y$  to  $W$  on  $XZ$ .  
Altitude  $YW$  bisects the base  $XZ$  so that  $XW = WZ = \frac{30}{2} = 15$ , as shown.



Since  $\angle YWX = 90^\circ$ ,  $\triangle YWX$  is right angled.

By the Pythagorean Theorem,  $17^2 = YW^2 + 15^2$  or  $YW^2 = 17^2 - 15^2$  or  $YW^2 = 289 - 225 = 64$ , and so  $YW = \sqrt{64} = 8$ , since  $YW > 0$ .

By reversing the process described in part (b), we rotate  $\triangle XWY$  clockwise  $90^\circ$  about  $W$  and similarly rotate  $\triangle ZWY$  counter-clockwise  $90^\circ$  about  $W$  to obtain a new isosceles triangle with the same area.

The new triangle formed has two equal sides of length 17 (since  $XY$  and  $ZY$  form these sides) and a third side having length twice that of  $YW$  or  $2 \times 8 = 16$  (since the new base consists of two copies of  $YW$ ).

3. (a)

Two-digit number	Step 1	Step 2	Step 3
68	$6 \times 8 = 48$	$4 \times 8 = 32$	$3 \times 2 = 6$

Beginning with 68, the process stops after 3 steps.

- (b) For the process to stop at 8, the preceding number must have two digits whose product is 8.

The factor pairs of 8 are:  $1 \times 8$ ;  $2 \times 4$ .

Thus, the only two-digit numbers whose digits have a product of 8 are 18, 81, 24, and 42.

Next, consider the factor pairs of 18, 81, 24, and 42.

The factor pairs of 18 are:  $1 \times 18$ ;  $2 \times 9$ ;  $3 \times 6$ .

Since the factors 1 and 18 cannot be used to form a two-digit number, the only two-digit numbers whose digits have a product of 18 are 29, 92, 36, 63.

The factor pairs of 81 are:  $1 \times 81$ ;  $3 \times 27$ ;  $9 \times 9$ .

Thus, the only two-digit number whose digits have a product of 81 is 99.

The factor pairs of 24 are:  $1 \times 24$ ;  $2 \times 12$ ;  $3 \times 8$ ;  $4 \times 6$ .

Thus, the only two-digit numbers whose digits have a product of 24 are 38, 83, 46, 64.

The factor pairs of 42 are:  $1 \times 42$ ;  $3 \times 14$ ;  $6 \times 7$ .

Thus, the only two-digit numbers whose digits have a product of 42 are 67, 76.

A summary is shown in the table below.

Two-digit number	Preceding number
18	29, 92, 36, 63
81	99
24	38, 83, 46, 64
42	67, 76

Therefore, the only two digit numbers for which the process stops at 8 after 2 steps are 29, 92, 36, 63, 99, 38, 83, 46, 64, 67, 76.

- (c) For the process to stop at 4, the preceding number must have two digits whose product is 4.

The factor pairs of 4 are:  $1 \times 4$ ;  $2 \times 2$ .

The only two-digit numbers whose digits have a product of 4 are 14, 41, and 22.

We now find all two-digit numbers whose digits have a product of 14, 41, or 22.

Next, consider the factor pairs of 14, 41 and 22.

The factor pairs of 14 are:  $1 \times 14$ ;  $2 \times 7$ .

The only two-digit numbers whose digits have a product of 14 are 27 and 72.

The only factor pair of 41 is  $1 \times 41$ .

Since 1 and 41 cannot be used to form a two-digit number, there are no two-digit numbers whose digits have a product of 41.

The factor pairs of 22 are:  $1 \times 22$ ;  $2 \times 11$ .

Thus, there are no two-digit numbers whose digits have a product of 22.

A summary is shown in the table below.

Two-digit number	Preceding number
14	27, 72
41	there are none
22	there are none

We now find all two-digit numbers whose digits have a product of 27 or 72.

This is summarized in the table below.

Two-digit number	Preceding number
27	39, 93
72	89, 98

Since there are no two-digit numbers whose digits have a product of 39, 93, 89 or 98, we have completed our list.

The only two digit numbers for which the process stops at 4 are 14, 41, 22, 27, 72, 39, 93, 89, 98.

- (d) Although systematic trial and error will find us *the* two-digit number for which the process stops after 4 steps (there is only one), some work done earlier in this question may save us some time.

Notice in part (b) that there are 11 different numbers for which the process stops at 8 after 2 steps.

Of these 11 numbers only 3 can be continued on backwards to previous steps.

These are 36, 63 and 64.

Since there is only one step that precedes 63, that is 79 or 97, we have not found a 4-step number.

Since there is only one step that precedes 64, that is 88, we have not found a 4-step number.

However, 49 precedes 36 and 77 precedes 49.

That is, beginning with 77 the process stops after 4 steps as shown below.

Two-digit number	Step 1	Step 2	Step 3	Step 4
77	$7 \times 7 = 49$	$4 \times 9 = 36$	$3 \times 6 = 18$	$1 \times 8 = 8$

Therefore, 77 is a two-digit number for which the process stops after 4 steps.

Can you show that 77 is the only two-digit number for which the process stops after 4 steps?

Can you show that there is no two-digit number for which the process stops after more than 4 steps?

4. (a) Initially, Ian has 365 two-dollar coins.  
 That is, Ian begins with  $365 \times \$2$  or \$730.  
 After 365 days, he has spent  $365 \times \$1.72$  or \$627.80.  
 Therefore, Ian will have  $\$730 - \$627.80$  or \$102.20 remaining in the coin jar after 365 days.
- (b) Since Ian starts with 365 \$2.00 coins and his tea costs \$1.72 each day for 365 days, then he will use at most one of the \$2.00 coins each day. This means that he always has \$2.00 coins left.  
 Since Ian pays with the least amount possible each day, then he never pays more than \$2.00; he will pay with less than \$2.00 if he has accumulated at least \$1.72 in “loose change” (that is, in coins valued less than \$2.00).  
 Since Ian pays each time with \$2.00 or less, then he receives at most  $\$2.00 - \$1.72 = \$0.28$  in change each time.  
 Since Ian receives at most \$0.28 in change each time, then he receives at most one 25¢ coin each time.  
 We will show that the maximum number of 25¢ coins that he can have is 7.  
 Suppose Ian had 8 or more 25¢ coins in his jar at one time.  
 In this case, he must have had 7 or more 25¢ coins in his jar on the previous day since he never gets more than one 25¢ coin back in change.  
 But if Ian had 7 or more 25¢ coins in his jar, then he has at least \$1.75 in loose change and so would give the cashier at most \$1.75 and so would get at most \$0.03 in change.  
 In this case, he would not get a 25¢ coin as change and so would never get to 8 25¢ coins.  
 Therefore, Ian can never have more than 7 25¢ coins in his jar.  
 Lastly, we show that the maximum number of 25¢ coins is indeed 7 by showing that this actually happens.  
 Note that when Ian gives the cashier a \$2.00 coin and receives \$0.28 in change, the change will come as 1 25¢ coin and 3 1¢ coins.  
 Starting on the first day, Ian gives the cashier a \$2.00 coin and receives 28¢ in loose change until he has accumulated more than \$1.72 in loose change.  
 Note that after 6 days, Ian will have received  $6 \times \$0.28 = \$1.68$  in loose change (6 25¢ coins and  $3 \times 6 = 18$  1¢ coins).  
 Since Ian still has less than \$1.72, he must again use a \$2.00 coin on day 7, and so he again receives 28¢ in loose change on day 7, giving him 7 25¢ coins and 21 1¢ coins in total.  
 Therefore, there is a point at which Ian has 7 25¢ coins and Ian can never have 8 25¢ coins.  
 Thus, the maximum number of 25¢ coins that he can have is 7.
- (c) Part (b) describes Ian giving the cashier a \$2.00 coin and receiving \$0.28 in change, the change coming as 1 25¢ coin and 3 1¢ coins.  
 This situation happens frequently, so we define it to be a *typical day*.  
 Typical days occur whenever Ian has less than \$1.72 in loose change and he must offer the cashier a \$2.00 coin as payment for the tea.  
 For each day listed in the table below, the number of each type of coin in the jar at the end of the day is given.  
 To condense the table, some typical days have been omitted.  
 However, on each of these typical days we know that Ian paid with a \$2.00 coin and received 1 25¢ coin and 3 1¢ coins as change.

Coin Type	Day 7	Day 8	Day 14	Day 15	Day 21	Day 22	Day 28	Day 29	Day 35	Day 36	Day 42	Day 43	Day 49	Day 50
\$2.00	358	358	352	352	346	346	340	340	334	334	328	328	322	322
25¢	7	0	6	0	6	0	6	0	6	0	6	0	6	0
1¢	21	24	42	20	38	16	34	12	30	8	26	4	22	0

To clarify the table somewhat, the omitted days 9 to 13 (for example) are typical days. That is, Ian will offer a \$2.00 coin as payment and the number of \$2.00 coins in the jar will decrease by 1 on each of these days.

Ian will receive exactly 1 25¢ coin and 3 1¢ coins as change on each of these days.

The table shows that beginning at day 8, every 7<sup>th</sup> day is not a typical day.

That is, on days 8, 15, 22, 29, 36, 43 and 49, Ian has at least \$1.72 in loose change that he offers the cashier for payment instead of using a \$2.00 coin.

All other days listed in the table and omitted from the table are typical days.

After 50 days, Ian has used exactly  $365 - 322 = 43$  \$2.00 coins and has only \$2.00 coins remaining in the coin jar.

This is the first day that the coin jar returns to its initial state of containing only \$2.00 coins.

Since we are starting again on day 51 with only \$2.00 coins, the cycle of coins offered by Ian and received from the cashier will repeat every 50 days.

That is, after 250 days Ian will have used exactly  $5 \times 43 = 215$  \$2.00 coins and the coin jar will contain  $365 - 215 = 150$  \$2.00 coins only (there will be no 25¢ or 1¢ coins).

Thus, the number of 25¢ and 1¢ coins contained in the jar on the 277<sup>th</sup> day will be the same as those in the jar on the 27<sup>th</sup> day.

Using the table above, the number of 25¢ and 1¢ coins on the 28<sup>th</sup> day is 6 and 34, respectively.

Therefore, the number of 25¢ and 1¢ coins on the 27<sup>th</sup> day is 5 and 31, respectively, since the 28<sup>th</sup> day is a typical day.

We also note that there are 341 \$2.00 coins in the jar on the 27<sup>th</sup> day.

Thus, Ian used  $365 - 341 = 24$  \$2.00 coins in the first 27 days.

As a result of the repeating pattern of coins in the jar, 24 \$2.00 coins will be used from the 251<sup>st</sup> day to the end of the 277<sup>th</sup> day.

Since there are 150 \$2.00 coins in the jar after 250 days, then there are  $150 - 24 = 126$  \$2.00 coins in the jar after 277 days.

After 277 days, there will be 126 \$2.00 coins, 5 25¢ coins, and 31 1¢ coins in the jar.

We can check if our final answer is reasonable.

After 277 days, Ian has spent  $277 \times \$1.72 = \$476.44$ .

Therefore Ian should have  $\$730 - \$476.44 = \$253.56$  remaining in the coin jar.

From the answer given, the coin jar contains  $126 \times \$2.00 + 5 \times 25¢ + 31¢ = \$253.56$ .

We can be reasonably confident that we have answered this question correctly.

It is worth noting that Ian initially has \$2.00 coins or 200¢ coins at his disposal and the price of each tea is 172¢.

The lowest common multiple of 200 and 172 is 8600 (verify this for yourself).

Can you explain how we may have used this lowest common multiple to find the number of days it takes for the jar to return to its initial state of containing only \$2.00 coins?





**Canadian  
Mathematics  
Competition**

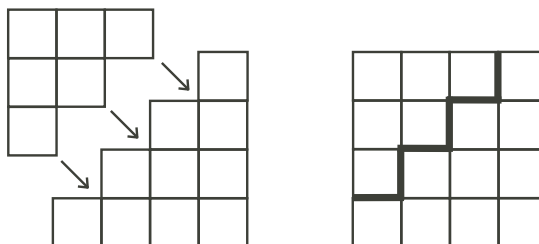
*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

***2010 Fryer Contest***

**Friday, April 9, 2010**

*Solutions*

1. (a) The piece on the right can be repositioned to rest on the left piece as shown. The resulting arrangement is a 4 by 4 square having 16 tiles.

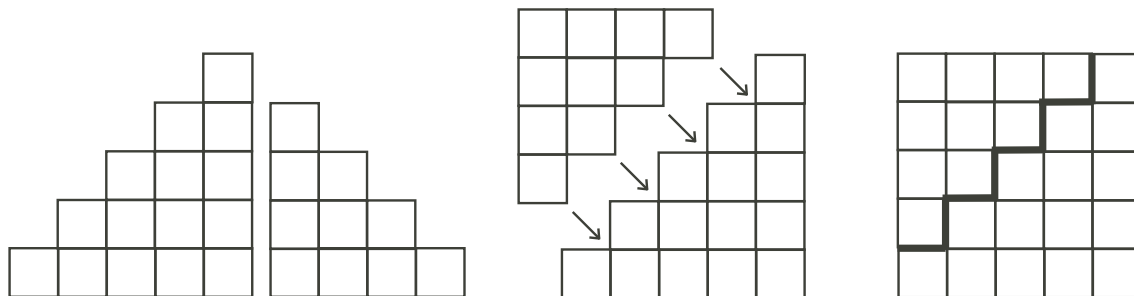


- (b) *Solution 1*

The top 4 rows of Figure 5 are identical to the 4 rows that form Figure 4. The bottom row of Figure 5 has two tiles more than the bottom row of Figure 4, or  $7 + 2 = 9$  tiles. Thus, Figure 5 has  $1 + 3 + 5 + 7 + 9 = 25$  tiles.

*Solution 2*

The bottom (5th) row of Figure 5 has two tiles more than the bottom row of Figure 4, or  $7 + 2 = 9$  tiles. Using the method of part (a), Figure 5 can be cut into two pieces and repositioned as shown. The resulting arrangement is a 5 by 5 square having 25 tiles.



- (c) We count the number of tiles in the bottom row of each of the first 5 figures and list the results in the following table:

Figure Number	Number of tiles in the bottom row
1	1
2	3
3	5
4	7
5	9

Since the number of tiles in the bottom row of any figure is two more than the number of tiles in the bottom row of the previous figure, we can continue the table.

Figure Number	Number of tiles in the bottom row
6	11
7	13
8	15
9	17
10	19

Therefore, Figure 10 has 19 tiles in its bottom row.

(d) *Solution 1*

The top 9 rows of Figure 11 are identical to the 9 rows that form Figure 9.

Since Figure 11 has 11 rows, the difference between the total number of tiles in Figure 11 and the total number of tiles in Figure 9 is equal to the sum of the number of tiles in the 10th and 11th rows of Figure 11.

The number of tiles in the 10th row of Figure 11 is equal to the number of tiles in the 10th (bottom) row of Figure 10, or 19.

From part (c), we also know that the bottom row of Figure 11 has 2 tiles more than the bottom row of Figure 10, or  $19 + 2 = 21$  tiles.

Therefore, the difference between the total number of tiles in Figure 11 and the total number of tiles in Figure 9 is  $19 + 21 = 40$  tiles.

*Solution 2*

Using the method of part (a), it can be shown that Figure 11 can be cut into 2 pieces and rearranged to form an 11 by 11 square having  $11 \times 11 = 121$  tiles.

Similarly, Figure 9 can be shown to have  $9 \times 9 = 81$  tiles.

Therefore, the difference between the total number of tiles in Figure 11 and the total number of tiles in Figure 9 is  $121 - 81 = 40$  tiles.

2. (a) The average of any set of integers is equal to the sum of the integers divided by the number of integers.

Thus, the average of the integers is  $\frac{71+72+73+74+75}{5} = \frac{365}{5} = 73$ .

- (b) (i) Simplifying,  $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10$ .

(ii) Since the sum of the 5 consecutive integers is  $5n + 10$ , the average of these integers is  $\frac{5n + 10}{5} = n + 2$ .

If  $n$  is an even integer, then  $n + 2$  is an even integer.

If  $n$  is an odd integer, then  $n + 2$  is an odd integer.

Thus for the average  $n + 2$  to be odd, the integer  $n$  must be odd.

- (c) Simplifying,  $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) + (n + 5) = 6n + 15$ .

Since the sum of the 6 consecutive integers is  $6n + 15$ , the average of these integers is  $\frac{6n + 15}{6} = n + \frac{15}{6} = n + \frac{5}{2}$ .

For every integer  $n$ ,  $n + \frac{5}{2}$  is never an integer.

Therefore, the average of six consecutive integers is never an integer.

3. (a) Travelling at a speed of 60 km/h for 9 hours, Train 1 travels a distance of  $60 \times 9 = 540$  km. If the distance from Amville to Batton is  $d$ , then  $\frac{2}{3}d = 540$  and  $d = \frac{540(3)}{2} = 810$  km. Therefore, the distance from Amville to Batton is 810 km.

- (b) The distance from Batton to Amville is 810 km.

Two-thirds of the distance from Batton to Amville is  $\frac{2}{3} \times 810 = 540$  km.

Train 2 travelled the 540 km in 6 hours.

Thus, Train 2 travels at a constant speed of  $\frac{540}{6} = 90$  km/h.

- (c) Let  $t$  hours be the time that it takes Train 1 to travel from Amville to Cuford.

Since Train 1 travels at 60 km/h, the distance from Amville to Cuford is  $60t$  km.

Train 2 leaves Batton  $3\frac{1}{2}$  hours after Train 1 leaves Amville.

Since the trains arrive at Cuford at the same time, Train 2 takes  $(t - 3\frac{1}{2})$  hours to travel from Batton to Cuford.

Since Train 2 travels at 90 km/h, the distance from Batton to Cuford is  $90(t - 3\frac{1}{2})$  km. The distance from Amville to Cuford added to the distance from Cuford to Batton is equal to the distance from Amville to Batton, 810 km.

Thus,  $60t + 90(t - 3\frac{1}{2}) = 810$  or  $60t + 90t - 315 = 810$  or  $150t = 1125$  and  $t = 7\frac{1}{2}$  hours. Therefore, Train 1 needed  $7\frac{1}{2}$  hours to travel from Amville to Cuford.

Since Train 1 arrived at Cuford at 9:00 p.m., Train 1 left Amville at 1:30 p.m.

4. (a) A palindrome less than 1000 must be either 1, 2 or 3 digits in length.

We consider each of these 3 possibilities as separate cases below.

Case 1: one-digit palindromes

All of the positive integers from 1 to 9 are palindromes since they are the same when read forwards or backwards. Thus, there are 9 one-digit palindromes.

Case 2: two-digit palindromes

We are looking for all of the palindromes from the positive integers 10 to 99.

To be a two-digit palindrome, the first digit must equal the second digit.

Thus, there are only 9 two-digit palindromes;  $\{11, 22, 33, 44, 55, 66, 77, 88, 99\}$ .

Case 3: three-digit palindromes

All three-digit palindromes are of the form  $aba$ , for integers  $a$  and  $b$ , where  $1 \leq a \leq 9$  and  $0 \leq b \leq 9$ .

That is, to create a three-digit palindrome the hundreds digit and the units digit must be equal to each other, but not equal to zero.

Since the hundreds digit  $a$  can be any of the positive integers from 1 to 9, there are 9 choices for the hundreds digit  $a$ .

Since the tens digit  $b$  can be any of the integers from 0 to 9, there are 10 choices for  $b$ .

The final digit (the units digit) must equal the hundreds digit, thus there is no choice for the units digit since it has already been chosen.

In total, there are  $9 \times 10$  choices for  $a$  and  $b$  and therefore, 90 three-digit palindromes.

There are 9 one-digit palindromes, 9 two-digit palindromes and 90 three-digit palindromes.

In total, there are  $9 + 9 + 90 = 108$  palindromes less than 1000.

- (b) All seven-digit palindromes are of the form  $abcdcba$ , for integers  $a, b, c$ , and  $d$ , where  $1 \leq a \leq 9$  and  $0 \leq b, c, d \leq 9$ .

Since the last 3 digits  $cba$  are determined by the first three digits  $abc$ , the number of seven-digit palindromes is dependent on the number of choices for the first 4 digits  $abcd$  only. There are 9 choices for the first digit  $a$ , 10 choices for the second digit  $b$ , 10 choices for the third digit  $c$ , and 10 choices for the fourth digit  $d$ .

In total, there are  $9 \times 10 \times 10 \times 10$  choices for  $abcd$  and therefore, 9000 seven-digit palindromes.

- (c) The seven-digit palindromes between 1000000 and 2000000 begin with a 1, and are of the form  $1bcdcb1$ .

As in part (b), there are 10 choices for each of  $b, c$  and  $d$ , and therefore  $10 \times 10 \times 10 = 1000$  palindromes between 1000000 and 2000000.

In the increasing list of seven-digit palindromes, these are the first 1000 palindromes.

Similarly, the next 1000 palindromes lie between 2000000 and 3000000, and the next 1000 lie between 3000000 and 4000000.

Thus, the 2125th palindrome must lie between 3000000 and 4000000 and must be of the form  $3bcdcb3$ .

The smallest palindromes of the form  $3bcdcb3$  are those of the form  $30cdc03$ .

Since there are 10 possibilities for each of  $c$  and  $d$ , then there are  $10 \times 10 = 100$  such

palindromes.

The largest of these (3099903) is the 2100th palindrome in the list.

Similarly, there are 100 palindromes of the form  $31cdc13$ .

The 2125th palindrome will thus be of this form.

The smallest palindromes of the form  $31cdc13$  are those of the form  $310d013$ .

Since there are 10 possibilities for  $d$ , then there are 10 such palindromes.

Similarly, there are 10 palindromes of the form  $311d113$ .

The largest of these (3119113) is the 2120th in the list.

Also, there are 10 palindromes of the form  $312d213$ .

The 2125th palindrome in the entire list is the 5th largest of these, or  $3124213$ .

- (d) All six-digit palindromes are of the form  $abccba$ , for integers  $a, b, c$ , where  $1 \leq a \leq 9$  and  $0 \leq b, c \leq 9$ .

The integer  $N$  formed by these digits  $abccba$  can be expressed by considering the place value of each of its six digits.

That is,  $N = 100000a + 10000b + 1000c + 100c + 10b + a = 100001a + 10010b + 1100c$ .

Recognize that the integers 100001, 10010 and 1100 have a common factor of 11, so then  $N = 11(9091a + 910b + 100c)$  or  $N = 11k$  where  $k = 9091a + 910b + 100c$ .

For  $N$  to be divisible by 91, the integer  $11k$  must be divisible by 91.

Since 11 is a prime number and not a divisor of 91, then  $N$  is divisible by 91 if

$k = 9091a + 910b + 100c$  is divisible by 91. We rewrite  $k$  as,

$$\begin{aligned} k &= 9091a + 910b + 100c \\ &= (9100a - 9a) + 910b + (91c + 9c) \\ &= (9100a + 910b + 91c) + (9c - 9a) \\ &= 91(100a + 10b + c) + 9(c - a) \end{aligned}$$

Since  $91(100a + 10b + c)$  is divisible by 91 for any choices of  $a, b, c$ , then  $k$  is divisible by 91 only if  $9(c - a)$  is divisible by 91.

Since 9 has no factors in common with 91,  $9(c - a)$  is only divisible by 91 if  $(c - a)$  is divisible by 91.

Recall that  $1 \leq a \leq 9$  and  $0 \leq c \leq 9$ , thus 91 divides  $(c - a)$  only if  $c - a = 0$  or  $c = a$ .

Therefore, the six-digit palindromes that are divisible by 91 are of the form  $abaaba$  where  $1 \leq a \leq 9$  and  $0 \leq b \leq 9$ .

Since there are 9 choices for  $a$  and 10 choices for  $b$ , there are a total of  $9 \times 10 = 90$  six-digit palindromes that are divisible by 91.



**Canadian  
Mathematics  
Competition**

*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

***2009 Fryer Contest***

**Wednesday, April 8, 2009**

*Solutions*

1. (a) The total cost to make 100 cups of lemonade is  
 $\$0.15 \times 100 + \$12.00 = \$15.00 + \$12.00 = \$27.00$ .
  - (b) The total money earned for selling 100 cups of lemonade is  $\$0.75 \times 100 = \$75.00$ .  
 The profit is the total money earned minus the total cost, or  $\$75.00 - \$27.00 = \$48.00$ .
  - (c) Let  $x$  represent the number of cups that Emily must sell to break even.  
 The total cost to make  $x$  cups of lemonade is  $\$0.15 \times x + \$12.00 = \$0.15x + \$12.00$ .  
 The total money earned by selling  $x$  cups is  $\$0.75 \times x$  or  $\$0.75x$ .  
 For a profit of \$0, total cost must equal total money earned.  
 Therefore,  $\$0.15x + \$12.00 = \$0.75x$  or  $\$12.00 = \$0.60x$  or  $x = 20$ .  
 Emily must sell 20 cups of lemonade to break even.
  - (d) Let  $n$  represent the number of cups that Emily must sell to make a profit of exactly \$17.00.  
 As in (c), the total cost to make  $n$  cups of lemonade is  $\$0.15n + \$12.00$ .  
 As in (c), the total money earned by selling  $n$  cups is  $\$0.75n$ .  
 For a profit of \$17.00, the total money earned minus the total cost must equal \$17.00.  
 Thus,  $\$0.75n - (\$0.15n + \$12.00) = \$17.00$  or  $\$0.60n - \$12.00 = \$17.00$  or  $\$0.60n = \$29.00$ .  
 This gives  $n = \frac{29}{0.6} = \frac{290}{6} = 48\frac{1}{3}$ .  
 Since  $n$  represents the number of cups that Emily sells,  $n$  must be a non-negative integer and as such, cannot equal  $48\frac{1}{3}$ .  
 Therefore, it is not possible for Emily to make a profit of exactly \$17.00.
2. (a) Evaluating,  $2\nabla 5 = \frac{2+5}{1+2 \times 5} = \frac{7}{11}$ .
  - (b) Evaluating the expression in brackets first,  
 $(1\nabla 2)\nabla 3 = \left(\frac{1+2}{1+1 \times 2}\right) \nabla 3 = \left(\frac{3}{3}\right) \nabla 3 = 1\nabla 3 = \frac{1+3}{1+1 \times 3} = 1$ .  
 (Note that for any  $b > 0$ ,  $1\nabla b = \frac{1+b}{1+1 \times b} = \frac{1+b}{1+b} = 1$ .)
  - (c) By definition,  $2\nabla x = \frac{2+x}{1+2x}$ . Thus,  

$$\begin{aligned} \frac{2+x}{1+2x} &= \frac{5}{7} \\ 7(2+x) &= 5(1+2x) \\ 14+7x &= 5+10x \\ 9 &= 3x \end{aligned}$$
 so  $x = 3$ .
  - (d) We have,  $x\nabla y = \frac{x+y}{1+xy}$ . Thus, from the given information,  $\frac{x+y}{1+xy} = \frac{x+y}{17}$ .  
 The numerator,  $x+y$ , of each of these two fractions is non-zero since  $x > 0$  and  $y > 0$ .  
 Two equivalent fractions having equal, non-zero numerators have equal denominators.  
 Thus,  $1+xy = 17$  or  $xy = 16$ . The possible ordered pairs of positive integers  $(x, y)$ , for which  $xy = 16$ , are  $(1, 16)$ ,  $(16, 1)$ ,  $(2, 8)$ ,  $(8, 2)$ , and  $(4, 4)$ .
3. (a) We know that  $OA$  and  $OB$  are each radii of the semi-circle with centre  $O$ .  
 Thus,  $OA = OB = OC + CB = 32 + 36 = 68$ .  
 Therefore,  $AC = AO + OC = 68 + 32 = 100$ .
  - (b) The semi-circle with centre  $K$  has radius  $AK = \frac{1}{2}(AC) = \frac{1}{2}(100) = 50$ .  
 Thus, this semi-circle has an area equal to  $\frac{1}{2}\pi(AK)^2 = \frac{1}{2}\pi(50)^2 = 1250\pi$ .

- (c) The shaded area is equal to the area of the largest semi-circle with centre  $O$ , minus the combined areas of the two smaller unshaded semi-circles with centres  $K$  and  $M$ .

The radius of the smaller unshaded circle is  $MB = \frac{1}{2}(CB) = \frac{1}{2}(36) = 18$ .

Therefore, the shaded area equals

$$\begin{aligned}
 & \frac{1}{2}\pi(OB)^2 - \left(\frac{1}{2}\pi(AK)^2 + \frac{1}{2}\pi(MB)^2\right) \\
 &= \frac{1}{2}\pi(68)^2 - \left(\frac{1}{2}\pi(50)^2 + \frac{1}{2}\pi(18)^2\right) \\
 &= \frac{1}{2}\pi(68^2 - 50^2 - 18^2) \\
 &= \frac{1}{2}\pi(4624 - 2500 - 324) \\
 &= \frac{1}{2}\pi(1800) \\
 &= 900\pi
 \end{aligned}$$

- (d) *Solution 1*

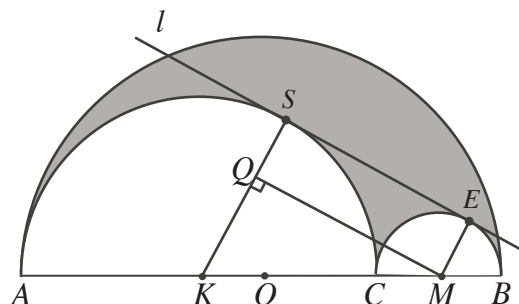
Construct line segments  $KS$  and  $ME$  perpendicular to line  $l$ .

Position point  $Q$  on  $KS$  so that  $MQ$  is perpendicular to  $KS$ , as shown.

In quadrilateral  $MQSE$ ,

$$\angle MQS = \angle QSE = \angle SEM = 90^\circ.$$

Hence, quadrilateral  $MQSE$  is a rectangle.



The larger unshaded semi-circle has radius 50, so  $KC = KS = 50$ .

The smaller unshaded semi-circle has radius 18, so  $ME = MC = MB = 18$ .

Thus,  $MK = MC + KC = 18 + 50 = 68$ .

The area of quadrilateral  $KSEM$  is the sum of the areas of rectangle  $MQSE$  and  $\triangle MKQ$ .

Since  $QS = ME = 18$ , then  $KQ = KS - QS = 50 - 18 = 32$ .

Using the Pythagorean Theorem in  $\triangle MKQ$ ,  $MK^2 = KQ^2 + QM^2$  or  $68^2 = 32^2 + QM^2$  or  $QM = \sqrt{68^2 - 32^2} = 60$  (since  $QM > 0$ ).

The area of  $\triangle MKQ$  is  $\frac{1}{2}(KQ)(QM) = \frac{1}{2}(32)(60) = 960$ .

The area of rectangle  $MQSE$  is  $(QM)(QS) = (60)(18) = 1080$ .

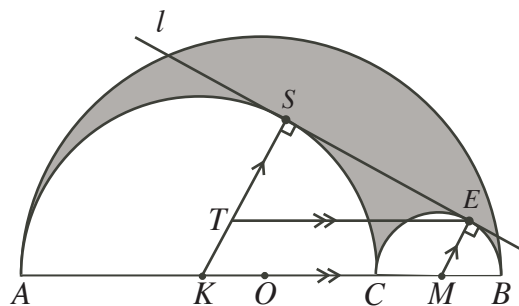
Thus, the area of quadrilateral  $KSEM$  is  $960 + 1080 = 2040$ .

*Solution 2*

Construct line segments  $KS$  and  $ME$  perpendicular to line  $l$ .

Since  $KS$  and  $ME$  are each perpendicular to line  $l$ , they are parallel to one another and thus,  $KSEM$  is a trapezoid.

Position point  $T$  on  $KS$  so that  $TE$  is parallel to  $KM$ , as shown.



The larger unshaded semi-circle has radius 50, so  $KC = KS = 50$ .

The smaller unshaded semi-circle has radius 18, so  $ME = MC = MB = 18$ .

Thus,  $KM = KC + CM = 50 + 18 = 68$ .

Since  $KTEM$  is a parallelogram  $TE = KM = 68$  and  $KT = ME = 18$ .

Thus,  $TS = KS - KT = 50 - 18 = 32$ .

Using the Pythagorean Theorem in  $\triangle TSE$ ,  $SE^2 = TE^2 - TS^2$  or  $SE^2 = 68^2 - 32^2$  or  $SE = \sqrt{68^2 - 32^2} = 60$  (since  $SE > 0$ ).

The area of trapezoid  $KSEM$  is  $\frac{(SE)(ME + KS)}{2} = \frac{(60)(18 + 50)}{2} = (30)(68) = 2040$ .



4. When we are calculating such a sum by hand, in each column starting at the right hand end, we add up the digits and add the carry to this to create a column total. We take this total, write down its unit digit and take the integer formed by the rest of the total to become the carry into the next column to the left.

For example, if a column total is 124, we write down the units digit 4 beneath this column and carry 12 into the next column to the left.

- (a) The sum of the digits in the ones column is  $2(101) = 202$ . Thus, the ones digit  $A$  is 2, and 20 is carried left to the tens column.
- (b) The sum of the digits in the tens column is  $2(100) = 200$  and the carry into the tens column is 20. Thus, the total in the tens column is  $200 + 20 = 220$ .  
Therefore, the tens digit  $B$  is 0 and 22 is carried left to the hundreds column.  
The sum of the digits in the hundreds column is  $2(99) = 198$ , which when added to the carry of 22 gives a column total of  $198 + 22 = 220$ .  
Thus, the hundreds digit  $C$  is also 0.

- (c) We show that the middle digit of the sum is 3 by using the following steps:

- Step 1: The sum has 101 digits
- Step 2: The total in the middle column is at least 113
- Step 3: The total in the middle column cannot be 114 or more

Steps 2 and 3 together will tell us that the total in the middle column is exactly 113, so the middle digit in the sum is the units digit of 113, which is 3. We will need to use the following Fact:

Fact: The carry from one column to the next is always at most 22.

We leave the explanation of this Fact to the end, but use it twice. We number the columns in the sum from left to right.

Step 1: The sum has 101 digits

The total in the 1st column is 2 plus the carry from the 2nd column. Thus, the total in the 1st column will have only 1 digit unless the carry from the 2nd column is at least 8. For the carry from the 2nd column to be at least 8, then the total in the 2nd column must be at least 80. The sum of the digits in the 2nd column is 4, so for a total of at least 80, the carry from the 3rd column would have to be at least 76.

But every carry is at most 22 (see the Fact below), so this is impossible.

Thus, the total in the 1st column has only 1 digit.

This means that the sum has exactly the same number of digits as the largest number among the numbers being added, so has 101 digits.

Step 2: The total in the middle column is at least 113

Since the sum has 101 digits, then the middle digit (or column) is the 51st column from the left. (There will be 50 digits to the left of this column and 50 digits to the right, giving 101 digits in total.)

The sum of the digits in the 51st column is  $2(51) = 102$ .

The sum of the digits in the 52nd column is  $2(52) = 104$ .

The sum of the digits in the 53rd column is  $2(53) = 106$ .

Thus, the carry from the 53rd column to the 52nd column is at least 10, so the total of the 52nd column is at least  $104 + 10 = 114$ .

Thus, the carry from the 52nd column to the 51st column is at least 11, so the total of the 51st column is at least  $102 + 11 = 113$ .

Step 3: The total in the middle column cannot be 114 or more

If the total in the 51st column was 114 or more, then the carry from the 52nd column would have to be at least  $114 - 102 = 12$ .

For the carry from the 52nd column to be at least 12, then the total in the 52nd column would have to be at least 120. For this total to be at least 120, then the carry from the 53rd column would have to be at least  $120 - 104 = 16$ .

For the carry from the 53rd column to be at least 16, then the total in the 53rd column would have to be at least 160. For this total to be at least 160, then the carry from the 54th column would have to be at least  $160 - 106 = 54$ .

But by the Fact (see below), the carry cannot be more than 22, so this is impossible.

Therefore, the total in the middle column cannot be 114 or more, so must be exactly 113.

It remains to look at the Fact.

Fact: The carry from one column to the next is always at most 22

We start from the rightmost column.

The sum of the digits in the 101st column is  $2(101) = 202$ , so 20 is carried to the 100th column.

The total of the 100th column is thus  $2(100) + 20 = 220$ , so 22 is carried to the 99th column.

The total of the 99th column is thus  $2(99) + 22 = 220$ , so 22 is carried to the 98th column. To this point, none of the carries is larger than 22.

Suppose that at some point the carry is at least 23.

If this is the case, then starting from the right end, there must be a first time that the carry is at least 23.

Let's suppose that this first time is the carry from  $n$ th column.

We know that  $n \leq 98$ , since the carries from the first three columns are all at most 22.

Also, we know that the carry into the  $n$ th column is at most 22, since the carry from the  $n$ th column is the first time that the carry is more than 22.

Let's look at this  $n$ th column.

The  $n$ th column includes  $n$  2's, so the sum of the digits in this column is  $2n$ .

Since  $n \leq 98$ , then the sum of the digits in the  $n$ th column is at most  $2(98) = 196$ .

For the carry from the  $n$ th column to be at least 23, the total in the  $n$ th column must be at least 230, which means that the carry into the  $n$ th column is at least  $230 - 196 = 34$ .

But we already know that the carry into the  $n$ th column is at most 22, since the carry from the  $n$ th column is the first time that the carry is at least 23.

This means that we have a contradiction, since the carry into the  $n$ th column cannot be both at most 22 and at least 34.

Thus, the only thing that can be wrong is our assumption that at some point the carry is at least 23.

Therefore, the carry is always at most 22.

This completes the argument, and shows that the total of the middle column is exactly 113, so the middle digit of the sum is 3.



**Canadian  
Mathematics  
Competition**

*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

***2008 Fryer Contest***

**Wednesday, April 16, 2008**

*Solutions*

1. (a) (i) The sum of the nine integers is

$$11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 = 135$$

- (ii) The nine numbers in the magic square add up to 135.

The sum of the numbers in each row is the same, so will be  $\frac{1}{3}$  of the total sum, or  $\frac{1}{3}(135) = 45$ .

Therefore, the magic constant is 45.

- (iii) The sum of the numbers in each row, in each column, and on each of the two main diagonals should be 45.

Since we are given two numbers in the first row and the top left to bottom right

diagonal, we can complete these to get 

18	11	16
	15	
		12

, since  $45 - 18 - 11 = 16$  and  $45 - 18 - 12 = 15$ .

We can now complete the second and third columns to get 

18	11	16
	15	17
	19	12

.

We can now finish the magic square to get 

18	11	16
13	15	17
14	19	12

.

- (b) (i) The sum of the sixteen integers is

$$1 + 2 + 3 + 4 + \cdots + 13 + 14 + 15 + 16 = 136$$

(We can pair up the integers 1 with 16, 2 with 15, and so on, to get 8 pairs, each of which add to 17.)

- (ii) The sixteen numbers in the magic square add up to 136.

The sum of the numbers in each row is the same, so will be  $\frac{1}{4}$  of the total sum, or  $\frac{1}{4}(136) = 34$ .

Therefore, the magic constant is 34.

- (iii) The sum of the numbers in each row, in each column, and on each of the two main diagonals should be 34.

Since we are given three numbers in the first and fourth rows, first, third and fourth

columns, and on both diagonals, we can complete these to get 

16	3	2	13
5	10	11	8
9		7	12
4	15	14	1

,

since  $34 - 16 - 3 - 13 = 2$ , and so on.

We can now complete the magic square to get 

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

.

2. (a) After playing 5 more games, the Sharks had played  $10 + 5 = 15$  games in total and had won  $8 + 1 = 9$  of these.

Their final winning percentage was  $\frac{9}{15} \times 100\% = 60\%$ .

- (b) The Emus played  $10 + x$  games in total and won  $4 + x$  games in total.

Since their final winning percentage was 70%, then  $\frac{4+x}{10+x} \times 100\% = 70\%$  or  $\frac{4+x}{10+x} = \frac{7}{10}$ .

Therefore,  $10(4 + x) = 7(10 + x)$ , so  $40 + 10x = 70 + 7x$  or  $3x = 30$  or  $x = 10$ .

Therefore, the Emus played  $10 + x = 10 + 10 = 20$  games in total.

- (c) Suppose the Pink Devils played and lost  $y$  more games so that they had won exactly  $\frac{2}{7}$  of their games at that point.

Then they had played  $7 + 3 + y = 10 + y$  games in total and won 7 of them.

Therefore,  $\frac{7}{10 + y} = \frac{2}{7}$  or  $7(7) = 2(10 + y)$  or  $2y + 20 = 49$  or  $2y = 29$ .

But  $y$  is not an integer, so there was no point when the Pink Devils had won exactly  $\frac{2}{7}$  of their games.

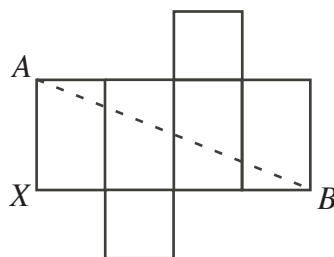
(We could have also argued starting with the equation  $7(7) = 2(10 + y)$  that the left side is even and the right side is odd, so there is no integer  $y$  that works.)

3. (a) The box that can be formed by folding the net in Figure 1 has dimensions 4 by 4 by 7, and so has volume  $4 \times 4 \times 7 = 112$ .

We can use the net more directly to calculate the surface area. The net shows that the box will have two faces that are 4 by 4, and four faces that are 4 by 7.

Therefore, the surface area is  $2(4 \times 4) + 4(4 \times 7) = 32 + 112 = 144$ .

- (b) We label point  $X$  on Figure 3.



Now  $AX$  is the height of the box, so  $AX = 6$ .

Also,  $\angle AXB = 90^\circ$ , since the box is rectangular.

Next, we see that  $XB = 2 + 2 + 2 + 2 = 8$ , since the four edges of the original box that form  $XB$  are those around the bottom face of the original box.

Therefore, by the Pythagorean Theorem,

$$AB^2 = AX^2 + XB^2 = 6^2 + 8^2 = 36 + 64 = 100$$

so  $AB = 10$ , since  $AB > 0$ .

- (c) We want to find the shortest path from  $A$  to  $G$  along the surface of the block. First, we make a few observations:

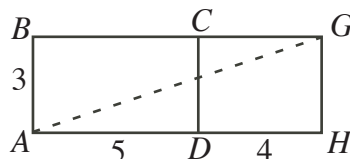
- Any path along the surface from  $A$  to  $G$  can be traced on an unfolded version of the block (that is, on a net like that in (a) or (b)).
- Given a particular way of unfolding the block, the shortest distance from  $A$  to  $G$  will be a straight line. The length of this straight line can be found using the Pythagorean Theorem, using the “rise” and the “run” from  $A$  to  $G$ , as in (b).
- In the given block, there is no single face that uses each of the vertices  $A$  and  $G$ . Thus, it is impossible for the caterpillar to travel from  $A$  to  $G$  along one face only, and so the caterpillar uses at least two faces.
- Any straight line path (and thus any path) that uses more than 2 faces will be longer than the possible straight line paths using exactly 2 faces. This is true because the rise and the run will be at least as long as in any of the 2 face paths below.

- Paths that include sections along edges do not need to be considered, as they will not form straight lines on the unfolded box.

So we need to examine all of the possible paths from  $A$  to  $G$  using exactly two faces. These combinations of faces are

- $ABCD$  then  $DCGH$
- $ABCD$  then  $BCGF$
- $ADHE$  then  $DCGH$
- $ADHE$  then  $EHGF$
- $ABFE$  then  $BCGF$
- $ABFE$  then  $EHGF$

Consider the straight line path across  $ABCD$  and  $DCGH$ .



The path is the hypotenuse of a right-angled triangle with legs of lengths 3 and 9, and so has length  $\sqrt{3^2 + 9^2} = \sqrt{90}$ .

In a similar way, the lengths of the paths are:

- $ABCD$  then  $DCGH$ :  $\sqrt{90}$
- $ABCD$  then  $BCGF$ :  $\sqrt{74}$
- $ADHE$  then  $DCGH$ :  $\sqrt{80}$
- $ADHE$  then  $EHGF$ :  $\sqrt{74}$
- $ABFE$  then  $BCGF$ :  $\sqrt{80}$
- $ABFE$  then  $EHGF$ :  $\sqrt{90}$

(We note that these lengths occur in three pairs of equal lengths, so we could have calculated three lengths and used symmetry to deal with the other three.)

Since the length of the shortest path is one of these lengths, then the shortest path has length  $\sqrt{74}$ .

4. (a) First, we examine the digits in a general palindrome.

If a palindrome  $P$  has an even number of digits (like 1221), then each digit from 0 to 9 will occur an even number of times in  $P$ , as each digit in the first half can be matched with a digit in the second half.

If a palindrome  $P$  has an odd number of digits (like 12321), then there will be one digit from 0 to 9 that occurs an odd number of times in  $P$  and every other digit will occur an even number of times. This is because  $P$  has a “middle” digit. If this digit is removed, the resulting number is a palindrome with an even number of digits and so has an even number of each of 0 to 9. Adding the “middle” digit back in, one of the digits occurs an odd number of times.

Any collection of digits, in which each occurs an even number of times, can be arranged to form a palindrome with an even number of digits by building the palindrome in pairs of digits from the centre out.

Similarly, any collection of digits in which each but one occurs an even number of times can be arranged to form a palindrome, by starting with one occurrence of the digit that

occurs the odd number of times in the middle, and building outwards.  
(No other collections of digits can be arranged to form a palindrome.)

Now, we examine  $x$ .

The integer  $x$  contains digits in the following distribution:

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	3	13	13	4	3	3	3	3	3	3

If we wanted to create a palindrome  $P$  with an even number of digits from the digits of  $x$ , we would need to reduce the number of occurrences of each of 0 to 9 so that each occurred an even number of times. To do this by removing the minimum number of digits, we can remove one each of 0, 1, 2, 4, 5, 6, 7, 8, 9. (To make an odd number even by subtracting the smallest amount, we subtract 1.) This ensures that each digit occurs an even number of times. Here, we have removed 9 digits in total.

If we wanted to create a palindrome  $P$  with an odd number of digits from the digits of  $x$ , we would need to reduce the number of occurrences of each of 0 to 9 so that all (but one) of them occurred an even number of times. To do this removing the minimum number of digits, we can remove 1 each of all but one of 0, 1, 2, 4, 5, 6, 7, 8, 9. (For the sake of argument, suppose that we do not remove a 9.) This ensures that each digit but 9 occurs an even number of times and 9 occurs an odd number of times. Here, we have removed 8 digits in total and formed a palindrome.

Therefore, the minimum number of digits that must be removed is 8.

- (b) From the chart above, the digits of  $x$  add to

$$3(0) + 13(1) + 13(2) + 4(3) + 3(5) + 3(6) + 3(7) + 3(8) + 3(9) = 168$$

To obtain a sum of 130, we must remove digits that have a sum of  $168 - 130 = 38$ .

We want to remove the minimum number of digits that have a sum of 38.

We try to do this by removing the largest digits first.

Removing three 9's, we have removed a total of 27.

Removing one 8, we have removed a total of 35.

We can now obtain a total removed of 38 by removing a 3.

Here, we have removed 5 digits.

We cannot remove 4 or fewer digits with a total of 38, as the maximum sum of 4 or fewer digits in theory is  $4(9) = 36$ .

Therefore, the minimum number of digits that we can remove is 5.

- (c) First, we enumerate the digits of  $y$ :

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	5	15	15	15	15	6	5	5	5	5

We can determine the sum of the digits by adding directly as above, or by noticing that each digit occurs at least 5 times, and grouping as follows:

$$5(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) + 10(1 + 2 + 3 + 4) + 5 = 330$$

We want to remove digits from  $y$  whose sum is  $330 - 210 = 120$  and in such a way that, among the digits that remain, at most one of the digits 0 to 9 occurs an odd number of times.

We use the strategy of removing digits in such a way that each of 0 to 9 occurs an even number of times, and the digits sum to at most 210, and then perhaps add back in a single

digit (giving the final palindrome an odd number of digits) to make the sum equal to 210. We have to make each digit occur an even number of times, so first we remove 1 each of 0, 1, 2, 3, 4, 6, 7, 8, 9, to obtain:

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	4	14	14	14	14	6	4	4	4	4

whose sum is 290.

We now want to remove digits in pairs, removing the smallest *number* of digits that makes the total sum less than or equal to 210.

We do this by removing the largest possible digits first. Removing four 8's and four 9's gives

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	4	14	14	14	14	6	4	4	0	0

whose sum is 222.

We need to remove at least two more digits to obtain a sum of less than or equal to 210. (We cannot remove a single digit equal to at least 12.)

We could do this for example by removing two 6's, obtaining

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	4	14	14	14	14	6	2	4	0	0

We could then add back in a single 0 to maintain the sum of 210.

In total, we have removed  $9 + 4 + 4 + 2 - 1 = 18$  digits.

This is the minimum number of digits because we have removed the minimum number of digits to form a palindrome with an even number of digits, and then added one digit back in.





# **Canadian Mathematics Competition**

*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

## ***2007 Fryer Contest***

**Wednesday, April 18, 2007**

*Solutions*

1. (a) Since Rectangle 3 has 4 rows of squares, then Rectangle 4 has 5 rows.  
 Since Rectangle 3 has 7 columns of squares, then Rectangle 4 has 9 columns.  
 Therefore, Rectangle 4 has  $5 \times 9 = 45$  squares.
- (b) Since Rectangle 4 has 5 rows, its height is 5.  
 Since Rectangle 4 has 9 columns, its width is 9.  
 Therefore, the perimeter of Rectangle 4 is  $2(5) + 2(9) = 28$ .
- (c) Since Rectangle 4 has 5 rows of squares, then Rectangle 7 has  $5 + 3 = 8$  rows.  
 Since Rectangle 4 has 9 columns of squares, then Rectangle 7 has  $9 + 2(3) = 15$  columns.  
 Since Rectangle 7 has 8 rows and 15 columns, it is 8 by 15, so has perimeter  $2(8) + 2(15) = 46$ .
- (d) *Solution 1*  
 Rectangle 7 is 8 by 15 and so has perimeter  $2(8) + 2(15) = 46$ .  
 Let us try a bigger rectangle.  
 Rectangle 17 has  $8 + 10 = 18$  rows and  $15 + 2(10) = 35$  columns, so is 18 by 35 and has perimeter  $2(18) + 2(35) = 106$ .  
 Let us try a rectangle that is still bigger.  
 Rectangle 27 has  $18 + 10 = 28$  rows and  $35 + 2(10) = 55$  columns, so is 28 by 55 and has perimeter  $2(28) + 2(55) = 166$ .  
 We are getting close!  
 Rectangle 28 has 29 rows and 57 columns, so has perimeter  $2(29) + 2(57) = 172$ .  
 Rectangle 29 has 30 rows and 59 columns, so has perimeter  $2(30) + 2(59) = 178$ .  
 Therefore,  $n = 29$ .  
 (Notice that this is the only answer as moving more steps along in the sequence makes the rectangles larger.)

*Solution 2*

Rectangle 7 is 8 by 15 and so has perimeter  $2(8) + 2(15) = 46$ .  
 When we move to Rectangle 8, the height increases by 1 and the width increases by 2.  
 This increase the perimeter by  $2(1) + 2(2) = 6$ .  
 The same increase in perimeter occurs at each step in the sequence.  
 Since  $178 - 46 = 132 = 22(6)$ , then we must move 22 steps beyond Rectangle 7 to get from a perimeter of 46 to a perimeter of 178.  
 Thus, Rectangle 29 has a perimeter of 178, so  $n = 29$ .

*Solution 3*

Rectangle 1 has 1 more row than its number, and the number of rows increases by 1 from each step to the next.  
 Therefore, Rectangle  $n$  has  $n + 1$  rows.  
 Rectangle 1 has 3 columns (which is 1 more than twice its number), and the number of columns increases by 2 from each step to the next.  
 Therefore, Rectangle  $n$  has  $2n + 1$  columns.  
 A formula for the perimeter of Rectangle  $n$  is  $2(n+1) + 2(2n+1) = 2n+2+4n+2 = 6n+4$ .  
 For a perimeter of 178, we must have  $6n + 4 = 178$  or  $6n = 174$  or  $n = 29$ .  
 Therefore, Rectangle 29 has a perimeter of 178.

2. (a) Jim buys  $5 + 2 + 3 = 10$  tickets.  
 The total cost of the tickets is  $5(\$25) + 2(\$10) + 3(\$5) = \$160$ .

Therefore, the average cost of the tickets was  $\frac{\$160}{10} = \$16$ .

- (b) Since Mike buys 8 tickets at an average cost of \$12, their total cost is  $8 \times \$12 = \$96$ .

When he buys 5 more platinum tickets, he pays  $5 \times \$25 = \$125$ .

In total, Mike thus pays  $\$96 + \$125 = \$221$  for 13 tickets.

Therefore, the new average cost of the tickets is  $\frac{\$221}{13} = \$17$ .

- (c) *Solution 1*

The first 10 tickets that Ophelia buys at an average cost of \$14 have a total cost of  $10 \times \$14 = \$140$ .

When she buys  $n$  more platinum tickets, she pays  $25n$  dollars for these additional tickets.

In total, she has now paid  $140 + 25n$  dollars for  $10 + n$  tickets.

Since the average price of the tickets that she has bought is \$20, then

$$\begin{aligned}\frac{140 + 25n}{10 + n} &= 20 \\ 140 + 25n &= 20(10 + n) \\ 140 + 25n &= 200 + 20n \\ 5n &= 60 \\ n &= 12\end{aligned}$$

so she buys 12 more platinum tickets.

*Solution 2*

For the first ten tickets that Ophelia buys, the average cost is \$6 less per ticket than the final average cost of \$20.

Therefore, she has paid  $10 \times \$6 = \$60$  less in total than she would have if she had paid \$20 on average for these tickets.

For a final average of \$20 per ticket, the new platinum tickets must cost in total \$60 more than an average of \$20.

Since each platinum ticket costs \$5 more on average than the final average, then she buys  $\$60 \div \$5 = 12$  more platinum tickets.

3. (a) For 992 466 1A6 to be divisible by 8, we must have 1A6 divisible by 8.

We check each of the possibilities, using a calculator or by checking by hand:

106 is not divisible by 8, 116 is not divisible by 8, 126 is not divisible by 8,

136 is divisible by 8,

146 is not divisible by 8, 156 is not divisible by 8, 166 is not divisible by 8,

176 is divisible by 8,

186 is not divisible by 8, 196 is not divisible by 8

Therefore, the possible values of  $A$  are 3 and 7.

- (b) For  $D 767 E 89$  to be divisible by 9, we must have  $D + 7 + 6 + 7 + E + 8 + 9 = 37 + D + E$  divisible by 9.

Since  $D$  and  $E$  are each a single digit then each is between 0 and 9, so  $D + E$  is between 0 and 18.

Therefore,  $37 + D + E$  is between 37 and 55.

The numbers between 37 and 55 that are divisible by 9 are 45 and 54.

If  $37 + D + E = 45$ , then  $D + E = 8$ .

If  $37 + D + E = 54$ , then  $D + E = 17$ .

Therefore, the possible values of  $D + E$  are 8 and 17.

- (c) For  $541G5072H6$  to be divisible by 72, it must be divisible by 8 and by 9.

It is easier to check for divisibility by 8 first, since this will allow us to determine a small number of possibilities for  $H$ .

For  $541G5072H6$  to be divisible by 8, we must have  $2H6$  divisible by 8.

Going through the possibilities as in part (a), we can find that  $2H6$  is divisible by 8 when  $H = 1, 5, 9$  (that is, 216, 256 and 296 are divisible by 8 while 206, 226, 236, 246, 266, 276, 286 are not divisible by 8).

We must now use each possible value of  $H$  to find the possible values of  $G$  that make  $541G5072H6$  divisible by 9.

First,  $H = 1$ . What value(s) of  $G$  make  $541G507216$  divisible by 9?

In this case, we need  $5 + 4 + 1 + G + 5 + 0 + 7 + 2 + 1 + 6 = 31 + G$  divisible by 9.

Since  $G$  is between 0 and 9, then  $31 + G$  is between 31 and 40, so must be equal to 36 if it is divisible by 9. Thus,  $G = 5$ .

Next,  $H = 5$ . What value(s) of  $G$  make  $541G507256$  divisible by 9?

In this case, we need  $5 + 4 + 1 + G + 5 + 0 + 7 + 2 + 5 + 6 = 35 + G$  divisible by 9.

Since  $G$  is between 0 and 9, then  $35 + G$  is between 35 and 44, so must be equal to 36 if it is divisible by 9. Thus,  $G = 1$ .

Last,  $H = 9$ . What value(s) of  $G$  make  $541G507296$  divisible by 9?

In this case, we need  $5 + 4 + 1 + G + 5 + 0 + 7 + 2 + 9 + 6 = 39 + G$  divisible by 9.

Since  $G$  is between 0 and 9, then  $39 + G$  is between 39 and 48, so must be equal to 45 if it is divisible by 9. Thus,  $G = 6$ .

Therefore, the possible pairs of values are  $H = 1$  and  $G = 5$ ,  $H = 5$  and  $G = 1$ , and  $H = 9$  and  $G = 6$ .

(Note that we could have combined the analysis of these last three cases.)

4. (a) *Solution 1*

By the Pythagorean Theorem,  $YZ^2 = YX^2 + XZ^2 = 60^2 + 80^2 = 3600 + 6400 = 10000$ , so  $YZ = 100$ .

(We could also have found  $YZ$  without using the Pythagorean Theorem by noticing that  $\triangle XYZ$  is a right-angled triangle with its right-angle at  $X$  and  $XY = 60 = 3(20)$  and  $XZ = 80 = 4(20)$ . This means that  $\triangle XYZ$  is similar to a 3-4-5 triangle, so has  $YZ = 5(20) = 100$ .)

Since  $\triangle YXZ$  is right-angled at  $X$ , its area is  $\frac{1}{2}(60)(80) = 2400$ .

Since  $XW$  is perpendicular to  $YZ$ , then the area of  $\triangle YXZ$  is also equal to  $\frac{1}{2}(100)(XW) = 50XW$ .

Therefore,  $50XW = 2400$ , so  $XW = 48$ .

By the Pythagorean Theorem,  $WZ^2 = 80^2 - 48^2 = 6400 - 2304 = 4096$ .

Thus,  $WZ = \sqrt{4096} = 64$ .

*Solution 2*

By the Pythagorean Theorem,  $YZ^2 = YX^2 + XZ^2 = 60^2 + 80^2 = 3600 + 6400 = 10000$ , so  $YZ = 100$ .

Let  $WZ = a$ . Then  $YW = 100 - a$ .

Let  $XW = h$ .

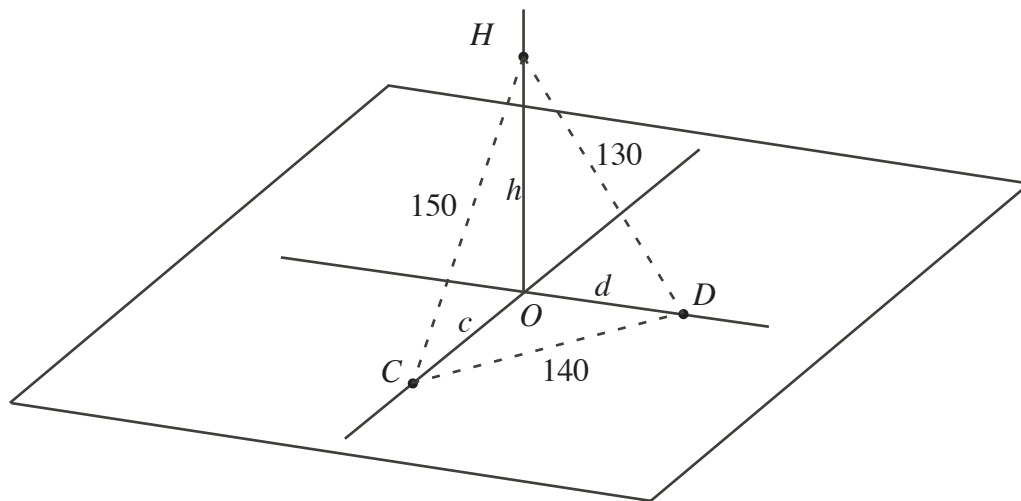
By the Pythagorean Theorem in  $\triangle XWY$ , we have  $(100 - a)^2 + h^2 = 60^2$ .

By the Pythagorean Theorem in  $\triangle XWZ$ , we have  $a^2 + h^2 = 80^2$ .  
 Subtracting the first of these equations from the second, we obtain

$$\begin{aligned} a^2 - (100 - a)^2 &= 80^2 - 60^2 \\ a^2 - (10000 - 200a + a^2) &= 6400 - 3600 \\ 200a - 10000 &= 2800 \\ 200a &= 12800 \\ a &= 64 \end{aligned}$$

Therefore,  $WZ = 64$ .

(b) Let  $OC = c$ ,  $OD = d$  and  $OH = h$ .

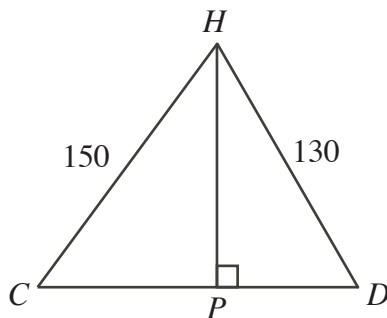


Note that  $OH$  is perpendicular to the field, so  $OH$  is perpendicular to  $OC$  and to  $OD$ .  
 Also, since  $OD$  points east and  $OC$  points south, then  $OD$  is perpendicular to  $OC$ .  
 Since  $HC = 150$ , then  $h^2 + c^2 = 150^2$  by the Pythagorean Theorem.  
 Since  $HD = 130$ , then  $h^2 + d^2 = 130^2$ .  
 Since  $CD = 140$ , then  $c^2 + d^2 = 140^2$ .  
 Adding the first two equations, we obtain  $2h^2 + c^2 + d^2 = 150^2 + 130^2$ .  
 Since  $c^2 + d^2 = 140^2$ , then

$$\begin{aligned} 2h^2 + 140^2 &= 150^2 + 130^2 \\ 2h^2 &= 150^2 + 130^2 - 140^2 \\ 2h^2 &= 19800 \\ h^2 &= 9900 \\ h &= \sqrt{9900} = 30\sqrt{11} \end{aligned}$$

Therefore, the height of the balloon above the field is  $30\sqrt{11} \approx 99.5$  m.

(c) To save the most rope, we must have  $HP$  having minimum length.  
 For  $HP$  to have minimum length,  $HP$  must be perpendicular to  $CD$ .



(Among other things, we can see from this diagram that sliding  $P$  away from the perpendicular position does make  $HP$  longer.)

In the diagram,  $HC = 150$ ,  $HD = 130$  and  $CD = 140$ .

Let  $HP = x$  and  $PD = a$ . Then  $CP = 140 - a$ .

By the Pythagorean Theorem in  $\triangle HPC$ ,  $x^2 + (140 - a)^2 = 150^2$ .

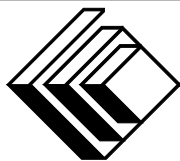
By the Pythagorean Theorem in  $\triangle HPD$ ,  $x^2 + a^2 = 130^2$ .

Subtracting the second equation from the first, we obtain

$$\begin{aligned} (140 - a)^2 - a^2 &= 150^2 - 130^2 \\ (19600 - 280a + a^2) - a^2 &= 5600 \\ 19600 - 280a &= 5600 \\ 280a &= 14000 \\ a &= 50 \end{aligned}$$

Therefore,  $x^2 + 90^2 = 150^2$  or  $x^2 = 150^2 - 90^2 = 22500 - 8100 = 14400$  so  $x = 120$ .

So the shortest possible rope that we can use is 120 m, which saves  $130 + 150 - 120 = 160$  m of rope.



# **Canadian Mathematics Competition**

*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

## ***2006 Fryer Contest***

**Thursday, April 20, 2006**

*Solutions*

1. (a) Her average mark in the seven courses is

$$\frac{94 + 93 + 84 + 81 + 74 + 83 + 79}{7} = \frac{588}{7} = 84$$

- (b) Samantha's highest possible average would be if she obtained a mark of 100 in French. In this case, her average in the eight courses would be

$$\frac{94 + 93 + 84 + 81 + 74 + 83 + 79 + 100}{8} = \frac{688}{8} = 86$$

- (c) *Solution 1*

If Samantha's final average over all eight courses is 85, then the sum of her marks in the eight courses is  $8 \times 85 = 680$ .

In part (a), we saw that the sum of her first seven marks is 588, so her mark in French is  $680 - 588 = 92$ .

*Solution 2*

Samantha's average in the first seven courses is 84, so if her French mark was 84, then her average would remain as 84.

If Samantha's mark in French was 100 (as in (b)), her average would be 86.

Since her average is 85 (half-way between these two averages), then her French mark is half-way between 84 and 100, or 92.

2. (a) The bottom layer of the cube is a 7 by 7 square of cubes, so uses  $7 \times 7 = 49$  cubes. The next layer of the cube is a 5 by 5 square of cubes, so uses  $5 \times 5 = 25$  cubes. The next layer of the cube is a 3 by 3 square of cubes, so uses  $3 \times 3 = 9$  cubes. The top layer consists of a single cube. Therefore, the total number of cubes used is  $49 + 25 + 9 + 1 = 84$ .

- (b) *Solution 1*

The cube in the top layer has 5 visible faces (only the bottom face is hidden).

In the second layer from the top, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there is 1 cube on each of the four sides of the layer with 2 visible faces (another 8 visible faces).

In the second layer from the bottom, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 3 cubes on each of the four sides of the layer with 2 visible faces (another  $4 \times 3 \times 2 = 24$  visible faces).

In the bottom layer, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 5 cubes on each of the four sides of the layer with 2 visible faces (another  $4 \times 5 \times 2 = 40$  visible faces).

Therefore, the total number of visible faces is  $5 + 12 + 8 + 12 + 24 + 12 + 40 = 113$ .

*Solution 2*

When we look at the pyramid from the top, we see a  $7 \times 7$  square of visible faces, or 49 visible faces. (This square is composed of faces from all of the levels of the pyramid.)

When we look at the pyramid from each of the four sides, we see  $1 + 3 + 5 + 7 = 16$  visible faces, so there are  $4(16) = 64$  visible faces on the sides.

Therefore, there are  $49 + 64 = 113$  visible faces in total.



(c) *Solution 1*

To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three or five numbers are visible, depending on its position.

For the top cube (with 5 visible faces), we position this cube with the “1” on the bottom face (and so is hidden).

The total of the numbers visible on this cube is  $2 + 3 + 4 + 5 + 6 = 20$ .

For the 4 corner cubes on each layer (each with 3 visible faces), we position these cubes with the 4, 5 and 6 all visible (this is possible since the faces with the 4, 5 and 6 share a vertex) and the 1, 2 and 3 hidden.

There are 12 of these cubes, so the total of the numbers visible on these cubes is  $12 \times (4 + 5 + 6) = 180$ .

For the cubes on the sides (that is, not at the corner) of each layer, we position the cubes with the 5 and 6 visible (this is possible since the faces with 5 and 6 share an edge) and the 1, 2, 3, and 4 hidden.

There are  $4 + 12 + 20 = 36$  of these cubes, so the total of the numbers visible on these cubes is  $36 \times (5 + 6) = 396$ .

Therefore, the overall largest possible total is  $20 + 180 + 396 = 596$ .

*Solution 2*

To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three or five numbers are visible, depending on its position.

As in Solution 2 to (b), view the pyramid from the top. Position the cubes so that each top faces which is visible is 6 (for a total of  $49 \times 6 = 294$ ).

Consider next the top cube. It has only one hidden face, which will be the 1, since the 6 is on top. (This does maximize the sum of the visible faces.) This adds  $2 + 3 + 4 + 5 = 14$  to the total of the visible faces thus far.

Consider lastly the faces visible on the sides (a total of  $4(3 + 5 + 7) = 60$  faces). To maximize the sum of the numbers on these faces, we would like to make them all 5 (since we have already used the 6s). This would give a total of  $5 \times 60 = 300$ . However, there are 12 corner cubes to which we have assigned two 5s, so we must change one of the 5s on each to a 4, decreasing the total by 12. (This is possible, since the 4, 5 and 6 meet at a vertex on each cube.)

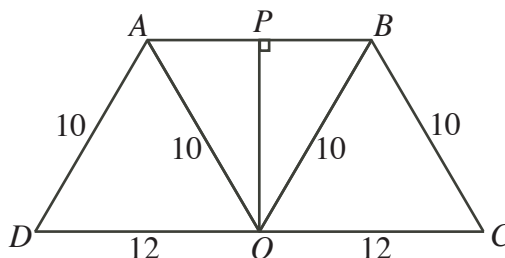
Therefore, the overall largest possible total is  $294 + 14 + 300 - 12 = 596$ .

3. (a) Since  $\triangle AOB$  is isosceles with  $AO = OB$  and  $OP$  is perpendicular to  $AB$ , then  $P$  is the midpoint of  $AB$ , so  $AP = PB = \frac{1}{2}AB = \frac{1}{2}(12) = 6$ .

By the Pythagorean Theorem,  $OP = \sqrt{AO^2 - AP^2} = \sqrt{10^2 - 6^2} = \sqrt{64} = 8$ .

(b) *Solution 1*

Trapezoid  $ABCD$  is formed from three congruent triangles, so its area is three times the area of one of these triangles.



Each triangle has a base of length 12 and a height of length 8 (from (a), since  $OP$  is one of these heights), so has area  $\frac{1}{2}(12)(8) = 48$ .

Therefore, the area of the trapezoid is  $3 \times 48 = 144$ .

*Solution 2*

Since  $ABCD$  is a trapezoid with height of length 8 ( $OP$  is the height of  $ABCD$ ) and parallel sides ( $AB$  and  $DC$ ) of length 12 and 24, then its area is

$$\frac{1}{2} \times \text{Height} \times \text{Sum of parallel sides} = \frac{1}{2}(8)(12 + 24) = 144$$

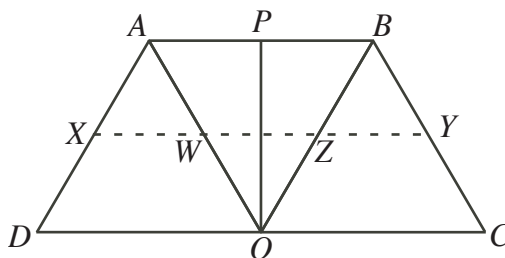
(c) *Solution 1*

Since  $XY$  cuts  $AD$  and  $BC$  each in half, then it also cuts the height  $PO$  in half:

Since  $XY$  is parallel to  $AB$  and  $DC$  and cuts each of  $AD$  and  $BC$  at its midpoint, then it must cut each of  $AO$  and  $BO$  at its midpoint ( $W$  and  $Z$ , respectively). Therefore,  $\triangle WZO$  is similar to  $\triangle ABO$  and its side lengths are half of those of  $\triangle ABO$ .

Thus, its height is half of  $PO$ .

In a similar way, the dimensions of  $\triangle AXW$  and  $\triangle BYZ$  are half of those of  $\triangle ADO$  and  $\triangle BCO$ .



Thus, each of the two smaller trapezoids has height 4.

Also, since  $XW = \frac{1}{2}DO$ ,  $WZ = \frac{1}{2}AB$  and  $ZY = \frac{1}{2}OC$ , then  $XY = 3(6) = 18$ .

Using the formula for the area of a trapezoid from (b), the area of trapezoid  $ABYX$  is  $\frac{1}{2}(4)(12 + 18) = 60$  and the area of trapezoid  $XYCD$  is  $\frac{1}{2}(4)(18 + 24) = 84$ .

Thus, the ratio of their areas is  $60 : 84 = 5 : 7$ .

*Solution 2*

Since  $XY$  cuts  $AD$  and  $BC$  each in half, then it also cuts the height  $PO$  in half.

Thus, each of the two smaller trapezoids has height 4.

Next, we find the length of  $XY$ .

From (b), we know how to compute the area of a trapezoid and we know that the sum of the areas of trapezoids  $ABYX$  and  $XYCD$  must equal that of trapezoid  $ABCD$ .

Therefore,

$$\begin{aligned}\frac{1}{2}(4)(AB + XY) + \frac{1}{2}(4)(XY + DC) &= 144 \\ 2(12 + XY) + 2(XY + 24) &= 144 \\ 4(XY) &= 72 \\ XY &= 18\end{aligned}$$

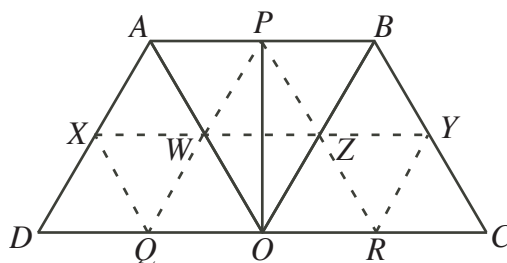
Thus, the area of trapezoid  $ABYX$  is  $\frac{1}{2}(4)(12+18) = 60$  and the area of trapezoid  $XYCD$  is  $\frac{1}{2}(4)(18 + 24) = 84$ .

Thus, the ratio of their areas is  $60 : 84 = 5 : 7$ .

### *Solution 3*

Let  $Q$  and  $R$  be the midpoints of  $DO$  and  $OC$ , respectively, and  $W$  and  $Z$  the points where  $XY$  crosses  $AO$  and  $BO$ , respectively.

Join  $X$  and  $W$  to  $Q$ ,  $Z$  and  $Y$  to  $R$ , and  $W$  and  $Z$  to  $P$ .



We have now divided trapezoid  $ABCD$  into 12 congruent triangles:

To show this, we consider the partitioning of  $\triangle AOD$  into four small triangles.

Because  $X$ ,  $W$  and  $Q$  are the midpoints of  $AD$ ,  $AO$  and  $OD$ , respectively, then  $XW$ ,  $WQ$  and  $QX$  are parallel to  $DO$ ,  $AD$  and  $OA$ , respectively.

Therefore, triangles  $AXW$ ,  $XDQ$ ,  $WQO$ , and  $QWX$  are congruent, because each has sides of length 5, 5 and 6.

The same is true for the partitioning of  $\triangle ABO$  and  $\triangle BOC$ .

Since  $ABYX$  is made up of 5 of these triangles and  $XYCD$  is made up of 7 of these triangles and each of these 12 triangles has equal area, then the ratio of the area of  $ABYX$  to the area of  $XYCD$  is  $5 : 7$ .

### 4. (a) *Solution 1*

There are 100 integers from 1 to 100.

Of these, 10 end with the digit 7: 7, 17, 27, 37, 47, 57, 67, 77, 87, 97.

Also, 10 begin with the digit 7: 70, 71, 72, 73, 74, 75, 76, 77, 78, 79.

But 77 is included in both lists, so we need to be careful not to count it twice. Therefore, we need to subtract  $10 + 10 - 1 = 19$  of the 100 integers.

Therefore, the number of these integers which do not contain 7 is  $100 - 19 = 81$ .

### *Solution 2*

The integer 100 does not contain the digit 7, nor does the integer 0, so we replace 100 with 0 in the list and count the number of integers from 0 to 99 which do not contain the

digit 7.

Each of these integers can be written as a two-digit integer (where we allow the first digit to be 0): 00, 01, 02, ..., 98, 99.

Since we would like all integers not containing the digit 7, there are 9 possibilities for the first digit (0 through 9, excluding 7) and for each of these possibilities, there are 9 possibilities for the second digit.

Therefore, there are  $9 \times 9 = 81$  integers in this range which do not contain the digit 7.

(b) *Solution 1*

From part (a), there are 81 integers not containing the digit 7 between 1 and 100.

Similarly, there will be 81 such integers in each of the ranges 101 to 200, 201 to 300, 301 to 400, 401 to 500, and 501 to 600. (We can conclude this since the first digit is not a 7, and the tens and units digits follow the same pattern as in (a).)

From 601 to 700, there will be 80 such integers (since 700 includes 7).

From 701 to 800, there is only 1 such integer (namely, 800 – all others contain a 7).

In each of the intervals 801 to 900, 901 to 1000, 1001 to 1100, 1101 to 1200, 1201 to 1300, 1301 to 1400, 1401 to 1500, and 1501 to 1600, there are 81 such integers.

From 1601 to 1700, there are 80; from 1701 to 1800, there is 1; from 1801 to 1900 and 1901 to 2000, there are 81.

Therefore, the total number of such integers is  $16(81) + 2(80) + 2(1) = 18(81) = 1458$ .

*Solution 2*

The integer 2000 does not contain the digit 7, nor does the integer 0, so we can replace 2000 with 0 in the list and count the number of integers from 0 to 1999 which do not contain the digit 7.

Each of these integers can be written as a four-digit integer  $\underline{a}\underline{b}\underline{c}\underline{d}$ , where the integer is allowed to begin with one or more zeros, with  $a$  allowed to be 0 or 1, and each of  $b$ ,  $c$  and  $d$  allowed to be 0, 1, 2, 3, 4, 5, 6, 8, or 9.

Thus, there are 2 possibilities for the first digit; for each of these choices, there are 9 choices for the second digit; for each of these choices, there are 9 choices for the third digit; for each of these choices, there are 9 choices for the fourth digit. The total number of such integers is thus  $2 \times 9 \times 9 \times 9 = 1458$ .

(c) *Solution 1*

In this solution, we will repeatedly use the fact that the sum of the integers from 1 to  $n$  is  $\frac{1}{2}n(n+1)$ .

Consider first the integers from 1 to 100.

The sum of these integers is  $\frac{1}{2}(100)(101) = 5050$ .

The integers in this set which do contain the digit 7 are 7, 17, 27, 37, 47, 57, 67, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 87, 97, whose sum is 1188.

Therefore, the sum of the integers from 1 to 100 which do not contain the digit 7 is  $5050 - 1188 = 3862$ .

There are 81 numbers from 101 to 200 not containing the digit 7 as well. Each of these is 100 more than a corresponding number between 1 and 100 which does not contain the digit 7, so the sum of these 81 numbers is  $3862 + 81(100)$ .

We can use this approach to determine the sum of the appropriate numbers in each range of 100, as shown in the table:

Range	Number of integers not containing 7	Sum
1 to 100	81	3862
101 to 200	81	$3862 + 81(100)$
201 to 300	81	$3862 + 81(200)$
301 to 400	81	$3862 + 81(300)$
401 to 500	81	$3862 + 81(400)$
501 to 600	81	$3862 + 81(500)$
601 to 700	80	$3862 + 81(600) - 700$
701 to 800	1	800
801 to 900	81	$3862 + 81(800)$
901 to 1000	81	$3862 + 81(900)$
1001 to 1100	81	$3862 + 81(1000)$
1101 to 1200	81	$3862 + 81(1100)$
1201 to 1300	81	$3862 + 81(1200)$
1301 to 1400	81	$3862 + 81(1300)$
1401 to 1500	81	$3862 + 81(1400)$
1501 to 1600	81	$3862 + 81(1500)$
1601 to 1700	80	$3862 + 81(1600) - 1700$
1701 to 1800	1	1800
1801 to 1900	81	$3862 + 81(1800)$
1901 to 2000	81	$3862 + 81(1900)$
2001 to 2006	6	$2001 + 2002 + 2003 + 2004 + 2005 + 2006$

Therefore, the overall sum is

$$\begin{aligned}
 & 18(3862) \\
 & \quad + 81(100)(1 + 2 + 3 + 4 + 5 + 6 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 18 + 19) \\
 & \quad - 700 + 800 - 1700 + 1800 + (2001 + 2002 + 2003 + 2004 + 2005 + 2006) \\
 & = 69516 + 8100(166) + 200 + 12021 \\
 & = 1426337
 \end{aligned}$$

### *Solution 2*

Consider first the integers from 000 to 999 that do not contain the digit 7. (We can include 000 in this list as it will not affect the sum.)

Since each of the three digits has 9 possible values, there are  $9 \times 9 \times 9 = 729$  such integers. If we fix any specific digit in any of the three positions, there will be exactly 81 integers with that digit in that position, as there are 9 possibilities for each of the remaining digits. (For example, there are 81 such integers ending in 0, 81 ending in 1, etc.)

We sum these integers by first summing the units digits, then summing the tens digits, and then summing the hundreds digits.

Since each of the 9 possible units digits occurs 81 times, the sum of the units digits column is

$$81(0) + 81(1) + 81(2) + 81(3) + 81(4) + 81(5) + 81(6) + 81(8) + 81(9) = 81(38)$$

Since each of the 9 possible tens digits occurs 81 times, the sum of the tens digits column is  $81(0 + 10 + 20 + 30 + 40 + 50 + 60 + 80 + 90) = 81(380)$ .

Similarly, the sum of the hundreds digits column is  $81(3800)$ .

Thus, the sum of the integers from 0 to 999 that do not contain the digit 7 is

$$81(38) + 81(380) + 81(3800) = 81(38)(1 + 10 + 100) = 81(38)(111) = 341\,658$$

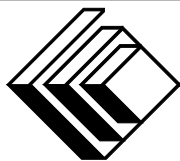
Each of the 729 integers from 1000 to 1999 which do not contain 7 is 1000 more than such an integer between 0 and 999. (There are again 729 of these integers as the first digit is fixed at 1, and each of the remaining three digits has 9 possible values.)

Thus, the sum of these integers from 1000 to 1999 is equal to the sum of the corresponding ones from 0 to 999 plus  $729(1000)$ , or  $729\,000 + 341\,658 = 1\,070\,658$ .

Each of the integers from 2000 to 2006 does not contain a 7, and their sum is

$$2000 + 2001 + 2002 + 2003 + 2004 + 2005 + 2006 = 7(2003) = 14\,021.$$

Therefore, the overall sum is  $341\,658 + 1\,070\,658 + 14\,021 = 1\,426\,337$ .



# **Canadian Mathematics Competition**

*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

## ***2005 Fryer Contest***

**Wednesday, April 20, 2005**

*Solutions*

1. (a) The area of a circle with radius  $r$  is  $\pi r^2$ .  
 So the area of the larger circle is  $\pi(10^2) = 100\pi$  and the area of the smaller circle is  $\pi(6^2) = 36\pi$ .  
 The area of the ring between the two circles is the difference of these two areas.  
 Therefore, the area of the ring is  $100\pi - 36\pi = 64\pi$ .
  - (b) The area of the inside circle (region  $X$ ) is  $\pi(4^2) = 16\pi$ .  
 Using a similar technique to part (a), the area of the middle ring (region  $Y$ ) is  $\pi(6^2) - \pi(4^2) = 36\pi - 16\pi = 20\pi$ .  
 Also, the area of the outer ring (region  $Z$ ) is  $\pi(7^2) - \pi(6^2) = 49\pi - 36\pi = 13\pi$ .  
 Therefore, region  $Y$  has the largest area.
  - (c) The area of the ring between the two largest circles is  $\pi(13^2) - \pi(12^2) = 169\pi - 144\pi = 25\pi$ .  
 Suppose that the radius of the smallest circle is  $r$ .  
 Thus, the area of the smallest circle is  $\pi r^2$ .  
 Since the area of the smallest circle is equal to the area of the ring between the two largest circles, then  $\pi r^2 = 25\pi$  so  $r^2 = 25$  and so  $r = 5$  since  $r > 0$ .  
 Therefore, the radius of the smallest circle is 5.
2. (a) Anh goes first and puts an A in the middle box.  
 According to the rules, Bharati must put her initial in one or two boxes which are next to each other. So Bharati can only put a B in one of the two empty boxes, since they are not next to each other.



This leaves one empty box, and Anh wins by putting an A in this box.  
 Therefore, Anh will win no matter what Bharati does.

(b) *Solution 1*

Suppose Anh puts an A in the rightmost box.



Then Bharati can only put a B in one of the two empty boxes, since they are not next to each other.

This leaves one empty box, and Anh will win by putting an A in this empty box.  
 Therefore, Anh is guaranteed to win if he puts an A in the rightmost box.

*Solution 2*

Suppose Anh puts an A in the second box from the right end.



Then Bharati can only put a B in one of the two empty boxes, since they are not next to each other.

This leaves one empty box, and Anh will win by putting an A in this empty box.

Therefore, Anh is guaranteed to win if he puts an A in the second box from the right.



*Solution 3*

Suppose Anh puts an A in the rightmost box.

B		A		A
---	--	---	--	---

We can now remove the leftmost and rightmost boxes, leaving us with three boxes with an A in the middle and it being Bharati's turn.

This leaves us in the situation of part (a), so Anh will be guaranteed to win.

(c) Possibility #1

Suppose Bharati puts a B in the two rightmost boxes.

	A		B	B
--	---	--	---	---

Then Anh can only put an A in one of the two empty boxes, since they are not next to each other.

This leaves one empty box, and Bharati wins by putting a B in this empty box.

Therefore, Bharati is guaranteed to win in this case.

Possibility #2

Suppose Bharati puts a B in the middle box and the box to its right.

	A	B	B	
--	---	---	---	--

Then Anh can only put an A in one of the two empty boxes, since they are not next to each other.

This leaves one empty box, and Bharati wins by putting a B in this empty box.

Therefore, Bharati is guaranteed to win in this case.

These are the two possible moves that Bharati can make next to guarantee she wins.

(The only other four possible moves are putting a B in any one of the empty boxes. Why do each of these four moves allow A to win?)

3. (a) *Solution 1*

Since the side lengths of a Nakamoto triangle are in the ratio 3 : 4 : 5, then the side lengths must be the products 3, 4 and 5 with the same integer.

For one of the sides to have a length of 28, it must be the multiple of 4, since 28 is not a multiple of 3 or 5.

Since  $28 = 4 \times 7$ , then the three side lengths must be  $3 \times 7 = 21$ , 28 and  $5 \times 7 = 35$ .

*Solution 2*

Since the side lengths of a Nakamoto triangle are three integers in the ratio 3 : 4 : 5, then the side lengths are  $3x$ ,  $4x$  and  $5x$ , for some positive integer  $x$ .

Since one of these three sides is 28, then we must have  $4x = 28$  (or  $x = 7$ ), because 28 is not a multiple of 3 or 5.

Therefore, the side lengths are  $3 \times 7 = 21$ , 28 and  $5 \times 7 = 35$ .

(b) *Solution 1*

Since the side lengths of a Nakamoto triangle are in the ratio  $3 : 4 : 5$ , then the side lengths must be the products 3, 4 and 5 with the same integer.

The Nakamoto triangle with the shortest sides is that with side lengths 3, 4 and 5, which has a perimeter of  $3 + 4 + 5 = 12$ .

Since we are given the Nakamoto triangle of perimeter 96 and  $96 = 12 \times 8$ , then we must multiply each of the side lengths of the triangle with sides 3, 4 and 5 by 8 to get a perimeter of 96.

Therefore, the side lengths are  $3 \times 8 = 24$ ,  $4 \times 8 = 32$  and  $5 \times 8 = 40$ .

*Solution 2*

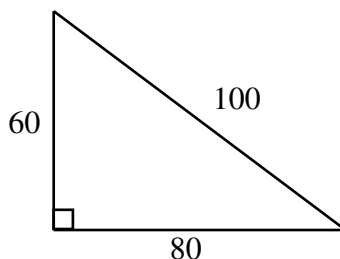
Since the side lengths of a Nakamoto triangle are three integers in the ratio  $3 : 4 : 5$ , then the side lengths are  $3x$ ,  $4x$  and  $5x$ , for some integer  $x$ .

Since the perimeter is 96, then  $3x + 4x + 5x = 96$  or  $12x = 96$  or  $x = 8$ .

Therefore, the side lengths are  $3 \times 8 = 24$ ,  $4 \times 8 = 32$  and  $5 \times 8 = 40$ .

- (c) Since 60 is divisible by 3, then there is a Nakamoto triangle with a side length of 60 in the “3” position. Since  $60 = 3 \times 20$ , then the side lengths of this triangle are 60,  $4 \times 20 = 80$  and  $5 \times 20 = 100$ .

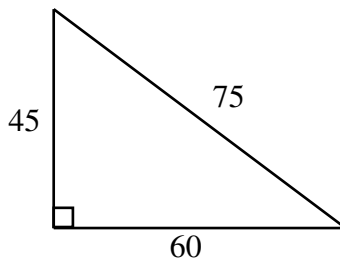
Since the ratio of the side lengths are  $3 : 4 : 5$ , then this triangle is right-angled. In fact, since  $60^2 + 80^2 = 100^2$ , then by the Pythagorean Theorem, the right angle is between the sides of lengths 60 and 80. (Alternatively, we could have said that since the triangle is right-angled and 100 is the longest side, then 100 must be the hypotenuse, so the right angle is between the sides of lengths 60 and 80.)



Therefore, the area of this triangle is  $\frac{1}{2}bh = \frac{1}{2}(60)(80) = 2400$ .

Since 60 is divisible by 4, then there is a Nakamoto triangle with a side length of 60 in the “4” position. Since  $60 = 4 \times 15$ , then the side lengths of this triangle are  $3 \times 15 = 45$ , 60 and  $5 \times 15 = 75$ .

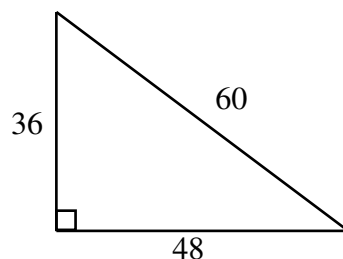
Since the triangle is right-angled and 75 is the longest side, then 75 must be the hypotenuse, so the right angle is between the sides of lengths 45 and 60.



Therefore, the area of this triangle is  $\frac{1}{2}bh = \frac{1}{2}(45)(60) = 1350$ .

Since 60 is divisible by 5, then there is a Nakamoto triangle with a side length of 60 in the “5” position. Since  $60 = 5 \times 12$ , then the side lengths of this triangle are  $3 \times 12 = 36$ ,  $4 \times 12 = 48$  and 60.

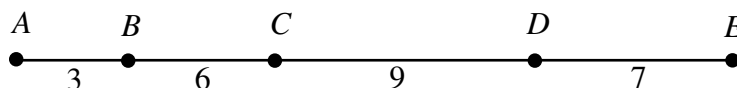
Since the triangle is right-angled and 60 is the longest side, then 60 must be the hypotenuse, so the right angle is between the sides of lengths 36 and 48.



Therefore, the area of this triangle is  $\frac{1}{2}bh = \frac{1}{2}(36)(48) = 864$ .

Thus, the possible areas of a Nakamoto triangle with a side length of 60 are 2400, 1350 and 864.

4. (a)



Here,  $AB = 3$ ,  $AC = 9$ ,  $AD = 18$ ,  $AE = 25$ ,  $BC = 6$ ,  $BD = 15$ ,  $BE = 22$ ,  $CD = 9$ ,  $CE = 16$ ,  $DE = 7$ .

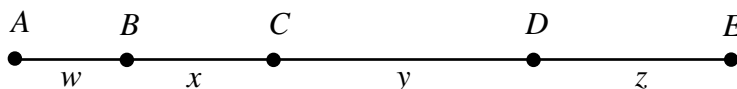
Therefore, the super-sum of  $AE$  is  $3 + 9 + 18 + 25 + 6 + 15 + 22 + 9 + 16 + 7 = 130$ .

(b) *Solution 1*

If the sub-segments have lengths 1 to 10, then the super-sum is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$$

Suppose that the basic sub-segments have lengths  $AB = w$ ,  $BC = x$ ,  $CD = y$  and  $DE = z$ .



Since all of the sub-segments have integer lengths, then all of the basic sub-segments must have integer length.

What are the lengths of the sub-segments in terms of  $w$ ,  $x$ ,  $y$  and  $z$ ?

We have  $AB = w$ ,  $AC = w + x$ ,  $AD = w + x + y$ ,  $AE = w + x + y + z$ ,  $BC = x$ ,  $BD = x + y$ ,  $BE = x + y + z$ ,  $CD = y$ ,  $CE = y + z$ , and  $DE = z$ .

Then the super-sum of  $AE$  is

$$w + (w + x) + (w + x + y) + (w + x + y + z) + x + (x + y) + (x + y + z) + y + (y + z) + z = 4w + 6x + 6y + 4z$$

Therefore, the super-sum is always an even number since

$$4w + 6x + 6y + 4z = 2(2w + 3x + 3y + 2z).$$

This tells us that the super-sum cannot be 55.

Thus, it is impossible for the line segment  $AE$  to have sub-segments of lengths 1 to 10.

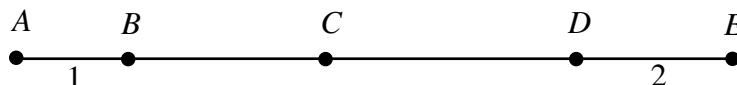
*Solution 2*

Since the longest sub-segment of  $AE$  is  $AE$ , then we must have  $AE = 10$ .

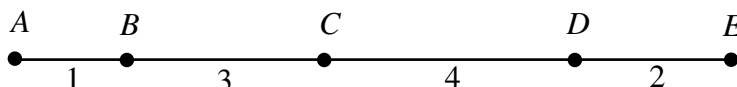
Since the sum of the lengths of the basic sub-segments is the length of  $AE$  and the basic sub-segments are included in the list of sub-segments, then the lengths of the sub-segments must be 1, 2, 3, and 4.

In order to get a sub-segment of length 9, we must have the 2, 3 and 4 adjacent, so the 1 must be on one end.

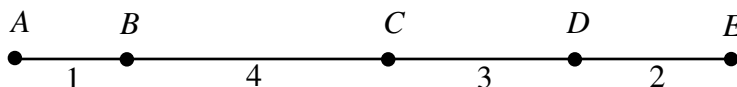
In order to get a sub-segment of length 8, we must have the 1, 3 and 4 adjacent, so the 2 must be on the other end.



This leaves us with two possibilities:



and



In the first possibility, there is no sub-segment of length 5 and two sub-segments of length 4 ( $AC$  and  $CD$ ).

In the second possibility, there is no sub-segment of length 6 and two sub-segments of length 5 ( $AC$  and  $CE$ ).

Having checked all possibilities, we see that it is impossible for the line segment  $AE$  to have sub-segments of lengths 1 to 10.

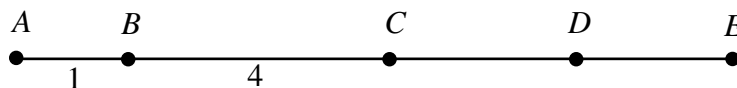
### *Solution 3*

Since the longest sub-segment of  $AE$  is  $AE$ , then we must have  $AE = 10$ .

Since the sum of the lengths of the basic sub-segments is the length of  $AE$  and the basic sub-segments are included in the list of sub-segments, then the lengths of the sub-segments must be 1, 2, 3, and 4.

In order to avoid repeating lengths of sub-segments, we cannot have the 2 or 3 next to the 1.

Therefore, we must have the 1 on one end with the 4 next to it.



But this means that the 2 and the 3 are next to each other, so we have two sub-segments of length 5, which is impossible.

Therefore, it is impossible for the line segment  $AE$  to have sub-segments of lengths 1 to 10.

### (c) *Solution 1*

What happens to the super-sum when we first add a basic sub-segment  $JK$  of length  $\frac{1}{10}$ ? The sub-segments of  $AK$  include all of the sub-segments of  $AJ$  (which have a combined length of 45), and then some additional sub-segments.

The additional sub-segments are ones which are sub-segments of  $AK$  but not of  $AJ$ . In other words, they are sub-segments which include the basic sub-segment  $JK$ . These are the sub-segments  $JK$ ,  $IK$ ,  $HK$ , and so on, all the way to  $BK$  and  $AK$ .

What are the lengths of these additional sub-segments?

$$JK = \frac{1}{10}$$

$$IK = IJ + JK = \frac{1}{9} + \frac{1}{10}$$

$$HK = HI + IJ + JK = \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$

$\vdots$

$$AK = AB + BC + \cdots + IJ + JK = 1 + \frac{1}{2} + \cdots + \frac{1}{9} + \frac{1}{10}$$

All 10 of these additional sub-segments contain the basic sub-segment  $JK$ , 9 contain  $IJ$ , 8 contain  $HI$ , and so on, with 2 containing  $BC$  and 1 containing  $AB$ .

Therefore, the super-sum of  $AK$  is

$$45 + 10 \left( \frac{1}{10} \right) + 9 \left( \frac{1}{9} \right) + \cdots + 2 \left( \frac{1}{2} \right) + 1(1) = 45 + 10 = 55$$

What happens when we add a basic sub-segment  $KL$  of length  $\frac{1}{11}$ ?

As above, this will add to the sub-segments already contained in  $AK$  an additional 11 sub-segments containing  $KL$ , 10 containing  $JK$ , and so on, with 1 containing  $AB$ .

Thus, the super-sum of  $AL$  will be

$$55 + 11 \left( \frac{1}{11} \right) + 10 \left( \frac{1}{10} \right) + \cdots + 1(1) = 55 + 11 = 66$$

Similarly, as we add the final four basic sub-segments to obtain  $AP$ , we will add 12, then 13, then 14, then 15 to the super-sum.

Therefore, the super-sum of  $AP$  is  $66 + 12 + 13 + 14 + 15 = 120$ .

### *Solution 2*

Each sub-segment of  $AP$  is made up of a number of neighbouring basic sub-segments. We can determine the super-sum of  $AP$  by counting the number of sub-segments in which each basic sub-segment occurs, and so determine the contribution of each basic sub-segment to the super-sum.

The basic sub-segment  $AB$  occurs in sub-segments  $AB, AC, \dots, AP$ , or 15 sub-segments in total.

By symmetry,  $OP$  will also occur in 15 sub-segments.

The basic sub-segment  $BC$  occurs in sub-segments  $BC, BD, \dots, BP$  (14 in total) and  $AC, AD, \dots, AP$  (another 14 in total), for a total of 28 sub-segments.

By symmetry,  $NO$  will also occur in 28 sub-segments.

Is there a better way to count these total number of sub-segments which contain a given basic sub-segment without having to list them all?

Consider  $BC$  again.

A sub-segment containing  $BC$  must have left endpoint  $A$  or  $B$  (2 possibilities) and right endpoint from  $C$  to  $P$  (14 possibilities). Each combination of left endpoint and right endpoint is possible, so there are  $2 \times 14 = 28$  possible sub-segments. (The same argument applies for  $NO$  with 14 possible left endpoints and 2 possible right endpoints.)

We make a table containing each of the remaining basic sub-segments, the number of possible left endpoints for a sub-segment containing it, the number of possible right endpoints, and the total number of sub-segments:

Basic sub-segment	# Possible L endpoints	# Possible R endpoints	Total # of sub-segments
$CD$	3	13	39
$DE$	4	12	48
$EF$	5	11	55
$FG$	6	10	60
$GH$	7	9	63
$HI$	8	8	64
$IJ$	9	7	63
$JK$	10	6	60
$KL$	11	5	55
$LM$	12	4	48
$MN$	13	3	39

Therefore, the super-sum is

$$15(1) + 28(\frac{1}{2}) + 39(\frac{1}{3}) + 48(\frac{1}{4}) + 55(\frac{1}{5}) + 60(\frac{1}{6}) + 63(\frac{1}{7}) + 64(\frac{1}{8}) + 63(\frac{1}{9}) + 60(\frac{1}{10}) + 55(\frac{1}{11}) + 48(\frac{1}{12}) + 39(\frac{1}{13}) + 28(\frac{1}{14}) + 15(\frac{1}{15})$$

or

$$15 + 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

Therefore, the super-sum of  $AP$  is 120.

# Canadian Mathematics Competition

An activity of The Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario

## *2004 Solutions* *Fryer Contest* (Grade 9)

1. (a) Since  $\xrightarrow{R}$  indicates that we take a reciprocal, then  $3 \xrightarrow{R} \frac{1}{3}$ .  
 Since  $\xrightarrow{A}$  indicates to take a reciprocal, then  $\frac{1}{3} \xrightarrow{A} \frac{1}{\frac{1}{3}} + 1 = \frac{4}{3}$ .  
 Continuing,  $\frac{4}{3} \xrightarrow{R} \frac{3}{4}$ ,  $\frac{3}{4} \xrightarrow{A} \frac{3}{4} + 1 = \frac{7}{4}$  and  $\frac{7}{4} \xrightarrow{R} \frac{4}{7}$ .  
 In summary,  $3 \xrightarrow{R} \frac{1}{3} \xrightarrow{A} \frac{4}{3} \xrightarrow{R} \frac{3}{4} \xrightarrow{A} \frac{7}{4} \xrightarrow{R} \frac{4}{7}$ .

(b) As in (a),

$$\begin{aligned} x &\xrightarrow{R} \frac{1}{x} \\ \frac{1}{x} &\xrightarrow{A} \frac{1}{x} + 1 = \frac{1}{x} + \frac{x}{x} = \frac{1+x}{x} \\ \frac{1+x}{x} &\xrightarrow{R} \frac{x}{1+x} \\ \frac{x}{1+x} &\xrightarrow{A} \frac{x}{1+x} + 1 = \frac{x}{1+x} + \frac{1+x}{1+x} = \frac{1+2x}{1+x} \\ \frac{1+2x}{1+x} &\xrightarrow{R} \frac{1+x}{1+2x} \end{aligned}$$

In summary,

$$x \xrightarrow{R} \frac{1}{x} \xrightarrow{A} \frac{1+x}{x} \xrightarrow{R} \frac{x}{1+x} \xrightarrow{A} \frac{1+2x}{1+x} \xrightarrow{R} \frac{1+x}{1+2x}$$

(Since we started by taking the reciprocal of  $x$ , we are assuming that  $x$  is not equal to 0.)

(c) *Solution 1*

By inspection, we observe that if  $\frac{1+x}{1+2x}$  equals  $\frac{14}{27}$ , then  $x+1=14$  or  $x=13$ . We check to see that the value  $x=13$  makes the denominator equal to 27, which it does.

(We note that we have to be careful to *check* that our answer works here, because it will not work for instance if  $\frac{1+y}{1+3y} = \frac{2}{5}$ , where the solution is  $y=3$ .)

*Solution 2*

From (b), if the operations begin with  $x$ , the final result is  $\frac{1+x}{1+2x}$ .

Since we want the final result to be  $\frac{14}{27}$ , then we set  $\frac{1+x}{1+2x} = \frac{14}{27}$  and solve for  $x$  to

determine the original input:

$$\frac{1+x}{1+2x} = \frac{14}{27}$$

$$27(1+x) = 14(1+2x)$$

$$27 + 27x = 14 + 28x$$

$$13 = x$$

Therefore, the input should be 13 to obtain a final result of  $\frac{14}{27}$ . (We can check this by actually doing the 5 operations starting with 13.)



*Solution 3*

We start with  $\frac{14}{27}$  and work backwards through the operations, using the “inverse” of each of the steps.

Since the fifth operation was taking a reciprocal to get  $\frac{14}{27}$ , then the number at the previous step must have been  $\frac{27}{14}$ .

Since the fourth operation was adding 1 to get  $\frac{27}{14}$ , then the number at the previous step must have been  $\frac{27}{14} - 1 = \frac{13}{14}$ .

Since the third operation was taking a reciprocal to get  $\frac{13}{14}$ , then the number at the previous step must have been  $\frac{14}{13}$ .

Since the second operation was adding 1 to get  $\frac{14}{13}$ , then the number at the previous step must have been  $\frac{14}{13} - 1 = \frac{1}{13}$ .

Since the first operation was taking a reciprocal to get  $\frac{1}{13}$ , then the original input must have been 13.

2. (a) Since at least one of each type of prize is given out, then these four prizes account for  $\$5 + \$25 + \$125 + \$625 = \$780$ .

Since there are five prizes given out which total \$905, then the fifth prize must have a value of  $\$905 - \$780 = \$125$ .

Thus, the Fryer Foundation gives out one \$5 prize, one \$25 prize, two \$125 prizes, and one \$625 prize.

- (b) As in (a), giving out one of each type of prize accounts for \$780.

The fifth prize could be a \$5 prize for a total of  $\$780 + \$5 = \$785$ .

The fifth prize could be a \$25 prize for a total of  $\$780 + \$25 = \$805$ .

The fifth prize could be a \$625 prize for a total of  $\$780 + \$625 = \$1405$ .

(We already added an extra \$125 prize in (a).)

- (c) *Solution 1*

Since at least one of each type of prize is given out, this accounts for \$780. So we must figure out how to distribute the remaining  $\$880 - \$780 = \$100$  using at most 5 of each type of prize. We cannot use any \$125 or \$625 prizes, since these are each greater than the remaining amount.

We could use four additional \$25 prizes to make up the \$100.

Could we use fewer than four \$25 prizes? If we use three additional \$25 prizes, this accounts for \$75, which leaves \$25 remaining in \$5 prizes, which can be done by using five additional \$5 prizes.

Could we use fewer than three \$25 prizes? If so, then we would need to make at least \$50 with \$5 prizes, for which we need at least ten such prizes. But we can use at most six \$5 prizes in total, so this is impossible.

Therefore, the two ways of giving out \$880 in prizes under the given conditions are:

- i) one \$625 prize, one \$125 prize, five \$25 prizes, one \$5 prize
- ii) one \$625 prize, one \$125 prize, four \$25 prizes, six \$5 prizes

We can check by addition that each of these totals \$880.

### *Solution 2*

We know that the possible total values using at least one of each type of prize and exactly five prizes are \$785, \$805, \$905 and \$1405.

We try starting with \$785 and \$805 to get to \$880. (Since \$905 and \$1405 are already larger than \$880, we do not need to try these.)

Starting with \$785, we need to give out an additional \$95 to get to \$880. Using three \$25 prizes accounts for \$75, leaving \$20 to be split among four \$5 prizes. (Using fewer than three \$25 prizes will mean we need more than six \$5 prizes in total.) So in this way, we need one \$625 prize, one \$125 prize, four \$25 prizes, and six \$5 prizes (since there were already two included in the \$785).

Starting with \$805, we need to give out an additional \$75 to get to \$880. Using three \$25 prizes will accomplish this, for a total of one \$625 prize, one \$125 prize, five \$25 prizes, and one \$5 prize. We could also use two \$25 prizes and five \$5 prizes to make up the \$75, for a total of one \$625 prize, one \$125 prize, four \$25 prizes, and six \$5 prizes (which is the same as we obtained above starting with \$785). If we use fewer than two additional \$25 prizes, we would need too many \$5 prizes.

Therefore, the two ways of giving out \$880 in prizes under the given conditions are:

- i) one \$625 prize, one \$125 prize, five \$25 prizes, one \$5 prize
- ii) one \$625 prize, one \$125 prize, four \$25 prizes, six \$5 prizes

We can check by addition that each of these totals \$880.

3. (a) If Bob places a 3, then the total of the two numbers so far is 8, so Avril should place a 7 to bring the total up to 15.

Since Bob can place a 3 in any of the eight empty circles, Avril should place a 7 in the circle directly opposite the one in which Bob places the 3. This allows Avril to win on her next turn.

- (b) As in (a), Bob can place any of the numbers 1, 2, 3, 4, 6, 7, 8, 9 in any of the eight empty circles. On her next turn, Avril should place a disc in the circle directly opposite the one in which Bob put his number. What number should Avril use? Avril should place the number that brings the total up to 15, as shown below:

<u>Bob's First Turn</u>	<u>Total so far</u>	<u>Avril's Second Turn</u>
1	6	9
2	7	8
3	8	7
4	9	6
6	11	4
7	12	3
8	13	2
9	14	1

Since each of these possibilities is available to Avril on her second turn (since 5 is not in the list and none is equal to Bob's number), then she can always win on her second turn.

- (c) Bob can place any of the numbers 1, 2, 4, 5, 7, 8 in any of the six empty circles. We can pair these numbers up so that the sum of the two numbers in the pair plus 6 is equal to 15:

1 and 8; 2 and 7; 4 and 5

So when Bob uses one of these numbers, Avril can use the other number from the pair, place it directly opposite the one that Bob entered, and the total of the three numbers on this line through the centre will be 15, so Avril will win the game.

4. (a) The 10<sup>th</sup> triangular number is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

The 24<sup>th</sup> triangular number is  $1 + 2 + 3 + \cdots + 23 + 24$ . We could add this up by hand or on a calculator to obtain 300. Instead, we could notice that if we pair up the numbers starting with the first and last, then the second and second last, and so on, we obtain:

$$(1 + 24) + (2 + 23) + (3 + 22) + \cdots + (11 + 14) + (12 + 13)$$

that is, 12 pairs, each adding to 25, giving a total of  $12 \times 25 = 300$ . (This pairing is the basis for a general method of finding a formula for  $1 + 2 + \cdots + n$ .)

- (b) *Solution 1*

Let the three consecutive triangle numbers be

$$1 + 2 + 3 + \cdots + (n-1) + n \quad (1),$$

$$1 + 2 + 3 + \cdots + (n-1) + n + (n+1) \quad (2), \text{ and}$$

$$1 + 2 + 3 + \cdots + (n-1) + n + (n+1) + (n+2) \quad (3).$$

We add these three numbers up, and bring the last term (the  $(n+2)$ ) from (3) to (1):

$$\begin{aligned}
& [1 + 2 + 3 + \cdots + (n-1) + n + (n+2)] \\
& + [1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] \\
& + [1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] \\
& = [1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] + 1 \quad (\text{writing } n+2 \text{ as } (n+1) + 1) \\
& + [1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] \\
& + [1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] \\
& = 3[1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] + 1
\end{aligned}$$

which is 1 more than three times the middle of these three numbers.

### Solution 2

This solution uses the formula  $1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$  (which we can obtain using a similar pairing argument that we saw in (a)).

So the three consecutive triangle numbers are

$$\begin{aligned}
1 + 2 + 3 + \cdots + (n-1) + n &= \frac{n(n+1)}{2}, \\
1 + 2 + 3 + \cdots + (n-1) + n + (n+1) &= \frac{(n+1)(n+2)}{2}, \text{ and} \\
1 + 2 + 3 + \cdots + (n-1) + n + (n+1) + (n+2) &= \frac{(n+2)(n+3)}{2}
\end{aligned}$$

Adding these three, we obtain

$$\begin{aligned}
& \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} + \frac{(n+2)(n+3)}{2} \\
& = \frac{n^2 + n}{2} + \frac{n^2 + 3n + 2}{2} + \frac{n^2 + 5n + 6}{2} \\
& = \frac{3n^2 + 9n + 8}{2} \\
& = \frac{3n^2 + 9n + 6}{2} + 1 \\
& = 3\left(\frac{n^2 + 3n + 2}{2}\right) + 1
\end{aligned}$$

which is 1 more than three times the middle number of the three consecutive triangular numbers.

(c) *Solution 1*

We are told that the 3<sup>rd</sup>, 6<sup>th</sup> and 8<sup>th</sup> triangular numbers are in arithmetic sequence, and that the 8<sup>th</sup>, 12<sup>th</sup> and 15<sup>th</sup> triangular numbers are in arithmetic sequence. This seems to suggest a pattern, since we add 3 to 3 to get 6 and then 2 to 6 to get 8, followed by 4 to 8 to get 12 and 3 to 12 to get 15.

This suggests that the 15<sup>th</sup>, 20<sup>th</sup> and 24<sup>th</sup> (adding 5 to 15 and then adding 4) triangular numbers are in arithmetic sequence. We know that the 15<sup>th</sup> triangular number is 120 (given) and that the 24<sup>th</sup> triangular number is 300 (from (a)). We can check using a calculator that the 20<sup>th</sup> triangular number is 210. This suggests that the pattern continues. (But this doesn't prove that the pattern works!)

Continuing the pattern we get the following groups of numbers:

24<sup>th</sup>, 30<sup>th</sup> and 35<sup>th</sup>

35<sup>th</sup>, 42<sup>nd</sup> and 48<sup>th</sup>

48<sup>th</sup>, 56<sup>th</sup> and 63<sup>rd</sup>

Let's try the third set:

$$1 + 2 + \cdots + 47 + 48 = (1 + 48) + (2 + 47) + \cdots + (24 + 25) = 24 \times 49 = 1176$$

so this is still too small, since we want all three to be bigger than 2004.

Continuing the pattern:

63<sup>rd</sup>, 72<sup>nd</sup>, 80<sup>th</sup>,

80<sup>th</sup>, 90<sup>th</sup> and 99<sup>th</sup>

Let's try this set. Using our pairing technique from above:

$$1 + 2 + \cdots + 79 + 80 = 40 \times 81 = 3240$$

$$1 + 2 + \cdots + 89 + 90 = 45 \times 91 = 4095$$

$$1 + 2 + \cdots + 98 + 99 = (1 + 2 + \cdots + 99 + 100) - 100 = 50 \times 101 - 100 = 4950$$

Checking these,  $4950 - 4095 = 855 = 4095 - 3240$ , so the 80<sup>th</sup>, 90<sup>th</sup> and 99<sup>th</sup> triangular numbers are in arithmetic sequence.

*Solution 2*

In the two given examples, the difference between the positions of the first two numbers (ie. the 3<sup>rd</sup> and 6<sup>th</sup> in the first example) is one more than the difference between the positions of the second two numbers.

Let's try this pattern and see if we can continue this.

Let's look at the  $n$ th triangular number, the  $(n + 12)$ th triangular number (12 positions further along) and the  $(n + 23)$ th triangular number (11 positions further along). Can we find a value of  $n$  which makes these in arithmetic sequence? (There was no special reason to choose  $(n + 12)$ ; we could try larger or smaller numbers to see if they work.) If these are in arithmetic sequence, then

$$\begin{aligned}
[1 + 2 + \cdots + (n + 12)] - [1 + 2 + \cdots + n] &= [1 + 2 + \cdots + (n + 23)] - [1 + 2 + \cdots + (n + 12)] \\
(n + 1) + (n + 2) + \cdots + (n + 12) &= (n + 13) + (n + 14) + \cdots + (n + 23) \\
12n + (1 + 2 + \cdots + 12) &= 11n + (13 + 14 + \cdots + 23) \\
n &= (13 + 14 + \cdots + 23) - (1 + 2 + \cdots + 11) - 12 \\
n &= 11(12) - 12 \\
n &= 120
\end{aligned}$$

So the 120<sup>th</sup>, 132<sup>nd</sup> and 143<sup>rd</sup> triangular numbers are in arithmetic progression.

We could calculate these numbers using the pairing idea from (a) to check our answer:

$$1 + 2 + \cdots + 119 + 120 = 60 \times 121 = 7260$$

$$1 + 2 + \cdots + 131 + 132 = 66 \times 133 = 8778$$

$$1 + 2 + \cdots + 142 + 143 = (1 + 2 + \cdots + 143 + 144) - 144 = 72 \times 145 - 144 = 10296$$

Checking these,  $8778 - 7260 = 1518 = 10296 - 8778$ , so the 120<sup>th</sup>, 132<sup>nd</sup> and 143<sup>rd</sup> triangular numbers are in arithmetic sequence.



# Canadian Mathematics Competition

An activity of The Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario

## *2003 Solutions* *Fryer Contest* (Grade 9)

1. (a) *Solution 1*

The average (mean) is equal to the sum of all of the marks, divided by the total number of marks.

Since we know already that there are 32 students (we can check this by looking at the graph), then the average is

$$\frac{1(10) + 2(30) + 2(40) + 1(50) + 4(60) + 6(70) + 9(80) + 4(90) + 3(100)}{32} = \frac{2240}{32} = 70$$

Therefore, the average mark was 70.

*Solution 2*

The average (mean) is equal to the sum of all of the marks, divided by the total number of marks.

Using the bar graph, we can list out all of the marks:

10, 30, 30, 40, 40, 50, 60, 60, 60, 60, 70, 70, 70, 70, 70, 70, 80, 80, 80, 80, 80, 80, 80, 80, 80, 90, 90, 90, 90, 100, 100, 100.

Adding these up using a calculator and dividing by 32, we find that the average mark is 70.

- (b) After his first 6 tests, since Paul's average is 86, then has gotten a total of  $6(86) = 516$  marks.

After getting 100 on his seventh test, Paul has gotten a total of  $516 + 100 = 616$  marks, so his new average is  $\frac{616}{7} = 88$ .

(c) *Solution 1*

If Mary gets 100, her average becomes 90.

If Mary gets 70, her average becomes 87.

So a difference of 30 marks on the test gives a difference of 3 marks in the average.

Since her average is her total number of marks divided by her total number of tests, and a difference of 30 in the total number of marks makes a difference of 3 in her average, then

she will have written  $\frac{30}{3} = 10$  tests.

*Solution 2*

Suppose that after the next test, Mary has written  $n$  tests.

If her average after getting 100 on the next test is 90, then Mary has earned  $90n$  marks in total after the first  $n$  tests, and so  $90n - 100$  before she writes the  $n$ th test.

If her average after getting 70 on the next test is 87, then Mary has gotten  $87n$  marks in total after the  $n$ th test, and so she will have earned  $87n - 70$  marks before the next test.

Therefore, since the number of marks before her next test is the same in either case,



$$87n - 70 = 90n - 100$$

$$30 = 3n$$

$$n = 10$$

So Mary will have written 10 tests.

### *Extension*

We start by using the given information to try to figure out some more things about the marks of the 32 students.

Since the median mark is the “middle mark” in a list of marks which is increasing, then there at least 16 students who have marks that are at least 80.

Since the difference between the highest and lowest marks is 40, and there are students who got at least 80, then the lowest mark in the class cannot be lower than 40.

Since the average mark in the class is 58, then the total number of marks is  $32(58) = 1856$ .

So what does this tell us?

Since at least 16 students got at least 80, then this accounts for at least 1280 marks, leaving  $1856 - 1280 = 576$  marks for the remaining 16 students.

But the lowest possible mark in the class was 40, so these remaining 16 students got at least 40 each, and so got at least  $16(40) = 640$  marks in total!

So we have an inconsistency in the data.

Thus, the teacher made a calculation error.

(There is a variety of different ways of reaching this same conclusion. As before, we know that 16 students will have a mark of at least 80, which accounts for 1280 marks. By the same reasoning, the other 16 students would account for the other 576 marks. The average for these 16 students is thus about 34, which implies that at least one of these lower students must have a mark of 34 or lower. This now contradicts the statement that the range is 40 since  $80 - 34 > 40$ .)

### 2. (a) *Solution 1*

If Xavier goes first and calls 4, then on her turn Yolanda can call any number from 5 to 14, since her number has to be from 1 to 10 greater than Xavier's.

But if Yolanda calls a number from 5 to 14, then Xavier can call 15 on his next turn, since 15 is from 1 to 10 bigger than any of the possible numbers that Yolanda can call.

So Xavier can call 15 on his second turn no matter what Yolanda calls, and is thus always guaranteed to win.

*Solution 2*

If Xavier goes first and calls 4, then Yolanda will call a number of the form  $4 + n$  where  $n$  is a whole number between 1 and 10.

On his second turn, Xavier can call 15 (and thus win) if the difference between 15 and  $4 + n$  is between 1 and 10. But  $15 - (4 + n) = 11 - n$  and since  $n$  is between 1 and 10, then  $11 - n$  is also between 1 and 10, so Xavier can call 15.

Therefore, Xavier's winning strategy is to call 15 on his second turn.

- (b) In (a), we saw that if Xavier calls 4, then he can guarantee that he can call 15.

Using the same argument, shifting all of the numbers up, to guarantee that he can call 50, he should call 39 on his previous turn.

(In this case, Yolanda can call any whole number from 40 to 49, and in any of these cases Xavier can call 50, since 50 is no more than 10 greater than any of these numbers.)

In a similar way, to guarantee that he can call 39, he should call 28 on his previous turn, which he can do for the same reasons as above.

To guarantee that he can call 28, he should call 17 on his previous turn.

To guarantee that he can call 17, he should call 6 on his previous turn, which could be his first turn.

Therefore, Xavier's winning strategy is to call 6 on his first turn, 17 on his second turn, 28 on his third turn, 39 on his fourth turn, and 50 on his last turn.

At each step, we are using the fact that Xavier can guarantee that his number on one turn is 11 greater than his number on his previous turn. This is because Yolanda adds 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 to his previous number, and he can then correspondingly add 10, 9, 8, 7, 6, 5, 4, 3, 2, or 1 to her number, for a total of 11 in each case.

*Extension*

In (b), we discovered that Xavier can always guarantee that the difference between his numbers on two successive turns is 11.

In fact, Yolanda can do the same thing, using exactly the same strategy as Xavier did.

If the target number is between 1 and 9, then Xavier will win on his very first turn by calling that number.

If the target number is then 11 greater than a number between 1 and 9, Xavier will win as in (b). Thus, Xavier wins for 12 through 20.

What about 10 and 11? In each of these cases, Yolanda can win by choosing 10 or 11 on her first turn, which she can do for any initial choice of Xavier's, since he chooses a number between 1 and 9.

Therefore, Yolanda will also win for 21 and 22, and so also for 32 and 33, and so on.

Since either Yolanda or Xavier can repeat their strategy as many times as they want, then Xavier can ensure that he wins if the target number is a multiple of 11 more than one of 1 through 9.

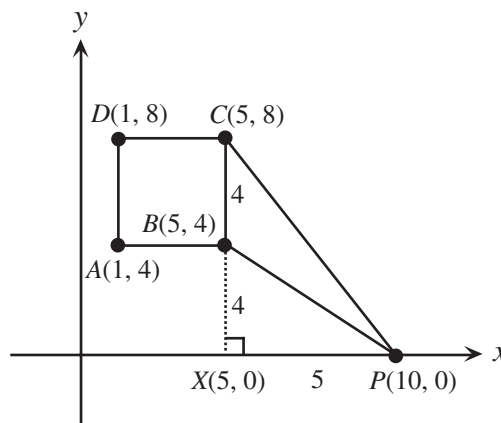
Similarly, Yolanda can ensure that she wins if the target number is a multiple of 11 more than 10 or 11, ie. if the target number is a multiple of 11, or 1 less than a multiple of 11.

3. (a) *Solution 1*

Since  $ABCD$  is a square and  $AD$  has side length 4, then each of the sides of  $ABCD$  has length 4. We can also conclude that  $B$  has coordinates  $(5,4)$  and  $C$  has coordinates  $(5,8)$ . (Since  $AD$  is parallel to the  $y$ -axis, then  $AB$  is parallel to the  $x$ -axis.)

If we turn  $\triangle PBC$  on its side, then we see that its base is  $BC$  which has length 4. Also the height of the triangle is the vertical distance from the line  $BC$  to  $P$ , which is 5. (We can see this by extending the line  $CB$  to a point  $X$  on the  $x$ -axis. Then  $X$  has coordinates  $(5,0)$  and  $PX$ , which has length 5, is perpendicular to  $CB$ .)

Therefore, the area of  $\triangle PBC$  is  $\frac{1}{2}bh = \frac{1}{2}(4)(5) = 10$ .



*Solution 2*

Since  $ABCD$  is a square and  $AD$  has side length 4, then all of the sides of  $ABCD$  have length 4. We can also conclude that  $B$  has coordinates  $(5,4)$  and  $C$  has coordinates  $(5,8)$ . (Since  $AD$  is parallel to the  $y$ -axis, then  $AB$  is parallel to the  $x$ -axis.)

Extend  $CB$  down to a point  $X$  on the  $x$ -axis. Point  $X$  has coordinates  $(5,0)$ .

Then the area of  $\triangle PBC$  is the difference between the areas of  $\triangle PCX$  and  $\triangle PBX$ .

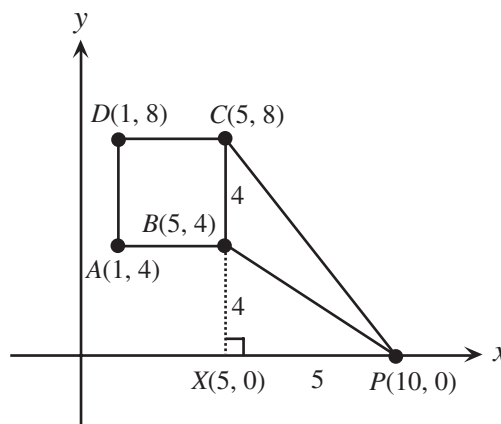
$\triangle PCX$  has base  $PX$  of length 5 and height  $CX$  of length 8.

$\triangle PBX$  has base  $PX$  of length 5 and height  $BX$  of length 4.

Therefore, the area of  $\triangle PBC$  is

$$\frac{1}{2}(4)(10) - \frac{1}{2}(4)(5) = 10$$

as required.



(b) *Solution 1*

Since triangle  $CBE$  lies entirely outside square  $ABCD$ , then the point  $E$  must be “to the right” of the square, ie.  $a$  must be at least 5.

Also, we need to know the area of the square. Since the side length of the square is 4, its area is 16.

Thus if we turn  $\triangle CBE$  on its side, then its we see that its base is  $BC$  which has length 4.

Also the height of the triangle is the vertical distance from the line  $BC$  to  $E$ , which is  $a - 5$  since  $E$  has coordinates  $(a, 0)$  and  $BC$  is part of the line  $x = 5$ .

Therefore, since the area of  $\triangle CBE$  is equal to the area of the square,

$$\frac{1}{2}(4)(a - 5) = 16$$

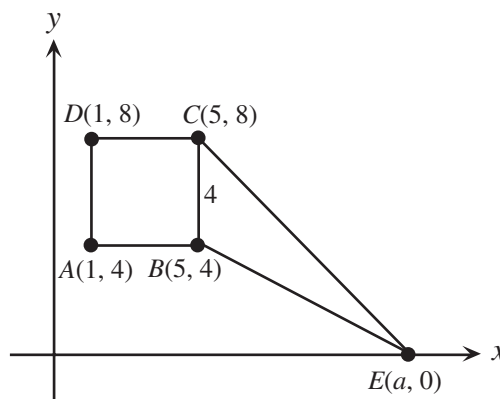
$$2a - 10 = 16$$

$$2a = 26$$

$$a = 13$$

Thus,  $a = 13$ .

(It is easy to verify that if  $a = 13$ , then the height of the triangle is 8 and its base is 4, giving an area of 16.)



### Solution 2

Since triangle  $CBE$  lies entirely outside square  $ABCD$ , then the point  $E$  must be “to the right” of the square, ie.  $a$  must be at least 5.

Also, we need to know the area of the square. Since the side length of the square is 4, its area is 16.

Extend  $CB$  down to a point  $X$  on the  $x$ -axis. Point  $X$  has coordinates  $(5, 0)$ .

Then the area of  $\triangle CBE$  is the difference between the areas of  $\triangle CXE$  and  $\triangle BXE$ .

$\triangle CXE$  has base  $EX$  of length  $a - 5$  and height  $CX$  of length 8.

$\triangle BXE$  has base  $EX$  of length  $a - 5$  and height  $BX$  of length 4.

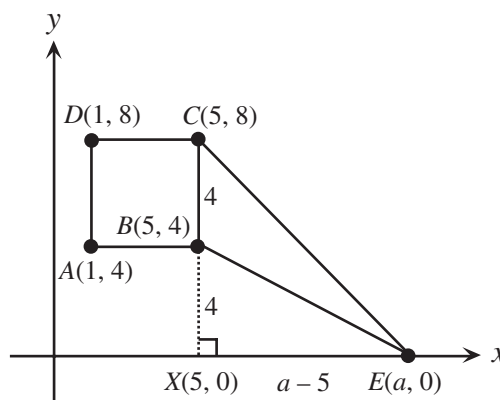
Therefore, since the area of  $\triangle CBE$  is equal to the area of the square,

$$\frac{1}{2}(a - 5)(8) - \frac{1}{2}(a - 5)(4) = 16$$

$$4(a - 5) - 2(a - 5) = 16$$

$$a - 5 = 8$$

$$a = 13$$



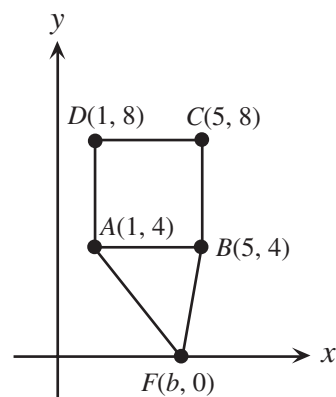
Thus,  $a = 13$ , as required.

(c) Suppose that  $F$  has coordinates  $(b, 0)$ .

Then triangle  $ABF$  has base  $AB$  of length 4.

The height of triangle  $ABF$  is the vertical distance from  $F$  to the line  $AB$ , which is always 4, no matter where  $F$  is.

Thus, the area of triangle  $ABF$  is always  $\frac{1}{2}bh = \frac{1}{2}(4)(4) = 8$ , which is not equal to the area of the square.



### Extension

Since triangle  $DCG$  lies entirely outside the square, then  $G$  is “above” the line through  $D$  and  $C$ , ie. the  $y$ -coordinate of  $G$  is at least 8.

Since the area of triangle  $DCG$  is equal to the area of the square, then the area of triangle  $DCG$  is 16.

Now triangle  $DCB$  has base  $DC$ , which has length 4, so  $\frac{1}{2}bh = \frac{1}{2}(4)h = 16$  or  $h = 8$ .

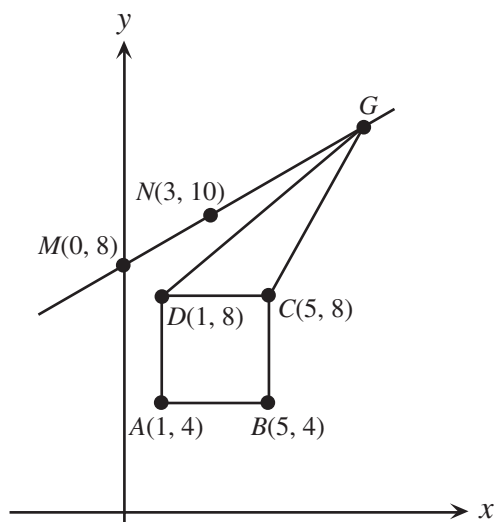
Since the height of triangle  $DCG$  is 8, then  $G$  has  $y$ -coordinate 16, since both  $D$  and  $C$  have  $y$ -coordinate 8.

So we must find the point on the line through  $M(0, 8)$  and  $N(3, 10)$  which has  $y$ -coordinate 16.

To get from  $M$  to  $N$ , we go 3 to the right and up 2.

To get from  $N$  to  $G$ , we go up 6, so we must go 9 to the right.

Therefore,  $G$  has coordinates  $G(12, 16)$ .



4. (a) The best approach here is to list the numbers directly. The possible totals are, from smallest to largest:

1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111

There 15 possible totals, and their sum (that is, the power-sum) is 8888.

(b) *Solution 1*

First, we consider the numbers that are sums of 1 or more of the numbers from  $\{1, 10, 100, 1000\}$ . In (a), we saw that the sum of these numbers is 8888.

What happens when we consider numbers that are sums of 1 or more of the numbers from  $\{1, 10, 100, 1000, 10\,000\}$ ? When we do this, we obtain all 15 of the numbers from the previous paragraph, we obtain the 15 numbers obtained by adding 10 000 to all of the numbers from the previous paragraph, and also the number 10 000. (Either our sum does not include 10 000 as a term, or it does; if it doesn't, it must be one of the numbers from (a); if it does, it could be 10 000 on its own, or it could be 10 000 plus one of the numbers from (a).)

Therefore, we have  $15 + 15 + 1 = 31$  numbers in total, whose sum is

$$8888 + [8888 + 15(10\,000)] + 10\,000 = 2(8888) + 160\,000 = 2(88\,888) = 177\,776$$

What happens when we consider the numbers that are sums of 1 or more of the numbers from  $\{1, 10, 100, 1000, 10\,000, 100\,000\}$ ? When we do this, we obtain all 31 of the numbers from (a), we obtain the 31 numbers obtained by adding 100 000 to all of the numbers from (a), and also the number 100 000.

Therefore, we have  $31 + 31 + 1 = 63$  numbers in total, whose sum is

$$177\,776 + [177\,776 + 31(100\,000)] + 100\,000 = 2(177\,776) + 3\,200\,000 \\ = 3\,555\,552$$

What happens when we add 1 000 000 to the set? We then obtain, as before,

$63 + 63 + 1 = 127$  numbers in total, whose sum is

$$3\,555\,552 + [3\,555\,552 + 63(1\,000\,000)] + 1\,000\,000 = 2(3\,555\,552) + 64\,000\,000 \\ = 71\,111\,104$$

Therefore, the power sum is 71 111 104.

### *Solution 2*

There are seven numbers in the given set. When we are considering sums of one or more numbers from the set, each of the seven numbers in the set is either part of the sum, or not part of the sum. So there are two choices ("in" or "out") for each of the 7 elements.

So we can proceed by first choosing the elements we want to add up, and then adding them up. Since for each of the two possibilities for the "1" (ie. chosen or not chosen), there are two possibilities for the "10", and there are two possibilities for the "100", and so on. In total, there will be  $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128$  ways of choosing elements. Notice that this includes the possibility of choosing no elements at all (since we could choose not to select each of the seven elements).

So there are 128 possible sums (including the sum which doesn't add up any numbers at all!).

In how many of these sums is the “1” chosen? If the “1” is chosen, then there are still 2 possibilities for each of the remaining six elements (either chosen or not chosen), so there are  $2^6 = 64$  sums with the 1 included, so the 1 contributes 64 to the power-sum.

In how many of these sums is the “10” chosen? Using exactly the same reasoning, there are 64 sums which include the 10, so the 10 contributes 640 to the power-sum.

Extending this reasoning, each of the 7 elements will contribute to 64 of the sums. (Note that including the “empty” sum doesn’t affect the power-sum.)

Therefore, the power sum is

$$\begin{aligned} &64(1 + 10 + 100 + 1000 + 10000 + 100000 + 1000000) \\ &= 64(1111111) \\ &= 71111104 \end{aligned}$$

### *Extension*

We start looking at small numbers to see if we can see a pattern.

Using the numbers 1, 2, 3, we can form the sums

$$1, 2, 3, 1 + 3 = 4, 1 + 2 = 3, 2 + 3 = 5, 1 + 2 + 3 = 6$$

If we include the 6, we can obtain all of these sums, as well as 6 plus each of these sums.

In other words, we obtain each of the numbers 1 through 12 as totals.

Then including the 12, we can obtain 13 through 24, so we now have each of 1 to 24 as totals.

Including the 24, we obtain all numbers up to 48.

Including the 48, we obtain all numbers up to 96.

Including the 96, we obtain all numbers up to 192.

Therefore, there are 192 different totals possible.

(We can check as well that the sum of the elements in the original set is 192.)