Lecture 11 - Part A: The Gauss Map

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Math 143:
Topics in Geometry

A Big Picture

Motivation

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Keys

dN_p is a self-adoint linear map $T_p(S) \to T_p(S)$. Thus, both dN_p and the second fundamental form (which relies on dN_p) can be diagonalized (i.e., there is an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that $dN_p(e_1) = -k_1 e_1$ and $dN_p(e_2) = -k_2 e_2$).

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For the rest of this lecture, S will denote a regular orientable surface in which an orientation (i.e., a differentiable field of unit normal vectors N) has been chosen; this will simply be called a surface S with an orientation N.

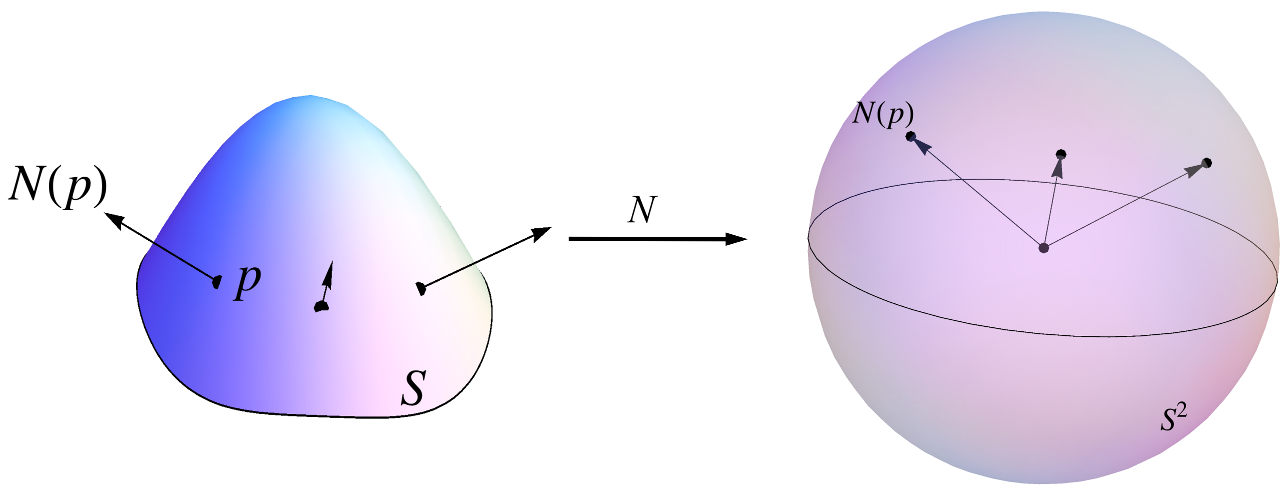
Defining the Gauss Map

Definition

Let $S \subset \mathbb{R}^3$ be a surface with an orientation N. The map $N : S \to \mathbb{R}^3$ takes its values in the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The map $N : S \to S^2$, thus defined, is called the *Gauss map* of S.



The Differential of the Gauss Map

Remark

It is straightforward to verify that the Gauss map is differentiable. The differential dN_p of N at $p \in S$ is a linear map from $T_p(S)$ to $T_{N(p)}(S^2)$. Since $T_p(S)$ and $T_{N(p)}(S^2)$ are parallel planes, dN_p can be viewed as a linear map on $T_p(S)$.

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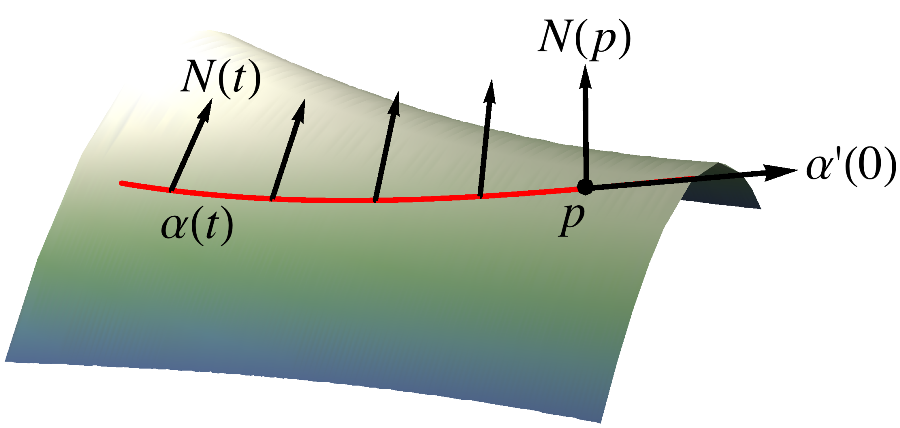
Remark

The tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in $T_p(S)$. It measures the rate of change of the normal vector N, restricted to the curve $\alpha(t)$, at $t = 0$. Thus, dN_p measures how N pulls away from $N(p)$ in a neighborhood of p.



The Differential of the Gauss Map

Proposition

The differential $dN_p : T_p(S) \to T_p(S)$ of the Gauss map is a self-adjoint linear map.

Proof.

$\square$

The Second Fundamental Form

Remark

The fact that $dN_p : T_p(S) \to T_p(S)$ is a self-adjoint linear map allows us to associate to dN_p a quadratic form Q in $T_p(S)$, given by $Q(v) = \langle dN_p v, v \rangle$, $v \in T_p(S)$. To obtain a geometric interpretation of this quadratic form, we need a few definitions. For reasons that will soon become clear, we shall use the quadratic form $-Q$.

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Definition

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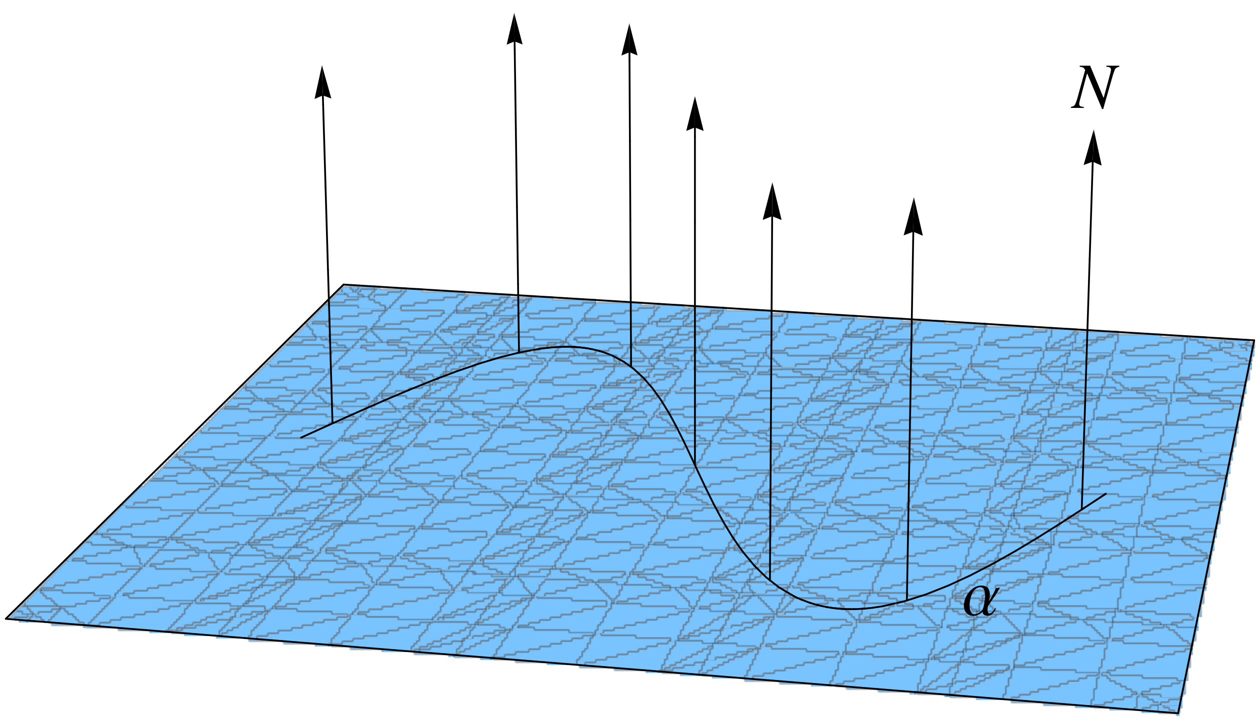
To give an interpretation of the second fundamental form II_p, consider a regular curve $C \subset S$ parametrized by $\alpha(s)$, where s is the arc length of C, and with $\alpha(0) = p$.

Examples

Example

For a plane P given by $ax + by + cz + d = 0$, the unit normal vector $N = (a, b, c)/\sqrt{a^2 + b^2 + c^2}$ is constant, and therefore $dN \equiv 0$.



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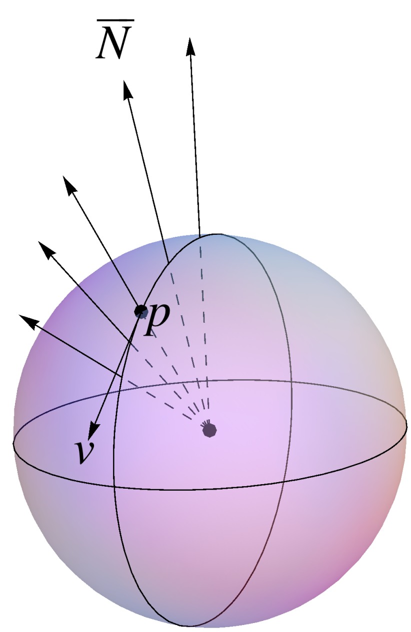
Example

Consider the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$



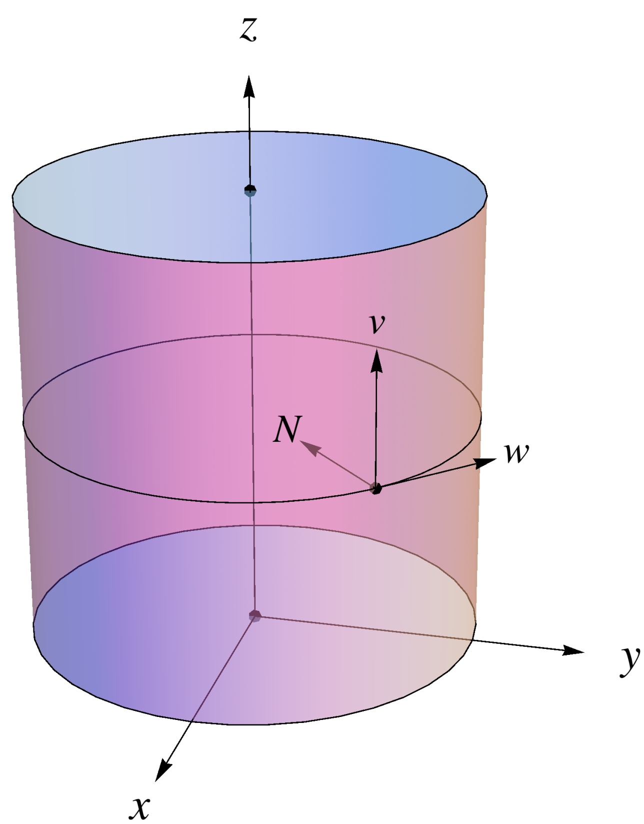
Examples

Example

Consider the cylinder
$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.
We conclude the following: If v is a vector tangent to the cylinder and parallel to the z axis, then

$$dN(v) = 0 = 0v;$$

if w
is a vector tangent to the cylinder and parallel to the xy plane, then $dN(w) = -w$. It follows that the vectors v and w are eigenvectors of dN with eigenvalues
0 and -1, respectively.



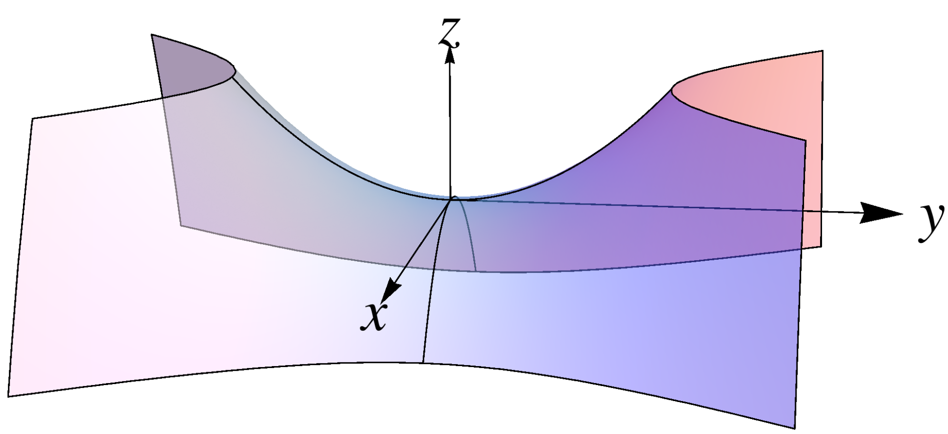
Examples

Example

Let us analyze the point $p = (0,0,0)$ of the hyperbolic paraboloid $z = y^2 - x^2$. For this, we consider a parametrization $\mathbf{x}(u,v)$ given by

$$\mathbf{x}(u,v) = (u, v, v^2 - u^2),$$

and compute the normal vector $N(u,v)$.



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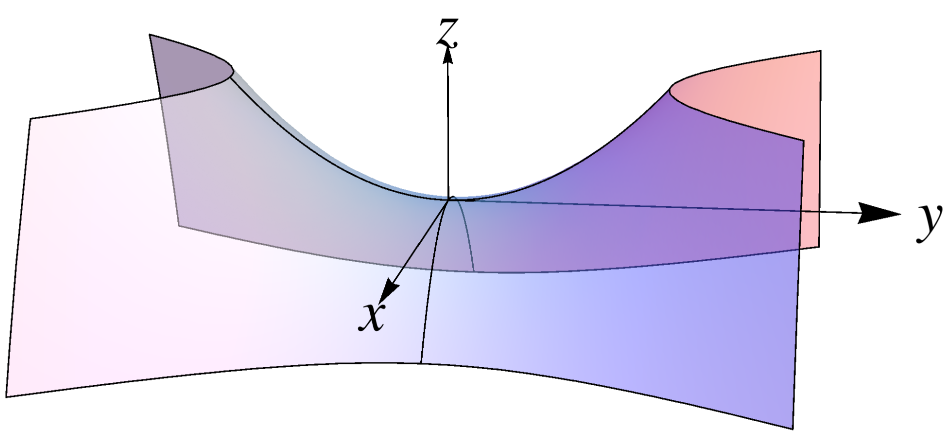
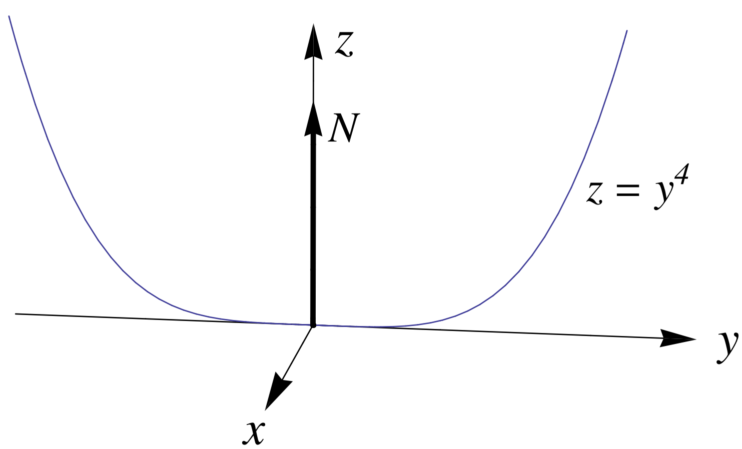


It follows that the vectors $(1,0,0)$ and $(0,1,0)$ are eigenvectors of dN_p with eigenvalues 2 and -2, respectively.

Examples

Example

Consider the surface of revolution obtained by rotating the curve $z = y^4$ about the z axis. We shall show that at $p = (0, 0, 0)$ the differential $dN_p = 0$.



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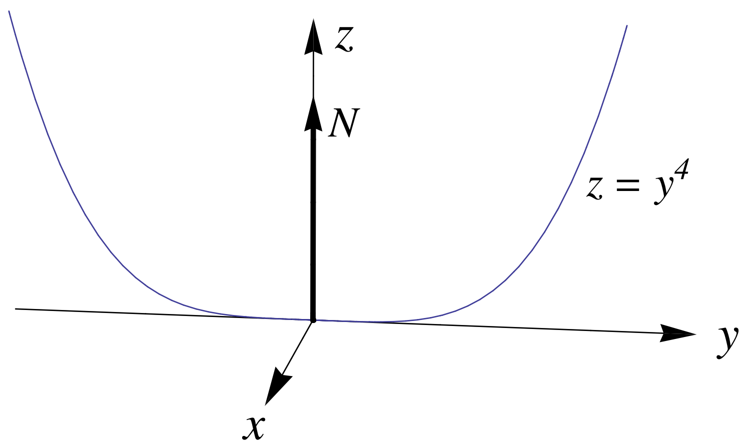

To see this, we observe that the curvature of the curve $z = y^4$ at p is equal to zero. Moreover, since the xy plane is a tangent plane to the surface at p, the normal vector $N(p)$ is parallel to the z axis. Therefore, any normal section at p is obtained from the curve $z = y^4$ by rotation; hence, it has curvature 0. It follows that all normal curvatures are zero at p, and thus $dN_p = 0$.