Topic: Surfaces and Their Applications

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There are many ways to represent surfaces

- Parametrized
- As a set but each associates a parametrization (This gives Regular Surface definition.)
- Landmarks
- Explicit
- Implicit
- Meshes (using control points)

Surfaces in Parametric Form (similar for parametric curves)

- **Parametric Form**
  - The parametric form of a curve expresses the value of each spatial variable for points on the curve in terms of an independent variable, ***u***, *the* parameter. In three dimensions, we have three explicit functions:

    $x = x(u)$, $y = y(u)$, $z = z(u)$.

  - One of the advantages of the parametric form is that it is the same in two and three dimensions. In the former case, we simply drop the equation for *z*.
  - Parametric surfaces require two parameters. We can describe a surface by three equations of the form : $\mathbf{x = x(u, v)}$, $\mathbf{y = y(u, v)}$, $\mathbf{z = z(u, v)}$,

Example: Parametrizing a sphere

$$x = r\cos(\theta)\sin(\varphi),$$
$$y = r\sin(\theta)\sin(\varphi),$$
$$z = r\cos(\varphi),$$

where θ is from 0 to 2π and φ is from 0 to π.

Note: If we require (0, 2\pi) and (0, \pi) open for the convenience of analysis on the sphere, then we can not cover the entire sphere!

Note: We are not just parametrizing a surface and finding its surface area. We want to use the surfaces to model a real world problem including big data problem.

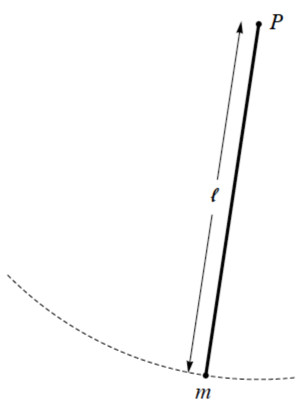
A Motivational Example

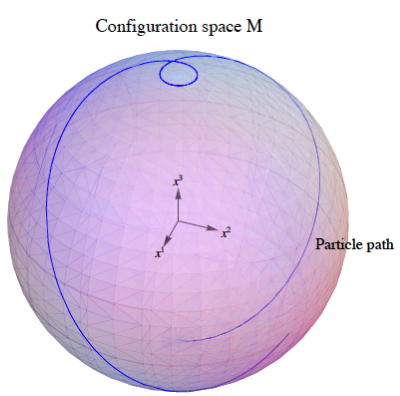
Mathematical Models and Physical Systems

When we wish to describe a physical system in a "mathematical" way we try to construct some sort of mathematical structure which, in some sense, "represents" those aspects of the system which are of interest to us. This structure is then a "mathematical model" of the physical system.

Example

A mass m is fixed on the end of a rigid rod of negligible mass having length ℓ. One end of the rod is fixed at a point P in space so that the mass can move about about P subject to the condition that it always be a distance ℓ from P. The sphere M (a *regular surface* or *manifold*) of all possible positions for m is called the *configuration space* of the system.





Example (cont'd)

Suppose we are only interested in the motion of the particle. Then we take, as the state of the particle, the pair of three-dimensional vectors (x, v), $x = (x^1, x^2, x^3)$, $v = (v^1, v^2, v^3)$, where x is the position vector of m and v is the velocity vector of m (with respect to some Cartesian coordinate system).

Since the mass must stay on the sphere M, we see v must be tangent to M. Thus our *state space* S does not consist of all pairs of 3-vectors but, instead, we have the *tangent bundle* of M (which can also be viewed as a manifold);

$$S = \{(x, v) \mid x \in M \text{ and } v \text{ is tangent to } M \text{ at } x\}.$$

Although S is not a Euclidean space, nor an open set in one, we shall see that S is a space on which notions such as tangent vector, vector field, and time-dependent vector field have meaning. If we have a force field then the force field will determine a vector field on the state space S.

Definitions and Examples

Note: View a regular surface as a set of points!

Definition

A subset $S \subset \mathbb{R}^3$ is a *regular surface* if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. $\mathbf{x}$ is differentiable (so we can use calculus).

2. $\mathbf{x}$ is a homeomorphism (so we can use analysis)

3. $\mathbf{x}$ is regular (so we can use linear algebra)

Remark

In contrast to our treatment of curves, we have *defined a surface as a subset* S of $\mathbb{R}^3$, and not as a map. This is achieved by covering S with the traces of parametrizations which satisfy conditions 1, 2, and 3.

$\mathbf{x}$ is differentiable

This means that if we write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U,$$

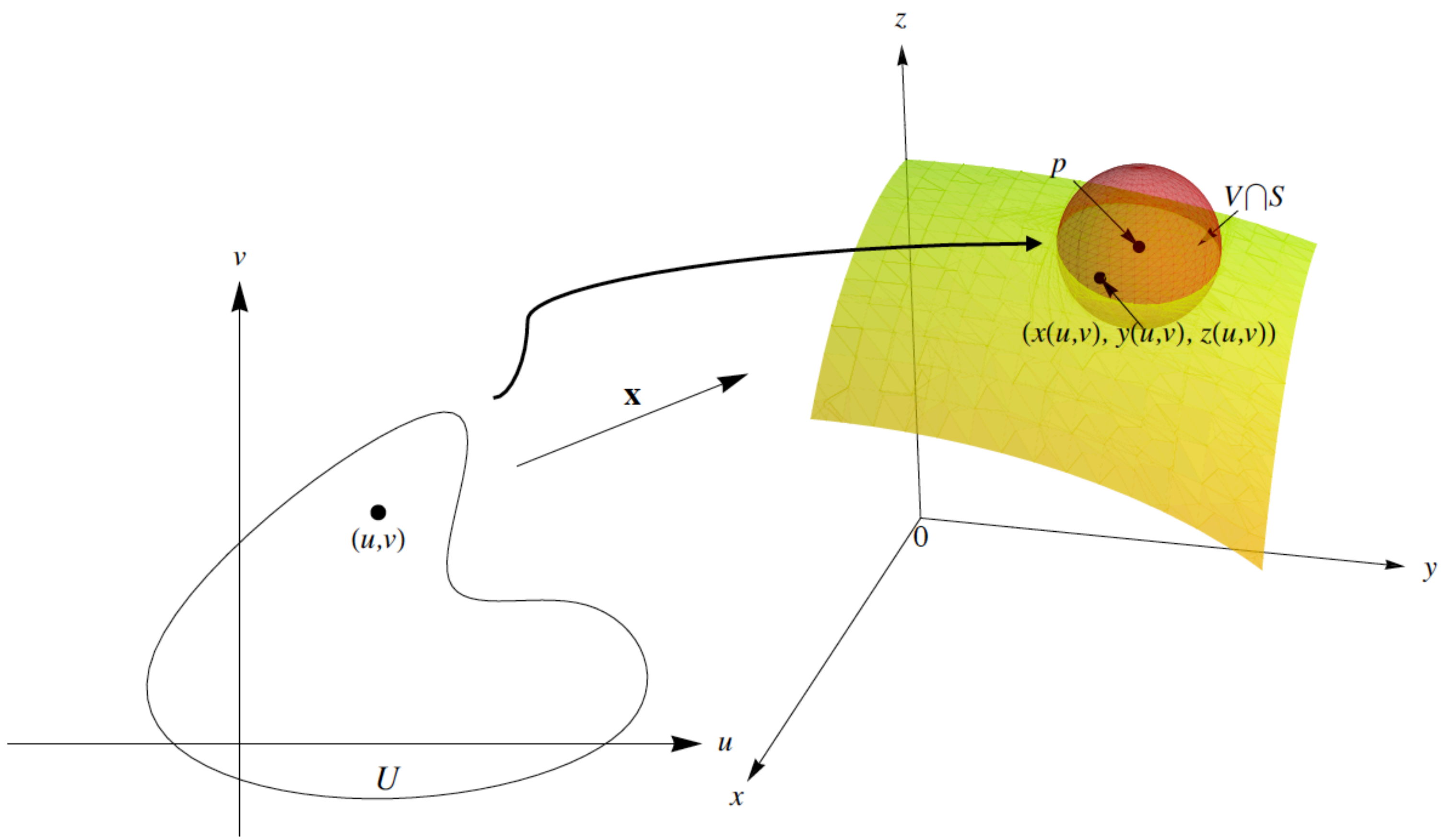
the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ have continuous partial derivatives of all orders.

$\mathbf{x}$ is a homeomorphism

Since $\mathbf{x}$ is continuous by condition 1, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous; that is, $\mathbf{x}^{-1}$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \to \mathbb{R}^2$ defined on an open set W containing $V \cap S$.

$\mathbf{x}$ is regular

For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.



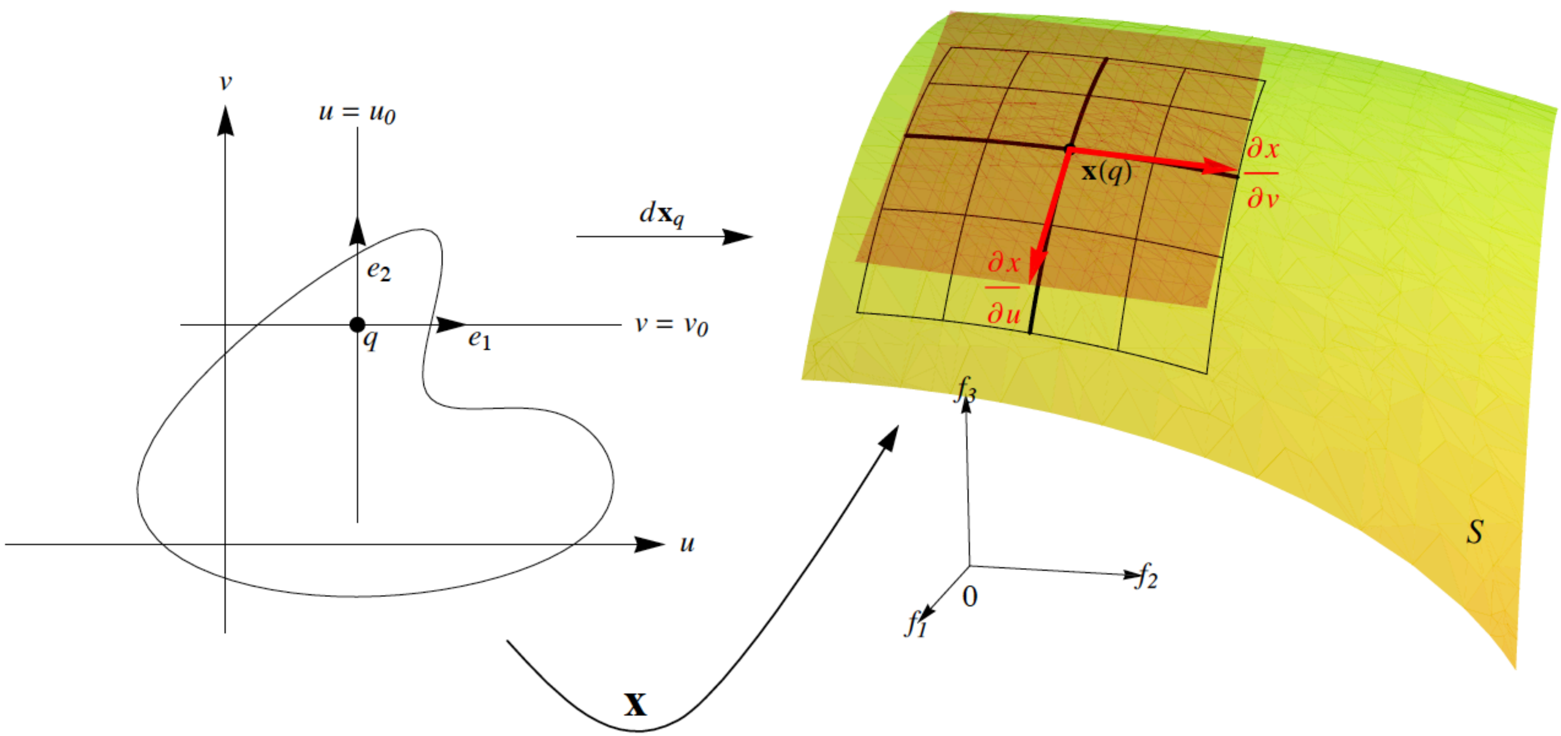
Definition

The mapping **x** is called a *parametrization* or a *system of (local) coordinates* in (a neighborhood of) p. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood*.

The Regularity Condition

An Illustrative Example

To give condition 3 a more familiar form, let us compute the matrix of the linear map $d\mathbf{x}_q$ in the canonical bases $e_1 = (1,0)$, $e_2 = (0,1)$ of $\mathbb{R}^2$ with coordinates u, v and $f_1 = (1,0,0)$, $f_2 = (0,1,0)$, $f_3 = (0,0,1)$ of $\mathbb{R}^3$, with coordinates (x,y,z).



The Regularity Condition

An Illustrative Example (cont'd)

Thus, the matrix of the linear map $d\mathbf{x}_q$ in the referred (standard) basis is

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

Condition 3 may now be expressed by requiring the two column vectors of this matrix to be linearly independent; or, equivalently, that the vector product $\partial \mathbf{x}/\partial u \wedge \partial \mathbf{x}/\partial v \neq 0$; or, in still another way, that one of the minors of order 2 of the matrix $d\mathbf{x}_q$, that is, one of the Jacobian determinants

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y,z)}{\partial(u,v)}, \quad \frac{\partial(x,z)}{\partial(u,v)},$$

be nonzero at q.

The Three Conditions

- Condition 1 is very natural if we expect to do some differential geometry on S.
- The one-to-oneness in condition 2 has the purpose of preventing self-intersections in regular surfaces. This is clearly necessary if we are to speak about, say, *the* tangent plane at a point $p \in S$. The continuity of the inverse in condition 2 has a more subtle purpose. For the time being, we shall mention that this condition is essential to proving that certain objects defined in terms of a parametrization do not depend on this parametrization but only on the set S itself.
- Finally, condition 3 will guarantee the existence of a "tangent plane" at all points of S.

Proving that a Set is a Regular Surface

Example

Let us show that the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

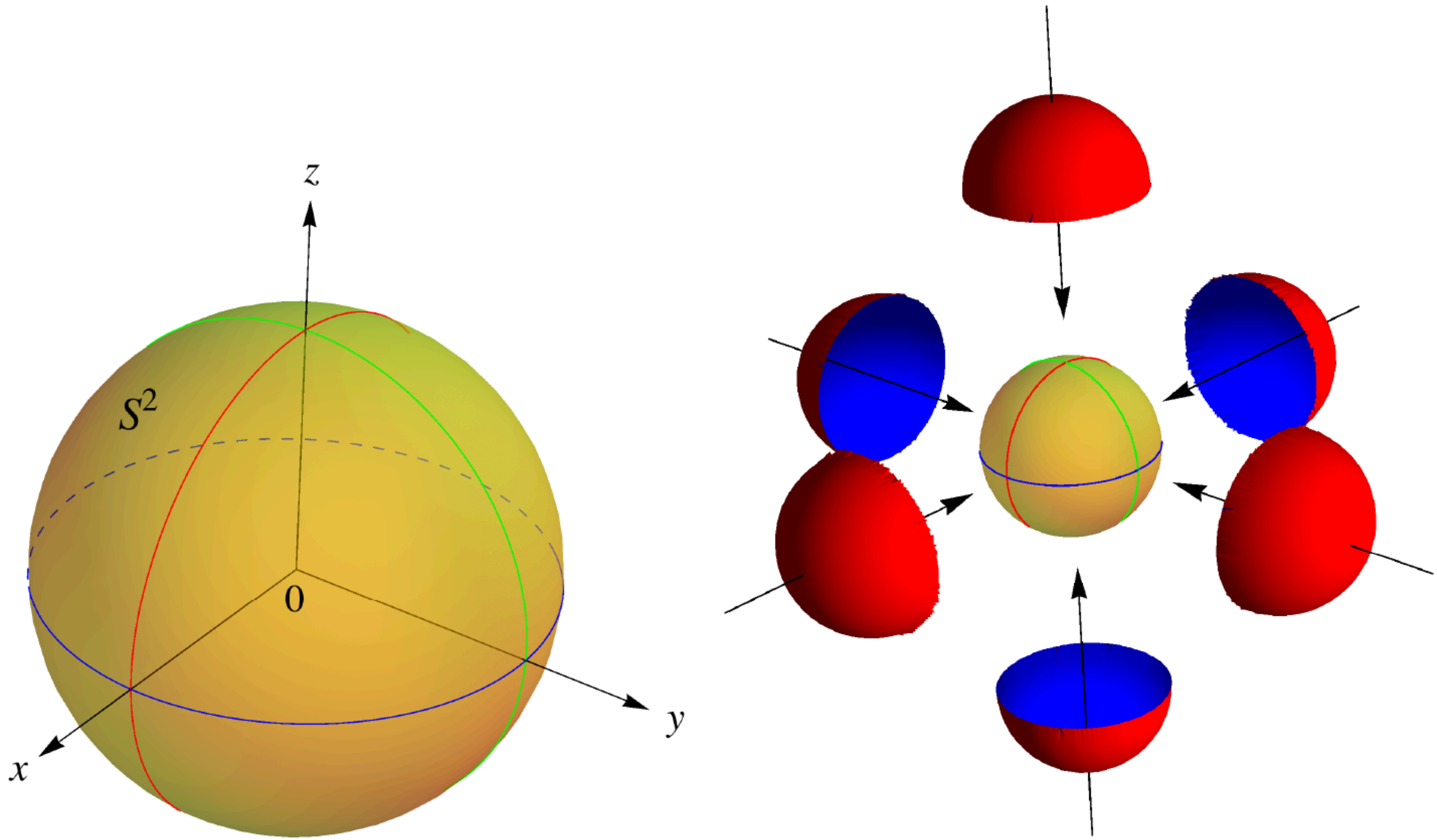
is a regular surface.

Method 1: Using Cartesian Coordinates

We first verify that the map $\mathbf{x}_1 : U \in \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{x}_1(x, y) = (x, y, +\sqrt{1 - (x^2 + y^2)}), \quad (x, y) \in U,$$

where $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ and
$U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is a parametrization of S^2.



z
S^2
0
x
y

Proving that a Set is a Regular Surface

We shall now cover the whole sphere with similar parametrizations as follows. we define $\mathbf{x}_2 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{x}_2(x, y) = (x, y, -\sqrt{1 - (x^2 + y^2)}),$$

check that $\mathbf{x}_2$ is a parametrization, and observe that $\mathbf{x}_1(U) \cup \mathbf{x}_2(U)$ covers S^2 minus the equator $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}$. Then, using the xz and zy planes, we define the parametrization

$$\begin{aligned}
\mathbf{x}_3(x, z) &= (x, +\sqrt{1 - (x^2 + z^2)}, z), \\
\mathbf{x}_4(x, z) &= (x, -\sqrt{1 - (x^2 + z^2)}, z), \\
\mathbf{x}_5(y, z) &= (+\sqrt{1 - (y^2 + z^2)}), y, z), \\
\mathbf{x}_6(y, z) &= (-\sqrt{1 - (y^2 + z^2)}), y, z),
\end{aligned}$$

which, together with $\mathbf{x}_1$ and $\mathbf{x}_2$, cover S^2 completely and shows that S^2 is a regular surface.

Proving that a Set is a Regular Surface

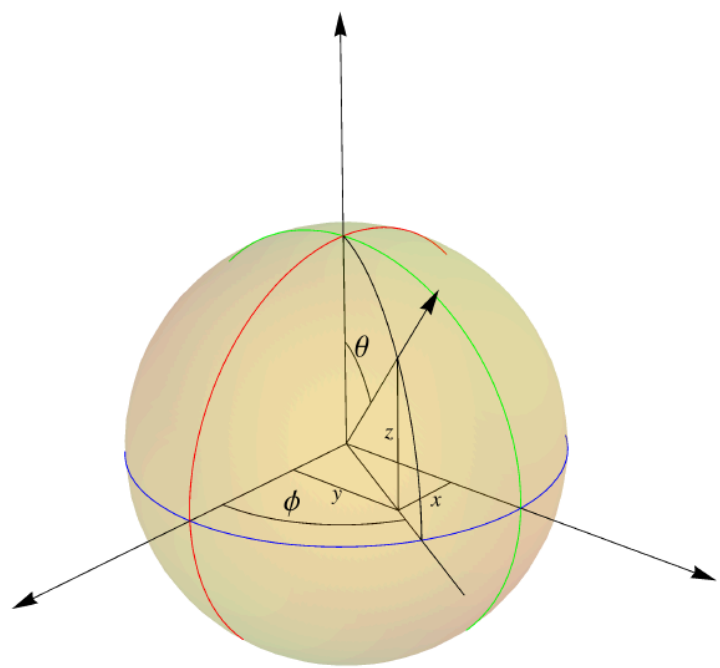
Method 2: Using Spherical Coordinates

Homework!

For most applications, it is convenient to relate parametrizations to the geographical coordinates on S^2. Let $V = \{(\theta, \varphi) \mid 0 < \theta < \pi, 0 < \varphi < 2\pi\}$ and let $\mathbf{x} : V \to \mathbb{R}^3$ be given by

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Clearly, $\mathbf{x}(V) \subset S^2$.



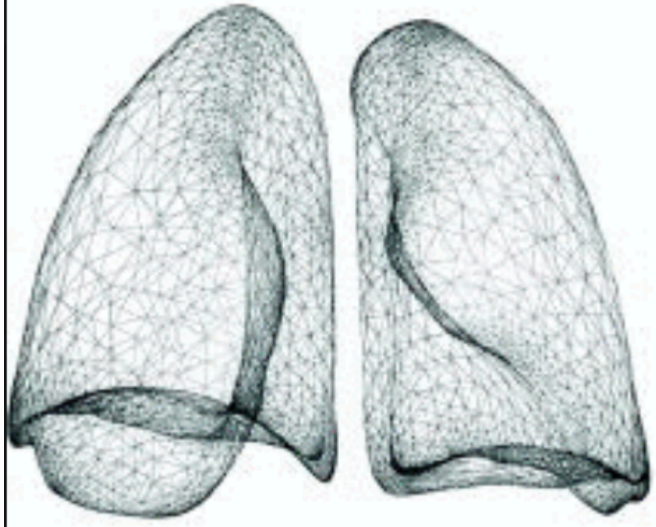
Proving that a Set is a Regular Surface

We shall prove that $\mathbf{x}$ is a parametrization of S^2.

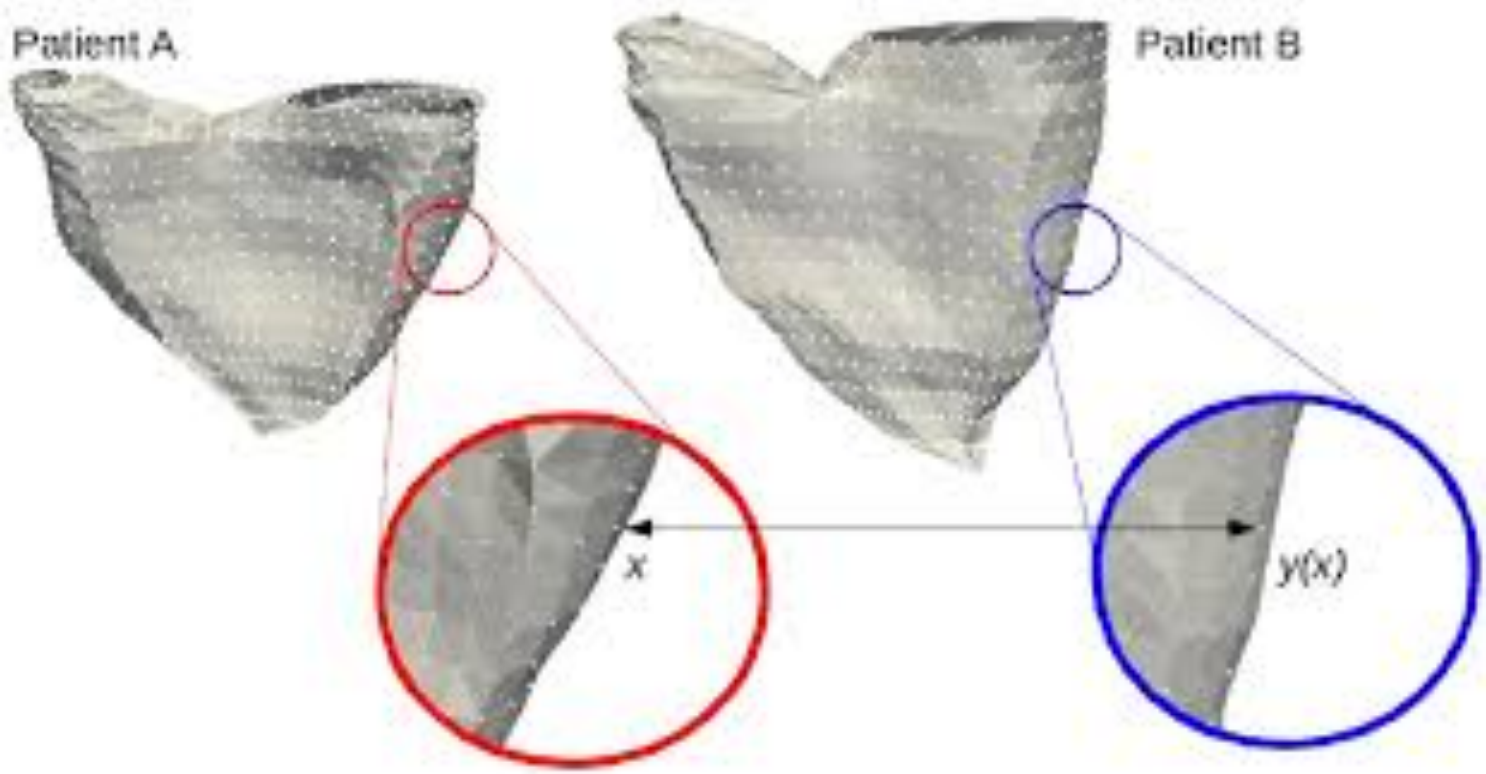
Next, we observe that given $(x, y, z) \in S^2 \setminus C$, where C is the semicircle $C = \{(x, y, z) \in S^2 \mid y = 0, x \geq 0\}$, θ is uniquely determined by $\theta = \cos^{-1} z$, since $0 < \theta < \pi$. By knowing θ, we find $\sin\varphi$ and $\cos\varphi$ from $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, and this determines φ uniquely $(0 < \varphi < 2\pi)$. It follows that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1}$. To complete the verification of condition 2, we should prove that $\mathbf{x}^{-1}$ is continuous. However, since we shall soon prove that this verification is not necessary provided we already know that the set S is a regular surface, we shall not do that here.

We remark that $\mathbf{x}(V)$ only omits a semicircle of S^2 (including the two poles) and that S^2 can be covered with the coordinate neighborhoods of two parametrizations of this type.

Another way to represent surfaces: Using Landmarks. (This gives Shape Manifold, Details later).

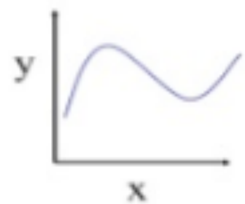


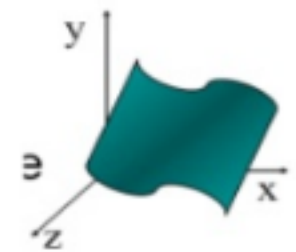
Initial situation: lung shape at EE



Recall: Explicit Representation of Surfaces (similar for curves)

- **Explicit Representation**
  - The explicit form of a curve in two dimensions
  - gives the value of one variable,
  - the dependent variable,
  - in terms of the other,
  - the independent variable
  - In *x, y space,* we might write $y = f(x)$.
  - a surface represented by an equation of the form $z = f(x, y)$





Those are called graphs.

Key: All Graphs are Regular Surfaces!

Recall: Implicit Representation of Surfaces (similar for curves)

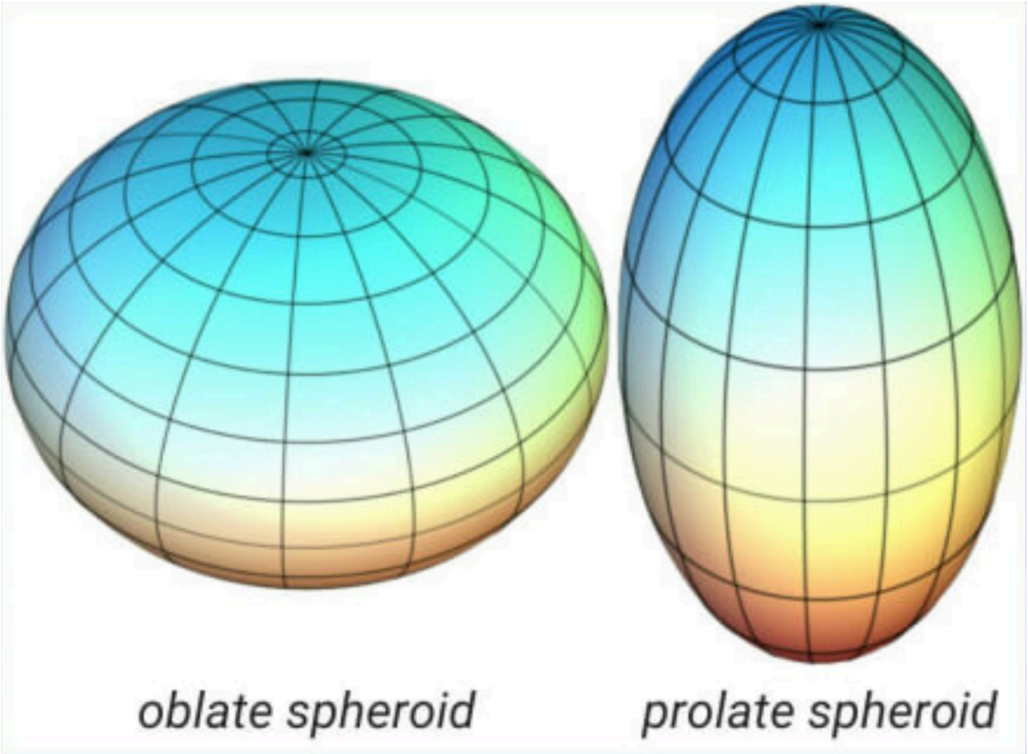
- **Implicit Representations**

  In two dimensions, an implicit curve can be represented by the equation $f(x, y) = 0$

  - The implicit form is less coordinate-system dependent than is the explicit form.
  - In three dimensions, the implicit form $f(x, y, z) = 0$
  - Curves in three dimensions are not as easily represented in implicit form.
  - We can represent a curve as the intersection, if it exists, of the two surfaces: $f(x, y, z) = 0, g(x, y, z) = 0.$

Example: An Ellipse solid in implicit form.

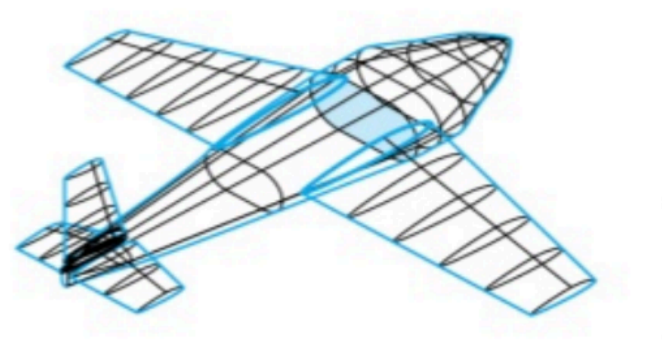
An ellipsoid is a regular surface!



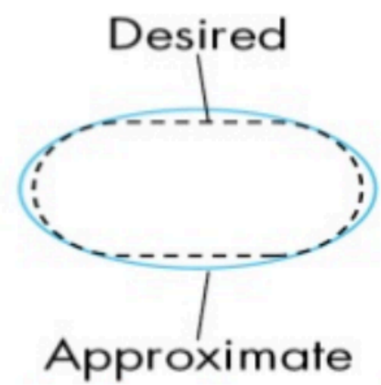
Key: Inverse image of regular value are Regular Surfaces!

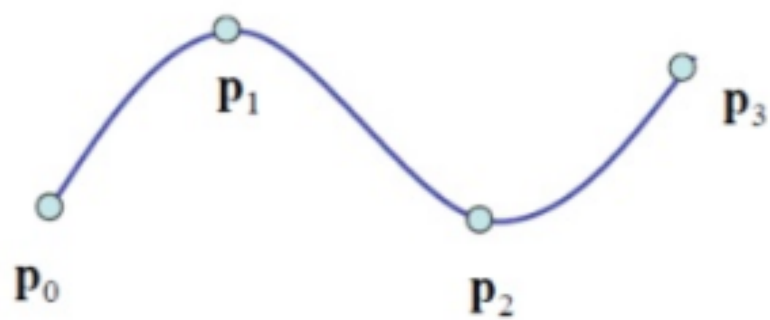
Surface Mash Representation

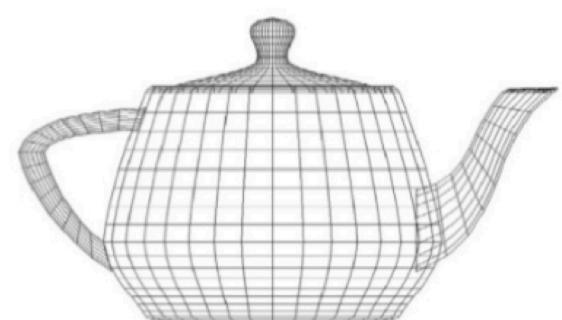
In order to understand it, we have to start with curves…



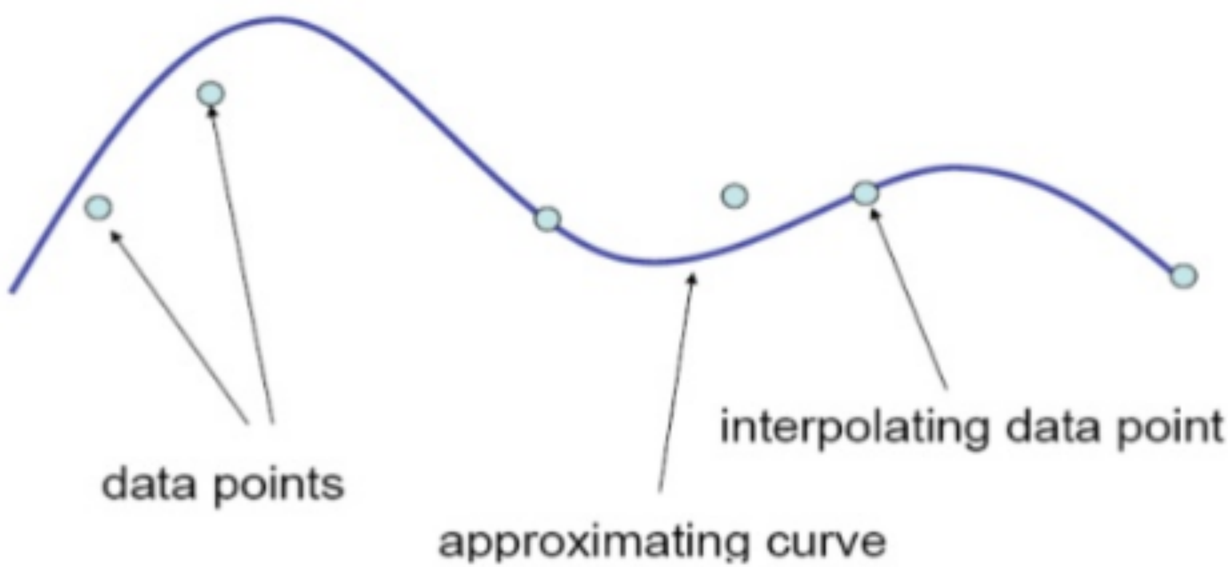
- Given 4 control points P0, P1, P2, P3
- Space 0 <= u <= 1 evenly
- P0 = P(0), P1 = P(1/3), P2 = P(2/3), P3 = P(1)







Motivation



- Do not want over-fitting or under-fitting of the data.
- Locally, choose a cubic curve often work well.

How to get a matrix representation?

$$p(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3$$

- Apply the interpolating conditions at u=0, 1/3, 2/3, 1

$$p_0 = p(0) = c_0$$

$$p_1 = p(\tfrac{1}{3}) = c_0 + \tfrac{1}{3}c_1 + (\tfrac{1}{3})^2 c_2 + (\tfrac{1}{3})^3 c_3$$

$$p_2 = p(\tfrac{2}{3}) = c_0 + \tfrac{2}{3}c_1 + (\tfrac{2}{3})^2 c_2 + (\tfrac{2}{3})^3 c_3$$

$$p_3 = p(1) = c_0 + c_1 + c_2 + c_3$$

- We can write these equations in matrix form as

p=Ac

I.e., c=A^{-1}p

$$\mathbf{p} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \mathbf{Ac} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{3}\right) & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \left(\frac{2}{3}\right) & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Solving for **c we find the *interpolation matrix***

$$\mathbf{M}_I = \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & -22.5 & 18 & -4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{bmatrix}$$

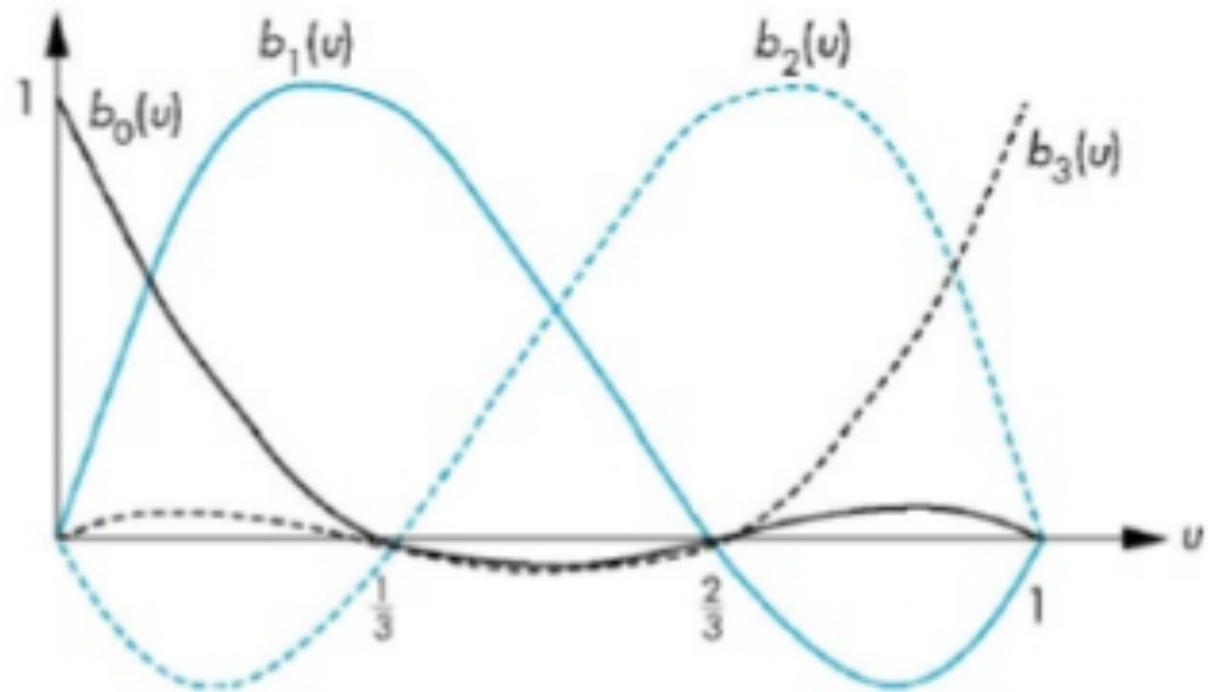
$$\mathbf{c} = \mathbf{M}_I \mathbf{p}$$

- Rewriting the equation for p(u) .

Read only!

$$\mathrm{p(u)}=\mathbf{u}^T\mathbf{c}=\mathbf{u}^T\mathbf{M}_I\mathbf{p} = \mathbf{b}(\mathrm{u})^T\mathbf{p}$$

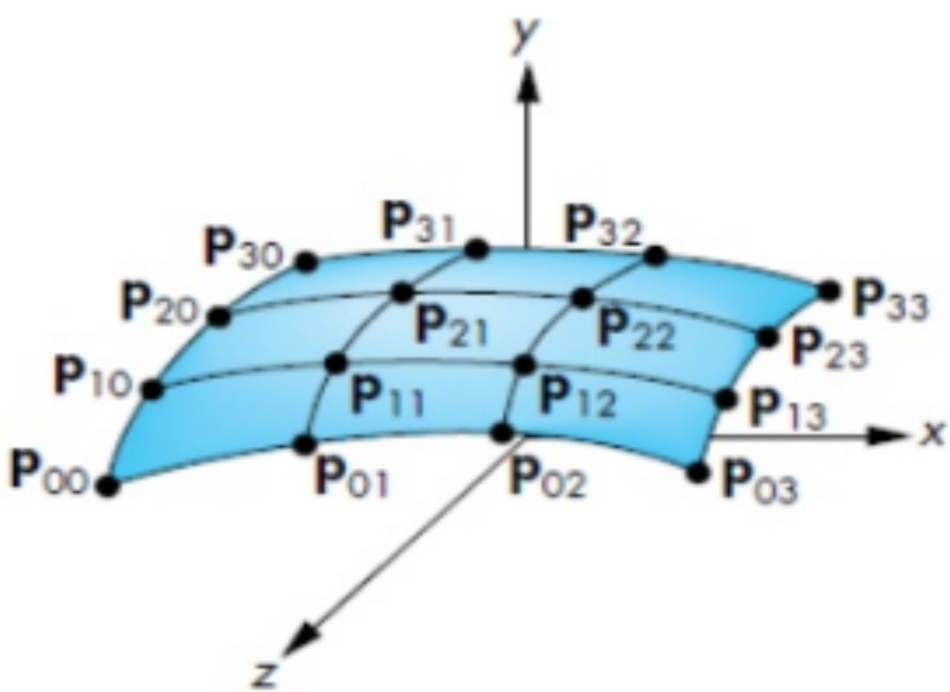
$$b(u) = (1 - u)\,P_0 + u\,P_3$$



- Shows that we can build and analyze surfaces from our knowledge of curves

Read only!

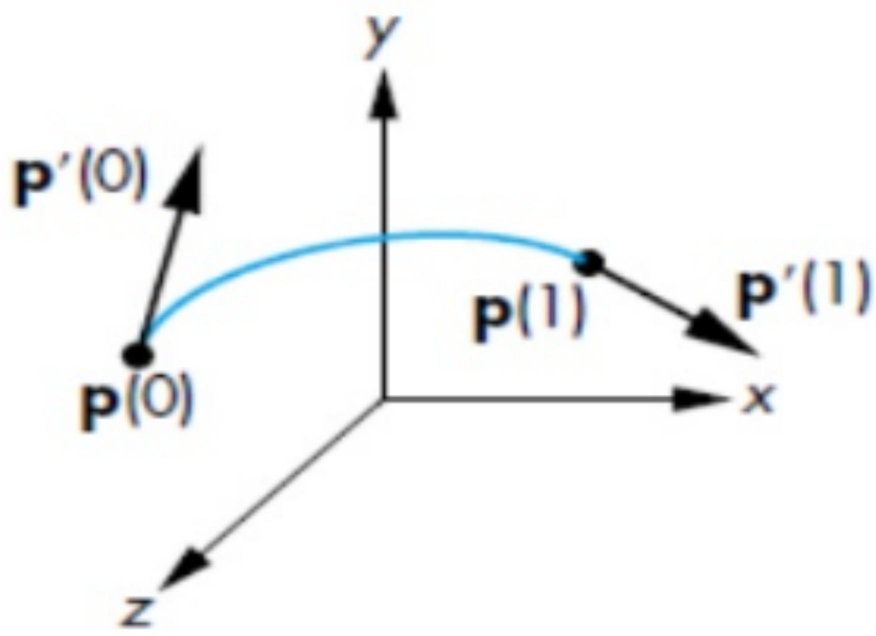
$$p(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} c_{ij} u^i v^j$$



HERMITE CURVES AND SURFACES

Read only!

- Another cubic polynomial curve
- Specify two endpoints and their tangents



The Hermite Form

Read only!

- As Before

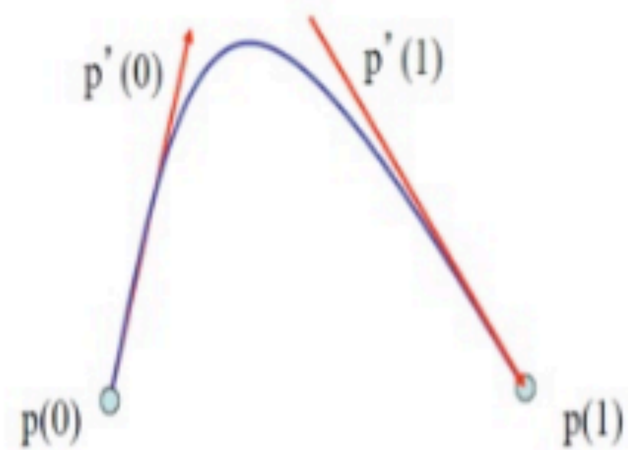
$$p(0) = p_0 = c_0$$
$$p(1) = p_1 = c_0 + c_1 + c_2 + c_3$$

- Calculate derivative $p(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3$

$$p'(u) = c_1 + 2uc_2 + 3u^2 c_3$$

- Yields

$$p'(0) = p'_0 = c_1$$
$$p'(1) = p'_1 = c_1 + 2c_2 + 3c_3$$



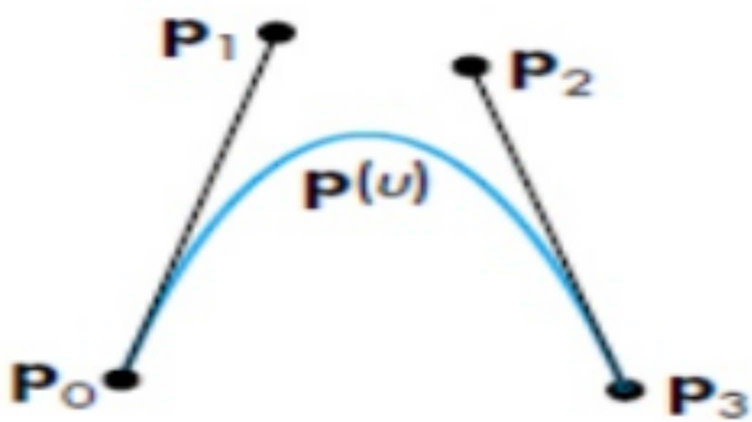
Bezier Curves

Read only!

- Widely used in computer graphics
- Approximate tangents by using control points

$$\mathbf{p}'(0) = 3(\mathbf{p}_1 - \mathbf{p}_0)$$

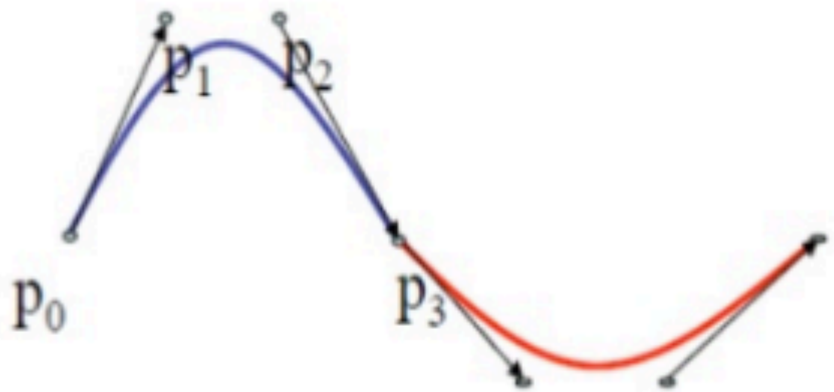
$$\mathbf{p}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2)$$



Read only!

Analysis Bezier form

- Is much better than the interpolating form
- But the derivatives are not continuous at join points

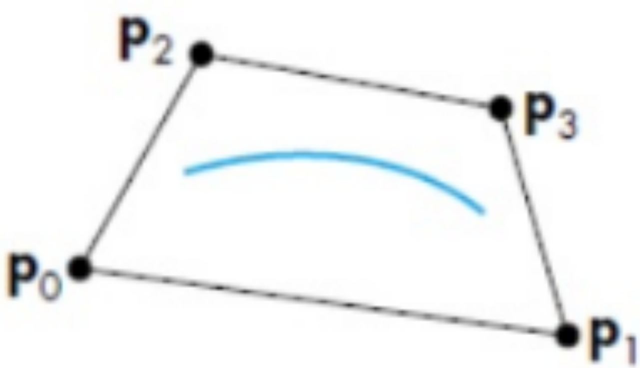


- What shall we do to solve this ?

Read only!

B-Splines

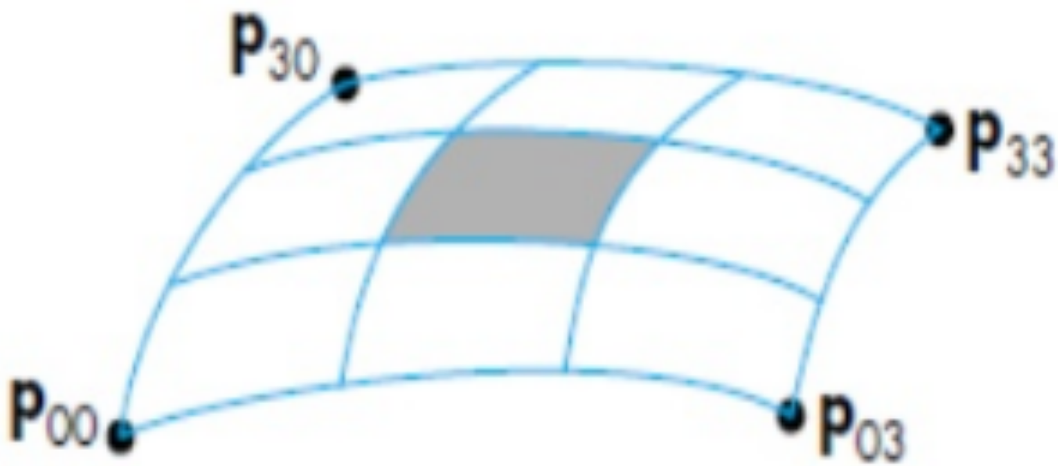
- Basis Splines
- Allows us to apply more continuity
- the curve must lie in the convex hull of the control points



Read only!

Spline Surfaces

- B-spline surfaces can be defined in a similar way



NURBS

Read only!

- Nonuniform Rational B-Spline curves and surfaces add a fourth variable w to x,y,z
  - Can interpret as weight to give more importance to some control data
  - Can also interpret as moving to homogeneous coordinate
- Requires a perspective division
  - NURBS act correctly for perspective viewing
- Quadrics are a special case of NURBS

Read only!

Rendering Curves and Surfaces

- Introduce methods to draw curves
- For explicit and parametric: we can evaluate the curve or surface at a sufficient number of points that we can approximate it with our standard flat objects
- For implicit surfaces: we can compute points on the object that are the intersection of rays from the center of projection through pixels with the object

Evaluating Polynomials

Read only!

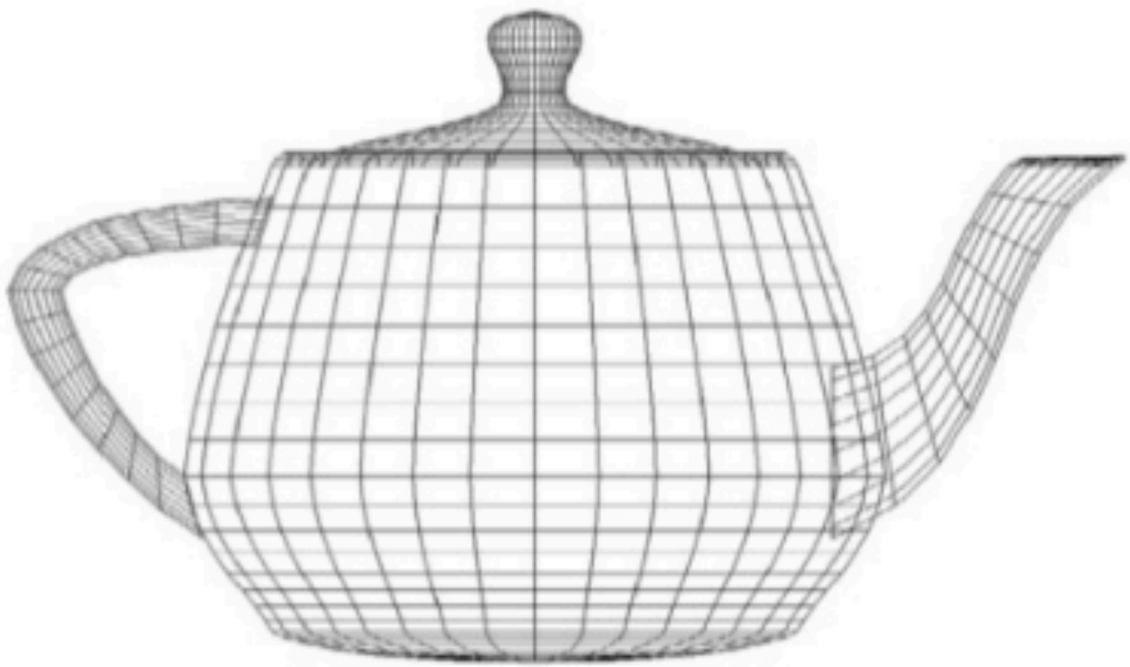
- Simplest method to render a polynomial curve is to evaluate the polynomial at many points and form an approximating polyline
- For surfaces we can form an approximating mesh of triangles or quadrilaterals
- Use Horner's method to evaluate polynomials

$$p(u)=c_0+u(c_1+u(c_2+uc_3))$$

Read only!

THE UTAH TEAPOT

- Most famous data set in computer graphics
- Widely available as a list of 306 3D vertices and the indices that define 32 Bezier patches



ALGEBRAIC SURFACES - Quadrics

Read only!

- Although quadrics can be generated as special case of NURBS curves
- Quadrics are described by implicit algebraic equations
- *Quadric can be written in the form* :

$$q(x, y, z) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{13}xz$$

$$+ b_1x + b_2y + b_3z + c = 0.$$

Quadrics

Read only!

- This class of surfaces includes ellipsoids, parabaloids, and hyperboloids
- We can write the general equation

$$\mathbf{p}^T \mathbf{A} \mathbf{p} + \mathbf{b}^T \mathbf{p} + c = 0,$$

Read only!

Mesh Subdivision

- A theory of **subdivision surfaces** has emerged that deals with both the theoretical and practical aspects of these ideas.
- We have two type of meshes:
  - triangles meshes.
  - quadrilaterals meshes.

Read only!

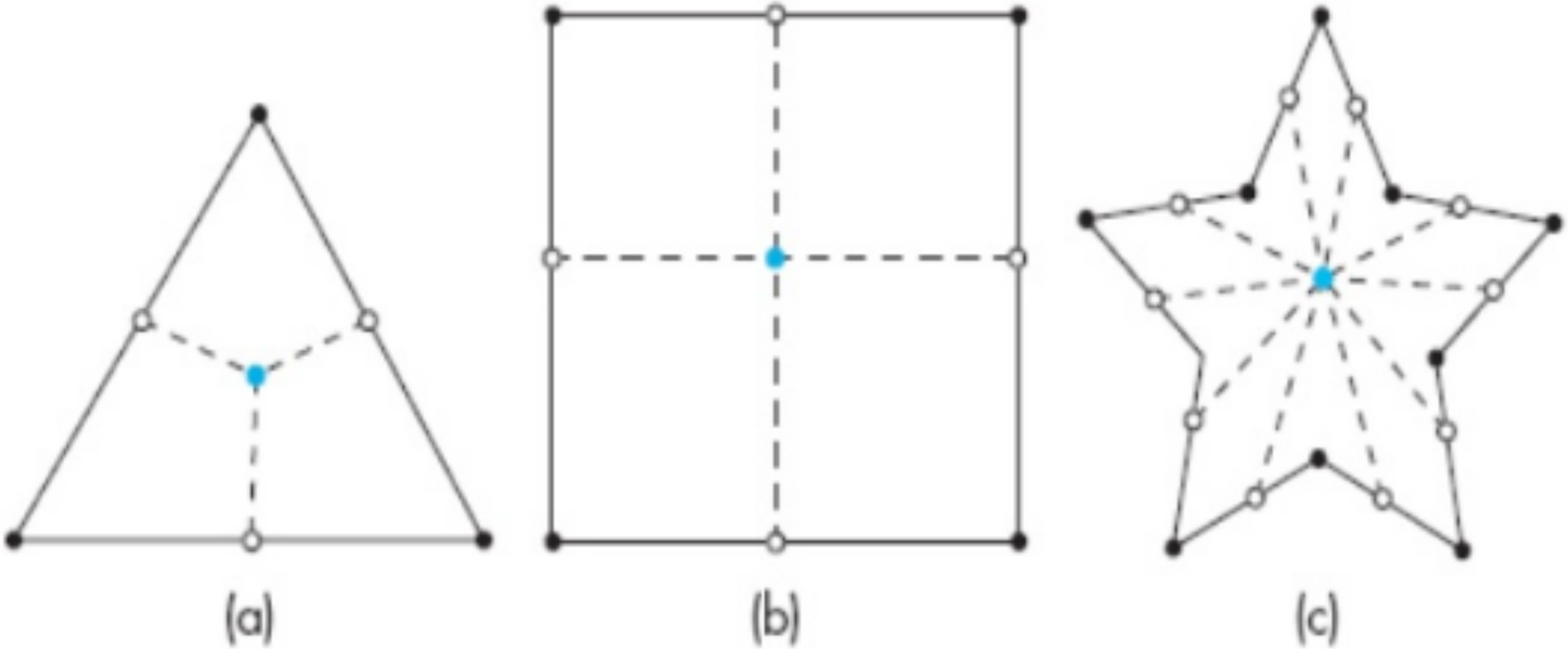
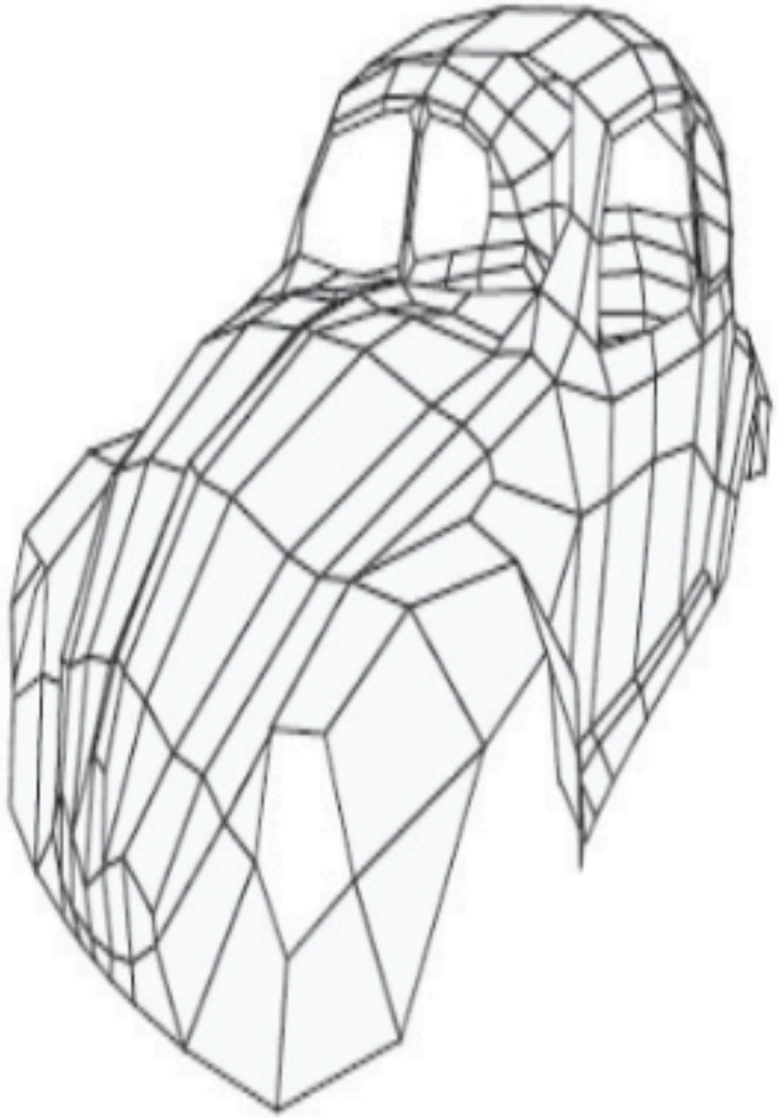
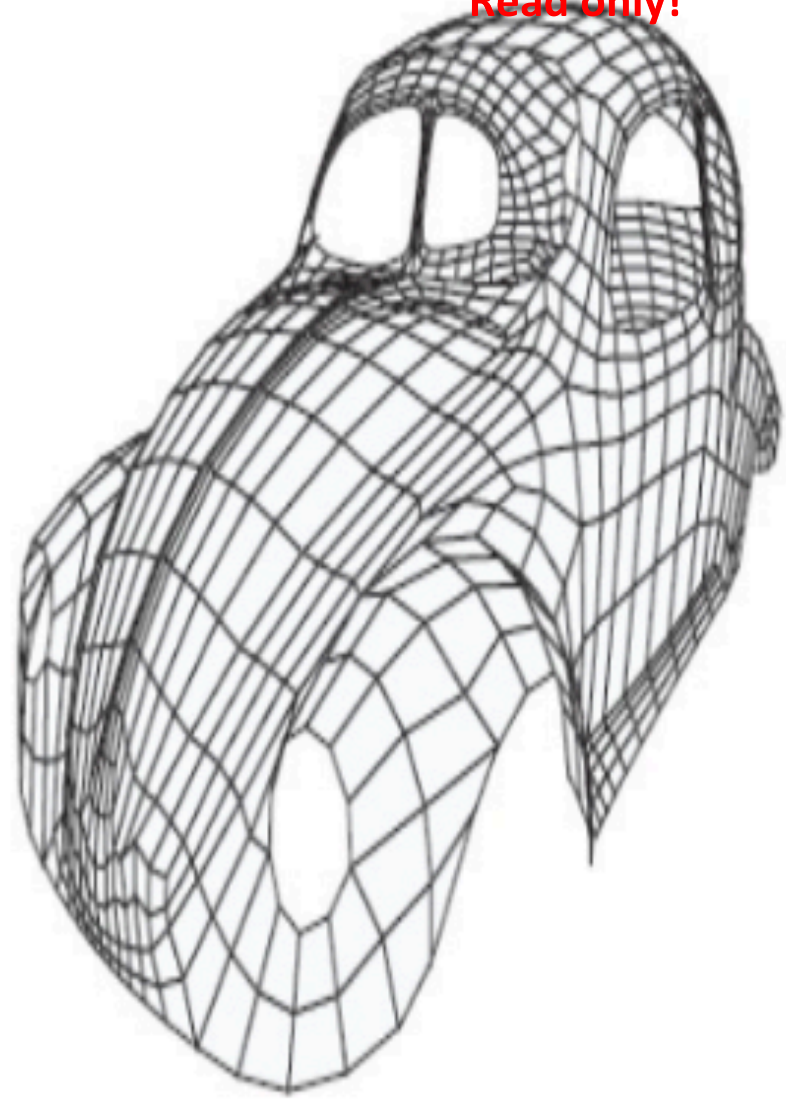


FIGURE 10.43 Polygon subdivision. (a) Triangle. (b) Rectangle. (c) Star-shaped polygon.





Mesh representations are needed before obtain rendered surfaces

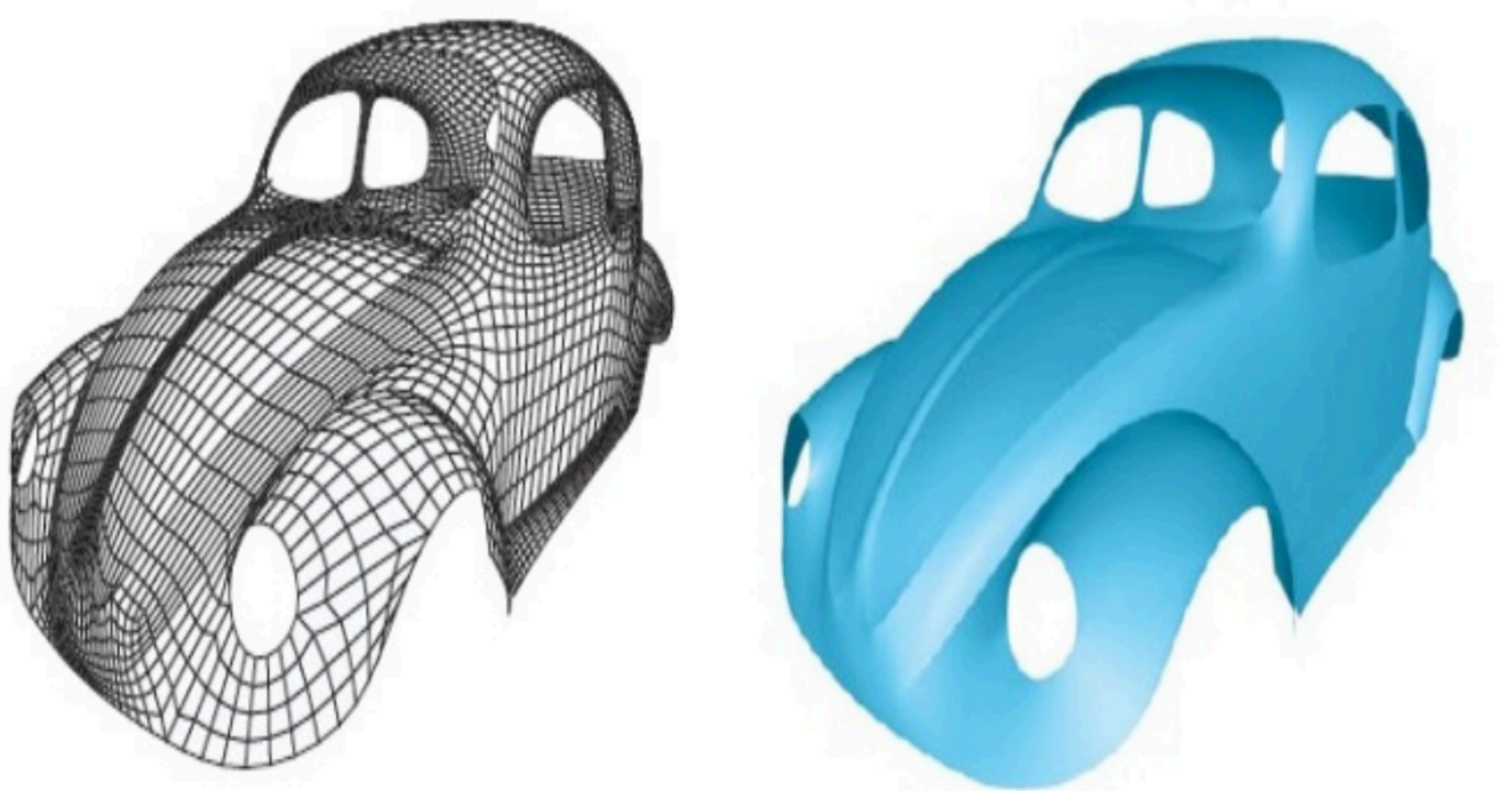


FIGURE 10.46 Successive subdivisions of polygonal mesh and rendered surface. (Images courtesy Caltech Multi-Res Modeling Group)

References

- https://www.slideshare.net/MohammedMahmoud/curves-and-surfaces