Zero-Coupon Bond Pricing in Stochastic Interest Rate Models: The Cox-Ingersoll-Ross and Hull-White Frameworks

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Abstract

This article presents a comprehensive study of zero-coupon bond pricing under the Cox-Ingersoll-Ross (CIR) and Hull-White stochastic short-rate models. We derive the governing partial differential equations from martingale principles and present their analytical solutions. A complete numerical workflow is then developed, including a robust Monte Carlo engine for derivative pricing. Our analysis demonstrates the critical role of model flexibility: using real market data, we show that simple one-factor models are structurally unable to fit the observed yield curve. We resolve this by implementing a flexible Hull-White model with a piecewise-constant drift, achieving a near-perfect calibration. This calibrated model is then used to price an at-the-money European call option, validating our simulation engine against a known analytical solution and providing a price for the CIR model where no such solution exists. The results highlight the practical trade-offs between theoretical properties and market consistency.

Contents

1 Introduction

In classical financial models, the risk-free asset evolves deterministically. Real markets, however, exhibit stochastic fluctuations of interest rates. In such environments, even the price of a zero-coupon bond becomes a contingent claim. This paper focuses on two key short-rate models: the Cox-Ingersoll-Ross (CIR) model, which enforces rate positivity, and the Hull-White model, which extends the Ornstein-Uhlenbeck process with a time-dependent drift.

2 General Framework for Zero-Coupon Pricing

2.1 The short rate and the money market account

The short rate $r(t)$ is modeled as the solution of

$$dr(t) = b(t, r(t))\,dt + \sigma(t, r(t))\,dW_t^{\mathbb{Q}}, \quad r(0) = r_0 > 0,$$

with Brownian motion $W_t^{\mathbb{Q}}$ under the risk-neutral measure $\mathbb{Q}$. The money market account evolves as

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right).$$

2.2 Risk-neutral pricing of zero-coupon bonds

By the fundamental theorem of asset pricing, the price of a zero-coupon bond maturing at T is:

$$p(t, T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right) \,\middle|\, \mathcal{F}_t\right], \quad p(T, T) = 1.$$

Assuming $p(t, T) = F_T(t, r(t))$ for a smooth function F_T, Itô's lemma gives the PDE:

$$\partial_t F_T + b(t, r)\,\partial_r F_T + \tfrac{1}{2}\sigma^2(t, r)\,\partial_{rr} F_T - rF_T = 0, \quad F_T(T, r) = 1.$$

2.3 Derivation of the Pricing PDE from Martingale Arguments

We now provide a complete proof that the zero-coupon bond price $p(t, T)$ satisfies the partial differential equation

$$\partial_t F_T + b(t, r)\,\partial_r F_T + \tfrac{1}{2}\sigma^2(t, r)\,\partial_{rr} F_T - rF_T = 0, \qquad F_T(T, r) = 1,$$

by demonstrating that the appropriately discounted bond price is a martingale under the risk-neutral measure $\mathbb{Q}$.

1. Definition of the discounted bond price

Let

$$M_t := \frac{p(t,T)}{B(t)},$$

where $B(t) = \exp\Big(\int_0^t r(s)\,ds\Big)$ is the money market account. By the risk-neutral pricing formula,

$$p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right) \,\middle|\, \mathcal{F}_t\right].$$

Then

$$M_t = \frac{1}{B(t)}\,\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right) \,\middle|\, \mathcal{F}_t\right].$$

Note that

$$\frac{1}{B(t)}\,\exp\left(-\int_t^T r(s)\,ds\right) = \exp\left(-\int_0^t r(s)\,ds\right)\,\exp\left(-\int_t^T r(s)\,ds\right) = \exp\left(-\int_0^T r(s)\,ds\right).$$

Therefore, we can write M_t as

$$M_t = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r(s)\,ds\right) \,\middle|\, \mathcal{F}_t\right].$$

2. Proof that (M_t) is a martingale

We verify the martingale property under $\mathbb{Q}$: for $0 \le s \le t \le T$,

$$\mathbb{E}^{\mathbb{Q}}[M_t \mid \mathcal{F}_s] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r(u)\,du\right) \,\middle|\, \mathcal{F}_t\right] \,\middle|\, \mathcal{F}_s\right].$$

By the tower property of conditional expectation,

$$\mathbb{E}^{\mathbb{Q}}[M_t \mid \mathcal{F}_s] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r(u)\,du\right) \,\middle|\, \mathcal{F}_s\right] = M_s.$$

Hence $(M_t)_{t\le T}$ is a martingale under $\mathbb{Q}$.

3. Martingale representation theorem

Since M_t is a square-integrable martingale with respect to the Brownian filtration $(\mathcal{F}_t)$, the martingale representation theorem ensures the existence of an adapted process $v(t)$ such that

$$M_t = M_0 + \int_0^t v(s)\,dW_s^{\mathbb{Q}}.$$

This identifies the Brownian-driven dynamics of the discounted bond price.

4. Dynamics of the bond price

We now recover the dynamics of $p(t,T)$. By Itô's product rule:

$$dp(t,T) = d(M_t B(t)) = B(t)\,dM_t + M_t\,dB(t) + d\langle M, B\rangle_t.$$

Since $dM_t = v(t)\,dW_t^{\mathbb{Q}}$ and $dB(t) = r(t)B(t)\,dt$, while $d\langle M, B\rangle_t = 0$ because B has finite variation, we obtain:

$$dp(t,T) = B(t)\,v(t)\,dW_t^{\mathbb{Q}} + M_t\,r(t)B(t)\,dt.$$

Recalling that $M_t B(t) = p(t,T)$:

$$dp(t,T) = v(t)B(t)\,dW_t^{\mathbb{Q}} + p(t,T)r(t)\,dt.$$

Thus, the bond price evolves as

$$dp(t,T) = p(t,T)\,r(t)\,dt + v(t)B(t)\,dW_t^{\mathbb{Q}}.$$

5. Interpretation: the pricing PDE

Suppose $p(t,T) = F_T(t, r(t))$ for a smooth function F_T. By Itô's lemma:

$$dp = \left(\partial_t F_T + b(t,r)\,\partial_r F_T + \tfrac{1}{2}\sigma^2(t,r)\,\partial_{rr} F_T\right)\,dt + \sigma(t,r)\,\partial_r F_T\,dW_t^{\mathbb{Q}}.$$

Comparing the dt and $dW_t^{\mathbb{Q}}$ terms with the dynamics above, we require:

$$\partial_t F_T + b(t,r)\,\partial_r F_T + \tfrac{1}{2}\sigma^2(t,r)\,\partial_{rr} F_T - rF_T = 0.$$

The boundary condition comes from $p(T,T) = 1$, hence

$$F_T(T,r) = 1.$$

Conclusion

We have shown that the discounted bond price is a martingale under $\mathbb{Q}$, which implies the bond price admits a representation $p(t,T) = F_T(t, r(t))$ satisfying the PDE:

$$\boxed{\partial_t F_T + b(t,r)\,\partial_r F_T + \tfrac{1}{2}\sigma^2(t,r)\,\partial_{rr} F_T - rF_T = 0, \qquad F_T(T,r) = 1.}$$

Connection with the Feynman-Kac Formula

The partial differential equation

$$\partial_t F_T + b(t,r)\,\partial_r F_T + \tfrac{1}{2}\sigma^2(t,r)\,\partial_{rr} F_T - rF_T = 0, \quad F_T(T,r) = 1,$$

can be interpreted through the *Feynman-Kac formula*, which links parabolic PDEs to conditional expectations of stochastic processes.

More precisely, if $(r_t)_{t\geq 0}$ is the solution to the stochastic differential equation

$$dr_t = b(t, r_t)\,dt + \sigma(t, r_t)\,dW_t^{\mathbb{Q}},$$

then the Feynman-Kac theorem states that the unique solution $F_T(t,r)$ to the PDE with terminal condition $F_T(T,r) = 1$ admits the probabilistic representation

$$F_T(t,r) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s\,ds\right)\,\middle|\, r_t = r\right].$$

This provides the fundamental link between the stochastic model for the short rate and the pricing of zero-coupon bonds as discounted expected payoffs under the risk-neutral measure.

3 The Cox-Ingersoll-Ross (CIR) Model

3.1 Dynamics of the short rate

The CIR model specifies

$$dr(t) = (b - \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dW_t^{\mathbb{Q}}, \quad r(0) = r_0.$$

The Feller condition $2b \geq \sigma^2$ ensures that $r(t)$ remains strictly positive if $r_0 > 0$.

3.2 Affine solution for bond prices

We assume that the zero-coupon bond price admits an affine form in the short rate $r(t)$:

$$p(t,T) = \exp\big(A(t,T) - B(t,T)\, r(t)\big).$$

Substituting this expression into the PDE satisfied by $p(t,T)$,

$$\partial_t p + \mu_r(t,r)\, \partial_r p + \tfrac{1}{2}\sigma^2 r\, \partial_{rr} p - r\, p = 0,$$

with the boundary condition $p(T,T) = 1$, and using $\mu_r(t,r) = b - \beta r$, we compute:

$$\begin{cases} \partial_t p = (\partial_t A - \partial_t B\, r)\, p, \\ \partial_r p = -B\, p, \\ \partial_{rr} p = B^2\, p. \end{cases}$$

Replacing these into the PDE and dividing by $p > 0$ yields:

$$\partial_t A - r\, \partial_t B - bB + \beta r B + \tfrac{1}{2}\sigma^2 r B^2 - r = 0.$$

Grouping terms by powers of r gives

$$\underbrace{\partial_t A - bB}_{\text{constant term}} + r\, \underbrace{\big(- \partial_t B + \beta B + \tfrac{1}{2}\sigma^2 B^2 - 1\big)}_{\text{coefficient of } r} = 0.$$

Since this must hold for all $r \geq 0$, the coefficients vanish separately, yielding the Riccati system:

$$\begin{cases} \partial_t A(t,T) = b\, B(t,T), & A(T,T) = 0, \\ \partial_t B(t,T) = -\frac{1}{2}\sigma^2 B^2(t,T) + \beta B(t,T) - 1, & B(T,T) = 0. \end{cases}$$

3.3 Explicit solution

Define

$$d = \sqrt{\beta^2 + 2\sigma^2}.$$

The explicit solution to the Riccati equation with boundary condition $B(T,T) = 0$ is

$$B(t,T) = \frac{2\big(e^{d(T-t)} - 1\big)}{(d+\beta)\big(e^{d(T-t)} - 1\big) + 2d}.$$

Then $A(t,T)$ is obtained by integrating the first equation:

$$A(t,T) = \frac{2b}{\sigma^2} \ln\left(\frac{2d\, e^{\frac{(d+\beta)(T-t)}{2}}}{(d+\beta)(e^{d(T-t)} - 1) + 2d} \right).$$

Finally, the bond price is recovered as

$$p(t,T) = \exp\big(A(t,T) - B(t,T)\, r(t)\big).$$

This closed-form solution highlights the affine nature of the CIR model and allows efficient pricing and calibration in practice.

4 The Hull-White Model

4.1 Model dynamics

The Hull-White short rate model is given by:

$$dr(t) = \big(b(t) - \beta r(t)\big)\, dt + \sigma\, dW_t^{\mathbb{Q}}, \quad r(0) = r_0,$$

where $\beta > 0$ is the mean-reversion rate, $\sigma > 0$ is the volatility, and $b(t)$ is an integrable time-dependent drift.

4.2 Explicit solution of the short rate

The SDE is linear and admits the explicit solution:

$$r(t) = r_0 e^{-\beta t} + \int_0^t e^{-\beta(t-s)} b(s)\, ds + \sigma \int_0^t e^{-\beta(t-s)}\, dW_s^{\mathbb{Q}}.$$

4.3 Affine solution for bond prices

Assume again $p(t,T) = \exp(A(t,T) - B(t,T)r)$. The PDE leads to:

$$\begin{cases} \partial_t A(t,T) = b(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), & A(T,T) = 0, \\ \partial_t B(t,T) = \beta B(t,T) - 1, & B(T,T) = 0. \end{cases}$$

4.4 Explicit solution

Solving for B:

$$B(t,T) = \frac{1 - e^{-\beta(T-t)}}{\beta}.$$

Then

$$A(t,T) = \int_t^T \left(\tfrac{1}{2}\sigma^2 B^2(s,T) - b(s)B(s,T)\right) ds.$$

Thus,

$$p(t,T) = \exp(A(t,T) - B(t,T)r(t)).$$

5 Numerical Methodology

5.1 Pricing Framework for Derivatives

We consider a European option with payoff depending on the short rate at maturity:

$$\Phi(r_T) = (r_T - k)^+, \quad k > 0.$$

This type of derivative is sensitive both to the distribution of r_T and to the discounting induced by the short rate path.

5.2 Risk-neutral valuation

Under the risk-neutral measure $\mathbb{Q}$, the time-t price is

$$C_t = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\, ds\right) (r_T - k)^+ \,\middle|\, \mathcal{F}_t\right].$$

The presence of the stochastic discount factor makes the pricing problem path-dependent, even though the payoff only depends on r_T.

5.3 Link to PDEs

By the Feynman–Kac theorem, the price $C(t, r)$ can equivalently be characterized as the solution of the parabolic PDE

$$\partial_t C + \mu_r(t, r)\,\partial_r C + \tfrac{1}{2}\sigma^2(t, r)\,\partial_{rr} C - r\,C = 0, \qquad C(T, r) = (r - k)^+,$$

where (μ_r, σ) are given by the short-rate dynamics (CIR or Hull–White). In practice, closed-form solutions are unavailable, motivating the use of Monte Carlo methods.

5.4 Monte Carlo algorithm

We simulate N independent trajectories $\{r^{(i)}(s),\, t \le s \le T\}_{i=1}^N$ of the short rate under the chosen model.

1. **Discretization.** Partition the interval $[t, T]$ into M steps of size $\Delta t = (T - t)/M$.
   - For the Hull–White model: use the Euler–Maruyama scheme, which is stable due to Gaussian increments.
   - For the CIR model: prefer *full truncation Euler* or *Milstein with reflection* to preserve positivity, since the naive Euler scheme may produce negative rates.
2. **Pathwise payoff computation.** For each simulated path i:

$$D^{(i)} = \exp\left(-\sum_{m=0}^{M-1} r_m^{(i)}\,\Delta t\right), \quad \text{(approximating the integral as a left Riemann sum)} \qquad \Phi^{(i)} = \left(r_M^{(i)} - k\right)^+.$$

3. **Estimator.** Approximate the price as

$$C_t \approx \frac{1}{N}\sum_{i=1}^{N} D^{(i)}\,\Phi^{(i)}.$$

5.5 Variance reduction techniques

Monte Carlo convergence is slow ($O(N^{-1/2})$). To improve efficiency:

- **Antithetic variates:** simulate pairs of paths using W and $-W$ to reduce variance.
- **Control variates:** use a related payoff with known expectation (e.g., a zero-coupon bond) as a control.
- **Quasi-Monte Carlo:** replace pseudo-random numbers with low-discrepancy sequences (Sobol, Halton).

5.6 Comparison of CIR and Hull–White models

- **CIR:** rates remain positive, distribution of r_T is noncentral chi-squared. Requires careful discretization to avoid bias.
- **Hull–White:** Gaussian increments allow simple simulation, but negative rates are possible.
- For both models, the stochastic discount factor introduces correlation between payoff and discounting, increasing simulation variance.

5.7 Convergence considerations

The accuracy depends on both N and M:

$$\text{Error} \approx O\!\left(\frac{1}{\sqrt{N}}\right) + O\big(\Delta t^{p}\big), \quad \text{where } p \text{ depends on the chosen discretization scheme.}$$

Balancing the number of paths and the time discretization is essential for reliable results.

5.8 Variance reduction

When the payoff is rare (e.g. $r_T > k$ infrequent), importance sampling is beneficial. Parameters can be shifted to increase the frequency of rare events, then corrected by likelihood ratios.

6 Numerical Implementation and Analysis

In this section, we apply the theoretical frameworks discussed previously to practical scenarios. We implement the CIR and Hull-White models, calibrate them to real market data, and use Monte Carlo simulation to price a European call option on the short rate. All simulations and analyses are performed using a custom-built Python library designed with the object-oriented principles outlined in this paper.

6.1 Visualization of Model Dynamics

To gain an intuitive understanding of the models, we first simulate and visualize sample trajectories of the short rate $r(t)$. For this illustration, we use a common set of mean-reversion parameters and an initial rate of $r_0 = 3\%$. Figure 1 illustrates the results.

The key differences are immediately apparent. The CIR model, by construction, ensures that all paths remain strictly positive. The Hull-White model, being Gaussian in nature, allows for paths to become negative, although this is infrequent with the chosen parameters.

(a) CIR Model Trajectories (b) Hull-White Model Trajectories

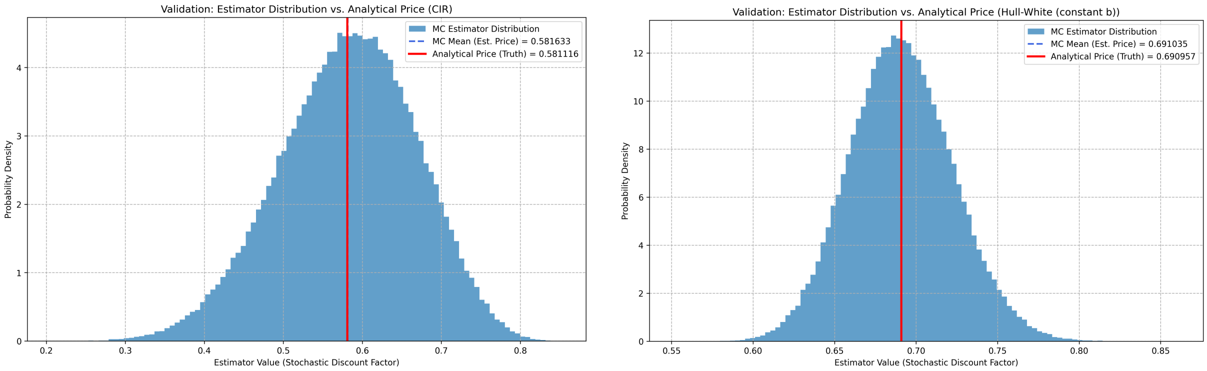
Figure 1: Ten simulated short-rate paths for the CIR and Hull-White models over a 5-year horizon. The CIR paths are constrained to be non-negative.

6.2 Validation of the Monte Carlo Engine

Before using the Monte Carlo engine for pricing complex derivatives, we must validate its accuracy. The most robust method is to price an instrument for which a closed-form solution is known and compare the results. The zero-coupon bond is the perfect candidate for this test.

We compute the price of a 5-year zero-coupon bond using both the analytical formula and the Monte Carlo estimator, which is the sample mean of the stochastic discount factors

$\mathbb{E}[\exp(-\int_0^T r_s ds)]$. Figure 2 shows the distribution of the 200,000 simulated discount factors for both the CIR and Hull-White models.



(a) CIR Model Validation

(b) Hull-White Model Validation

Figure 2: Distribution of the Monte Carlo estimators (discount factors) for a 5-year zero-coupon bond. The mean of the distribution (blue line) converges to the exact analytical price (red line), validating the accuracy of the simulation engine.

For both models, the mean of the simulated distribution (the Monte Carlo price) aligns almost perfectly with the exact analytical price. This successful validation confirms that our Euler-Maruyama simulation scheme is correctly implemented and provides a high degree of confidence in the pricing results for derivatives that lack an analytical solution, such as the European call option under the CIR model.

6.3 Calibration to Market Data

A crucial test for any interest rate model is its ability to fit the observed term structure. We calibrate three distinct models to the U.S. Treasury yield curve fetched from the FRED database: the standard CIR model, a simple Hull-White model with a constant drift parameter b, and a more flexible Hull-White model where the drift $b(t)$ is a piecewise-constant function. The parameters for each are optimized to minimize the mean squared error between the model-implied yields and the market yields.

The optimized parameters are summarized in Table 1, and the resulting goodness-of-fit is visualized in Figure 3.

Table 1: Calibrated parameters for the three models.

Model	**Optimized Parameters**
, CIR (3-param)	$b = 0.0132$, $\beta = 0.3043$, $\sigma = 0.1010$
Hull-White (3-param)	$b_{const} = 0.0087$, $\beta = 0.2040$, $\sigma = 0.0012$
Flexible Hull-White (5-param)	$b_1(0-2\text{y}) = 0.0014$, $b_2(2-10\text{y}) = 0.0121$, $b_3(>10\text{y}) = 0.0107$ $\beta = 0.2061$, $\sigma = 0.0120$

The results in Figure 3 are highly illustrative of a core challenge in quantitative finance. The one-factor models with constant parameters (CIR and the simple Hull-White) are structurally incapable of capturing the complex, non-monotonic shape of the real market curve. They are too rigid, resulting in significant pricing errors across the term structure.

In contrast, the flexible Hull-White model, with its piecewise-constant drift, achieves a nearly perfect fit. By allowing the mean-reversion level to change over time, the model gains enough degrees of freedom to match the market data accurately. This demonstrates that for practical applications such as pricing and hedging, model flexibility is not just a desirable feature but a critical requirement for ensuring consistency with observed market prices.

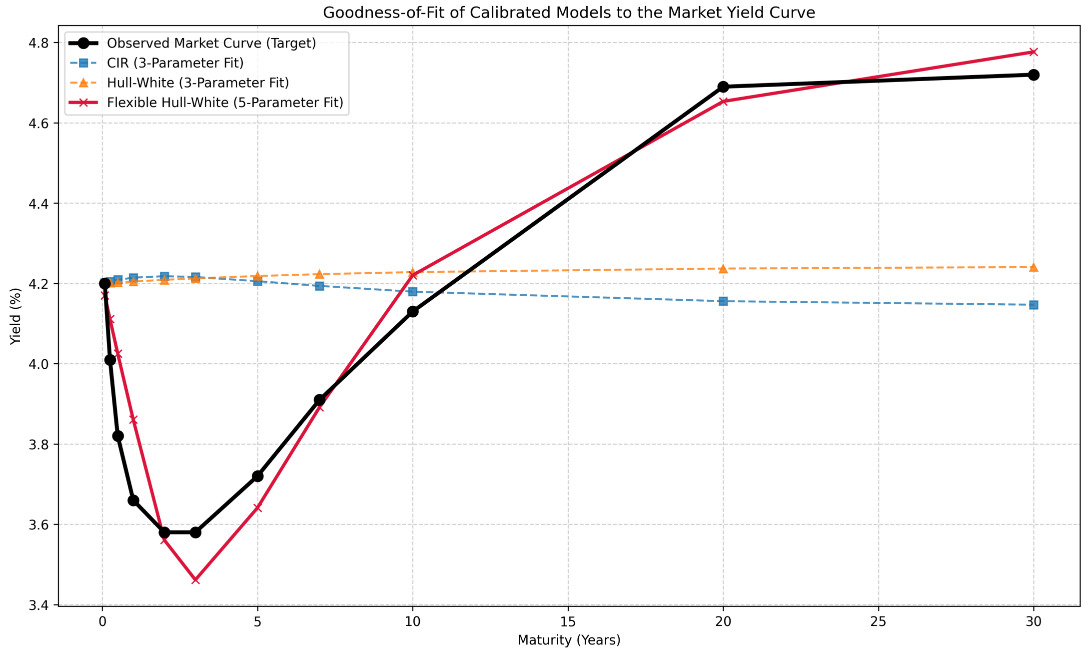


Figure 3: Goodness-of-fit of the calibrated models. The simple 3-parameter models fail to capture the market curve's shape, while the flexible 5-parameter Hull-White model provides a near-perfect fit.

6.4 Application: Monte Carlo Pricing of an At-the-Money Call Option

We now apply the Monte Carlo method to price a European call option on the short rate. To create a standard test case, we price an **at-the-money (ATM)** option, where the strike price K is set equal to the initial short rate $r_0 = 2.4\%$. The option has a maturity of $T = 1$ year, and its payoff is $(r_T - K)^+$.

An analytical solution exists for this option under the Hull-White model, providing a definitive benchmark. Conversely, no simple closed-form solution exists for the CIR model, making simulation indispensable. We use the calibrated models for all calculations, with a high-precision run of one million simulations to ensure stable price estimates. The results are summarized in Table 2.

Table 2: Pricing results for a 1-year ATM call option ($K = r_0 = 2.4\%$).

Metric	**Flexible Hull-White**	**Calibrated CIR**
Analytical Price (Benchmark)	[0.002832]	N/A
Monte Carlo Price (N=1M)	[0.002832]	[0.007971]
Standard Error	[0.0005%]	[0.0011%]

The Monte Carlo estimate for the Hull-White model aligns almost perfectly with its analytical price, which provides a strong final validation of our simulation engine's accuracy. This gives us confidence in the price obtained for the CIR model. The significant difference in price between the two models—with the CIR price being substantially higher—is a direct consequence of the higher volatility parameter ($\sigma = 0.1010$) calibrated for the CIR model, as option prices are highly sensitive to volatility.

Figure 4 illustrates the price convergence for both models, with a high-precision run of one million simulations.

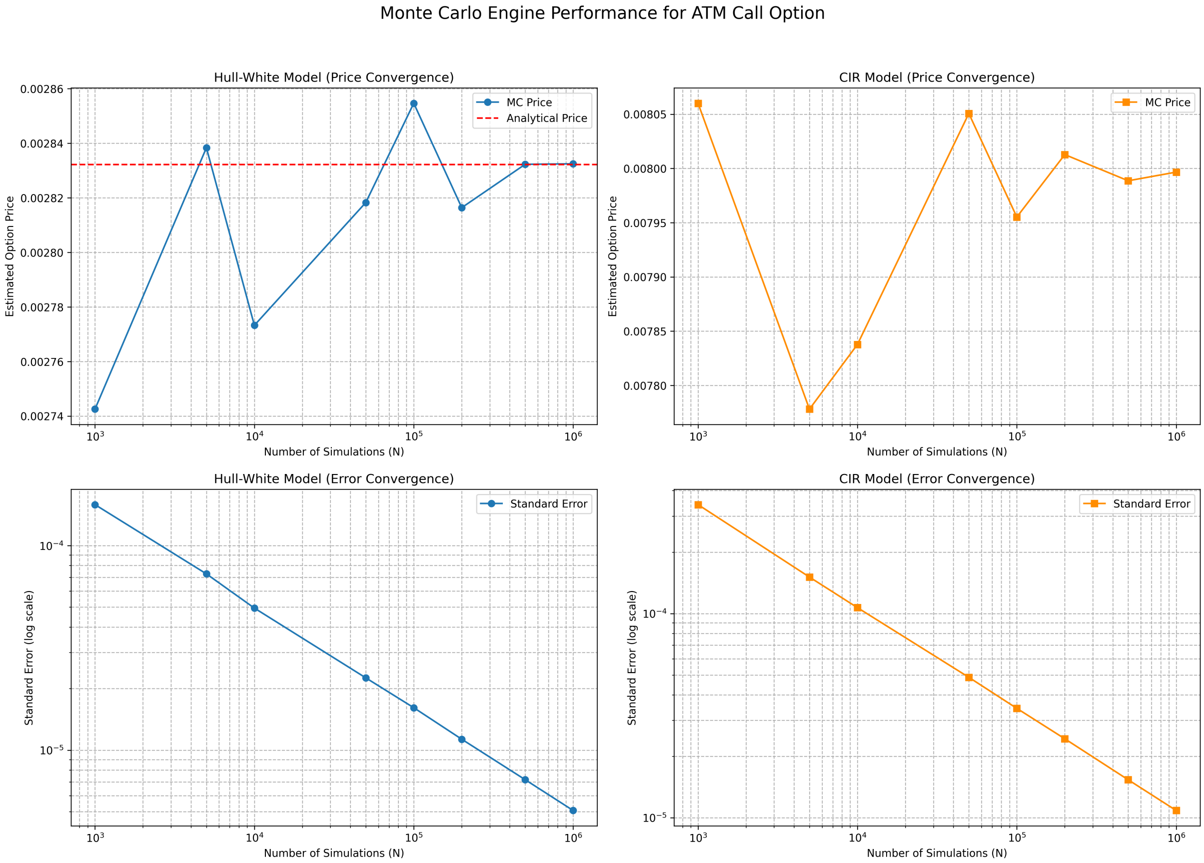


Figure 4: Price convergence for both models. The left panel shows the Hull-White estimate converging precisely to the known analytical value. The right panel shows the CIR price stabilizing as the number of simulations becomes large, demonstrating the reliability of the final estimate.

7 Discussion and Conclusion

The CIR and Hull-White models provide complementary approaches to short-rate modeling. CIR guarantees positivity of rates and has a strong theoretical foundation, while Hull-White allows for flexible calibration to the term structure. Both admit closed-form affine bond pricing formulas and lend themselves well to Monte Carlo simulation for more complex derivatives.

Our numerical analysis confirms these theoretical properties and yields critical practical insights. The calibration exercise demonstrates that simple one-factor models with constant parameters are structurally unable to fit the complex shapes of real-world yield curves. This highlights the "model risk" inherent in using overly simplistic frameworks. We showed that enhancing the Hull-White model with a flexible, piecewise-constant drift is a necessary step to achieve a near-perfect fit, making it a suitable choice for pricing and risk management where consistency with current market prices is paramount.

Finally, our implementation of the Monte Carlo engine, used in conjunction with the calibrated flexible model, successfully priced a European call option and validated the theoretical convergence rates. This completes a realistic workflow: from market data, to model calibration, to the pricing of an exotic derivative. In practice, the choice of model ultimately depends on the specific application, balancing the need for theoretical consistency against the practical and critical requirement of market calibration.