Zero-Coupon Bond Pricing in Stochastic Interest Rate Models: The Cox-Ingersoll-Ross and Hull-White Frameworks

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Abstract

This article presents a comprehensive study of zero-coupon bond pricing in two major stochastic short-rate models: the Cox-Ingersoll-Ross (CIR) and Hull-White models. We begin with the general martingale framework under the risk-neutral measure, derive the associated partial differential equations (PDEs), and solve them explicitly for each model. Numerical methods, including Euler discretization and Monte Carlo simulation, are introduced for both models. We conclude with a critical discussion of their theoretical features and practical applicability.

Contents

1 Introduction

In classical financial models, the risk-free asset evolves deterministically. Real markets, however, exhibit stochastic fluctuations of interest rates. In such environments, even the price of a zero-coupon bond becomes a contingent claim. This paper focuses on two key short-rate models: the Cox-Ingersoll-Ross (CIR) model, which enforces rate positivity, and the Hull-White model, which extends the Ornstein-Uhlenbeck process with a time-dependent drift.

2 General Framework for Zero-Coupon Pricing

2.1 The short rate and the money market account

The short rate $r(t)$ is modeled as the solution of

$$dr(t) = b(t, r(t))\, dt + \sigma(t, r(t))\, dW_t^{\mathbb{Q}}, \quad r(0) = r_0 > 0,$$

with Brownian motion $W_t^{\mathbb{Q}}$ under the risk-neutral measure $\mathbb{Q}$. The money market account evolves as

$$B(t) = \exp\left(\int_0^t r(s)\, ds\right).$$

2.2 Risk-neutral pricing of zero-coupon bonds

By the fundamental theorem of asset pricing, the price of a zero-coupon bond maturing at T is:

$$p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\, ds\right) \,\middle|\, \mathcal{F}_t\right], \quad p(T,T) = 1.$$

Assuming $p(t,T) = F_T(t, r(t))$ for a smooth function F_T, Itô's lemma gives the PDE:

$$\partial_t F_T + b(t,r)\, \partial_r F_T + \tfrac{1}{2}\sigma^2(t,r)\, \partial_{rr} F_T - rF_T = 0, \quad F_T(T,r) = 1.$$

2.3 Derivation of the Pricing PDE from Martingale Arguments

We now provide a complete proof that the zero-coupon bond price $p(t,T)$ satisfies the partial differential equation

$$\partial_t F_T + b(t,r)\, \partial_r F_T + \tfrac{1}{2}\sigma^2(t,r)\, \partial_{rr} F_T - rF_T = 0, \qquad F_T(T,r) = 1,$$

by demonstrating that the appropriately discounted bond price is a martingale under the risk-neutral measure $\mathbb{Q}$.

1. Definition of the discounted bond price

Let

$$M_t := \frac{p(t,T)}{B(t)},$$

where $B(t) = \exp\left(\int_0^t r(s)\, ds\right)$ is the money market account. By the risk-neutral pricing formula,

$$p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\, ds\right) \,\middle|\, \mathcal{F}_t\right].$$

Then

$$M_t = \frac{1}{B(t)} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right) \,\middle|\, \mathcal{F}_t\right].$$

Note that

$$\frac{1}{B(t)} \exp\left(-\int_t^T r(s)\,ds\right) = \exp\left(-\int_0^t r(s)\,ds\right) \exp\left(-\int_t^T r(s)\,ds\right) = \exp\left(-\int_0^T r(s)\,ds\right).$$

Therefore, we can write M_t as

$$M_t = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r(s)\,ds\right) \,\middle|\, \mathcal{F}_t\right].$$

2. Proof that (M_t) is a martingale

We verify the martingale property under $\mathbb{Q}$: for $0 \le s \le t \le T$,

$$\mathbb{E}^{\mathbb{Q}}[M_t \mid \mathcal{F}_s] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r(u)\,du\right) \,\middle|\, \mathcal{F}_t\right] \middle| \mathcal{F}_s\right].$$

By the tower property of conditional expectation,

$$\mathbb{E}^{\mathbb{Q}}[M_t \mid \mathcal{F}_s] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r(u)\,du\right) \,\middle|\, \mathcal{F}_s\right] = M_s.$$

Hence $(M_t)_{t \le T}$ is a martingale under $\mathbb{Q}$.

3. Martingale representation theorem

Since M_t is a square-integrable martingale with respect to the Brownian filtration $(\mathcal{F}_t)$, the martingale representation theorem ensures the existence of an adapted process $v(t)$ such that

$$M_t = M_0 + \int_0^t v(s)\,dW_s^{\mathbb{Q}}.$$

This identifies the Brownian-driven dynamics of the discounted bond price.

4. Dynamics of the bond price

We now recover the dynamics of $p(t,T)$. By Itô's product rule:

$$dp(t,T) = d(M_t B(t)) = B(t)\,dM_t + M_t\,dB(t) + d\langle M, B\rangle_t.$$

Since $dM_t = v(t)\,dW_t^{\mathbb{Q}}$ and $dB(t) = r(t)B(t)\,dt$, while $d\langle M, B\rangle_t = 0$ because B has finite variation, we obtain:

$$dp(t,T) = B(t)\,v(t)\,dW_t^{\mathbb{Q}} + M_t\,r(t)B(t)\,dt.$$

Recalling that $M_t B(t) = p(t,T)$:

$$dp(t,T) = v(t)B(t)\,dW_t^{\mathbb{Q}} + p(t,T)r(t)\,dt.$$

Thus, the bond price evolves as

$$dp(t,T) = p(t,T)\,r(t)\,dt + v(t)B(t)\,dW_t^{\mathbb{Q}}.$$

5. Interpretation: the pricing PDE

Suppose $p(t,T) = F_T(t, r(t))$ for a smooth function F_T. By Itô's lemma:

$$dp = \left(\partial_t F_T + b(t,r)\,\partial_r F_T + \tfrac{1}{2}\sigma^2(t,r)\,\partial_{rr} F_T\right)\,dt + \sigma(t,r)\,\partial_r F_T\,dW_t^{\mathbb{Q}}.$$

Comparing the dt and $dW_t^{\mathbb{Q}}$ terms with the dynamics above, we require:

$$\partial_t F_T + b(t,r)\,\partial_r F_T + \tfrac{1}{2}\sigma^2(t,r)\,\partial_{rr} F_T - rF_T = 0.$$

The boundary condition comes from $p(T,T) = 1$, hence

$$F_T(T,r) = 1.$$

Conclusion

We have shown that the discounted bond price is a martingale under $\mathbb{Q}$, which implies the bond price admits a representation $p(t,T) = F_T(t, r(t))$ satisfying the PDE:

$$\boxed{\partial_t F_T + b(t,r)\,\partial_r F_T + \tfrac{1}{2}\sigma^2(t,r)\,\partial_{rr} F_T - rF_T = 0, \qquad F_T(T,r) = 1.}$$

Connection with the Feynman-Kac Formula

The partial differential equation

$$\partial_t F_T + b(t,r)\,\partial_r F_T + \tfrac{1}{2}\sigma^2(t,r)\,\partial_{rr} F_T - rF_T = 0, \quad F_T(T,r) = 1,$$

can be interpreted through the *Feynman-Kac formula*, which links parabolic PDEs to conditional expectations of stochastic processes.

More precisely, if $(r_t)_{t\geq 0}$ is the solution to the stochastic differential equation

$$dr_t = b(t, r_t)\,dt + \sigma(t, r_t)\,dW_t^{\mathbb{Q}},$$

then the Feynman-Kac theorem states that the unique solution $F_T(t,r)$ to the PDE with terminal condition $F_T(T,r) = 1$ admits the probabilistic representation

$$F_T(t,r) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s\,ds\right)\,\middle|\, r_t = r\right].$$

This provides the fundamental link between the stochastic model for the short rate and the pricing of zero-coupon bonds as discounted expected payoffs under the risk-neutral measure.

3 The Cox-Ingersoll-Ross (CIR) Model

3.1 Dynamics of the short rate

The CIR model specifies

$$dr(t) = (b - \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dW_t^{\mathbb{Q}}, \quad r(0) = r_0.$$

The Feller condition $2b \geq \sigma^2$ ensures that $r(t)$ remains strictly positive if $r_0 > 0$.

3.2 Affine solution for bond prices

We assume that the zero-coupon bond price admits an affine form in the short rate $r(t)$:

$$p(t,T) = \exp\big(A(t,T) - B(t,T)\, r(t)\big).$$

Substituting this expression into the PDE satisfied by $p(t,T)$,

$$\partial_t p + \mu_r(t,r)\, \partial_r p + \tfrac{1}{2}\sigma^2 r\, \partial_{rr} p - r\, p = 0,$$

with the boundary condition $p(T,T) = 1$, and using $\mu_r(t,r) = b - \beta r$, we compute:

$$\begin{cases} \partial_t p = (\partial_t A - \partial_t B\, r)\, p, \\ \partial_r p = -B\, p, \\ \partial_{rr} p = B^2\, p. \end{cases}$$

Replacing these into the PDE and dividing by $p > 0$ yields:

$$\partial_t A - r\, \partial_t B - bB + \beta r B + \tfrac{1}{2}\sigma^2 r B^2 - r = 0.$$

Grouping terms by powers of r gives

$$\underbrace{\partial_t A - bB}_{\text{constant term}} \; + r\, \underbrace{\big(- \partial_t B + \beta B + \tfrac{1}{2}\sigma^2 B^2 - 1\big)}_{\text{coefficient of } r} = 0.$$

Since this must hold for all $r \geq 0$, the coefficients vanish separately, yielding the Riccati system:

$$\begin{cases} \partial_t A(t,T) = b\, B(t,T), & A(T,T) = 0, \\ \partial_t B(t,T) = -\tfrac{1}{2}\sigma^2 B^2(t,T) + \beta B(t,T) - 1, & B(T,T) = 0. \end{cases}$$

3.3 Explicit solution

Define

$$d = \sqrt{\beta^2 + 2\sigma^2}.$$

The explicit solution to the Riccati equation with boundary condition $B(T,T) = 0$ is

$$B(t,T) = \frac{2\big(e^{d(T-t)} - 1\big)}{(d+\beta)\big(e^{d(T-t)} - 1\big) + 2d}.$$

Then $A(t,T)$ is obtained by integrating the first equation:

$$A(t,T) = \frac{2b}{\sigma^2} \ln\left(\frac{2d\, e^{\frac{(d+\beta)(T-t)}{2}}}{(d+\beta)(e^{d(T-t)} - 1) + 2d} \right).$$

Finally, the bond price is recovered as

$$p(t,T) = \exp\big(A(t,T) - B(t,T)\, r(t)\big).$$

This closed-form solution highlights the affine nature of the CIR model and allows efficient pricing and calibration in practice.

3.4 Numerical simulation

Using Euler discretization:

$$r_{n+1} = r_n + (b - \beta r_n)\Delta t + \sigma\sqrt{r_n}\Delta W_n^{\mathbb{Q}}.$$

If the Feller condition is not met, r_n may become negative, so schemes such as Alfonsi's are often preferred.

4 The Hull-White Model

4.1 Model dynamics

The Hull-White short rate model is given by:

$$dr(t) = \big(b(t) - \beta r(t)\big)\,dt + \sigma\,dW_t^{\mathbb{Q}}, \quad r(0) = r_0,$$

where $\beta > 0$ is the mean-reversion rate, $\sigma > 0$ is the volatility, and $b(t)$ is an integrable time-dependent drift.

4.2 Explicit solution of the short rate

The SDE is linear and admits the explicit solution:

$$r(t) = r_0 e^{-\beta t} + \int_0^t e^{-\beta(t-s)} b(s)\,ds + \sigma \int_0^t e^{-\beta(t-s)}\,dW_s^{\mathbb{Q}}.$$

4.3 Affine solution for bond prices

Assume again $p(t,T) = \exp(A(t,T) - B(t,T)r)$. The PDE leads to:

$$\begin{cases} \partial_t A(t,T) = \frac{1}{2}\sigma^2 B^2(t,T) + b(t)B(t,T), & A(T,T) = 0, \\ \partial_t B(t,T) = \beta B(t,T) - 1, & B(T,T) = 0. \end{cases}$$

4.4 Explicit solution

Solving for B:

$$B(t,T) = \frac{1 - e^{-\beta(T-t)}}{\beta}.$$

Then

$$A(t,T) = \int_t^T \left(\tfrac{1}{2}\sigma^2 B^2(s,T) + b(s)B(s,T)\right)\,ds.$$

Thus,

$$p(t,T) = \exp(A(t,T) - B(t,T)r(t)).$$

4.5 Numerical simulation

Euler discretization applies:

$$r_{n+1} = r_n + (b(t_n) - \beta r_n)\Delta t + \sigma\Delta W_n^{\mathbb{Q}}.$$

Different choices of $b(t)$ (constant, seasonal, mean-reverting, etc.) directly influence the yield curve. Unlike CIR, the Hull-White model allows negative interest rates.

5 Monte Carlo Pricing of Derivatives

5.1 Derivative payoff

We consider a European option with payoff depending on the short rate at maturity:

$$\Phi(r_T) = (r_T - k)^+, \quad k > 0.$$

This type of derivative is sensitive both to the distribution of r_T and to the discounting induced by the short rate path.

5.2 Risk-neutral valuation

Under the risk-neutral measure $\mathbb{Q}$, the time-t price is

$$C_t = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)(r_T - k)^+ \,\middle|\, \mathcal{F}_t\right].$$

The presence of the stochastic discount factor makes the pricing problem path-dependent, even though the payoff only depends on r_T.

5.3 Link to PDEs

By the Feynman–Kac theorem, the price $C(t,r)$ can equivalently be characterized as the solution of the parabolic PDE

$$\partial_t C + \mu_r(t,r)\,\partial_r C + \tfrac{1}{2}\sigma^2(t,r)\,\partial_{rr} C - r\,C = 0, \qquad C(T,r) = (r-k)^+,$$

where (μ_r, σ) are given by the short-rate dynamics (CIR or Hull–White). In practice, closed-form solutions are unavailable, motivating the use of Monte Carlo methods.

5.4 Monte Carlo algorithm

We simulate N independent trajectories $\{r^{(i)}(s),\, t \le s \le T\}_{i=1}^N$ of the short rate under the chosen model.

1. **Discretization.** Partition the interval $[t,T]$ into M steps of size $\Delta t = (T-t)/M$.
   - For the Hull–White model: use the Euler–Maruyama scheme, which is stable due to Gaussian increments.
   - For the CIR model: prefer *full truncation Euler* or *Milstein with reflection* to preserve positivity, since the naive Euler scheme may produce negative rates.
2. **Pathwise payoff computation.** For each simulated path i:

$$D^{(i)} = \exp\left(-\sum_{m=0}^{M-1} r_m^{(i)}\,\Delta t\right), \quad \text{(approximating the integral as a left Riemann sum)} \qquad \Phi^{(i)} = \big(r_M^{(i)} - k\big)^+.$$

3. **Estimator.** Approximate the price as

$$C_t \approx \frac{1}{N}\sum_{i=1}^N D^{(i)}\,\Phi^{(i)}.$$

5.5 Variance reduction techniques

Monte Carlo convergence is slow ($O(N^{-1/2})$). To improve efficiency:

- **Antithetic variates:** simulate pairs of paths using W and $-W$ to reduce variance.
- **Control variates:** use a related payoff with known expectation (e.g., a zero-coupon bond) as a control.
- **Quasi-Monte Carlo:** replace pseudo-random numbers with low-discrepancy sequences (Sobol, Halton).

5.6 Comparison of CIR and Hull–White models

- **CIR:** rates remain positive, distribution of r_T is noncentral chi-squared. Requires careful discretization to avoid bias.
- **Hull–White:** Gaussian increments allow simple simulation, but negative rates are possible.
- For both models, the stochastic discount factor introduces correlation between payoff and discounting, increasing simulation variance.

5.7 Convergence considerations

The accuracy depends on both N and M:

$$\text{Error} \approx O\left(\frac{1}{\sqrt{N}}\right) + O\big(\Delta t^p\big), \quad \text{where } p \text{ depends on the chosen discretization scheme.}$$

Balancing the number of paths and the time discretization is essential for reliable results.

5.8 Variance reduction

When the payoff is rare (e.g. $r_T > k$ infrequent), importance sampling is beneficial. Parameters can be shifted to increase the frequency of rare events, then corrected by likelihood ratios.

6 Discussion and Conclusion

The CIR and Hull-White models provide complementary approaches to short-rate modeling. CIR guarantees positivity of rates and has a strong theoretical foundation, while Hull-White allows flexible calibration through its time-dependent drift $b(t)$, at the cost of permitting negative rates. Both admit closed-form affine bond pricing formulas and lend themselves well to Monte Carlo simulation for more complex derivatives. In practice, model choice depends on the calibration requirements and market conditions.