On Empty Convex Polygons in a Planar Point Set*

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Abstract

Let P be a set of n points in general position in the plane. Let $X_k(P)$ denote the number of empty convex k-gons determined by P. We derive, using elementary proof techniques, several equalities and inequalities involving the quantities $X_k(P)$ and several related quantities. Most of these equalities and inequalities are new, except for a few that have been proved earlier using a considerably more complex machinery from matroid and polytope theory, and algebraic topology. Some of these relationships are also extended to higher dimensions. We present several implications of these relationships, and discuss their connection with several long-standing open problems, the most notorious of which is the existence of an empty convex hexagon in any point set with sufficiently many points.

1 Introduction

Let P be a set of n points in general position in the plane. How many empty convex k-gons must P always determine, for $k = 3, 4, 5, \ldots$? The interest in this class of problems arose after Horton had shown 20 years ago [17] that there exist sets of arbitrarily large size that do not contain empty convex 7-gons (and thus no empty convex k-gons for any $k \geq 7$). It is still a notoriously hard open problem whether every set with sufficiently many points must contain an empty convex hexagon. The size of the largest known set that does not contain an empty convex hexagon is 29, as found by Overmars [22] (see also [23]). In this paper we develop machinery that might be useful for tackling this problem.

In contrast, any set with sufficiently many points must contain many empty triangles, convex quadrilaterals, and convex pentagons. Specifically, Bárány and Füredi [3] have

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shown that any n-point set must determine at least $n^2 - O(n \log n)$ empty triangles, at least $\frac{1}{2}n^2 - O(n \log n)$ empty convex quadrilaterals, and at least $\lfloor n/10 \rfloor$ empty convex pentagons, where the latter bound can be improved to $\lfloor (n-4)/6 \rfloor$ (see [4]). The bound on the number of empty convex pentagons follows from a result of Harborth [15], which shows that among any 10 points there are 5 that form an empty convex pentagon. Three interrelated open problems (see [4]) are to show that

(P3) the number of empty triangles is always at least $(1+c)n^2$, for some constant $c > 0$,

(P4) the number of empty convex quadrilaterals is always at least $\left(\frac{1}{2} + c\right) n^2$, for some constant $c > 0$, and

(P5) the number of empty convex pentagons is always at least cn^2, for some constant $c > 0$.

In general, the lower bounds cannot be super-quadratic, as has been noted in several papers [5, 8]. The construction with the best upper bounds is due to Bárány and Valtr [5]; it produces n-point sets with roughly $1.62n^2$ empty triangles, $1.94n^2$ empty convex quadrilaterals, $1.02n^2$ empty convex pentagons, and $0.2n^2$ empty convex hexagons. Both constructions in [5, 8] use Horton's construction as the main building block.

In this paper we obtain a variety of results concerning the number of empty convex polygons in planar point sets (and of empty convex polytopes in higher dimensions). Our first set of results consists of linear equalities in the numbers $X_k(P)$ of empty convex k-gons in an n-element planar point set P, for $k = 3, 4, 5, \ldots$. All these equalities involve the alternating sums

$$M_0(P) = \sum_{k \geq 3} (-1)^{k+1} X_k(P), \quad \text{and}$$

$$M_r(P) = \sum_{k \geq 3} (-1)^{k+1} \frac{k}{r} \binom{k-r-1}{r-1} X_k(P), \quad \text{for } r \geq 1,$$

and express these sums in closed form, relating them to certain geometric parameters of the point set P. We refer to $M_r(P)$ as the *r-th alternating moment* of $\{X_k(P)\}_{k \geq 3}$. The coefficient of $X_k(P)$ in the expression for $M_r(P)$ is the number of ways to choose r elements from a circular list of k elements, so that no two adjacent elements are chosen.[1]

For example, we show that

$$\begin{aligned}
M_0(P) &= \binom{n}{2} - n + 1, \\
M_1(P) &= \sum_{k \geq 3} (-1)^{k+1} k X_k(P) = 2\binom{n}{2} - H(P), \\
M_2(P) &= \sum_{k \geq 4} (-1)^{k+1} \frac{k(k-3)}{2} X_k(P) = -T_2(P),
\end{aligned}$$

where $H(P)$ is the number of edges of the convex hull of P, and where $T_2(P)$ is the number of pairs of edges ab, cd, that are delimited by four distinct points of P, lie in convex position,

[1] This is known as Cayley's problem; see, e.g., Exercise 2.3.23 in [21].

and are such that the wedge bounded by their supporting lines and containing both of them does not contain any point of P in its interior. See Figure 2(i).

In fact, our general bound can be written as follows. Set $X_0(P) = 1$, $X_1(P) = n$, and $X_2(P) = \binom{n}{2}$. Intuitively, this says that the empty set is regarded as an empty convex 0-gon, each point of P is regarded as an empty convex 1-gon, and each edge spanned by P is regarded as an empty convex 2-gon. Define $T_r(P)$, for $r \geq 2$, to be the number of r-tuples of vertex-disjoint edges $e_1, \ldots, e_r$ spanned by P that lie in convex position, and are such that the region $\tau(e_1, \ldots, e_r)$, formed by the intersection of the r halfplanes that are bounded by the lines supporting $e_1, \ldots, e_r$ and containing the other edges, has no point of P in its interior. See Figure 3. We also extend this definition by putting $T_0(P) = 0$ and $T_1(P) = H(P)$. Then our equalities can be written in the form

$$M_r^*(P) := \sum_{k \geq 2r} (-1)^k \frac{k}{r} \binom{k-r-1}{r-1} X_k(P) = T_r(P),$$

for each $r \geq 0$. However, we will use the former set of expressions, because the resulting analysis is somewhat more natural, and also because M_0 and M_1 have been used in previous works. We note that although we will consider sets of points in general position, a more delicate analysis can show that the same arguments are valid to sets of points in degenerate position as well (see Section 6 for more details).

The first equality (for $M_0(P)$), given in Theorem 2.1 (as well as its extension to higher dimensions—see below), has been earlier obtained by Edelman and Jamison in their survey on convex geometries [9] (cited as an unpublished result of J. Lawrence, and independently proven by the authors), and it also follows from a more general recent result of Edelman et al. [11]. The second equality (for $M_1(P)$), given in Theorem 2.2 below, has been recently obtained by Ahrens et al. [1], using tools from matroid/greedoid theory specific to the convex geometry defined by point sets in the plane. Nevertheless, they use elementary geometric arguments (different from those in the present note). (Actually, the quantity $M_1 + n - 2\binom{n}{2}$, which, by Theorem 2.2, is equal to the number of points of P interior to its convex hull, is known as *Crapo's beta invariant* for convex geometries arising from Euclidean point configurations in a d-dimensional space.) Ahrens et al. conjectured the extension of Theorem 2.2 to higher dimensions, as formulated in Theorem 4.2 below, and this was later proved by Edelman and Reiner [10], using tools from algebraic topology, and independently by Klain [20], using the theory of valuations on lattices of high-dimensional polytopes. It should be emphasized that the results by Ahrens et al., Edelman and Reiner, and Klain also apply to point configurations which are not in general position. A short discussion of this case is given in the concluding Section 6.

In contrast, our proofs are simple and elementary, and can be extended to derive the entire system of equalities for all the moments $M_r(P)$. This is done in Theorem 2.3. A similar proof technique applies also to point sets in higher dimensions, and we demonstrate this extension in Theorems 4.1 and 4.2. (As just discussed, these theorems, which extend Theorems 2.1 and 2.2 to higher dimensions, were already obtained in [9, 10, 20], with considerably more complicated proofs.) However, the proof technique for higher-order moments does not extend so far to higher dimensions. We have recently learned that Valtr, in an unpublished work [26], has also proved Theorems 2.1 and 2.2 using arguments similar to ours.

As far as we can tell, bounding $T_2(P)$ (or, for that matter, $T_r(P)$ for any $r \geq 3$) is a problem that has not been considered before, and we regard it as a significant by-product of our paper, to highlight this problem and to provide compelling motivation for its study (this motivation will be discussed in more detail later).

We show that $T_2(P) \le n(n-1) - 2H(P)$ (Theorem 3.1). Our analysis shows that any upper bound on $T_2(P)$ of the form $(1-c)n^2$, for any fixed $c > 0$, would yield improved bounds for all three open problems (P3)–(P5) mentioned above (although it does not seem to imply the existence of an empty convex hexagon).

An even more interesting problem is to bound the number $T_2^*(P)$ of convex empty quadrilaterals that cannot be extended into a convex empty pentagon by adding a vertex from P. Note that a quadrilateral *abcd* is counted in $T_2^*(P)$ if and only if both pairs of opposite edges are T_2-configurations; see Figure 2(ii). We show that $T_2^*(P) \le \binom{n}{2} - H(P)$. We also establish several inequalities that involve $T_2^*(P)$ and the $X_k(P)$'s, and use them to show that any upper bound on $T_2^*(P)$ of the form $\left(\frac{1}{2} - c\right) n^2$, for any fixed $c > 0$, will yield improved lower bounds for $X_k(P)$, for $k = 3, 4, 5, 6$, that are related to problems (P3)–(P5). We also provide the worst-case lower bounds $\frac{3}{4}n^2 - O(n)$ for $T_2(P)$, and $\frac{1}{4}n^2 - O(n)$ for $T_2^*(P)$.

Next, we derive *inequalities* involving the quantities $X_k(P)$. The main group of inequalities are related to the moments $M_r(P)$. They assert that all the tails of the series defining $M_r(P)$ are non-negative, for any $r \ge 0$. More precisely, we have

$$X_t(P) - X_{t+1}(P) + X_{t+2}(P) - \cdots \ge 0,$$

$$tX_t(P) - (t+1)X_{t+1}(P) + (t+2)X_{t+2}(P) - \cdots \ge 0,$$

for any $t \ge 3$, and

$$\frac{t}{r}\binom{t-r-1}{r-1}X_t(P) - \frac{t+1}{r}\binom{t-r}{r-1}X_{t+1}(P) + \frac{t+2}{r}\binom{t-r+1}{r-1}X_{t+2}(P) - \cdots \ge 0,$$

for $r \ge 2$ and for any $t \ge 2r$.

Combining these inequalities with the closed-form expressions for the full series, we obtain equivalent inequalities involving prefixes of these series. For example, we obtain that $X_3(P) - X_4(P) + \cdots - X_t(P) \le \binom{n}{2} - n + 1$ when $t \ge 4$ is even, and $X_3(P) - X_4(P) + \cdots + X_t(P) \ge \binom{n}{2} - n + 1$ when $t \ge 3$ is odd.

Another collection of inequalities involves the first three numbers $X_3(P), X_4(P), X_5(P)$. Many, but not all of them, are obtained as direct implications of the prefix inequalities noted above. The most significant among them are

$$\begin{aligned} X_4(P) &\ge X_3(P) - \frac{n^2}{2} - O(n), \quad \text{and} \\ X_5(P) &\ge X_3(P) - n^2 - O(n). \end{aligned}$$

They provide a strong connection (stronger than the one noted in [4]) between the three problems (P3)–(P5). In particular, the constants c in (P4) and (P5) are at least as large as the constant in (P3). In addition, we derive similar inequalities that also involve $T_2(P)$ and $T_2^*(P)$, and show, as promised above, that any upper bound on $T_2(P)$ of the form $(1-c)n^2$ would solve the three problems (P3)–(P5), and that a similar improvement in the upper bound for $T_2^*(P)$ would have similar implications.

In spite of all the equalities and inequalities that we have derived in this paper, the problems (P3)–(P5), as well as the problem of the existence of an empty convex hexagon, remain open. Nevertheless, it is our hope that the techniques that we have developed will eventually facilitate progress on these hard problems.

2 The Vector of Empty Convex Polygons and its Moments

Let P be a set of n points in the plane in general position. For each $k \geq 3$, let $X_k(P)$ denote the number of empty convex k-gons spanned by P. Recall that the *r-th alternating moment* of P, for any $r \geq 0$, is defined as

$$M_0(P) = \sum_{k \geq 3} (-1)^{k+1} X_k(P), \quad \text{and}$$

$$M_r(P) = \sum_{k \geq 3} (-1)^{k+1} \frac{k}{r} \binom{k-r-1}{r-1} X_k(P), \quad \text{for } r \geq 1.$$

In this section we derive explicit expressions for all these moments. To simplify our notations, we will usually drop P from them, and write $X_k(P)$ simply as X_k, and $M_r(P)$ as M_r (and similarly for the other notations $H(P)$, $T_2(P)$, etc.). The expressions for M_0 (Theorem 2.1) and for M_1 (Theorem 2.2) are already known [1, 9, 10, 20]. However, as discussed in the introduction, the proofs in [1, 9, 10, 20] are considerably more involved and do not use elementary combinatorial techniques. In contrast, our proofs are much simpler and elementary. The expressions for the M_r with $r \geq 2$ (Theorem 2.3) are new, with the same elementary proof technique.

Theorem 2.1. *$M_0 = \binom{n}{2} - n + 1$.*

Proof: We claim that any continuous motion of the points of P which is sufficiently generic does not change the value of M_0. By "sufficiently generic" we mean that the points of P remain distinct and in general position during the motion, except at a finite number of critical times where exactly one triple of points becomes collinear. Clearly, until such a collinearity occurs, M_0 does not change.

Suppose that $p, q, r \in P$ become collinear, with r lying between p and q. The only convex polygons spanned by P whose emptiness (or convexity) status may change are those that have both p and q (and possibly also r) as vertices, either just before or just after the collinearity. Let Q be such a convex k-gon that does not have r as a vertex. See Figure 1. If Q was empty before the collinearity and r is about to enter Q, then Q stops being empty, and the $(k+1)$-gon Q', obtained by replacing the edge pq of Q by the polygonal path prq, which was convex and empty just before the collinearity, stops being convex. Since the sizes of Q and of Q' differ by 1, their combined contribution to M_0 is 0 before the collinearity and 0 afterwards, so they do not affect the value of M_0. Symmetrically, if r is about to exit Q and is the only point in Q before the collinearity, then Q becomes newly empty, and Q' becomes newly convex and empty. Again, this does not affect the value of M_0. There is no other kind of events that may affect the value of M_0 at this critical configuration.

We may thus obtain the value of M_0 by computing it for the case where P is in convex position. In this case, we have $X_k = \binom{n}{k}$, for any $k \geq 3$. Hence,

$$M_0 = \binom{n}{3} - \binom{n}{4} + \binom{n}{5} - \cdots = \binom{n}{2} - n + 1,$$

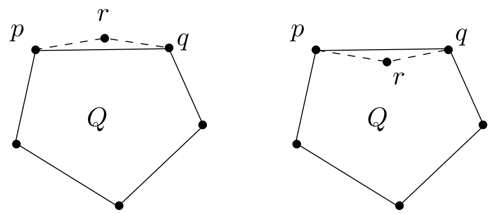


Figure 1: The continuous motion argument in the proof of Theorem 2.1.

as asserted. $\Box$

In other words, M_0 does not depend on the shape of P but only on its size. The situation is not as simple for higher-order alternating moments, although it is still reasonably under control:

Theorem 2.2. *$M_1 = 2\binom{n}{2} - H$, where H is the number of edges of the convex hull of P.*

Proof: Fix a directed edge $e = pq$ whose endpoints belong to P, and define, for each $k \geq 3$, $X_k(e)$ to be the number of empty convex k-gons that contain e as an edge and lie to the left of e. Define

$$M_0(e) = \sum_{k\geq 3} (-1)^{k+1} X_k(e).$$

It is easy to see that $\sum_e M_0(e) = 3X_3 - 4X_4 + 5X_5 - \cdots = M_1$. This follows from the observation that each empty convex k-gon Q is counted exactly k times in $\sum_e M_0(e)$, once for each of its edges. Moreover, arguing as in the proof of Theorem 2.1, the value of $M_0(e)$ depends only on the number of points of P that lie to the left of e. This follows by a similar continuous motion argument, in which the points to the left of e move in a sufficiently generic manner, without crossing the line supporting e, while the endpoints of e, as well as the points on the other side of e, remain fixed. If there are m points to the left of e, then when these points, together with p and q, are in convex position, they satisfy

$$M_0(e) = \binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots,$$

which is 1 if $m > 0$, and 0 if $m = 0$, that is, if e is a clockwise-directed edge of the convex hull of P. Since the total number of directed edges spanned by P is $2\binom{n}{2}$, it follows that

$$M_1 = \sum_e M_0(e) = 2\binom{n}{2} - H,$$

as asserted. $\Box$

Theorem 2.3. *$M_r = -T_r$, for any $r \geq 2$.*

Proof: The proof is similar to that of Theorem 2.2. Here we fix r edges $e_1, \ldots, e_r$ that are spanned by P, have distinct endpoints, and are in convex position. For each choice of $e_1, \ldots, e_r$ with these properties and for each $k \geq 2r$, define $X_k(e_1, \ldots, e_r)$ to be the number of empty convex k-gons that contain $e_1, \ldots, e_r$ as edges. Note that this definition is void

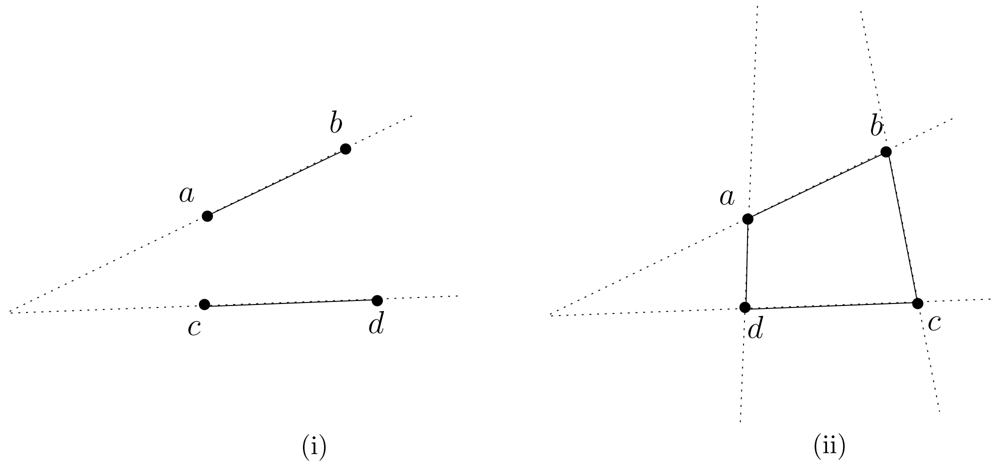


Figure 2: (i) A pair of edges (ab, cd) that is counted in T_2: They lie in convex position and define an empty wedge. (ii) A quadrilateral $abcd$ that is counted in T_2^*: Both wedges are empty.

for $k < 2r$. Note also that we do not have to consider $e_1, \ldots, e_r$ as directed edges (as we did in the proof of Theorem 2.2). Define

$$M_0(e_1, \ldots, e_r) = \sum_{k \geq 2r} (-1)^{k+1} X_k(e_1, \ldots, e_r).$$

Then, arguing in complete analogy to the case of M_1, it follows from the definition of M_r that $\sum_{e_1,\ldots,e_r} M_0(e_1, \ldots, e_r) = M_r$, where the sum is over all *unordered* r-tuples of distinct edges with distinct endpoints in convex position. This follows from the fact that each empty convex k-gon Q, for $k \geq 2r$, is counted exactly $\frac{k}{r}\binom{k-r-1}{r-1}$ times in $\sum_{e_1,\ldots,e_r} M_0(e_1, \ldots, e_r)$, once for each (unordered) r-tuple of vertex-disjoint edges of Q.

Moreover, as above, the value of $M_0(e_1, \ldots, e_r)$ depends only on the number m of points of P that lie in the region $\tau(e_1, \ldots, e_r)$, as defined in the introduction. Again, this follows by a continuous motion argument, in which the points in $\tau(e_1, \ldots, e_r)$ move in a sufficiently generic manner, without crossing any of the lines bounding this region, while the endpoints of $e_1, \ldots, e_r$, as well as the points of P outside $\tau(e_1, \ldots, e_r)$, remain fixed. If m is positive, placing at least one of these m points in the interior of the convex hull of $e_1, \ldots, e_r$ shows that $M_0(e_1, \ldots, e_r) = 0$, and if $m = 0$ then $M_0(e_1, \ldots, e_r) = -1$, because in this case we have $X_{2r}(e_1, \ldots, e_r) = 1$ and $X_k(e_1, \ldots, e_r) = 0$ for all other values of k. Hence

$$M_r = \sum_{e_1,\ldots,e_r} M_0(e_1, \ldots, e_r) = -T_r,$$

where T_r is as defined in the introduction. $\square$

Remark: We can obtain closed-form expressions for any alternating sum of the form $\sum_{k\geq 3}(-1)^{k+1} C_r(k) X_k$, where r is an integer, and $C_r(k)$ is a polynomial of degree r in k, by expressing any such series as a linear combination of $M_0, M_1, \ldots, M_r$. Alternating sums for which the corresponding linear combination has only non-negative coefficients are of particular interest, because of the inequalities that we will later derive in Section 5, which will then yield similar inequalities for the new sums too.

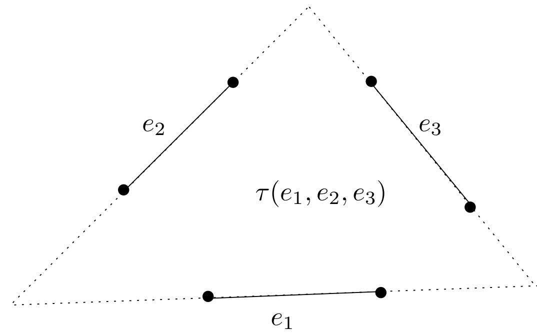


Figure 3: A T_3-configuration.

Some initial implications. One can solve the linear equations for M_0, M_1, M_2, so as to eliminate X_3, X_4, X_5, and obtain the following three expressions for X_3, X_4, X_5:

$$\begin{aligned} X_3 &= 2(n^2 - 6n + 5) + 3H - T_2 + \sum_{k \geq 6} (-1)^k \frac{(k-4)(k-5)}{2} X_k, \\ X_4 &= \frac{5}{2}(n^2 - 7n + 6) + 5H - 2T_2 + \sum_{k \geq 6} (-1)^k (k-3)(k-5) X_k, \\ X_5 &= n^2 - 7n + 6 + 2H - T_2 + \sum_{k \geq 6} (-1)^k \frac{(k-3)(k-4)}{2} X_k. \end{aligned} \tag{1}$$

When $X_6 = 0$ (and thus $X_k = 0$ for every $k \geq 6$), the solution becomes

$$\begin{aligned} X_3 &= 2(n^2 - 6n + 5) + 3H - T_2, \\ X_4 &= \frac{5}{2}(n^2 - 7n + 6) + 5H - 2T_2, \\ X_5 &= n^2 - 7n + 6 + 2H - T_2. \end{aligned} \tag{2}$$

In this case, since $X_5 \geq 0$, we have

$$T_2 \leq n^2 - 7n + 6 + 2H \leq n^2 - 5n + 6.$$

(We will shortly derive a similar bound for T_2 that holds in general.) Substituting this in the expressions for X_3, X_4, we obtain (using the trivial estimate $H \geq 3$)

$$\begin{aligned} X_3 &\geq n^2 - 5n + 4 + H \geq n^2 - 5n + 7, \\ X_4 &\geq \frac{1}{2}(n^2 - 7n + 6) + H \geq \frac{1}{2}(n^2 - 7n + 12). \end{aligned}$$

As mentioned in the introduction, similar lower bounds (with slightly worse lower-order terms) have been obtained by Bárány and Füredi [3] for the general case.

Another immediate implication of Theorems 2.1 and 2.2 is the following equality, which holds when $X_6 = 0$.

$$2X_3 - X_4 = 5M_0 - M_1 = \frac{(3n-10)(n-1)}{2} + H. \tag{3}$$

Remarks. (1) One can also consider the elimination of X_3, X_4, X_5, X_6 from the four equations for M_0, M_1, M_2, M_3. The resulting equations are:

$$\begin{aligned}
X_3 &= 2(n^2-6n+5)+3H-T_2+T_3+\sum_{k\geq 7}(-1)^{k+1}\frac{(k-4)(k-5)(k-6)}{12}X_k, \\
X_4 &= \frac{5}{2}(n^2-7n+6)+5H-2T_2+\frac{5}{2}T_3+\sum_{k\geq 7}(-1)^{k+1}\frac{(k-2)(k-5)(k-6)}{4}X_k, \quad (4) \\
X_5 &= n^2-7n+6+2H-T_2+2T_3+\sum_{k\geq 7}(-1)^{k+1}\frac{(k-1)(k-4)(k-6)}{4}X_k, \\
X_6 &= \frac{1}{2}T_3+\sum_{k\geq 7}(-1)^{k+1}\frac{k(k-4)(k-5)}{12}X_k.
\end{aligned}$$

However, this does not lead to any further significant implication. In particular, so far this approach does not appear to be productive for establishing the existence of a convex empty hexagon (in any sufficiently large point set). However, since $X_6 \geq \frac{1}{2}T_3$, the following inequality always holds:

$$\sum_{k\geq 7}(-1)^{k+1}\frac{k(k-4)(k-5)}{12}X_k \geq 0.$$

This is the tail of the series for M_3, starting with the X_7-term. This is a special case of a general family of similar inequalities that we will derive in Section 5.

(2) The relation (3) provides a simple and fast one-sided test for the existence of an empty convex hexagon in a given set P. That is, if the equality does not hold then P contains an empty convex hexagon. Verifying the equality (3) can be done in time close to n^4, and perhaps further improvements are also possible. This may be a useful ingredient for a program that searches for sets that do not contain an empty convex hexagon.

(3) As shown by Edelman et al. [11] (as a special case of a more general result), one can construct a simplicial cell complex from the empty convex sets of any finite point set in $\mathbb{R}^d$, and show that this complex is homotopy equivalent to a point. This allows us to interpret Theorem 2.1 as the Euler relation on that complex. This connection between convex empty polygons spanned by a point set and simplicial complexes deserves further study.

3 An Upper Bound for T_2 and Related Bounds

An upper bound for T_2.

Theorem 3.1. $T_2 \leq n(n-1) - 2H$.

Proof: Let au and bv be two segments with distinct endpoints $a, b, u, v \in P$ and in convex position, so that the clockwise order of their endpoints along their convex hull is either a, u, b, v or a, v, b, u. Assume that this pair of edges forms an empty wedge, that is, (au, bv) forms a T_2-*configuration*. Assuming a generic coordinate system, we charge this configuration to the diagonal (ab or uv) whose right endpoint is the rightmost among a, u, b, v. Assume that this diagonal is ab and that a is its right endpoint, as illustrated in Figure 4.

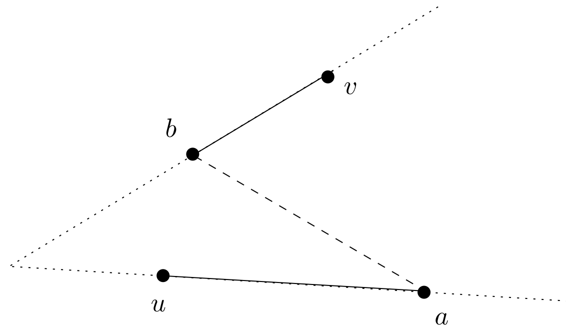


Figure 4: A T_2-configuration and the charged diagonal ab.

In the configuration depicted in the figure, $\vec{au}$ lies counterclockwise to $\vec{ab}$, and $\vec{bv}$ lies counterclockwise to $\vec{ba}$. The segment ab can also be charged by configurations for which $\vec{au}$ lies clockwise to $\vec{ab}$, and $\vec{bv}$ lies clockwise to $\vec{ba}$. We refer to the first type of configurations as *counterclockwise charges* (of the configuration to ab), and to the second type as *clockwise charges*.

We claim that a segment ab can receive at most one clockwise charge and at most one counterclockwise charge by a T_2-configuration of which it is the diagonal with the rightmost right endpoint. In addition, segments ab that are edges of the convex hull of P cannot receive any charge. The claim thus implies that

$$T_2 \leq 2\left(\binom{n}{2} - H\right) = n(n-1) - 2H.$$

In the proof of the claim, we assume to the contrary that ab receives two, say, counterclockwise charges, and denote the two charging configurations as (au, bv), (au', bv'). It is easily verified that $u \neq u'$ and $v \neq v'$ (in fact it suffices to verify that either $u \neq u'$ or $v \neq v'$).

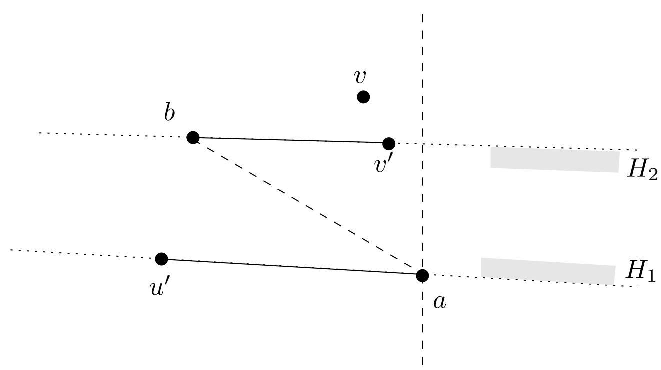


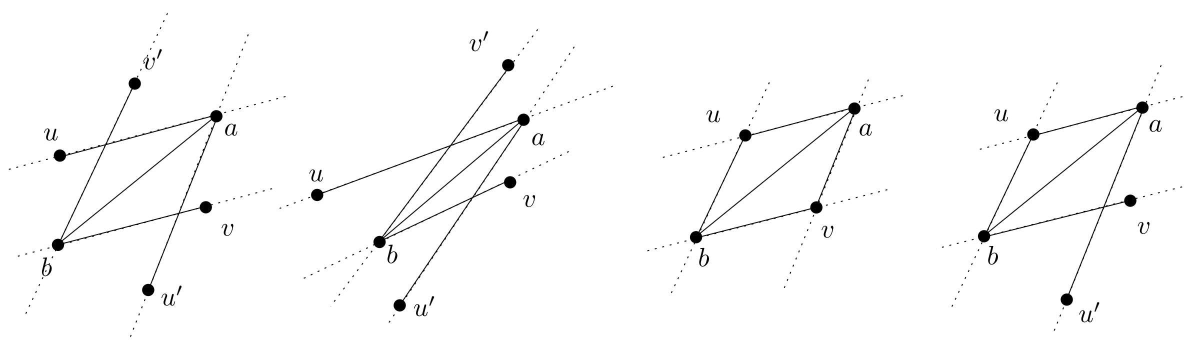
Figure 5: v has to lie in H_1.

The halfplane H_1 to the right of $\vec{au'}$ has to contain v, because u' and v lie to the left of a, and H_1 contains b. See Figure 5. Hence, the halfplane H_2 to the right of $\vec{bv'}$ cannot contain v (or else v would lie in the wedge determined by au' and bv'). Since H_2 contains a, v' must lie in the wedge between $\vec{ba}$ and $\vec{bv}$, and since v' lies to the left of a, it must lie in the wedge determined by au and bv, a contradiction. □

Attempting to improve the bound. An attempt to strengthen Theorem 3.1 proceeds as follows. Let ab be an edge that receives both a clockwise charge and a counterclockwise charge as the diagonal with the rightmost endpoint in two respective T_2-configurations (au, bv), (au', bv'). It is easily seen that, because of the properties of T_2-configurations, au and bv' must cross each other (including the possibility that $u = v'$), and similarly for au' and bv.

We obtain either the situation shown in Figure 6(a), in which a, v', u, b, u', v form a convex hexagon, or the situation in Figure 6(b), in which a, v', u, b, v and a, v', b, u', v are convex pentagons, or the situations in Figure 6(c,d) discussed below.

Indeed, we first claim that the line $\ell_{vv'}$ that supports vv' separates a and b. This follows since both v and v' lie to the left of a and on different sides of ab. The only situation in which $\ell_{vv'}$ does not separate a and b is when the quadrilateral $av'bv$ is not convex at b, as shown in Figure 7. But then, since au and bv' intersect, $aubv$ would not be convex, a contradiction that implies the claim.



(a) (b) (c) (d)

Figure 6: The various cases in the refined analysis of T_2.

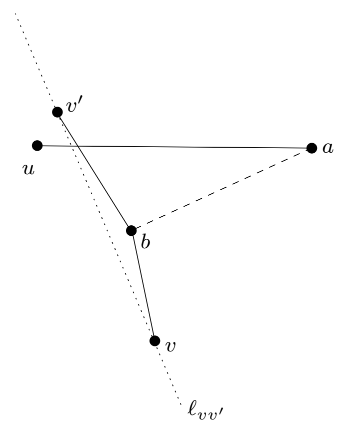


Figure 7: Showing that $\ell_{vv'}$ must separate a and b.

Now the situation depends on whether the line $\ell_{uu'}$ that supports uu' separates a and b. If it does (as shown in Figure 6(a)) then we get a convex hexagon. If $\ell_{uu'}$ does not separate a and b (as shown in Figure 6(b)) then we get the above two convex pentagons. It is also possible that either $u = v'$ or $v = u'$ or both; see Figure 6(c,d). If both coincidences occur (Figure 6(c)), $aubv$ is an empty convex quadrilateral that cannot be extended to an empty convex pentagon, so it forms a T_2^*-configuration. If only one of these coincidences occurs, say $u = v'$ (Figure 6(d)), then $aubu'v$ is a convex pentagon.

Although the convex hexagon in case (a), or the two convex pentagons in case (b), or the single convex pentagon in case (d), need not be empty, we claim that they can be replaced by empty ones. Consider for example the situation in Figure 6(a), reproduced in Figure 8. Any point of P that is interior to $av'ubu'v$ must lie in one of the triangles $\Delta(uxv')$ and $\Delta(vyu')$ (where x is the intersection point of the segments bv' and au, and y is the intersection point of bv and au'). Suppose that $\Delta(uxv')$ does contain a point of P in its interior, and consider the convex hull of all the points of P in the interior of $\Delta(uxv')$, including u and v'. Let $u''v''$ be any edge of that hull, other than $v'u$. Apply a symmetric argument to $\Delta(vyu')$ to obtain an edge $u'''v'''$ of the corresponding hull (assuming it to be nonempty). It is now easy to check that $av''u''bu'''v'''$ is an empty convex hexagon, having ab as a main diagonal and a as the rightmost vertex, and we charge our T_2-configuration to this hexagon. (The cases where one of the two hulls in $\Delta(uxv')$ and $\Delta(vyu')$ is empty, or both are empty, are handled in exactly the same manner.)

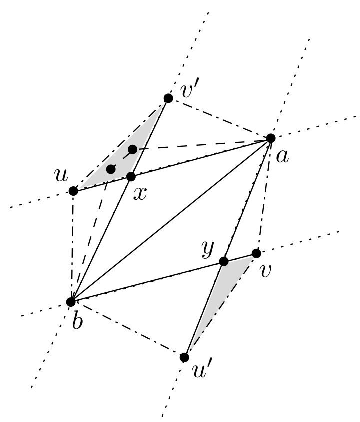


Figure 8: Charging the configuration in case (a) to an empty hexagon.

In a completely analogous manner, any of the pentagons in cases (b) and (d), if nonempty, can be replaced by an empty convex pentagon that has ab as a diagonal and a as the rightmost vertex.

This analysis allows us to "redirect" one of the clockwise and counterclockwise charges made to ab, to the resulting empty hexagon, to the one or two resulting empty pentagons, or to a T_2^*-configuration. Clearly, each empty hexagon is charged in this manner at most once (because it has only one main diagonal that emanates from its rightmost vertex), each empty pentagon is charged at most twice (once for each of the two diagonals that emanate from its rightmost vertex), and each T_2^*-configuration is charged once. We thus conclude:

$$T_2 \le \binom{n}{2} - H + 2X_5 + X_6 + T_2^*. \tag{5}$$

An interesting consequence of (5) is the following result, obtained by plugging (5) into the expression for X_5 in (1):

Corollary 3.2.

$$X_5 \geq \frac{1}{3}\left[\frac{n^2 - 13n + 12}{2} + 3H - X_6 - T_2^* - \sum_{k\geq 6}(-1)^{k+1}\frac{(k-3)(k-4)}{2}X_k\right].$$

In particular, if $X_7 = 0$ then

$$X_5 \geq \frac{1}{3}\left[\frac{n^2 - 13n + 12}{2} + 3H + 2X_6 - T_2^*\right].$$

Thus, any upper bound for T_2^* that is significantly smaller than $\binom{n}{2}$ (compare with (6) below) would result in a quadratic lower bound for X_5 for point sets with no empty convex heptagons, such as the Horton sets. Later, in Section 5, we will obtain a similar result, without having to assume that $X_7 = 0$.

An easy upper bound (in view of the proof of Theorem 3.1) for T_2^* is

$$T_2^*(P) \leq \frac{1}{2}T_2(P) \leq \binom{n}{2} - H. \qquad (6)$$

As already mentioned, we will later show that improving the constant in the quadratic term in this bound would lead to improved lower bounds involving X_3, X_4, and X_5, and several other implications. An observation that perhaps makes the analysis of T_2^* particularly interesting is that $T_2^*(P) = 0$ when P is a set of $n \geq 5$ points in convex position. In other words, in the situation where the parameters $X_k(P)$ attain their *maximum values*, $T_2^*(P)$ attains its *minimum* value 0.

Lower bounds. Figure 9 depicts a set P of an even number n of points for which $T_2^*(P) = \frac{1}{4}(n-2)^2$ and $T_2(P) = \frac{1}{2}(n-2)^2 + \frac{1}{4}(n-4)(n-6) = \frac{3}{4}n^2 - \frac{9}{2}n + 8$: There are $\frac{1}{4}(n-2)^2$ quadrilaterals spanned by a pair of edges, one on the lower hull of the points on the upper curve and one on the upper hull of the points on the lower curve. Each such quadrilateral gives rise to one T_2^*-configuration and to two T_2-configurations. In addition, each chain has $\frac{n}{2} - 1$ edges, and every vertex-disjoint pair of them yields a T_2-configuration, for a total of

$$2 \cdot \binom{\frac{n}{2} - 2}{2} = \frac{1}{4}(n-4)(n-6)$$

additional T_2-configurations.

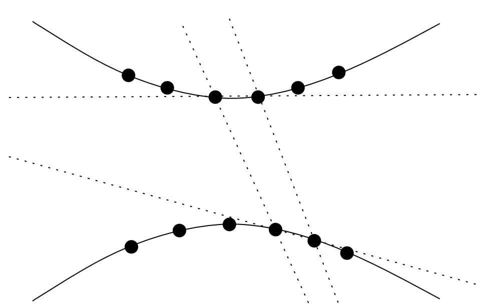


Figure 9: Lower bounds for T_2 and T_2^*.

4 Higher Dimensions

We next show that Theorems 2.1 and 2.2 can be extended to point sets in any dimension $d \geq 3$.

Let P be a set of n points in $\mathbb{R}^d$ in general position. For each $k \geq d+1$, let $X_k = X_k(P)$ denote the number of empty convex k-vertex polytopes spanned by P; these are convex polytopes with k vertices, all belonging to P, such that their interiors contain no point of P. Similarly to the planar case, define the *r-th alternating moment* of P, this time only for $r = 0, 1$, to be

$$M_0 = M_0(P) = \sum_{k \geq d+1} (-1)^{k+d+1} X_k,$$

$$M_1 = M_1(P) = \sum_{k \geq d+1} (-1)^{k+d+1} k X_k.$$

It is not clear what is the most natural way of defining higher-order moments in d-space. Besides, so far our analysis does not extend to higher-order moments.

Theorem 4.1. $M_0 = \binom{n}{d} - \binom{n}{d-1} + \cdots + (-1)^{d-1} n + (-1)^d$.

Proof: As in the planar case, we claim that any continuous motion of the points of P which is sufficiently generic does not change the value of M_0. By "sufficiently generic" we mean here that the points of P remain distinct and in general position during the motion, except at a finite number of critical times where $d+1$ points get to lie in a common hyperplane (but do not lie in any common lower-dimensional flat), and no other point lies on this hyperplane. Clearly, until such a criticality occurs, M_0 does not change.

Suppose that $p_1, p_2, \ldots, p_{d+1} \in P$ get to lie in a common hyperplane h_0. By Radon's theorem, there exists a partition of the set $P_0 = \{p_1, \ldots, p_{d+1}\}$ into two nonempty subsets $A \cup B$, so that $\text{conv}(A) \cap \text{conv}(B) \neq \emptyset$. Suppose first that neither A nor B is a singleton. We claim that in this case the set of empty convex polytopes spanned by P does not change, except that the face lattices of some of these polytopes may change. This follows from the observation that, unless A or B is a singleton, P_0 is in convex position within h_0, since no point lies in the convex hull of the other d points.

So assume, without loss of generality, that p_{d+1} becomes interior to the $(d-1)$-simplex σ spanned by $p_1, \ldots, p_d$. Let K be a convex polytope spanned by P, some of whose vertices belong to P_0. It can be checked that the only case where the emptiness or convexity of K can be affected by the critical event is when all the points $p_1, \ldots, p_d$ are vertices of K, and, with the possible exception of p_{d+1}, it contains no other point of P. Assume that p_{d+1} is not a vertex of K. Let K' denote the polytope obtained by adding p_{d+1} to K as a vertex, and by replacing σ by the d simplices that connect p_{d+1} to the facets of σ. Then, if p_{d+1} crosses the relative interior of σ into (respectively, out of) K then K stops (respectively, starts) being empty. Moreover, if K starts being empty, then so does K' (which has just become convex), and if K stops being empty, then K' stops being convex altogether. In either case, we obtain two convex polytopes that differ in one vertex, which are simultaneously added to the set of empty convex polytopes or simultaneously removed from that set. In either case, M_0 does not change.

Since M_0 does not change during such a continuous motion, it suffices to calculate its value when P is in convex position. Thus

$$M_0 = \binom{n}{d+1} - \binom{n}{d+2} + \binom{n}{d+3} - \cdots =$$

$$\binom{n}{d} - \binom{n}{d-1} + \cdots + (-1)^{d-1}n + (-1)^d,$$

as asserted. □

In other words, as in the planar case, M_0 does not depend on the shape of P but only on its size. Next, we generalize Theorem 2.2 to the higher-dimensional case.

Theorem 4.2. *For a set P of n points in $\mathbb{R}^d$ in general position, we have*

$$M_1 = d\binom{n}{d} - (d-1)\binom{n}{d-1} + \ldots + (-1)^{d+1}n + I,$$

where I is the number of points of P that are interior to the convex hull of P.

Proof: Fix an oriented $(d-1)$-simplex $f = p_1p_2 \ldots p_d$ spanned by P, and define $X_k(f)$, for each $k \geq d+1$, to be the number of empty convex k-vertex polytopes that contain f as a facet and lie in the positive side of f. Define

$$M_0(f) = \sum_{k \geq d+1} (-1)^{k+d+1} X_k(f).$$

Arguing as in the proof of Theorems 2.1 and 4.1, the value of $M_0(f)$ depends only on the number of points of P that lie in the positive side of f. This follows by a similar continuous motion argument, in which the points in the positive side of f move in a sufficiently generic manner, without crossing the hyperplane supporting f, while the vertices of f, as well as the points in the negative side of f, remain fixed. If there are m points in the positive side of f then, when they lie in convex position together with the vertices of f, they satisfy

$$M_0(f) = \binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots,$$

which is 1 if $m > 0$, and 0 if $m = 0$, that is, if f is a negatively-oriented facet of the convex hull of P.

We perform a sufficiently generic continuous motion of the points of P, and keep track of the changes in the value of M_1 as the points move. We claim that the value of $M_1 - I$ does not change during the motion.

Clearly, the set of empty convex polytopes of P does not change until some $d+1$ points of P, say, $p_1, \ldots, p_{d+1}$, get to lie in a common hyperplane h. Arguing as in the proof of Theorem 4.1, the set of empty convex polytopes in P changes only if one of the points, say p_{d+1}, lies in the interior of the $(d-1)$-simplex f defined by $p_1, \ldots, p_d$. Observe that as long as this does not happen, I also remains unchanged. Hence, consider a critical event of the above kind, and assume that p_{d+1} crosses f from its negative side to its positive side. As argued in the proof of Theorem 4.1, the only convex k-vertex polytopes whose emptiness or convexity status may change at this criticality are those that have $p_1, \ldots, p_d$ as vertices.

Let K be such a k-vertex polytope which does not have p_{d+1} as a vertex. Then f must be a facet of K, for otherwise K would contain p_{d+1} in its interior both before and after the crossing of f by p_{d+1}.

If K was empty before the crossing, then K must lie in the positive halfspace determined by f, and it stops being empty after the crossing. Moreover, in this case K', as defined in the proof of Theorem 4.1, was an empty convex $(k+1)$-polytope before the crossing, and stops being convex after the crossing, so it is no longer counted in M_1 after the crossing. Therefore, K causes each of X_k and X_{k+1} to change by -1 , and thus causes M_1 to change by $-((-1)^{k+d+1}k + (-1)^{k+d+2}(k+1)) = (-1)^{k+d+1}$.

If K becomes empty after the crossing, then K lies in the negative halfspace determined by f. Observe that K contained p_{d+1} in its interior before the crossing, and thus was not empty then. Moreover, K' is a newly generated empty convex $(k+1)$-polytope after the crossing. Therefore, K causes each of X_k and X_{k+1} to change by $+1$, and thus causes M_1 to change by $((-1)^{k+d+1}k + (-1)^{k+d+2}(k+1)) = (-1)^{k+d}$.

It follows that the crossing causes the value of M_1 to change by $M_0(f^+) - M_0(f^-)$, where $f^+ = f$ and f^- is the oppositely oriented copy of f, and where both $M_0(f^-)$ and $M_0(f^+)$ are calculated with respect to $P \setminus \{p_{d+1}\}$.

If p_{d+1} is an internal point of $\mathrm{conv}(P)$, both before and after the crossing, then there are points of $P \setminus \{p_{d+1}\}$ on both sides of f, so that both $M_0(f^-)$ and $M_0(f^+)$ are 1, implying that M_1 remains unchanged by the crossing, and clearly so does I. Hence $M_1 - I$ remains unchanged.

If p_{d+1} was an extreme point of $\mathrm{conv}(P)$ before the crossing, then there are points of $P \setminus \{p_{d+1}\}$ only on the positive side of f or on f itself. Hence we have $M_0(f^-) = 0$ and $M_0(f^+) = 1$ and so M_1 increases by 1. However, I also increases at the same time by 1 since p_{d+1} becomes an interior point after the crossing (we ignore the easy case where P is a simplex in $\mathbb{R}^d$). Therefore, $M_1 - I$ remains unchanged in this case too. A completely symmetric analysis handles the case where p_{d+1} becomes an extreme point of P after the crossing.

It is easy to check that if the points of P are in convex position then $I = 0$ and $M_1 = d\binom{n}{d} - (d-1)\binom{n}{d-1} + \ldots + (-1)^{d+1}n$. This completes the proof of the theorem. $\square$

Remarks: (1) An interesting open problem is to extend Theorems 4.1 and 4.2 to higher-order moments. The current proof technique does not seem to yield such an extension.

(2) Consider the following variant of the problem, in which $X_k(P)$ is the number of empty convex polytopes spanned by P that have k facets (rather than k vertices). Can one obtain equalities similar to those in Theorems 4.1 and 4.2 for this setup? In the plane, any polygon with k vertices also has k edges (facets), and vice versa. In three dimensions, assuming general position, the number of facets is always $2k - 4$, where k is the number of vertices. Hence, Theorems 4.1 and 4.2 extend easily to the case where we count k-facet empty convex polytopes. However, in higher dimensions, the connection between the number of vertices and the number of facets of a convex polytope is much less constrained; see, e.g., [27].

5 Inequalities Involving the X_k's

In this section we derive a variety of inequalities that involve the parameters $X_k(P)$. The main collection of inequalities involves tails and prefixes of the series that define the moments $M_0, M_1, \ldots$. For simplicity of presentation, we first consider inequalities related to M_0 and M_1, and then study the general case involving M_r, for $r \geq 2$.

5.1 Head and tail inequalities for M_0 and M_1

Theorem 5.1. *For any finite point-set P in general position in the plane, and for each $t \geq 3$, we have*

$$\begin{aligned} X_t(P) - X_{t+1}(P) + X_{t+2}(P) - \cdots &\geq 0, \\ tX_t(P) - (t+1)X_{t+1}(P) + (t+2)X_{t+2}(P) - \cdots &\geq 0, \end{aligned} \tag{7}$$

with equality holding, in either case, if and only if $X_t(P) = 0$.

Recalling Theorems 2.1 and 2.2, an equivalent formulation of the theorem is given by

Theorem 5.2. *For any finite point-set P in general position in the plane, we have, for each $t \geq 3$ odd,*

$$\begin{aligned} X_3(P) - X_4(P) + X_5(P) - \cdots + X_t(P) &\geq \binom{n}{2} - n + 1, \\ 3X_3(P) - 4X_4(P) + 5X_5(P) - \cdots + tX_t(P) &\geq 2\binom{n}{2} - H, \end{aligned} \tag{8}$$

and for each $t \geq 4$ even,

$$\begin{aligned} X_3(P) - X_4(P) + X_5(P) - \cdots - X_t(P) &\leq \binom{n}{2} - n + 1, \\ 3X_3(P) - 4X_4(P) + 5X_5(P) - \cdots - tX_t(P) &\leq 2\binom{n}{2} - H, \end{aligned} \tag{9}$$

with equality holding, in either case, if and only if $X_{t+1}(P) = 0$.

We will prove the latter Theorem 5.2. The proof is based on the following lemma.

Lemma 5.3. *Let $p, q \in P$ be two distinct points, and let $e = \vec{pq}$ be the directed segment that they span. Assume that there is at least one point of P to the left of e. For each $k \geq 3$, let $X_k(e)$ denote the number of empty convex k-gons that are contained in the closed halfplane to the left of e, and have e as an edge. Then $X_3(e) - X_4(e) + \cdots - X_t(e) \leq 1$, if $t \geq 4$ is even, and $X_3(e) - X_4(e) + \cdots + X_t(e) \geq 1$, if $t \geq 3$ is odd. Moreover, in both cases equality holds if and only if $X_{t+1}(e) = 0$.*

Proof: First, we have shown in the proof of Theorem 2.2 that the "infinite" sum $X_3(e) - X_4(e) + X_5(e) - \cdots = 1$ (for edges e with at least one point of P to their left). Therefore, if $X_{t+1}(e) = 0$, then $X_j(e) = 0$ for all $j \geq t+1$, and the equality in the lemma follows.

We prove the lemma by induction on t. For $t = 3$ we have $X_3(e) \geq 1$ because there is at least one point of P to the left of e. Moreover, if $X_3(e) = 1$, then $X_4(e) = 0$, for otherwise the two vertices of an empty convex quadrilateral "sitting" on e would give rise to two empty triangles sitting on e. The converse argument, that $X_4(e) = 0$ implies $X_3(e) = 1$, will follow from the treatment of general values of t, given below. Since the induction argument relates X_t to X_{t-2}, we also need to establish the lemma for $t = 4$, which will be done shortly, after preparing the required machinery.

Let $t \geq 4$, and assume that the lemma holds for all $t' < t$. Let $P_{pq}^{+} = \{y_1, \ldots, y_m\}$ denote the set of all points y_i of P that lie to the left of e, and are such that the triangle pqy_i is empty (note that $m = X_3(e)$). Observe that if K is an empty k-gon that lies to the left of e and has e as an edge, then the other vertices of K must belong to P_{pq}^{+}. It is easy to see that the set P_{pq}^{+} is linearly ordered so that $y_i \prec y_j$ if y_j lies in the right wedge with apex y_i formed by the lines py_i and qy_i (i.e., y_j lies to the right of the directed lines $\vec{py_i}$ and $\vec{qy_i}$). We assume without loss of generality that the points of P_{pq}^{+} are enumerated as $y_1, \ldots, y_m$ in this order. See Figure 10.

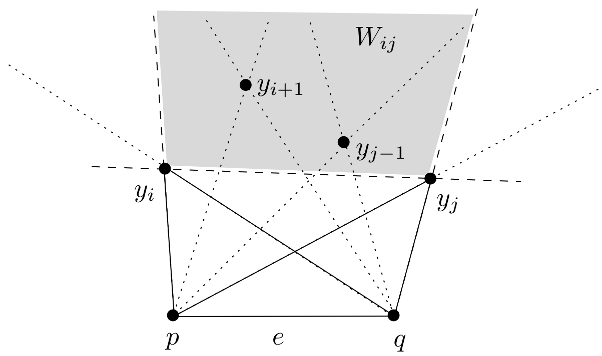


Figure 10: The region W_{ij}

Note first that for any $i < m$, $py_iy_{i+1}q$ is a convex empty quadrilateral. Hence we have $X_4(e) \geq X_3(e) - 1$. In particular, this establishes the inequality asserted in the lemma for $t = 4$. If $X_4(e) = X_3(e) - 1$ then we must have $X_5(e) = 0$, for otherwise we can obtain at least one additional empty convex quadrilateral, involving non-consecutive vertices y_i, y_j, from an empty convex pentagon "sitting" on e; see Figure 11.

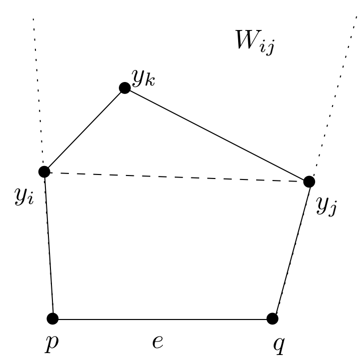


Figure 11: An empty convex pentagon yields an empty convex quadrilateral py_iy_jq, with y_i, y_j non-consecutive.

For each $1 \le i < j \le m$, let W_{ij} denote the open region formed by the intersection of the three halfplanes lying respectively to the right of $\vec{py_i}$ and to the left of $\vec{y_i y_j}$ and $\vec{qy_j}$. See Figure 10. Let K be any empty convex k-gon which lies to the left of e and has e as an edge. If K is not a triangle, let y_i (respectively, y_j) be the vertex of K that is adjacent to p (respectively, to q). Clearly, py_iy_jq forms an empty convex quadrilateral. Moreover, the other vertices of K belong to P^+_{pq}, lie in W_{ij}, and together with y_i, y_j they form an empty convex $(k-2)$-gon. The converse is also true, namely, if py_iy_jq is an empty convex quadrilateral, then there is a one-to-one correspondence between empty convex k-gons in which y_i, p, q, y_j are consecutive vertices (in counterclockwise order), and empty convex $(k-2)$-gons formed by points of $P^+_{pq} \cap W_{ij}$ and having y_i, y_j as (consecutive) vertices. As a matter of fact, in this case the points of P^+_{pq} that are contained in W_{ij} are precisely $y_{i+1}, \ldots, y_{j-1}$. Indeed, $W_{ij} = W_i^{(R)} \cap W_j^{(L)} \setminus Q$, where $W_i^{(R)}$ is the right wedge with apex y_i formed between the lines py_i and qy_i, $W_j^{(L)}$ is the left wedge with apex y_j formed between the lines py_j and qy_j, and Q is the quadrilateral y_ipqy_j; see Figure 10. The claim is then immediate from the definition of the linear order and from the fact that Q is empty.

For each pair of indices $i < j$, let $X_k^{(i,j)}$ denote the number of empty convex k-gons whose vertices belong to $P^+_{pq} \cap W_{ij}$ and that have y_i, y_j as vertices. Put $F_{ij} = X_3^{(i,j)} - X_4^{(i,j)} + \cdots + (-1)^{t-1} X_{t-2}^{(i,j)}$. Then

$$X_5(e) - X_6(e) + X_7(e) + \cdots + (-1)^{t+1} X_t(e) = \sum_{i,j} F_{ij}, \qquad (10)$$

where the sum extends over all $i < j$ such that the quadrilateral py_iy_jq is empty.

Case 1: t is even.

By induction hypothesis, if $P^+_{pq} \cap W_{ij}$ is nonempty, then $F_{ij} \le 1$. If $P^+_{pq} \cap W_{ij} = \emptyset$ then $F_{ij} = 0$, by definition.

There are exactly $X_4(e)$ pairs y_iy_j such that the quadrilateral py_iy_jq is empty and convex. Among these, exactly $X_4(e) - (X_3(e) - 1)$ are such that $i < j - 1$; this follows from the fact, already noted above for the case $t = 4$, that *all* quadrilaterals $py_iy_{i+1}q$ are empty, for $i < m$. Note that, for an empty quadrilateral py_iy_jq, $i < j - 1$ if and only if $P^+_{pq} \cap W_{ij}$ is nonempty. Hence, the left-hand side of (10) is at most the number of empty quadrilaterals py_iy_jq with $i < j - 1$; that is, it is at most $X_4(e) - (X_3(e) - 1)$.

Before continuing, we note that this argument implies that when $X_4(e) = 0$ we must have $X_3(e) = 1$, which is the missing ingredient in the proof of the lemma for $t = 3$. Note also that if $X_5(e) = 0$ then $X_3^{(i,j)} = 0$ for every $i < j$ for which py_iy_jq is empty. Hence the only such empty quadrilaterals are those with $i = j - 1$. By the preceding argument, this implies that $X_4(e) = X_3(e) - 1$, which is the missing ingredient in the proof for $t = 4$.

Hence, we have

$$X_3(e) - X_4(e) + X_5(e) - \cdots - X_t(e) \le$$
$$X_3(e) - X_4(e) + (X_4(e) - (X_3(e) - 1)) = 1.$$

If equality holds, then $F_{ij} = 1$ whenever $P^+_{pq} \cap W_{ij}$ is nonempty and py_iy_jq is empty. By the induction hypothesis, $X_{t-1}^{(i,j)} = 0$ for all such i, j. If $P^+_{pq} \cap W_{ij}$ is empty, then clearly $X_{t-1}^{(i,j)} = 0$. Therefore, $X_{t-1}^{(i,j)} = 0$ for every $i < j$ for which py_iy_jq is empty. This, in turn, implies that $X_{t+1} = 0$ (since the existence of an empty convex $(t+1)$-gon of this kind would imply that $X_{t-1}^{(i,j)} > 0$ for some i and j of this kind).

Case 2: t is odd.

By induction hypothesis, if $P_{pq}^{+} \cap W_{ij}$ is nonempty, then $F_{ij} \geq 1$. Hence, in complete analogy to Case 1,

$$X_3(e) - X_4(e) + X_5(e) - \cdots + X_t(e) \geq$$
$$X_3(e) - X_4(e) + (X_4(e) - (X_3(e) - 1)) = 1.$$

The case of equality is handled in the same way as in Case 1. □

We next proceed to prove Theorem 5.2 (and Theorem 5.1), in two steps.

Proof of the M_0-inequalities in Theorem 5.2: First, observe that, by Theorem 2.1, if $X_{t+1} = 0$ then equality holds in our theorem.

Let K be an empty convex k-gon. Let p be the lowest vertex of K, and let a, b be the vertices of K adjacent to p. The triangle pab is clearly empty, and the $(k-1)$-gon obtained from K by removing the vertex p is contained in the wedge W_{pab} whose apex is p and which is delimited by the rays $\vec{pa}$ and $\vec{pb}$. See Figure 12. The converse is also true, namely, there is a one-to-one correspondence between the empty convex k-gons whose lowest vertex is p, and the empty convex $(k-1)$-gons that have two consecutive vertices a, b that lie above p, so that their remaining vertices are contained in the wedge W_{pab} and pab is an empty triangle.

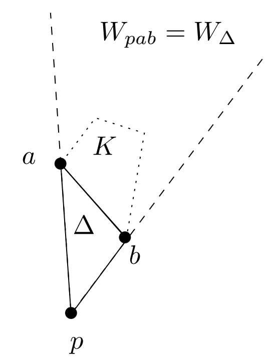


Figure 12: Shaving off the lowest triangle from an empty convex k-gon.

For each empty triangle $\Delta = pab$, let $X_k^{(\Delta)}$ denote the number of empty convex k-gons contained in $W_\Delta = W_{pab}$, having the upper edge of Δ as an edge, and separated from p by that edge. Put $F(\Delta) = X_3^{(\Delta)} - X_4^{(\Delta)} + \cdots + (-1)^t X_{t-1}^{(\Delta)}$. Then, by the one-to-one correspondence that we have just argued, $-X_4 + X_5 - X_6 + \cdots + (-1)^{t+1} X_t = -\sum_\Delta F(\Delta)$.

Case 1: t is odd.

We apply Lemma 5.3 to $P' = P \cap W_\Delta$, for each empty triangle Δ such that W_Δ contains at least one point of P in its interior, using the clockwise-directed top edge of Δ as the edge e in the lemma. We thus conclude that $F(\Delta) \leq 1$ for any such triangle Δ. We claim that there are exactly $1 + 2 + 3 + \cdots + (n-2) = \binom{n}{2} - n + 1$ empty triangles Δ such that W_Δ does not contain any additional point of P. Indeed, sort the points of P in decreasing

y-order, and enumerate them as $p_1, \ldots, p_n$ in this order. Fix a point p_i, and sort the *higher* points $p_1, \ldots, p_{i-1}$ in angular order about p_i. The empty triangles Δ with p_i as their lower vertex and with $W_\Delta \cap P = \emptyset$, are precisely those whose other two vertices are consecutive points in this angular order, and their number is thus $i-2$. Summing over all $i = 3, \ldots, n$, we obtain the claim. Then we have

$$X_3 - X_4 + X_5 - X_6 + \cdots + X_t = X_3 - \sum_\Delta F(\Delta) \geq$$

$$X_3 - |\{\Delta \mid \ \Delta \text{ empty and } W_\Delta \text{ not empty}\}| = \binom{n}{2} - n + 1.$$

If equality holds, then $F(\Delta) = 1$ for every empty triangle Δ with W_Δ nonempty. By Lemma 5.3 (applied to $P' = P \cap W_\Delta$), $X_t^{(\Delta)} = 0$ for any such Δ. Clearly, $X_t^{(\Delta)} = 0$ for an empty triangle Δ with W_Δ empty. Therefore, $X_t^{(\Delta)} = 0$ for every empty triangle Δ, which implies that $X_{t+1} = 0$ (since every empty convex $(t+1)$-gon gives rise to an empty triangle Δ with $X_t^{(\Delta)} > 0$).

Case 2: t is even.

Applying Lemma 5.3 for each empty triangle Δ such that W_Δ contains at least one additional point, as in the case where t is odd, we conclude that $F(\Delta) \geq 1$ for any such triangle Δ. As in Case 1, there are exactly $\binom{n}{2} - n + 1$ empty triangles Δ such that W_Δ does not contain any additional point of P. Then

$$X_3 - X_4 + X_5 - X_6 + \cdots + X_t = X_3 - \sum_\Delta F(\Delta) \leq$$

$$X_3 - |\{\Delta \mid \ \Delta \text{ empty and } W_\Delta \text{ not empty}\}| = \binom{n}{2} - n + 1.$$

The case of equality is handled in the same way as in Case 1. □

Proof of the M_1-inequalities of Theorem 5.2: First, observe that if $X_{t+1} = 0$, then Theorem 2.2 implies that equality holds.

Let $p, q \in P$ be two distinct points and let $e = \overrightarrow{pq}$ be the directed segment that they span. Let $X_k(e)$ denote, as in Lemma 5.3, the number of empty convex k-gons which have e as an edge and are to the left of e.

As in the proof of Theorem 2.2, it is easy to see that

$$3X_3 - 4X_4 + 5X_5 - \cdots + (-1)^{t+1} t X_t =$$

$$\sum_e \left(X_3(e) - X_4(e) + X_5(e) - \cdots + (-1)^{t+1} X_t(e)\right).$$

Case 1: t is odd.

By Lemma 5.3, $X_3(e) - X_4(e) + \cdots + X_t(e) \geq 1$, if there is at least one point of P to the left of e, or in other words, if e is not an edge of the convex hull of P (with P lying to its right). If e is such a hull edge, then of course $X_3(e) - X_4(e) + \cdots + X_t(e) = 0$. Hence,

$$3X_3 - 4X_4 + 5X_5 - \cdots + (-1)^{t+1} t X_t =$$

$$\sum_e (X_3(e) - X_4(e) + X_5(e) - \cdots + X_t(e)) \geq 2\binom{n}{2} - H.$$

If equality holds, then $X_3(e) - X_4(e) + \ldots + X_t(e) = 1$ for every edge e which is not an edge of the convex hull of P. By Lemma 5.3, $X_{t+1}(e) = 0$ for these edges. It follows that $X_{t+1}(e) = 0$ for every edge e and consequently $X_{t+1} = 0$.

Case 2: t is even.

By Lemma 5.3, $X_3(e) - X_4(e) + \ldots - X_t(e) \le 1$, if e is not an edge of the convex hull of P (with P lying to its right). Otherwise the sum is 0. The proof now proceeds exactly as in the case of odd t, except that the direction of the inequalities is reversed. □

5.2 Head and tail inequalities for general M_r

Theorems 5.1 and 5.2 can be extended to sums related to higher order moments. Specifically, we have:

Theorem 5.4. *For any finite point-set P in general position in the plane, for any $r \ge 2$ and for any $t \ge 2r$, we have*

$$\sum_{k\ge t} (-1)^{k+t} \frac{k}{r} \binom{k-r-1}{r-1} X_k(P) \ge 0,$$

with equality holding if and only if $X_t(P) = 0$. Alternatively,

$$\sum_{k=2r}^{t} (-1)^{k} \frac{k}{r} \binom{k-r-1}{r-1} X_k(P) \le T_r, \quad \text{for } t \ge 2r+1 \text{ odd, and}$$

$$\sum_{k=2r}^{t} (-1)^{k} \frac{k}{r} \binom{k-r-1}{r-1} X_k(P) \ge T_r, \quad \text{for } t \ge 2r \text{ even,}$$

with equality holding, in either case, if and only if $X_{t+1}(P) = 0$.

The proof uses an appropriate extension of Lemma 5.3 that involves r edges instead of one. To make it easier to follow the analysis, we first give the extension to $r = 2$ edges, use it to prove the theorem for this special case, and then analyze the general case.

The case $r = 2$.

Lemma 5.5. *Let $e_1 = ab$ and $e_2 = cd$ be a fixed pair of edges with endpoints $a, b, c, d \in P$, such that e_1 and e_2 are in convex position, with their endpoints lying in counterclockwise order a, b, c, d, and such that they span an empty convex quadrilateral Q. Assume further that the wedge $\tau(e_1, e_2)$ bounded by the lines supporting e_1 and e_2 and containing these edges, has at least one point of P in its interior. For each $k \ge 4$, let $X_k(e_1, e_2)$ denote the number of empty convex k-gons that have e_1 and e_2 as edges. Then*

$$X_4(e_1, e_2) - X_5(e_1, e_2) + \cdots + X_t(e_1, e_2) \ge 0, \quad \text{for } t \ge 6 \text{ even, and}$$
$$X_4(e_1, e_2) - X_5(e_1, e_2) + \cdots - X_t(e_1, e_2) \le 0, \quad \text{for } t \ge 5 \text{ odd.}$$

For $t = 4$, the sum is 1. Moreover, equality holds, in either case, if and only if $X_{t+1}(e_1, e_2) = 0$.

Proof: As shown in the proof of Theorem 2.3, the "infinite" sum $X_4(e_1,e_2)-X_5(e_1,e_2)+\cdots$ is 0 (when $\tau(e_1,e_2)$ is nonempty). Hence if $X_{t+1}(e_1,e_2)=0$ then equality holds (in either case).

Let τ_{cb} denote the portion of $\tau(e_1,e_2)$ that lies to the left of $\vec{cb}$, and let τ_{ad} denote the portion of $\tau(e_1,e_2)$ that lies to the left of $\vec{ad}$. See Figure 13. Put $P_{cb}=P\cap\tau_{cb}$ and $P_{ad}=P\cap\tau_{ad}$.

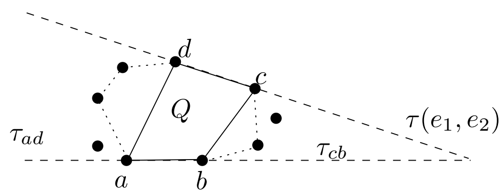


Figure 13: The structure in the proof of Lemma 5.5.

For any convex k-gon K that has e_1 and e_2 as edges, its vertices are a,b,c,d, j points of P_{cb}, for some $0\le j\le k-4$, that, together with b and c, span an empty convex $(j+2)$-gon, and $k-4-j$ points of P_{ad} that, together with a and d, span an empty convex $(k-j-2)$-gon. Conversely, any pair of an empty convex $(j+2)$-gon K_1, whose vertices are b,c and j points of P_{cb}, and an empty convex $(k-j-2)$-gon K_2, whose vertices are a,d and $k-j-4$ points of P_{ad}, are such that $K_1\cup Q\cup K_2$ is an empty convex k-gon.

Borrowing the notations of Lemma 5.3, we thus have

$$X_k(e_1,e_2)=\sum_{j=0}^{k-4}X_{j+2}(cb)X_{k-j-2}(ad),$$

where $X_{j+2}(cb)$ is computed only with respect to the points in P_{cb}, and similarly for $X_{k-j-2}(ad)$. We use here the convention that $X_2(e)=1$ for any edge e; that is, we regard e as an empty convex 2-gon. Hence

$$\begin{aligned}S_t:=\sum_{k=4}^{t}(-1)^kX_k(e_1,e_2)&=\sum_{k=4}^{t}(-1)^k\sum_{j=0}^{k-4}X_{j+2}(cb)X_{k-j-2}(ad)\\&=\sum_{j=0}^{t-4}\left[(-1)^{j+2}X_{j+2}(cb)\sum_{k=j+4}^{t}(-1)^{k-j-2}X_{k-j-2}(ad)\right].\end{aligned}$$

We replace k by $k'+j+2$, and then replace j by $j'-2$, to obtain

$$\begin{aligned}S_t&=\sum_{j=0}^{t-4}\left[(-1)^{j+2}X_{j+2}(cb)\sum_{k'=2}^{t-j-2}(-1)^{k'}X_{k'}(ad)\right]\\&=\sum_{j'=2}^{t-2}\left[(-1)^{j'}X_{j'}(cb)\sum_{k'=2}^{t-j'}(-1)^{k'}X_{k'}(ad)\right].\end{aligned}$$

By assumption, at least one of the sets P_{cb}, P_{ad} is nonempty. Without loss of generality, assume that $P_{ad} \neq \emptyset$. By Lemma 5.3, we have

$$\sum_{k'=2}^{t-j'} (-1)^{k'} X_{k'}(ad) \quad \begin{cases} \geq 0 & t-j' \geq 4 \text{ is even,} \\ \leq 0 & t-j' \geq 3 \text{ is odd,} \\ = 1 & t-j' = 2. \end{cases}$$

Suppose now that $t \geq 6$ is even. Then the parity of $t-j'$ is the same as that of j'. This is easily seen to imply that all terms in the main sum (on j') are non-negative, and hence $S_t \geq 0$, as asserted. Using a fully symmetric argument, one shows that $S_t \leq 0$ when $t \geq 5$ is odd. (We note that for $t = 4$ the sum is always 1.)

If $S_t = 0$ then all terms in the main sum (on j') are 0. Suppose to the contrary that $X_{t+1}(e_1, e_2) \neq 0$. Then there exists $2 \leq j' \leq t-1$ such that $X_{j'}(cb) X_{t+1-j'}(ad) > 0$. If $j' \leq t-2$ then the j'-th term in the sum is positive, because $X_{j'}(cb) > 0$ and $\sum_{k'=2}^{t-j'} (-1)^{k'} X_{k'}(ad) > 0$; the latter inequality follows from Lemma 5.3, since $X_{t+1-j'}(ad) > 0$. Hence the total main sum is positive, a contradiction. The case $j' = t-1$ is handled by interchanging the roles of cb and ad, as it is easy to check.

This completes the proof of the lemma. $\square$

Proof of Theorem 5.4 for $r = 2$: As above, it suffices to prove only the head inequalities. Here the coefficients are $\frac{1}{2}k(k-3)$, so we write them as such. As in the proof of Theorem 2.3, we have

$$\sum_{k=4}^{t} (-1)^k \frac{k(k-3)}{2} X_k(P) = \sum_{e_1, e_2} \sum_{k=4}^{t} (-1)^k X_k(e_1, e_2).$$

Case 1. $t \geq 5$ is odd. By Lemma 5.5, $\sum_{k=4}^{t} (-1)^k X_k(e_1, e_2) \leq 0$, when the wedge formed by e_1 and e_2 contains at least one point of P in its interior. If this is not the case, then (e_1, e_2) is a T_2-configuration, and the sum is equal to 1. Hence,

$$\sum_{k=4}^{t} (-1)^k \frac{k(k-3)}{2} X_k(P) \leq T_2,$$

as asserted.

Case 2. $t \geq 6$ is even. By Lemma 5.5, $\sum_{k=4}^{t} (-1)^k X_k(e_1, e_2) \geq 0$, when (e_1, e_2) is not a T_2-configuration, and is 1 otherwise. Hence,

$$\sum_{k=4}^{t} (-1)^k \frac{k(k-3)}{2} X_k(P) \geq T_2,$$

as asserted.

Case 3. $t = 4$. In this case we need to show that $2X_4 \geq T_2$, which is obvious, since each T_2-configuration (e_1, e_2) spans an empty convex quadrilateral, and each such quadrilateral can be obtained from at most two T_2-configurations.

If equality holds then $\sum_{k=4}^{t} (-1)^k X_k(e_1, e_2) = 0$ for every pair (e_1, e_2) that is not a T_2-configuration. By Lemma 5.5, $X_{t+1}(e_1, e_2) = 0$ for every such pair of edges, and of course $X_{t+1}(e_1, e_2) = 0$ also for pairs that are T_2-configurations. This implies, as above, that $X_{t+1}(P) = 0$, and thus completes the proof of the theorem for $r = 2$. $\square$

The general case. We now turn to the case of arbitrary $r \geq 2$, and begin with extending Lemma 5.5:

Lemma 5.6. *Let $r \geq 3$, and let $e_1, e_2, \ldots, e_r$ be r vertex-disjoint edges that are spanned by P, lie in convex position, and span an empty convex $(2r)$-gon Q. Assume further that the region $\tau(e_1, \ldots, e_r)$, as defined in the introduction, has at least one point of P in its interior. For each $k \geq 2r$, let $X_k(e_1, \ldots, e_r)$ denote the number of empty convex k-gons that have $e_1, \ldots, e_r$ as edges. Then*

$$X_{2r}(e_1,\ldots,e_r) - X_{2r+1}(e_1,\ldots,e_r) + \cdots + X_t(e_1,\ldots,e_r) \geq 0, \quad \text{for } t \geq 2r+2 \text{ even, and}$$
$$X_{2r}(e_1,\ldots,e_r) - X_{2r+1}(e_1,\ldots,e_r) + \cdots - X_t(e_1,\ldots,e_r) \leq 0, \quad \text{for } t \geq 2r+1 \text{ odd.}$$

The sum is equal to 1 for $t = 2r$.
Moreover, equality holds, in either case, if and only if $X_{t+1}(e_1, \ldots, e_r) = 0$.

Proof: The "infinite" sum $X_{2r}(e_1,\ldots,e_r) - X_{2r+1}(e_1,\ldots,e_r) + \cdots$ is 0 (when $\tau = \tau(e_1,\ldots,e_r)$ is nonempty), as shown in the proof of Theorem 2.3. Hence, if $X_{t+1}(e_1,\ldots,e_r) = 0$ then equality holds (in either case).

Assume without loss of generality that $e_1, \ldots, e_r$ appear in the clockwise order along the boundary of Q (or along the boundary of τ). For each i, let W_i denote the connected component of $\tau \setminus Q$ that has an endpoint of e_i and an endpoint of e_{i+1} (where e_{r+1} is taken to be e_1) on its boundary. Let $\bar{e}_i$ denote the edge of W_i that also bounds Q. (In other words, the edges of Q are $e_1, \bar{e}_1, e_2, \bar{e}_2, \ldots, e_r, \bar{e}_r$, in clockwise order.) Put $P_i = P \cap W_i$, for $i = 1, \ldots, r$. See Figure 14.

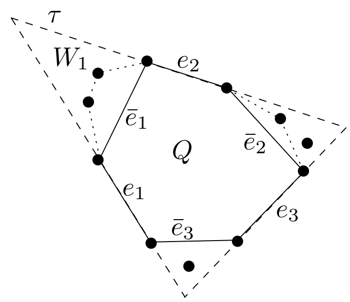


Figure 14: The structure in the proof of Lemma 5.6.

For any convex k-gon K that has $e_1, \ldots, e_r$ as edges, its vertices are the $2r$ endpoints of the edges $e_1, \ldots, e_r$, and $k - 2r$ additional points that are grouped in r disjoint subsets $V_1, \ldots, V_r$, where V_i is a subset of P_i of size $j_i \geq 0$, such that it forms an empty convex $(j_i + 2)$-gon with the endpoints of $\bar{e}_i$. We have $\sum_{i=1}^{r}(j_i + 2) = k$. Conversely, for any choice of sets V_i with the above properties, the union of these sets, together with the endpoints of the e_i's, is the vertex set of an empty convex k-gon that has $e_1, \ldots, e_r$ as edges.

As in Lemma 5.5, we thus have (where we replace the preceding quantities $j_i + 2$ by j_i)

$$X_k(e_1,\ldots,e_r) = \sum_{\substack{j_1 \geq 2, \ldots, j_r \geq 2 \\ j_1 + \cdots + j_r = k}} \prod_{i=1}^{r} X_{j_i}(\bar{e}_i),$$

where, as in Lemma 5.5, $X_{j_i}(\bar{e}_i)$ is computed only with respect to the points in P_i (we orient $\bar{e}_1$ so that P_i lies in the half-plane to its left), and where we use the convention that $X_2(e) = 1$ for any edge e. Hence

$$S_t := \sum_{k=2r}^{t} (-1)^k X_k(e_1, \ldots, e_r) = \sum_{k=2r}^{t} (-1)^k \sum_{\substack{j_1 \geq 2, \ldots, j_r \geq 2 \\ j_1 + \cdots + j_r = k}} \prod_{i=1}^{r} X_{j_i}(\bar{e}_i).$$

We now proceed by induction on r. We have already established the lemma for $r = 2$. Assume then that $r \geq 3$, and that the lemma holds for all $r' < r$. We can rewrite S_t as

$$S_t = \sum_{k=2r}^{t} \sum_{j_r=2}^{k-2r+2} \left[(-1)^{j_r} X_{j_r}(\bar{e}_r) \cdot (-1)^{k-j_r} \sum_{\substack{j_1 \geq 2, \ldots, j_{r-1} \geq 2 \\ j_1 + \cdots + j_{r-1} = k - j_r}} \prod_{i=1}^{r-1} X_{j_i}(\bar{e}_i) \right]$$

$$= \sum_{j_r=2}^{t-2r+2} (-1)^{j_r} X_{j_r}(\bar{e}_r) \cdot \left[\sum_{k=j_r+2r-2}^{t} (-1)^{k-j_r} \sum_{\substack{j_1 \geq 2, \ldots, j_{r-1} \geq 2 \\ j_1 + \cdots + j_{r-1} = k - j_r}} \prod_{i=1}^{r-1} X_{j_i}(\bar{e}_i) \right]$$

We replace k in the expression in the brackets by $k' + j_r$, so this expression becomes

$$\sum_{k'=2r-2}^{t-j_r} (-1)^{k'} \sum_{\substack{j_1 \geq 2, \ldots, j_{r-1} \geq 2 \\ j_1 + \cdots + j_{r-1} = k'}} \prod_{i=1}^{r-1} X_{j_i}(\bar{e}_i),$$

which, by the induction hypothesis, is non-negative for $t - j_r \geq 2r$ even, non-positive for $t - j_r \geq 2r - 1$ odd, and 1 for $t - j_r = 2r - 2$.

Suppose now that $t \geq 2r + 2$ is even. Then the parity of $t - j_r$ is the same as that of j_r. Hence all terms in the main sum (on j_r) are non-negative, and hence $S_t \geq 0$, as asserted. Using a fully symmetric argument, one shows that $S_t \leq 0$ when $t \geq 2r + 1$ is odd, and we note that for $t = 2r$ the sum is always 1.

The proof that equality implies that $X_{t+1}(P) = 0$ is carried out exactly as in the preceding proofs, and we omit the details. This completes the proof of the lemma. □

Proof of Theorem 5.4 for arbitrary r: As above, it suffices to prove only the head inequalities. As in the proof of Theorem 2.3, we have

$$\sum_{k=2r}^{t} (-1)^k \frac{k}{r} \binom{k-r-1}{r-1} X_k(P) = \sum_{e_1, \ldots, e_r} \sum_{k=2r}^{t} (-1)^k X_k(e_1, \ldots, e_r).$$

Case 1. $t \geq 2r + 1$ is odd.

By Lemma 5.6, $\sum_{k=2r}^{t} (-1)^k X_k(e_1, \ldots, e_r) \leq 0$, when $\tau(e_1, \ldots, e_r)$ contains at least one point of P in its interior. If this is not the case, then $(e_1, \ldots, e_r)$ is a T_r-configuration, and the sum is equal to 1. Hence,

$$\sum_{k=2r}^{t} (-1)^k \frac{k}{r} \binom{k-r-1}{r-1} X_k(P) \leq T_r,$$

as asserted.

Case 2. $t \geq 2r+2$ is even.

By Lemma 5.6, $\sum_{k=2r}^{t}(-1)^k X_k(e_1,\ldots,e_r) \geq 0$, when $(e_1,\ldots,e_r)$ is not a T_r-configuration, and is 1 otherwise. Hence,

$$\sum_{k=2r}^{t}(-1)^k \frac{k}{r}\binom{k-r-1}{r-1} X_k(P) \geq T_r,$$

as asserted.

Case 3. $t = 2r$. In this case we need to show that $2X_{2r} \geq T_r$, which is obvious, since each T_r-configuration $(e_1,\ldots,e_r)$ spans an empty convex $(2r)$-gon, and each such polygon can be obtained from at most two T_r-configurations.

This completes the proof of the theorem for arbitrary $r \geq 2$. $\square$

5.3 Inequalities involving X_3, X_4, and X_5

Inequalities for X_4. One important application of Theorem 5.2 is for $t=4$, which yields the following pair of inequalities:

$$\begin{aligned} X_3(P) - X_4(P) &\leq \binom{n}{2} - n + 1, \\ 3X_3(P) - 4X_4(P) &\leq n(n-1) - H, \end{aligned}$$

or, equivalently,

$$X_4(P) \geq \max\left\{X_3(P) - \binom{n}{2} + n - 1,\ \frac{3}{4}X_3(P) - \frac{n(n-1)-H}{4}\right\}. \tag{11}$$

As mentioned in the introduction, Bárány and Füredi [3] have shown that

$$X_3(P) \geq n^2 - O(n \log n).$$

It follows from (11) that this lower bound implies the lower bound

$$X_4(P) \geq \frac{1}{2}n^2 - O(n \log n).$$

This was also established in [3], but the explicit inequalities relating X_3 to X_4 that are given above make the connection between the two lower bounds more direct. Note also that the first term in (11) dominates (if at all) the second term when $X_3(P) \geq (n-1)(n-4) + H$. In view of the lower bound in [3], the second term in (11) dominates only in a small range of values of X_3, between $n^2 - O(n \log n)$ and $(n-1)(n-4) + H$.

Inequalities for X_5. In the formulation of the following theorem, we introduce a new quantity $H' = H'(P)$, which is defined as follows. For each $p \in P$, let P_p^+ denote the set of all points of P that lie above (the horizontal line through) p, and let C_p^+ denote the convex hull of P_p^+. Then H' is equal to the number of points $p \in P$ for which the two tangents from p to C_p^+ meet it at two *consecutive* vertices. See Figure 15.

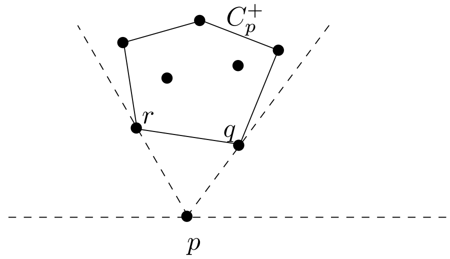


Figure 15: A point p counted in H'.

Theorem 5.7.

$$X_5(P) \geq \max\left\{X_3(P) - (n-2)(n-3) - H',\ \frac{3}{5}X_3(P) - \frac{n(n-1)-H}{5} - \frac{2}{5}T_2\right\}.$$

Proof: We start with the proof of the first inequality. For each convex empty pentagon Q spanned by P we generate an empty triangle, whose vertices are the lowest vertex p of Q and the two vertices of Q not adjacent to p; see Figure 16.

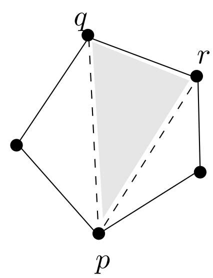


Figure 16: Charging an empty pentagon to an empty triangle.

Clearly, each empty convex pentagon generates a unique empty triangle. However, not all empty triangles are generated in this manner: Let $\Delta = pqr$ be an empty triangle spanned by P, so that p is its lowest vertex and r lies to the right of $\vec{pq}$. Associate with Δ the wedge $w(\Delta)$, consisting of the points that lie above (the horizontal line passing through) p and to the right of the directed line $\vec{qr}$. The triangle Δ is contained in $w(\Delta)$ and partitions it into three subregions: Δ itself, the portion Δ_L lying to the left of $\vec{pq}$, and the portion Δ_R lying to the right of $\vec{pr}$; see Figure 17.

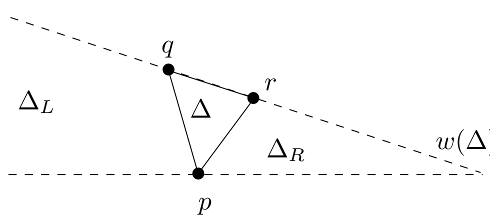


Figure 17: An empty triangle Δ and the partition of its associated wedge.

It follows that Δ is not generated from an empty pentagon if and only if either Δ_L or Δ_R is empty.

We estimate the size of the set E_L of triangles Δ for which Δ_L is empty; analyzing the set of triangles for which Δ_R is empty is done in a fully symmetric fashion. Fix a point $p \in P$, and consider the set $E(p)$ of edges qr spanned by P such that $pqr \in E_L$. Note that both q and r lie above p. We view $E(p)$ as the edge set of a graph on the set P_p^+ of points that lie above p, and claim that $E(p)$ does not contain any cycle. Indeed, suppose to the contrary that $E(p)$ did contain a cycle, and let q be the vertex in the cycle such that $\vec{pq}$ forms the smallest angle with the positive x-direction. Since q is the rightmost point of this cycle, $E(p)$ contains two edges qu, qv emanating from q, such that both pu and pv lie counterclockwise to pq, with, say, qu lying clockwise to qv. See Figure 18. But then either the triangle $\Delta = pqu$ or its associated left region Δ_L would contain v, contrary to the definition of $E(p)$. Hence $E(p)$ is a forest, and so it contains at most $|P_p^+| - 1$ edges. Consequently, the overall number of triangles Δ for which $\Delta_L = \emptyset$ is at most $\sum_{k=3}^{n}(k-2) = \binom{n-1}{2}$. Symmetrically, the number of triangles Δ for which $\Delta_R = \emptyset$ is also at most $\binom{n-1}{2}$. Therefore, the number of empty triangles that are not generated from an empty pentagon in the manner prescribed above is at most $(n-1)(n-2)$.

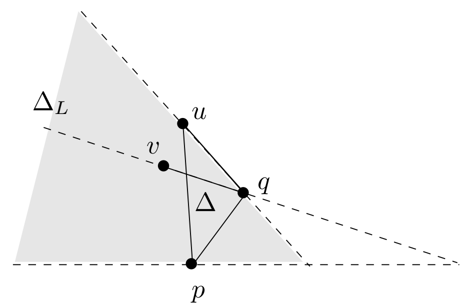


Figure 18: $E(p)$ does not contain a cycle.

We can improve the bound further by noting that we have doubly counted empty triangles Δ for which both Δ_L and Δ_R are empty. We can obtain a lower bound for the number of such triangles, as follows. Let $\Delta = pqr$ be such a triangle, where p is the lowest vertex. In the notation preceding the theorem, qr is an edge of C_p^+, with the property that the line through qr separates p from C_p^+. The converse is also easily seen to hold: Any edge qr of C_p^+ with this property gives rise to a doubly counted triangle pqr. These edges qr are precisely those that lie along the boundary of C_p^+ between the two contact points of the tangents from p to C_p^+. By definition, the number of such edges is at least two, unless p is counted in H', in which case this number is 1. Hence, the overall number of doubly counted triangles is at least $2(n-2) - H'$. Then, the total number of triangles that are not generated from an empty pentagon is at most

$$(n-1)(n-2) - 2(n-2) + H' = (n-2)(n-3) + H',$$

It thus follows that $X_5 \geq X_3 - (n-2)(n-3) - H'$, as asserted.

We next prove the second inequality of the theorem. For each empty convex pentagon Q spanned by P we generate five empty triangles, whose vertices are obtained by removing a pair of nonadjacent vertices of Q (as in Figure 16).

A triangle Δ may be generated in this manner in at most three different ways, in each of which the generating empty convex pentagon has a different pair of edges of Δ as diagonals. We associate each of these possibilities with the vertex v of Δ that is common to the two edges, and refer to the pair (Δ, v) as a *pointed triangle*.

Clearly, there exist pointed triangles (Δ, v) for which such an extension is impossible. Let p, q be the two other vertices of Δ, so that v lies to the left of the directed line $\vec{pq}$. Let P^+_{pq} denote the subset of the points of P that lie in the halfplane to the left of the line $\vec{pq}$. Then (Δ, v) admits no extension into an empty convex pentagon if and only if v is an extreme point of P^+_{pq} such that the triangle pqv is empty. Let $t = t_{pq}$ denote the number of such points v. First note that if P^+_{pq} is nonempty then $t \geq 1$ and if P^+_{pq} is empty then $t = 0$. Moreover, if $t > 1$ then these points form a chain of consecutive vertices of the convex hull of P^+_{pq}, and for each of the $t-1$ pairs (u, v) of consecutive vertices among them, (p, q, u, v) is a T_2-configuration. See Figure 19.

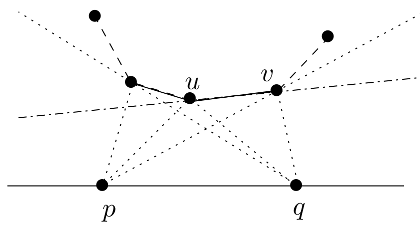


Figure 19: Pointed triangles with edge pq that cannot be extended to an empty convex pentagon.

There are $n(n-1)$ ordered pairs p, q, and H of them satisfy $t_{pq} = 0$ (these are the directed edges of the convex hull of P that contain P on their right side). Each of the remaining pairs defines at least one pointed triangle that admits no extension, and any additional such triangle can be charged to a T_2-configuration, where any such configuration is charged exactly twice. This implies that the number of 'bad' pointed triangles is at most $n(n-1) - H + 2T_2$, and this is easily seen to imply the second part of the theorem. $\Box$

Some implications. (i) Theorem 5.4 implies, for $r = 2$, that $2X_4(P) - 5X_5(P) \leq T_2(P)$. Combining this inequality with (5), we obtain

$$2X_4 - 5X_5 \leq \binom{n}{2} - H + 2X_5 + X_6 + T_2^*.$$

Substituting the lower bound of [3] for X_4, we obtain

$$\frac{n^2}{2} \leq 7X_5 + X_6 + T_2^* + O(n \log n).$$

Hence, any improved upper bound on T_2^* of the form $(\frac{1}{2} - c)n^2$ would imply that

$$7X_5 + X_6 \geq cn^2 - O(n \log n).$$

Hence, it would imply that every sufficiently large set either contains quadratically many empty convex pentagons, or quadratically many empty convex hexagons.

(ii) Curiously, plugging the lower bound of [3] for X_3 into Theorem 5.7, and using Theorem 3.1, we do not obtain a quadratic lower bound for $X_5(P)$. Still, any improvement of the coefficient of the quadratic term in the upper bound for T_2 would lead to a quadratic lower bound for X_5.

(iii) Any improvement of the coefficient of the quadratic term in the lower bound for X_3 would lead to an improvement, by the same amount, of the quadratic lower bounds for X_4 and X_5. These explicit relations are more quantitative than what has been earlier observed by Bárány and others [2].

(iv) Comparing the above inequalities with the explicit expressions for X_3 and X_5 given in (1), we obtain the following corollaries:

$$\begin{aligned} \textstyle\sum_{k\geq 6}(-1)^k(k-4)X_k &\geq H - H' - 2, \\ \textstyle\sum_{k\geq 6}(-1)^k k(k-4)X_k &\geq 0. \end{aligned}$$

The first inequality implies that, when $H > H' + 2$, P contains an empty convex hexagon.

Note that these inequalities are tail inequalities in the series for $M_1 - 4M_0$ and $2M_2 - M_1$, respectively. Because these linear combinations involve *negative* coefficients, they cannot be deduced from the tail inequalities derived in Sections 5.1 and 5.2 (cf. the remark following Theorem 2.3).

6 Discussion and Open Problems

This paper raises several new open problems and also leaves unsolved several old ones.

One problem is to generalize the formulas for the moments M_r, for $r > 1$, to dimension $d > 3$. One of the difficulties here is that the number of facets of a convex simplicial polytope with k vertices in R^d, is not determined by k. Nevertheless, we believe that a solution to this problem is possible, using techniques similar to those that we have introduced.

Two other interesting open problems involve the parameters T_2 and T_2^*. The main questions here are: (i) Is it true that $T_2 < (1-c)n^2$, for some constant $c > 0$? (ii) Is it true that $T_2^* < (1-c)n^2/2$, for some constant $c > 0$? As we have seen earlier, an affirmative answer to any of these problems leads to sharper lower bounds on the number of empty triangles, convex quadrilaterals, and convex pentagons determined by a set of n points in general position in the plane.

Clearly, the main open problem that our analysis so far still has not settled is whether every set of sufficiently many points in general position in the plane contains an empty convex hexagon. The other main open problems are to improve the constants in the lower bounds on the number of empty triangles, convex quadrilaterals, and convex pentagons, as discussed earlier in detail.

We note that the results in this paper can be generalized to the case where the set P of points is not in general position, so that more than two points may be collinear. In this case, we define $X_k(P)$ to be the number of k-tuples of points of P that lie in strictly convex position and the intersection of their convex hull with P consists of exactly these k points. In this case, it is important to consider X_2 explicitly as well, since it may be different from $\binom{n}{2}$.

It is easy to see, for example, that Theorem 2.1 remains true verbatim. Indeed, assume that during the continuous motion a point x becomes collinear between two other points a and b. It is easily seen that the emptiness and convexity status of a polygon Q can change if and only if ab newly becomes (or used to be) an edge of Q. An inspection of this situation shows that the alternating sum M_0^* remains unchanged. Here we also have to include X_2 in the analysis, to cater to situations where an empty triangle xab is destroyed, say, during the collinearity, but so is the empty segment ab.

Another important class of problems concerns the number of k-*lines* spanned by a set P of n points in the plane, that is, lines that contain exactly k points of P. Denote these numbers by $t_k(P)$, or just t_k, for short. The goal is to obtain linear equalities and inequalities involving these numbers. This setup is somewhat similar to the one studied in this paper, because we can regard a k-line as a degenerate form of an empty convex k-gon.

For example, one always has, trivially, $\sum_{k\geq 2}\binom{k}{2}t_k = \binom{n}{2}$, and there is in fact a variant of the continuous motion argument that proves this equality. Furthermore, $\sum_{k\geq 2} t_k$ is the total number of lines spanned by P, and $\sum_{k\geq 2} kt_k$ is the total number of incidences between these lines and the points of P.

There are several known important inequalities. The first is Melchior's inequality [7]:

$$t_2 \geq 3 + t_4 + 2t_5 + 3t_6 + \cdots, \text{ if } t_n = 0,$$

which is a simple consequence of the Euler formula. The second is Hirzebruch's inequality [16]:

$$t_2 + \frac{3}{4}t_3 \geq n + t_5 + 3t_6 + 5t_7 + \cdots, \text{ if } t_n = t_{n-1} = t_{n-2} = 0,$$

whose only known proof uses difficult tools from enumerative algebraic geometry.

A simple application of the Szemerédi–Trotter theorem [24] on the number of point-line incidences implies an interesting tail inequality:

$$t_k + t_{k+1} + t_{k+2} + \cdots \leq 16.875\frac{n^2}{(k-1)^3} \text{ for } k < n^{1/3}, n > n_0(k),$$

which is asymptotically best possible (here $n_0(k)$ is an absolute constant which depend only on k).

Related inequalities are the Erdős–Purdy inequalities [13], which state that if $t_n = 0$, then $\max(t_2, t_3) \geq n - 1$, and $\max(t_2, t_3) \geq t_i$, for all i. Several additional inequalities are derived there too.

One of the goals for future research is to develop continuous motion proofs of the above inequalities on the parameters t_k. We also hope that this approach might be useful for the famous "orchard"-type problems, originated by Sylvester [7]: what is the maximum number $t_k^{\text{orchard}}(n)$ of k-lines in a set of n points in the plane that does not contain $k+1$ collinear points? Some partial results on this problem can be found in [6, 12, 14, 18].

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References

[1] C. Ahrens, G. Gordon and E.W. McMahon, Convexity and the beta invariant, *Discrete Comput. Geom.* 22 (1999), 411–424.

[2] I. Bárány, personal communication.

[3] I. Bárány and Z. Füredi, Empty simplices in Euclidean space, *Canadian Math. Bull.* 30 (1987), 436–445.

[4] I. Bárány and G. Károlyi, Problems and results around the Erdős-Szekeres convex polygon theorem, *Proc. Japan Conference on Discrete and Comp. Geom. (JCDCG)* 2000, Tokyo, Japan, *Lecture Notes in Computer Science* 2098 (J. Akiyama, M. Kano, and M. Urabe, Eds.), Springer Verlag (2001), 91–105.

[5] I. Bárány and P. Valtr, Planar point sets with a small number of empty convex polygons, *Stud. Sci. Math. Hung.* 41 (2004), 243–266.

[6] P. Brass, On point sets without k collinear points, in *Discrete Geometry – in Honor of W. Kuperberg's 65th Birthday*, A. Bezdek, ed., Marcel Dekker, 2003, 185–192.

[7] P. Brass, W. Moser and J. Pach, *Research Problems in Discrete Geometry*, Springer Verlag, New York, in press.

[8] A. Dumitrescu, Planar sets with few empty convex polygons, *Stud. Sci. Math. Hung.* 36 (2000), 93–109.

[9] P.H. Edelman and R. Jamison, The theory of convex geometries, *Geometriae Dedicata* 19 (1987), 247–270.

[10] P.H. Edelman and V. Reiner, Counting the interior of a point configuration, *Discrete Comput. Geom.* 23 (2000), 1–13.

[11] P.H. Edelman, V. Reiner and V. Welker, Convex, acyclic, and free sets of an oriented matroid, *Discrete Comput. Geom.* 27 (2002), 99–116.

[12] P. Erdős, Research Problem 36, *Periodica Math. Hungarica* 15 (1984), 101–103.

[13] P. Erdős and G. Purdy, Extremal problems in combinatorial geometry, in: *Handbook of Combinatorics*, Vol. 1, R. L. Graham et al., eds., North-Holland/Elsevier 1995, 809–874.

[14] Z. Füredi and I. Palasti, Arrangements of lines with large number of triangles, *Proc. Amer. Math. Soc.* 92 (1984), 561–566.

[15] H. Harborth, Konvexe Fünfecke in ebenen Punktmengen, *Elem. Math.* 33 (1978), 116–118.

[16] F. Hirzebruch, Singularities of algebraic surfaces and characteristic numbers, in: *Proc. Lefschetz Centennial Conference*, Mexico City, 1984, Part I, *AMS Contemporary Mathematics Series* 58 (1986), 141–150.

[17] J.D. Horton, Sets with no empty convex 7-gons, *Canadian Math. Bull.* 26 (1983), 482–484.

[18] D. Ismailescu, Restrcited point configurations with many collinear k-tuplets, *Discrete Comput. Geom.* 28 (2002), 571–575.

[19] M. Katchalski and A. Meir, On empty triangles determined by points in the plane, *Acta. Math. Hung.* 51 (1988), 323–328.

[20] D. Klain, An Euler relation for valuations of polytopes, *Adv. Math.* 147 (1999), 1–34.

[21] J. Matoušek and J. Nešetřil, *Invitation to Discrete Mathematics*, Clarendon Press, Oxford, 1998.

[22] M. Overmars, Finding sets of points without empty convex 6-gons, *Discrete Comput. Geom.* 29 (2003), 153–158.

[23] M. Overmars, B. Scholten and I. Vincent, Sets without empty convex 6-gons, *Bull. European Assoc. Theor. Comp. Sci.* 37 (1989), 160–168.

[24] E. Szemerédi and W. T. Trotter, Extremal problems in discrete geometry, *Combinatorica* 3 (1983), 381–392.

[25] P. Valtr, On the minimum number of empty polygons in planar point sets, *Stud. Sci. Math. Hung.* 30 (1995), 155–163.

[26] P. Valtr, personal communication.

[27] G.M. Ziegler, *Lectures on Polytopes*, Springer Verlag, New York, 1995.