Proximity Problems on Line Segments Spanned by Points*

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1 Abstract

We address proximity problems for lines and line segments spanned by points in the plane.

Closest (farthest) line segment from point. Given a set $S = \{p_1, p_2, \ldots, p_n\}$ of n points in the plane, and another point q, find an extremal (closest, farthest) line segment from q among the set E of $O(n^2)$ line segments defined by the points in S.

The solutions to this problem are related to efficient solutions for the following problems.

Closest (farthest) line from point. Given a set $S = \{p_1, p_2, \ldots, p_n\}$ of n points in the plane, and another point q, find an extremal (closest, farthest) line from q among the set L of $O(n^2)$ lines defined by the points in S.

There are a number of motivations for studying these proximity problems. For example, consider a vertex q that is to be inserted in a geometric graph G and assume one would like to place a label at q. For clarity in visualization, it is desirable the label does not intersect any edge that is not adjacent to q. If new edges can be created dynamically, a suitable test for deciding whether to place q at a particular location might include finding the closest edge to q among the edges of the complete graph G_c defined by the vertices of G.

The problems we study are related to the well known slope selection and distance selection problems, and to a number of optimization problems such as computing the largest empty disk or the smallest enclosing circle of S. The slope selection problem is to find the line in L with the k-th smallest slope and has been solved deterministically in $O(n \log n)$ time [2, 6]. A simple, randomized $O(n \log n)$ time algorithm has been presented in [3]. The distance selection problem is to find the k-th smallest distance between the points in S and can be solved in $O(n \log n + n^{2/3} k^{1/3} \log^{5/3} n)$ expected time [1] or in $O(n^{4/3} \log^2 n)$ deterministic time [6]. However, computing the smallest distance (the closest pair of S) or the largest distance (the diameter of S) are textbook problems in computational geometry and can be solved optimally in $O(n \log n)$ time and $O(n)$ space [8].

Computing the farthest or closest line of L from a point q is closely related to counting the number of lines in L that are intersected by a disk $\mathcal{D}$ centered at q. Since a line $l \in L$ that intersects $\mathcal{D}$ can be sandwiched between two lines (not necessarily in L) parallel to l and tangent to $\mathcal{D}$, using a standard

*This research was partially supported by NSF grant CCF-0430366.

point-line duality transform, the duals of the lines intersecting $\mathcal{D}$ correspond to points inside the *stabbing region* $\mathcal{D}^*$ of $\mathcal{D}$ [9]. The boundary $\partial\mathcal{D}^*$ of $\mathcal{D}^*$ is the set of points dual to the tangents of $\mathcal{D}$, and $\mathcal{D}^*$ is bounded from above by a convex x-monotone curve and from below by a concave x-monotone curve (see [9], page 255; in case of disks these curves are the two branches of a hyperbola [5]). Thus, to count the number of lines in L that are intersected by $\mathcal{D}$ it is enough to count the number of intersection points of the dual lines of S that are within $\mathcal{D}^*$. This can be done in $O(n \log n)$ time using Mount and Netanyahu's [7] algorithm for counting the number of line intersections inside a bounded region (two vertical lines corresponding to the smallest and largest slopes of the lines in L can be added). Since the only step of the algorithm in [7] that depends on the radius of $\mathcal{D}$ is the sorting step, one can use parametric search to compute the smallest radius disk centered at q and intersecting k lines of L, for any integer $1 \leq k \leq \binom{n}{2}$. The parametric search part is essentially the same as that for selecting vertices in arrangements (see [4], page 687), and takes $O(n \log^2 n)$ time. Then, the k closest lines from q can be reported in $O(n \log^2 n + k)$ time and $O(n)$ space.

Results. As with the distance selection problem, it would be interesting to see if computing an extremal line (or line segment) from q can be done faster than computing the k-th closest line (or line segment). We present $O(n \log n)$ time, $O(n)$ space algorithms for these problems. Our main result is summarized in the theorem below.

Theorem 1 *Given a set S of n points in the plane and another point q, the closest (farthest) line segment from q among those defined by S can be found in $O(n \log n)$ time and $O(n)$ space.*

Our solutions are based on simple geometric primitives and are easy to implement. A simple application of our techniques for computing the closest and farthest lines from q leads to an $O(n \log n)$ time, $O(n)$ space solution for the following problem.

Minimum (maximum) area anchored triangle. Given a set S of n points in the plane and an anchor point q, compute the minimum (maximum) area triangle defined by q with $S \setminus \{q\}$.

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