Theoretische Physik II
Elektrodynamik

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Kapitel 1

Mathe

1.1 Grundbegriffe

Ladung

$$\text{diskrete Ladungsverteilung} \qquad Q = \sum_{i=1}^{n} q_i$$

$$\text{kontinuierliche Ladungsverteilung} \qquad Q = \int_V \rho(\boldsymbol{r})\, \mathrm{d}^3 r$$

$$\text{Punktladung} \qquad \rho(\boldsymbol{r}) = q\delta(\boldsymbol{r} - \boldsymbol{r}_0)$$

$$n \text{ Punktladungen} \qquad \rho(\boldsymbol{r}') = \sum_{j=1}^{n} q_j\, \delta(\boldsymbol{r}' - \boldsymbol{r}_j)$$

Coulomb'sches Gesetz

$$\text{zwei Punktladungen} \qquad \boldsymbol{F}_{12} = \frac{1}{4\pi\varepsilon_0}\, q_1 q_2\, \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} = -\boldsymbol{F}_{21}$$

$$n \text{ Punktladungen} \qquad \boldsymbol{F}_1 = \frac{1}{4\pi\varepsilon_0}\, q_1 \sum_{j=2}^{n} q_j\, \frac{\boldsymbol{r} - \boldsymbol{r}_j}{|\boldsymbol{r} - \boldsymbol{r}_j|^3}$$

$$\text{Beziehung zur } E\text{-Feld} \qquad \boldsymbol{F}(\boldsymbol{r}) = q\, \boldsymbol{E}(\boldsymbol{r})$$

$$\text{Beziehung zur Potential} \qquad \boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla}\, q\, \varphi(\boldsymbol{r})$$

Elektrisches Feld

$$\text{im bel. Raumpunkt} \qquad \boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0}\, \frac{\boldsymbol{r} - \boldsymbol{r}_0}{|\boldsymbol{r} - \boldsymbol{r}_0|^3}$$

$$\text{diskrete Ladungsverteilung} \qquad \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{j=2}^{n} q_j\, \frac{\boldsymbol{r} - \boldsymbol{r}_j}{|\boldsymbol{r} - \boldsymbol{r}_j|^3}$$

$$\text{kontinuierliche Ladungsverteilung} \qquad \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\boldsymbol{r}')\, \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\, \mathrm{d}^3 r'$$

$$\text{(Identität)} \qquad \Big\downarrow \quad \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = -\nabla_r \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$\text{Beziehung zur Potential} \qquad \boldsymbol{E}(\boldsymbol{r}) = -\nabla\, \varphi(\boldsymbol{r})$$

$$\Longrightarrow \qquad \boldsymbol{\nabla} \times q\boldsymbol{E} = 0 \qquad \text{d.h., die Coulomb-Kraft ist konservativ}$$

Skalare Elektrische Potential

im bel. Raumpunkt kontinuierlich $\quad \varphi(\boldsymbol{r}) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int \dfrac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}$

im bel. Raumpunkt diskret $\quad \varphi(\boldsymbol{r}) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\sum_{j=1}^{n} \dfrac{q_j}{|\boldsymbol{r}-\boldsymbol{r}_j|}$

Spannung / Potentialdifferenz $\quad U(\boldsymbol{r},\, \boldsymbol{r}_0) = \varphi(\boldsymbol{r}) - \varphi(\boldsymbol{r}_0) = - \displaystyle\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{E}(\boldsymbol{r}')\, d\boldsymbol{r}'$

Operator Nomenklatur

$$\begin{aligned} \operatorname{div} \boldsymbol{A} &= \boldsymbol{\nabla} \cdot \boldsymbol{A} \\ \operatorname{grad} \boldsymbol{A} &= \boldsymbol{\nabla} \boldsymbol{A} \\ (\textit{eng.}\ \text{curl}) \quad \operatorname{rot} \boldsymbol{A} &= \boldsymbol{\nabla} \times \boldsymbol{A} \end{aligned}$$

1.2 Einführung

Dirac'sche Delta-Funktion

Definition

$$\int_V \delta(\boldsymbol{r}-\boldsymbol{r}_0)\, \mathrm{d}^3 r := \begin{cases} 1\,, & r_0 \in V \\ 0\,, & \text{sonst} \end{cases}$$

$$\delta(\boldsymbol{r}-\boldsymbol{r}_0) = 0 \qquad \forall\, \boldsymbol{r} \neq \boldsymbol{r_0}$$

Bemerkung: Die δ-Funktion ist keine Funktion im üblichen mathematischen Sinne. Man bezeichnet sie deshalb als **uneigentliche Funktion** oder als **Distribution**. Heuristisch:

$$\delta(x) = \begin{cases} +\infty\,, & x = 0 \\ 0\,, & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Formeln

$$\int_a^b f(x)\, \delta(x-x_0)\, dx = \begin{cases} f(x_0)\,, & a < x_0 < b \\ \frac{1}{2} f(x_0)\,, & x_0 = a \vee b \\ 0 & \text{sonst} \end{cases}$$

$$\text{“ } f(x)\, \delta'(x-x_0) = -f'(x)\, \delta(x-x_0) \text{ ”} \qquad \text{(heuristisch)}$$

$$\delta(x-x_0) = \frac{d}{dx} \Theta(x-x_0) \qquad (\Theta \ \text{ sei die Stufenfunktion})$$

Mehrdimensionale Delta-Funktion

Kartesisch (x,y,z) $\quad \delta(\boldsymbol{r}-\boldsymbol{r}_0) = \delta(x-x_0)\, \delta(y-y_0)\, \delta(z-z_0)$

Kugel (r,θ,φ) $\quad \delta(\boldsymbol{r}-\boldsymbol{r}_0) = \dfrac{1}{r_0^2 \sin\theta_0}\, \delta(r-r_0)\, \delta(\theta-\theta_0)\, \delta(\varphi-\varphi_0)$

Zylinder (ρ,ϕ,z) $\quad \delta(\boldsymbol{r}-\boldsymbol{r}_0) = \dfrac{1}{\rho_0}\, \delta(\rho-\rho_0)\, \delta(\varphi-\varphi_0)\, \delta(z-z_0)$

Produktformeln

f, g sind skalare Felder, $\boldsymbol{F}, \boldsymbol{G}$ sind vektor Felder:

$$
\begin{aligned}
\boldsymbol{\nabla}(fg) &= f\boldsymbol{\nabla}(g) + g\boldsymbol{\nabla}(f) \\
\boldsymbol{\nabla} \cdot (f\boldsymbol{G}) &= f\boldsymbol{\nabla} \cdot (\boldsymbol{G}) + \boldsymbol{G} \cdot \boldsymbol{\nabla}(f) \\
\boldsymbol{\nabla}(\boldsymbol{F} \times \boldsymbol{G}) &= \boldsymbol{G} \cdot \boldsymbol{\nabla} \times (\boldsymbol{F}) - \boldsymbol{F} \cdot \boldsymbol{\nabla} \times (\boldsymbol{G}) \\
\boldsymbol{\nabla} \times (f\boldsymbol{G}) &= f\boldsymbol{\nabla} \times (\boldsymbol{G}) - \boldsymbol{G} \times \boldsymbol{\nabla}
\end{aligned}
$$

Identitäten

$$
\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\boldsymbol{F}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \Delta\boldsymbol{F} \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) &= \boldsymbol{0} \\
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) &= 0 \\
\boldsymbol{\nabla} \times (\boldsymbol{a} \times \boldsymbol{\nabla} f) &= \boldsymbol{a}\Delta f - \boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{\nabla} f)
\end{aligned}
$$

Kugelkoordinaten

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \sin\phi \cos\theta \\ \rho \sin\phi \sin\theta \\ \rho \cos\phi \end{pmatrix}, \qquad \int\limits_{-\infty}^{+\infty} f(\boldsymbol{r}) \ \mathrm{d}r = \int\limits_{0}^{2\pi} \int\limits_{0}^{\pi} \int\limits_{0}^{\infty} f(r, \phi, \theta) r^2 \sin\phi \ \mathrm{d}r \mathrm{d}\phi \mathrm{d}\theta
$$

Gradienten eines skalaren Feldes

$$
\boldsymbol{\nabla} g(\boldsymbol{r}) = \sum_i \boldsymbol{e}_i \frac{1}{\left|\frac{\partial \boldsymbol{r}}{\partial u_i}\right|} \frac{\partial}{\partial u_i} g(\boldsymbol{r})
$$

1.3 Integralsätze

Gaußscher Satz

$$
\int_V \boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}) \ \mathrm{d}^3 r = \oint_F \boldsymbol{E} \ \mathrm{d}\boldsymbol{f}
$$

Stokes Satz

$$
\int_F \boldsymbol{\nabla} \times \boldsymbol{a} \ \mathrm{d}\boldsymbol{f} = \int_{\partial F} \boldsymbol{a} \ \mathrm{d}\boldsymbol{r}
$$

Greensche Identitäten

$$
\int_V \boldsymbol{\nabla} g(\boldsymbol{r}) \cdot \boldsymbol{\nabla} h(\boldsymbol{r}) + g(\boldsymbol{r})\Delta h(\boldsymbol{r}) \ \mathrm{d}^3 r = \oint_{\partial V} g(\boldsymbol{r})\boldsymbol{\nabla} h(\boldsymbol{r}) \ \mathrm{d}\boldsymbol{f}
$$

$$
\int_V g(\boldsymbol{r})\Delta h(\boldsymbol{r}) g(\boldsymbol{r}) - h(\boldsymbol{r})\Delta g(\boldsymbol{r}) \ \mathrm{d}^3 r = \oint_{\partial V} g(\boldsymbol{r})\boldsymbol{\nabla} h(\boldsymbol{r}) - h(\boldsymbol{r})\boldsymbol{\nabla} g(\boldsymbol{r}) \ \mathrm{d}\boldsymbol{f}
$$

1.4 Incomplete/Unassigned:

$$\boldsymbol{\nabla}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\boldsymbol{\nabla}'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$\int_a^b u(x)v'(x)\mathrm{d}x = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\mathrm{d}x$$

Dirichlet Randwertproblem $$\Delta_r\mathcal{G}(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{\varepsilon_0}\delta(\boldsymbol{r}-\boldsymbol{r}')$$

$$\mathcal{G}(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{q}\Phi(\boldsymbol{r}$$

Poisson-Gleichung $$\Delta\Phi(\boldsymbol{r}) = -\frac{1}{\varepsilon_0}\rho(\boldsymbol{r})$$

$$H(x) \coloneqq \int_{-\infty}^{x}\delta(s)\mathrm{d}s = \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } x\geq 0 \end{cases}$$