Lecture3-1 Geometric Shapes (Point & Curve)

1. Introduction

Geometry has no color and texture

- Points – 0 degree of freedom shape
- Curves – 1 degree of freedom shape
- Surfaces – 2 degree of freedom shape
- Solid objects – 3 degree of freedom shape

2 and 3 dimensional spaces

Time is yet another dimension however different

At the display level, drawn as

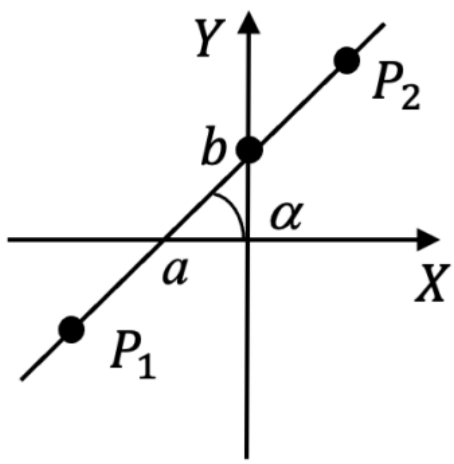
- pixels (picture elements)
- connected segments (polylines)
- shaded polygons (polygon meshes)

Learning Objectives

- To understand how **points** and **curves** can be used in solving **data visualization problems**
- To understand curves as objects with **1 degree of freedom**
- To understand what **mathematical representation** is the most efficient for defining and displaying **curves**
- To understand how **different coordinate systems** can be used together for deriving mathematical representations of curves

Problems met

Axis dependency



- $y = kx + b$
- But how to represent vertical line: $x = c$?

Multi-valued functions

$$x^2 + y^2 = r$$
$$y = \pm\sqrt{r^2 - x^2}$$

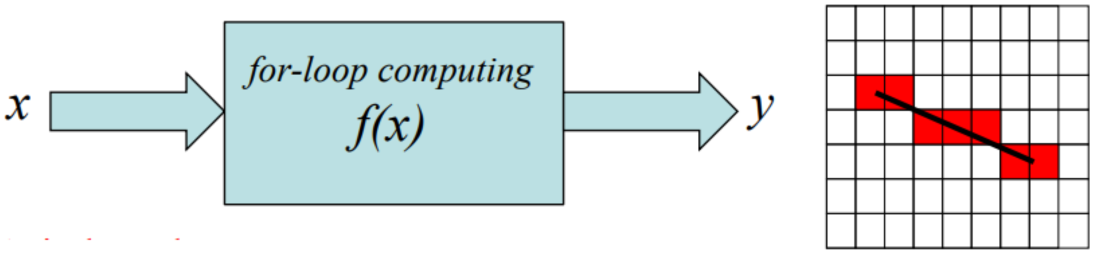
- Two values for each input while only a positive value is computed by function $sqrt()$
- $arcsin, arcos$, ...: **infinite number** of function values for each input while math libraries only compute the 'main' value

Arcs

- How to use a simple min/max domain of x-coordinates to define an arc of a circle?

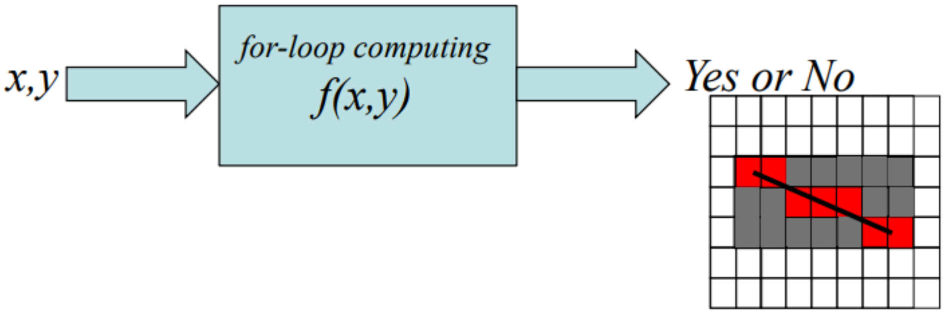
How to render image according to different types of functions

Explicit Functions



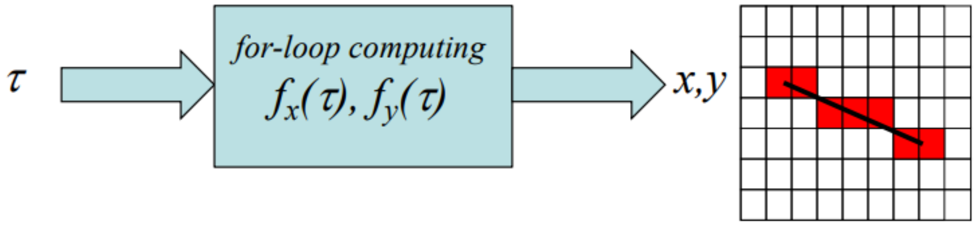
- $y = f(x), x = f(y)$
- **Domain** of the **argument** and range of the function values
- Sample the **domain** to obtain the other coordinate
- All the points with the computed coordinates will belong to the shape
- Advantage
    - Fast computation
- Disadvantage (Can not solve)
    - Axis dependency
    - Multivalued functions
    - Arcs

Implicit Functions



- $f(x,y)=0$
- Domain of the **two arguments**
- To sample the domain to **check of the sampled point belongs to the curve**
- Most of the sampled points will **not belong to the curve**
- Advantage
  - Axis independent
- Disadvantage
  - Multivalued functions
  - Arcs

Parametric Functions of Coordinates



- $x=f_x(t), y=f_y(t), z=f_z(t), t\in[t_1,t_2]$
- Domain of **one argument** (parameter)
- To sample the domain to compute the coordinates of the points that belong to the curve
- **All the computed points** will belong to the curve
- Advantage
  - Fast computation
  - Can represent arc (Converted from Polar Coordinate System)
  - May solve multivalued functions
    - Use other way of obtaining the parametric functions

Normalize the domain

In parametric function representation, typically in this course, we often constraint the domain of the parameter in $[0,1]$

Based on the parameter representation of the line, we can obtain, if the original domain of the parameter p is $[a,b]$, then

$$p = a + (b - a)u, u \in [0, 1]$$

2. Points

- Individual points
- Reference points
    - Defined by Cartesian coordinates (x, y, z), polar (r, α), spherical (r, α, β) or cylindrical (h, r, α) coordinates
- Point rendering
- Splats rendering
    - Use three points to rendering a splats(by interpolation)
- 2D pixels (picture elements) and 3D voxels (volume elements)

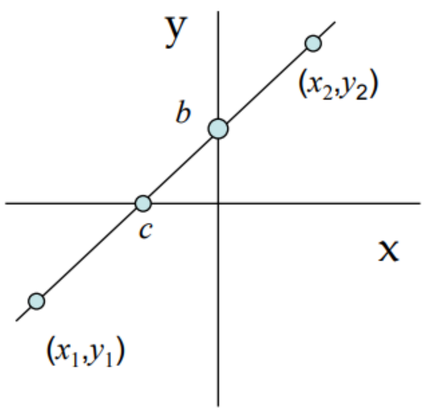
3. 2D Curves

Summary on 2D curves

- 2D and 3D
- Polylines(折线) – **interpolation** by connected **straight line segments**
- Implicit (only 2D)
    - $f(x, y) = 0$
- Explicit (only 2D)
    - $y = f(x)$ or $x = f(y)$
- Parametric (2D and 3D)
    - $x = x(t), y = y(t), t = [t1, t2]$
    - $x = x(t), y = y(t), z = z(t), t = [t1, t2]$

Straight Line (Segment, Ray)

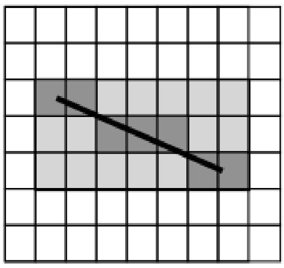
Implicit Representation



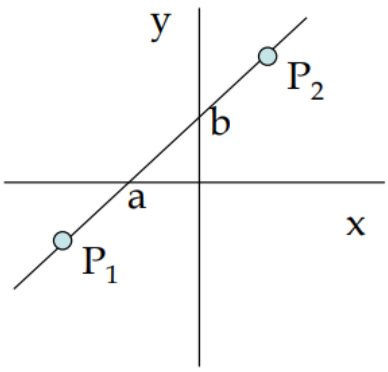
$$Ax + By + C = 0$$
$$\frac{y - y_1}{x - x_1} - \frac{y - y_2}{x - x_2} = 0$$

Straight Line	Segment	Ray
$x, y \in (-\infty, \infty)$	$x \in [x_1, x_2], y \in [y_1, y_2]$	$x \in [x_1, \infty], y \in [y_1, \infty]$

- Drawing is done by sampling points (pixels) within the x and y domains. It is **slow** since most of the points within the domain do not belong to the segment



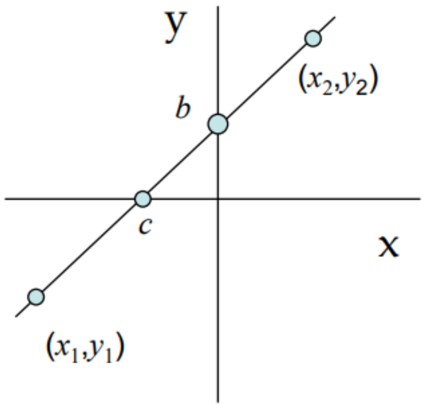
- There is another methods, if we know the points a and b that the straight line is **intercept** with x and t axis, then we can write this implicit representation



$$\frac{x}{a} + \frac{y}{b} = 1 \to \frac{x}{a} + \frac{y}{b} - 1 = 0$$

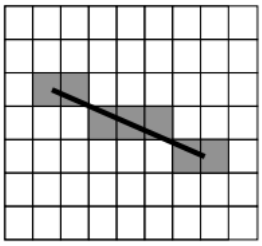
- Equation in intercepts

Explicit Representation



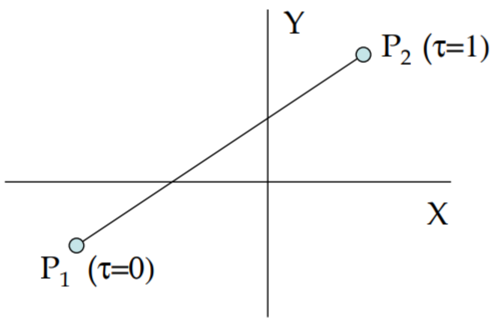
$$y = ax + b \text{ or } x = dy + c$$
$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Straight Line	Segment	Ray
$x, y \in (-\infty, \infty)$	$x \in [x_1, x_2], y \in [y_1, y_2]$	$x \in [x_1, \infty], y \in [y_1, \infty]$



- Drawing is done by incrementing x or y and obtaining y and x, respectively. **Fast**. Integer version used for drawing segments in all computers.
- **Axes dependency**: special cases for drawing vertical and horizontal lines $x = c, y = b$
- Check whether it is a straight line!

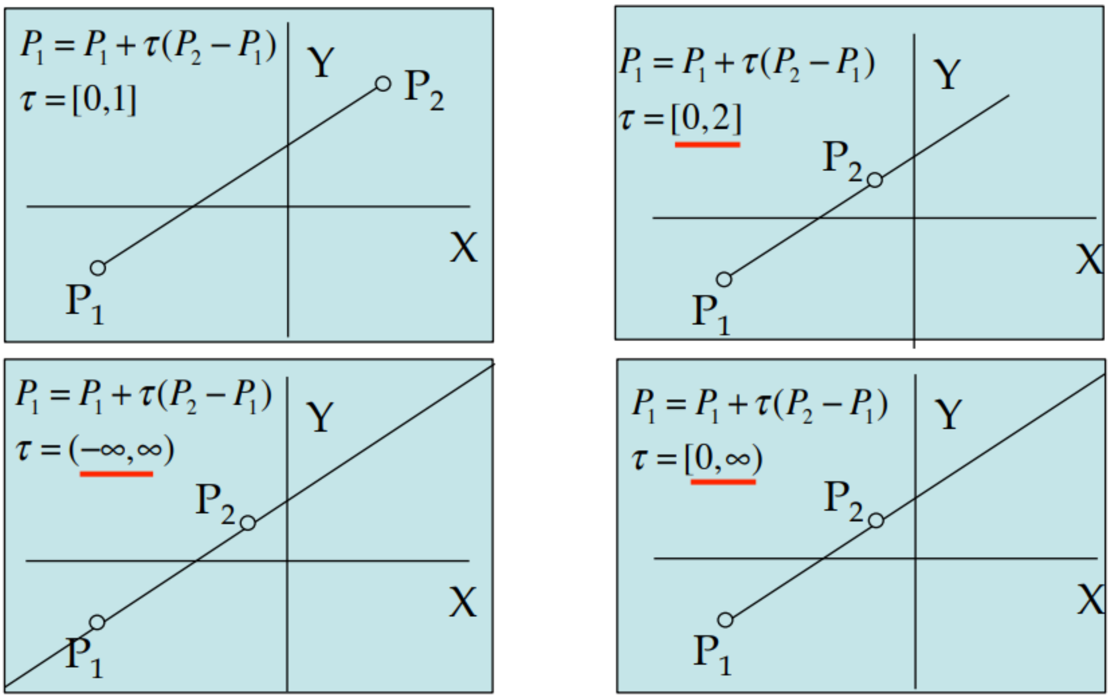
Parametric Representation



$$P = P_1 + \tau\,(P_2 - P_1)$$
$$x = x_1 + \tau\,(x_2 - x_1) = x_1(1-\tau) + \tau x_2$$
$$y = y_1 + \tau\,(y_2 - y_1) = y_1(1-\tau) + \tau y_2$$
$$\tau = [0, 1]$$

Straight Line	Segment	Ray
$\tau = (-\infty, \infty)$	$\tau \in [0, 1]$	$\tau = [0, \infty)$ $\tau = (-\infty, 1]$

- Drawing is done by incrementing parameter τ and obtaining x and y
- **Axes independent. Fast.**
- You can freely change the range of τ



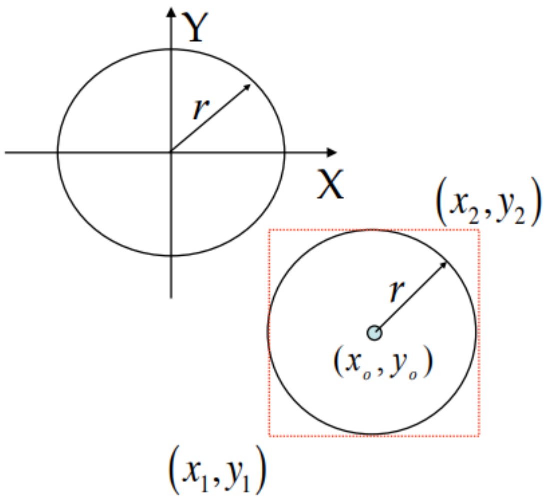
Summary

2D straight lines, segments and rays can be defined mathematically by

- Implicit functions $f(x, y) = 0$
  - Slow for rendering
- Explicit functions $y = f(x)$ or $x = f(y)$
  - Fast but axes dependent
- Parametric functions One parameter only $x = x(t), y = y(t), t = [t_1, t_2]$
  - Fast and axes independent

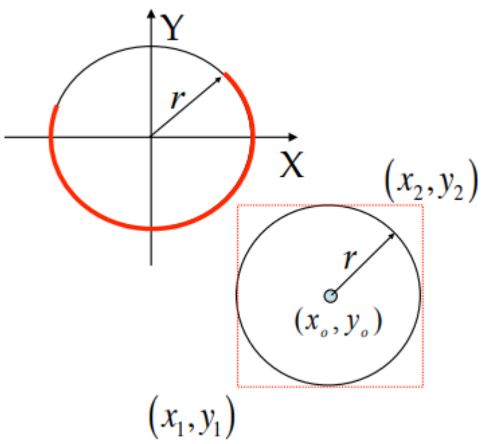
Circle (Arc)

Implicit Representation



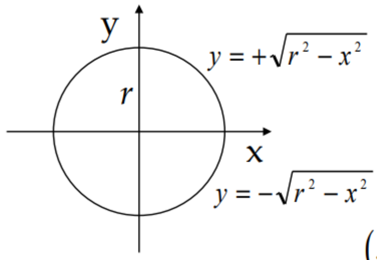
$$r^2 - x^2 - y^2 = 0$$
$$r^2 - (x - x_0)^2 - (y - y_0)^2 = 0$$
$$x \in [x_1, x_2], y \in [y_1, y_2]$$

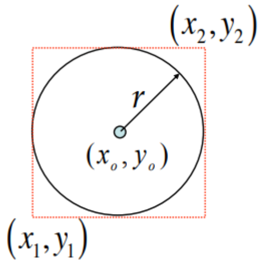
- We have to find the domain of x and y
- Drawing is done by sampling points within the x and y domains. **Slow**.
- If we want to define an arc, it is **impossible** to do it using only $x \in [x_1, x_2], y \in [y_1, y_2]$



- Requires **angular values** as in **polar coordinates**

Explicit Representation

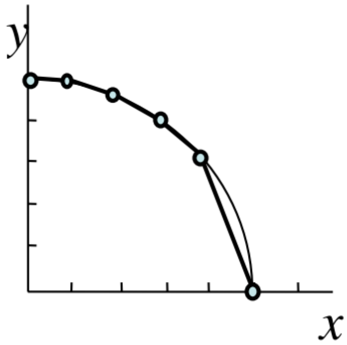




$$y = \pm\sqrt{r^2 - x^2}$$
$$y = \pm\sqrt{r^2 - (x - x_0)^2} + y_0$$

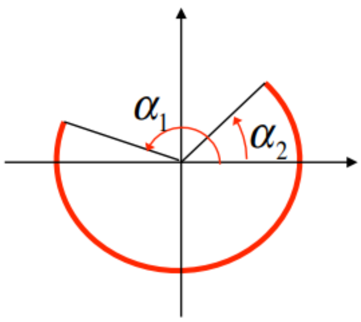
It is not exactly a mathematical function

- Axes dependency: 2 formulas for the upper and lower semicircles
- Drawing is done by incrementing x or y and obtaining y and x, respectively.
- It is **fast** but with **irregular segment length interpolation**.



- Impossible to define **arc** domain with only $x \in [x_1, x_2], y \in [y_1, y_2]$
- Requires **angular values** as in polar coordinates

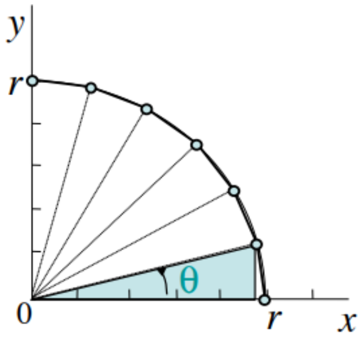
Explicit Representation in Polar Coordinates



- In polar coordinates: $r = r(\alpha)$
- Original-centered circle: $r = \text{constant radius}, \alpha \in [0, 2\pi]$
- Arc is defined by the domain of $\alpha \in [\alpha_1, \alpha_2]$
- Fast drawing is done by incrementing angle α and obtaining radius r
- **Other (not origin centered)** circle-arc locations are problematic to define in polar coordinates

Parametric Representation

Conversion from polar coordinates $r(\alpha)$ to Cartesian with a constant r



$$x = rcos(\theta)$$
$$y = rsin(\theta)$$
$$\theta = \alpha$$

$$\begin{aligned} x &= r\cos(\theta) + x_0 \quad 0 \le \theta \le 2\pi \quad \text{for a circle} \\ y &= r\sin(\theta) + y_0 \quad \theta_1 \le \theta \le \theta_2 \quad \text{for an arc} \end{aligned}$$

Drawing is done by incrementing parameter θ and obtaining x and y.

It is axes **independent**, **fast** and with **a uniform length** of the segments interpolating the circle

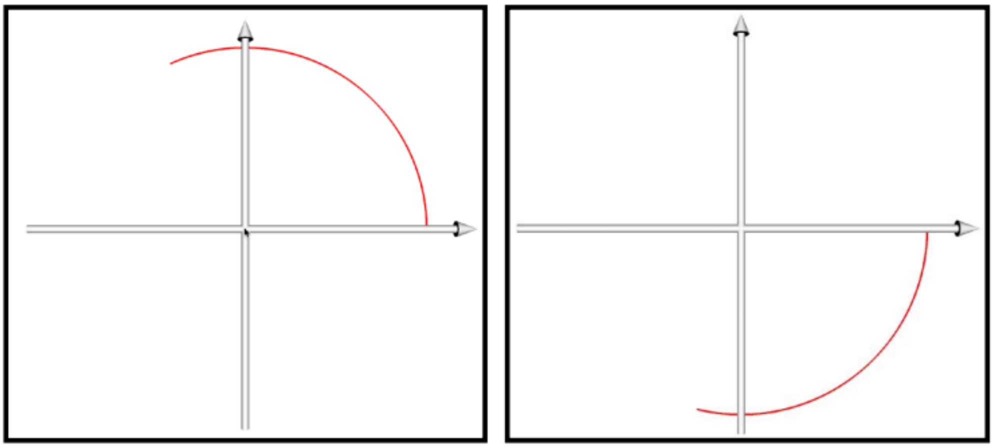
By transform the domain of the parameter, you can therefore obtain

$$x = rcos(2\pi u)$$
$$y = rsin(2\pi u)$$
$$\alpha = 2\pi u$$

$$\begin{aligned} x &= r\cos(u) + x_0 \quad 0 \le u \le 1 \quad \text{for a circle} \\ y &= r\sin(u) + y_0 \quad u_1 \le u \le u_2 (u_1, u_2 \in [0,1]) \quad \text{for an arc} \end{aligned}$$

Sample Rate(Resolution), Period and Offset

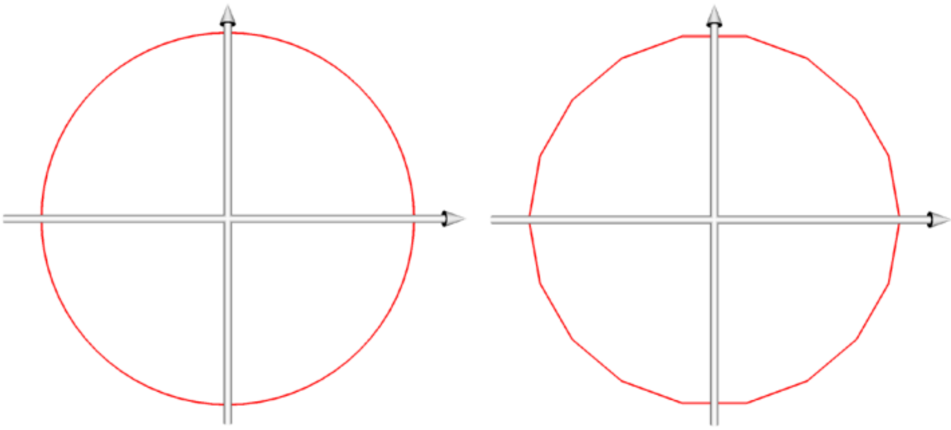
The rendering sequences are different



$x = r\cos(\theta) \quad y = r\sin(\theta)$
$\theta \in [0, 2\pi]$

$x = r\cos(\theta) \quad y = r\sin(\theta)$
$\theta \in [0, -2\pi]$

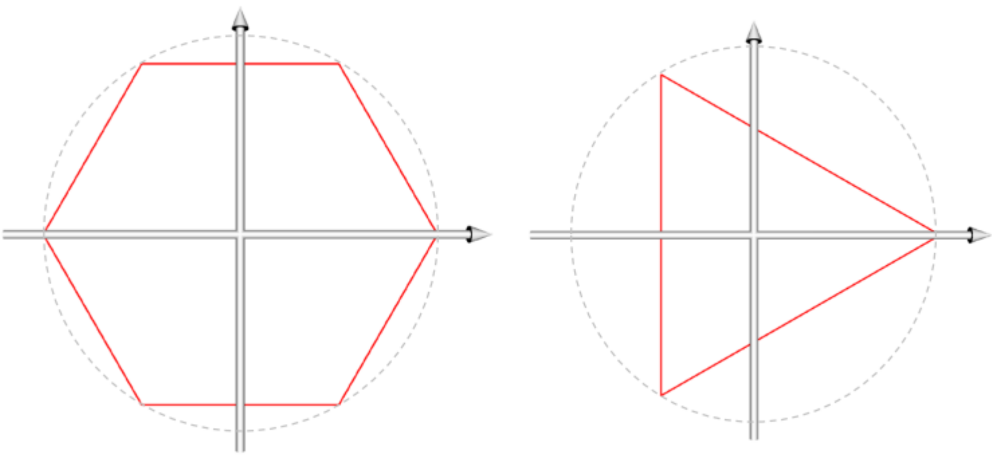
The sample rates are different



72 sampling points (segments)

18 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta) \quad \theta \in [0, 2\pi]$$

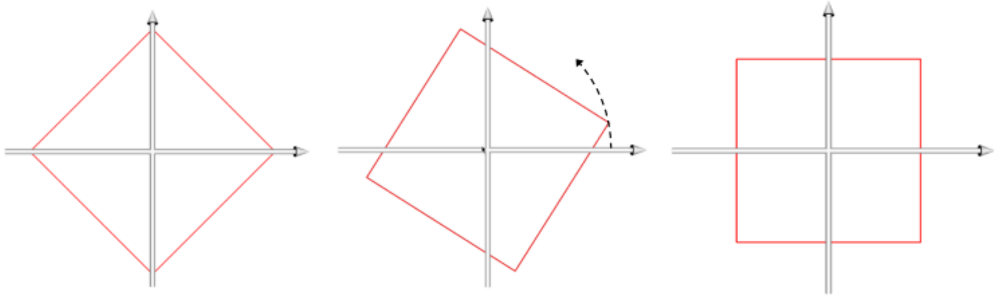


6 sampling points

3 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta) \quad \theta \in [0, 2\pi]$$

Add the offset will rotate the graphic

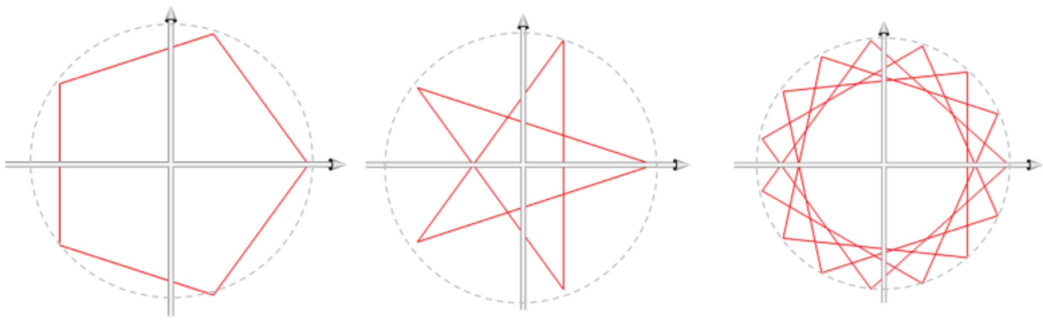


4 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$
$$\theta \in [0, 2\pi]$$

4 sampling points and offset

$$x = r\cos(\theta + \frac{\pi}{4}) \quad y = r\sin(\theta + \frac{\pi}{4})$$
$$\theta \in [0, 2\pi]$$



5 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$
$$\theta \in [0, 2\pi]$$

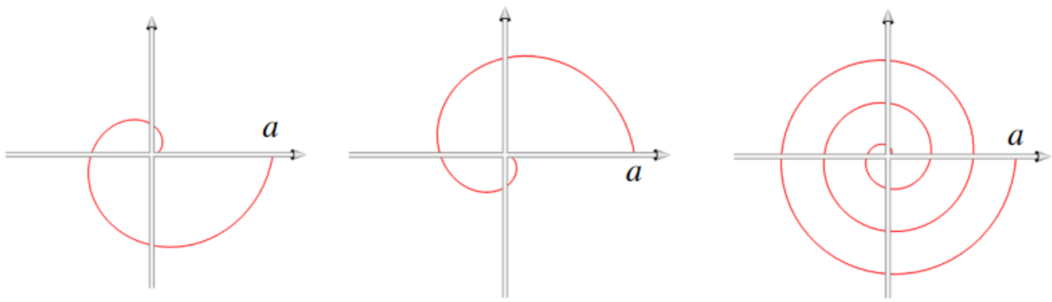
5 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$
$$\theta \in [0, 4\pi]$$

15 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$
$$\theta \in [0, 8\pi]$$

Spiral Curve

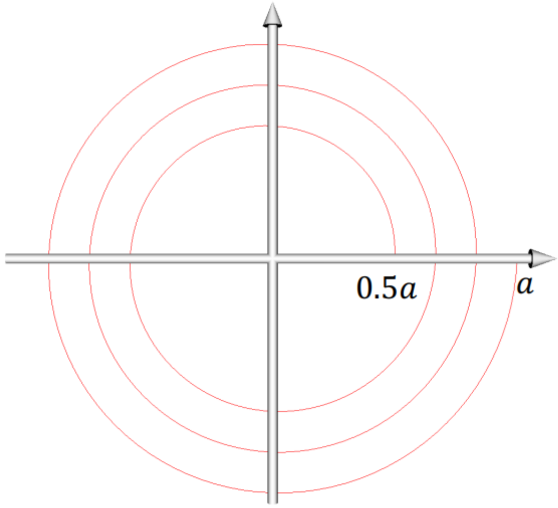


$$x = a \cdot u \cdot \cos(2\pi u)$$
$$y = a \cdot u \cdot \sin(2\pi u)$$
$$u \in [0, 1]$$

$$x = a \cdot u \cdot \cos(-2\pi u)$$
$$y = a \cdot u \cdot \sin(-2\pi u)$$
$$u \in [0, 1]$$

$$x = a \cdot u \cdot \cos(6\pi u)$$
$$y = a \cdot u \cdot \sin(6\pi u)$$
$$u \in [0, 1]$$

- Fig1: counter clock wise, one rotation
- Fig2: clock wise, one rotation
- Fig3: counter clock wise, three rotations



$$x = a \cdot \underline{u} \cdot \cos(6\pi u)$$
$$y = a \cdot \underline{u} \cdot \sin(6\pi u)$$
$$u \in [0,1]$$

$$\Downarrow$$

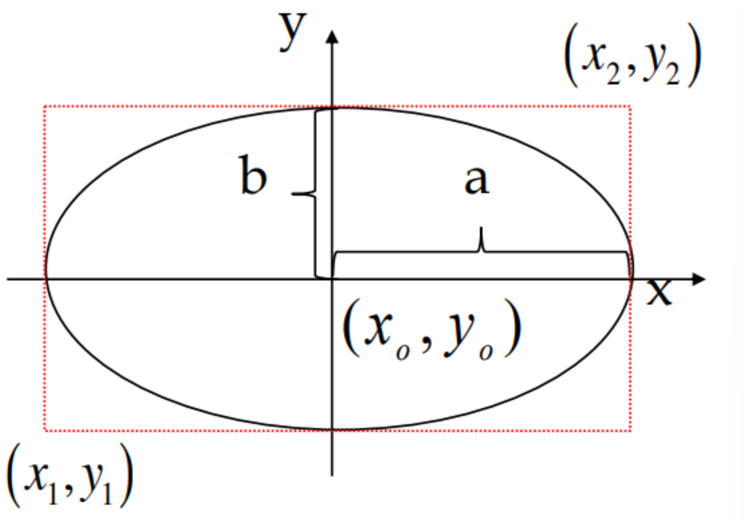
$$x = a \cdot \underline{(0.5 + (1 - 0.5) \cdot u)} \cdot \cos(6\pi u)$$
$$y = a \cdot \underline{(0.5 + (1 - 0.5) \cdot u)} \cdot sin(6\pi u)$$

$$\Downarrow$$

$$x = a \cdot (0.5 + 0.5 \cdot u) \cdot \cos(6\pi u)$$
$$y = a \cdot (0.5 + 0.5 \cdot u) \cdot si\,n(6\pi u)$$
$$u \in [0,1]$$

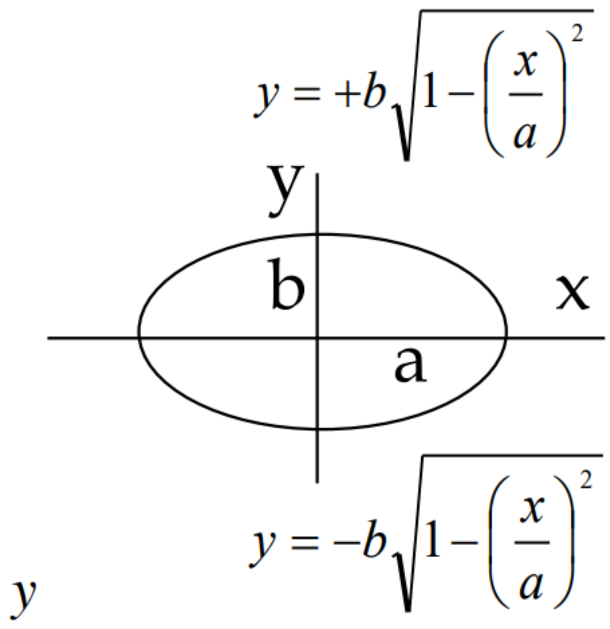
Ellipse (Arc)

Implicit Representation



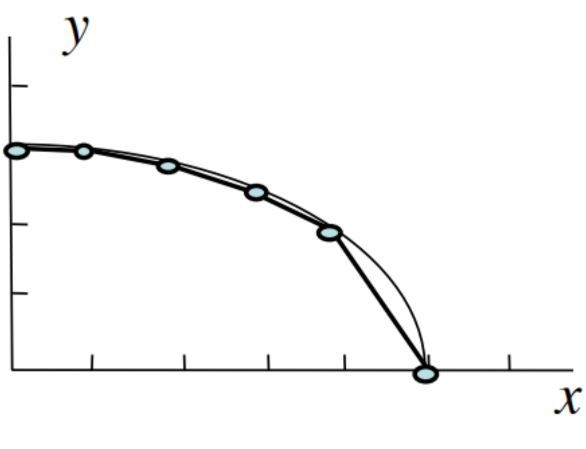
$$1 - (\frac{x}{a})^2 - (\frac{y}{b})^2 = 0$$
$$1 - (\frac{x - x_0}{a})^2 - (\frac{y - y_0}{b})^2 = 0$$
$$x \in [x_1, x_2], y \in [y_1, y_2]$$

Explicit Representation

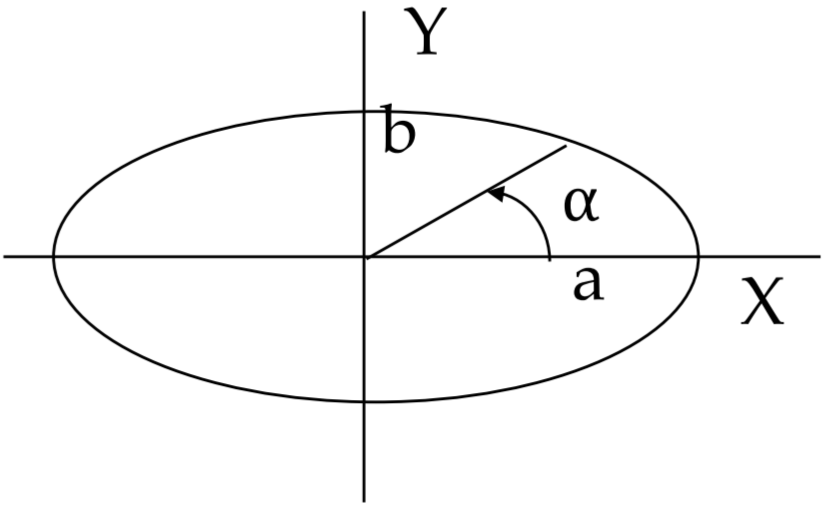


$$y = \pm b\sqrt{1-(\frac{x}{a})^2}$$
$$y = \pm b\sqrt{1-(\frac{x-x_0}{a})^2}+y_0$$
$$x \in [x_1, x_2], y \in [y_1, y_2]$$

- Drawing is done by incrementing x or y and obtaining y and x, respectively.
- It is fast but with irregular segment length interpolation



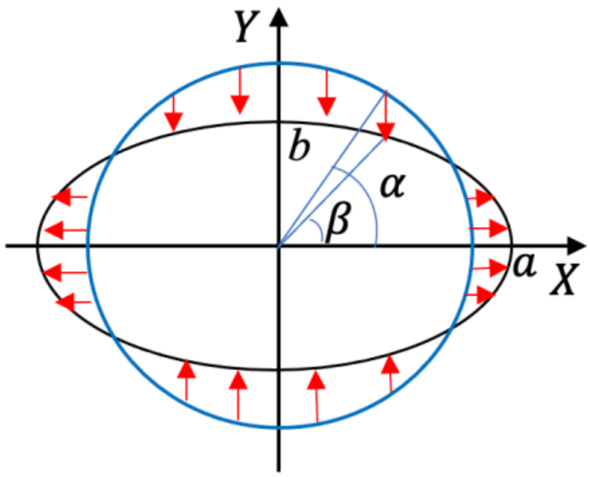
Explicit Representation in Polar Coordinates



In polar coordinate system, we have $x = rcos(\alpha), y = rsin(\alpha)$, thus

- $1 - (\frac{x}{a})^2 - (\frac{y}{b})^2 = 0$ can be converted into $1 - (\frac{rcos(\alpha)}{a})^2 - (\frac{rsin(\alpha)}{b})^2 = 0$
- $r = \frac{1}{\sqrt{(\frac{cos(\alpha)}{a})^2 + (\frac{som(\alpha)}{b})^2}}, 0 \le \alpha \le 2\pi$
- We only need positive values of the square root

Parametric Representation

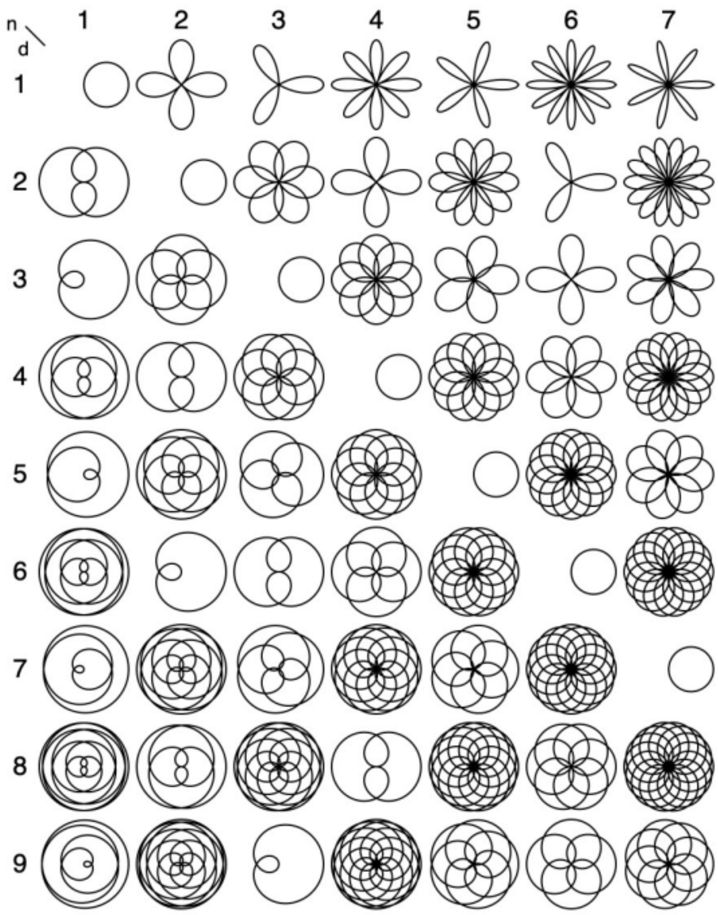


$$\begin{aligned} x &= a\cos(\theta) + x_0 \quad & 0 \le \theta \le 2\pi \quad & \text{for a ellipse} \\ y &= b\sin(\theta) + y_0 \quad & \theta_1 \le \theta \le \theta_2 \quad & \text{for an arc} \end{aligned}$$

- Parameter θ is not a polar angle α!
- Parameter $\theta = 2\pi u$ is not a polar angle α!

Find a representation of a curve

Conversion from Polar Coordinates



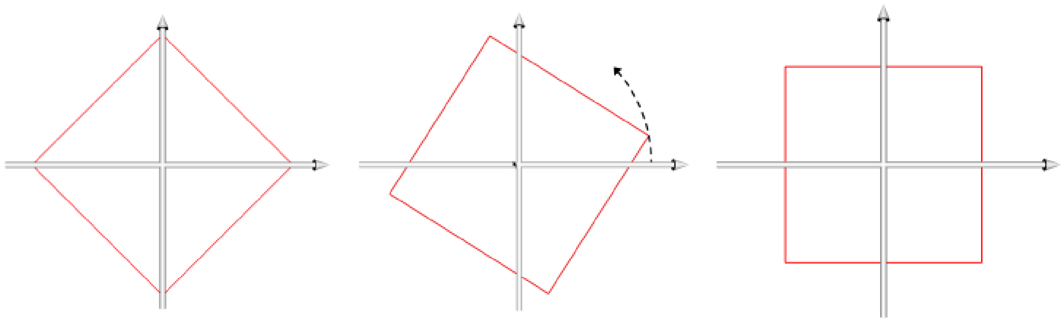
Rose curves defined by

$$r = cos(k\alpha), \alpha \in [0, 2\pi]$$

- for various values of $k = \frac{n}{d}$

$$x = rcos(\alpha) = cos(k\alpha)cos(\alpha)$$
$$y = rsin(\alpha) = cos(k\alpha)sin(\alpha)$$
$$\alpha \in [0, 2d\pi]$$

Rotation by adding an offset



4 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$
$$\theta \in [0, 2\pi]$$

4 sampling points and offset

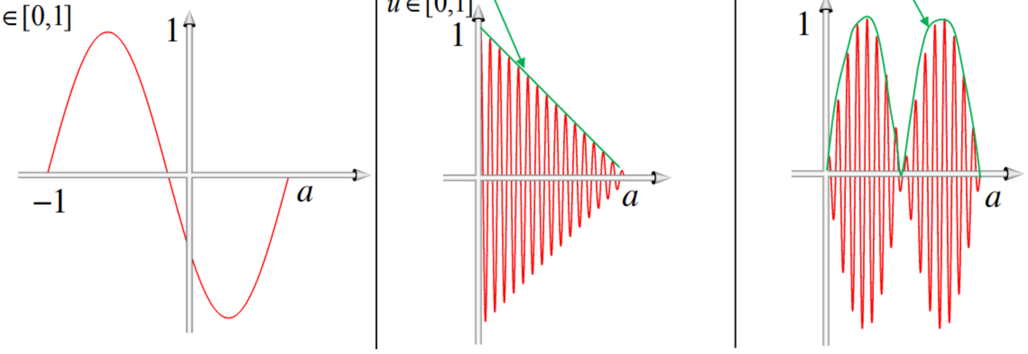
$$x = r\cos(\theta + \frac{\pi}{4}) \quad y = r\sin(\theta + \frac{\pi}{4})$$
$$\theta \in [0, 2\pi]$$

Parameterisation, Modulation

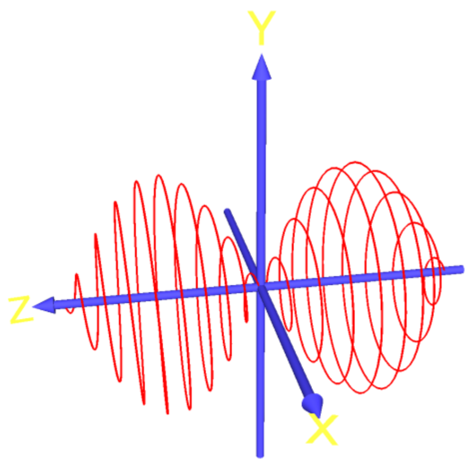
$$P = P_1 + u*(P_2 - P_1), P_1 = -1, P_2 = a$$
$$x = -1 + u\cdot(a - (-1))$$
$$y = \sin(u2\pi)$$
$$u \in [0,1]$$

$$x = u\cdot a$$
$$amplitude = 1 + u*(0-1) = 1-u$$
$$y = (1-u)\cdot\sin(16\cdot 2\pi\cdot u)$$
$$u \in [0,1]$$

$$x = u\cdot a$$
$$y = |\sin(2\pi\cdot u)|\cdot\sin(16\cdot 2\pi\cdot u)$$
$$u \in [0,1]$$



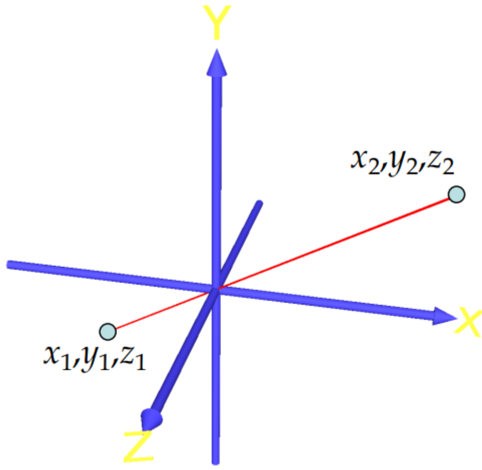
4. 3D Curves



- Can be only defined **parametrically**
- Explicit and implicit functions exists only for plane curves

$$
\begin{aligned}
x &= f_x(\tau) \\
y &= f_y(\tau) \\
z &= f_z(\tau) \\
\tau &= [\tau_1, \tau_2]
\end{aligned}
$$

Straight Line

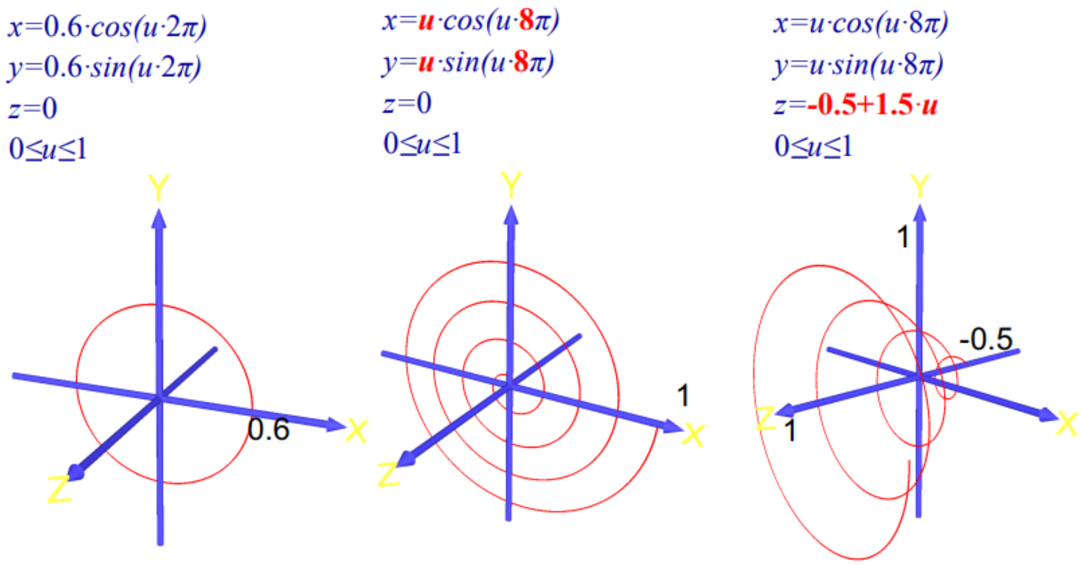


$$
\begin{aligned}
x &= x_1 + \tau\,(x_2 - x_1) \\
y &= y_1 + \tau\,(y_2 - y_1) \\
z &= z_1 + \tau\,(z_2 - z_1)
\end{aligned}
$$

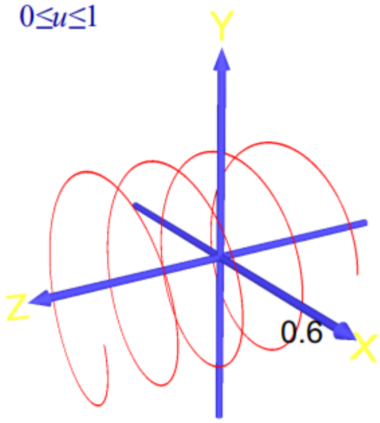
$$
\begin{array}{ll}
\tau = [0, 1] & \text{Segment} \\
\tau = [0, \infty) & \text{Ray} \\
\tau = (-\infty, \infty) & \text{Straight line}
\end{array}
$$

- No explicit and implicit representation

Parametric Curves



$x = \mathbf{0.6} \cdot cos(u \cdot 8\pi)$
$y = \mathbf{0.6} \cdot sin(u \cdot 8\pi)$
$z = -0.5 + 1.5 \cdot u$
$0 \le u \le 1$



$x = 0.6|\mathbf{sin(u*2\pi)}| \cdot cos(u \cdot 36\pi)$
$y = 0.6|\mathbf{sin(u*2\pi)}| \cdot sin(u \cdot 36\pi)$
$z = -1 + 2 \cdot u$
$0 \le u \le 1$