Lecture1 Introduction to Computer Graphics and Foundation Mathematics

1. Definitions

Computer Graphics

Computer Graphics concerns the pictorial synthesis of real or imaginary objects from their computer-based models

- **Visualization**: To form an image of something; envisage; to make a physical 3D model
- **Image Processing**: The analysis of scenes, or reconstruction of models of 2D or 3D objects from their pictures

It is about:

- Making images with a computer
- Printing 2D images and 3D models with a computer
- Visualization: from digital model of geometry to colors visible on the monitor or hardcopy
- Direct implementation of analytic geometry in computers
- Digital computers approximate since limited by precision of data representation

We have to use analytical geometry to do the following things:

- How to **define geometry** using **mathematical formulas, algorithms, procedures**
- Specify which **particular domain of coordinates** to be worked with
- How to **sample the domain** to eventually come up with coordinates of points to be displayed

Hardware - Graphics Display



- **Pixel** – picture element
- **Resolution:** number of pixels which can be displayed horizontally and vertically.

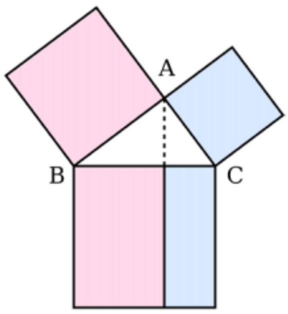
- **Number of colors:** Maximum actual number of colors depends on the graphics display device and the video card. Device independent colors are programed as real numbers between 0 and 1
- **RGB color model**: addition of **red+green+blue=white** used with graphics displays (used in this course)
- **CMYK color model**: subtraction of **cyan+magenta+yellow=black (key)** used when printing (not used in the course)
- **Polygonization:** to calculate polygons interpolating surfaces of shapes
- **Shading**: filling in surface of polygons with colored pixels

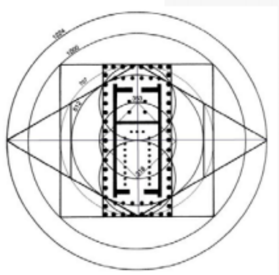
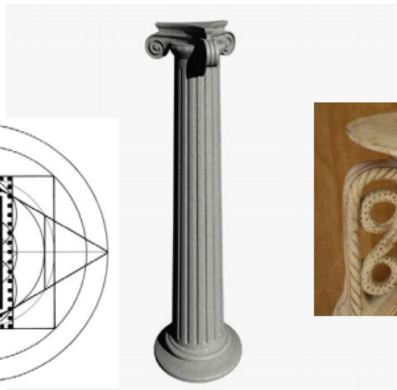
2. Visualization Steps

1. **Define Objects**: 3D shapes, defined by polygonal surfaces (triangle sets)
2. **Define a Viewpoint** (viewer position and orientation) and viewing parameters
    - tilt angle, view angle, etc.
3. **Define light source(s)**: ambient, point, directional, etc.
4. **Define visible material properties**

3. Geometry History

Greeks Geometry




- For the ancient Greek mathematicians, geometry was the crown jewel of their sciences, reaching a completeness and perfection of methodology that no other branch of their knowledge had attained
- More than a field of mathematics but rather an attempt to explain the universe

Romans Geometry

Roman Numerals

- I, II, III, IV, V, VI, VII, VIII, IX, X, L, C, D, M
- Numbers are formed by combining symbols and adding the values, e.g. CCVII is 207
- Replaced in 14th century by Hindu-Arabic numerals
- Still used by plumbers and in mechanical and aerospace engineering (e.g., 1 ½” pipe), as well as in use in monarchs and Popes names, hours, years, chapters marks, etc.

Weakness of Roman Numerals

- No zero
- No negative numbers
- Though decimal for numbers, duodecimal (dozenal) fractional notations were used
    - 1/12 – ounce, 2/12=1/6 – sixth, 3/12=1/4 – quarter, 4/12=1/3 – third, 5/12 – five ounces, 6/12=1/2 – half, … , 12/12=1 – unit

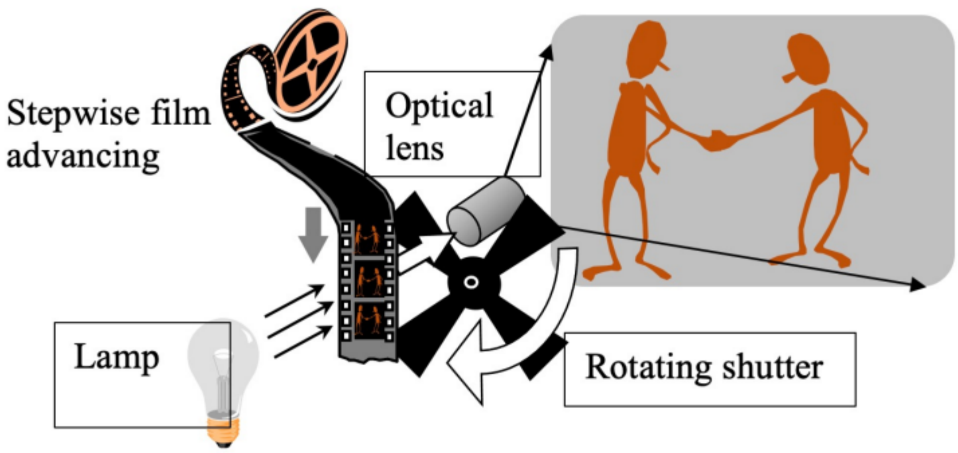
Descartes (笛卡尔)

- Rene Descartes, "La Geometrie" (1637). Started with **geometric curves** and produces their **equations**
- Revolutionized mathematics by providing the first systematic link between **Euclidean geometry** ("Elements" 300 BC) and **algebra**
- Bridge between **algebra and geometry** was built
- Cartesian coordinate system, polar coordinates, etc.
    - uniquely determine position of points on a plane

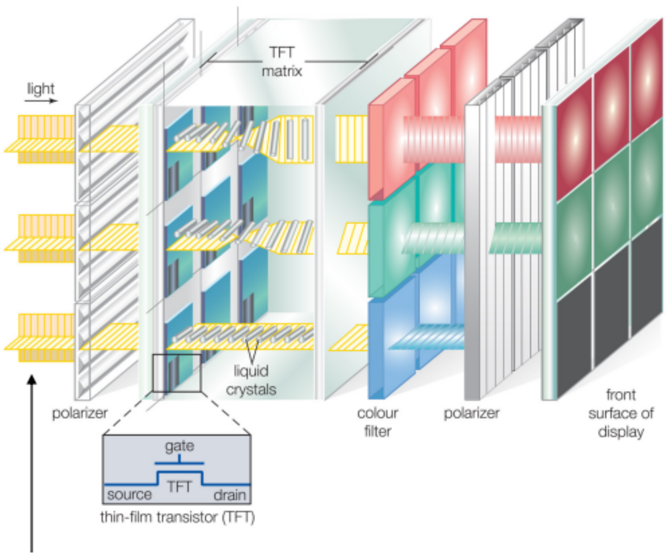
4. Foundation Mathematics

Displaying images

Film projectors



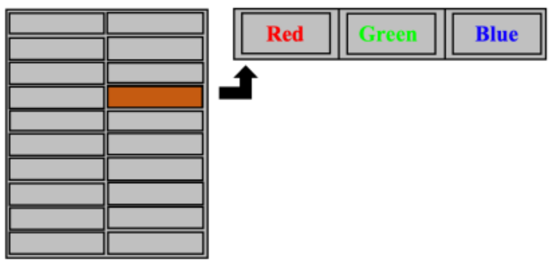
LCD monitors



- CCFL (Cold-cathode Fluorescent Lamp) back light
- LED (Light-Emitting Diode) back light, direct LED
- QLED (Quantum Light-emitting Diode) uses tiny nanoparticles called quantum dots (as in Samsung TVs)
- QLED LCDs compete with OLEDs (Organic LED). Burn-in problem in OLED matrices.

Graphics memory

Frame buffer in a true color mode

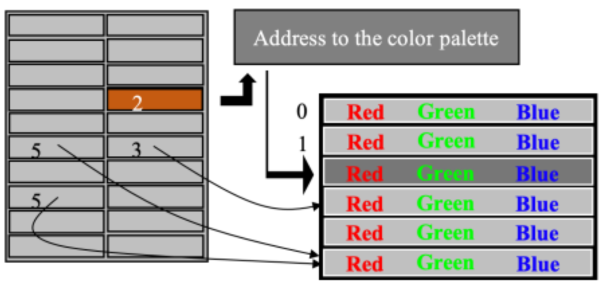


Each frame store 24bits, 8-8-8 -> r-g-b

So there are 16,777,216 colors are made of different mixes of red blue and green colors

Frame buffer with a Color Palette

Each frame record the address to the color palette



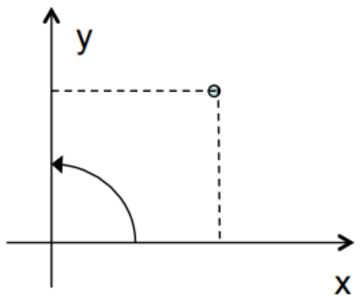
Coordinate systems

Coordinates are signed numbers used to **uniquely determine the position of a point** or other geometric elements in the modeling space

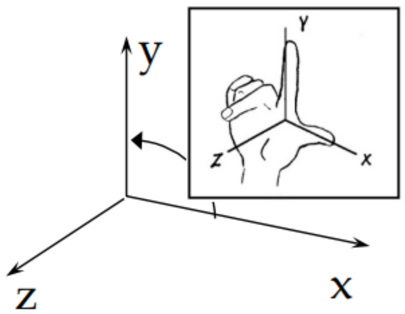
- **Number of coordinates = dimension of space**
  - plane – 2 coordinates
  - 3D space – 3 coordinates

Cartesian Coordinate System (笛卡尔坐标系)

- Cartesian Coordinate System (2D and 3D)
- Cartesius is a Latinized name of Descartes
- Coordinate **axes** are located at **90 degrees** to each other
- Cartesian coordinates are the foundation of analytic geometry linear algebra, complex analysis, differential geometry, multivariate calculus, group theory, and more
- **Right-handed** 2D coordinate system
  - If rotation of the first axis towards the second axis is **counterclockwise**

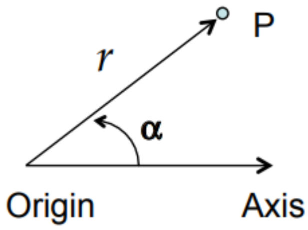


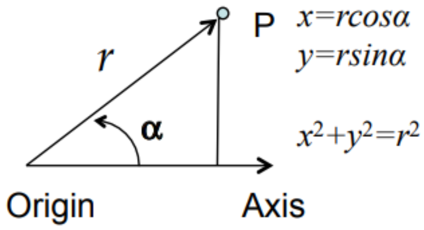
- **Right-handed** 3D coordinate system
  - While curling fingers from the first to the second axis, the extended thumb will show the direction of the third axis



- Advantage
  - Computer always uses Cartesian Coordinate System to display images. So all the rest of the coordinate systems must be converted into Cartesian Coordinate System

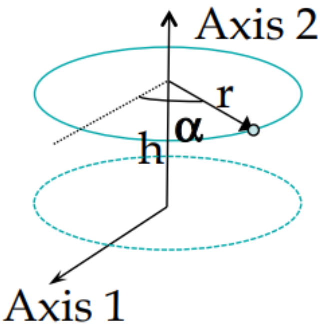
Polar Coordinate System (极坐标系)





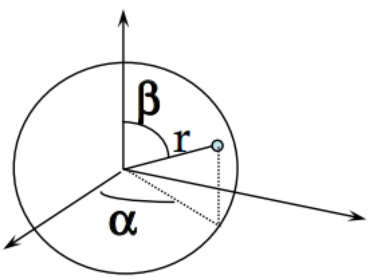
- Defined by an axis and an origin point on it
- Location of a 2D point P is defined as a **distance** r from the origin to the point and an **angle** α between the axis and the vector cast towards the point
  - Usually, distance r is **positive**, however it depends on the problem
  - Usually, angle α is from 0 to 2π, however it depends on the problem
  - Usually, positive α is measured in a **counter-clockwise** way
- Advantage
  - Design curve
  - Simple to convert polar coordinate system to cartesian coordinate system

Cylindrical Coordinate System (柱坐标系)



- Extension of polar system to 3D
- Defined by two orthogonal axes and an origin which is their intersection
- Location of a 3D point P is defined as a displacement h along the second axis towards the plane orthogonal to the two axes and containing the point P, a distance r from the second axis to the point, an angle α between the first axis and the vector cast towards the point
  - Distance h can be **positive and negative**
  - Distance r is **positive**
  - Angle α is from 0 to 2π
  - Positive α is measured **counter-clockwise** as seen towards the origin

Spherical Coordinate System (球面坐标系)



- Extension of polar system to 3D
- Defined by three orthogonal axes and an origin which is their intersection
- Location of a 3D point P is defined by a **distance** r from the origin to the point, an **two angles** α (azimuth 偏振角) and β (zenith 顶角) between the first axis and the vector cast towards the point
    - Distance r is positive
    - Angle α is from 0 to 2π
    - Angle β is from 0 to 1π
- Application
    - GPS define a location on earth
    - Haptic(触觉) device
- Advantage
    - Define everything that may be looked like a sphere

Analytic Functions

Mathematical Functions

- In mathematics, a function associates one quantity, the **argument** of the function, also known as the **input** in computer science, with another quantity, the **value** of the function, also known as the **output** in computer science
- **A function assigns exactly one output to each input**
- Values from the input domain map to the values in the function range
- In computer graphics, we try to avoid multiple outputs, in order to build a pipeline where we assign a certain output given by a specific input

Explicit way of function definition

- $y = f(x)$
- $z = f(x, y)$
- $g = f(x, y, z)$

Conversion from explicit $y = f(x)$ to implicit $f(x, y) = 0$

$$y - f(x) = 0$$

Conversion from explicit $y = f(x)$ to parametric $x = f_x(u)$ $y = f_y(u)$

$$\begin{aligned} x &= u \\ y &= f(u) \end{aligned}$$

Implicit way of function definition

- $f(x) = 0$
- $f(x, y) = 0$
- $f(x, y, z) = 0$

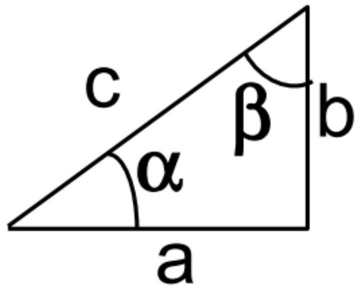
Parametric way of function definition

- $x = f_x(\mathbf{t}), y = f_y(\mathbf{t}), z = f_z(\mathbf{t}), \mathbf{t} = [\mathbf{t_1}, \mathbf{t_2}]$
- $x = f_x(\mathbf{u}, \mathbf{v}), y = f_y(\mathbf{u}, \mathbf{v}), z = f_z(\mathbf{u}, \mathbf{v}), \mathbf{u} = [\mathbf{u_1}, \mathbf{u_2}], \mathbf{v} = [\mathbf{v_1}, \mathbf{v_2}]$
- $x = f_x(\mathbf{u}, \mathbf{v}, \mathbf{w}), y = f_y(\mathbf{u}, \mathbf{v}, \mathbf{w}), z = f_z(\mathbf{u}, \mathbf{v}, \mathbf{w}), \mathbf{u} = [\mathbf{u_1}, \mathbf{u_2}], \mathbf{v} = [\mathbf{v_1}, \mathbf{v_2}], \mathbf{w} = [\mathbf{w_1}, \mathbf{w_2}]$
- It is an easy way to define a transform of one coordinate system/parameter space to another

Conversion from parametric $x = f_x(u)$ $y = f_y(u)$ to explicit $y = f(x)$ or implicit $f(x, y) = 0$

- by expressing parameter u as **a function of** x from the first equation and then by substituting it into the second equation
- by **eliminating parameter** u while doing algebraic manipulations with the two equations (raising to power, multiplications, additions, subtractions, divisions, etc.)

Pythagorean Theorem (勾股定理)

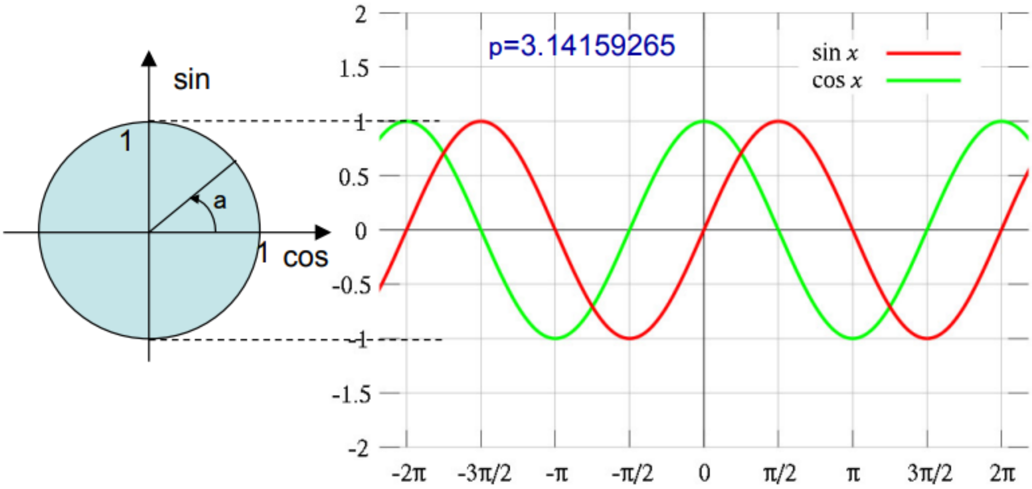


$$c^2 = a^2 + b^2$$
$$a = c \cdot cos\alpha \quad b = c \cdot sin\alpha$$
$$b = c \cdot cos\beta \quad a = c \cdot sin\beta$$
$$b = a \cdot tan\alpha \quad a = b \cdot tan\beta$$

Distance d between two points with coordinates (x_1, y_1) and (x_2, y_2) as a consequence of the theorem

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Trigonometry (三角函数)



Properties

- $sin^2\alpha + cos^2\alpha = 1$
- $sin(-\alpha) = -sin(\alpha)$
- $cos(\alpha) = cos(-\alpha)$
- $sin(\alpha + \beta) = sin\alpha cos\beta + cos\alpha sin\beta$
- $cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$

Matrices (矩阵)

A matrix (plural matrices) is a rectangular array of numbers denoted as

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

Addition

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix} = \begin{bmatrix} a+g & b+h & c+i \\ d+j & e+k & f+l \end{bmatrix}$$

Scalar multiplication

$$r \cdot \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} r \cdot a & r \cdot b & r \cdot c \\ r \cdot d & r \cdot e & r \cdot f \end{bmatrix}$$

Transpose

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

Matrix multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix} = \begin{bmatrix} ag+bi+ck & ah+bj+cl \\ dg+ei+fk & dh+ej+fl \end{bmatrix}$$

Determinant (行列式)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & k \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & k \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Vectors

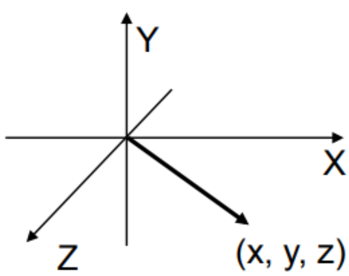
Vector is a geometric object that has both a **magnitude** (or length) and **direction**

A vector is visually represented by an arrow, connecting an initial point A with a terminal point B, and denoted by $\vec{AB}$ or $\mathbf{AB}$

A vector can be represented by identifying the **coordinates** of its **initial and terminal point**. For instance, the points $A = (a, b, c)$ and $B = (d, e, f)$

Vector Coordinates

$$\mathbf{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathbf{a} = \begin{bmatrix} x & y & z \end{bmatrix}$$

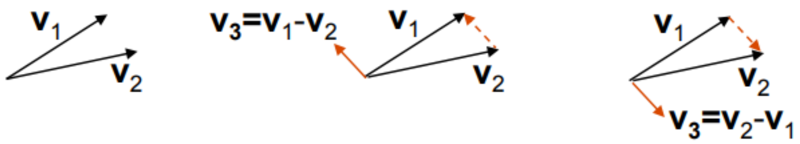


- To calculate with vectors, they are defined by the **coordinates of their endpoints** assuming that the tail of the vector coincides with the **origin**
- The endpoint coordinates are arranged into **column or raw vectors**, particularly when dealing with matrices

Sum of vectors

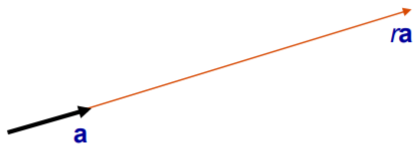
$$\mathbf{a} = [a_1, a_2, a_3] \;\; \mathbf{b} = [b_1, b_2, b_3]$$
$$\mathbf{c} = \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

Subtraction of vectors



$$\mathbf{a} = [a_1, a_2, a_3] \;\; \mathbf{b} = [b_1, b_2, b_3]$$
$$\mathbf{c} = \mathbf{a} - \mathbf{b} = -(\mathbf{b} - \mathbf{a}) = [a_1 - b_1, a_2 - b_2, a_3 - b_3]$$

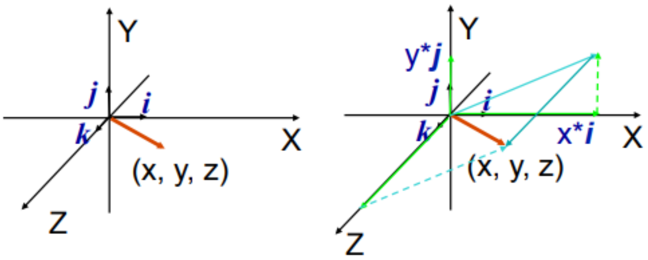
Scalar multiplication



$$\mathbf{a} = [a_1, a_2, a_3]$$
$$r\mathbf{a} = [ra_1, ra_2, ra_3]$$

Another way to define a vector

Another way to represent a vector is using **unit length vectors** defining X, Y, and Z coordinate axes and coordinates x, y, z



Vector magnitude

$$\mathbf{a} = [a_1, a_2, a_3]$$
$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Normalized vector

$$\mathbf{a} = [a_1, a_2, a_3]$$
$$\mathbf{a}_n = [\frac{a_1}{||\mathbf{a}||}, \frac{a_2}{||\mathbf{a}||}, \frac{a_3}{||\mathbf{a}||}]$$

Dot product / Scalar product

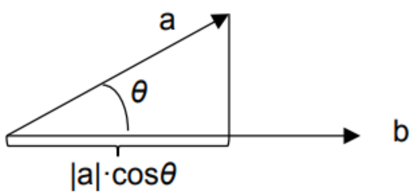
The result of dot product is a **number** or **scalar**

For **unit vectors**, the dot product is **a cos of the angle between them**

$$\mathbf{a} = [a_1, a_2, a_3] \;\; \mathbf{b} = [b_1, b_2, b_3]$$

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cdot cos\theta$$

- where θ is the measure of the angle between $\mathbf{a}$ and $\mathbf{b}$



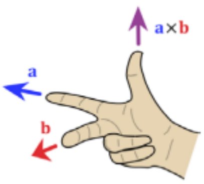
$$\mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$$

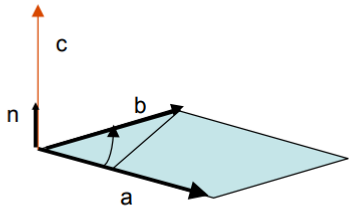
Vector product / Cross product

$$\mathbf{a} = [a_1, a_2, a_3] \;\; \mathbf{b} = [b_1, b_2, b_3]$$

$$\mathbf{c} = \mathbf{a}{\times}\mathbf{b} = (||\mathbf{a}|| \cdot ||\mathbf{b}|| \cdot sin\theta) \cdot \mathbf{n}$$

- θ is the measure of the angle between $\mathbf{a}$ and $\mathbf{b}$
- $\mathbf{n}$ is the normal vector which perpendicular to $\mathbf{a}$ and $\mathbf{b}$ produced by right-hand rules





$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \boldsymbol{i} \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - \boldsymbol{j} \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \boldsymbol{k} \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

5. Summary

- Computer graphics makes images with computers
- Visualization requires: **object model (geometry + material), light source(s), observer**
- Visualization requires to define some **mathematical model** to be rendered into images
- Shapes consist of **geometry, colors, image textures, and geometrical textures**
- All shape components can be defined in their **own coordinate systems** and merged together into one object
- Shapes can be further transformed and eventually **grouped into one application coordinate system**
- **Viewer and light sources** have to be defined to render the scene
- **Vector and matrix algebra** is used intensively in computer graphics to make it **dimension independent** and **computationally efficient**