Lecture3-2 Geometric Shapes (Surface)

1. Introduction

Learning objectives

- To understand how surfaces can be used in solving data visualization problems
- To understand surfaces as objects with **2 degree of freedom**
- To understand what mathematical representations are the most efficient for defining and displaying **surfaces**
- To understand how different coordinate systems can be used together for deriving mathematical representations of surfaces
- To understand surfaces as objects created by moving curves

2. Polygonal Representation

Three elements

- List of vertices
- List of polygons formed by the vertices
- List of normal vectors built at the vertices (optional)

Pay attention that

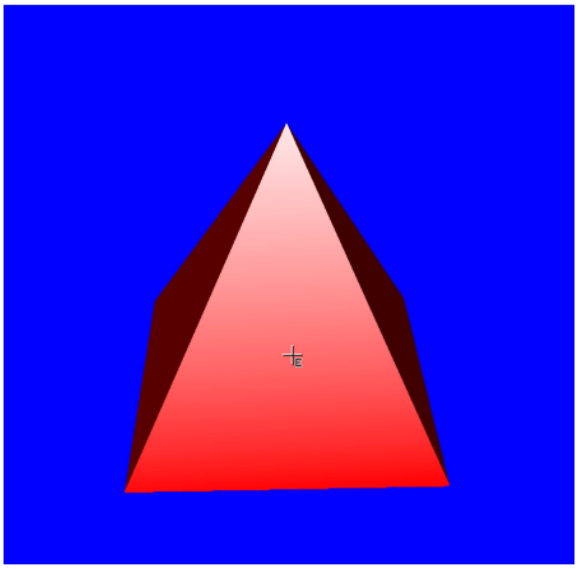
- Order of vertices is important
- Usually only one visible side where the normal is pointing out from

Common polygon mesh data format

- **.obj** (Wavefront)
- **.stl** (STereoLithography, a.k.a. "Standard Triangle Language" and "Standard Tessellation Language")
- Indexed FaceSet in VRML and X3D

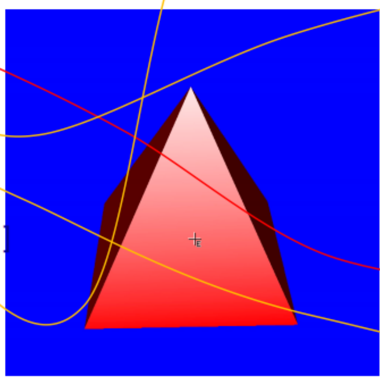
VRML Polygon Mesh

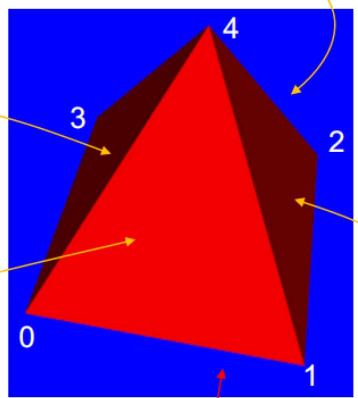
```
coord Coordinate { point [
 0,000000  0.000000   0.000000,
-0.005000 -0.505000 -1.619200,
-0.105000 -0.485000 -1.619200,
. . . . . . . . . . . . . . . . . . . . . . . . . ]  }
normal Normal { vector [
 1.000000  0.000000  0.000000
 0.176000 -0.601000 -0.780000
-0.306000 -0.340000 -0.889000
. . . . . . . . . . . . . . . . . . . . . . . . . ]  }
coordIndex [
70 65 62 -1
23 63 6 -1
18 19 17 -1
. . . . . . . . . . . . . . . . . . . . . . . . . ] }
```



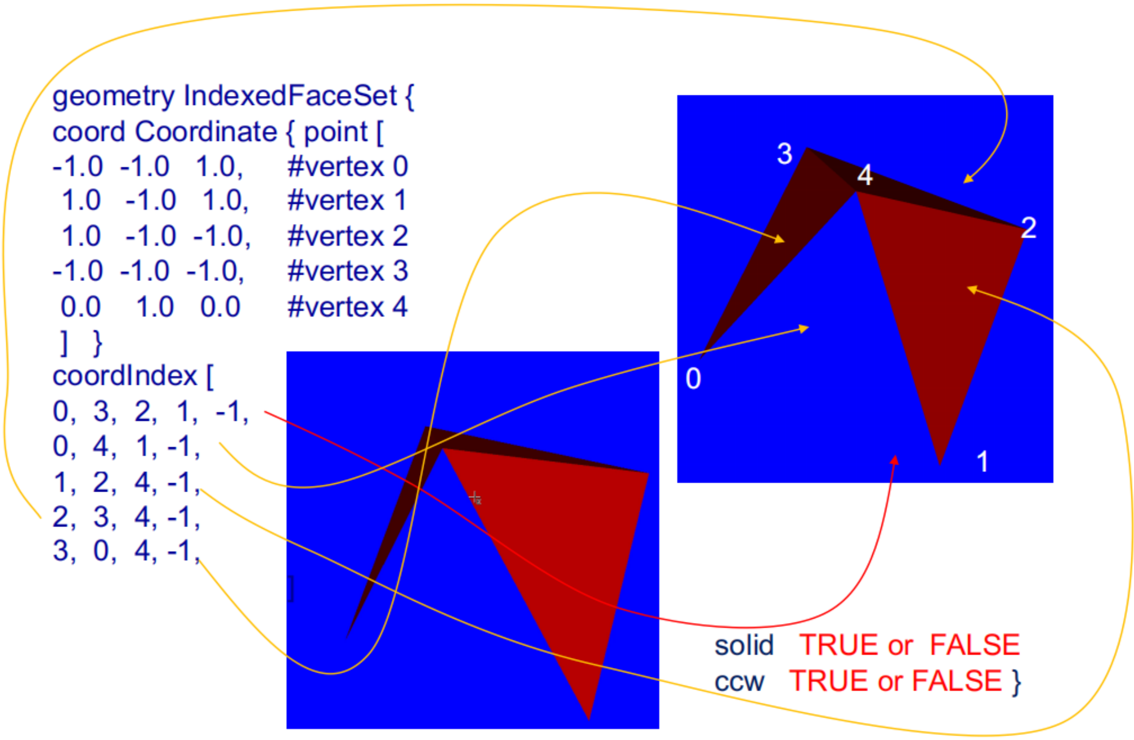
```
geometry IndexedFaceSet {
coord Coordinate { point [
-1.0  -1.0   1.0,      #vertex 0
 1.0   -1.0   1.0,     #vertex 1
 1.0   -1.0  -1.0,     #vertex 2
-1.0  -1.0  -1.0,      #vertex 3
 0.0    1.0   0.0      #vertex 4
 ]  }
coordIndex [
0,  3,  2,  1,  -1,
0,  1,  4, -1,
1,  2,  4, -1,
2,  3,  4, -1,
3,  0,  4, -1,

     }
```





- The coordIndex follow the counter clock wise sequence, it will render the face which is counter clock wize. If you look at the face which is clock wise, computer graphic will not render it



3. 3D Surfaces

Plane Surface

Definition

- Three non-colinear points (points not on a single line)
- One point and a line not containing this point
- Two intersecting lines
- Two parallel lines
- One point and a normal vector

Implicit function

$$Ax + By + Cz + D = 0$$

The values of the coefficients can be obtained by solving a set of three plane equations using the coordinates for three non-collinear points in the plane and by eliminating D

$$\frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z + \frac{D}{D} = A'x + B'y + C'z + 1 = 0$$

> Example
>
> Find the equation of the plane that passes through the points:

$$P(1,0,2) \quad Q(-1,1,2) \quad R(5,0,3)$$

We can write an equation that

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

- (x_0, y_0, z_0) is a point on plane
- $[a, b, c]$: perpendicular to plane
- $A=a, B=b, C=c, D=ax_0+by_0+cz_0$

Then we can use the determinant to solve the problem

https://www.youtube.com/watch?v=0qYJfKG-3l8

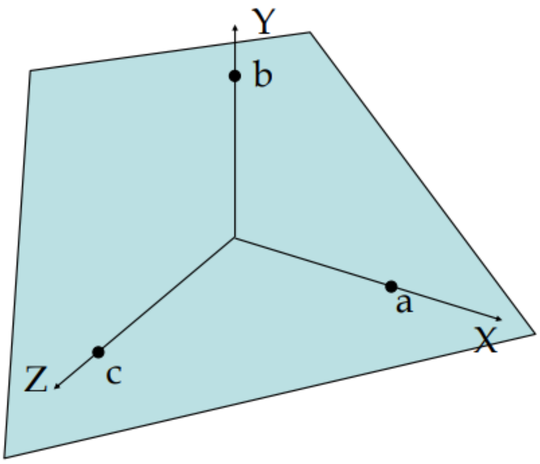
So the normal vector is $\vec{N}=[1,2,-4]$. So the equation of this line is:

$$(x-x_0)+2(y-y_0)-4(z-z_0)=0$$

The distance of one point (x, y, z) to the plane will be

$$\frac{|Ax+By+Cz+D|}{\sqrt{A^2+B^2+C^2}}$$

Implicit equation in intercepts



$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0$$

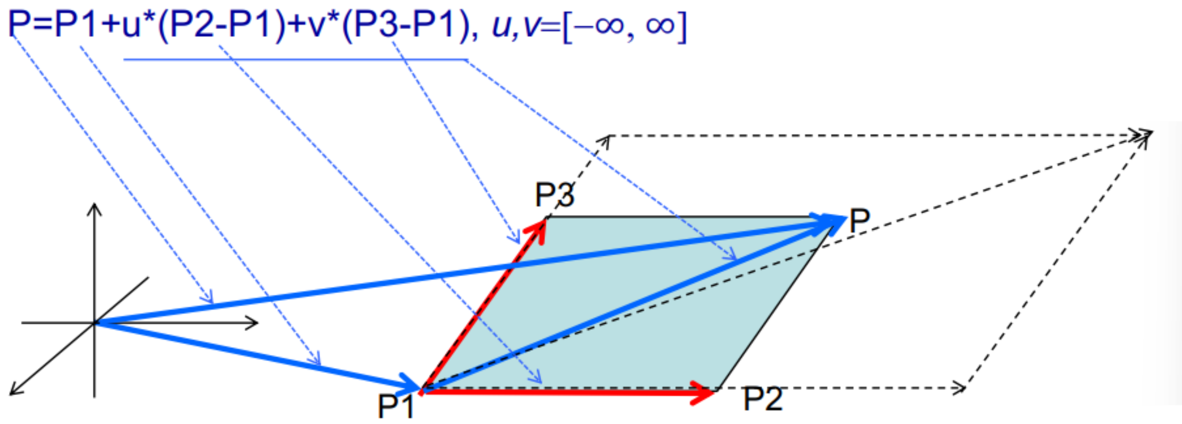
Explicit function

Explicit functions can be derived from it but are seldom used for drawing since it is axis dependent

Plane Parametrically

Suppose there are three non-collinear points $P1, P2, P3$ in the plane

Then the equation will be

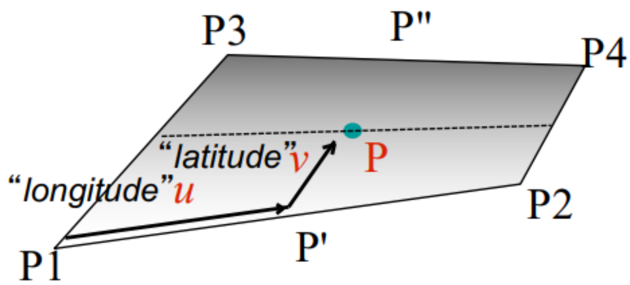


- $u, v \in [-\infty, \infty]$: a plane
- $u, v \in [0, 1]$: a parallelogram

Bilinear Surface (双线性曲面)

Parametric Representation

Suppose there are four non-parallel points $P1, P2, P3, P_4$ in the plane



$$P' = P_1 + u(P_2 - P_1)$$
$$P'' = P_3 + u(P_4 - P_3)$$
$$P = P' + v(P'' - P')$$

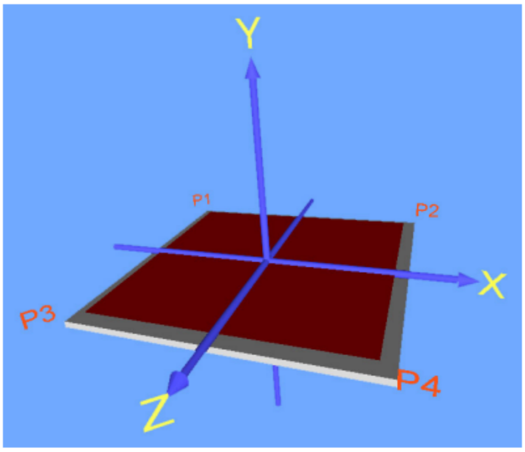
Thus, we can deduce that

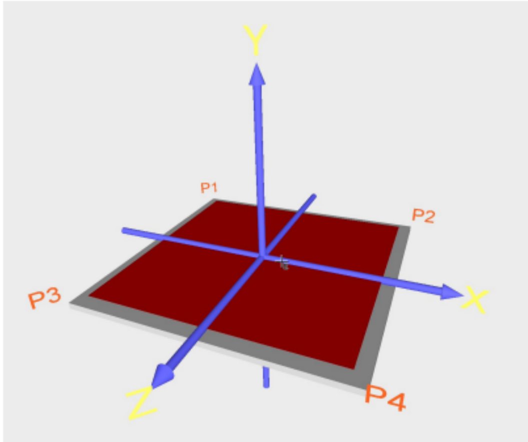
$$P = P_1 + u(P_2 - P_1) + v(P_3 - P_1 + u(P_4 - P_3 – (P_2 - P_1)))$$

- Point P_1, P_2, P_3 and P_4 may be any points in a 3D space so that even "twisted" surfaces may result

P_1	P_2	P_3	P_4
$x_1=-1$	$x_2=1$	$x_3=-1$	$x_4=1$
$y_1=0$	$y_2=0$	$y_3=0$	$y_4=0$
$z_1=-1$	$z_2=-1$	$z_3=1$	$z_4=1$

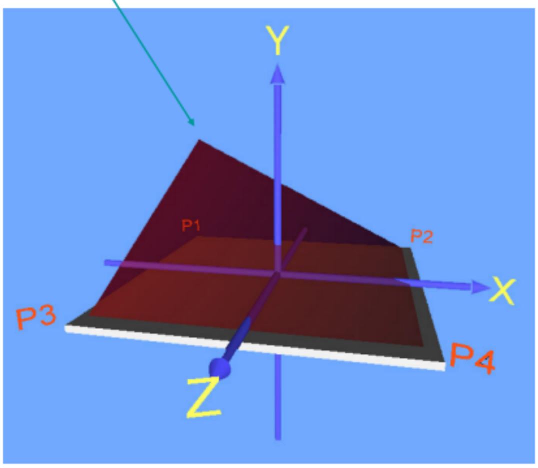
$$x(u,v)=x_1+u\cdot(x_2-x_1)+v\cdot(x_3-x_1+u\cdot(x_4-x_3-x_2+x_1))$$
$$y(u,v)=y_1+u\cdot(y_2-y_1)+v\cdot(y_3-y_1+u\cdot(y_4-y_3-y_2+y_1))$$
$$z(u,v)=z_1+u\cdot(z_2-z_1)+v\cdot(z_3-z_1+u\cdot(z_4-z_3-z_2+z_1))$$

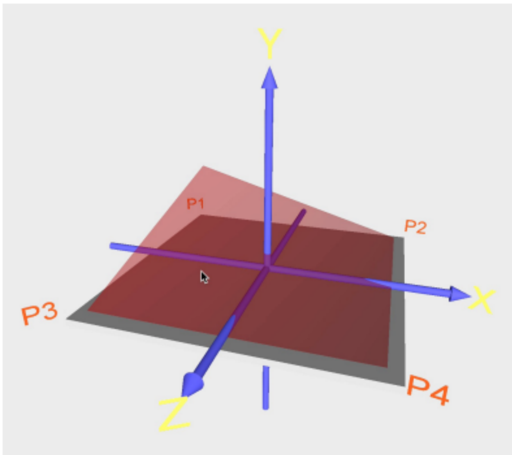




P_1	P_2	P_3	P_4
$x_1=-1$	$x_2=1$	$x_3=-1$	$x_4=1$
$y_1=\mathbf{1}$	$y_2=0$	$y_3=0$	$y_4=0$
$z_1=-1$	$z_2=-1$	$z_3=1$	$z_4=1$

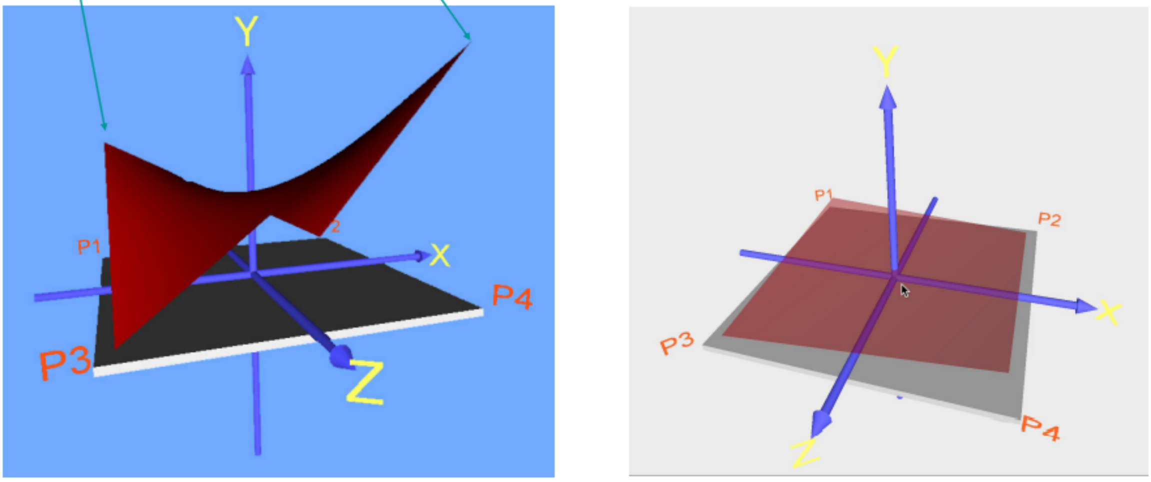
$$x(u,v)=x_1+u\cdot(x_2-x_1)+v\cdot(x_3-x_1+u\cdot(x_4-x_3-x_2+x_1))$$
$$y(u,v)=y_1+u\cdot(y_2-y_1)+v\cdot(y_3-y_1+u\cdot(y_4-y_3-y_2+y_1))$$
$$z(u,v)=z_1+u\cdot(z_2-z_1)+v\cdot(z_3-z_1+u\cdot(z_4-z_3-z_2+z_1))$$





P_1	P_2	P_3	P_4
$x_1=-1$	$x_2=1$	$x_3=-1$	$x_4=1$
$y_1=1$	$y_2=0$	$y_3=0$	$y_4=1.5$
$z_1=-1$	$z_2=-1$	$z_3=1$	$z_4=1$

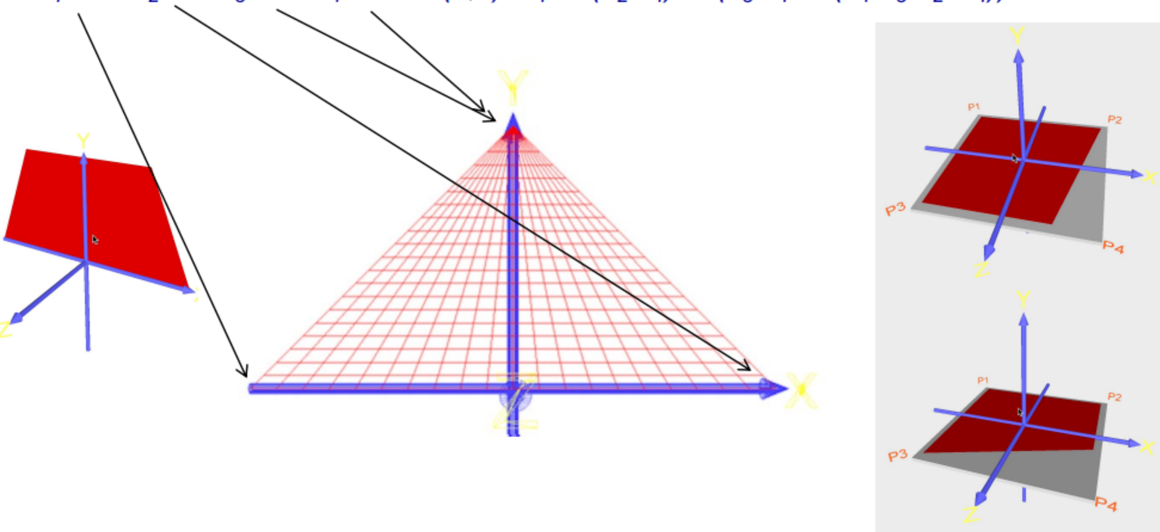
$$x(u,v)=x_1+u\cdot(x_2-x_1)+v\cdot(x_3-x_1+u\cdot(x_4-x_3-x_2+x_1))$$
$$y(u,v)=y_1+u\cdot(y_2-y_1)+v\cdot(y_3-y_1+u\cdot(y_4-y_3-y_2+y_1))$$
$$z(u,v)=z_1+u\cdot(z_2-z_1)+v\cdot(z_3-z_1+u\cdot(z_4-z_3-z_2+z_1))$$



Triangular Polygon

P_1	P_2	P_3 =	P_4
$x_1=-1$	$x_2=1$	$x_3=0$	$x_4=0$
$y_1=0$	$y_2=0$	$y_3=1$	$y_4=1$
$z_1=0$	$z_2=0$	$z_3=0$	$z_4=0$

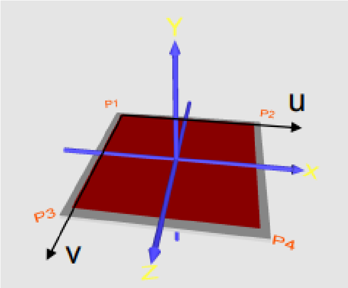
$$x(u,v)=x_1+u\cdot(x_2-x_1)+v\cdot(x_3-x_1+u\cdot(x_4-x_3-x_2+x_1))$$
$$y(u,v)=y_1+u\cdot(y_2-y_1)+v\cdot(y_3-y_1+u\cdot(y_4-y_3-y_2+y_1))$$
$$z(u,v)=z_1+u\cdot(z_2-z_1)+v\cdot(z_3-z_1+u\cdot(z_4-z_3-z_2+z_1))$$



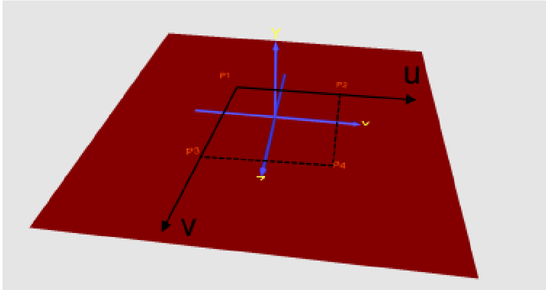
Plane Problem

Can it be used for defining planes if we allow the parameters to take any values at all

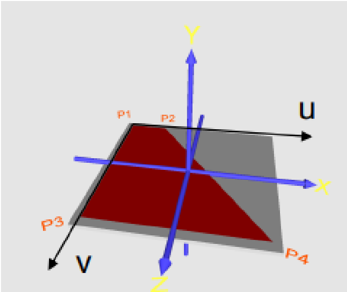
i.e. $u \in [-\infty, \infty], v \in [-\infty, \infty]$?



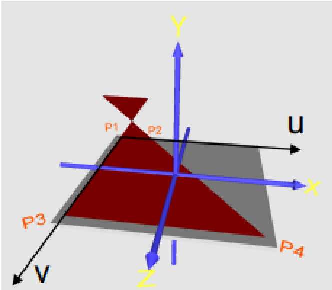
$u \in [0,1]; v \in [0,1]$



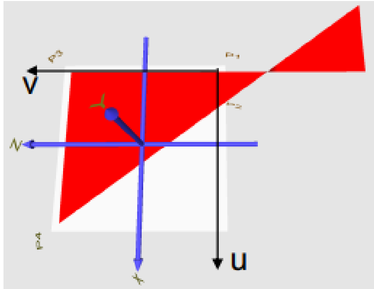
$u \in [-1,2]; v \in [-1,2]$



$u \in [0,1]; v \in [0,1]$



$u \in [0,1]; v \in [-1,1]$



- No, it may cause some fault
- Bilinear surfaces cannot be generally used for defining planes

4. Quadric Surfaces

Sphere

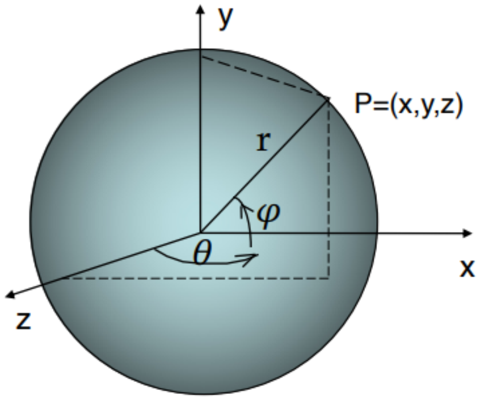
Implicit function

$$r^2 - x^2 - y^2 - z^2 = 0$$

Explicit function

$$z = \pm\sqrt{r^2 - x^2 - y^2}$$

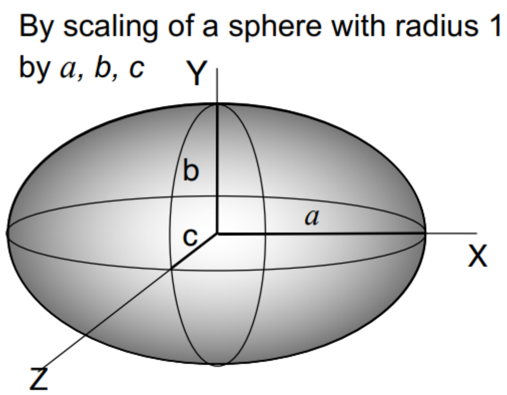
Parametric function



$$
\begin{aligned}
&x = r\cos\varphi\sin\theta \\
&y = r\sin\varphi \\
&z = r\cos\varphi\cos\theta \\
&-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\
&-\pi \le \theta \le \pi
\end{aligned}
$$

- Pay attention to the formula of the sphere may **changes** due to the management of different axies

Ellipsoid



Implicit function

$$1 - (\frac{x}{a})^2 - (\frac{y}{b})^2 - (\frac{z}{c})^2 = 0$$

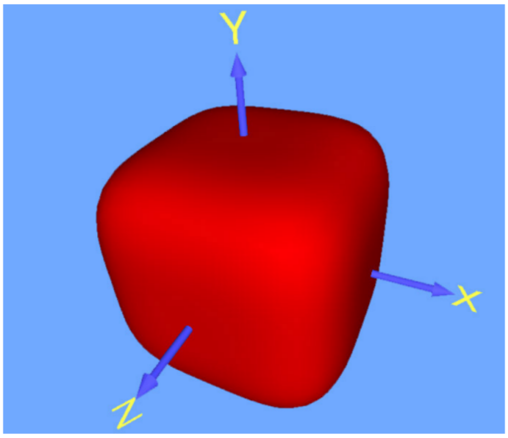
Explicit function

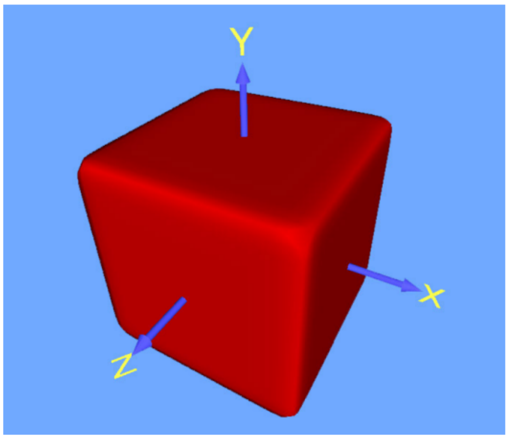
$$z = \pm c\sqrt{1 - (x/a)^2 - (y/b)^2}$$

Parametric function

$$
\begin{aligned}
&x = a * \cos\varphi \sin\theta \\
&y = b * \sin\varphi \\
&z = c * \cos\varphi \cos\theta \\
&-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\
&-\pi \le \theta \le \pi
\end{aligned}
$$

Super-ellipsoid

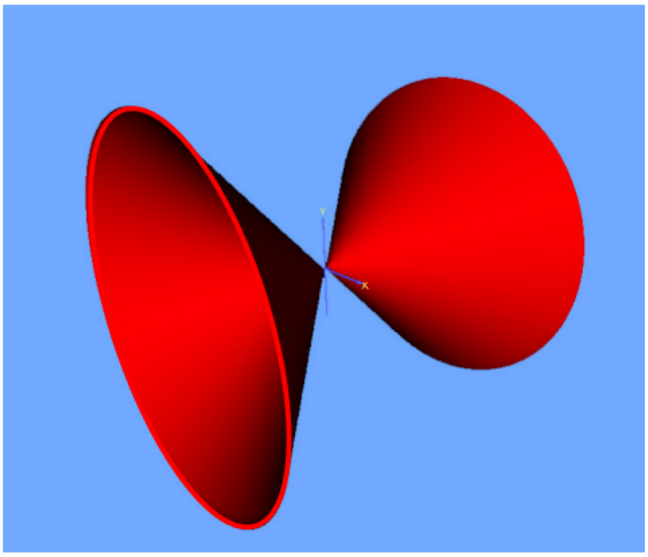




$1^2 - x^4 - y^4 - z^4 = 0$

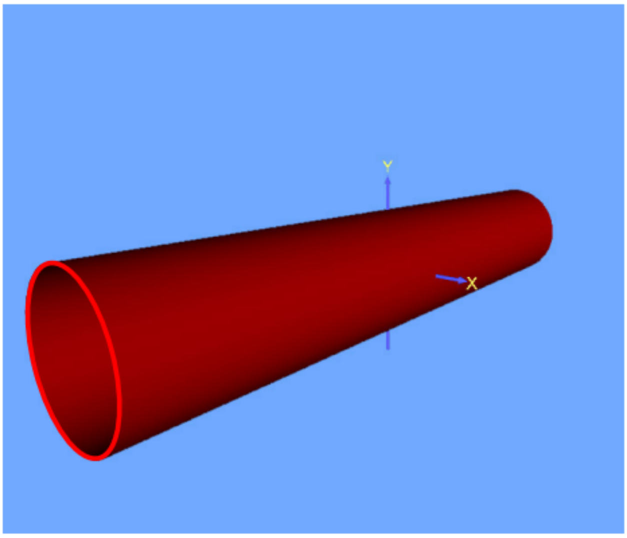
$1^2 - x^{16} - y^{16} - z^{16} = 0$

Cone



$$(\frac{z}{a})^2 - (\frac{x}{b})^2 - (\frac{y}{c})^2$$

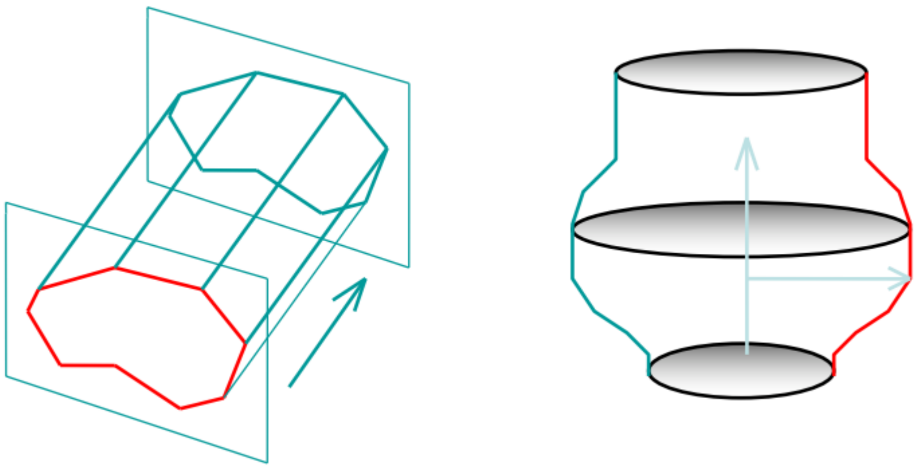
Elliptic Cylinder



$$1-(\frac{x}{b})^2-(\frac{y}{c})^2=0$$

5. Sweeping

Shapes are created by a curve moving along some path



Two particular cases of sweeping

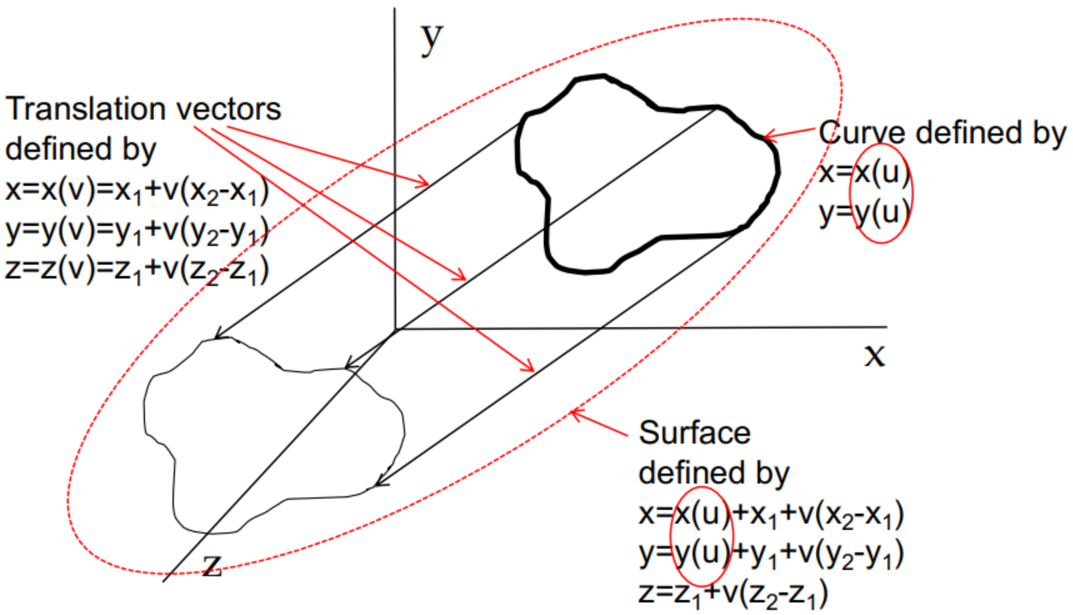
- translational sweeping
- rotational sweepingcan

be easily defined parametrically based on

- equation of a segment: $P=P_1+u(P_2-P_1)$
- equation of a circle: $x=rcos(\alpha),y=rsin(\alpha)$

More complicated cases of sweeping will be defined with **matrix transformations**, which are, however, still based on the same formulas

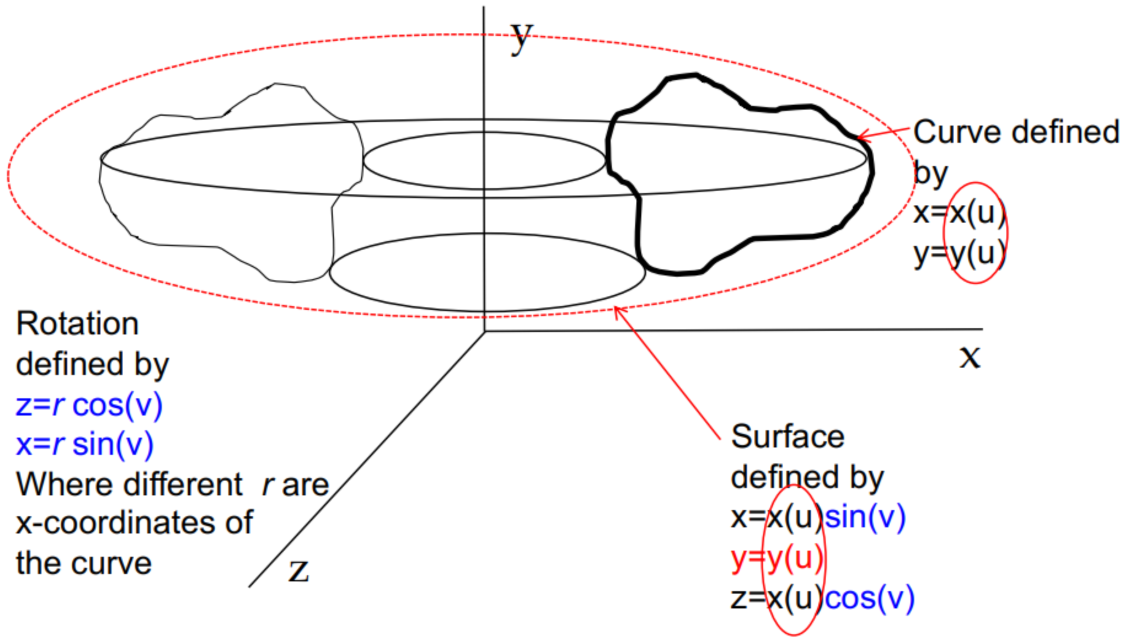
Translational Sweeping



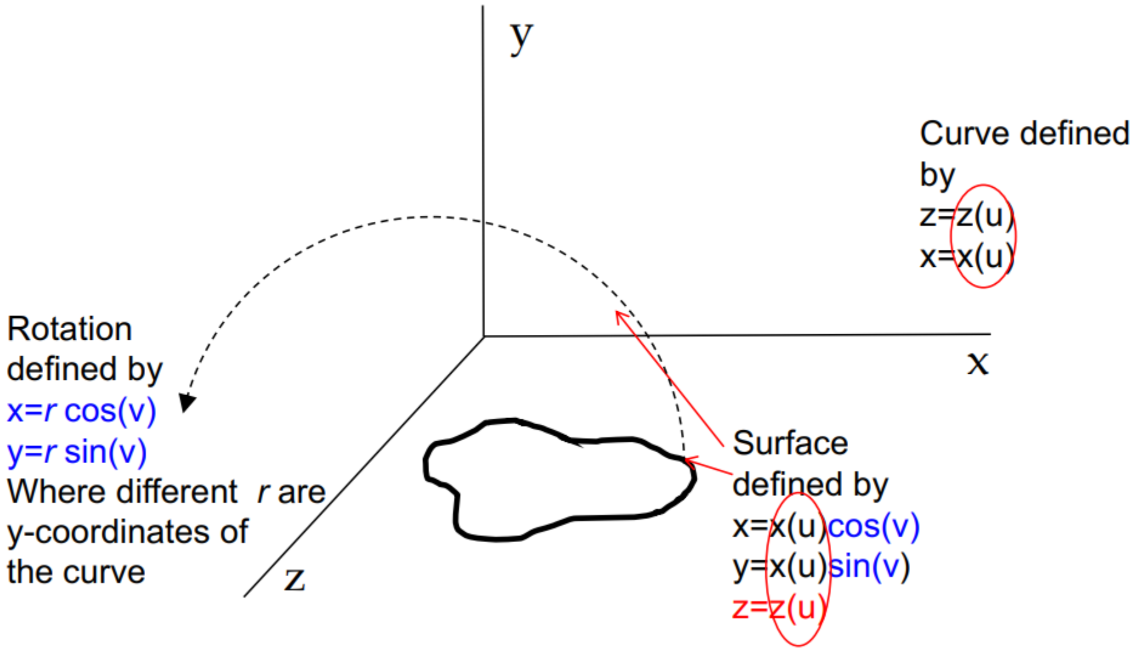
Rotational Sweeping

despite the rotation axis, typically in this course, the rotation follow the **right hand rules**

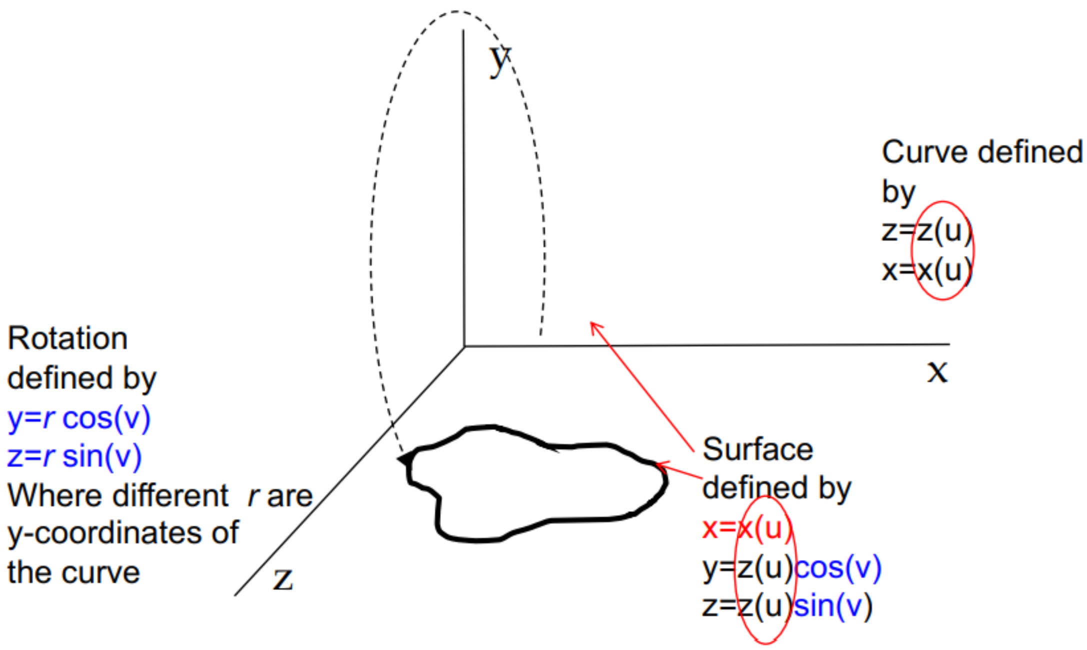
XY about Y



ZX about Z


y
Curve defined by
z=z(u)
x=x(u)
Rotation defined by
x=r cos(v)
y=r sin(v)
Where different r are y-coordinates of the curve
x
Surface defined by
x=x(u)cos(v)
y=x(u)sin(v)
z=z(u)
z

ZX about X


y
Curve defined by
z=z(u)
x=x(u)
Rotation defined by
y=r cos(v)
z=r sin(v)
Where different r are y-coordinates of the curve
x
Surface defined by
x=x(u)
y=z(u)cos(v)
z=z(u)sin(v)
z