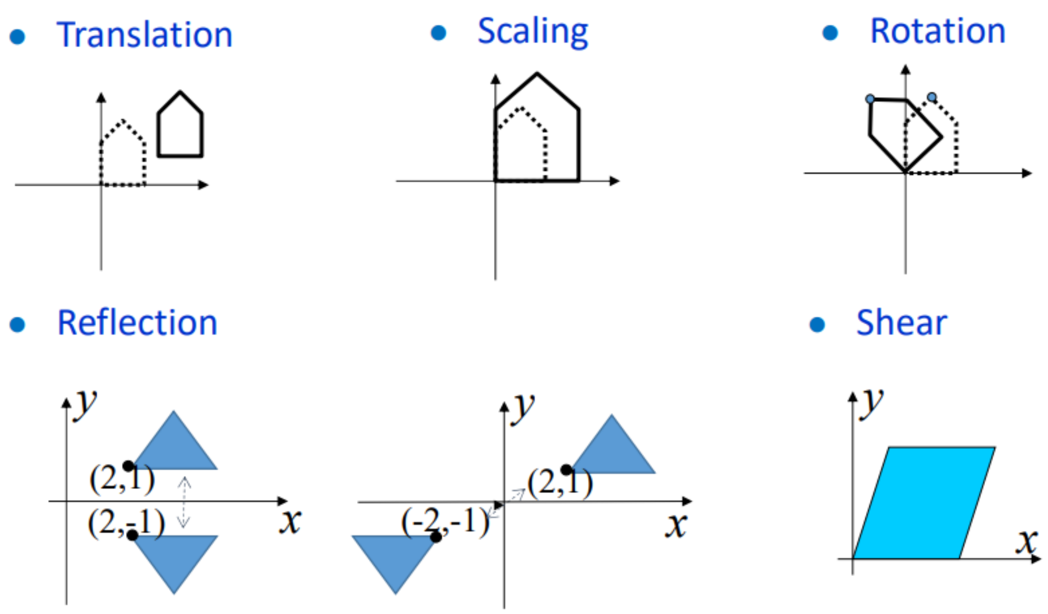
Lecture4 2D Transformation

1. Basic 2D Transformations



Representation

How to represent these transformations in computer

- Transformation: matrix M
- Point: column vector $\vec{p}$

Then the transformed point can be obtained by

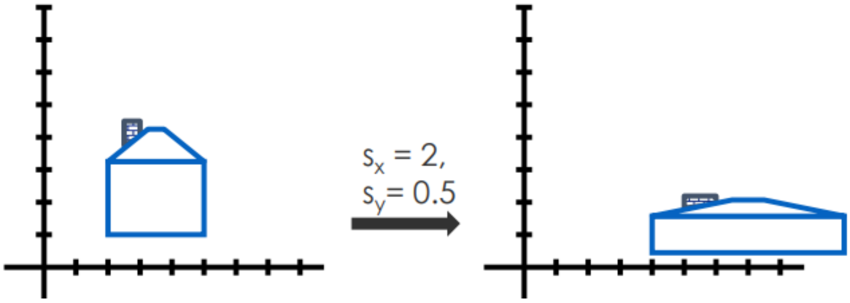
$$\vec{p'} = M\vec{p}$$

- Note that M goes on the left of p

Scaling Matrix

Scaling about the **origin**, with scaling factors (s_x, s_y)

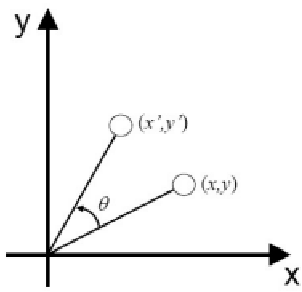
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Rotation Matrix

Rotation about the origin, with an angle θ

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



- the rotation is **counterclockwise**

How to deal with **clockwise** rotation?

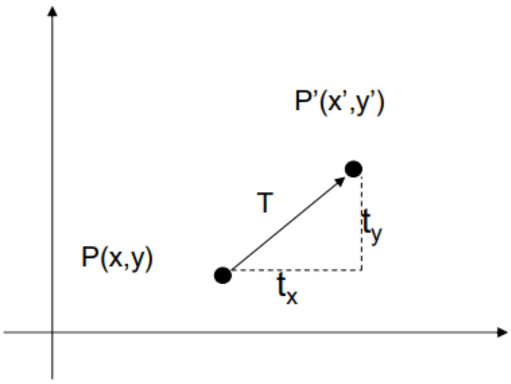
- Replace θ by $2\pi - \theta$
- Simply use $-\theta$

Translation (no matrix multiplication)

Translation is represented by the sum of two vectors, instead of matrix product

- Moves a point to a new location by adding translation amounts to the coordinates of the point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$



Why matrix?

- Calculate all the transform matrix, then apply the single product matrix to each of 1000 points
- Matrix operations can be highly optimized and carried out efficiently on graphics hardware (GPU)

2. Homogeneous coordinate

Definition

Expand 2D Cartesian coordinates (x, y) to 3-element (x_h, y_h, h), where h is a nonzero value satisfying

$$x = \frac{x_h}{h}, y = \frac{y_h}{h}$$

- (x_h, y_h, h) is called homogeneous coordinates of point (x, y)
- n 2D/3D transformation, we simply set $h = 1$ in general

Translation Matrix

Using homogeneous coordinates, translation can be represented by matrix product, with a 3×3 matrix:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling Matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation Matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection Matrix

Reflection over x-axis

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection over y-axis

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection over origin

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3. 2D Affine transformations

Definition

- Affine transformations are composites of four transformations:
  - translation
  - rotation
  - scaling
  - shear
- Affine transformations map straight lines to straight lines and preserve **ratios** of distances along straight lines
- Affine transformations preserve **parallelism** of lines but **not lengths and angles**

Affine transformations can always be represented by

$$x' = ax + by + m$$
$$y' = cx + dy + n$$

- a, b, c, d, m, n are constants
- (x, y) are the coordinates of the point to be transformed
- (x', y') are the coordinates of the transformed point

The general matrix form of affine transformations is

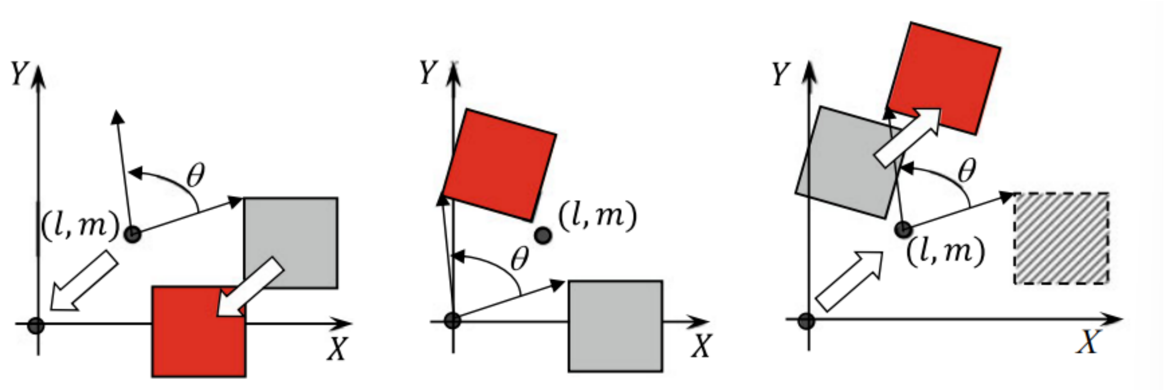
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & m \\ c & d & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Composition of transformations

- An affine transformation can be defined as **a composition of basic transformations**, which provides a way to define an affine transformation
- The **order** of transformations DOES matter

Rotation/Scaling/Reflection about a point

Rotation/Scaling/Reflection about any arbitrary point can be defined with **three** transformations



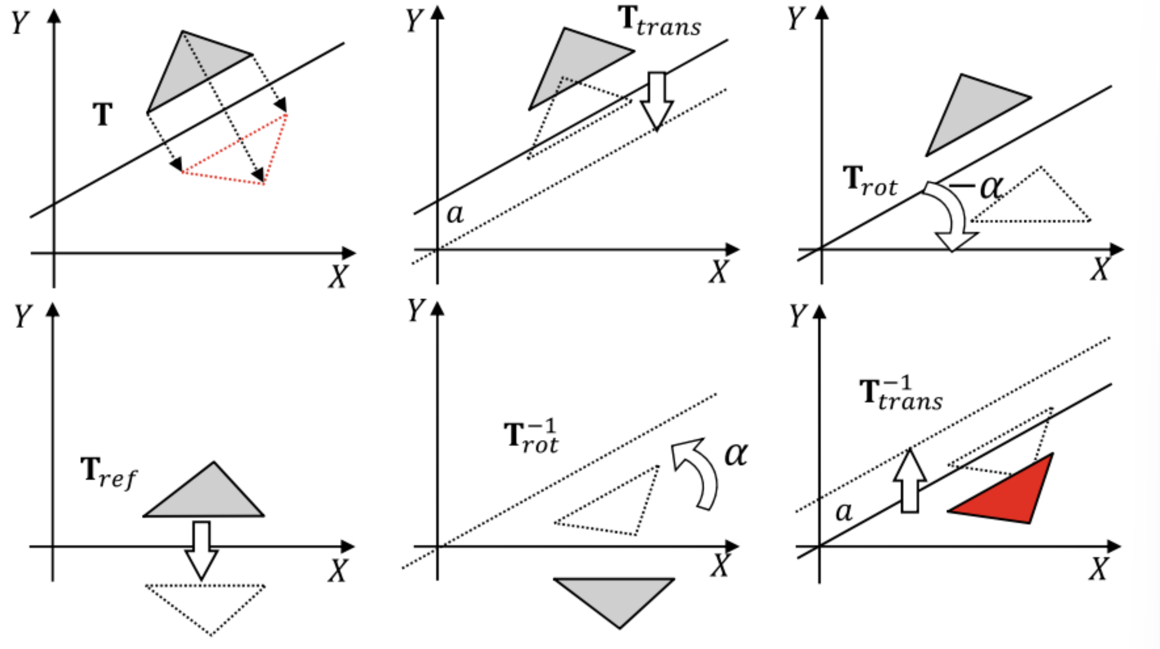
1. translation of the center of rotation toward the origin
2. rotation/scaling/reflection about the origin
3. inverse translation

$$\mathbf{P}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -l \\ 0 & 0 & -m \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -l \\ 0 & 0 & -m \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling/Reflection about a line

Scaling/Reflection about any arbitrary line can be defined with **five** transformations



1. translation the **intersection point of the line and the y-axis** to the origin
2. **align the line to the x-axis**
3. scaling about the origin / reflection about the **x-axis**
4. inverse align
5. inverse translation

$$\mathbf{T} = \mathbf{T}_{\text{trans}}^{-1}\,\mathbf{T}_{\text{rot}}^{-1}\,\mathbf{T}_{\text{ref}}\,\mathbf{T}_{\text{rot}}\,\mathbf{T}_{\text{trans}}$$

$$\mathbf{T}_{\text{trans}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{T}_{\text{rot}} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_{\text{ref}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{T}_{\text{rot}}^{-1} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{T}_{\text{trans}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$$