COS30019 - Introduction to Artificial Intelligence
Tutorial Problems Week 10

Task 1: Show from first principles that P(a|b∧a) = 1.

Given the definition of Conditional Probability (from "first principles"):

$$P(A|B) = \frac{P(A \wedge \mathrm{B})}{P(B)}$$

We get:

$$P(a|a \wedge \mathrm{b}) = \frac{P(a \wedge (\mathrm{b} \wedge \mathrm{a}))}{P(b \wedge \mathrm{a})} = \frac{P(b \wedge \mathrm{a})}{P(b \wedge \mathrm{a})} = 1$$

(because conjunctions is commutative and associative and $A \wedge \mathrm{A} \Leftrightarrow \mathrm{A}$)

Task 2: After your yearly checkup, the doctor has bad news and good news. The bad news is that you tested positive for serious disease and that the test is 99% accurate (i.e., the probability of testing positive when you do have the disease is 0.99, as is the probability of testing negative when you don't have the disease). The good news is that this is a rare disease, striking only 1 in 10,000 people of your age. Why is it good news that the disease is rare? What are the chances that you actually have the disease?

We investigate the effect of the probability of a disease, the prior probability, on the posterior probability.

TP – test positive
D – has disease

We have the following probabilities:

Definition	Notation	Probability
Probability of testing positive given one has the disease	$P(\mathrm{TP}\|D)$	0.99
Probability of testing negative given one does not have the disease	$P(\neg\mathrm{TP}\|\neg D)$	0.99
(Prior) probability that a random person has the disease	$P(D)$	10^{-4}

Probability of testing negative given one has the disease:

$P(\neg TP|D) = 1 - P(TP|D) = 0.01$

Probability of testing positive given one does not have the disease:

$P(TP|\neg D) = 1 - P(\neg TP|\neg D) = 0.01$

(Prior) probability that a random person does not have the disease:
$P(\neg D) = 1 - P(D) = 1 - 10^{-4} = 0.9999$

We know that this person was tested positive for the disease. We are also given the **probabilities of the disease** and the **conditional probabilities** according to the above table. We are interested in finding the probability of that person actually having the disease, given positive results of his/her test. That is, we want to find out $P(D|TP)$.

We apply Bayes' Rule $P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$, we get:

$$P(D|TP) = \frac{P(TP|D) \times P(D)}{P(TP)} \qquad (1)$$

Now, applying **Conditioning** (see the lecture):
$P(A) = P(A|B) \times P(B) + P(A|\neg B) \times P(\neg B)$
we have:

$$P(TP) = P(TP|D) \times P(D) + P(TP|\neg D) \times P(\neg D)$$

Substituting $P(TP)$ in (1), we get:

$$P(D|TP) = \frac{P(TP|D) \times P(D)}{P(TP|D) \times P(D) + P(TP|\neg D) \times P(\neg D)}$$

Finally, we get:

$$P(D|TP) = \frac{0.99 \times 10^{-4}}{0.99 \times 10^{-4} + 0.01 \times 0.9999} = 0.0098 \approx 0.01$$

$$P(D|TP) \approx 1\%$$

So, why is that???
The person is tested positive for the disease and the likelihood that he/she does have the disease is only about 1%!

Even though the accuracy of the test is quite high (99%), it is not high enough: IF we put 10,000 healthy people who don't have the disease through this test, about 100 of them will still come out with a POSITIVE test result. The fact that the disease is rare is very good news here: From the population of 10,000 people, there is only about 1 person having the disease. The person in the question is **likely** to be one of the 99 healthy people who don't have the disease and come out of the test with a POSITIVE test result! The chance that the person in the question happens to be the one person with the disease is thus only 1%! Clearly, the POSITIVE test result is still bad news: Before the test, this person comes in with a probability of having the disease of 0.01%; now, after the test, that probability has come up to almost 1%!

Task 3: (Exercise 13.15 from AIMA textbook) Suppose you are a witness to a night-time hit-and-run accident involving a taxi in Athens. All taxis in Athens are blue or green. You swear, under oath, that the taxi was blue. Extensive testing shows that, under the dim lighting conditions, discrimination between blue and green is 75% reliable. Is it possible to calculate the most likely colour for the taxi? (Hint: distinguish carefully between the proposition that the taxi is blue and the proposition that it appears blue.)
What about now, given that 9 out of 10 Athenian taxis are green?

B = taxi was blue
LB = taxi looked blue to you

$$P(LB|B) = 0.75$$
$$P(\neg LB|\neg B) = 0.75$$

We want to know the probability that the taxi was blue, given that it looked blue (because you tell the truth in court that you *saw a blue taxi*).

$$P(B|LB) = \frac{P(LB|B) \times P(B)}{P(LB)} = \frac{0.75 \times P(B)}{P(LB)}$$

$$P(\neg\mathrm{B}|\mathrm{LB}) = \frac{P(LB|\neg\mathrm{B}) \times \mathrm{P}(\neg\mathrm{B})}{P(LB)} = \frac{0.25 \times (1 - P(B))}{P(LB)}$$

We cannot decide the probability without some information about the prior probability of blue taxis, $P(B)$.

Given 9 out of 10 taxis are green and assuming the taxi in question is drawn randomly from the taxi population, $P(B) = 0.1$ and $P(\neg B) = 0.9$.

Using **conditioning**:
P(LB) = $P(LB|B) \times P(B) + P(LB|\neg B) \times P(\neg B)$ = 0.75*0.1 + 0.25*0.9

Therefore:

$$P(B|LB) = \frac{0.75 * 0.1}{0.75 * 0.1 + 0.25 * 0.9} = \frac{0.75 * 0.1}{0.75 * 0.1 + 0.75 * 0.3} = \frac{0.1}{0.1 + 0.3} = 0.25 = 25\%$$

Task 4: (The Monty Paradox and the TV game shows: How to Win?**)**
Apparently, this paradox or probability problem was inspired by the Monty Hall's TV game show *Let's Make A Deal*. The host, Monty Hall, offers the player the opportunity to win what is behind one of three doors. Typically there was a really nice prize (i.e. a car) behind one of the doors and not-so-nice prizes (i.e. a rabbit) behind the other two. After selecting a door, Monty would then proceed to open one of the doors you didn't select. It is important to note here that Monty would NOT open the door that concealed the car. Thus, the host always ELIMINATES one of the losing cases. At this point, he would then ask you if you wanted to switch your selection to another door.

C_i – initial player's choice
W – winning door

$P(W|C_i) = \frac{1}{3}$ – probability of winning given the initial choice
$P(\neg W|C_i) = \frac{2}{3}$ – probability of losing given the initial choice

Not switching – can win with $\frac{1}{3}$ probability as you will only win when your initial choice was correct (= $P(W|C_i)$)
Switching – can win with $\frac{2}{3}$ probability as you will win when your initial choice was wrong (= $P(\neg W|C_i)$)

Illustrative example:
Let's imagine that door 1 is the winning choice. Given three doors we get 6 permutations:

Player's Choice	Monty's Choice (crossed)		Not Switching	Switching
1	~~2~~	3	W	L
1	~~3~~	2	W	L
2	**1**	~~3~~	L	**W**
2	~~3~~	**1**	L	**W**
3	**1**	~~2~~	L	**W**
3	~~2~~	**1**	L	**W**

No matter what losing door Monty removes, the Player will always have 2 of 6 winning choices (1/3) when not switching, and 4 of 6 winning choices when switching (2/3).

What if Monte doesn't know where the price is and opens the door randomly? The probability of winning would be ½.