Lecture: Set fundamentals

COS10003 Computer Logic and Essentials (Hawthorn)

Semester 1 2021

Today

1. Sets
2. Set relations
3. Operations
4. Theorems
5. Counting

How we define collections of objects

Some approaches for proving simple statements

The principles for working with finite sets

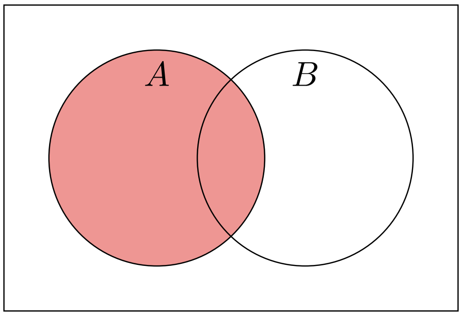
Overview

- Sets are collections of objects.
- The elements of a set can be physical objects, abstract objects or other sets.
- Sets are a key foundation of discrete maths and also computing.

Set definitions

- Sets can be empty (denoted as $\varnothing$)
- Sets can be equal, meaning two sets have the same elements.
- Sets can be ordered, however in computing they are generally treated as unordered.
- Elements of a set should be distinct.
- We sometimes work with a universal set U: the set containing everything of interest.

Visualising sets



Each disk or circle represents a set. Coloured sets represent the area we are interested in. The rectangular border represents the universal set: this is often denoted by U.

Defining sets

Enumerating all elements

- $\{1, 2, 3, 4, 5, 6\}$
- {Hawthorn, Croydon, Lilydale}

Providing a common property

- Natural numbers between 1 and 6
- Locations of Swinburne campuses in Australia

Notation

Natural numbers between 1 and 6

$$A = \{x : x \in \mathbb{N}, x \leq 6\}$$

$$A = \{x \in \mathbb{N} : x \leq 6\}$$

$$A = \{1, 2, 3, 4, 5, 6\}$$

Sometimes you might see a vertical bar in place of a colon.

Inclusion

We have $\in$ to denote membership of a set.

We also have two approaches to inclusion:

- $\subseteq$ where the first set is included in or equal to the second set
- $\subset$ where the first set is included in but not equal to the second set

The formal definition is that when $A \subseteq B$, for all $x \in A$, then $x \in B$.

Fill in the blank

Which symbol (membership or subset) is the best to use for the following expressions?

- $a \_ \{a, b, c\}$
- $f \_ \{a, b, c\}$
- $\{a, b\} \_ \{a, b, c\}$
- $\{a, b, d\} \_ \{a, b, c\}$
- $\{a, b, d\} \_ \{a, b, c\}$
- $\{a, b, d\} \_ \{a, b, c, d\}$
- $\{a, b, c\} \_ a$
- $\{a, b, c\} \_ \{a, b, c\}$

Extensionality

When both $A \subseteq B$ and $B \subseteq A$, then the sets A and B are identical.

Reasoning about sets

Would you say the following statement is true or false?

Whenever $A \subseteq B$ and $B \subseteq C$, $A \subseteq C$

Assume the left hand side is true: for any $x \in A$, then $x \in B$ from the inclusion definition, and if $x \in B$, then $x \in C$. For the right hand side, we know that if $x \in A$, then $x \in C$, which checks out.

The logic behind the reasoning

1. Be clear what you want to show: this is not straightforward as the goal will need to be broken down, which is good practice for programming problems
2. Use what you can: this could be axioms, other things you have already shown, and definitions
3. Be flexible: be prepared to climb down the mountain and try another approach

Makinson, Chapter 1

Another example

Would you say the following statement is true or false?

Whenever $A \subseteq B$ and $C \subseteq B$, $A \subseteq C$

This is false. The best approach here is to find a counterexample: some sets where the idea does not hold.For example, $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{4, 5\}$. So $A \subseteq B$ and $C \subseteq B$ but $A \not\subseteq C$.

The empty set

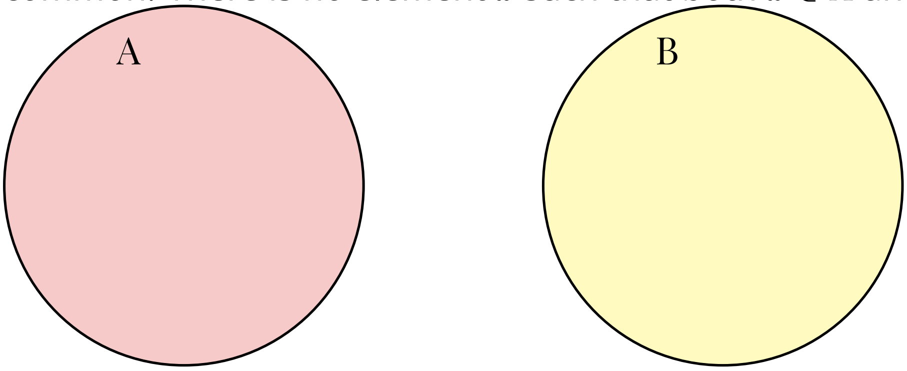
Sets can be empty. The trickiest concept to understand is the statement

$\varnothing \subseteq A$ for every set A

To prove this, we start with inclusion again, so if $x \in \varnothing$, then $x \in A$. But there is no x in $\varnothing$ and so this statement is true.

Disjoint sets

Two sets are mutually exclusive or disjoint if they have no elements in common. There is no element x such that both $x \in A$ and $x \in B$.



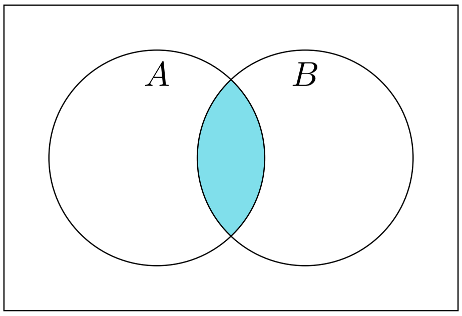
Assume we have two sets

You might like to look at the following set operations with these sets in mind.

- $A = \{1, 2, 3, 4, 5\}$ or $\{x : x \in \mathbb{N}, x \leq 5\}$
- $B = \{2, 4, 6, 8, 10, 12\}$ or $\{x : x \in \mathbb{N},$ x is even $, x \leq 12\}$
- $U = \{1...12\}$

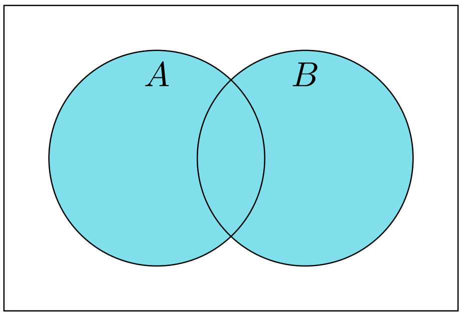
Intersection

$$x \in A \cap B \text{ iff } x \in A \text{ and } x \in B$$



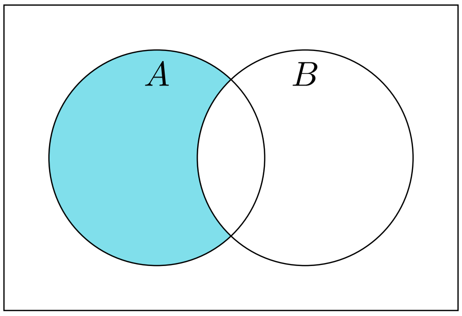
Union

$$x \in A \cup B \text{ iff } x \in A \text{ or } x \in B$$



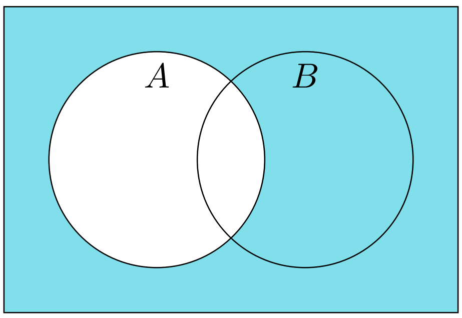
Difference

$$x \in A \setminus B \text{ iff } x \in A \text{ and } x \notin B$$



Complement

$$x \in A^C \text{ iff } x \notin A$$



With these sets

- $A = \{1, 2, 3, 4, 5\}$ or $\{x : x \in \mathbb{N}, x \leq 5\}$
- $B = \{2, 4, 6, 8, 10, 12\}$ or $\{x : x \in \mathbb{N},$ x is even $, x \leq 12\}$
- $U = \{1...12\}$

- What is $A \cap B$?
- What is $A \cup B$?
- What is $A \setminus B$?
- What is A^C?

With these sets again

- $A = \{1, 2, 3, 4, 5\}$ or $\{x : x \in \mathbb{N}, x \leq 5\}$
- $B = \{2, 4, 6, 8, 10, 12\}$ or $\{x : x \in \mathbb{N},$ x is even $, x \leq 12\}$
- $U = \{1...12\}$

- What is $A \cap B$? $x \leq 5$ and x is even
- What is $A \cup B$?
- What is $A \setminus B$?
- What is A^C?

To ponder

Would you say the following statement is true or false?

$$A \setminus B = A \cap B^C$$

For this, two parts need to be shown:

- $A \setminus B \subseteq A \cap B^C$
- $A \cap B^C \subseteq A \setminus B$

Both of these can then be defined using inclusion and the definitions for set difference and intersection.

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General rules about sets

Recall when talking about proofs, it was mentioned that you could use established rules?

Fundamental laws

Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$
Associative: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity: $A \cup \varnothing = A$, $A \cap U = A$
Complement: $A \cup A^C = U$, $A \cap A^C = \varnothing$

More laws

Idempotent: $A \cup A = A$, $A \cap A = A$

Identity or domination or null: $A \cup U = U$, $A \cap \varnothing = \varnothing$

Absorption: $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

De Morgan's: $(A \cup B)^C = A^C \cap B^C$, $(A \cap B)^C = A^C \cup B^C$

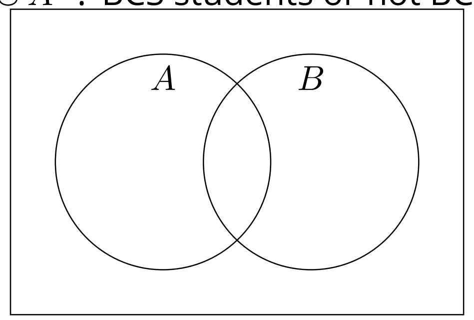
Involution or double complement: $(A^C)^C = A$

And some more complements: $U^C = \varnothing$, $\varnothing^C = U$

Complement law

Let A = BCS students, B = identify as female

What is $A \cup A^C$? BCS students or not BCS students?

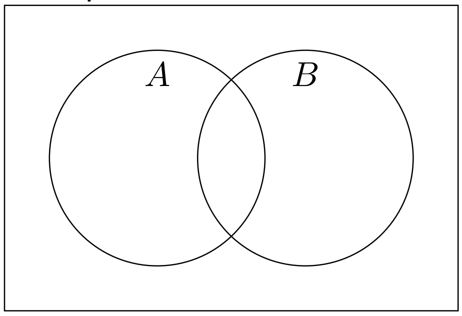


$A \cup A^C = U$

De Morgan's law

Let A = BCS students, B = identify as female

What is $A^C \cup B^C$ equal to? Not male or not BCS students?



$$(A \cap B)^C = A^C \cup B^C$$

Manipulating set expressions

How can we show that $(A \cap A^C)^C = U$?

One way:

$A \cap A^C = \varnothing$

complement

$\varnothing^C = U$

complement again

Another way:

$(A \cap A^C)^C = A^C \cup (A^C)^C$

de Morgan's

$(A^C)^C = A$

involution or double complement

$A^C \cup A = U$

complement

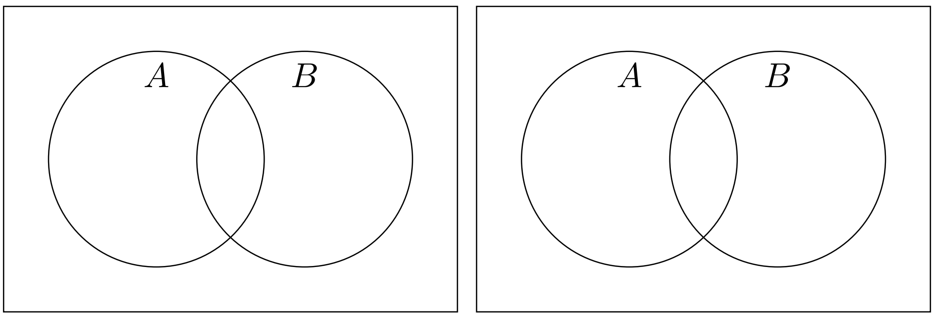
Another proof

Show that $(A \cup B) \cap (A^C \cap B)^C$ is equal to A.

$$\begin{aligned}(A \cup B) \cap (A^C \cap B)^C &= (A \cup B) \cap ((A^C)^C \cup B^C) \text{ De Morgan's} \\ &= (A \cup B) \cap (A \cup B^C) \text{ Involution} \\ &= A \cup (B \cap B^C) \text{ Distributive} \\ &= A \cup \varnothing \text{ Complement} \\ &= A \text{ Identity}\end{aligned}$$

Another proof

Show that $(A \cup B) \cap (A^C \cap B)^C$ is equal to A.



Proof techniques so far

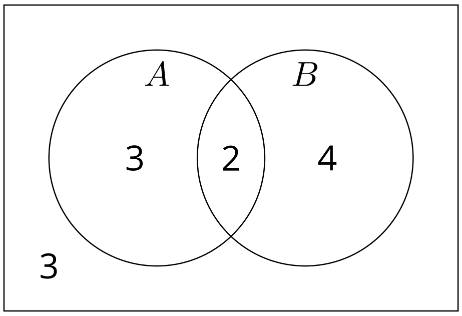
- Proof by element (if $x \in A$ etc.)
- (dis)Proof by counterexample
- Proof using set laws
- Proof using Venn diagram

Counting items

- Sometimes we have sets with a discrete number of items in them.
- Counting the items in different parts of the sets is a useful skill.

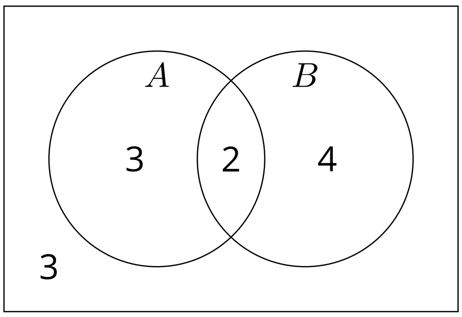
Some simple counting

- $A = \{1, 2, 3, 4, 5\}$ or $\{x : x \in \mathbb{N}, x \leq 5\}$
- $B = \{2, 4, 6, 8, 10, 12\}$ or $\{x : x \in \mathbb{N}$, x is even $, x \leq 12\}$
- $U = \{1...12\}$



Inclusion/exclusion

$$\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid$$



Example

- 100 students at a university ($|U|$)
- 50 enrolled in programming ($|P|$)
- 60 enrolled in music ($|M|$)
- 25 enrolled in both subjects ($|M \cap P|$)

- How many in programming or music? **85**
- How many not in programming or music? **15**
- How many in only programming?
- How many in only music?

Generalising, sort of

How does this work for three sets?

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

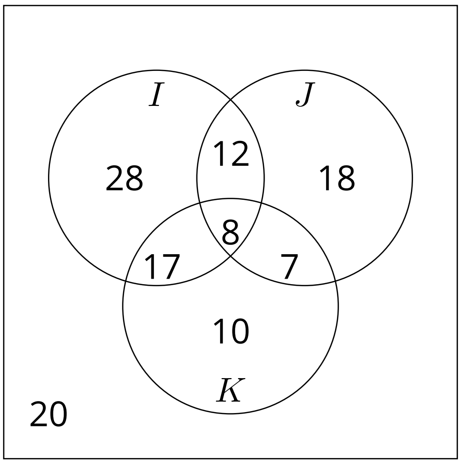
Another example

100 of 120 computing students also study a language:

- 65 Italian (I)
- 45 Japanese (J)
- 42 Korean (K)
- 20 Italian and Japanese
- 25 Italian and Korean
- 15 Japanese and Korean

How many study all three languages?

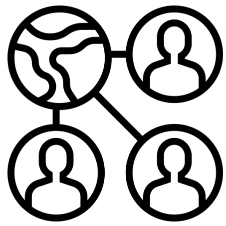
Basic counting



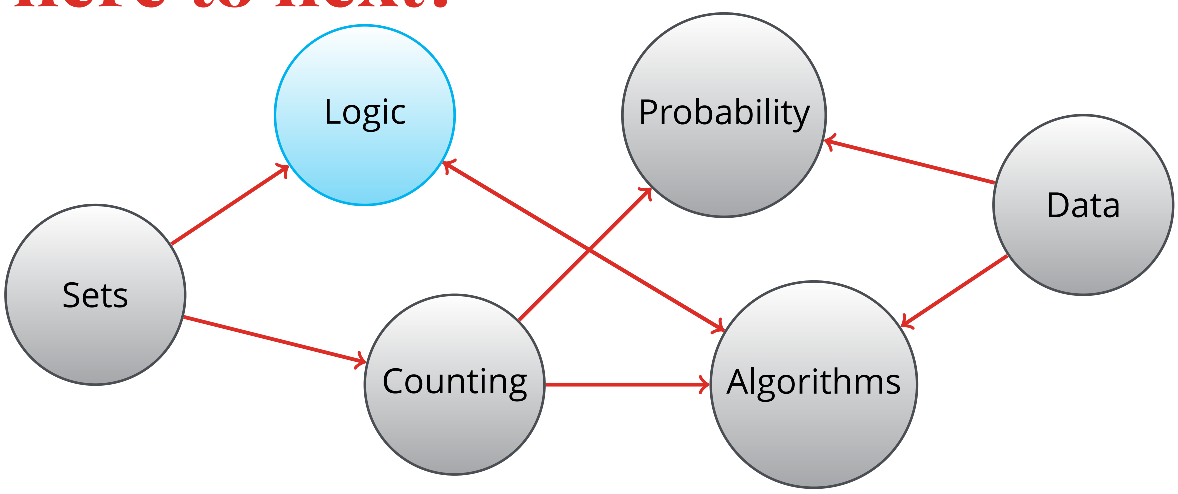
100 = 65 + 45 + 42 - 20 - 25 - 15 + #study all three languages
= 8 study all 3 languages (Why?)
Therefore:
20 - 8 = 12 I and J but not K
25 - 8 = 17 I and K but not J
15 - 8 = 7 J and K but not I
65 - 12 - 8 - 17 = 28 I only
45 - 12 - 8 - 7 = 18 J only
42 - 17 - 8 - 7 = 10 K only
120 - 100 = 20 None
28 + 18 + 10 = 56 One language only

Reflecting

- How can we define collections of objects?
- What is one approach to proving statements?
- What is the key principle for working with cardinalities of finite sets?



Where to next?



In which we look at the fundamentals of logic.

Questions I still have

Topics I need to review