Lecture 5
Relations and functions

COS10003 Computer Logic and Essentials (Hawthorn)

Semester 1 2021

Today

1. Sets revision
2. Relations
3. Properties
4. Composition
5. Functions
6. Function properties

How objects can be related

How similar objects can be grouped

How objects can be transformed (and back again)

Previously ...

- We looked at set theory.

Cartesian products

- The (Cartesian) product of two sets A and B is defined as $A \times B = \{(a, b) : a \in A \wedge b \in B\}$. $\times$ should be read as "cross".
- (a, b) is an ordered pair, in that it is not the same as (b, a): the order of the elements matters.
- We can also have ordered triples and so on, if we take the product of more than two sets.

Quick quiz

If $C = \{1, 3\}$ and $D = \{2, 4, 5\}$:

- what is $C \times D$?
- what is $C \times C$?
- is $(3, 2) \in D \times D$?
- what is the cardinality of $D \times C$?

What is a relation?

A relation R from A to B assigns to each ordered pair (a, b) in $A \times B$ one of the following statements:

1. a is related to b
2. a is not related to b

The relation is some sort of rule or criterion that is either satisfied or not satisfied.
For example, the statement ”x is less than y” is a relation on any set of real numbers. For any ordered pair (a, b) it is either satisfied or nor satisfied.

First go

If $A = \{1, 2, 3\}$ and $R \subseteq A \times A$ what are the following relations?
Start with enumerating
$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

- $R = \{(a, b) : b = a + 1\}$ $\{(1, 2), (2, 3)\}$
- $R = \{(a, b) : a$ is less than $b\}$
- $R = \{(a, b) : a$ is less than or equal to $b\}$
- $R = \{(a, b) : a + b = 4\}$

Note a valid relation on A needs to be a subset of $A \times A$.

Domain and range

The domain of a relation R is the set of all first elements of the ordered pairs, which belong to R. The range is the set of second elements.

$$R = \{(a, b) : b = a + 1\} = \{(1, 2), (2, 3)\}$$
$$domain(R) = \{1, 2\}$$
$$range(R) = \{2, 3\}$$

Inverse

If R is a relation from A to B, then the inverse of R (R^{-1}) is the relation from B to A. The inverse relation is composed of those ordered pairs, which when reversed belong to R:

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly the domain and range would then be swapped, that is $domain(R) = range(R^{-1})$ and $range(R) = domain(R^{-1})$.
Note that the domain of R^{-1} equals the range of R and the range of R^{-1} equals the domain of R.

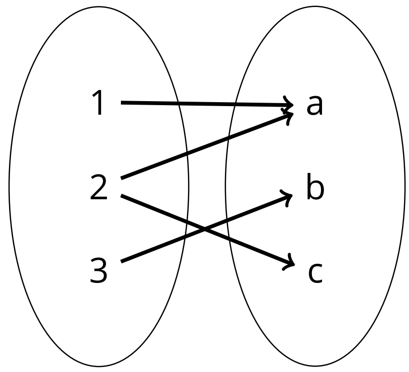
For you to do

For $A = \{1..10\}$, $B = \{1..10\}$ and $R = \{(a, b) : a - 2b = 0\}$, what is:

1. The relation (enumerated)?
2. The domain?
3. The range?
4. The inverse (enumerated)?

Illustrations

	a	b	c
1	1	0	0
2	1	0	1
3	0	1	0



What is this relation? $\{(1,a),(2,a),(2,c),(3,b)\}$

A relation

- A relation over integers ($\mathbb{Z}$) is that of **parity**: the two values are both even, or the two values are both odd.
- Enumerate?
- Domain? Range?

Reflexivity

Relations (for example, R over $A \times A$) that are reflexive comply with:

$$\forall a \in A, (a, a) \in R$$

For example, x is greater than or equal to y is reflexive, while x is greater than y is not (counter example: (1,1) is not in that relation).

Symmetry

Relations that are symmetric comply with:

$$\forall a, b \in A, (a, b) \in R \leftrightarrow (b, a) \in R$$

For example, x is equal to y is symmetric, but x is greater than y is not (counter example: 4 $>$ 3 but 3 $\not>$ 4).

Transitivity

Relations that are transitive comply with:

$$\forall a, b, c \in A, ((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R$$

For example, x is greater than or equal to y is transitive, while x is 3 greater than y is not (counter example: (7,4) and (4,1) means (7,1) should also be present, which it is not).

Equivalence: our example

A relation over integers ($\mathbb{Z}$) is that of **parity**: the two values are both even, or the two values are both odd.

- reflexive For every value x, (x, x) is part of the relation, e.g., $(1, 1), (3, 3)$
- symmetric For every pair (x, y), (y, x) is part of the relation, e.g. $(1, 3), (3, 1)$
- transitive For every x, y, z if (x, y) and (y, z) are part of the relation, so is (x, z), e.g., $(1, 3), (3, 5)$ so $(1, 5)$ is also included

Classes

An equivalence class is defined as:

$$[a]_R = \{b : (a, b) \in R\}$$

- $\forall a \in A, a \in [a]_R$ (reflexive)
- If $a \in [b]_R$, then $b \in [a]_R$ and $[a]_R = [b]_R$ (symmetry)
- For any $a, b \in A$, either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \varnothing$

Classes

- Take 1 from the parity relation. In $[1]_R$ we find $\{..., -3, -1, 1, 3, 5, 7, 9...\}$
- Take 2 from the parity relation. In $[2]_R$ we find $\{..., -4, -2, 0, 2, 4, 6, 8, 10...\}$
- Take 3 from the parity relation. In $[3]_R$ we find $\{..., -3, -1, 1, 3, 5, 7, 9...\}$
- ...

Partitions

A partition occurs when we can divide all the elements in a set S into distinct groups. The partition is a number of subsets ($A, B... \in P$) such that:

- $A, B... \in P$ are not empty
- For any $A, B... \in P$, either $A = B$ or $A \cap B = \varnothing$
- The union of all sets $A, B... \in P$ = S

Equivalence classes lead us to the partition.

Bringing people together

- Given certain conditions, we can combine relations using composition.
- Given relations $R \subseteq A \times B$ and $S \subseteq B \times C$, the composition $S \circ R$ is the set of ordered pairs (a, c) such that $(a, x) \in R$ and $(x, c) \in S$, where x is the same element in both pairs.
- Formally $S \circ R = \{(a, c) \in A \times C \mid \exists x \in B : (a, x) \in R \wedge (x, c) \in S\}$.

Previously ...

- We looked at relations.
- Relations define how objects are related to each other.
- Equivalence classes and partitions allow us to group similar objects.
- Composing relations allows us to build more information about how objects are related.

What is a function?

- Given two sets A and B, if each element of set A can be assigned a unique element of set B, the collection of such assignments is called a function from A into B.
- Set A is described as the domain of the function and set B is called the codomain. If f denotes a function from A into B, we can write $f : A \to B$.
- Formally a function f is a relation from A to B (that is, a subset of $A \times B$) such that each $a \in A$ belongs to a unique ordered pair (a, b) in f.

Which is a function?

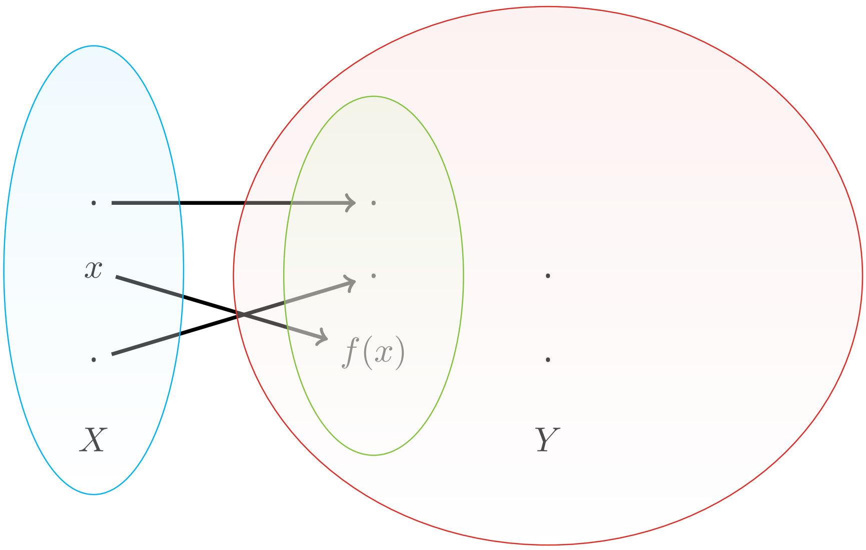
Given the sets $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, which is a function?

1. $\{(1, a), (2, c)\}$
2. $\{(1, a), (2, b), (3, c), (2, c)\}$
3. $\{(1, a), (2, c), (3, b)\}$
4. $\{(1, a), (2, c), (3, c)\}$

Terminology

Given $f : X \to Y, f(x)$:

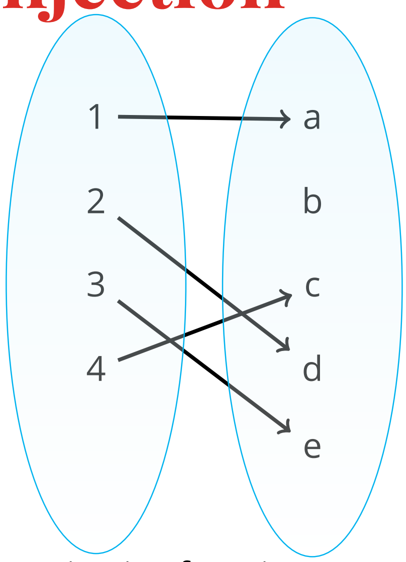
- Domain X
- Co-domain Y
- Image $f(x)$



Partial functions

A partial function is where not all of the domain is used, only a subset.
Outputs for the rest of the domain are undefined.
Division is a partial function over $\mathbb{R}$, as dividing by 0 is undefined.

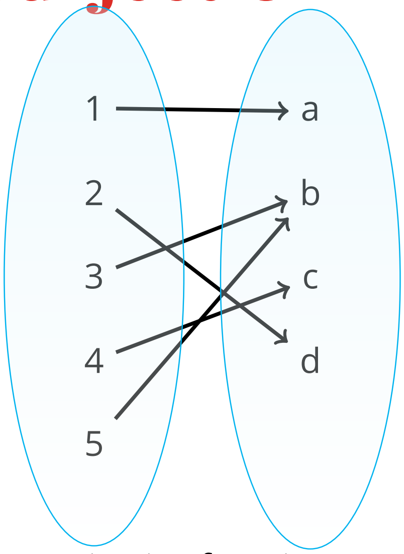
Injection



$\forall a, b \in X, f(a) = f(b) \rightarrow a = b$
where X is the domain of f

Injective functions are where each value in the co-domain is used
at most once.

Surjection



$\forall b \in Y, \exists a \in X, f(a) = b$
where X is the domain of f and Y is the co-domain

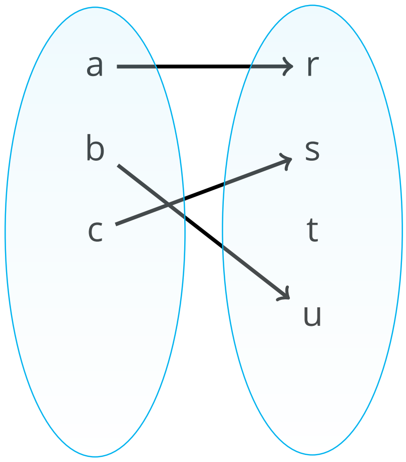
Surjective functions are where each value in the co-domain is used at least once.

Examples

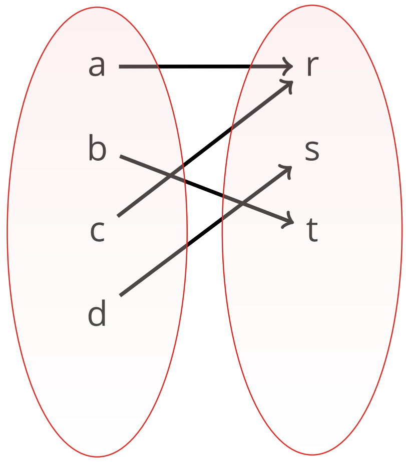
- $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = 2x$ is injective as the values in the co-domain are used 0 or 1 times.
- $f : \mathbb{R} \rightarrow \mathbb{Z}, f(x) = floor(x)$ is surjective as the values in the co-domain are used for more than 1 input.
- $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ is neither as the values in the co-domain less than 0 are not used at all (fails surjection) and the values above 0 are used twice for a^2 and $(-a)^2$ have the same answer (fails injection).

More examples

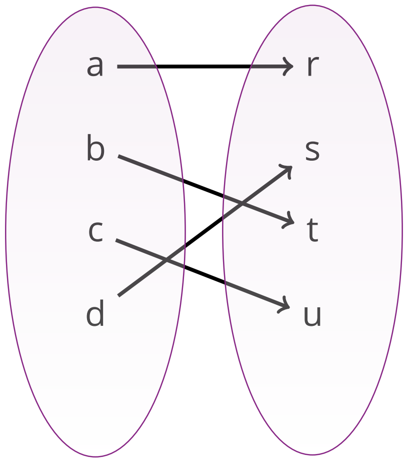
Injective



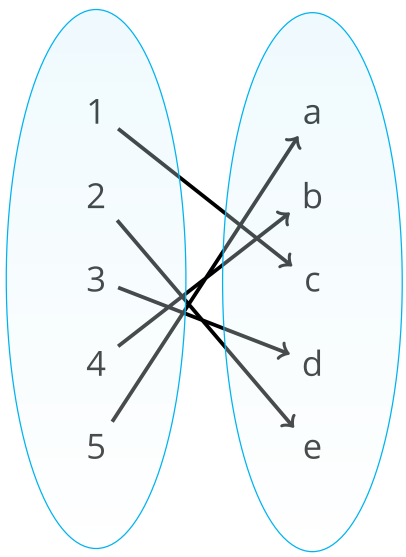
Surjective



Both!



Bijections

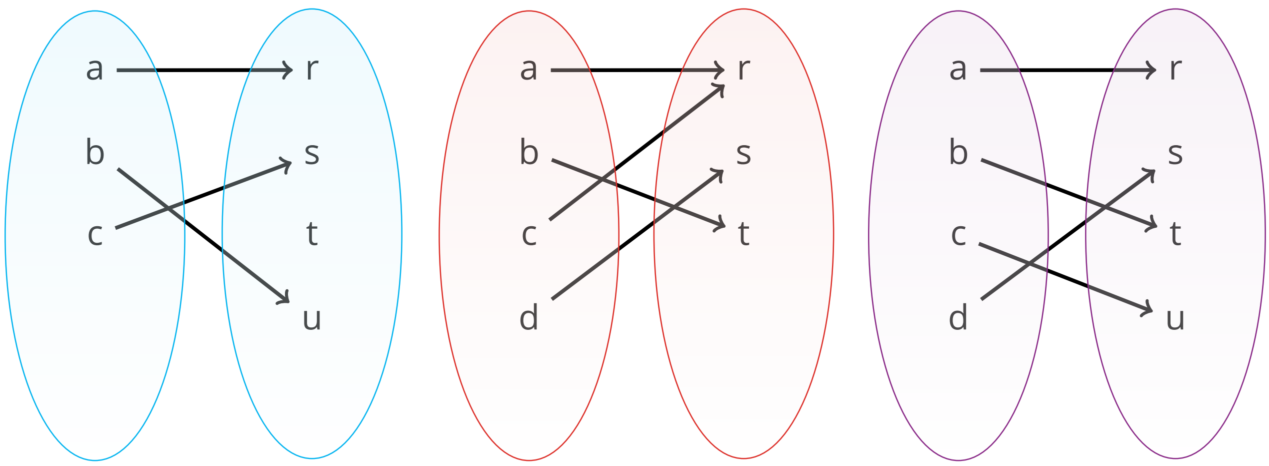


Bijective functions are where the function is both injective and surjective.

Why is bijection important?

- Only bijective functions have an inverse.
- This is important for many computing applications, including encryption.

Inverses?



How to calculate inverse

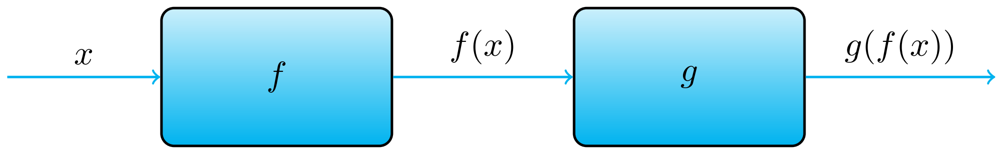
Substitute f^{-1} for the dependent variable and solve for x.

Example: $f : \mathbb{R} \to \mathbb{R}, f(x) = 5x + 1$

Substitute: $x = 5f^{-1}(x) + 1$, $f^{-1}(x) = \frac{x-1}{5}$

Composition

When the co-domain of f is equal to the domain of g, f and g can be composed. This produces $g(f(x))$ or $g \circ f(x)$.



Two functions

If $f : A \to B$ and $g : B \to C$, the composite function of f and g is
$g \circ f : A \to C, g \circ f(x) = g(f(x))$
For example: if $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = 2x - 1$, we get:
$g \circ f : \mathbb{R} \to \mathbb{R}, g \circ f(x) = g(f(x)) = g(x^3) = 2x^3 - 1$

Two functions

If $f : R \to R, f(x) = x^3$ and $g : R \to R, g(x) = 2x - 1$, what is $f \circ g$?

Is this the same as $g \circ f$?

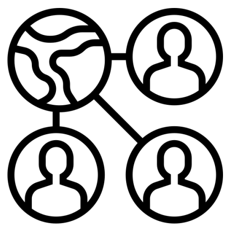
Going further

- If two functions are injective, is their composition injective?
- If two function are surjective, is their composition surjective?
- If a composition of two functions is surjective, what can we say about the two underlying functions?

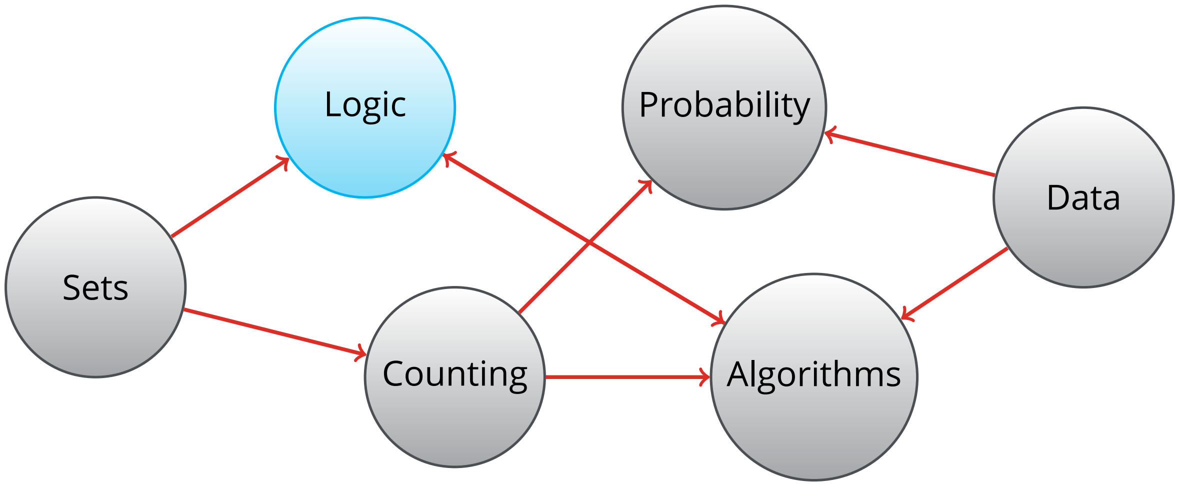
Answers on a postcard.

Reflecting

- How do we define how objects are related to each other?
- What is an approach to grouping similar objects together?
- How do we transform objects?
- And which property is needed to return an object to its original state?



Where to next?



In which we apply logic to circuits.

Sets revision Relations Properties Composition Functions Function properties Next

Lecture 5

Relations and functions

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SWINBURNE UNIVERSITY OF TECHNOLOGY

Semester 1 2021

Questions I still have

Topics I need to review