Lecture 4
Logic

COS10003 Computer Logic and Essentials (Hawthorn)

SWINBURNE UNIVERSITY OF TECHNOLOGY

Semester 1 2021

Today

1. Propositions
2. Connectives
3. Truth tables
4. Laws
5. Arguments
6. Predicate

The key elements of propositional logic

Working with symbolic statements

Resolving basic arguments

Propositions

- A proposition is simply a statement which is true or false.
- These are known as truth values, and variables can take one of these values.
- Propositional logic is used when the truth of a statement can be immediately determined.

Which are propositions?

Propositions:

- It is warm.
- Today is Wednesday.
- ATC is blue.

Not propositions:

- What is the time?
- x is greater than 0.
- Walk this way.

Propositions need to be answerable by either true or false.

Combining propositions

- Propositions by themselves can only get us so far.
- What if we want to say today is Wednesday and ATC is blue?
- How about if today is Wednesday, then ATC is blue?
- What about ATC is not blue?
- We need a way of combining propositions.

Negation $\neg$

- ATC is blue is **false**.
- ATC is not blue is **true**.
- ATC is white is **true**.
- ATC is not white is **false**.

Negation $\neg$

p	$\neg p$
false	**true**
true	**false**

Negation negates a statement, effectively putting a not in front of it. It is similar to complement in sets.

Or $\vee$

- Today is Wednesday or ATC is white is **true**.
- Today is Wednesday or ATC is blue is **true**.
- Today is Saturday or ATC is white is **true**.
- Today is Saturday or ATC is blue is **false**.

Or is used when we need at least one of many things to be true. It is similar to union in sets. Note in English we often use OR to mean either, not both. In logic both is acceptable.

Or $\vee$

Given two variables p and q:

p	q	$p \vee q$
true	true	**true**
true	false	**true**
false	true	**true**
false	false	**false**

And $\wedge$

- Today is Wednesday and ATC is white is **true**.
- Today is Wednesday and ATC is blue is **false**.
- Today is Saturday and ATC is white is **false**.
- Today is Saturday and ATC is blue is **false**.

And is used when we need all statements to be true. It is similar to intersection in sets.

And $\wedge$

p	q	$p \wedge q$
true	true	**true**
true	false	**false**
false	true	**false**
false	false	**false**

Using connectives

Let "it is cold" be denoted by C and "it is sunny" be denoted by S.

1. It is cold and sunny.
   - $C \wedge S$
2. It is not cold.
3. It is cold or sunny.
4. It is cold and not sunny.

Using connectives

Let “it is warm” be denoted by W and “it is raining” be denoted by R.

1. $R \wedge \neg W$
   - It is raining and not warm.
2. $\neg R$
3. $W \vee R$
4. $\neg(W \vee R)$

Material implication $\rightarrow$

- Implication is used for if-then statements (“if p then q” = $p \rightarrow q$).
- Note this is very different to if-statements when programming.
- In logic it is a promise of sorts.
- Note that $q \rightarrow p$ does not have the same truth value as $p \rightarrow q$.

Implication

- If today is Wednesday, then it is lecture day is **true**.
- If today is Wednesday, then it is not lecture day is **false**.
- If today is not Wednesday, then it is lecture day is **true**.
- If today is not Wednesday, then it is not lecture day is **true**.

Material implication $\rightarrow$

p	q	$p \rightarrow q$
true	true	**true**
true	false	**false**
false	true	**true**
false	false	**true**

Recall that impication is only false for true $\rightarrow$ false.

Some other properties

Sticking with if it is cloudy, I bring my umbrella ($C \rightarrow B$)

- This is **not** the same as $B \rightarrow C$ (converse)
- This is **the same** as $\neg B \rightarrow \neg C$ (contrapositive)
- This is **not** the same as $\neg C \rightarrow \neg B$ (inverse)

Biconditional $\leftrightarrow$

- Biconditional is used for if-and-only-if statements ("p if and only if q", $p \leftrightarrow q$).

Biconditional $\leftrightarrow$

- Today is Wednesday if and only if it is lecture day is **true**.
- Today is Wednesday if and only if it is not lecture day is **false**.
- Today is not Wednesday if and only if it is lecture day is **false**.
- Today is not Wednesday if and only if it is not lecture day is **true**.

Biconditional $\leftrightarrow$

p	q	$p \leftrightarrow q$
true	true	**true**
true	false	**false**
false	true	**false**
false	false	**true**

Biconditional is true when both statements are true or both statements are false.

Using connectives

Let "it is cold" be denoted by C and "it is sunny" be denoted by S.

1. If it is cold, then it is sunny.
    - $C \to S$
2. If it is sunny, then it is not cold.
3. It is sunny if and only if it is not cold.

Using connectives

Let "it is warm" be denoted by W, "it is raining" be denoted by R, and "it is sunny" be denoted by S.

1. $R \to W$
    - If it is raining, then it is warm.
2. $\neg W \to \neg R$
3. $W \leftrightarrow S$
4. $S \to W \vee R$

Summary: symbols for connectives

This is also the order of precedence, from highest to lowest.

- $\neg$ (not)
- $\wedge$ (and)
- $\vee$ (or)
- $\rightarrow$ (implication, if-then)
- $\leftrightarrow$ (biconditional, if and only if)

Lecture 4
Logic

COS10003 Computer Logic and Essentials (Hawthorn)

Semester 1 2021

Formalising calculating expressions

- So far we have looked at statements with one or two propositions, but in the real world we work with many statements and more complex statements.
- For this purpose we use truth tables.

Refreshing connectives

p	$\neg p$
false	**true**
true	**false**

And another connective

p	q	
true	true	**true**
true	false	**true**
false	true	**true**
false	false	**false**

Look carefully at the final column to determine which connective this is.

And for those at home

What is the logical formula shown in this truth table?

p	q	
true	true	**true**
true	false	**false**
false	true	**false**
false	false	**false**

Size of a truth table

- What is noticable about the number of rows in these truth tables?
- It is dependent on the number of variables n, in particular it is 2^n.
- Note that so far we are only looking at one propositional statement; let's look at some more complex examples.

Working with truth tables

How would I work with something like $\neg(p \wedge q) \vee (\neg p \wedge q)$?

- Write out columns for variables.
- Rule up enough rows (for n variables, it's 2^n rows) plus one for a header.
- Fill in the variables, ideally in order (I like starting from all true and ending at all false). Remember your binary counting!
- Set up intermediate values as columns and fill in the truth values.
- The last column should be the final value of the statement.

Working with truth tables

How would I work with something like $\neg(p \wedge q) \vee (\neg p \wedge q)$?

p	q	$(p \wedge q)$	$\neg(p \wedge q)$	$\neg p$	$(\neg p \wedge q)$	$\neg(p \wedge q) \vee (\neg p \wedge q)$
T	T	T	F	F	F	F
T	F	F	T	F	F	T
F	T	F	T	T	T	T
F	F	F	T	T	F	T

Tautology

Take a statement such as ATC is white or ATC is not white.

This can be represented as $a \vee \neg a$.

The truth table for this looks like:

a	$\neg a$	$a \vee \neg a$
true	false	true or false = **true**
false	true	false or true = **true**

A tautology is always true, regardless of the variable values.

Contradiction

Take a statement such as today is Wednesday and today is not Wednesday.
This can be represented as $a \wedge \neg a$.
The truth table for this looks like:

a	$\neg a$	$a \wedge \neg a$
true	false	true and false = **false**
false	true	false and true = **false**

A contradiction is always false, regardless of the variable values.

For you to do

Are these tautologies or contradictions or neither?

- $F \rightarrow a$
- $p \wedge (q \vee p)$
- $e \leftrightarrow \neg e$

Equivalency

Two statements are equivalent if their truth values are the same for the same variable values.

If today is Wednesday, then it is lecture day: $W \to L$

W	L	$W \to L$
true	true	**true**
true	false	**false**
false	true	**true**
false	false	**true**

It is not Wednesday or it is lecture day: $\neg W \lor L$

W	L	$\neg W \lor L$
true	true	**true**
true	false	**false**
false	true	**true**
false	false	**true**

Simplifying expressions

- Sometimes we are given a complex expression that could be simplified.
- This is where the laws of logic come in useful.

An example

- It isn't not sunny == It is sunny.
- $\neg\neg S \equiv S$
- This is known as the law of double negation.

S	$\neg S$	$\neg\neg S$
true	false	**true**
false	true	**false**

Another example

- It is not warm nor raining == It is not warm and not raining.
- $\neg(W \vee R) \equiv \neg W \wedge \neg R$
- This is known as De Morgan's Law.

W	R	$W \vee R$	$\neg(W \vee R)$	$\neg W \wedge \neg R$
true	true	true	**false**	**false**
true	false	true	**false**	**false**
false	true	true	**false**	**false**
false	false	false	**true**	**true**

Laws of logic (1)

- Commutative: $a \lor b \equiv b \lor a$, $a \land b \equiv b \land a$
- Associative: $a \lor (b \lor c) \equiv (a \lor b) \lor c$, $a \land (b \land c) \equiv (a \land b) \land c$
- Distributive: $a \lor (b \land c) \equiv (a \lor b) \land (a \lor c)$, $a \land (b \lor c) \equiv (a \land b) \lor (a \land c)$
- Identity: $a \land T \equiv a$, $a \lor F \equiv a$
- Domination: $a \land F \equiv F$, $a \lor T \equiv T$
- Idempotent: $a \land a \equiv a$, $a \lor a \equiv a$

Laws of logic (2)

- Complement or negation: $a \lor \neg a \equiv T$, $a \land \neg a \equiv F$
- Absorption: $a \lor (a \land b) \equiv a$, $a \land (a \lor b) \equiv a$
- De Morgan's: $\neg(a \lor b) \equiv \neg a \land \neg b$, $\neg(a \land b) \equiv \neg a \lor \neg b$
- Involution: $\neg\neg a \equiv a$
- Implication law: $a \to b \equiv \neg a \lor b$
- Equivalence law: $a \leftrightarrow b \equiv (a \to b) \land (b \to a)$

For you to do

Given the laws that are in the handout, which law fits best?

$\neg\neg A \equiv A$	$A \wedge (A \vee B) \equiv A$	$A \wedge B \equiv B \wedge A$	$A \wedge \text{True} \equiv A$
absorption	domination	double negation	commutative

Simplifying statements

We want to reduce logical expressions to a simpler form by removing redundancies.

$$p \vee \neg(\neg p \rightarrow q)$$

implication $p \vee \neg(\neg\neg p \vee q)$

double negation $p \vee \neg(p \vee q)$

De Morgan's $p \vee (\neg p \wedge \neg q)$

distributive $(p \vee \neg p) \wedge (p \vee \neg q)$

complement $T \wedge (p \vee \neg q)$

commutative $(p \vee \neg q) \wedge T$

identity $p \vee \neg q$

Simplifying statements

There are no guidelines for this, although $\leftrightarrow$ and $\rightarrow$ should be elimated first.

This expression should end up at $\neg p \vee \neg q$.

$$\neg(p \wedge q) \vee (\neg p \wedge q)$$

De Morgan's $(\neg p \vee \neg q) \vee (\neg p \wedge q)$

Resolving arguments

- One of the key uses for logic is the resolution of arguments.

Given a problem

- Jim is enrolled in COS10003 or he is not enrolled in business studies.
- If Jim is enrolled in business studies then he is not enrolled in COS10003.
- Therefore Jim is enrolled in COS10003.

Premises and conclusions

- The statement usually at the end that begins with ”therefore” is the conclusion.
- The other statements leading up to that point are the premises.
- For an argument to be valid, if the premises are true, then the conclusion must be true.

Given a problem

- Jim is enrolled in COS10003 or he is not enrolled in business studies. ($P_1 = e \vee \neg b$)
- If Jim is enrolled in business studies then he is not enrolled in COS10003. ($P_2 = b \rightarrow \neg e$)
- Therefore Jim is enrolled in COS10003. ($C = e$)

We need to show that $P_1 \wedge P_2 \models C$: this can be done using truth tables and showing that $(P_1 \wedge P_2) \rightarrow C$ is always true.

Returning to our problem

e	b	$e \vee \neg b$	$b \rightarrow \neg e$	e	$P_1 \wedge P_2 \rightarrow C$
T	T	T	F	T	T
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	T	F	F

No dice. The argument does not hold.

Altering the problem

- Jim is enrolled in COS10003 or he is not enrolled in business studies. ($P_1 = e \vee \neg b$)
- If Jim is enrolled in business studies then he is not enrolled in COS10003. ($P_2 = b \rightarrow \neg e$)
- Therefore Jim is not enrolled in business studies. ($C = \neg b$)

Returning to our problem

e	b	$e \vee \neg b$	$b \rightarrow \neg e$	$\neg b$	$P_1 \wedge P_2 \rightarrow C$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T

The argument does hold.

Another approach

Obviously this works with a small number of variables; larger numbers of variables means a larger truth table. In that case, you might need to use another approach.
If the argument can be expressed as $P_1 \wedge P_2 \wedge P_3 ... \wedge P_n \rightarrow C$, then this could be simplified to get to a smaller truth table.
Another approach is to falsify the conclusion, and then try to find a contradiction in the premises. If no contradiction can be found, this means that the premises can be true when the conclusion is false, so the argument is invalid.

A preview and more notation

- Propositional logic is known as zeroth-order logic.
- The next level is first-order logic or predicate logic.
- This uses many of the concepts from propositional logic and will return in second or third year.

Predicate

- A predicate is a statement containing one or more variables.
- If values are assigned to all the variables in a predicate, the statement becomes a proposition.
- Therefore $x > 10$ is a predicate, however if we give x a value, the resulting statement is a proposition.
- Alternatively we can use quantifiers to determine truth values.

Quantifiers

Universal quantifier $\forall$
For every single possible value, the truth value is consistent.
$\forall x > 0, \text{absolute}(x) = x$

Existential quantifier $\exists$
There is a value for which the statement holds true.
$\exists x > 0, \text{prime}(x)$

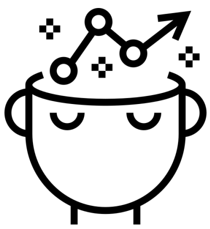
Which of the following are true?

Given the natural numbers, $x > 0$:

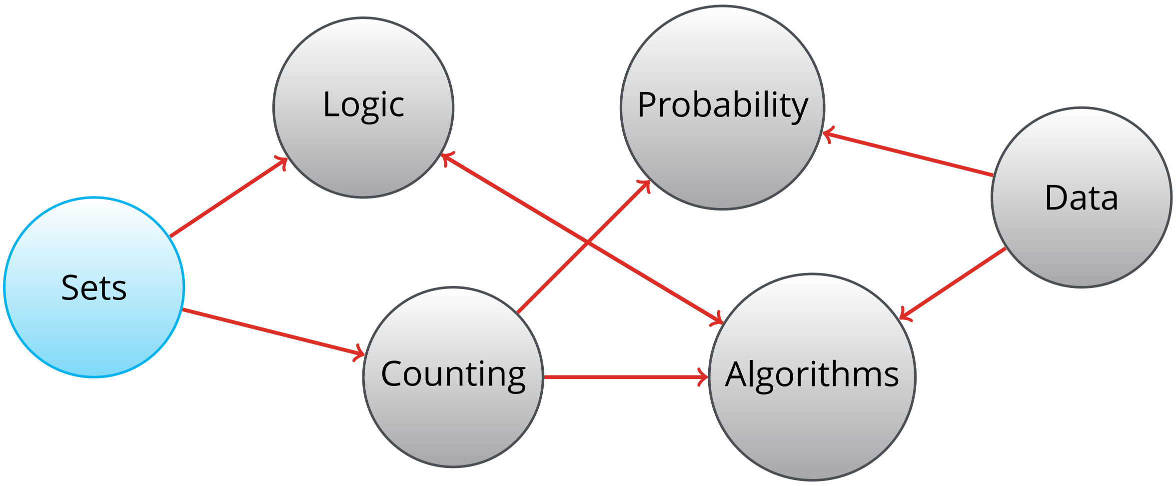
- $\forall x, x < 10$
- $\forall x, x > -10$
- $\exists x, x < 10$
- $\forall x \forall y, x < y$
- $\exists x \exists y, x + y = 5$

Reflecting

- What is a proposition?
- What are the connectives used in logic?
- What are the key approaches to showing equivalency of two logic statements?
- What needs to happen for an argument to hold?



Where to next?



In which we look at relations and functions.

Lecture 4

Logic

COS10003 Computer Logic and Essentials (Hawthorn)

SWINBURNE UNIVERSITY OF TECHNOLOGY

Semester 1 2021

Questions I still have

Topics I need to review