Lecture 7
Counting

COS10003 Computer Logic and Essentials (Hawthorn)

SWINBURNE UNIVERSITY OF TECHNOLOGY

Semester 1 2021

Today

1. Counting
2. Combinations
3. Permutations
4. Pascal's triangle
5. Identical objects
6. Pigeonhole

The principles of counting

Different approaches to counting objects

Overview

- Counting means finding out how many of a particular thing there are, without counting each one separately.
- Counting is on the surface rather easy, however can get tricky.
- This topic is also known as combinatorics or combinatorial analysis or enumeration.
- It has applications to many computing related fields, including cryptography, algorithms, and probability.

What you need to know

- What power sets are
- How to calculate a factorial (n!)
- The formulas for calculating combinations and permutations

Power sets

- Power set: the collection of all subsets of a set. For example: the power set of $A = \{1, 2, 3\}$ is $\mathcal{P}(A) = [\varnothing, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2\}, \{1, 2, 3\}]$
- The size of a power set $\mathcal{P}(A)$ is $2^{|A|}$.
- This is an first go at combinations.

Factorial

The factorial of a number n is defined as
$n! = 1 \times 2 \times 3 \times ... \times (n–2) \times (n - 1) \times n$

Problems

Let's start with a few problems.

You are having some friends over (say for gaming or TV) and need to go to the supermarket to get supplies.

Problem 0

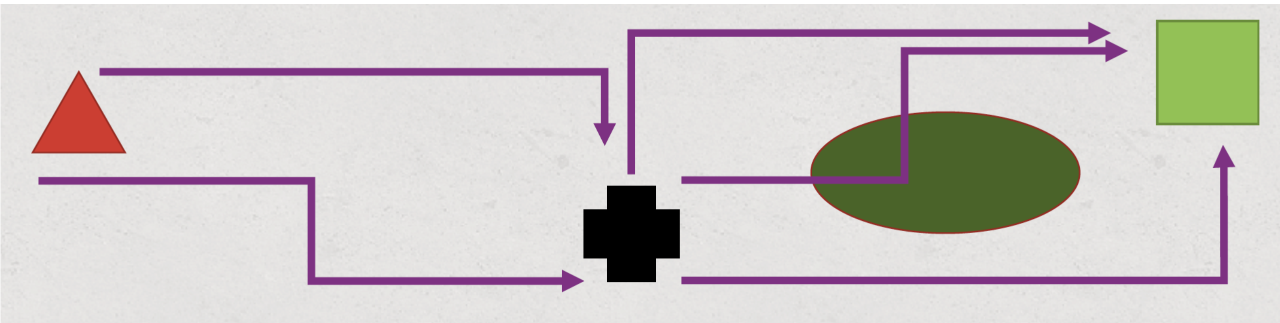
You have 2 hackathon tshirts and 5 Scandinavian metal tshirts that are clean enough to wear. How many tshirts do you have to choose from?

Problem 1

Either getting in your car or going out on the street, you see some cars. As you travel along, you ponder the big question: how many license plates are possible?

Problem 2

En route to the supermarket, there are several routes you can follow. How many different routes can you take to get to the supermarket?



Problem 3

You have reached the supermarket and have a list with five items. How many different ways can you collect the items that you need?

Problem 4

Your ”friends” have only chipped in enough money to purchase 6 bottles of drink. If there are 10 different bottles of drink to choose from at the supermarket, how many different combinations can you purchase?

Problem 5

When you get home, there is only just enough space for three people on the couch. How many different ways are there to arrange the couch seating if there are five people?

Sum principle

The number of elements in disjoint finite sets is the sum of their sizes.
This applies to Problem 0. So 2 + 5 = 7 shirts to choose from.

Product principle

The number of elements in the product of finite sets is the product of their sizes.
This applies to Problems 1 and 2, where you are making a collection or sequence of decisions.

Repetition and ordering

Repetition is where elements could be reused.
Ordering is where the order of elements matters when groups are made.
Problem 5 requires ordering, as who people are sitting next to matters, and so does Problem 3.

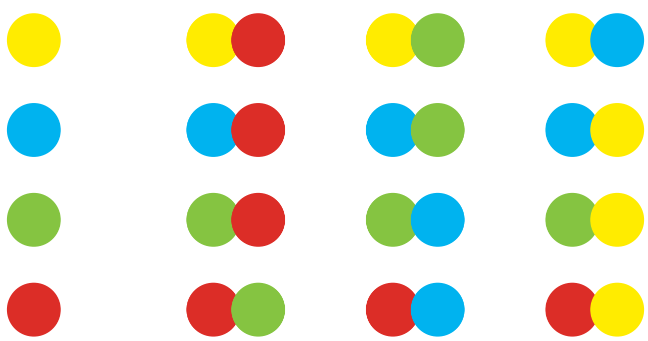
Tips for working with counting problems

- Read the problem carefully, looking for elements of sum and product principles, and repetition and ordering.
- Enumerate the problem, if only partially or a smaller problem.
- Work out what you need to calculate, and which rules are most appropriate.

Attempt 1

Let's select one of our problems, that of selecting drinks.

If we had four bottles of drink to choose two from, how many possibilities are there?



Selecting drinks

One way to think about it would be to select one, then another from what is left over. So four ways of selecting one, then three ways of selecting the rest. Which becomes 4 x 3 = 12.
But a basket with a red and blue bottle is the same as a basket with a blue and red bottle. So we can eliminate those as well, so we get 4 x 3 / 2 = 6 distinct baskets with two bottles.

Combinations

The meaning of $C(n, r)$ is that there are n items and we need to take r items from the n items. Then, the number of outcomes is $C(n, r)$.
The binomial coefficient $C(n, r)$ is defined as:
$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$

Attempt 1 again

Choose 2 bottles from 4.

$$
\begin{aligned}
C(4,2) = \binom{4}{2} &= \frac{4!}{(4-2)!2!} \\
=& \frac{4\times 3\times 2\times 1}{2\times 1\times 2\times 1} \\
=& \frac{4\times 3}{2\times 1} = \frac{12}{2} \\
=& 6
\end{aligned}
$$

Another take

We are assuming only one bottle of each type. What if there are many (unlimited) bottles of each type?

This means we can have multiple of the same bottle in each group, so our equation changes slightly.

$$C(n+r-1,r) = \frac{(n+r-1)!}{r!(n-1)!}$$

Choose 2 from 4 with repetition:

$$\frac{(4+2-1)!}{2!(4-1)!} = \frac{5!}{2!3!} = 10$$

For you to do

If there are 10 different bottles of drink to choose from at the supermarket, how many different combinations are there if you can only purchase 6?

What about with repetition?

Another variation

On the shelf there are 8 cylinder bottles and 6 cube bottles. How many ways can I select 4 bottles? $C(14, 4)$
How about if I wanted 2 cylinder bottles and 2 cube bottles? $C(8, 2) \times C(6, 2)$

And another variation

How about if we wanted at least one cylinder bottle in our set of 4?
We could add up 1 cylinder bottle combinations, 2 cylinder bottle combinations, etc.

Or we could take the opposite (complement?) and subtract 0 cylinder bottle combinations.

For you to do

If there are 3 green bottles and 7 red bottles to choose from at the supermarket, how many different combinations are there if you want 5 bottles:

- Of any colour?
- 2 green and 3 red?
- At most 2 green?

- definitely 3 green?

Note: use C notation as practice for the exam.

Returning to problem 1

Either getting in your car or going out on the street, you see some cars. As you travel along, you ponder the big question: how many license plates are possible?

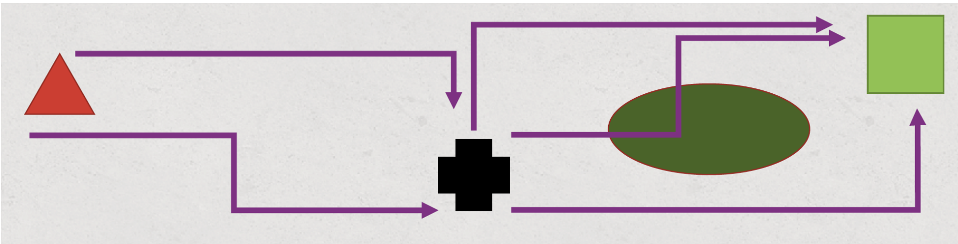
New plates were introduced in 2013, with one digit with two letters plus one digit with two letters. The two digits cannot be 0.

To ponder

Compare this to the old plate system (three letters following by three numbers), how many numberplates were possible?

Problem 2 again

En route to the supermarket, there are several routes you can follow. How many different routes can you take to get to the supermarket?



There are two paths to the crossing, then three paths to the supermarket, so $2 \times 3 = 6$ different paths.
Note the independence of the different options (two separate decisions) so we use multiplication.

Problem 3 again

You have reached the supermarket. How many different ways can you collect the five items that you need?
Note the ordering here matters.

Permutations

A permutation is defined as any arrangement of a set of n objects in a given order.

The formula

The logic used to determine the total number of permutations is as follows:

- The first element of the permutation can be any one of the n elements.
- The second element can be any of the remaining elements ($n-1$ choices).
- the third element is any of the remaining elements ($n-2$ choices).
- etc.

The total number of permutations is the product of the possibilities of each decision, that is, $n \times (n-1) \times (n-2) \times ...2 \times 1$. Hopefully this looks familar to you: it is indeed $n!$.

Problem 3 again

You have reached the supermarket. How many different ways can you collect the items that you need?

Subsets of permutations

We are often interested in taking only a subset to all the possible permutations.
An arrangement of any $r \leq n$ is called an r-permutation or a permutation of n objects taken r at a time. This is often represented symbolically as either $P(n, r)$ or P_r^n.
This is calculated as

$$\frac{n!}{(n-r)!}$$

Problem 5 again

When you get home, there is only just enough space for three people on the couch. How many different ways are there to arrange the couch seating for five people?

If we have 5 people and 3 spots on the couch, then we want $P(5,3) = 5!/2! = 120/2 = 60.$

Counting Combinations Permutations Pascal's triangle Identical objects Pigeonhole Next

For you to try

You have made a shortlist of 6 TV shows/movies to watch/games to play but will only have time for 2 of them. How many different ways can you choose and watch your selections?

I would say ordering matters in this case.

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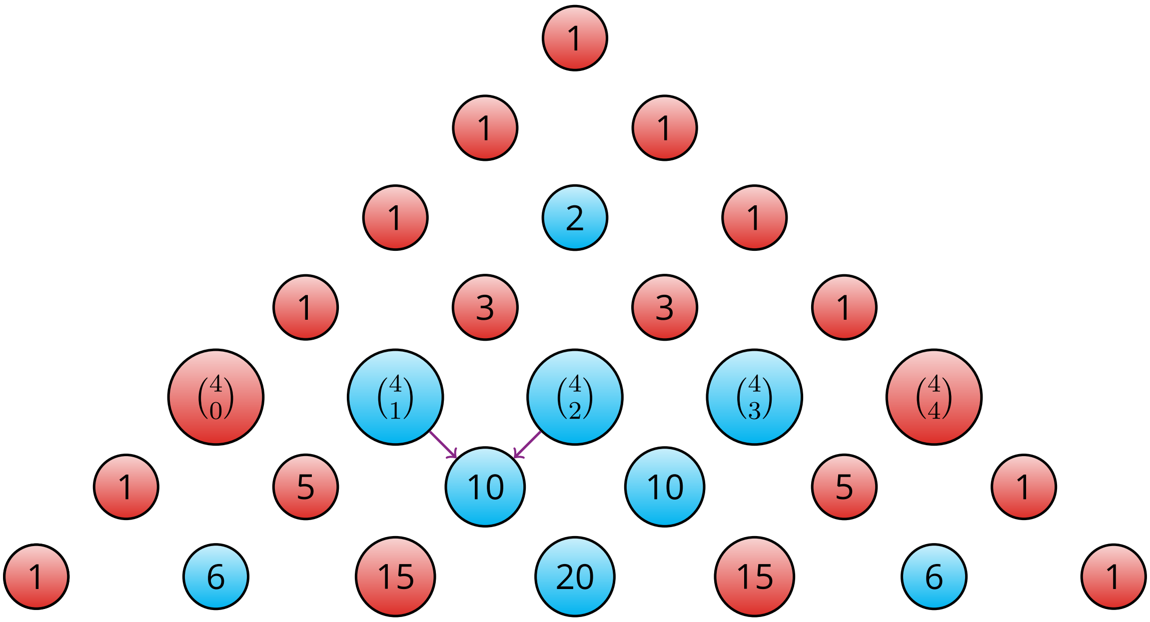
Pascal's triangle

$C(n, r)$ are called binomial coefficients because they are coefficients in the expansion of $(a + b)^n$.

They also appear in Pascal's triangle.



Properties

Fun fact:

$$C(n,r) = C(n-1,r-1) + C(n-1,r)$$

e.g.,$C(5,2) = 10$

$$= C(4,1) + C(4,2) = 4 + 6$$

Another fun fact:

$$C(n,r) = C(n,n-r)$$

e.g.,$C(5,2) = 10, C(5,3) = 10$

Different sorts of proofs

- Algebraic, where we substitute factorial values
- Combinatoric, where we work with combinations

DMOI is a good source for this: see
`http://discrete.openmathbooks.org/dmoi3/sec_comb-proofs.html`

Identical objects

In some cases, we are interested in the number of permutations of some objects that are alike or can be grouped in same way.

Problem 6

You have the usual student mismatched crockery. How many ways can two yellow, one green, one blue and one red plate be shared amongst your guests?

Problem 7

At the supermarket you could only get one box of 12 different cupcakes. How many ways can these be divided amongst your guests if two people get three and three people get two cupcakes?

Identical objects

In general, the number of permutations of n objects of which n_1 are alike in one respect, n_2 are alike in another respect, and so on up to r, is given by the following formula:

$$\frac{n!}{n_1!n_2!...r!}$$

Playing with words

The correctness of this formula can be illustrated by considering a word such as "lolly" in which a letter is repeated.
There are 5! = 120 permutations of the objects l_1, l_2, l_3, o_4, y_5 where the three l's are distinguished. Consider the following permutations.
$l_1l_2l_3oy$, $l_2l_1l_3oy$, $l_3l_1l_2oy$, $l_1l_3l_2oy$, $l_2l_3l_1oy$, $l_3l_2l_1oy$
Each of the above permutations result in the same word when the subscripts are removed, which means there are only 5!/3! = 20 different 5 letter words.

Problem 6 again

You have the usual student mismatched crockery. How many ways can two yellow, one green, one blue and one red plate be shared amongst your guests? Note that two yellow plates are the same, and the others are different. So giving the yellow plates to Gandalf and Lucy or Lucy and Gandalf is the same outcome.

Partitions

This principle also applies when items have to be partitioned in some way, e.g., into groups of particular sizes.
For problem 7 (dividing cupcakes), we get $\frac{12!}{3!\times 3!\times 2!\times 2!\times 2!} = 1663200$ ways of diving up the cupcakes.
Or you could use C(n,r) notation to get $C(12,3)\times C(9,3)\times C(6,2)\times C(4,2)\times C(2,2)$
$= 220\times 84\times 15\times 6\times 1 = 1663200$

Counting socks

If I have a drawer full of sock (let's say black and red) and I take one sock out at a time, how many socks do I need to take before I get a pair?

Pigeonhole principle

If n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

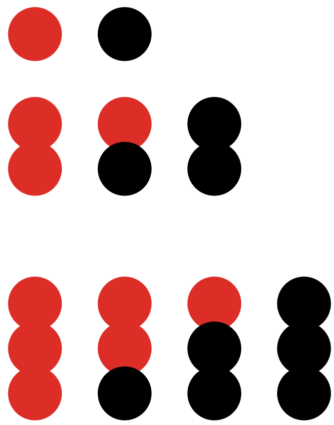
Proving this idea

See Example 3.2.9 in DMOI
(http://discrete.openmathbooks.org/dmoi3/sec_logic-proofs.html).

Formula

For natural numbers k and m, if $n = km + 1$ objects are distributed amongst m sets, then at least one of the sets will contain $k + 1$ or $\lceil \frac{n}{m} \rceil$ objects.

Returning to socks



$k+1$ is 2, so k is 1; m is 2 as there are two pigeonholes, so n is 3.

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Another problem

How many people do we need in a room to be guaranteed at least 5 people born on the same day of the week?

There are seven days in a week, so seven pigeonholes (m). We want $\lceil \frac{n}{m} \rceil$ to be 5, so 29 is the lowest number that makes that work.

Check with $29 = km + 1$: $28 = km$, $k = 4$.

How is this useful?

- Password setting: determining combinations given restrictions or requirements for particular characters
- Algorithm design: how many ways can we proceed through the code?
- Optimisation: developing heuristics for solving combinatorial problems

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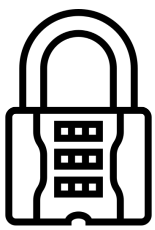
Reflecting

- When would you count combinations rather than permutations?
- Which rule is used when you have identical objects?
- Which principle is useful when you have a series of decisions to make?



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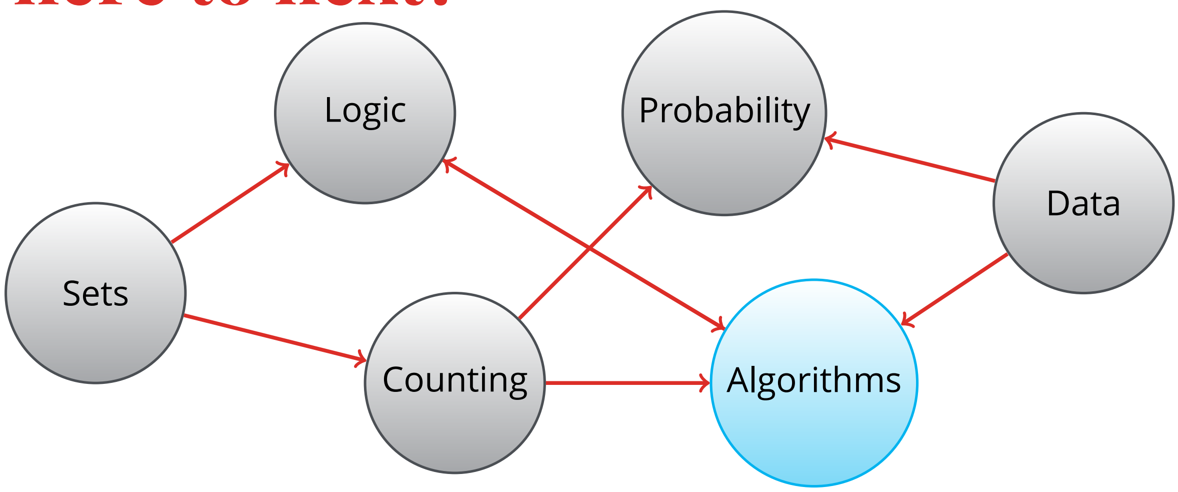
Where to next?



In which we look at algorithmic efficiency.

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Questions I still have

Topics I need to review