Tutorial: Relations and functions

Aim

The aim of this tutorial is for students to be able to work confidently with functions and relations.

Questions

1. If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$, and the relation R from A to B is $R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$:

b) Determine the matrix of the relation.
c) Draw the arrow diagram.

2. Consider $A = 1, 2, 3, 4, 6$. Let R be the relation "x divides y $(x|y)$ if and only if there exists an integer z such that $x \times z = y$". That is, x is a factor of y.

a) Write R as a set of ordered pairs.
b) Draw the directed graph of R.

3. For the following relations, identify which are equivalence relations. Recall $\mathbb{Z}$ is all integers, $\mathbb{Q}$ is rational numbers, and $\mathbb{N}$ is non-negative integers.

a) $R : \{(a, b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}, a + b \text{ is even}\}$
b) $S : \{(a, b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}, a + b \text{ is odd}\}$
c) $T : \{(a, b) \mid a \in \mathbb{Q} \setminus 0, b \in \mathbb{Q} \setminus 0, a \times b > 0\}$
d) $U : \{(a, b) \mid a \in \mathbb{N}, b \in \mathbb{N}, \mid a - b \mid \leq 2\}$

4. Which of the following is a valid partition on the set $S = \{a, b, c, d\}$?

a) $\{\{b, c\}, \{c, d\}, \{a\}\}$
b) $\{\{b, d\}, \{a, c\}, \varnothing\}$
c) $\{\{b, d\}, \{a, c\}\}$
d) $\{\{b, d\}, \{c\}\}$

5. For each of the diagrams in figure Question 5, Digraphs; determine whether it depicts a function from $A = \{a, b, c\}$ to $B = \{x, y, z\}$. Give your reasons.
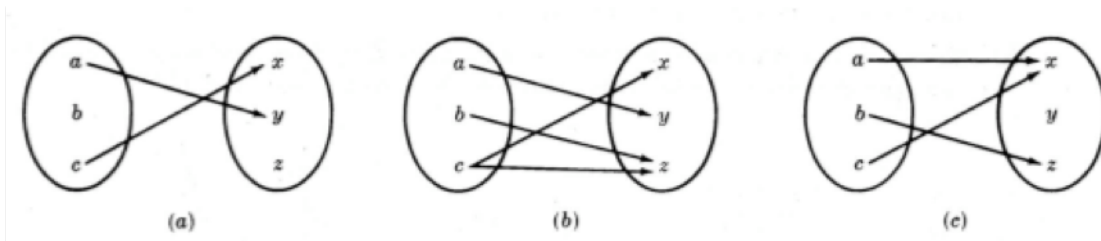

Figure 1: Question 5, Digraphs

6. Identify whether the following are partial or total functions. Give reasons why. $\mathbb{N}$ is the set of natural numbers, the non-negative integers.

a) $f : \mathbb{N} \to \mathbb{N}, f(x) = x/2$
b) $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, g(x, y) = x + y$
c) $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, h(x, y) = x - 2y$

7. Find the inverse functions of the following functions, if they exist. Note $\mathbb{R}$ is the set of all real numbers.

a) $f : \mathbb{R} \to \mathbb{R}, f(x) = 3x + 2$
b) $g : \mathbb{R} \to \mathbb{R}, g(x) = x^2 - 25$
c) $h : \mathbb{R} \to \mathbb{R}, h(x) = |x|$

8. A function $f : \{1, 2, 3, 4, 5\} \to \{0, 1, 2, 3, 4\}$ is defined as $f(n)$ is the remainder of $3n/5$.

a) Draw the arrow diagram.
b) Is f an injection, surjection or both?

9. Consider the following function definitions:

$$f : \mathbb{R} \to \mathbb{R}, f(x) = 4x - 3$$
$$g : \mathbb{R} \to \mathbb{R}, g(x) = x^2 + 1$$
$$h : \mathbb{R} \to \mathbb{R}, h(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\mathbb{R}$ is the set of all real numbers. Determine:

a) $f \circ f$
b) $f \circ g$
c) $h \circ f$

10. Let A be the set of all finite non-null strings of characters.

Let `upper`: $A \to A$ be a function that convert all lower case letters to upper case, leaving all other characters unchanged. The function `lower`: $A \to A$, converts all upper case letters to lower case.

a) Evaluate `upper("Computer Logic Essentials")`.
b) Evaluate `upper` $\circ$ `lower("Computer Logic Essentials")`.