ELTRA: An Embedding Method based on Learning-to-Rank to Preserve Asymmetric Information in Directed Graphs

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APPENDIX A

In this Appendix, we prove that the CRW scores are asymmetric, bounded, monotonic, unique, and always existent.

(1) **Asymmetry**: for every node-pair (u, v) where $u \neq v$, $S(u, v) \neq S(v, u)$.

Proof: According to Equation (2), if $u \neq v$, $S_k(u, v)$ is computed by considering O_u and I_u, while $S_k(v, u)$ is computed by considering O_v and I_v; since $O_u \neq O_v$ and $I_u \neq I_v$, then $S(u, v) \neq S(v, u)$

(2) **Bounding**: for all k, $0 \leq S_k(u, v) \leq 1$.

Proof: According to Equation (2), if $u \neq v$, then $S_0(u, v) = 0$, otherwise $S_0(u, v) = 1$; therefore $0 \leq S_0(u, v) \leq 1$. It means the property holds for $k = 0$. Now, we assume that the property holds for k, which means $0 \leq S_k(u, v) \leq 1$ for *any* node-pairs (u, v); according to the assumption $S_k(u, v) \geq 0$, thus

$$
\begin{aligned}
S_{k+1}(u, v) &= \frac{C}{2} \cdot \Big(\frac{\sum_{i \in O_u} S_k(i, v)}{|O_u|} + \frac{\sum_{j \in I_u} S_k(j, v)}{|I_u|}\Big) \\
&\geq \frac{C}{2} \cdot \Big(\frac{\sum_{i \in O_u}(0)}{|O_u|} + \frac{\sum_{j \in I_u}(0)}{|I_u|}\Big) \\
&\geq 0
\end{aligned}
$$

also, according to the assumption $S_k(u, v) \leq 1$, thus

$$
\begin{aligned}
S_{k+1}(u, v) &= \frac{C}{2} \cdot \Big(\frac{\sum_{i \in O_u} S_k(i, v)}{|O_u|} + \frac{\sum_{j \in I_u} S_k(j, v)}{|I_u|}\Big) \\
&\leq \frac{C}{2} \cdot \Big(\frac{\sum_{i \in O_u}(1)}{|O_u|} + \frac{\sum_{j \in I_u}(1)}{|I_u|}\Big) \\
&\leq \frac{C}{2} \cdot (1 + 1) \\
&\leq C
\end{aligned}
$$

since $0 < C < 1$, then $S_{k+1}(u, v) \leq 1$.

(3) **Monotonicity**: for every node-pair (u, v), the sequence $\{S_k(u, v)\}$ is non-decreasing as k increases.

Proof: If $u = v$, $S_0(u, v) = S_1(u, v) = \cdots = 1$; thus, the property holds. If $u \neq v$, according to Equation (2), $S_0(u, v) = 0$ and by the bounding property, $0 \leq S_1(u, v) \leq 1$; therefore, $S_0(u, v) \leq S_1(u, v)$, which means the property holds for $k = 0$. We assume that the property holds for all k where $S_{k-1}(u, v) \leq S_k(u, v)$ for *any* node-pairs (u, v), which means $S_k(u, v) - S_{k-1}(u, v) \geq 0$; we show the property holds for $k+1$ as follows:

$$
\begin{aligned}
S_{k+1}(u, v) - S_k(u, v) &= \frac{C}{2} \cdot \Big(\frac{\sum_{i \in O_u} S_k(i, v)}{|O_u|} + \frac{\sum_{j \in I_u} S_k(j, v)}{|I_u|}\Big) - \frac{C}{2} \cdot \Big(\frac{\sum_{i \in O_u} S_{k-1}(i, v)}{|O_u|} + \frac{\sum_{j \in I_u} S_{k-1}(j, v)}{|I_u|}\Big) \\
&= \frac{C}{2} \cdot \Big(\frac{\sum_{i \in O_u} S_k(i, v) - S_{k-1}(i, v)}{|O_u|} + \frac{\sum_{j \in I_u} S_k(j, v) - S_{k-1}(j, v)}{|I_u|}\Big) \\
&\geq \frac{C}{2} \cdot \Big(\frac{\sum_{i \in O_u}(0)}{|O_u|} + \frac{\sum_{j \in I_u}(0)}{|I_u|}\Big) = 0
\end{aligned}
$$

according to the assumptions, $S_k(u, v) - S_{k-1}(u, v) \geq 0$ and we already know that $C > 0$ therefore, $S_{k+1}(u, v) - S_k(u, v) \geq 0$, which means $S_{k+1}(u, v) \geq S_k(u, v)$.

(4) **Existence**: the fixed points $S(*,*)$ of the CRW equation always exists.

***Proof*:** By the bounding and monotonicity properties, for any node-pairs (u,v), $S_k(u,v)$ is bounded and non-decreasing as k increases. A sequence $S_k(u,v)$ converges to $\lim S(u,v) \in [0,1]$, according to the Completeness Axiom of calculus. $\lim\limits_{k\to\infty} S_{k+1}(u,v) = \lim\limits_{k\to\infty} S_k(u,v) = S(u,v)$ and the limit of a sum is identical to the sum of the limits, therefore

$$
\begin{aligned}
S(u,v) = \lim_{k\to\infty} S_{k+1} &= \frac{C}{2} \cdot \Big(\frac{\lim\limits_{k\to\infty} \sum\limits_{i\in O_u} S_k(i,v)}{|O_u|} + \frac{\lim\limits_{k\to\infty} \sum\limits_{j\in I_u} S_k(j,v)}{|I_u|} \Big) \\
&= \frac{C}{2} \cdot \Big(\frac{\sum_{i\in O_u} \lim\limits_{k\to\infty} S_k(i,v)}{|O_u|} + \frac{\sum_{j\in I_u} \lim\limits_{k\to\infty} S_k(j,v)}{|I_u|} \Big) \\
&= \frac{C}{2} \cdot \Big(\frac{\sum_{i\in O_u} S(i,v)}{|O_u|} + \frac{\sum_{j\in I_u} S(j,v)}{|I_u|} \Big) \\
&= S(u,v)
\end{aligned}
$$

(5) **Uniqueness**: the solution for the fixed-point $S(*,*)$ is always unique.

***Proof*:** Suppose that $S(*,*)$ and $S'(*,*)$ are two solutions for the CRW equation. Also, for *all* node-pairs (u,v), let $\delta(u,v) = S(u,v) - S'(u,v)$ be the difference between these two solutions. Let $M = \max\limits_{(u,v)} |\delta(u,v)|$ be the maximum absolute value of all differences observed for some nod-pairs (u,v) (i.e., $|\delta(u,v)| = M$). We need to prove that $M=0$. If $u=v$, $M=0$ since $S(u,v) = S'(u,v) = 1$. If $u \neq v$, $S(u,v)$ and $S'(u,v)$ are computed by CRW, and we have

$$
\begin{aligned}
\delta(u,v) &= S(u,v) - S'(u,v) \\
&= \frac{C}{2} \cdot \Big(\frac{\sum_{i\in O_u} S(i,v) - S'(i,v)}{|O_u|} + \frac{\sum_{j\in I_u} S(j,v) - S'(j,v)}{|I_u|} \Big) \\
&= \frac{C}{2} \cdot \Big(\frac{\sum_{i\in O_u} \delta(i,v)}{|O_u|} + \frac{\sum_{j\in I_u} \delta(j,v)}{|I_u|} \Big)
\end{aligned}
$$

thus,

$$
\begin{aligned}
M = |\delta(u,v)| &= \left| \frac{C}{2} \cdot \Big(\frac{\sum_{i\in O_u} \delta(i,v)}{|O_u|} + \frac{\sum_{j\in I_u} \delta(j,v)}{|I_u|} \Big) \right| \\
&\leq \frac{C}{2} \cdot \Big(\frac{\sum_{i\in O_u} |\delta(i,v)|}{|O_u|} + \frac{\sum_{j\in I_u} |\delta(j,v)|}{|I_u|} \Big) \\
&\leq \frac{C}{2} \cdot \Big(\frac{\sum_{i\in O_u} M}{|O_u|} + \frac{\sum_{j\in I_u} M}{|I_u|} \Big) \\
&= \frac{C}{2} \cdot (M+M) = C \cdot M
\end{aligned}
$$

Since $0 < C < 1$, surely $M=0$.