

Lecture 3: Planning by Dynamic Programming

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Outline

- 1 Introduction
- 2 Policy Evaluation
- 3 Policy Iteration
- 4 Value Iteration
- 5 Extensions to Dynamic Programming
- 6 Contraction Mapping

What is Dynamic Programming?

Dynamic sequential or temporal component to the problem

Programming optimising a “program”, i.e. a policy

- c.f. linear programming
- A method for solving complex problems
- By breaking them down into subproblems
 - Solve the subproblems
 - Combine solutions to subproblems

Requirements for Dynamic Programming

Dynamic Programming is a very general solution method for problems which have two properties:

- Optimal substructure
 - *Principle of optimality* applies
 - Optimal solution can be decomposed into subproblems
- Overlapping subproblems
 - Subproblems recur many times
 - Solutions can be cached and reused
- Markov decision processes satisfy both properties
 - Bellman equation gives recursive decomposition
 - Value function stores and reuses solutions

Planning by Dynamic Programming

- Dynamic programming assumes full knowledge of the MDP
- It is used for *planning* in an MDP
- For prediction:
 - Input: MDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$ and policy π
 - or: MRP $\langle \mathcal{S}, \mathcal{P}^\pi, \mathcal{R}^\pi, \gamma \rangle$
 - Output: value function v_π
- Or for control:
 - Input: MDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$
 - Output: optimal value function v_*
 - and: optimal policy π_*

Other Applications of Dynamic Programming

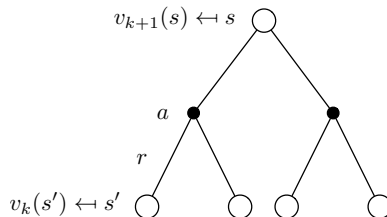
Dynamic programming is used to solve many other problems, e.g.

- Scheduling algorithms
- String algorithms (e.g. sequence alignment)
- Graph algorithms (e.g. shortest path algorithms)
- Graphical models (e.g. Viterbi algorithm)
- Bioinformatics (e.g. lattice models)

Iterative Policy Evaluation

- Problem: evaluate a given policy π
- Solution: iterative application of Bellman expectation backup
- $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_\pi$
- Using *synchronous* backups,
 - At each iteration $k + 1$
 - For all states $s \in \mathcal{S}$
 - Update $v_{k+1}(s)$ from $v_k(s')$
 - where s' is a successor state of s
- We will discuss *asynchronous* backups later
- Convergence to v_π will be proven at the end of the lecture

Iterative Policy Evaluation (2)



$$v_{k+1}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

$$\mathbf{v}^{k+1} = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi \mathbf{v}^k$$

Evaluating a Random Policy in the Small Gridworld



actions

| | | | |
|----|----|----|----|
| | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | |

$r = -1$
on all transitions

- Undiscounted episodic MDP ($\gamma = 1$)
- Nonterminal states 1, ..., 14
- One terminal state (shown twice as shaded squares)
- Actions leading out of the grid leave state unchanged
- Reward is -1 until the terminal state is reached
- Agent follows uniform random policy

$$\pi(n|\cdot) = \pi(e|\cdot) = \pi(s|\cdot) = \pi(w|\cdot) = 0.25$$

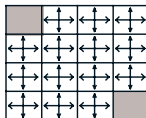
Iterative Policy Evaluation in Small Gridworld

v_k for the
Random Policy

$k = 0$

| | | | |
|-----|-----|-----|-----|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

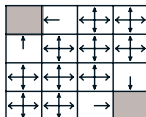
Greedy Policy
w.r.t. v_k



random
policy

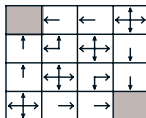
$k = 1$

| | | | |
|------|------|------|------|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |



$k = 2$

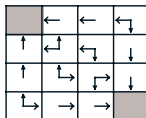
| | | | |
|------|------|------|------|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |



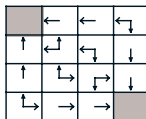
Iterative Policy Evaluation in Small Gridworld (2)

 $k = 3$

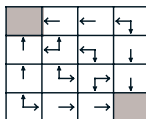
| | | | |
|------|------|------|------|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |


 $k = 10$

| | | | |
|------|------|------|------|
| 0.0 | -6.1 | -8.4 | -9.0 |
| -6.1 | -7.7 | -8.4 | -8.4 |
| -8.4 | -8.4 | -7.7 | -6.1 |
| -9.0 | -8.4 | -6.1 | 0.0 |


 $k = \infty$

| | | | |
|------|------|------|------|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |



optimal
policy

How to Improve a Policy

- Given a policy π
 - **Evaluate** the policy π

$$v_{\pi}(s) = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \dots | S_t = s]$$

- **Improve** the policy by acting greedily with respect to v_{π}

$$\pi' = \text{greedy}(v_{\pi})$$

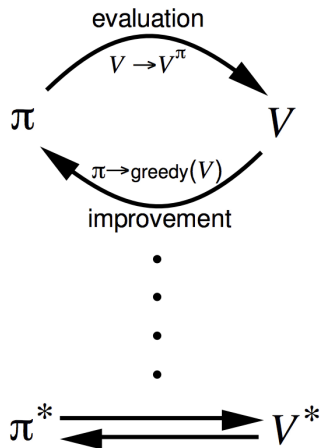
- In Small Gridworld improved policy was optimal, $\pi' = \pi^*$
- In general, need more iterations of improvement / evaluation
- But this process of **policy iteration** always converges to π^*

Policy Iteration



Policy evaluation Estimate v_π
Iterative policy evaluation

Policy improvement Generate $\pi' \geq \pi$
Greedy policy improvement

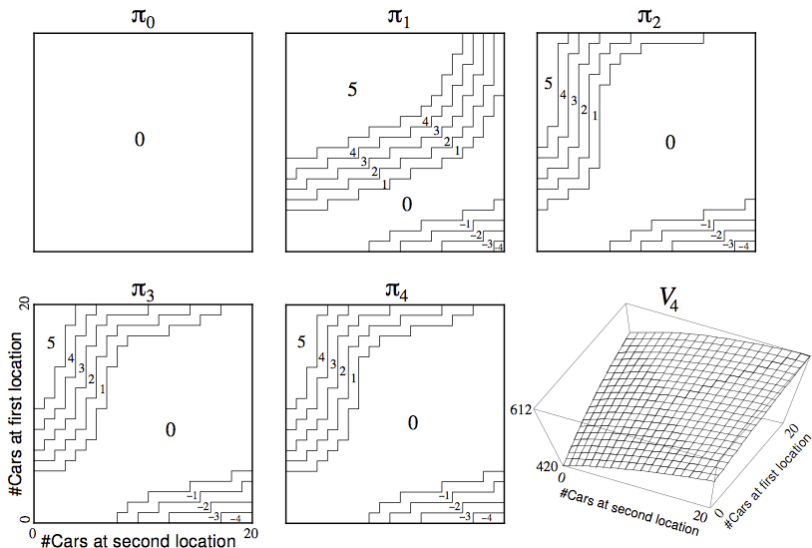


Jack's Car Rental



- States: Two locations, maximum of 20 cars at each
- Actions: Move up to 5 cars between locations overnight
- Reward: \$10 for each car rented (must be available)
- Transitions: Cars returned and requested randomly
 - Poisson distribution, n returns/requests with prob $\frac{\lambda^n}{n!} e^{-\lambda}$
 - 1st location: average requests = 3, average returns = 3
 - 2nd location: average requests = 4, average returns = 2

Policy Iteration in Jack's Car Rental



Policy Improvement

- Consider a deterministic policy, $a = \pi(s)$
- We can *improve* the policy by acting greedily

$$\pi'(s) = \operatorname{argmax}_{a \in \mathcal{A}} q_{\pi}(s, a)$$

- This improves the value from any state s over one step,

$$q_{\pi}(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_{\pi}(s, a) \geq q_{\pi}(s, \pi(s)) = v_{\pi}(s)$$

- It therefore improves the value function, $v_{\pi'}(s) \geq v_{\pi}(s)$

$$\begin{aligned} v_{\pi}(s) &\leq q_{\pi}(s, \pi'(s)) = \mathbb{E}_{\pi'} [R_{t+1} + \gamma v_{\pi}(S_{t+1}) \mid S_t = s] \\ &\leq \mathbb{E}_{\pi'} [R_{t+1} + \gamma q_{\pi}(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s] \\ &\leq \mathbb{E}_{\pi'} [R_{t+1} + \gamma R_{t+2} + \gamma^2 q_{\pi}(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s] \\ &\leq \mathbb{E}_{\pi'} [R_{t+1} + \gamma R_{t+2} + \dots \mid S_t = s] = v_{\pi'}(s) \end{aligned}$$

Policy Improvement (2)

- If improvements stop,

$$q_{\pi}(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_{\pi}(s, a) = q_{\pi}(s, \pi(s)) = v_{\pi}(s)$$

- Then the Bellman optimality equation has been satisfied

$$v_{\pi}(s) = \max_{a \in \mathcal{A}} q_{\pi}(s, a)$$

- Therefore $v_{\pi}(s) = v_{*}(s)$ for all $s \in \mathcal{S}$
- so π is an optimal policy

Modified Policy Iteration

- Does policy evaluation need to converge to v_π ?
- Or should we introduce a stopping condition
 - e.g. ϵ -convergence of value function
- Or simply stop after k iterations of iterative policy evaluation?
- For example, in the small gridworld $k = 3$ was sufficient to achieve optimal policy
- Why not update policy every iteration? i.e. stop after $k = 1$
 - This is equivalent to *value iteration* (next section)

Generalised Policy Iteration

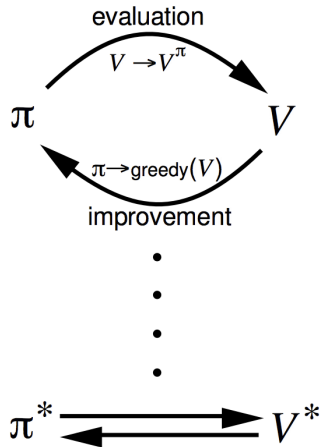


Policy evaluation Estimate v_π

Any policy evaluation algorithm

Policy improvement Generate $\pi' \geq \pi$

Any policy improvement algorithm



Principle of Optimality

Any optimal policy can be subdivided into two components:

- An optimal first action A_*
- Followed by an optimal policy from successor state S'

Theorem (Principle of Optimality)

A policy $\pi(a|s)$ achieves the optimal value from state s , $v_\pi(s) = v_(s)$, if and only if*

- *For any state s' reachable from s*
- *π achieves the optimal value from state s' , $v_\pi(s') = v_*(s')$*

Deterministic Value Iteration

- If we know the solution to subproblems $v_*(s')$
- Then solution $v_*(s)$ can be found by one-step lookahead

$$v_*(s) \leftarrow \max_{a \in \mathcal{A}} \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

- The idea of value iteration is to apply these updates iteratively
- Intuition: start with final rewards and work backwards
- Still works with loopy, stochastic MDPs

Example: Shortest Path

| | | | |
|---|--|--|--|
| g | | | |
| | | | |
| | | | |
| | | | |

Problem

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

 V_1

| | | | |
|----|----|----|----|
| 0 | -1 | -1 | -1 |
| -1 | -1 | -1 | -1 |
| -1 | -1 | -1 | -1 |
| -1 | -1 | -1 | -1 |

 V_2

| | | | |
|----|----|----|----|
| 0 | -1 | -2 | -2 |
| -1 | -2 | -2 | -2 |
| -2 | -2 | -2 | -2 |
| -2 | -2 | -2 | -2 |

 V_3

| | | | |
|----|----|----|----|
| 0 | -1 | -2 | -3 |
| -1 | -2 | -3 | -3 |
| -2 | -3 | -3 | -3 |
| -3 | -3 | -3 | -3 |

 V_4

| | | | |
|----|----|----|----|
| 0 | -1 | -2 | -3 |
| -1 | -2 | -3 | -4 |
| -2 | -3 | -4 | -4 |
| -3 | -4 | -4 | -4 |

 V_5

| | | | |
|----|----|----|----|
| 0 | -1 | -2 | -3 |
| -1 | -2 | -3 | -4 |
| -2 | -3 | -4 | -5 |
| -3 | -4 | -5 | -5 |

 V_6

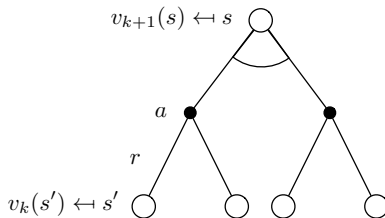
| | | | |
|----|----|----|----|
| 0 | -1 | -2 | -3 |
| -1 | -2 | -3 | -4 |
| -2 | -3 | -4 | -5 |
| -3 | -4 | -5 | -6 |

 V_7

Value Iteration

- Problem: find optimal policy π
- Solution: iterative application of Bellman optimality backup
- $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_*$
- Using synchronous backups
 - At each iteration $k + 1$
 - For all states $s \in \mathcal{S}$
 - Update $v_{k+1}(s)$ from $v_k(s')$
- Convergence to v_* will be proven later
- Unlike policy iteration, there is no explicit policy
- Intermediate value functions may not correspond to any policy

Value Iteration (2)



$$v_{k+1}(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

$$\mathbf{v}_{k+1} = \max_{a \in \mathcal{A}} \mathbf{R}^a + \gamma \mathbf{P}^a \mathbf{v}_k$$

Example of Value Iteration in Practice

<http://www.cs.ubc.ca/~poole/demos/mdp/vi.html>

Synchronous Dynamic Programming Algorithms

| Problem | Bellman Equation | Algorithm |
|------------|--|-----------------------------|
| Prediction | Bellman Expectation Equation | Iterative Policy Evaluation |
| Control | Bellman Expectation Equation + Greedy Policy Improvement | Policy Iteration |
| Control | Bellman Optimality Equation | Value Iteration |

- Algorithms are based on state-value function $v_{\pi}(s)$ or $v_{*}(s)$
- Complexity $O(mn^2)$ per iteration, for m actions and n states
- Could also apply to action-value function $q_{\pi}(s, a)$ or $q_{*}(s, a)$
- Complexity $O(m^2n^2)$ per iteration

Asynchronous Dynamic Programming

- DP methods described so far used *synchronous* backups
- i.e. all states are backed up in parallel
- *Asynchronous DP* backs up states individually, in any order
- For each selected state, apply the appropriate backup
- Can significantly reduce computation
- Guaranteed to converge if all states continue to be selected

Asynchronous Dynamic Programming

Three simple ideas for asynchronous dynamic programming:

- *In-place* dynamic programming
- *Prioritised sweeping*
- *Real-time* dynamic programming

In-Place Dynamic Programming

- Synchronous value iteration stores two copies of value function
for all s in \mathcal{S}

$$v_{new}(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_{old}(s') \right)$$

$$v_{old} \leftarrow v_{new}$$

- In-place value iteration only stores one copy of value function
for all s in \mathcal{S}

$$v(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right)$$

Prioritised Sweeping

- Use magnitude of Bellman error to guide state selection, e.g.

$$\left| \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right) - v(s) \right|$$

- Backup the state with the largest remaining Bellman error
- Update Bellman error of affected states after each backup
- Requires knowledge of reverse dynamics (predecessor states)
- Can be implemented efficiently by maintaining a priority queue

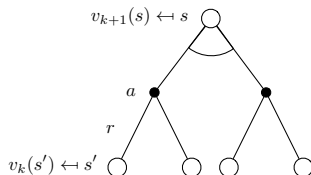
Real-Time Dynamic Programming

- Idea: only states that are relevant to agent
- Use agent's experience to guide the selection of states
- After each time-step S_t, A_t, R_{t+1}
- Backup the state S_t

$$v(S_t) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_{S_t}^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{S_t s'}^a v(s') \right)$$

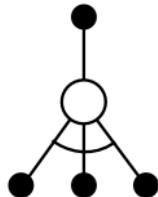
Full-Width Backups

- DP uses *full-width* backups
- For each backup (sync or async)
 - Every successor state and action is considered
 - Using knowledge of the MDP transitions and reward function
- DP is effective for medium-sized problems (millions of states)
- For large problems DP suffers Bellman's *curse of dimensionality*
 - Number of states $n = |\mathcal{S}|$ grows exponentially with number of state variables
- Even one backup can be too expensive



Sample Backups

- In subsequent lectures we will consider *sample backups*
- Using sample rewards and sample transitions
 $\langle S, A, R, S' \rangle$
- Instead of reward function \mathcal{R} and transition dynamics \mathcal{P}
- Advantages:
 - Model-free: no advance knowledge of MDP required
 - Breaks the curse of dimensionality through sampling
 - Cost of backup is constant, independent of $n = |\mathcal{S}|$



Approximate Dynamic Programming

- Approximate the value function
- Using a *function approximator* $\hat{v}(s, \mathbf{w})$
- Apply dynamic programming to $\hat{v}(\cdot, \mathbf{w})$
- e.g. Fitted Value Iteration repeats at each iteration k ,
 - Sample states $\tilde{\mathcal{S}} \subseteq \mathcal{S}$
 - For each state $s \in \tilde{\mathcal{S}}$, estimate target value using Bellman optimality equation,

$$\tilde{v}_k(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \hat{v}(s', \mathbf{w}_k) \right)$$

- Train next value function $\hat{v}(\cdot, \mathbf{w}_{k+1})$ using targets $\{\langle s, \tilde{v}_k(s) \rangle\}$

Some Technical Questions

- How do we know that value iteration converges to v_* ?
- Or that iterative policy evaluation converges to v_π ?
- And therefore that policy iteration converges to v_* ?
- Is the solution unique?
- How fast do these algorithms converge?
- These questions are resolved by *contraction mapping theorem*

Value Function Space

- Consider the vector space \mathcal{V} over value functions
- There are $|\mathcal{S}|$ dimensions
- Each point in this space fully specifies a value function $v(s)$
- What does a Bellman backup do to points in this space?
- We will show that it brings value functions *closer*
- And therefore the backups must converge on a unique solution

Value Function ∞ -Norm

- We will measure distance between state-value functions u and v by the ∞ -norm
- i.e. the largest difference between state values,

$$\|u - v\|_{\infty} = \max_{s \in \mathcal{S}} |u(s) - v(s)|$$

Bellman Expectation Backup is a Contraction

- Define the *Bellman expectation backup operator* T^π ,

$$T^\pi(v) = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v$$

- This operator is a γ -contraction, i.e. it makes value functions closer by at least γ ,

$$\begin{aligned} \|T^\pi(u) - T^\pi(v)\|_\infty &= \|(\mathcal{R}^\pi + \gamma \mathcal{P}^\pi u) - (\mathcal{R}^\pi + \gamma \mathcal{P}^\pi v)\|_\infty \\ &= \|\gamma \mathcal{P}^\pi(u - v)\|_\infty \\ &\leq \|\gamma \mathcal{P}^\pi\| \|u - v\|_\infty \\ &\leq \gamma \|u - v\|_\infty \end{aligned}$$

Contraction Mapping Theorem

Theorem (Contraction Mapping Theorem)

For any metric space \mathcal{V} that is complete (i.e. closed) under an operator $T(v)$, where T is a γ -contraction,

- *T converges to a unique fixed point*
- *At a linear convergence rate of γ*

Convergence of Iter. Policy Evaluation and Policy Iteration

- The Bellman expectation operator T^π has a unique fixed point
- v_π is a fixed point of T^π (by Bellman expectation equation)
- By contraction mapping theorem
- Iterative policy evaluation converges on v_π
- Policy iteration converges on v_*

Bellman Optimality Backup is a Contraction

- Define the *Bellman optimality backup operator* T^* ,

$$T^*(v) = \max_{a \in \mathcal{A}} \mathcal{R}^a + \gamma \mathcal{P}^a v$$

- This operator is a γ -contraction, i.e. it makes value functions closer by at least γ (similar to previous proof)

$$\|T^*(u) - T^*(v)\|_\infty \leq \gamma \|u - v\|_\infty$$

Convergence of Value Iteration

- The Bellman optimality operator T^* has a unique fixed point
- v_* is a fixed point of T^* (by Bellman optimality equation)
- By contraction mapping theorem
- Value iteration converges on v_*