Computer Graphics

Curve and Surface Modeling 1

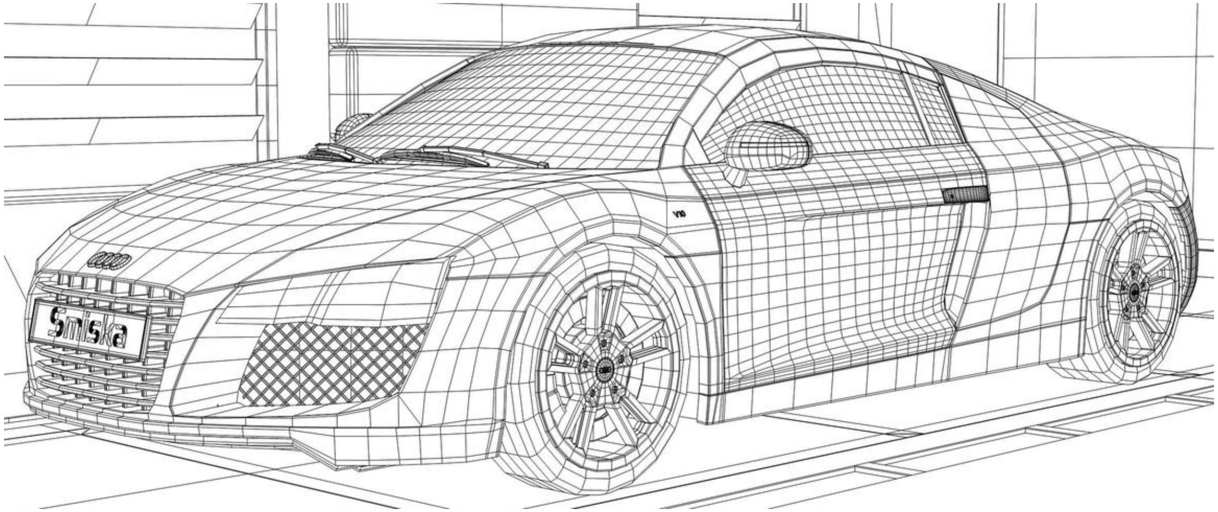
Teacher: Dr. Zhuo SU (苏卓)

E-mail: suzhuo3@mail.sysu.edu.cn

School of Data and Computer Science

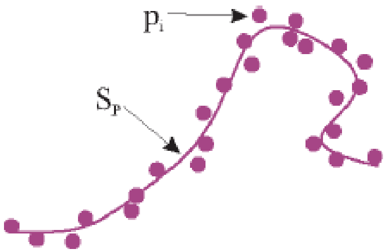
Outline

- Interpolation and Approximation
- Curve Modeling
  - Parametric curve
  - Bézier curve
- Surface Modeling
  - Bézier surface

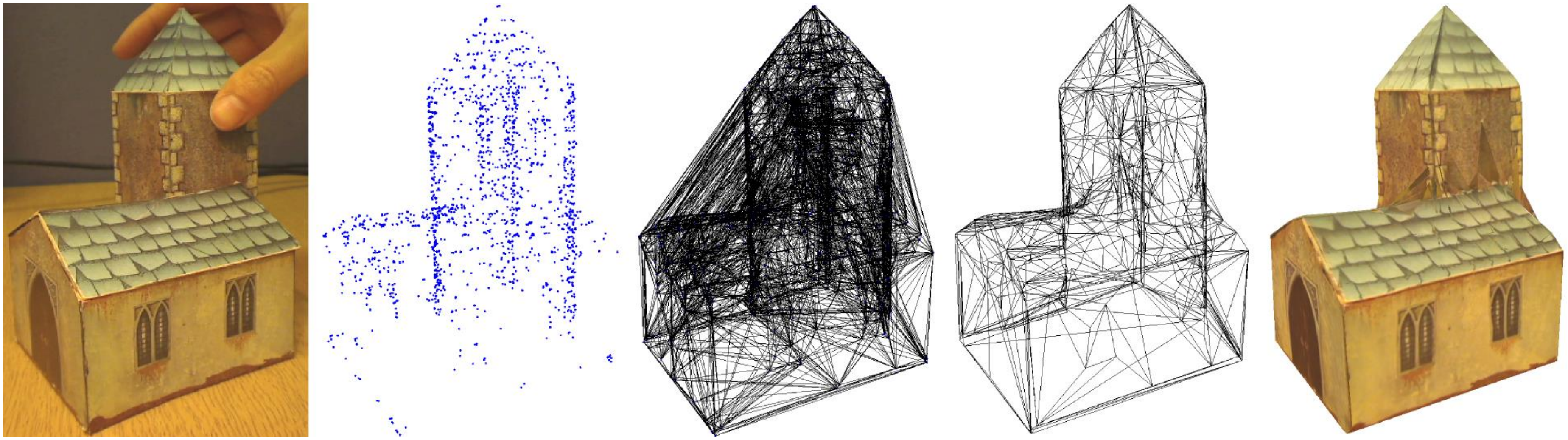


Introduction

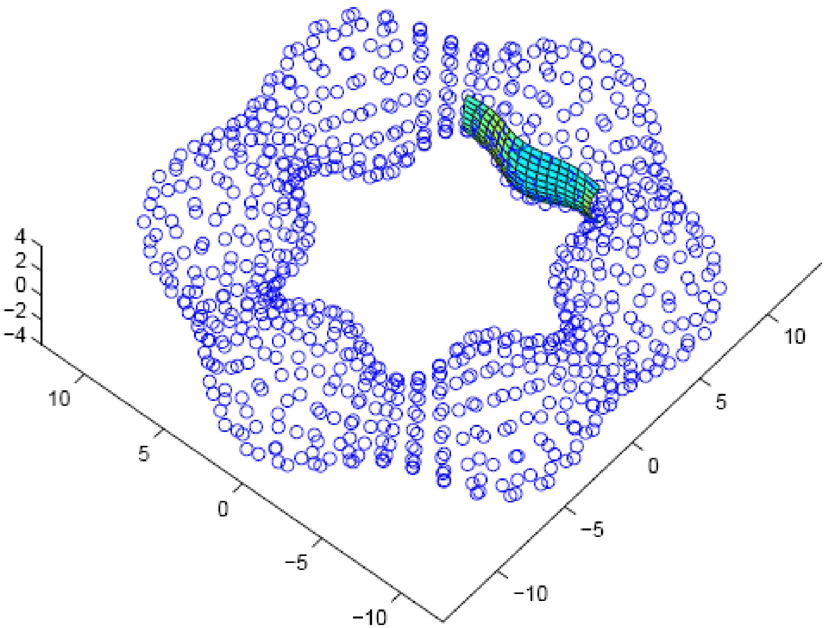
- Raw data is very popular in many experimental study and usually it need fitting before it can be understand well.

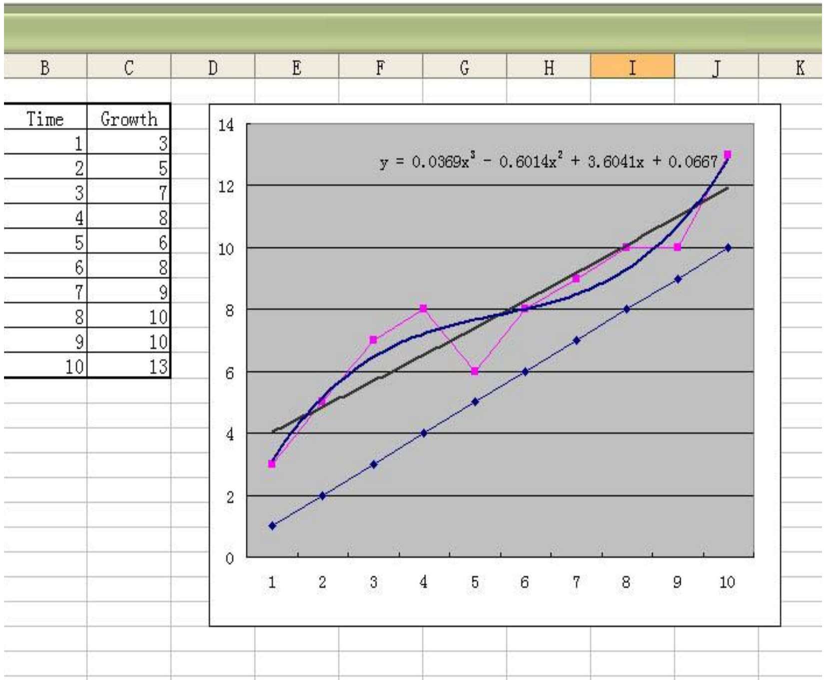


Introduction



Introduction



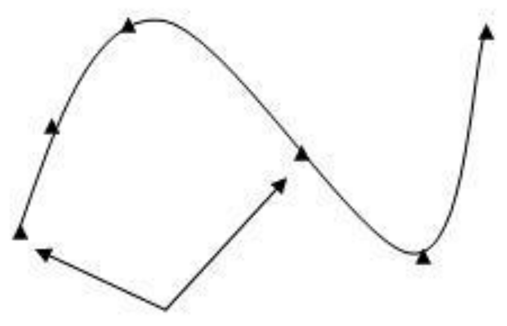


Surface fitting to 3D points

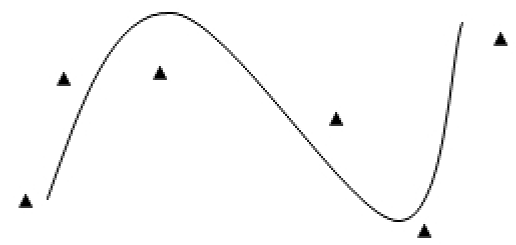
Chart by Microsoft Excel

Interpolation and approximation

- Interpolation: When the curve passes through all the control points then it is called as Interpolation.



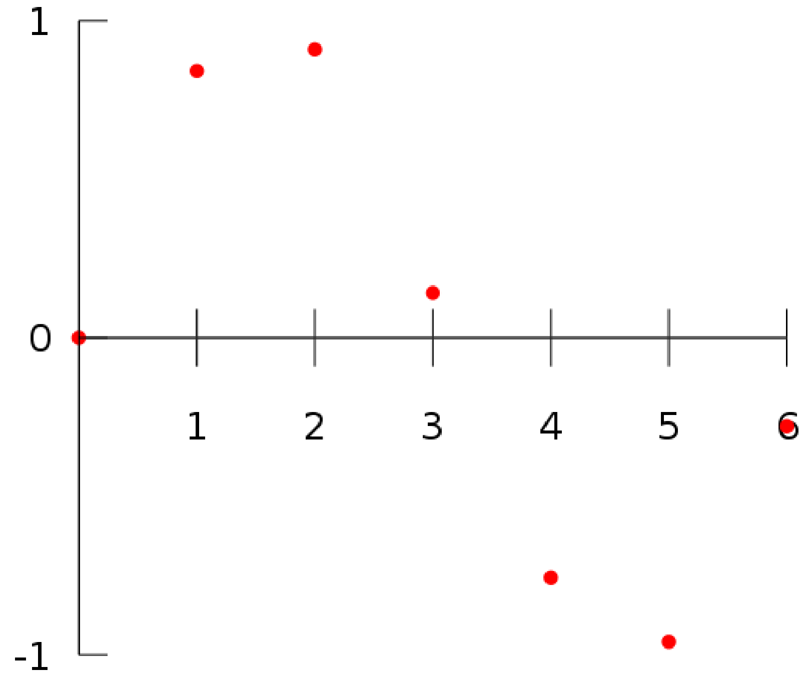
- Approximation: When the curve does not passes through the control points then it is called as Approximation.



Interpolation case

- For example, suppose we have a table like this, which gives some values of an unknown function *f*.

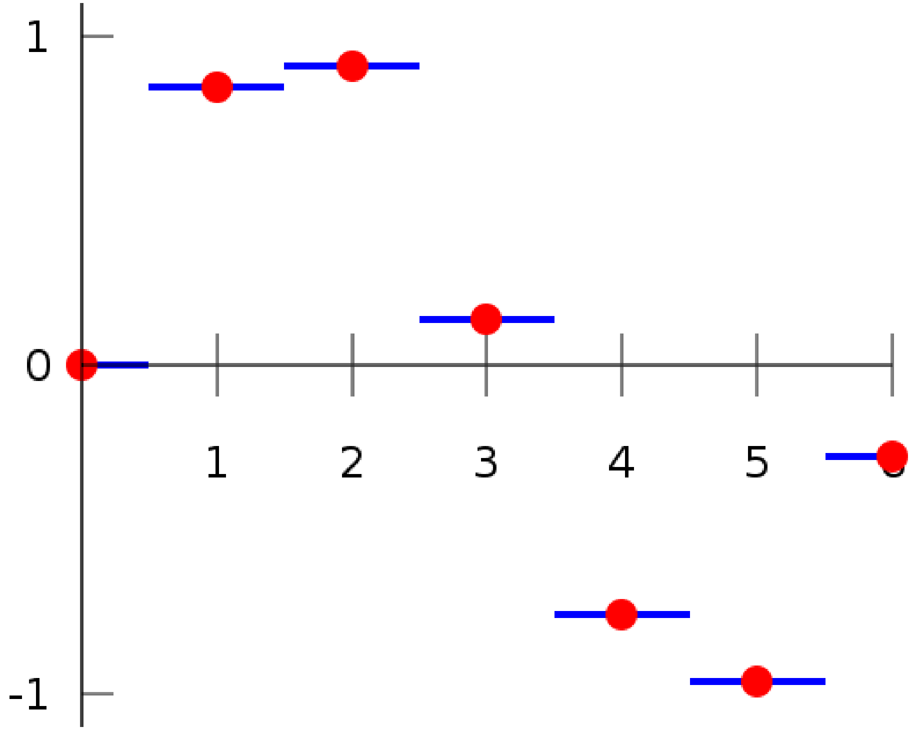
x	$f(x)$
0	0
1	0.8415
2	0.9093
3	0.1411
4	−0.7568
5	−0.9589
6	−0.2794



Interpolation provides a means of estimating the function at intermediate points, such as x = 2.5.

Piecewise constant interpolation

- The simplest interpolation method is to locate the nearest data value, and assign the same value.



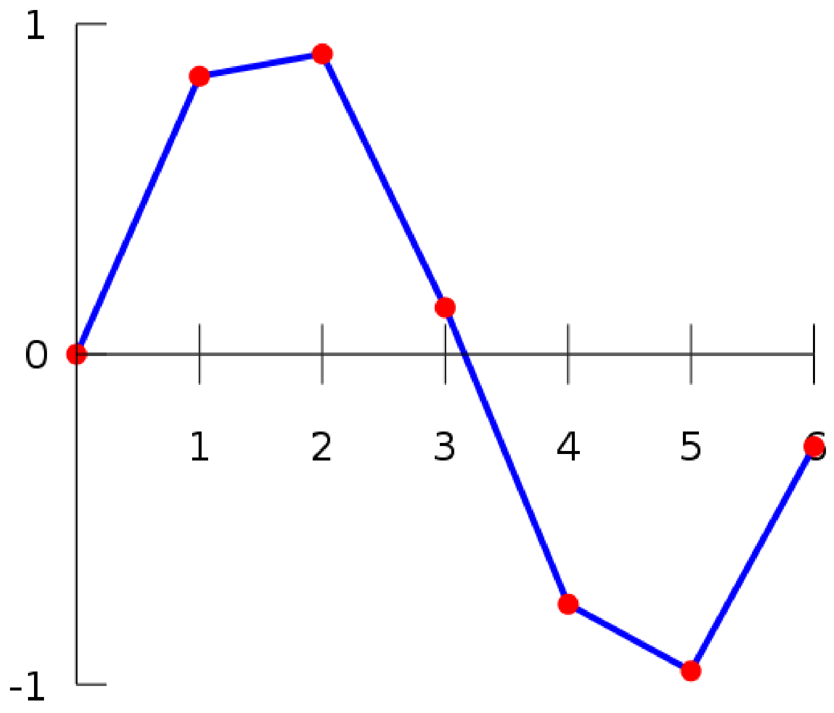
Linear interpolation

- Generally, linear interpolation takes two data points, say (x_a,y_a) and (x_b,y_b), and the interpolant is given by:

$$y = y_a + (y_b - y_a)\,\frac{x - x_a}{x_b - x_a} \text{ at the point } (x, y)$$

$$\frac{y - y_a}{y_b - y_a} = \frac{x - x_a}{x_b - x_a}$$

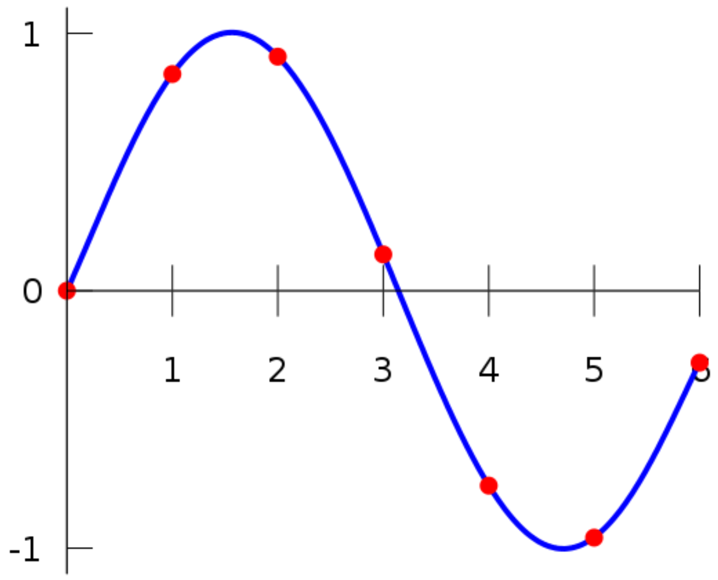
$$\frac{y - y_a}{x - x_a} = \frac{y_b - y_a}{x_b - x_a}$$



Polynomial interpolation

- Polynomial interpolation is a generalization of linear interpolation. Note that the linear interpolation is a linear function. We now replace this interpolation with a polynomial of higher degree.
- The following sixth degree polynomial goes through all the seven points:

$$
\begin{aligned}
f(x) = & -0.0001521x^6 + 0.003130x^5 \\
& +0.07321x^4 - 0.3577x^3 \\
& +0.2255x^2 + 0.9038x
\end{aligned}
$$

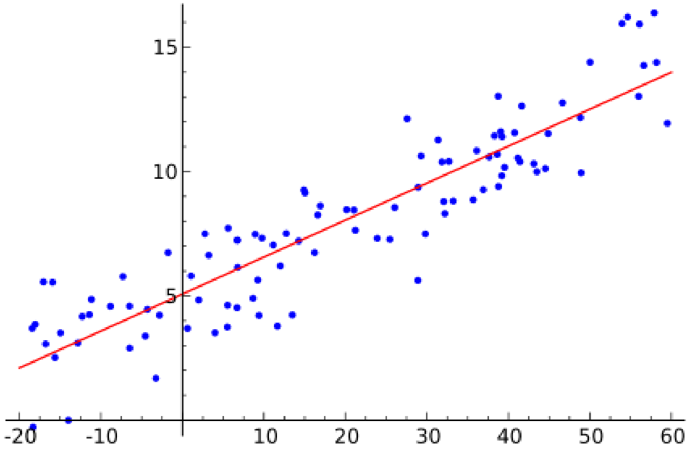


Approximation – Least squares fitting

- Linear least squares
- A fitting model is a linear one when the model comprises a linear combination of the parameters, i.e.,

$$f(x,\beta) = \sum_{j=1}^{m} \beta_j \phi_j(x),$$

where the function ϕ_j is a function of x .



- Letting

$$X_{ij} = \frac{\partial f(x_i,\beta)}{\partial \beta_j} = \phi_j(x_i),$$

- we can then see that in that case the least square estimate (or estimator, in the context of a random sample), β is given by

$$\hat{\beta} = (X^T X)^{-1} X^T \mathbf{y}.$$

Least squares fitting example

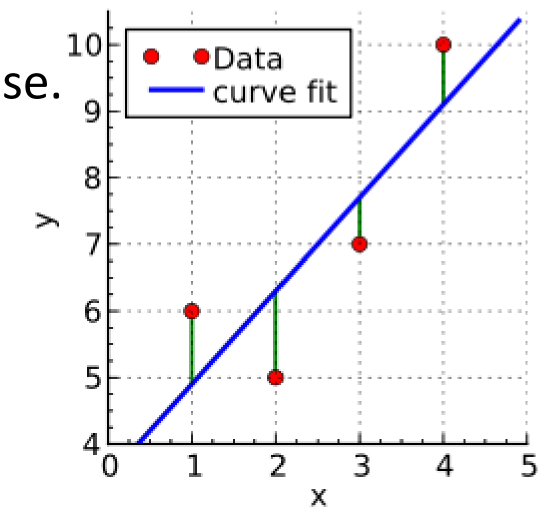
- As a result of an experiment, four (x, y) data points were obtained, (1, 6), (2, 5), (3, 7), and (4, 10).
- We hope to find a line $y = \beta_1 + \beta_2 x$ that best fits these four points. In other words, we would like to find the numbers β_1 and β_2 that approximately solve the over-determined linear system

$$\beta_1 + 1\beta_2 = 6, \quad \beta_1 + 2\beta_2 = 5,$$
$$\beta_1 + 3\beta_2 = 7, \quad \beta_1 + 4\beta_2 = 10.$$

of four equations in two unknowns in some "best" sense.

- A **residual** is defined as the difference between the actual value of the dependent variable and the value predicted by the model.

$$r_i = y_i - f(x_i, \beta).$$



Least squares fitting example

- The "error", at each point, between the curve fit and the data is the difference between the right- and left-hand sides of the equations above. The least squares approach to solving this problem is to try to make the sum of the squares of these errors as small as possible; that is, to find the minimum of the function

$$\begin{aligned} S(\beta_1,\beta_2) &= \left[6-(\beta_1+1\beta_2)\right]^2+\left[5-(\beta_1+2\beta_2)\right]^2 \\ &\quad +\left[7-(\beta_1+3\beta_2)\right]^2+\left[10-(\beta_1+4\beta_2)\right]^2 \\ &= 4\beta_1^2+30\beta_2^2+20\beta_1\beta_2-56\beta_1-154\beta_2+210 \end{aligned}$$

- The minimum is determined by calculating the partial derivatives of $S(\beta_1, \beta_2)$ with respect to β_1 and β_2 and setting them to zero

$$\frac{\partial S}{\partial \beta_1}=0=8\beta_1+20\beta_2-56, \quad \frac{\partial S}{\partial \beta_2}=0=20\beta_1+60\beta_2-154.$$

- This results in a system of two equations in two unknowns, called the normal equations, which give, when solved $\beta_1 = 3.5, \beta_2 = 1.4$,

and the equation $y = 3.5 + 1.4x$ of the line of best fit.

Approximation – Quadratic least squares fitting

- Importantly, in "linear least squares", we are not restricted to using a line as the model as in the above example. For instance, we could have chosen the restricted quadratic model $y = \beta_1 x^2$. This model is still linear in the β_1 parameter, so we can still perform the same analysis, constructing a system of equations from the data points: $6 = \beta_1 (1)^2, \quad 5 = \beta_1 (2)^2$

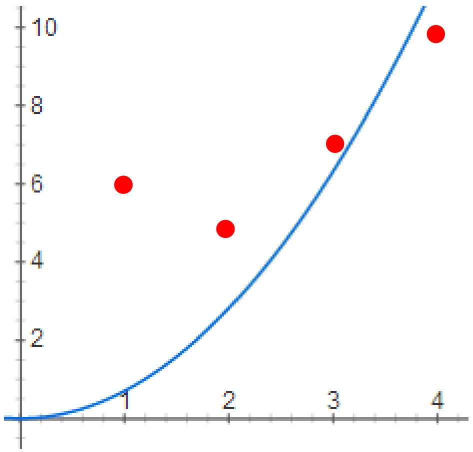
  $7 = \beta_1 (3)^2, \; 10 = \beta_1 (4)^2$

- The partial derivatives with respect to the parameters (this time there is only one) are again computed and set to 0:

  $$\frac{\partial S}{\partial \beta_1} = 0 = 708\beta_1 - 498$$

  and solved $\beta_1 = 0.703x^2$

- leading to the resulting best fit model $y = 0.703x^2$



General - Least Square Methods

- How to draw a curve approximately fitting to raw data?
  - Raw data usually has noise. The values of dependent variables vary even though all the independent variables are constant. Therefore, the estimation of the trend the dependent variables is needed. This process is called regression or curve fitting.
  - The estimated equation (matrix) satisfy the raw data. However, the equation is not usually unique, and the equation or curve with a minimal deviation from all data points is desirable.
  - This desirable best-fitting equation can be obtained by least square method which uses the minimal sum of the deviations squared (偏差的平方和) from a given set of data.

Least square formulation

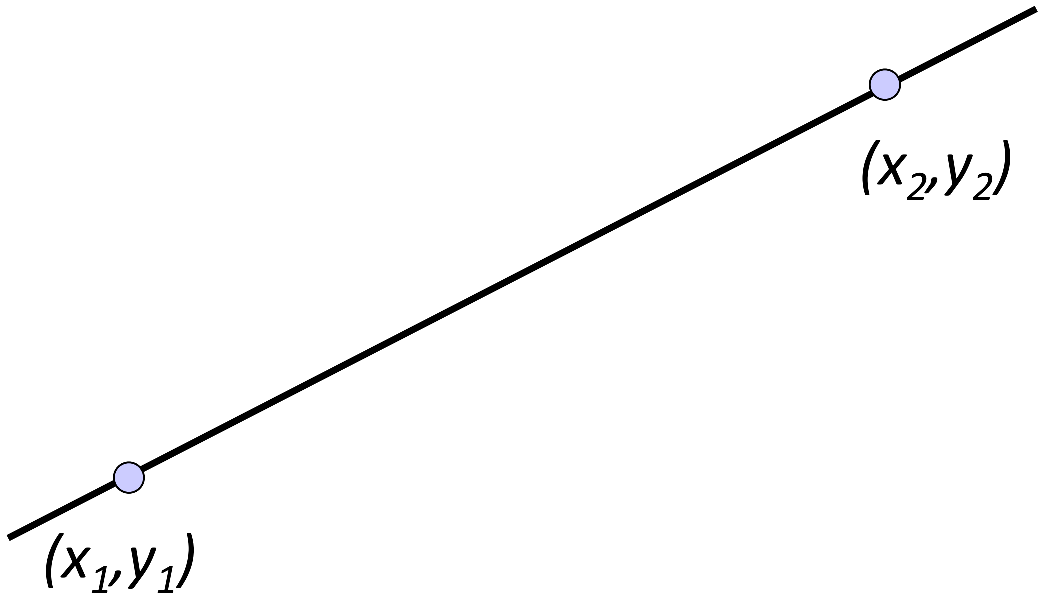
- If you have a data set *(x_1,y_1), (x_2, y_2) , ... , (x_n, y_n)* and the best curve *$f(x)$* should be with the property as follows

Minimum Least Square error

$$E = \sum_{i=1}^{n} (f(x_i) - y_i)^2$$

Least square line

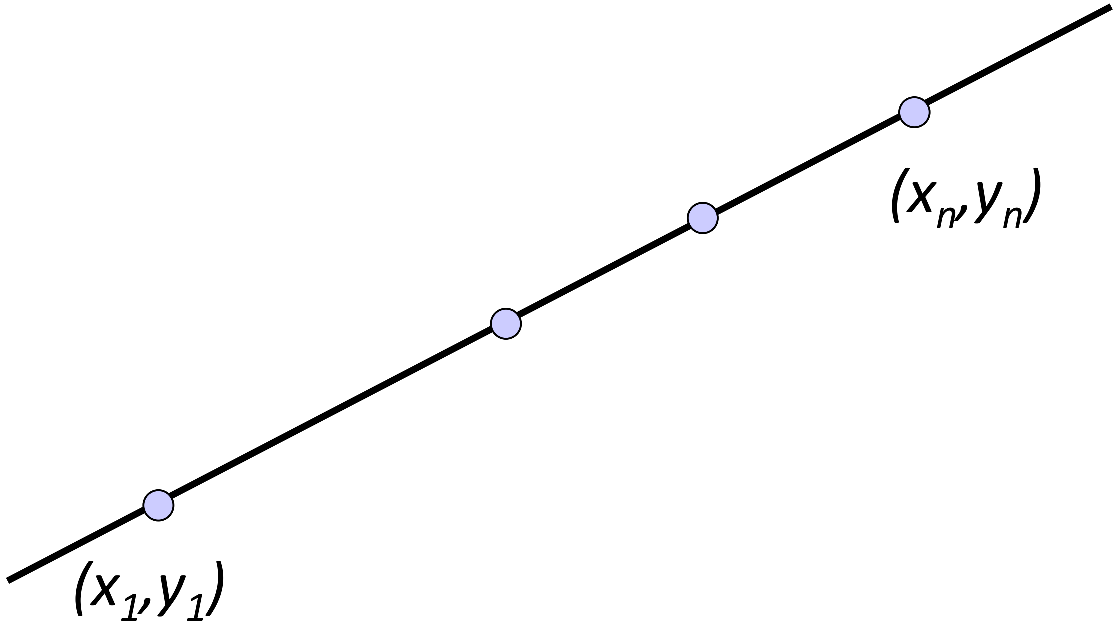
- When $n = 2$, $E = 0$



Least square line

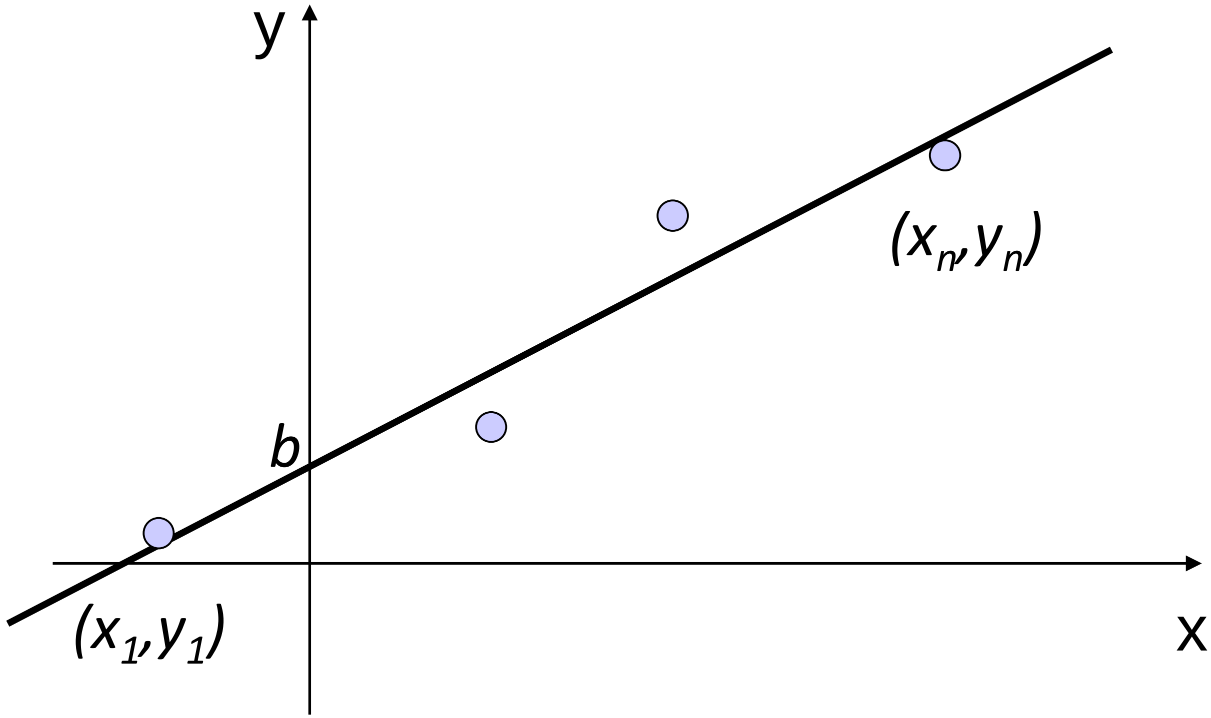
- When $n>2$, if (x_1,y_1), (x_2, y_2) , ... , (x_n, y_n) are collinear,

  $E = 0$



Least square line

- Line equation $y = kx + b$



Least square line

- How to get *k* and *b* ?

$$\begin{cases} kx_1 + b = y_1 \\ kx_2 + b = y_2 \\ \quad\cdots \\ kx_n + b = y_n \end{cases} \quad \text{or} \quad \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & 1 \\ x_n & 1 \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Mostly an approximation solution can exist, when the rank of the coefficient matrix is 2, which is the column number.

Least square line

- How to get k and b ?

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{bmatrix}$$

Least square line

- How to get *k* and *b* ?

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{bmatrix}$$

The unique solution of this system *k* and *b* can satisfy the following condition and a least square line is obtained.

Minimum $E = \sum_{i=1}^{n} (f(x_i) - y_i)^2$

Least square line

- When the coefficient matrix is singular, for example $x_1 = x_2 = \cdots = x_n$, k and b cannot be solved directly

$$\cancel{y = kx + b}$$

$$\cancel{x = ky + b}$$

Or even general parametric form

$$\begin{cases} x = x(t) = a_0 + a_1 t \\ y = y(t) = b_0 + b_1 t \end{cases}$$

Least square line

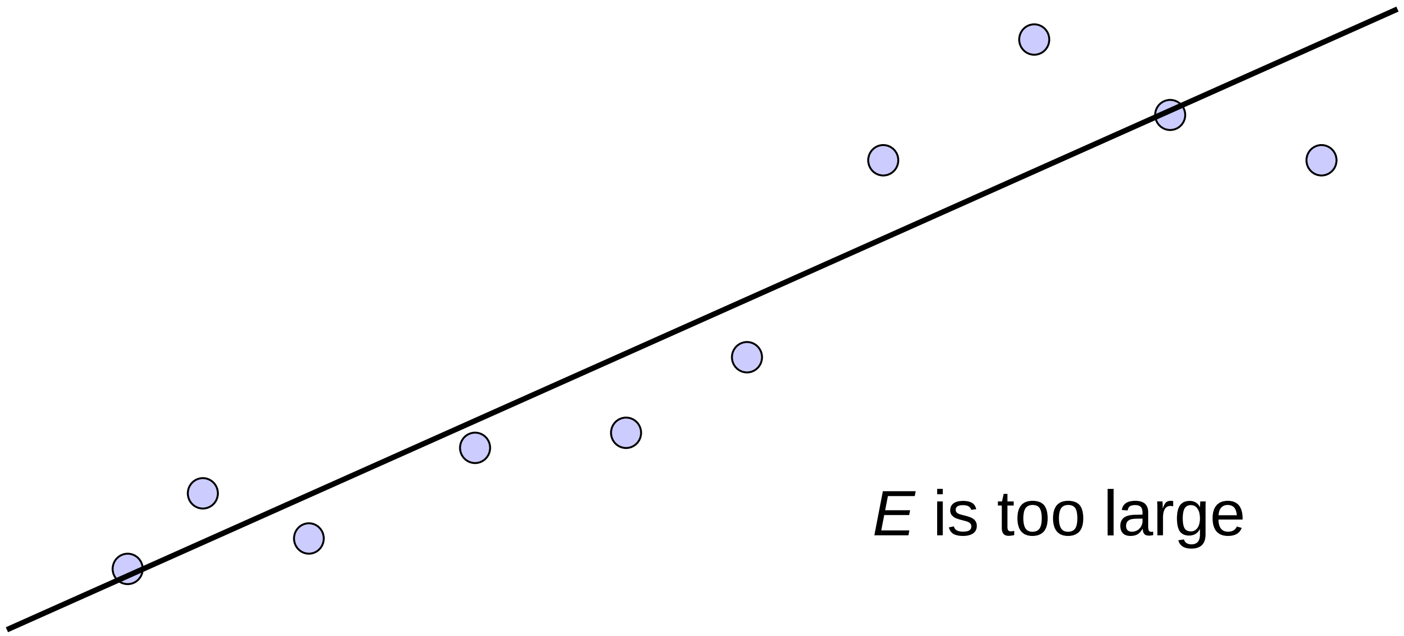
- For parametric definition, the least square line problem is now to find a_0, a_1, b_0, b_1 satisfying

$$\text{Minimum} \quad E = \sum_{i=1}^{n} \left[(x(t_i) - x_i)^2 + (y(t_i) - y_i)^2 \right]$$

- Ways to choose $t_1, t_2, \cdots, t_n$ will affect the result.

Least square curve

- Why we need least square curve? When raw data is too complicated, least square line is not good enough.



Least square curve

- In more general, $f(x) \in P_k$ can be a polynomial of degree k

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$$

- The problem becomes now to find $a_0, a_1, \cdots, a_k$ satisfying the following

Minimum $E = \sum_{i=1}^{n} (f(x_i) - y_i)^2$

Least square curve

- Systems to be solved are

$$
\begin{aligned}
a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_k x_1^k &= y_1 \\
a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_k x_2^k &= y_2 \\
\cdots \qquad \cdots & \\
a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_k x_n^k &= y_n
\end{aligned}
$$

And

$$
\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}
$$

The coefficient matrix $\boldsymbol{M}$: $\boldsymbol{n}\ \boldsymbol{x}\ (\boldsymbol{k}+\mathbf{1})$

Least square curve

- When $n > k$ and rank of M is $k+1$, we solve the following system to get the least square curve

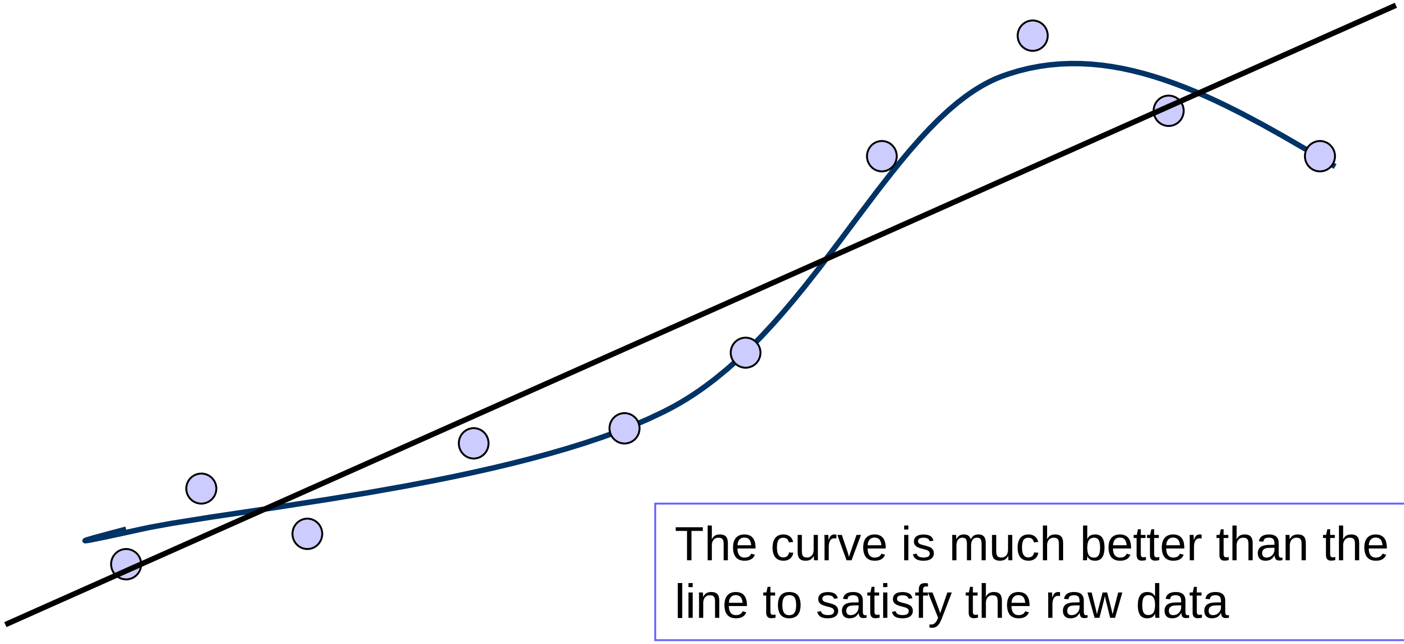
$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k & x_3^k & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k & x_3^k & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$(M^T M)X = M^T D$$

The coefficient matrix $\boldsymbol{M^T M}$: $(\boldsymbol{k}+\boldsymbol{1})\ \boldsymbol{x}\ (\boldsymbol{k}+\boldsymbol{1})$

Least square curve

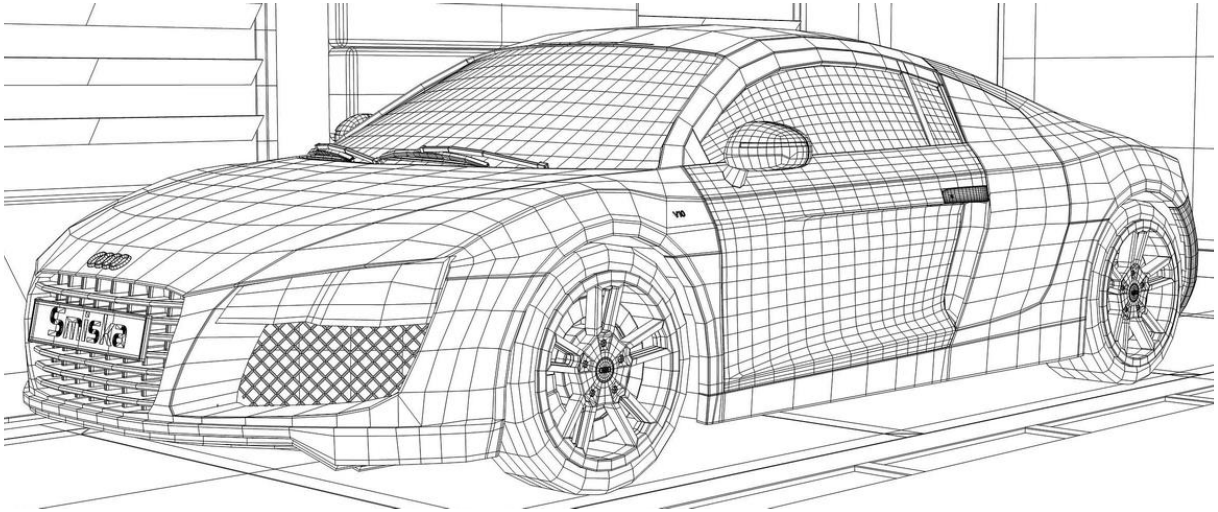
- If $x_1 \neq x_2 \neq \cdots \neq x_n$ and $n > k$, we can always find the unique solution of the system, and it will be the least square solution to the original system.



The curve is much better than the line to satisfy the raw data

Outline

- Interpolation and Approximation
- Curve Modeling
  - Parametric curve
  - Bézier curve
- Surface Modeling
  - Bézier surface



Classification of curves

$$y = x^2 + 5x + 3 \longrightarrow y = f(x)$$

(**explicit curve**)

$$(x - x_c)^2 + (y - y_c)^2 - r^2 = 0 \longrightarrow g(x, y) = 0$$

(**implicit curve**)

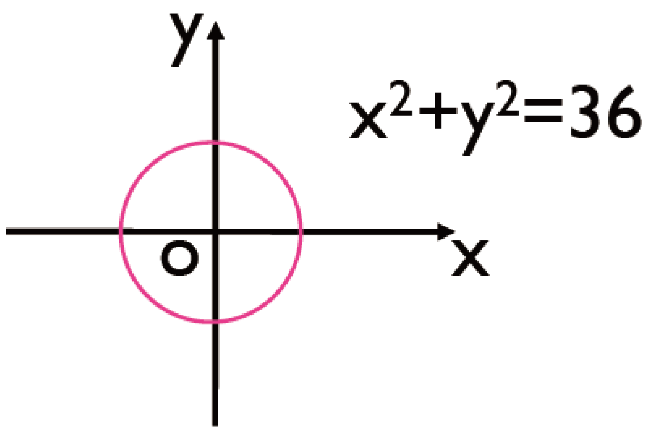
$$\begin{aligned} x &= x_c + r \cdot \cos\theta \\ y &= y_c + r \cdot \sin\theta \end{aligned} \longrightarrow \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

(**parametric curve**)

Classification of curves

implicit curve

- planar: $f(x,y)=0$:
  $x^2+y^2-36=0$

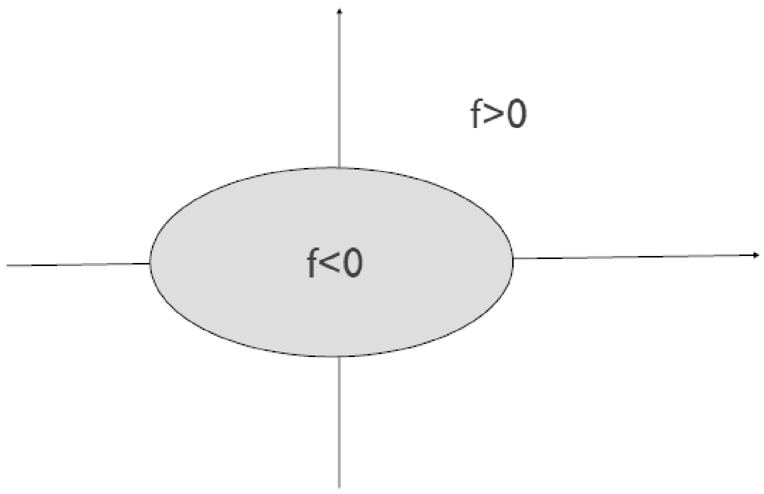


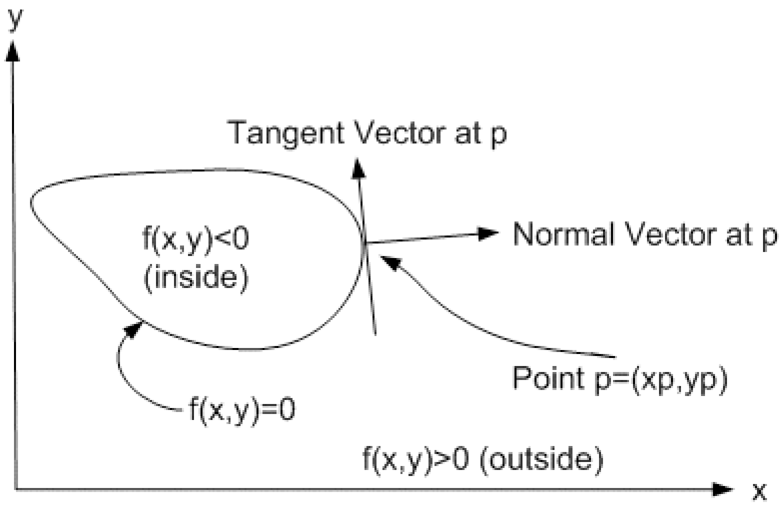
- 3D curves

$$\begin{cases} f(x,y,z)=0, \\ g(x,y,z)=0. \end{cases}$$

Implicit curves

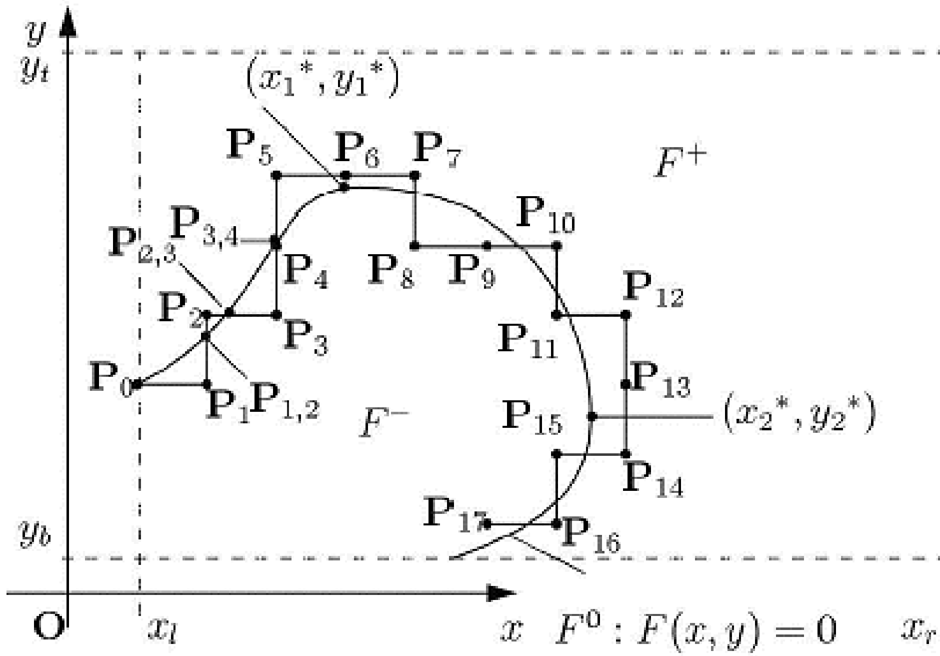
- **Advantage** of implicit curve:
  - To a point (x,y), it is easy to detect whether f(x,y) is >0 ,<0 or =0.
- **Disadvantage** of implicit curve:
  - To a curve f(x,y)=0, it is difficult to find the point on it.





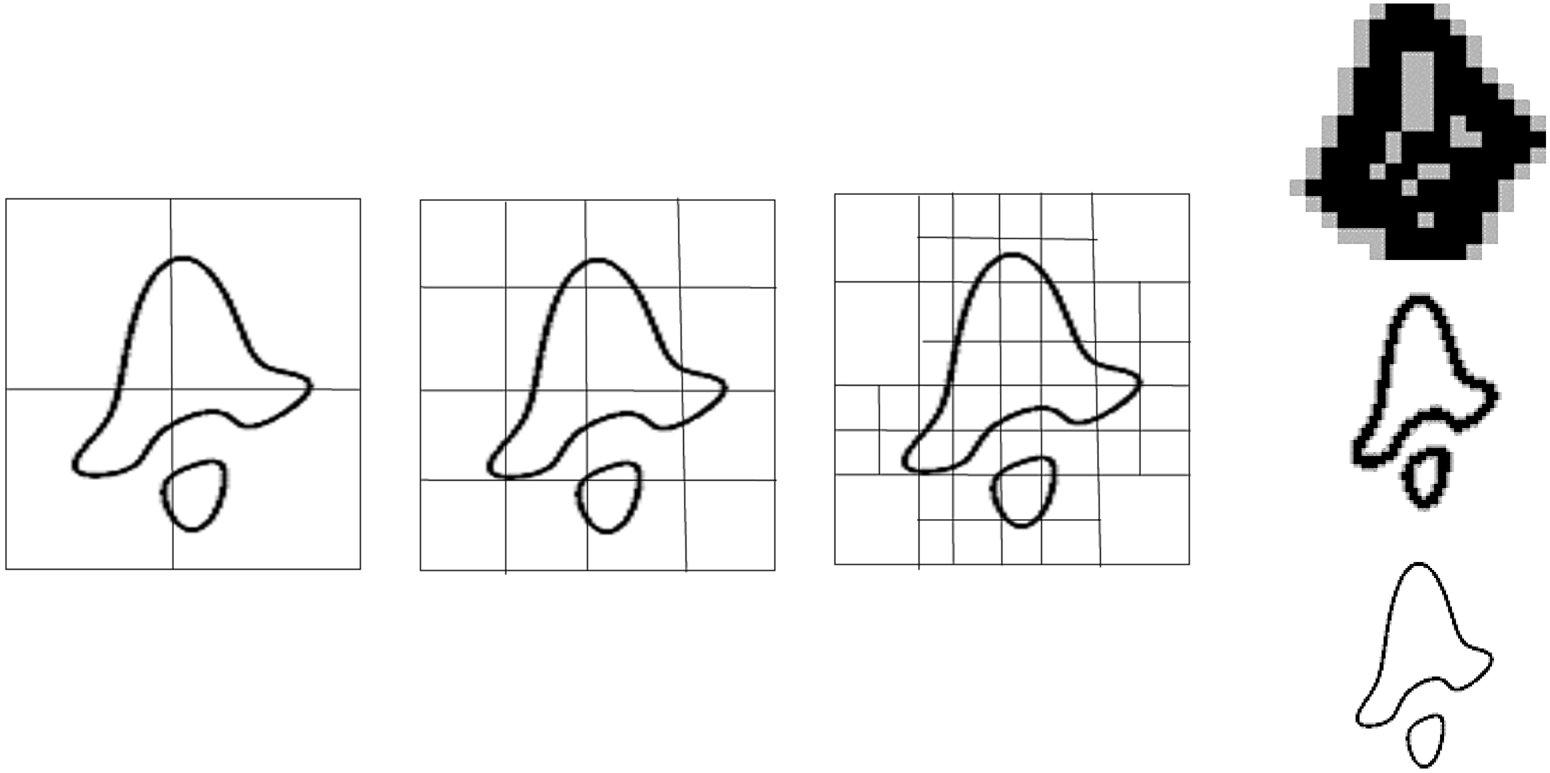
Implicit curves

- Display of implicit curves---chain coding



Implicit curves

- Display of implicit curves -- subdivision

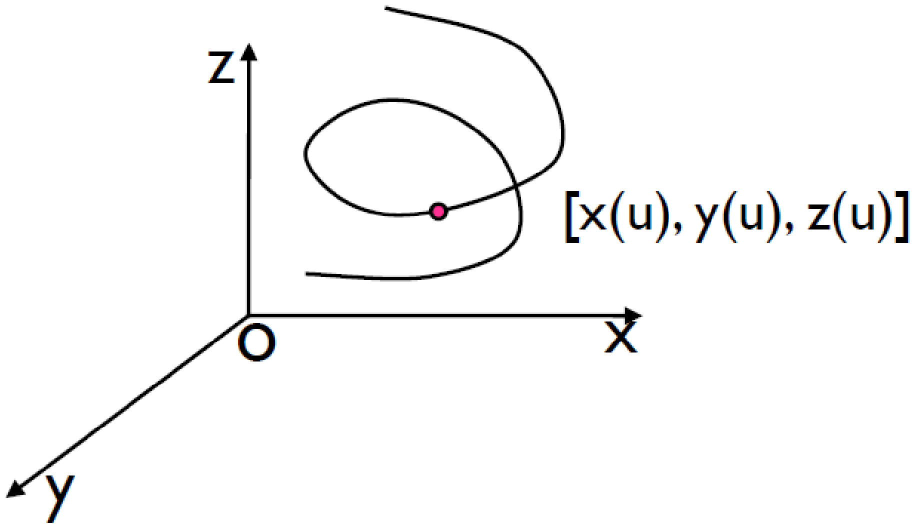


Parametric curves

- Variable is a scalar, and function is a vector:

$$C = C(u) = [x(u), y(u), z(u)],$$

- Every element of the vector is a function of the variable (the parameter)



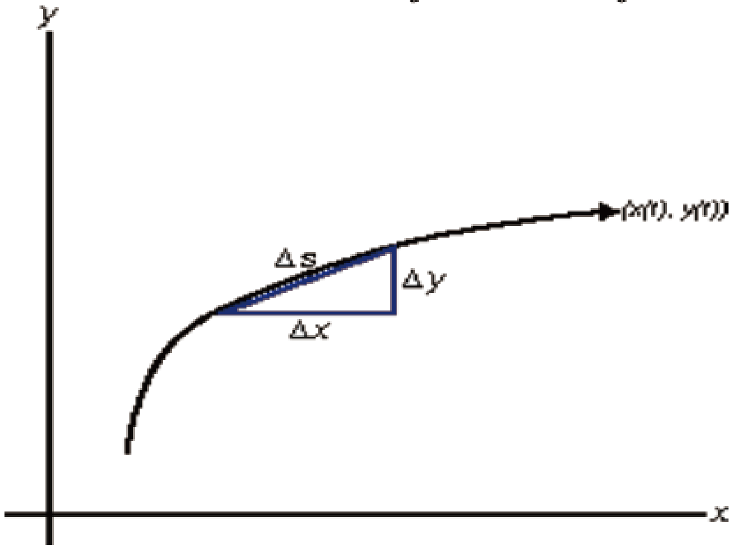
Parametric curves

given a curve **C**(u), its tangent is ***T***=***C***'(u).

difference of arc length:

$$(ds)^2=(dx)^2+(dy)^2+(dz)^2=((x')^2+(y')^2+(z')^2)d^2u$$

- Arc length: $s=\int_{u_0}^{u} ds=\int_{u_0}^{u}\sqrt{(x')^2+(y')^2+(z')^2}\,du$



Least square parametric curve

- Parametric definition of the curve (3D)

$$\begin{cases} x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k \\ y(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k \\ z(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_k t^k \end{cases}$$

- Square Error

$$E = \sum_{i=1}^{n} \left[(x(t_i) - x_i)^2 + (y(t_i) - y_i)^2 + (z(t_i) - z_i)^2 \right]$$

Least square curve – general case

- General method to solve the problem is based on the following

$$\frac{\partial E}{\partial a_i} = 0, \frac{\partial E}{\partial b_i} = 0, \frac{\partial E}{\partial c_i} = 0, i = 0, \cdots, k$$

- The least square solution can be got by solving a related linear system

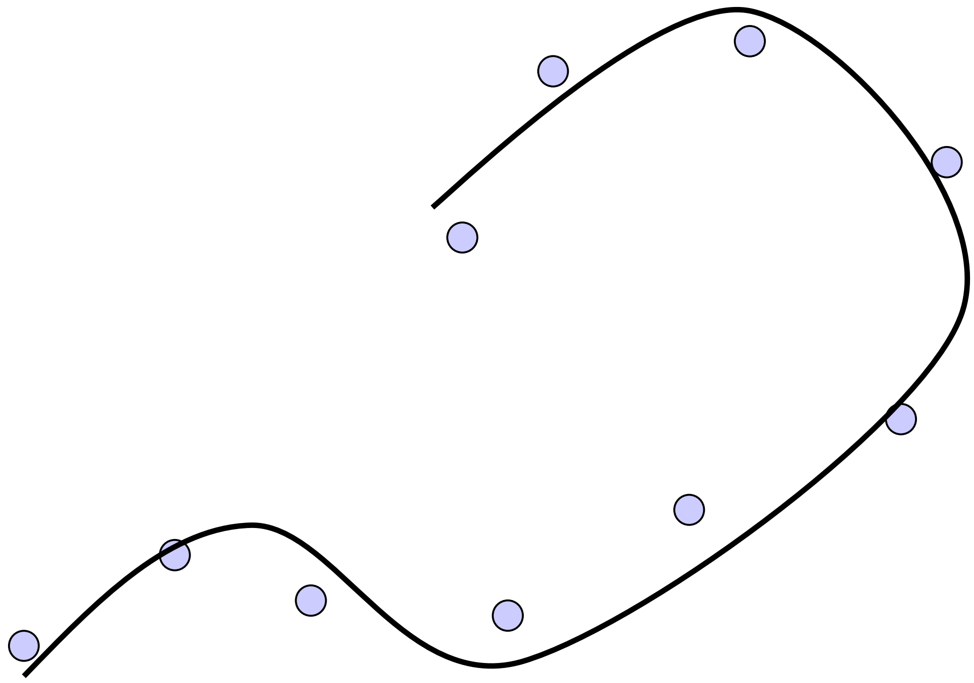
Least square parametric curve

- *Remark: the different choice of t_1, t_2, …, t_n will lead different result.*
- Chord length (弦长) parameter is one of the best.

$$t_1 = 0$$

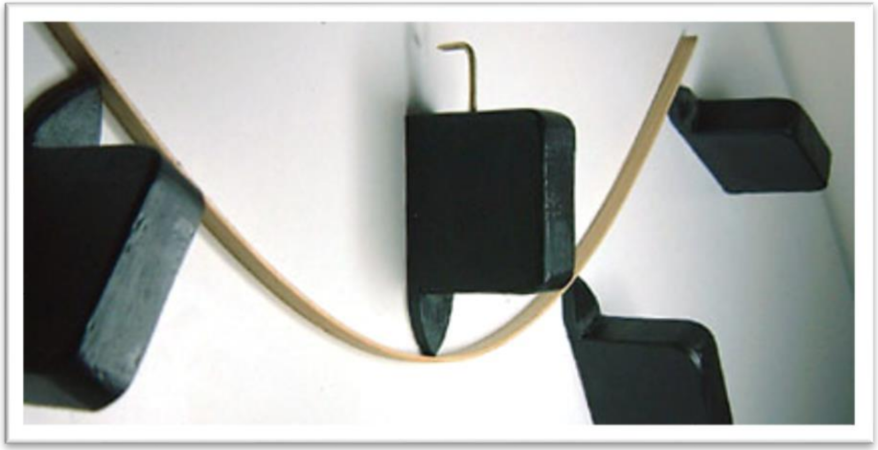
$$t_i = t_{i-1} + \left\| P_i - P_{i-1} \right\|$$

$$i = 2, \cdots, n$$



Parametric curves and splines

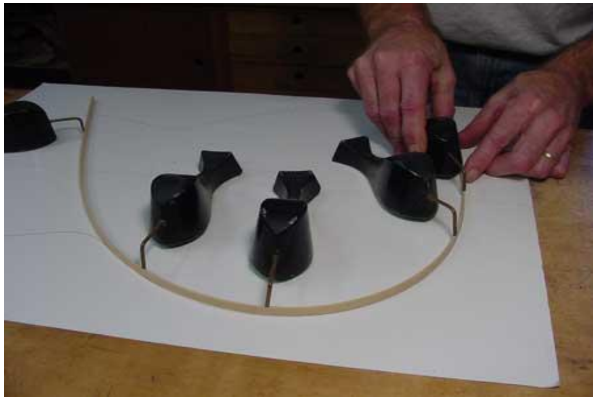
- Cubic Hermite interpolation (Hermite Spline)
- Bezier curves



Splines - History

- Draftsman use ‘ducks’ and strips of wood (splines) to draw curves
- Wood splines have second-order continuity
- And pass through the control points

A Duck (weight)



Ducks trace out curve

Spline in industry

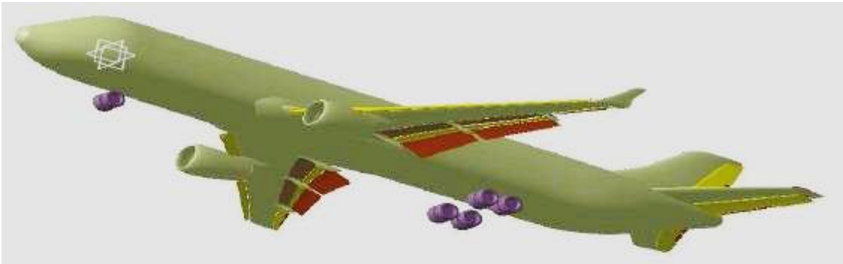


图 4 翼梢小翼参数化模型

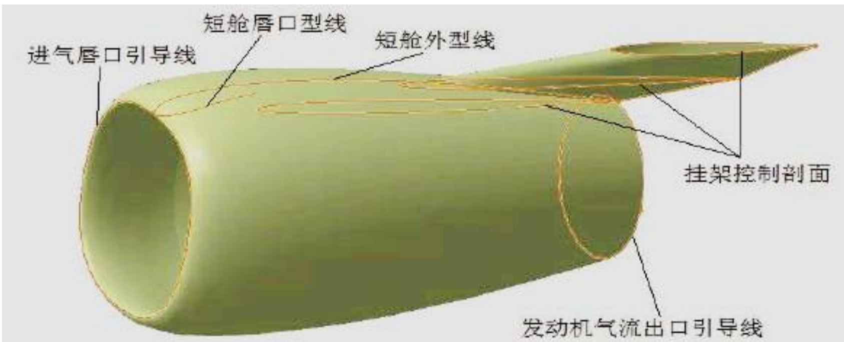


图 5 短舱及挂架参数化模型

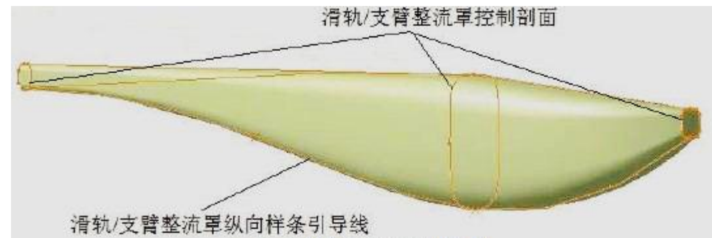


图 6 滑轨整流罩参数化模型

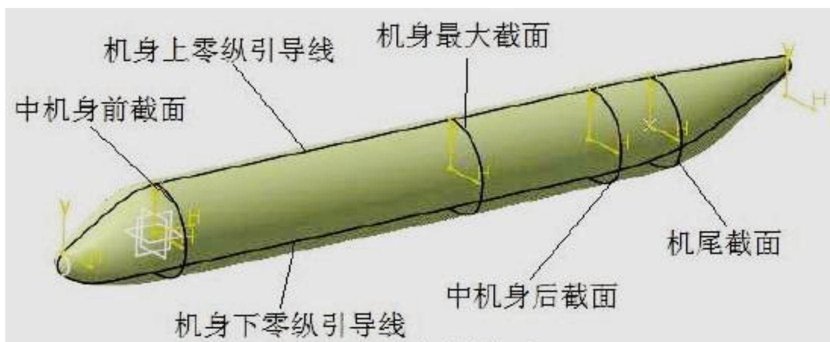
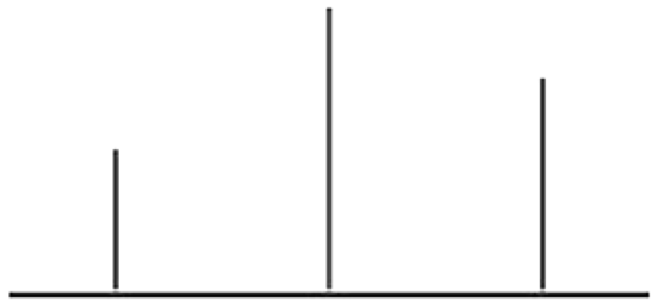


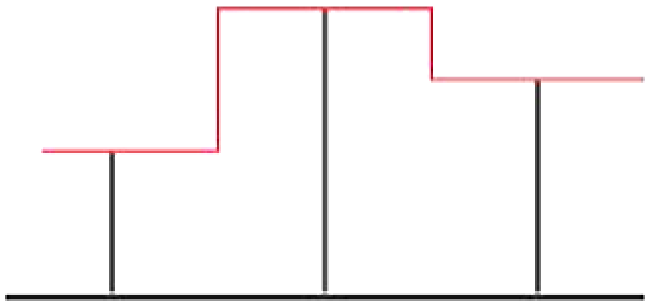
图 3 机身参数模型

Interpolation

- Goal: interpolate values

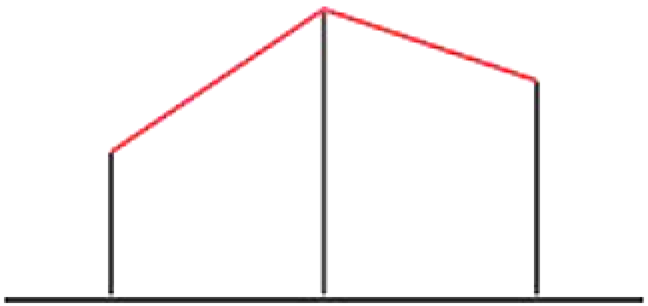


Nearest neighbor interpolation



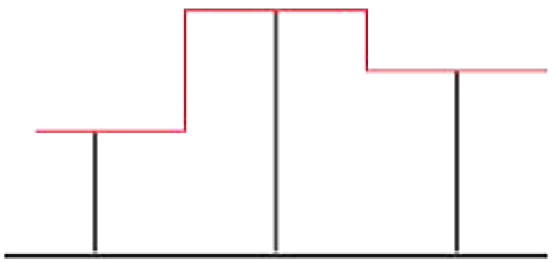
Problem: values not continuous

Linear interpolation

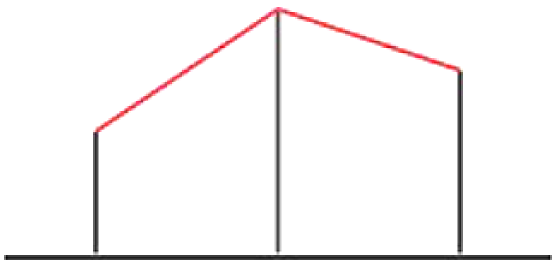


Problem: derivatives not continuous

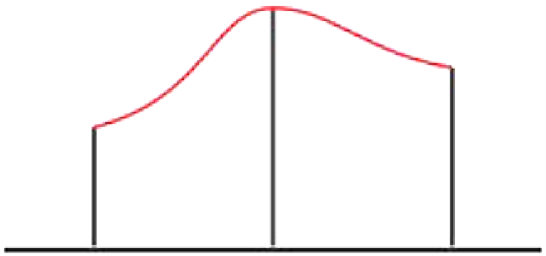
Smooth interpolation?



NN interpolation

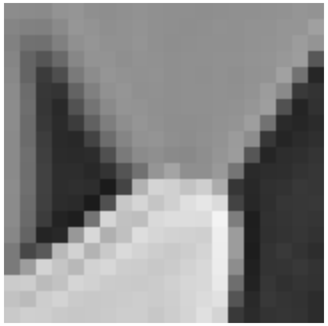


Linear interpolation

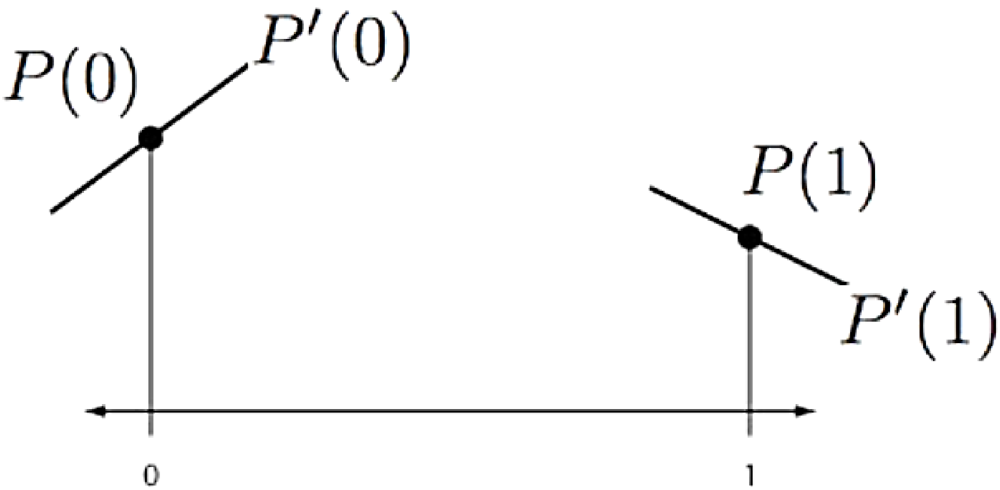


Smooth curve

20 x 20 pixel



Cubic Hermite Interpolation



Given: value and derivatives at 2 points

Hermite 曲线是通过给定曲线的两个端点的位置矢量P(0)、P(1)以及两个端点处的切线矢量P’(0)、P’(1)来描述曲线。

Cubic Hermite Interpolation

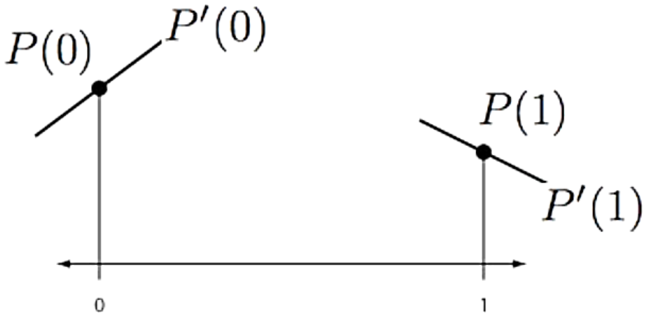
- Assume Cubic polynomial

$$P(t) = a\,t^3 + b\,t^2 + c\,t + d$$

- Solve for coefficients:

$$P(0) = h_0 = d$$

$$P(1) = h_1 = a + b + c + d$$



Cubic Hermite Interpolation

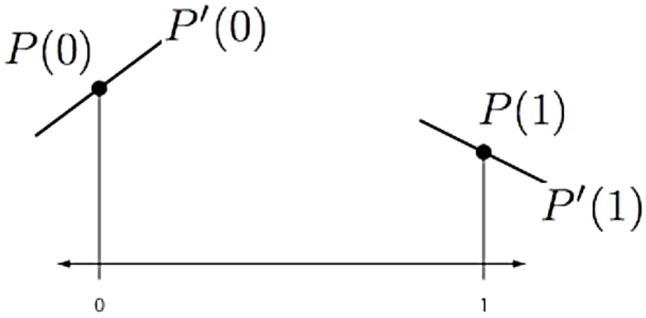
- Cubic polynomial

$$P(t) = a\,t^3 + b\,t^2 + c\,t + d$$
$$P'(t) = 3a\,t^2 + 2b\,t + c$$

- Solve for coefficients:

$$P(0) = h_0 = d$$
$$P(1) = h_1 = a + b + c + d$$

$$P'(0) = h_2 = c$$
$$P'(1) = h_3 = 3a + 2b + c$$



Matrix Representation

$$h_0 = d$$
$$h_1 = a + b + c + d$$
$$h_2 = c$$
$$h_3 = 3a + 2b + c$$

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

- Matrix Inverse: Solve for a, b, c, d

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

Transpose $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

$$\left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right)^T = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

- Cubic polynomial

$$P(t) = a\,t^3 + b\,t^2 + c\,t + d$$

$\boxed{t^0=1}$

$$P(t) = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Matrix Transformation

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad P(t) = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\overbrace{\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}}^{\begin{bmatrix} h_0 & h_1 & h_2 & h_3 \end{bmatrix}} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Insert identity matrix

Change Basis

$$\underbrace{\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} h_0 & h_1 & h_2 & h_3 \end{bmatrix}} \underbrace{\begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}}_{\begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ H_3(t) \end{bmatrix}}$$

$$\begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ H_3(t) \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Hermite Basis Functions

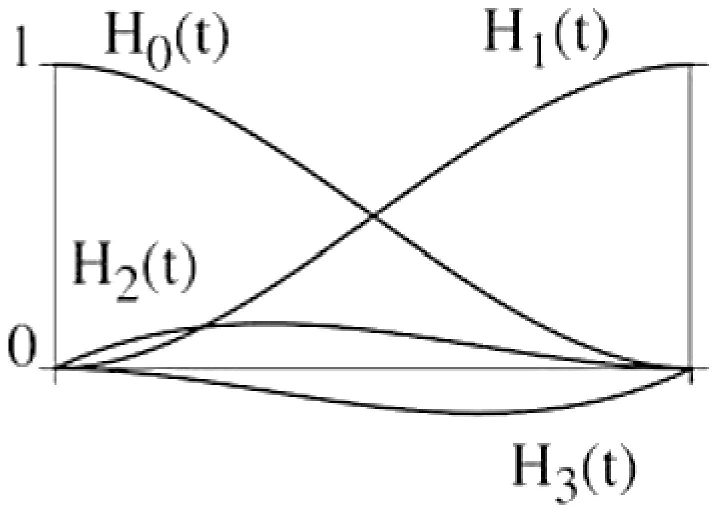
$$H_0(t) = 2t^3 - 3t^2 + 1$$

$$H_1(t) = -2t^3 + 3t^2$$

$$H_2(t) = t^3 - 2t^2 + t$$

$$H_3(t) = t^3 - t^2$$

Hermite Basis Functions



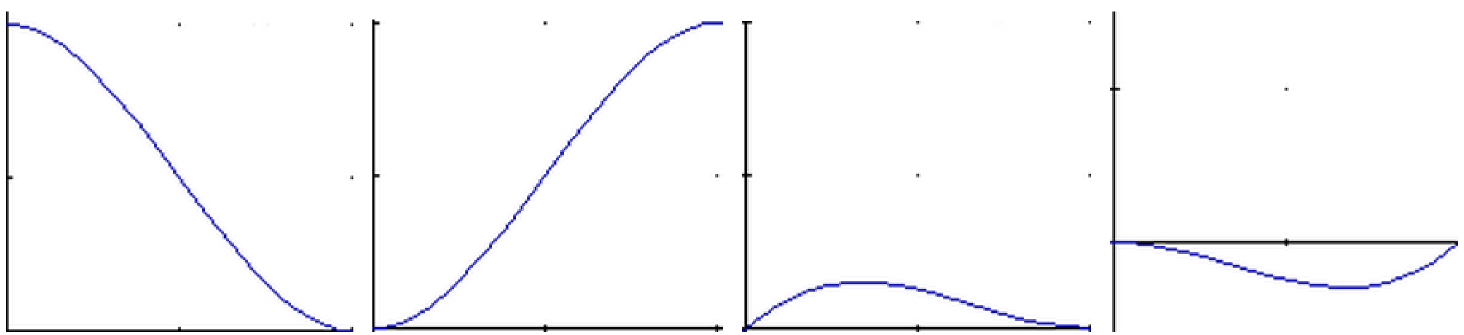
$$H_0(t) = 2t^3 - 3t^2 + 1$$

$$H_1(t) = -2t^3 + 3t^2$$

$$H_2(t) = t^3 - 2t^2 + t$$

$$H_3(t) = t^3 - t^2$$

Below are the 4 graphs of the 4 functions (from left to right: h1, h2, h3, h4).



(all graphs except the 4th have been plotted from 0,0 to 1,1)

Hermite Basis Functions

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ H_3(t) \end{bmatrix}$$

$$\Downarrow$$

$$P(t) = \sum_{i=0}^{3} h_i H_i(t)$$

$$P(0) = h_0 = d$$
$$P(1) = h_1 = a + b + c + d$$
$$P'(0) = h_2 = c$$
$$P'(1) = h_3 = 3a + 2b + c$$

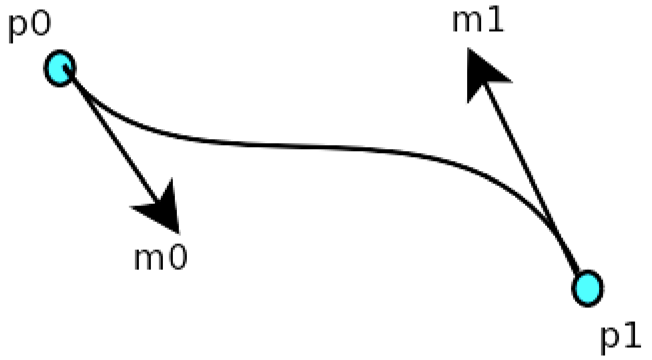
Case

- $\mathbf{P}(t) = (2t^3 - 3t^2 + 1)\ \mathbf{p}0$

  $+ (t^3 - 2t^2 + t)\ \mathbf{m}0$

  $+ (-2t^3 + 3t^2)\ \mathbf{p}1$

  $+ (t^3 - t^2)\ \mathbf{m}1$



$$t \in [0, 1]$$

Case

- The derivatives and the shape of Hermite curves