ALJABAR LINEAR

Pertemuan ke-18 dan 19 - Dekomposisi QR dan Gram-Schmidt

Oleh:

Annastya Bagas Dewantara, S.T., M.T.

(email: annastya.bagas@ui.ac.id)

Fakultas Teknik

Universitas Indonesia

Daftar Paparan

Dekomposisi QR

Definisi

Inner product dikatakan **orthogonal** ketika:

$$\langle v_1, v_2 \rangle = 0$$

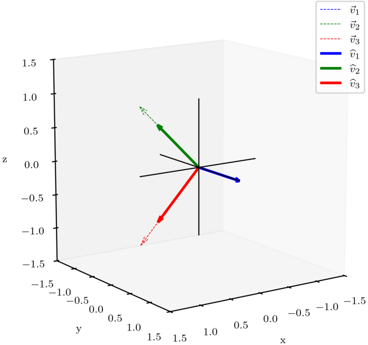
Jika tiap vektor, memiliki panjang satu, maka disebut **orthonormal**.

$$\|v_1\| = \|v_2\| = 1,$$

Motivasi:

1. Pada **Euclidean Vector Space**, basis dapat dibentuk dari sembarang himpunan vektor yang **linearly independent** dan mampu **men-spanning ruang vektor** tersebut.
2. Pada **Inner Product Space**, perhitungan menjadi lebih mudah jika basis terdiri dari vektor-vektor **orthogonal**.
3. Himpunan vektor orthogonal dapat dinormalisasi menjadi **orthonormal basis**, yang menjadi dasar bagi **Gram–Schmidt Process** dan **QR-Decomposition**.

Definisi



Jika diketahui vektor-vecor berupa:

$$v_1 = (0, 1, 0), \quad v_2 = (1, 0, 1), \quad v_3 = (1, 0, -1)$$

Melalui **Euclidian Inner Product** , diperoleh:

$$\langle v_1, v_2 \rangle = \langle v_1, v_3 \rangle = \langle v_2, v_3 \rangle = 0$$

Sehingga $S = \{v_1, v_2, v_3\}$ merupakan himpunan ortogonal.

$$\|v_1\| = 1, \quad \|v_2\| = \sqrt{2}, \quad \|v_3\| = \sqrt{2}$$

Dengan menormalkan setiap vektor, diperoleh:

$$u_1 = \frac{v_1}{\|v_1\|} = (0, 1, 0), \quad u_2 = \frac{v_2}{\|v_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad u_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

Koordinat terhadap Basis Ortogonal dan Ortonormal

Misal $S = \{v_1, v_2, \ldots, v_n\}$ adalah basis ortogonal dan setiap vektor u dinyatakan sebagai:

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

$$\langle u, v_i \rangle = \langle \underbrace{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n}_{u}, v_i \rangle \implies \boxed{\textit{Additivity}}$$

$$\langle u, v_i \rangle = c_1 \langle v_1, v_i \rangle + \cdots + c_n \langle v_n, v_i \rangle \implies \boxed{\textit{Homogenity}}$$

Karena S ortogonal, $\langle v_j, v_i \rangle = 0$ untuk $j \neq i$, maka:

$$\langle u, v_i \rangle = c_i \langle v_i, v_i \rangle = c_i \|v_i\|^2$$

Sehingga diperoleh:

$$\boxed{c_i = \frac{\langle u, v_i \rangle}{\|v_i\|^2}}$$

Jika S ortonormal, maka $\|v_i\| = 1$, sehingga:

$$\boxed{c_i = \langle u, v_i \rangle}$$

Koordinat vektor u terhadap basis ortogonal $S = \{v_1, v_2, \ldots, v_n\}$ adalah:

$$(u)_S = \left(\frac{\langle u, v_1 \rangle}{\|v_1\|^2}, \frac{\langle u, v_2 \rangle}{\|v_2\|^2}, \ldots, \frac{\langle u, v_n \rangle}{\|v_n\|^2} \right)$$

Sedangkan terhadap basis ortonormal, koordinatnya menjadi:

$$(u)_S = (\langle u, v_1 \rangle, \langle u, v_2 \rangle, \ldots, \langle u, v_n \rangle)$$

Contoh 1: Koordinat terhadap Basis Ortonormal

Diberikan basis ortonormal pada $\mathbb{R}^3$:

$$v_1 = (0, 1, 0), \quad v_2 = \left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right), \quad v_3 = \left(\tfrac{3}{5}, 0, \tfrac{4}{5}\right)$$

Nyatakan $u = (1, 1, 1)$ sebagai kombinasi linear dari $S = \{v_1, v_2, v_3\}$ dan tentukan koordinat $(u)_S$.

Karena S ortonormal, maka:

$$u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \langle u, v_3\rangle v_3$$

Hitung hasil kali dalam:

$$\langle u, v_1\rangle = (1, 1, 1)\cdot(0, 1, 0) = 1,$$

$$\langle u, v_2\rangle = (1, 1, 1)\cdot\left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right) = -\tfrac{1}{5},$$

$$\langle u, v_3\rangle = (1, 1, 1)\cdot\left(\tfrac{3}{5}, 0, \tfrac{4}{5}\right) = \tfrac{7}{5}.$$

Substitusi ke ekspresi u:

$$u = 1v_1 - \tfrac{1}{5}v_2 + \tfrac{7}{5}v_3$$

$$(1, 1, 1) = (0, 1, 0) - \tfrac{1}{5}\left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right) + \tfrac{7}{5}\left(\tfrac{3}{5}, 0, \tfrac{4}{5}\right)$$

Vektor koordinat:

$$(u)_S = \left(1,\ -\tfrac{1}{5},\ \tfrac{7}{5}\right)$$

Contoh 2: Koordinat terhadap Basis Ortogonal

Diberikan

$$w_1 = (0,2,0), \quad w_2 = (3,0,3), \quad w_3 = (-4,0,4), \qquad u = (1,2,4).$$

$$\begin{aligned} \langle w_1, w_2 \rangle &= (0,2,0)\cdot(3,0,3) = 0, \\ \langle w_1, w_3 \rangle &= (0,2,0)\cdot(-4,0,4) = 0, \\ \langle w_2, w_3 \rangle &= (3,0,3)\cdot(-4,0,4) = -12+12 = 0. \end{aligned}$$

Normalisasi:

$$\|w_1\| = 2, \quad \|w_2\| = 3\sqrt{2}, \quad \|w_3\| = 4\sqrt{2}$$

$$\widehat{w}_1 = \frac{w_1}{\|w_1\|} = (0,1,0),$$

$$\widehat{w}_2 = \frac{w_2}{\|w_2\|} = \left(\frac{3}{3\sqrt{2}}, 0, \frac{3}{3\sqrt{2}}\right) = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right),$$

$$\widehat{w}_3 = \frac{w_3}{\|w_3\|} = \left(\frac{-4}{4\sqrt{2}}, 0, \frac{4}{4\sqrt{2}}\right) = \left(-\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right).$$

$$\langle u, \widehat{w}_1 \rangle = (1,2,4)\cdot(0,1,0) = 2,$$

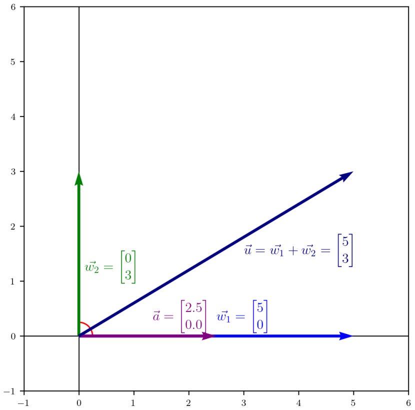
$$\langle u, \widehat{w}_2 \rangle = (1,2,4)\cdot\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right) = \frac{1+4}{\sqrt{2}} = \frac{5}{\sqrt{2}},$$

$$\langle u, \widehat{w}_3 \rangle = (1,2,4)\cdot\left(-\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right) = \frac{-1+4}{\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

$$u = 2\widehat{w}_1 + \frac{5}{\sqrt{2}}\widehat{w}_2 + \frac{3}{\sqrt{2}}\widehat{w}_3.$$

$$(u)_S = \left(2, \frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right).$$

Projeksi Ortogonal pada $\mathbb{R}^n$



Jika $\vec{u}$ dan $\vec{a}$ terletak $\mathbb{R}^n$, dan $\vec{a} \neq 0$, dan $\vec{u}$ dapat dinyatakan dengan $\vec{u} = \vec{w_1} + \vec{w_2}$, di mana $\vec{w_1} = k\vec{a}$ dan $\vec{w_2}$ ortogonal terhadap $\vec{a}$ ($\vec{a} \perp \vec{w_2}$).

$$\vec{u} = \vec{w_1} + \vec{w_2}$$
$$\vec{u} = k\vec{a} + \vec{w_2}$$

Maka dot product:

$u \cdot a = (ka + w_2) \cdot a = ka \cdot a + w_2 \cdot a$

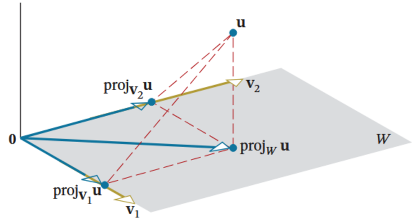
$$= ka \cdot a + w_2 \cdot a = k\|a\|^2 \leftarrow \boxed{w_2 \cdot a = 0, \quad (w_2 \perp a)}$$

$$k = \frac{u \cdot a}{\|a\|^2}$$

Sehingga:

$$\boxed{\text{proj}_{\vec{a}}\,\vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\,\vec{a} = \vec{w_1}} \qquad \boxed{\vec{w_2} = \vec{u} - \text{proj}_{\vec{a}}\,\vec{u} = \vec{u} - \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\,\vec{a}}$$

Projeksi Ortogonal pada Inner Product Space



Definisi

Jika W adalah finite-dimensional subspace dari inner product space V, maka setiap vektor $u \in V$ dapat dinyatakan secara unik sebagai:

$$\vec{u} = \vec{w_1} + \vec{w_2}$$
$$\vec{u} = \text{proj}_W\, u + \text{proj}_{W^\perp}\, u$$
$$\vec{u} = \text{proj}_W\, u + (u - \text{proj}_W\, u)$$

dengan $w_1 \in W$ dan $w_2 \in W^\perp$.

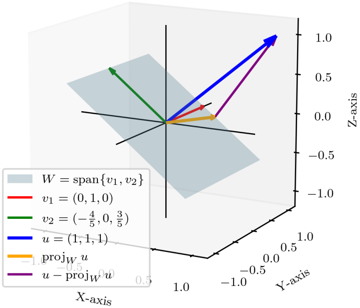
(a) Jika $\{v_1, v_2, \ldots, v_r\}$ adalah basis ortogonal W:

$$\text{proj}_W u = \frac{\langle u, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2\rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle u, v_r\rangle}{\|v_r\|^2} v_r$$

(b) Jika $\{v_1, v_2, \ldots, v_r\}$ adalah basis ortonormal W:

$$\text{proj}_W u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \cdots + \langle u, v_r\rangle v_r$$

Contoh 1: Projeksi Ortogonal



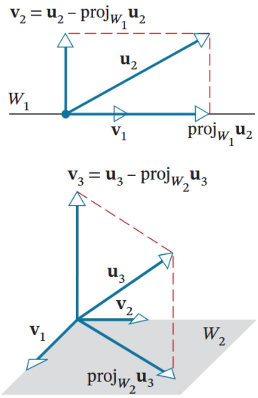
Misalkan $\mathbb{R}^3$ memiliki hasil kali dalam Euclidean, dan W dibentang oleh dua vektor ortonormal:

$$v_1 = (0, 1, 0), \quad v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

Untuk $u = (1, 1, 1)$, proyeksi ortogonalnya pada W adalah:

$$\begin{aligned}
\text{proj}_W u &= \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 \\
&= (1)(0, 1, 0) + \left(-\frac{1}{5}\right)\left(-\frac{4}{5}, 0, \frac{3}{5}\right) \\
&= \left(\frac{4}{25}, 1, -\frac{3}{25}\right)
\end{aligned}$$

Proses Gram–Schmidt



Proses Gram–Schmidt adalah metode untuk mengubah basis menjadi basis ortogonal atau ortonormal. Misalkan $S = \{u_1, u_2, \ldots, u_n\}$ adalah basis dalam ruang produk dalam V. Proses Gram–Schmidt menghasilkan himpunan ortogonal $S' = \{v_1, v_2, \ldots, v_n\}$ sebagai berikut:

$$
\begin{aligned}
v_1 &= u_1, \\
v_2 &= u_2 - \text{proj}_{v_1} u_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1, \\
v_3 &= u_3 - \text{proj}_{v_1} u_3 - \text{proj}_{v_2} u_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2, \\
&\vdots \\
v_n &= u_n - \sum_{j=1}^{n-1} \text{proj}_{v_j} u_n = u_n - \sum_{j=1}^{n-1} \frac{\langle u_n, v_j\rangle}{\|v_j\|^2} v_j.
\end{aligned}
$$

Tahapan Gram–Schmidt

Ubah basis $\{u_1, u_2, \dots, u_r\}$ menjadi basis ortogonal $\{v_1, v_2, \dots, v_r\}$ dengan langkah berikut:

1. $v_1 = u_1$
2. $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$
3. $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$
4. $v_4 = u_4 - \frac{\langle u_4, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_4, v_2 \rangle}{\|v_2\|^2} v_2 - \frac{\langle u_4, v_3 \rangle}{\|v_3\|^2} v_3$
5. $\dots$ hingga v_r

Langkah opsional: Normalisasi setiap vektor ortogonal untuk memperoleh basis ortonormal $\{q_1, q_2, \dots, q_r\}$:

$$q_i = \frac{v_i}{\|v_i\|}, \quad i = 1, 2, \dots, r$$

Contoh 1: Proses Gram–Schmidt

Ruang vektor: $\mathbb{R}^3$ dengan hasil kali dalam Euclidean. Basis awal:

$$u_1 = (1,1,1), \quad u_2 = (0,1,1), \quad u_3 = (0,0,1)$$

Transformasikan menjadi basis ortogonal $\{v_1, v_2, v_3\}$ dan ortonormal $\{q_1, q_2, q_3\}$.

$$\begin{aligned}
v_1 &= u_1 = (1,1,1) \\
v_2 &= u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1 \\
&= (0,1,1) - \frac{2}{3}(1,1,1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\
v_3 &= u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2 \\
&= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{3}(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \\
&= (0, -\frac{1}{2}, \frac{1}{2})
\end{aligned}$$

$$\|v_1\| = \sqrt{3}, \quad \|v_2\| = \frac{\sqrt{6}}{3}, \quad \|v_3\| = \frac{1}{\sqrt{2}}$$

Basis ortonormal:

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

QR-Decomposition

QR-decomposition adalah faktorisasi matriks $A \in \mathbb{R}^{m \times n}$ berbasis proses Gram–Schmidt, dengan u_i adalah basis vektor ke-i dan q_i adalah basis ortonomal ke-i, sehingga:

$$\underbrace{\left[u_1 \mid u_2 \mid \cdots \mid u_n\right]}_{A} = \underbrace{\left[q_1 \mid q_2 \mid \cdots \mid q_n\right]}_{Q} \underbrace{\begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \dots & \langle u_n, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \dots & \langle u_n, q_2 \rangle \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & \langle u_n, q_n \rangle \end{bmatrix}}_{R}$$

Dengan properti Gram–Schmidt:

$$u_i = \sum_{j=1}^{i} \langle u_i, q_j \rangle q_j, \quad q_j \perp q_1, \dots, q_{j-1}, \quad r_{jj} = \left\| u_j - \sum_{k=1}^{j-1} \mathrm{proj}_{q_k} u_j \right\| \neq 0$$

Linear Regression via QR-Decomposition

Diberikan data: $x = [1, 2, 3, 4, 5]^T$, $y = [2, 3, 4, 5, 6]^T$ dan model regresi linier:

$$y = \beta_0 + \beta_1 x, \quad Y = X\beta, \quad X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

1. QR-decomposition (Gram–Schmidt):

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad q_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = u_2 - \mathrm{proj}_{q_1} u_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{10}} \end{bmatrix}, \quad R = Q^T X = \begin{bmatrix} \sqrt{5} & 3\sqrt{5} \\ 0 & \sqrt{10} \end{bmatrix}$$

2. Solve via R:
Dari $Y = X\beta = QR\beta$:

$$Q^T Y = R\beta \implies \beta = R^{-1} Q^T Y$$

Dengan

$$R^{-1} = \begin{bmatrix} 1/\sqrt{5} & \frac{-3\sqrt{10}}{10} \\ 0 & \frac{\sqrt{10}}{10} \end{bmatrix}$$

$$Q^T Y = \begin{bmatrix} \frac{1}{\sqrt{5}}(2 + 3 + 5 + 4 + 6) \\ \frac{1}{\sqrt{10}}(-2 \cdot 2 - 1 \cdot 3 + 0 + 1 \cdot 5 + 2 \cdot 6) \end{bmatrix} = \begin{bmatrix} 4\sqrt{5} \\ \sqrt{10} \end{bmatrix}$$

Maka

$$\beta = R^{-1} Q^T Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

model regresi: $\hat{y} = 1 + 1x$

Tugas

Tugas ke-6

6.1 No. 3, 5, 9

6.2 No. 2, 11, 13

6.3 No. 2, 20, 30, 34

(Deadline: 25/11/2025 Offline di kelas, Portofolio)

Daftar Pustaka

Daftar Pustaka I