1. Solve the system of equations (10)

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 4 & -1 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ -9 \end{bmatrix}$$

   manually using Gaussian Elimination assuming that the calculation is being done on a decimal computer capable of carrying only two floating-point digits.

2. For each of the following computational physics tasks, suggest a library function that you could use in either C or Python:
   (a) Computing Fourier transform of a sample (1)
   (b) Obtaining the QR decomposition of a matrix (1)
   (c) Obtaining a million random numbers from a lognormal PDF (1)
   (d) Solving an ODE initial value problem using an 8th-order Runge-Kutta Method (1)
   (e) Obtaining the singular value decomposition of a matrix (1)
   (f) Sampling a 548-dimensional PDF (1)
   (g) Solving an initial value problem for an ODE using adaptive step-size control (1)
   (h) Integrating a 9-dimensional function using a Monte Carlo method (1)
   (i) Solving a boundary value problem for 3 coupled ODEs (1)
   (j) Computing the eigenvalues and eigenvectors of a 10×10 complex matrix (1)

3. Show that if the $n \times n$ matrix $\mathbf{A}$ is tridiagonal, then a solution to the system of linear equations $\mathbf{Ax} = \mathbf{b}$ can be obtained in $\mathcal{O}(n)$ steps. (10)

4. Generate 1024 uniformly distributed random numbers between 0 and 1. Then
   (a) Plot your sample. (1)
   (b) Compute the power spectrum for this sample. (3)
   (c) What is the minimum and maximum value of the wavevector k? (1)
   (d) Plot the power spectrum in five uniform k bins. (2)
   (e) Present a short, intuitive, verbal argument for why your plot is correct (3)

5. You are in the market for a software library that you could use for a computational physics task. You find five libraries from different developers. Mention three criteria by which you will choose one of these libraries for your work. (10)

6. Solve the initial value problem (10)

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = 32y_1 + 66y_2 + \frac{2}{3}x + \frac{2}{3}$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = -66y_1 - 133y_2 - \frac{1}{3}x - \frac{1}{3}$$

   with $0 \le x \le 0.5$, $y_1(0) = 1/3$ and $y_2(0) = 1/3$. Plot the solution. Present an argument for why the solution is correct.

7. A linear congruential pseudorandom number generator has four parameters: modulus, multiplier, increment, and seed. The first number in the pseudorandom sequence generated by such a generator is equal to the given seed. Give an example of a linear congruential generator in which the seed appears again at some point in the output sequence. Then give an example generator in which the seed never appears again in the sequence. (10)

8. Solve the boundary value problem (10)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 4(y - x)$$

with $0 \le x \le 1$, $y(0) = 0$, and $y(1) = 2$. The correct solution is given by

$$y(x) = e^2(e^4 - 1)^{-1}(e^{2x} - e^{-2x}) + x.$$

Tabulate the relative error in your computed solution in percentage points.

9. Obtain the singular values of the matrix (10)

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

and the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

10. Compute the Fourier transform of a box function (10)

$$f(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Make a figure showing the graph of this function. Also plot its Fourier transform for three different values of the sampling rate.