Numerical Linear Algebra

Mid-term test

Fall 2014

HINT: If some questions seem too challenging skip them and try another ones.

Variant 1

1. (1 pt) What packages in Python are often used for numerical purposes (name at least 2)? Give an example of Python code that creates an $n \times m$ matrix with all elements equal to 1.

2. (2 pts)

   - What is the complexity of matrix-by-vector multiplication?
   - Is it possible to reduce this complexity for general matrix and vector? Why?

3. (1 pt) What is the advantage of using block versions of standard matrix algorithms?

4. (1 pt) Give examples of standard matrix norms. How can vector norms help to define matrix norms?

5. (2 pts) Find the third singular value $\sigma_3(A)$ of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Explain the answer.

6. (2 pts) Find $\mathrm{cond}_2 \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$, where $\epsilon \in \mathbb{R}$. Note: subscript 2 in cond_2 means that second norm is used.

7. (2 pts) Suppose that you want to calculate the best low-rank approximation of a certain matrix with absolute precision ϵ. What should you do? How many parameters are in the rank-r approximation?

8. (2 pts) Is it a good idea to find eigenvalues of large matrices via characteristic polynomial? Why? What analytical methods for estimating eigenvalues do you know?

9. (1 pt) What is a normal matrix? Give an example of normal matrix. What is the crucial property of normal matrices?

10. (3 pts) Why does LU decompostion fail on the matrix $\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$ when ϵ is small enough? How can this problem be solved? Bonus: what is the exact value of ϵ when it starts to fail?

11. (2 pts) How to calculate QR decompostion via Gram matrices? Is it a good idea for a numerical algorithm? Name 2 approaches you know that are used in practice.

12. (1 pt) What is the QR algorithm about?

13. (4 pts) The goal of compressed sensing is to find the sparsest solution x of an undetermined linear system $y = Ax$ where $A \in \mathbb{R}^{n \times m}$, $n < m$. In order to achieve it one could try to find solution which has minimal first norm. Intuition behind this fact is quite simple in 2D:

    - (1 pt) Draw disks $\|x\| = \text{const}$ in $1, 2$ and ∞ norms
    - (3 pts) Find graphically solutions of $y = Ax$, $\|x\|_* \to \min$, where $A \in \mathbb{R}^{1 \times 2}$ and $* = \{1, 2, \infty\}$. Which norm yields the sparsest solution?