Numerical Linear Algebra

Mid-term test

Fall 2014

HINT: If some questions seem too challenging skip them and try another ones.

Variant 2

1. (1 pt) What package in Python is often used for plotting purposes? Give example of Python code that creates an $n \times m$ random matrix.
2. (2 pts)
   - What is the complexity of direct matrix-by-matrix multiplication?
   - Is it possible to reduce this complexity for general matrices? Give example of a faster algorithm.
3. (1 pt) What is the purpose of block algorithms in BLAS and LAPACK?
4. (1 pt) Give examples of standard vector norms. How can they help to define a matrix norm?
5. (2 pts) Find the third singular value $\sigma_3(A)$ of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Explain the answer.
6. (2 pts) Find $\text{cond}_2 \begin{bmatrix} \epsilon^2 & 0 \\ 0 & \epsilon \end{bmatrix}$, where $\epsilon \in \mathbb{R}$. Note: subscript 2 in cond_2 means that second norm is used.
7. (2 pts) Suppose that you want to calculate the best low-rank approximation of a certain matrix with relative precision ϵ. What should you do? How many parameters are in the rank-r approximation?
8. (2 pts) Is it a good idea to find eigenvalues of large matrices via characteristic polynomial? Why? What method for estimating the largest eigenvalue do you know?
9. (1 pt) What is lower triangular matrix? Let L be lower triangular. Specify what type of structure it has if L is a normal matrix.
10. (3 pts) Why does LU decompostion fail on the matrix $\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$ when ϵ is small enough? How can this problem be solved? Bonus: what is the exact value of ϵ when it starts to fail?
11. (2 pts) How to calculate QR decompostion via Gram-Shmidt procedure? Is it a good idea for a numerical algorithm? Name 2 approaches you know that are used in practice.
12. (1 pt) How to solve full eigenvalue problem via QR decomposition? What is the name of the algorithm?
13. (4 pts) The goal of compressed sensing is to find the sparsest solution x of an undetermined linear system $y = Ax$ where $A \in \mathbb{R}^{n \times m}$, $n < m$. In order to achieve it one could try to find solution which has minimal first norm. Intuition behind this fact is quite simple in 2D:
    - (1 pt) Draw disks $\|x\| = \text{const}$ in $1, 2$ and ∞ norms
    - (3 pts) Find graphically solutions of $y = Ax$, $\|x\|_* \to \min$, where $A \in \mathbb{R}^{1 \times 2}$ and $* = \{1, 2, \infty\}$. Which norm yields the sparsest solution?